Introduction to Dynamic Programming

ScPo Graduate Labor

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Introduction

- This lecture will introduce you to a powerful technique called *dynamic programming (DP)*.
- This set of slides is very similar to the one from your grad macro course. (same teacher.)
- We will repeat much of that material today.
- Then we will talk about a paper that uses DP to solve a dynamic lifecycle model.

References

- Before we start, some useful references on DP:
  1. Adda and Cooper (2003): Dynamic Economics.
  2. Ljungqvist and Sargent (2012) (LS): Recursive Macroeconomic Theory.
  3. Lucas and Stokey (1989): Recursive Methods in Economics Dynamics.
- They are ordered in increasing level of mathematical rigor. Adda and Cooper is a good overview, LS is rather short.

Cake Eating

- You are given a cake of size W_1 and need to decide how much of it to consume in each period $t = 1, 2, 3, \ldots$
- Cake consumption valued as $u(c)$, u is concave, increasing, differentiable and $\lim_{c \to 0} u'(c) = \infty$.
- Lifetime utility is

$$U = \sum_{t=1}^{T} \beta^{t-1} u(c_t), \beta \in [0, 1] \quad (1)$$

- Let's assume the cake does not depreciate/perish, s.t. the *law of motion* of cake is

$$W_{t+1} = W_t - c_t, t = 1, 2, ..., T \quad (2)$$

  i.e. cake in $t+1$ is this cake in t minus whatever you have of it in t.
- How to decide on the optimal consumption sequence $\{c_t\}_{t=1}^{T}$?

A sequential Problem

- This problem can be written as

$$
\begin{aligned}
v\left(W_1\right) &= \max_{\{W_{t+1},c_t\}_{t=1}^{T}} \sum_{t=1}^{T} \beta^{t-1} u(c_t) \\
& s.t. \\
W_{t+1} &= W_t - c_t \\
c_t, W_{t+1} &\geq 0 \text{ and } W_1 \text{ given.}
\end{aligned}
\tag{3}
$$

- Notice that the law of motion (2) implies that

$$
\begin{aligned}
W_1 &= W_2 + c_1 \\
&= \left(W_3 + c_2\right) + c_1 \\
&= \ldots \\
&= W_{T+1} + \sum_{t=1}^{T} c_t
\end{aligned}
\tag{4}
$$

Solving the sequential Problem

- Formulate and solve the Lagrangian for (3) with (4):

Solving the sequential Problem

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$$L = \sum_{t=1}^{T} \beta^{t-1} u(c_t) + \lambda \left[W_1 - W_{T+1} - \sum_{t=1}^{T} c_t \right] + \phi \left[W_{T+1} \right]$$

- First order conditions:

$$\frac{\partial L}{\partial c_t} = 0 \implies \beta^{t-1} u'(c_t) = \lambda \ \forall t \tag{5}$$

$$\frac{\partial L}{\partial W_{T+1}} = 0 \implies \lambda = \phi \tag{6}$$

  - ϕ is lagrange multiplier on non-negativity constraint for W_{T+1}.
  - we ignore the constraint $c_t \geq 0$ by the Inada assumption.

Interpreting the sequential solution

- From (5) we know that $\beta^{t-1}u'(c_t) = \lambda$ holds in each t.
- Therefore

$$\begin{aligned} \beta^{t-1}u'(c_t) &= \lambda \\ &= \beta^{(t+1)-1}u'(c_{t+t}) \end{aligned}$$

  i.e. we get the Euler Equation

$$u'(c_t) = \beta u'(c_{t+1}) \tag{7}$$

- along an optimal sequence $\{c_t^*\}_{t=1}^T$, each adjacent period $t, t+1$ must satisfy (7).
- If (7) holds, one cannot increase utility by moving some c_t to c_{t+1}.
- What about deviation from $\{c_t^*\}_{t=1}^T$ between t and $t+2$?

Is the Euler Equation enough?

- Is the Euler Equation sufficient for optimality?

Is the Euler Equation enough?

- Is the Euler Equation sufficient for optimality?
- No! We could satisfy (7), but have $W_T > c_T$, i.e. there is some cake left.
- What does this remind you of?
- Discuss how this relates to the value of multipliers λ, ϕ.
- Solution is given by initial condition (W_1), terminal condition $W_{T+1} = 0$ and path in EE.
- Call this solution the **value function**

$$v\left(W_1\right)$$

- $v\left(W_1\right)$ is the maximal utility flow over T periods given initial cake W_1.

The Dynamic Programming approach with $T = \infty$

- Let's consider the case $T = \infty$.
- In other words

$$\max_{\{W_{t+1},c_t\}_{t=1}^{\infty}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \tag{8}$$

$$s.t.$$

$$W_{t+1} = W_t - c_t \tag{9}$$

- Under some conditions, this can be written as

$$v(W_t) = \max_{c_t \in [0, W_t]} u(c_t) + \beta v(W_t - c_t) \tag{10}$$

- Some Definitions:
  - Call W the **state variable,**
  - and c the **control variable**.
  - (9) is the **law of motion** or **transition equation.**

The Dynamic Programming approach with $T = \infty$

- Note that t is irrelevant in (10). Only W matters.
- Substituting $c = W - W'$, where x' is next period's value of x

$$v(W) = \max_{W' \in [0,W]} u\left(W - W'\right) + \beta v(W') \tag{11}$$

- This is the **Bellman Equation** after Richard Bellman.
- It is a functional equation (v is on both sides!).
- Our problem has changed from finding $\{W_{t+1}, c_t\}_{t=1}^{\infty}$ to finding the function v.

This is called a fixed point problem:

Find a function v such that plugging in W on the RHS and doing the maximization, we end up with *the same* v on the LHS.

