

# Report of Implementation of FUSINTER Algorithm for Practical Work in AI

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## Abstract

The goal of this project is the implementation of the FUSINTER discretization algorithm as it is described in [ZRR98]. It will consist out of three phases: The first one should be a naive implementation of the steps given in the paper in the Python Programming Language. The second phase will also be implemented in Python, but with optimizations to decrease the run time of the algorithm. The third and last phase is an implementation in C++ to further improve the run time. This last part is expected to be executable within Python.

## 1 Introduction

Discretization methods take continuous features and split them up into regions. As a basic example we can take the ages of a group of people (22, 85, 2, 30, 14, 5, 54). Now we can set a split point to separate minors from adults. In many countries that would be the age of 18. The discretized features are (adult, adult, minor, adult, minor, minor, adult).

Splitting by predefined values is probably the most basic method for discretization followed by those seeking to give either equal width between splitting points or equal frequency within each interval. In contrast to FUSINTER these algorithms don't use information provided by the labeling of data points. FUSINTER can be described as a supervised method.

Further we can distinguish between discretization algorithm that start with a set of splitting points and then iteratively remove them and those starting without any such points and iteratively adding them. The first ones are called bottom-up the later one top-down algorithms. FUSINTER falls into the first category.

In the following I will first give a description of the steps of FUSINTER. Then I will go into the details of my implementation of the first phase of the project. Then I will show some applications of the algorithm on different datasets. After that descriptions of the optimizations for the second part will follow. After an examination of the implementation in C++ I will end with some benchmarks.

The code of this project can be found at <https://github.com/floxo115/FUSINTER>.

## 2 The FUSINTER Algorithm

The following steps of the FUSINTER algorithm can be found in [ZRR98, p. 315f]. We expect  $\bar{x}_0$  to be a vector of real values and  $\bar{y}_0$  to be a vector of corresponding labels with unique values  $1, \dots, m$

1. We sort the components of  $\bar{x}_0$  in ascending order such that we get a new data vector  $\bar{x}$  and corresponding label vector  $\bar{y}$ .
2. From left to right we sweep over the sorted data and form intervals for runs of examples with the same labels.
3. If there are multiple examples for the same value having different labels, we close the current interval and create a new interval from the value with the mixed labels up to but not including the next example that has a bigger value.
4. The boundary points of the intervals are the splitting points of the FUSINTER algorithm.
5. From the intervals  $1, \dots, k$  we calculate the table  $T$  with columns  $T_1, \dots, T_k$ . The value of the  $i$ -th rows indicate the number of occurrences of examples labeled with  $i$ . If we would have two intervals with two classes we would write that as:

$$T = \begin{bmatrix} 3 & 0 \\ 2 & 2 \end{bmatrix}$$

This would mean that in the first interval we have 5 examples with 3 of label 1 and 2 of label 2 as well as 2 examples in interval 2 where all have label 2.

6. If we merge the  $i$ -th and the  $(i + 1)$ -th column of  $T$  we denote the new table with  $T_i$ . With the help of the real valued function  $\phi$  we can calculate the index  $t$  such that

$$t = \arg \max_i \phi(T) - \phi(T_i)$$

7. if  $\phi(T) - \phi(T_k) > 0$  we remove the  $k$ -th split and set  $T := T_k$
8. We do this again from step 2 until there are no more splits or the criterion in 7) is not met.

