Firm Modelling

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0.1 Introduction

The goal of this study is constructing a model which simulates the behavior of a company and especially its behavior towards the distribution of its revenue between investment and dividends.

Chapter 1

The Model

1.1 Definitions :

Let us consider a firm. Every T days, that firm will have to decide what to do with their cash flow. X_n is that specific flow at time $t = T \times n$.
The company has to allocate this money between investment I_n (how much money will be integrated in the firm's capital) and dividends D_n (how much money will be distributed between the shareholders).
To sum up we have the equation : $X_n = I_n + D_n$.
We have to add the fact that dividends can't be negative, so :
$X_n < 0 \implies (D_n = 0)$ & $(I_n = X_n < 0)$.
The capital of the company, which we'll be considering being the value of all its possessions. Then, we will make the hypothesis that this capital is easily transferable into money, so that if the cash flow X_n is negative the firm has no special fee to sell part of its capital in order to "pay this cash flow".
With time, the capital value diminishes with the given rate δ.

We can sum up all this information : $C_{n+1} = (1 - \delta) \times C_n + I_n$.

1.2 Modeling the cash flow :

First of all, the cash flow of a period (T days) is a sum of T daily cash flows $x_{n,j}$ (où $j \in \mathbb{N} \cap [1; T]$) and $X_n = \sum_{j=1}^{T} x_{n;j}$.
We will consider that for a fixed n, the $x_{n,j}$ are i.i.d. and their expected value is a constant, called μ_n, for those T days. The variance of this law represents the risk taken by the firm and we will call it σ^2 and it will be a general constant.

$$\begin{cases} \mathbb{E}[Xn] = \mu_n \times T \\ Var(X_n) = \sigma^2 T \end{cases}$$

Example : A simple model would be : $X_n \sim \mathcal{N}(\mu_n T, \sigma^2 T)$

μ_n should reflect the effectiveness of the firm. More a company invests, more the expected value of the cash flow is high. That's why we will consider the following sequence : $\mu_n = \mu_{n-1} \times \frac{I_{n-1}}{I_{n-2}}$.

A simple result is this relation :

$$\mu_n = \mu_0 \times \frac{I_{n-1}}{I_0}$$

Remark : μ_0 will be a constant defined initially whereas I_0 wil be determined later.

1.3 The priorities :

The first priority of the company is to invest at least what I will call the objective (or objective of investment) O_n. The objective is define so that if the past investment is superior to its objective, the expected value of the future capital will be higher than the previous capital. I call the rest the profit of the company P_n. So :

$$I_{n-1} > O_{n-1} \implies \mathbb{E}[C_{n+1}|C_n] > C_n$$
$$P_n = (X_n - O_n)^+$$
$$X_n < O_n \implies (P_n = 0) \;\&\; (I_n = X_n)$$

Remark : 1) the gap in the subscript in the first equation will be justified later.
2) $A^+ = Max(A, 0)$.
3) $A^- = Max(-A, 0)$.
4) $A = A^+ - A^-$.

The second priority is to give a dividend whose minimal value will be fixed at the constant D_0. So, we have the relation :

$$O_n < X_n < O_n + D_0 \implies (I_n = O_n) \;\&\; (D_n = I_n - X_n)$$

1.4 The allocation :

After respecting the previously described priorities, the firm has to decide what to do of the rest of its profit : how much is going to the shareholders D_n and how much is going in the capital as an investment bonus ΔI_n. We will use a utility function defined like this :

$$U(D, \Delta I) = \frac{1}{\alpha} log(D - D_0) + log(\Delta I)$$

Remark : α is a parameter representing how important for them it is to invest rather than giving their money away as dividends. Let us call α the allocation parameter.

As a result we can obtain those different equations :

$$\begin{cases} I_n = O_n + \Delta I_n \\ X_n = I_n + D_n \\ P_n = D_n + \Delta I_n \\ D_n = \frac{\alpha D_0 + P_n}{1+\alpha} \end{cases}$$

Remarks : 1) The last equation is only valid if $P_n > D_0$.
2) $\alpha \to 0 \implies D_n \to P_n$.
3) $\alpha \to +\infty \implies D_n \to D_0$.