Value Function and Policy Function

- Great! We have reduced an infinite-length sequential problem to a one-dimensional maximization problem.
- But we have to find 2(!) unknown functions! Why two?
- The maximizer of the RHS of (11) is the **policy function**, $g(W) = c^*$.
- This function gives the optimal value of the control variable, given the state.
- It satisfies

$$v(W) = u(g(W)) + \beta v(g(W)) \tag{12}$$

(you can see that the $\max$ operator vanished, because $g(W)$ is the optimal choice)
- In practice, finding value and policy function is the one operation.

Using Dynamic Programming to solve the Cake problem

- Let's pretend that we knew v for now:

$$v(W) = \max_{W' \in [0,W]} u\left(W - W'\right) + \beta v(W')$$

- Assuming v is differentiable, the FOC wrt W'

$$u'(c) = \beta v'\left(W'\right) \tag{13}$$

- Taking the partial derivative w.r.t. the state W, we get the *envelope condition*

$$v'(W) = u'(c) \tag{14}$$

- This needs to hold in each period. Therefore

$$v'(W') = u'(c') \tag{15}$$

Using Dynamic Programming to solve the Cake problem

- Combining (13) with (15)

$$
\begin{aligned}
u'(c) &\overset{(13)}{=} \beta v'\left(W'\right) \\
&\overset{(15)}{=} \beta u'(c')
\end{aligned}
$$

  we obtain the usual euler equation.
- Any solution $v$will satisfy this necessary condition, as in the sequential case.

Using Dynamic Programming to solve the Cake problem

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  we obtain the usual euler equation.
- Any solution $v$will satisfy this necessary condition, as in the sequential case.
- So far, so good. But we still don't know v!

Finding v

- Finding the Bellman equation v and associated policy function g is not easy.
- In general, it is impossible to find an analytic expression, i.e. to do it by hand.
- Most of times you will use a computer to solve for it.
- **preview:** The rationale for why we can find it has to do with the fixed point nature of the problem. We will see that under some conditions we can **always** find that fixed point.
- We will look at a particular example now, that we can solve by hand.

Finding v: an example with closed form solution

- Let's assume that $u(c) = \ln c$ in (11).
- Also, let's conjecture that the value function has the form

$$v(W) = A + B \ln W \tag{16}$$

- We have to find A, B such that (16) satisfies (11).
- Plug into (11):

$$A + B \ln W = \max_{W'} \ln\left(W - W'\right) + \beta\left(A + B \ln W'\right) \tag{17}$$

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- FOC wrt W':

$$\begin{aligned} \frac{1}{W - W'} &= \frac{\beta B}{W'} \\ W' &= \beta B (W - W') \\ W' &= \frac{\beta B}{1 + \beta B} W \\ &\equiv g(W) \end{aligned}$$

Finding v: an example with closed form solution

- Let's use this policy function in (17):

$$\begin{aligned} v(W) &= \ln\left(W - g(W)\right) + \beta\left(A + B\ln g(W)\right) \\ &= \ln\frac{W}{1+\beta B} + \beta\left(A + B\ln\left[\frac{\beta B}{1+\beta B}W\right]\right) \end{aligned}$$

- Now we collect all terms $\ln W$ on the RHS, and put all else into the constant A:

$$\begin{aligned} v(W) &= A + \ln W + \beta B \ln W \\ &= A + (1+\beta B)\ln W \end{aligned}$$

- We conjectured that $v(W) = A + B\ln W$. Hence

$$\begin{aligned} B &= (1+\beta B) \\ B &= \frac{1}{1-\beta} \end{aligned}$$

- Policy function: $g(W) = \beta W$

The Guess-and-Verify method

- Note that we guessed a functional form for v.
- And then we verified that it consitutues a solution to the functional equation.
- This method (guess and verify) would in principle always work, but it's not very practical.

Solving the Cake problem with $T < \infty$

- When time is finite, solving this DP is fairly simple.
- If we know the value in the final period, we can simply go backwards in time.
- In period T there is no point setting $W' > 0$. Therefore

$$v_T(W) = u(W) \tag{18}$$

- Notice that we index the value function with time in this case:
  - it's not the same to have W in period 1 as it is to have W in period T. Right?
- But if we know v_T for all values of W, we can construct v_{T-1}!

Backward Induction and the Cake Problem

- We know that

$$
\begin{aligned}
v_{T-1}\left(W_{T-1}\right) &= \max_{W_T \in [0, W_{T-1}]} u\left(W_{T-1} - W_T\right) + \beta v_T\left(W_T\right) \\
&= \max_{W_T \in [0, W_{T-1}]} u\left(W_{T-1} - W_T\right) + \beta u\left(W_T\right) \\
&= \max_{W_T \in [0, W_{T-1}]} \ln\left(W_{T-1} - W_T\right) + \beta \ln W_T
\end{aligned}
$$

- FOC wrt W_T:

$$
\begin{aligned}
\frac{1}{W_{T-1} - W_T} &= \frac{\beta}{W_T} \\
W_T &= \frac{\beta}{1+\beta} W_{T-1}
\end{aligned}
$$

- Thus the value function in $T-1$ is

$$
v_{T-1}\left(W_{T-1}\right) = \ln\left(\frac{W_{T-1}}{\beta}\right) + \beta \ln\left(\frac{\beta}{1+\beta} W_{T-1}\right)
$$

Backward Induction and the Cake Problem

- Correspondingly, in $T-2$:

$$\begin{aligned} v_{T-2}\left(W_{T-2}\right) &= \max_{W_{T-1}\in[0,W_{T-2}]} u\left(W_{T-2}-W_{T-1}\right)+\beta v_{T-1}\left(W_{T-1}\right) \\ &= \max_{W_{T-1}\in[0,W_{T-2}]} u\left(W_{T-2}-W_{T-1}\right) \\ &+ \beta\left[\ln\left(\frac{W_{T-1}}{\beta}\right)+\beta\ln\left(\frac{\beta}{1+\beta}W_{T-1}\right)\right] \end{aligned}$$

- FOC wrt W_{T-2}.
- and so on until $t=1$.
- Again, without log utility, this quickly get intractable. But your computer would proceed in the same backwards iterating fashion.
- Notice that with T finite, there is *no fixed point problem* if we do backwards induction.