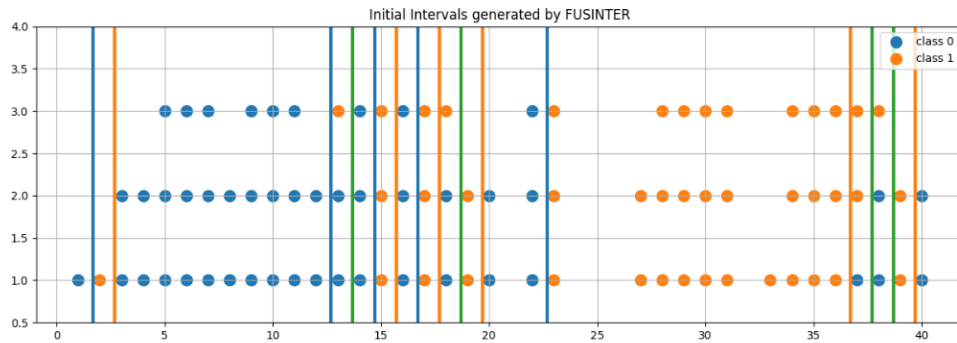


Figure 1: Initial splits created by steps 1-3 of the FUSINTER algorithm

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The  $\phi$  function estimates the quality of the discretization. It is a function from a  $m \times k$  matrix to the positive real numbers. In the original paper Shannon's Entropy and Quadratic Entropy is used [ZRR98, p. 318].

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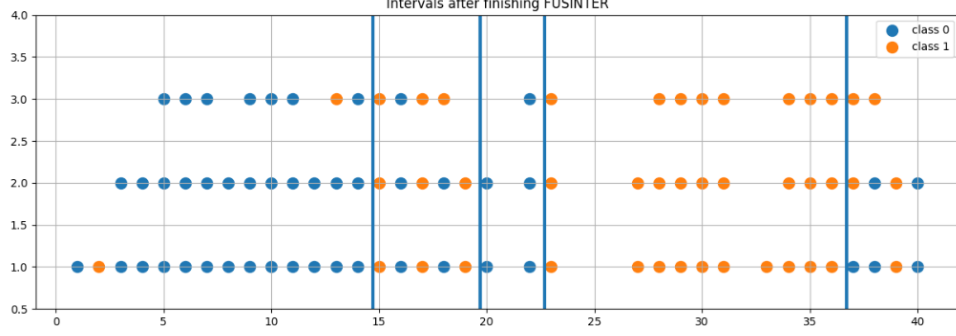


Figure 2: Final splits after completing FUSINTER algorithm

$$\text{Shannon's Entropy} = \phi_1(T) := \sum_{j=1}^k \left\{ \alpha \frac{n_j}{n} \left( - \sum_{i=1}^m \frac{n_{ij} + \lambda}{n_j + m\lambda} \log_2 \frac{n_{ij} + \lambda}{n_j + m\lambda} \right) + (1 - \alpha) \frac{m\lambda}{n_j} \right\}$$

$$\text{Quadratic Entropy} = \phi_2(T) := \sum_{j=1}^k \left\{ \alpha \frac{n_j}{n} \left( \sum_{i=1}^m \left( \frac{n_{ij} + \lambda}{n_j + m\lambda} \right) \left( 1 - \frac{n_{ij} + \lambda}{n_j + m\lambda} \right) \right) + (1 - \alpha) \frac{m\lambda}{n_j} \right\}$$

The variables in the preceding formulas are defined as:

$k$  : number of splits in table

$n$  : number of examples in table

$m$  : number of unique labels

$n_j$  : number of examples in  $j$ -th column

$n_{ij}$  : value of cell in row  $i$  and column  $j$

$\lambda$  and  $\alpha$  : tuning parameters

### 3 Phase 1: Naive Implementation in Python

For implementing the algorithm the Numpy library was used since it is the de facto standard for handling data in machine learning.

The code mainly consists out of three classes (Splitter, TableManager, FUSINTERDiscretizer) that will be described in the following. After that applications to data and benchmarks will be shown.

#### 3.1 Classes

##### 3.1.1 Splitter

The responsibility of the Splitter is to generate the initial split given already sorted data. So it constitutes the most important part for steps 1 to 4. Its interface only consists of a single public method `apply()` that runs these steps on data.

The most important part of the Splitter is the while loop inside the `apply()` method: Here we iterate over all examples. At each step we check if there are multiple examples at the same value. And if the label of the current value are of the same type (1...m) or if they are mixed (-1). The label of the current value and the next index is returned by `_get_label_of_next_value(index)`. If the label of the current value is different from the label of the last value or if it is mixed, we create a new split point. If the label stays the same, we go on with the loop.