1.5 Setting the objective :

Initialization : We will suppose that before our model started, the firm was in a stationary state where $X_0 = \mu_0 T$, $C = C_0$, $I_0 = O_0$ and $D = D_0$.
In order to have such a state we need to choose a certain δ :

$$C_0 = (1-\delta)C_0 + I_0 \iff \delta = \frac{I_0}{C_0} \tag{1.1}$$

Remark : With those conditions : $I_0 = O_0 = \mu_0 T - D_0$.

Conditions for developement :

$$\begin{aligned} & C_{n+1} > C_n \\ & \iff (1-\delta) \times C_n + I_n > C_n \\ & \iff -\delta \times C_n + X_n - D_n > 0 \\ & \iff D_n < X_n - \delta C_n \end{aligned}$$

Here we have a limitation in the dividends allowed that we always need to respect. But dividends can't be negative and are chosen after X_n is determined, so the last equation to verify is : $X_n > \delta C_n$.
We can observe that at $t = n$, we can't really have an effect on this equation as C_n and the parameters of X_n are already established at the end of $t = n-1$. So, we will try to see what should a company do at $t = n-1$. As X_n is a random variable, what we can do is replace X_n by its expected value, so that, at least the risk-free company ($\sigma^2 = 0$) will grow.

$$
\begin{aligned}
&\mu_n T > \delta((1-\delta)C_{n-1} + I_{n-1}) \\
&\iff \mu_0 \times \tfrac{I_{n-1}}{I_0} \times T > \delta(1-\delta)C_{n-1} + \delta I_{n-1} \\
&\iff I_{n-1} \times (\tfrac{\mu_0 T}{I_0} - \delta) > \delta(1-\delta)C_{n-1} \\
&\iff I_{n-1} > \tfrac{\delta(1-\delta)}{\frac{\mu_0 T}{I_0} - \delta} C_{n-1}
\end{aligned}
$$

Result :

$$
O_{n-1} = \frac{\delta(1-\delta)}{\frac{\mu_0 T}{I_0} - \delta} C_{n-1} \tag{1.2}
$$

1.6 Conclusion :

Initialisation : We first choose T, μ_0 and D_0 which will be public and general information about the firm that we are modelling.

Behavior parameters : The decisions that the company will make will be completely defined by α and σ^2 that we also need to choose initially. As this information is private, it will be hard to model an existing company with this model.

Chapter 2

Upgrades

2.1 Adaptive investment

Instead of considering that only the last investment has an effect on the expected value of the cash flow or the effectiveness of the firm. We will consider that it is the last τ investments that do (here we'll use a moving average (MA) but we can use the weighted coefficients of our choice) . So, the equation of μ changes and becomes :

$$\mu_n = \frac{\mu_0}{I_0\tau} \times \sum_{t=n-\tau}^{n-1} I_t = \frac{\mu_0}{I_0\tau} MA^{\tau}_{t=n-1}(I_t) \tag{2.1}$$

We only have to reconsider the objective equation. So, to sum up we want to make sure that the expected value of the cash flow will increase between $t = n$ and $t = n + 1$. We suppose that the investments before $t = n - 1$ are all fixed and we have to calculate the minimum investment for $t = n - 1$:

$$\begin{aligned}
&\mu_n T > \delta((1-\delta)C_{n-1} + I_{n-1}) \\
&\iff \tfrac{\mu_0 T}{I_0} \times (\tfrac{I_{n-1}}{\tau} + \tfrac{\tau-1}{\tau} MA^{\tau-1}_{t=n-2}) > \delta(1-\delta)C_{n-1} + \delta I_{n-1} \\
&\iff I_{n-1} \times (\tfrac{\mu_0 T}{I_0\tau} - \delta) > \delta(1-\delta)C_{n-1} - \tfrac{\mu_0 T(\tau-1)}{I_0\tau} MA^{\tau-1}_{t=n-2} \\
&\iff I_{n-1} > \frac{\delta(1-\delta)C_{n-1} - \frac{\mu_0 T(\tau-1)}{I_0\tau} MA^{\tau-1}_{t=n-2}}{\frac{\mu_0 T}{I_0\tau} - \delta}
\end{aligned}$$