Dynamic Programming Theory

- Let's go back to the infinite horizon problem.
- Let's define a general DP as follows.
- Payoffs over time are

$$U = \sum_{t=1}^{\infty} \beta^t \tilde{u}(s_t, c_t)$$

  where $\beta < 1$ is a discount factor, s_t is the state, c_t is the control.
- The state (vector) evolves as $s_{t+1} = h(s_t, c_t)$.
- All past decisions are contained in s_t.

DP Theory: more assumptions

- Let $c_t \in C(s_t), s_t \in S$ and assume $\tilde{u}$ is bounded in $(c, s) \in C \times S$.
- Stationarity: neither payoff $\tilde{u}$ nor transition h depend on time.
- Modify $\tilde{u}$ to u s.t. in terms of s' (as in cake: $c = W - W'$):

$$v(s) = \max_{s' \in \Gamma(s)} u(s, s') + \beta v(s') \tag{19}$$

- $\Gamma(s)$ is the constraint set (or feasible set) for s' when the current state is s:
    - before that was $\Gamma(W) = [0, W]$
- We will work towards one possible set of sufficient conditions for the existence to the functional equation. Please consult Stokey and Lucas for greater detail.

Proof of Existence

Theorem

Assume that $u(s, s')$ is real-valued, continuous, and bounded, that $\beta \in (0, 1)$, and that the constraint set $\Gamma(s)$ is nonempty, compact, and continuous. Then there exists a unique function $v(s)$ that solves (19).

Proof.

Stokey and Lucas (1989, theoreom 4.6). □

The Bellman Operator $T(W)$

- Define an *operator* T*on function* W *as* $T(W)$*:*

$$T(W)(s) = \max_{s' \in \Gamma(s)} u(s, s') + \beta W(s') \tag{20}$$

- The Bellman operator takes a guess of current value function W, performs the maximization, and returns the next value function.
- Any $v(s) = T(v)(s)$ is a solution to (19).
- So we need to find a fixed point of $T(W)$.
- This argument proceeds by showing that $T(W)$ is a **contraction**.
- Info: This relies on the *Banach (or contraction) mapping theorem*.
- There are two sufficiency conditions we can check: Monotonicity, and Discounting.

The Blackwell (1965) sufficiency conditions: Monotonicity

- Need to check Monotonicity and Discounting of the operator $T(W)$.
- **Monotonicity** means that

$$W(s) \geq Q(s) \quad \Longrightarrow \quad T(W)(s) \geq T(Q)(s), \forall s$$

- Let $\phi_Q(s)$ be the policy function of

$$Q(s) = \max_{s' \in \Gamma(s)} u(s, s') + \beta Q(s')$$

and assume $W(s) \geq Q(s)$. Then

$$\begin{aligned} T(W)(s) &= \max_{s' \in \Gamma(s)} u(s, s') + \beta W(s') \geq u(s, \phi_Q(s)) + \beta W(\phi_Q(s)) \\ &\geq u(s, \phi_Q(s)) + \beta Q(\phi_Q(s)) \equiv T(Q)(s) \end{aligned}$$

Show example with $W(s) = \log(s^2), Q(s) = \log(s), s > 0$

The Blackwell (1965) sufficiency conditions: Discounting

- Adding constant a to W leads $T(W)$ to increase less than a.
- In other words

$$T(W+a)(s) \leq T(W)(s) + \beta a, \beta \in [0,1)$$

- *discounting* because $\beta < 1$.
- To verify on the Bellman operator:

$$T(W+a)(s) = \max_{s' \in \Gamma(s)} u(s,s') + \beta \left[W(s') + a \right] = T(W)(s) + \beta a$$

- Intuition: the discounting property is key for a **contraction**.
- In successive iterations on $T(W)$ we add only a fraction β of W.

Contraction Mapping Theorem (CMT)

- The CMT tells us that for a function of type $T(\cdot)$
  1. There is a unique fixed point. (from previous Stokey-Lucas proof.)
  2. This fixed point can be reached by iterating on T in (20) **using an arbitrary starting point.**
- Very useful to find a solution to (19):
  1. Start with an initial guess $V_0(s)$.
  2. Apply the Bellman operator to get $V_1 = T(V_0)$
     1. if $V_1(s) = V_0(s)$ we have a solution, done.
     2. if not, continue:
  3. Apply the Bellman operator to get $V_2 = T(V_1)$
  4. etc until $T(V) = V$.
- Again: if $T(V)$ is a contraction, this **will** converge.
- This technique is called **value function iteration.**

Value Function inherits Properties of u

Theorem

*Assume $\underline{u(s,s')}$ is real-values, continuous, **concave** and bounded, $0 < \beta < 1$, that S is a convex subset of $\mathbb{R}^k$ and that the constraint set $\Gamma(s)$ is non-empty, compact-valued, convex, and continuous. Then the unique solution to (19) is **strictly concave.** Furthermore, the policy $\phi(s)$ is a continuous, single-valued function.*

Proof.