This approach is able to create an array of splitting points in linear time since each example is examined exactly one time.

The code can be found at [here](#)

### 3.1.2 TableManager

The responsibility of the TableManager is to create a table  $T$  as described in step 6 and also to merge such tables like in step 6. For this it uses the `create` and `compress_table` methods respectively.

The `create` method takes the data and the splits generated by the Splitter and creates a  $m \times k$  matrix for data with  $m$  labels and  $k$  initial splits. The table is generated by iterating over the data and counting the appearances of all labels in the current interval and then setting them as columns in the matrix. This process examines each example exactly once and hence runs in linear time.

The `compress_table` method takes a matrix like the one generated with the `create` function and an index to create a new table but with the columns  $i$  and  $i + 1$  merged together. So the table

$$T_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

with  $i = 1$  becomes

$$T_2 = \begin{bmatrix} 1 & 5 & 4 \\ 5 & 13 & 8 \end{bmatrix}$$

Also this function looks at each example exactly once and hence runs in linear time.

The code can be found [here](#)

### 3.1.3 FUSINTERDiscretizer

The main part of FUSINTER is implemented here. After generating initial splits and an initial table  $T$  we iteratively compute which split points to remove. This is done in the `apply()` method.

Here we iteratively compute the value from step 6 for all possible merges and then test if the maximum value is strictly positive like in step 7.

The implementation is as close as possible to the algorithm described above and is in that sense naive. We run many computation multiple times and allocate many unnecessary tables. For  $n$  iterations we calculate  $n + (n - 1) + (n - 2) + \dots + 1$  table merges in the worst case. That gives a run time complexity of  $O(n^2)$ . This will be a major point for optimization in phase 2 of the project.

Further the implementation of the Shannon's Entropy and the Quadratic Entropy although they are of linear complexity do heavily rely on loops, which is also bad for performance in Python.

The code can be found [here](#).

## 3.2 Applications

In the following we are applying the FUSINTER algorithm to various datasets and observe the behavior given different values for the parameters  $\alpha, \lambda$ .

### 3.3 Dataset From the Original Paper

This dataset is taken from the original paper [ZRR98, p. 315 (Figure 5)]. It consists out of integer values being labeled with 2 classes.

We first give results with  $\lambda = 1$ :