Remark : To have this last equation we need to respect this condition at the denominator :

$$\begin{aligned}
&\tfrac{\mu_0 T}{I_0\tau} - \delta > 0 \\
&\iff \mu_0 T > \delta I_0 \tau \\
&\iff \tau < \tfrac{\mu_0 T}{\delta I_0} = \tfrac{\mu_0 T C_0}{I_0^2} = \tau^*
\end{aligned}$$

Conclusion :

$$\tau < \tau^* \implies O_{n-1} = \frac{\delta(1-\delta)C_{n-1}I_0\tau - \mu_0 T(\tau-1)MA_{t=n-2}^{\tau-1}}{\mu_0 - \delta I_0 \tau} \tag{2.2}$$

2.2 Behavior management :

In those models the behavior of the company was depending on two parameters α in the dividends distribution, and σ the risk taken by a given firm. Those constant can be easily modified through time. For instance, after a long period of good results the firm can allocate more money as dividends whereas when the times are hard, even if there is a good cash flow after T days, giving less than the usual $D_n(\alpha)$ would be more responsible. A first way to solve this issue would be to consider this relation :

$$\alpha_n = \alpha_0 \times \frac{C_n}{C_0} \tag{2.3}$$

Remark : The same relation could be established for σ^2

One problem remain. If $\mu_n << 0$, it will be very difficult to reversed this negative tendency because no positive investment would be possible to do. So, we can create a savings variable S_n which will be money "put in the safe of the firm" (so to speak). What I mean is that this money could be used as investment during "bad times". It will be triggered when this condition is no longer respected (for instance) $O_n^+ - I_n^+ < W \times I_n^+$ where W is the warning variable. And, if not used, this money could be remunerated at the risk-free rate R_n^f or the market average rate R_n^M (dayly) for T days (n indicates the time, here, this rate is not depending on the firm). Then we have to introduce β which is the equivalent to α in the utility function but this time for S_n.If S_n is negative, the new priority between the investment one and the dividend one would be to pay (at least) the interest over the last T days.

$$\begin{cases} P_n = I_n + D_n + \Delta S_n \\ S_{n+1} = (1+R_n)^T \times S_n + \Delta S_n \\ O_n^+ - I_n^+ \geq W \times I_n^+ \implies \Delta S_n = -(O_n^+ + I_n^-) \\ U(D, \Delta I, \Delta S) = \frac{1}{\alpha} log(D - D_0) + log(\Delta I) + \frac{1}{\beta} log(\Delta S - R_n \times S_n^-) \\ C_n = (1-\delta)C_{n-1} + X_n - D_n - \Delta S_n \end{cases}$$

Remark : 1) The third relation implies that ΔS is negative as the company "is in bad shape". If not, the firm will maximize its utility function, this implies a positive ΔS.
2) If S is negative, this variable will represent a loan taken by the firm.

Utility optimization :

$$\begin{cases} D = D_0 + (P - D_0 - R_n \times S_n^-) \times \frac{1}{1+\alpha(1+\frac{1}{\beta})} \\ \Delta S = (P - D_0 - R_n \times S_n^-) \times \frac{1}{1+\beta(1+\frac{1}{\alpha})} \\ \Delta I = (P - D_0 - R_n \times S_n^-) \times \frac{1}{1+\frac{1}{\alpha}+\frac{1}{\beta}} \end{cases}$$

2.3 Market Modeling:

When you need to model a whole market there are certain correlation that you need to put in your model. For instance, all the firms that sells cars have cash flows that are highly correlated to the automobile market and, as a result, to their business rivals. Here, this model makes you able to model it using random vectors.
For instance, if we consider N firms, we can define the cash flows per days as :

$$\begin{cases} x_t \sim \mathcal{N}(\mu_n, \Sigma) \\ x_t, \mu_n \in \mathbb{R}^N \\ \Sigma \in \mathcal{S}_N^+ \end{cases}$$

Remark : For the company k we have :

$$X_n^k = \sum_{t=nT_k+1}^{(n+1)T_k} x_t^k$$

The method for each company is then the same that the one explained in the chapter 2. This model can help us analyse the difference of evolution of 2 different companies with high correlated results but with different behavior parameters (previously defined α and σ) given a certain situation or evolution in the market.

Remark : For every company a (if possible unique) initialisation would have to be made.