See theorem 4.8 in Stokey and Lucas (1989). $\square$

Value Function inherits Properties of u

- proof shows that if V is concave, so is $T(V)$.
- Given $u(s, s')$ is concave, let the initial guess be

$$V_0(s) = \max_{s' \in \Gamma(s)} u(s, s')$$

  and therefore $V_0(s)$ is concave.
- Since T preserves concavity, $V_1 = T(V_0)$ is concave etc.

Stochastic Dynamic Programming

- There are several ways to include uncertainty into this framework - here is one:
- Let's assume the existence of a variable ϵ, representing a *shock*.
- Assumptions:
  1. ϵ_t affects the agent's payoff in period t.
  2. ϵ_t is exogenous: the agent cannot influence it.
  3. ϵ_t depends only on ϵ_{t-1} (and not on ϵ_{t-2}. although we could add ϵ_{t-1} as a state variable!)
  4. The distribution of $\epsilon' | \epsilon$ is time-invariant.
- Defined in this way, we call ϵ a *first-order Markov process*.

The Markov Property

Definition

A stochastic process $\{x_t\}$ is said to have the *Markov property* if for all $k \geq 1$ and all t,

$$\Pr\left(x_{t+1} | x_t, x_{t-1}, \ldots, x_{t-k}\right) = \Pr\left(x_{t+1} | x_t\right).$$

We assume that $\{\epsilon_t\}$ has this property, and characterize it by a *Markov Chain*.

Markov Chains

Definition

A time-invariant n**-State Markov Chain** consists of:

1. n vectors of size $(n, 1)$: $e_i, i = 1, \ldots, n$ such that the i-th entry of e_i is one and all others zero,
2. one (n, n) **transition matrix** P, giving the probability of moving from state i to state j, and
3. a vector $\pi_{0i} = \Pr(x_0 = e_i)$ holding the probability of being in state i at time 0.

- $e_1 = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix}', e_2 = \begin{bmatrix} 0 & 1 & \ldots & 0 \end{bmatrix}', \ldots$ are just a way of saying "x is in state i".
- The elements of P are

$$P_{ij} = \Pr\left(x_{t+1} = e_j | x_t = e_i\right)$$

Assumptions on P and π_0

1. For $i = 1, \ldots, n$, the matrix P satisfies

$$\sum_{j=1}^{n} P_{ij} = 1$$

2. The vector π_0 satisfies

$$\sum_{i=1}^{n} \pi_{0i} = 1$$

- In other words, P is a *stochastic matrix*, where each row sums to one:
  - row i has the probabilities to move to any possible state j. A valid probability distribution must sum to one.
- P defines the probabilities of moving from current state i to future state j.
- π_0 is a valid initial probability distribution.

Transition over two periods

- The probability to move from i to j over two periods is given by P^2_{ij}.
- Why:

$$\begin{aligned} \Pr\left(x_{t+2} = e_j | x_t = e_i\right) &= \\ \sum_{h=1}^{n} \Pr\left(x_{t+2} = e_j | x_{t+1} = e_h\right) \Pr\left(x_{t+1} = e_h | x_{t+1} = e_i\right) &= \\ \sum_{h=1}^{n} P_{ih} P_{hj} &= P^{(2)}_{ij} \end{aligned}$$

- Show 3-State example to illustrate this.

Conditional Expectation from a Markov Chain

- What is expected value of x_{t+1} given $x_t = e_i$?
- Simple:

$$
\begin{aligned}
E\left[x_{t+1} | x_t = e_i\right] &= \text{values of x} \times \text{Prob of those values} \\
&= \sum_{j=1}^{n} e_j \times \Pr\left(x_{t+1} = e_j | e_i\right) \\
&= \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} (P_i)'
\end{aligned}
$$

where P_i is the i-th row of P, and $(P_i)'$ is the transpose of that row (i.e. a column vector).

- What is the conditional expectation of a function $f(x)$, i.e. what is

$$E\left[f(x_{t+1}) | x_t = e_i\right]?$$

Back to Stochastic DP

- With the Markovian setup, we can rewrite (19) as

$$v(s,\epsilon) = \max_{s' \in \Gamma(s,\epsilon)} u(s,s',\epsilon) + \beta E\left[v(s',\epsilon')|\epsilon\right] \tag{21}$$

Theorem

If $u(s,s',\epsilon)$ is real-valued, continuous, concave, and bounded, if $\beta \in (0,1)$, and constraint set is compact and convex, then

1. *there exists a unique value function $v(s,\epsilon)$ that solves (21).*
2. *there exists a stationary policy function $\phi(s,\epsilon)$.*

Proof.

This is a direct application of Blackwell's sufficiency conditions:

1. with $\beta < 1$ discounting holds for the operator on (21).
2. Monotonicity can be established as before.

Optimality in the Stochastic DP

- As before, we can derive the first order conditions on (21):

$$u_{s'}(s, s', \epsilon) + \beta E\left[V_{s'}(s', \epsilon')|\epsilon\right] = 0$$

- differentiating (21) w.r.t. s to find $V_{s'}(s', \epsilon')$ we find

$$u_{s'}(s, s', \epsilon) + \beta E\left[u_{s'}(s', s'', \epsilon')|\epsilon\right] = 0$$

DP Application 1: The Deterministic Growth Model

- We will now solve the deterministic growth model with dynamic programming.
- Remember:

$$V(k) = \max_{c=f(k)-k' \geq 0} u(c) + \beta V(k') \tag{22}$$

- Assume $f(k) = k^{\alpha}, u(c) = \ln c$.
- We will use *discrete state DP*. We cannot hope to know V at all $k \in \mathbb{R}_+$. Therefore we compute V at a finite set of points, called a *grid*.
- Hence, we must also choose those grid points.