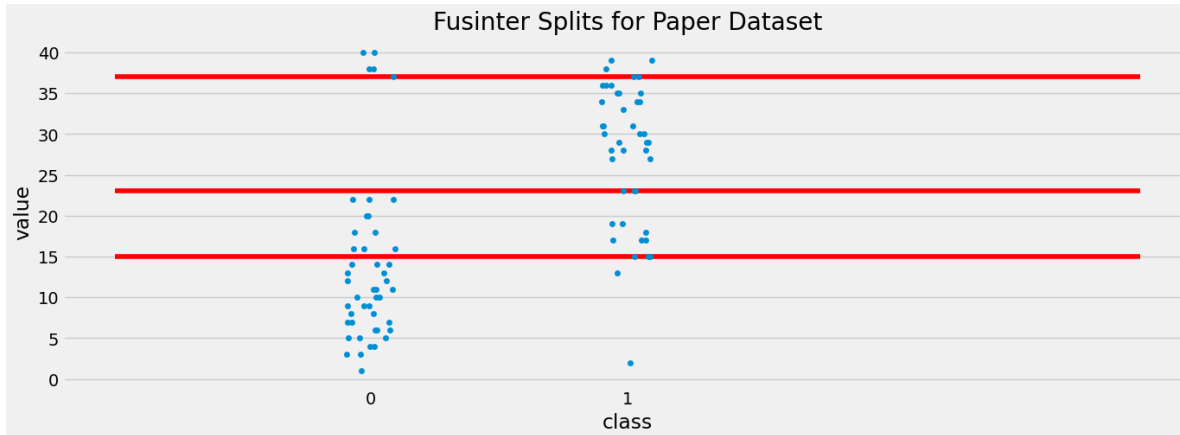


Figure 3: Splits with  $\lambda = 1$  and  $\alpha = 0.95$

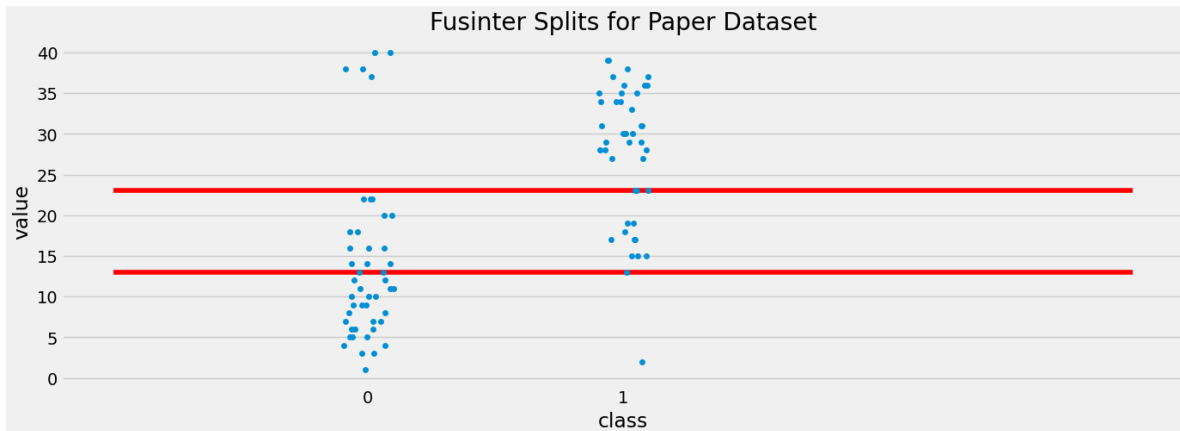


Figure 4: Splits with  $\lambda = 1$  and  $\alpha = 0.85$

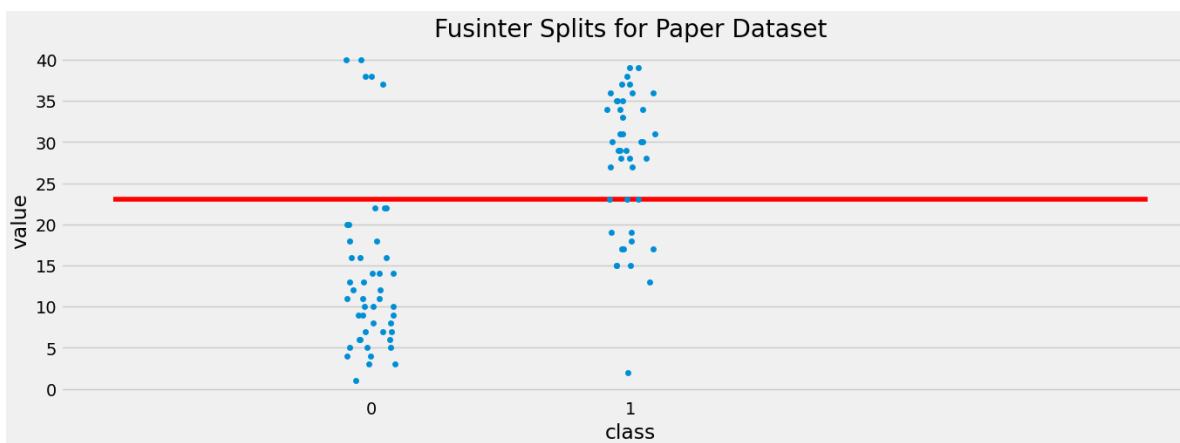


Figure 5: Splits with  $\lambda = 1$  and  $\alpha = 0.8$

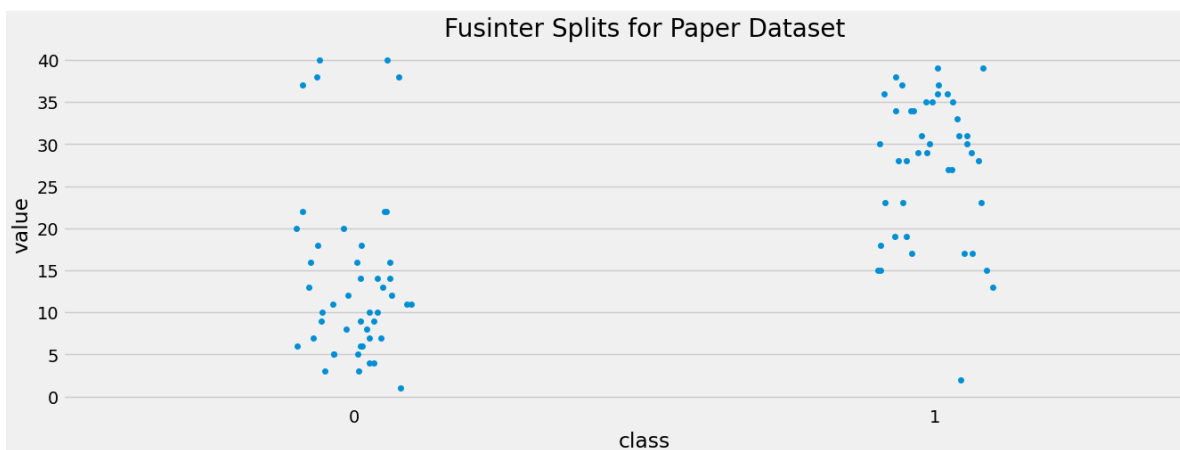