DP Application: Discretize state and solution space

- There are many ways to approach this problem:

$$V(k) = \max_{k' \in [0,k^\alpha]} \ln(k^\alpha - k') + \beta V(k') \tag{23}$$

- Probably the easiest goes like this:
  1. Discretize V onto a grid of n points $\mathcal{K} \equiv \{k_1, k_2, \ldots, k_n\}$.
  2. Discretize control k': change $\max_{k' \in [0,k^\alpha]}$ to $\max_{k' \in \mathcal{K}}$, i.e. choose k' from the discrete grid.
  3. Guess an initial function $V_0(k)$.
  4. Iterate on (23) until $d\left(V_{t+1} - V_t\right) < \varepsilon$, where $d()$ is a measure of distance, and $\varepsilon > 0$ is a tolerance level chosen by you.

Application to Labor Supply: French (2005)

- How does labor supply relate to the **Social Security** and **Pension systems**?
- How does health enter this picture?
- We need a $\boxed{\text{dynamic}}$ model here: you will work differently if you expect to get a pension at age 60.

French (2005) Structure

- People in the model can save (not borrow).
- There is random income each period.
- People probabilistically age (i.e. if they don't die, they get a year older)
- Health evolves over this lifecycle in a state-dependent fashion.

Policy Simulations

- Once estimated, use the model for policy experiments
- Change retirement age from 62 to 63.
- Reduce Social Security (SS) Benefits by 20 %
- Eliminate tax wage from SS meanstest.

Model

- Choose consumption and hours C_t, H_t as well as whether to apply for SS, $B_t \in \{0, 1\}$

- Health status is M_t, which enters utility:

$$U(C_t, H_t, M_t)$$

- There is a **bequest function** $b(A_T)$, where A_t are assets.

- Denote s_t the probability of being alive at t, given alive at $t-1$.

- Hence, $S(j,t) = \frac{1}{s_t}\Pi_{k=t}^{j} s_k$ is prob of living through j, given alive at t

- You die for sure in $T+1$, hence $s_{T+1} = 0$.

Model

- At time t, lifetime utility is

$$U(C_t, H_t, M_t) + \mathbb{E}_t \left[\sum_{j=t+1}^{T+1} \beta^j S(j-1,t)(s_j U(C_j, H_j, M_j) + (1-s_j) b(A_j) \right] \quad (24)$$

- Applying for SS sets $B_t = 1$
- The $\boxed{\text{problem of the consumer}}$ is to choose a plan $\{C_j, H_j, B_j\}_{t=j}^{T+1}$ that maximizes (24), subject to
  1. mortality determination
  2. health determination
  3. wage determination
  4. spousal income
  5. and a budget constraint.

Utility Function

Within period utility is

$$U(C_t, H_t, M_t) = \frac{1}{1-\nu}\left(C_t^{\gamma}\left(L - H_t - \theta_P P_t - \phi \mathbf{1}\{M = \text{bad}\}\right)^{1-\gamma}\right)^{1-\nu} \tag{25}$$

where

1. leisure is $L - H_t - \theta_P P_t - \phi \mathbf{1}\{M = \text{bad}\}$
2. Health is good/bad when $M = \{0, 1\}$.
3. If participate in labor force, $P_t = 1$, and pay fixed cost θ_P
4. Retirement: set $H = 0$. Can re-enter labor at any time.

Role of Utility

The distribution of annual hours of work is clustered around both 2000 and 0 hours of work, a regularity in the data that standard utility functions have a difficult time replicating. Fixed costs of work are a common way of explaining this regularityin the data (Cogan, 1981). Fixed costs of work generate a reservation wage for a given marginal utility of wealth. Below the reservation wage, hours worked is zero. Slightly above the reservation wage, hours worked may be large. Individual level labour supplyi s highly responsive aroundthis reservation wage level although wage increases above the reservation wage result in a smaller labour supply response.

Bequest, Health and Survival

- The bequest function is

$$b(A_t) = \theta_B \frac{(A_t + K)^{(1-\nu)\gamma}}{1-\nu}$$

- Survival is a function of age and health

$$s_{t+1} = s(M_t, t+1)$$

- Health evolves according to a markov chain

$$\pi_{\text{good,bad},t+1} = \Pr(M_{t+1} = \text{good} | M_t = \text{bad}, t+1)$$

Wages

- Wages are given by

$$\ln W_t = \alpha \ln H_t + W(M_t, t) + AR_t \tag{26}$$

- $W(\cdot)$ is an age profile, AR an autoregressive component.

$$AR_t = \rho AR_{t-1} + \eta_t, \eta \sim N(0, \sigma_\eta^2)$$

- Spousal income is

$$ys_t = ys(W_t, t) \tag{27}$$

Budget constraint

The budget constraint is

$$A_{t+1} = A_t + Y(x_t, \tau) + B_t \times ss_t - C_t \geq 0 \tag{28}$$

with the function Y measuring post tax income, r interest, pb pension, ss social security, and

$$x_t = rA_t + W_t H_t + ys_t + pb_t + \epsilon_t \tag{29}$$

- Social Security available only after age 62. It lasts till death.
- Similarly, pension benefits are paid out after age 62.

Social Security and Retiring at 65

- SS benefits depend on Averaged Indexed Monthly Earnings, **AIME**. This is average earnings in the 35 highest-paying years.
- Incentives to start SS at age 65: from 62-65, every early year reduces benefits by 6.7%. From 65-70, every additional work year increases benefits only by 3%.
- Under 70 year-olds who work while on SS are taxed heavily. Earnings above 6000$ per year are taxed at 50% (plus federal, state and payroll tax).

Pensions

- Pensions are similar. Pension wealth is typically illiquid before a certain age.
- Also a function of AIME.
- Focus on **defined benefit pension** plans. The pension formula is *defined and known* in advance. The pension does not depend on stock market returns, for example.

Model Solution

- The state vector is $X_t = (A_t, W_t, B_t, M_t, AIME_t)$
- Preferences are $\theta = (\gamma, \nu, \theta_P, \theta_B, \phi, L, \beta)$
- The value function solves

$$V_t(X_t) = \max_{C_t,H_t,B_t} \{U(C_t, H_t, M_t)$$
$$+\beta s_{t+1} \sum_{M \in \text{good,bad}} \int V_{t+1}(X_{t+1}) dF(W_{t+1}|M_{t+1}, W_t, t) \pi(M_{t+1}|M_t, t)$$
$$+\beta(1 - s_{t+1} b(A_{t+1}))\} \tag{30}$$

- Backward induction over grids $\mathcal{X}$ and $\mathcal{C}, \mathcal{H}$

Estimation: 2 Steps

Estimation proceeds in two steps:

1. Estimate laws of motion for exogenous state variables outside of the model, and calibrate others to reasonable values.

2. Use those processes as given in the model, and find a set of preference parameters that generates model output as close as possible to observed data.

Simulated Method of Moments as Extremum Estimator

Definition: Extremum Estimator

An estimator $\hat{\theta}$ of dimension p is called and **extremum estimator** if there is a scalar objective function $Q_n : \mathbb{R}^p \mapsto \mathbb{R}$ and a parameter space $\Theta \subset \mathbb{R}^p$ such that

$$\hat{\boldsymbol{\theta}} = \arg\max_{\boldsymbol{\theta} \in \Theta} Q_n(\boldsymbol{\theta})$$

- OLS
- GMM/SMM
- IV
- MLE

GMM moment function

The $\boxed{\text{Generalized Method of Moments}}$ relies on a set of K moment conditions which provide limited information about a parameter θ, given a set of data w_i:

$$\mathbb{E}[g(w_i; \underbrace{\theta}_{p\times 1})] = \underbrace{\mathbf{0}}_{K\times 1} \tag{31}$$

This has **sample analog**:

$$\boldsymbol{g_n(\theta)} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{g}(w_i; \theta) \tag{32}$$

Can we solve (31)?

GMM Estimator

Definition: GMM Estimator

The **GMM estimator** $\hat{\theta}(\hat{W})$ of θ satisfying

$$\mathbb{E}[g(w_i; \underbrace{\theta}_{p\times 1})] = \underbrace{\mathbf{0}}_{K\times 1}$$

with $\boldsymbol{\theta} \in \Delta \subset \mathbb{R}^p$ is defined as

$$\hat{\boldsymbol{\theta}}(\hat{W}) = \arg\min_{\boldsymbol{\theta}\in\Delta} n g_n(\boldsymbol{\theta})' \hat{W} g_n(\boldsymbol{\theta}) \tag{33}$$

where $\hat{W}$ is a $K \times K$ weighting matrix that may depend on the data and is symmetric and p.d.

GMM $\rightarrow$ SMM

- $\boxed{\text{Simulated Method of Moments}}$ uses simulated data $\tilde{w}_i(\theta)$ instead of w_i.
- Notice the main difference: the (simulated!) data now depend on the model parameters θ.
- The problem is still to choose θ such that (32) is a close to zero as possible. We only modify (32) slightly to read

$$\boldsymbol{g}_n(\boldsymbol{\theta}) = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{g}(\tilde{w}_i(\theta)) \tag{34}$$

- Making this change, the estimator is still defined by (33)

SMM Practicalities

- Notice that obtaining a sample of simulated data $\{\tilde{w}_i(\theta)\}_{i=1}^{N}$ involves

  1. Solving the dynamic program (30), $V_t(X_t, \theta)$
  2. Using the optimal policies, simulating a panel of data.

  where step 1 usually is costly in terms of computation time.

- The precise form of the moment function g is, to some extent, at the researcher's discretion.

- **Which moments to include in g** is a similar question to **which variables to include in my regression**.

French (2005): Moment Function g

The moments the model must match are:

1. Median assetes
2. Mean assets
3. Mean participation conditional on health
4. Mean hours conditional on health

See Equations (14)-(17) in the paper for how those are computed.

Estimating Lifecycle Profiles from Data

- If t indexes calendar time and i individuals, define a generic **lifecycle profile** of variable x by

$$\mathbb{E}[x_{it}|\text{age}_i] \tag{35}$$

- Written as in (35), this contains individual fixed effects, year effects and family size effects. A complicated object to interpret.
- Probably better would be

$$\mathbb{E}[x_{it}|\text{age}_i, t, i, M_{it}, \text{famsize}_{it}]$$

Question: How is this conditional expectation identified?

Estimating Lifecycle Profiles from Data

- To address those issues, French proposes

$$Z_{it} = f_i + \sum_{m \in \text{good,bad}} \sum_{k=1}^{T} \Pi_{mk} \mathbf{1}[\text{age}_{it} = k | m]$$
$$+ \sum_{j=1}^{J} \mathbf{1}[\text{famsize}_{it} = j] + \Pi_U U_t + u_{it}$$

  where Z stands for assets, hours, participation or wages.

- The required age profiles are the sequences of paramters $\sum_{k=1}^{T} \Pi_{mk}$.

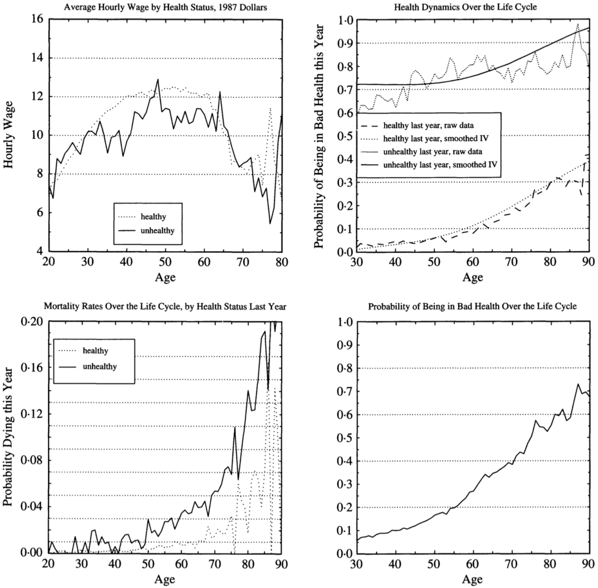
Selected Wage Data

- A problem with this is that wage data is only observed for workers.
- **Classical Question**: What about the wage of non-workers? This is really like in Heckman (1977).
- If wage growth were the same for workers and non-workers, this would not be a problem – the fixed effect estimator identifies wage growth.
- French (2005) proposes to compare selection bias in real and simulated data in an iterative proceedure.

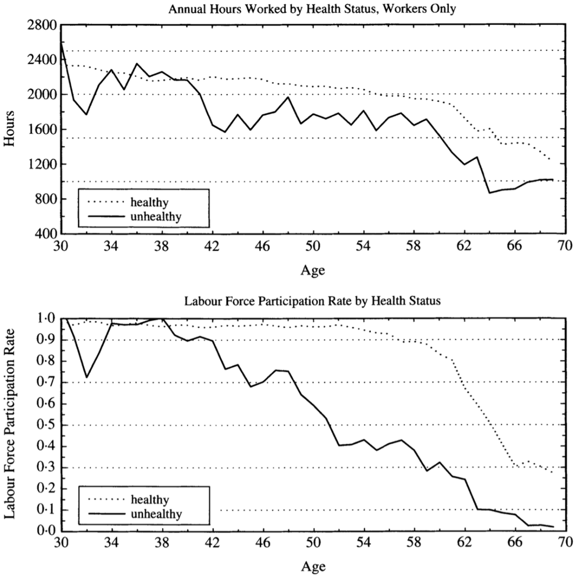
Data

- Panel Study of Income Dynamics (PSID) 1968–1997.
- Rich income data, assets in several waves, labor supply, and also health data are included.
- PSID is a great dataset to work with.
- `https://cran.r-project.org/web/packages/psidR/index.html` is a great package to build it.

Results: Exogenous profiles 1



Results: Exogenous profiles 2



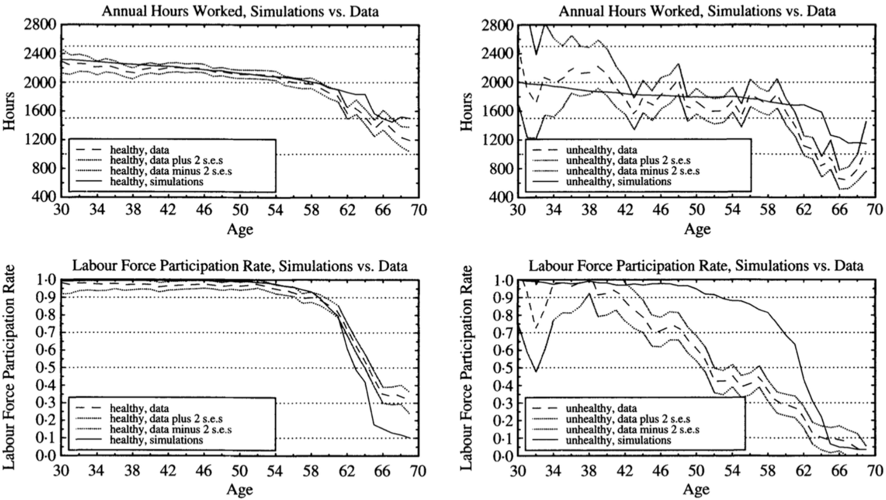
Results: Parameter Estimates

Parameter and definition	Specification (1)	(2)	(3)	(4)
γ Consumption weight	0·578 (0·003)	0·602 (0·003)	0·533 (0·003)	0· 615 (0·004)
ν Coefficient of relative risk aversion, utility	3·34 (0·07)	3·78 (0·07)	3·19 (0·05)	7·69 (0·15)
β Time discount factor	0·992 (0·002)	0·985 (0·002)	0·981 (0·001)	1·04 (0·004)
L Leisure endowment	4466 (30)	4889 (32)	3900 (24)	3399 (28)
ϕ Hours of leisure lost, bad health	318 (9)	191 (7)	196 (8)	202 (6)
θ_P Fixed cost of work, in hours	1313 (14)	1292 (15)	335 (7)	240 (6)
θ_B Bequest weight	1·69 (0·05)	2·58 (0·07)	1·70 (0·04)	0·037 (0·001)
χ^2 Statistic: (233 degrees of freedom)	856	880	830	1036
$\epsilon_{h,W}(40)$ Labour supply elasticity, age 40	0·37	0·37	0·35	0·19
$\epsilon_{h,W}(60)$ Labour supply elasticity, age 60	1·24	1·33	1·10	1·04
Reservation hours level, age 62	885	916	1072	1051
Coefficient of relative risk aversion	2·35	2·68	2·17	5·11

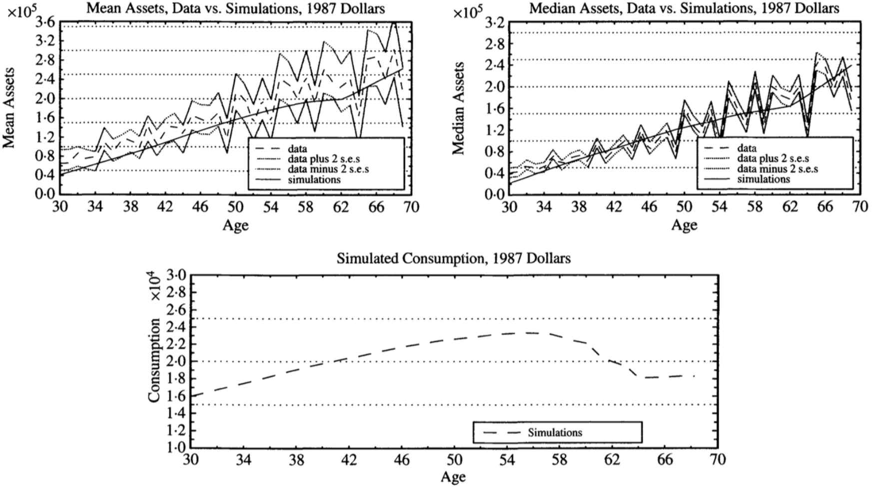
Standard errors in parentheses

Specifications described below:
(1) Does not account for selection or tied wage-hours offers
(2) Accounts for selection but not tied wage-hours offers
(3) Accounts for tied wage-hours offers but not selection
(4) Accounts for selection and tied wage-hours offers

Results: Model Fit 1



Results: Model Fit 2



Experiments

Baseline (1987 policy environment) vs

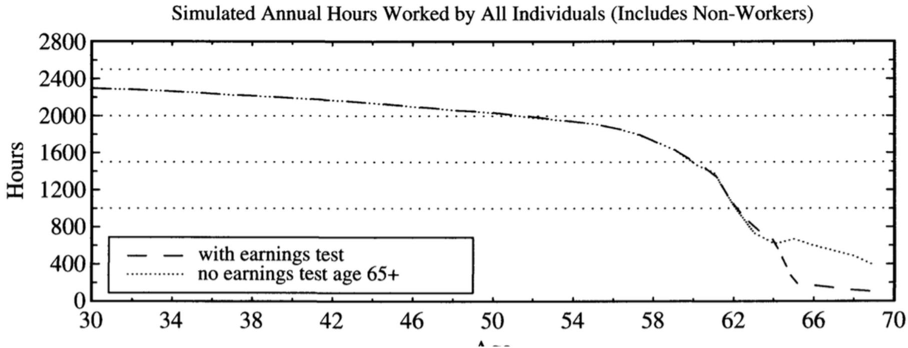
1. Redues SS benefits by 20%
2. Increase early retirement age from 62 to 63.
3. Eliminate SS earnings test for over 65 year olds.
4. Earnings test was in fact abolished in 2000 - look at model predictions.

Results: Policy Experiments

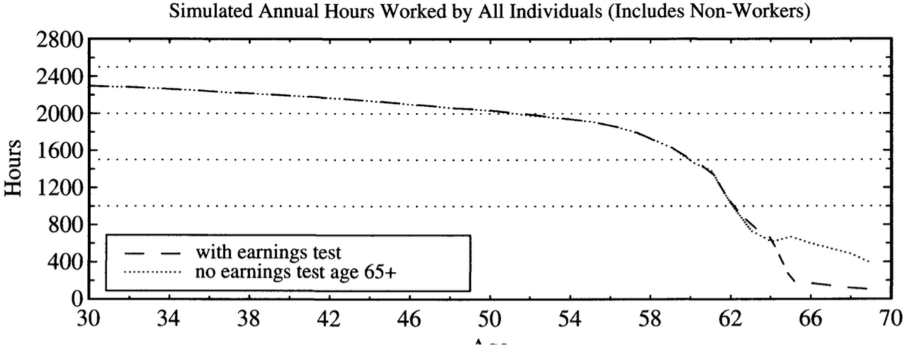
	Years worked	Hours worked per year	PDV of labour income ($)	PDV of consumption ($)	Assets at age 62 ($)	
With borrowing constraints						
1987 policies	32·60	2097	1781	1583	190	
Reduce benefits action after 62	32·83	$\boxed{2099}$	1789	1569	200	
Reduce benefits, reduce taxes	33·00	2115	1803	1586	203	no income effect
Shift early retirement age to 63	32·62	2096	1781	1584	190	borrow. constr. ?
Eliminate earnings test, age 65+	$\boxed{33·62}$	2085	1799	1594	188	subst. effect
Without borrowing constraints						
1987 policies	32·39	2067	1764	1603	158	
Reduce benefits	32·58	2063	1770	1587	168	
Reduce benefits, reduce taxes	32·68	2078	1781	1602	170	
Shift early retirement age to 63	32·39	2067	1764	1603	158	
Eliminate earnings test, age 65+	33·46	2063	1784	1616	154	

PDV stands for present discounted value.
Consumption, labour income, and assets are measured in thousands.

Results: Model Fit 2



Results: Model Fit 2



- Removing earnings test increases lifetime wealth.
- Leisure being normal good, this increases demand for C as well as L
- Hence participate for more years.

Conclusions

- French (2005) constructs an estimable lifecycle model with labour supply, retirement, savings and social security.
- People would spend 3 months more in work if SS were decreased by 30%.
- The sharp drop in participation at age 65 is explained by actuarial unfairness of pension and SS around that age.

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