Figure 6: Splits with  $\lambda = 1$  and  $\alpha = 0.25$

We see that the higher the value of  $\alpha$  the more splits we are going to produce.

Now we keep the value  $\alpha = 0.95$  and vary the  $\lambda$ :

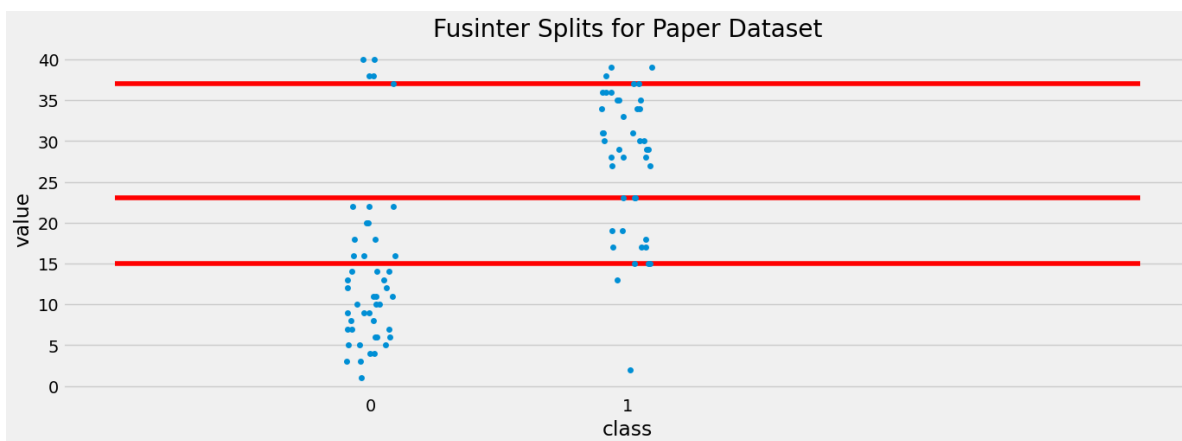


Figure 7: Splits with  $\lambda = 1$  and  $\alpha = 0.95$

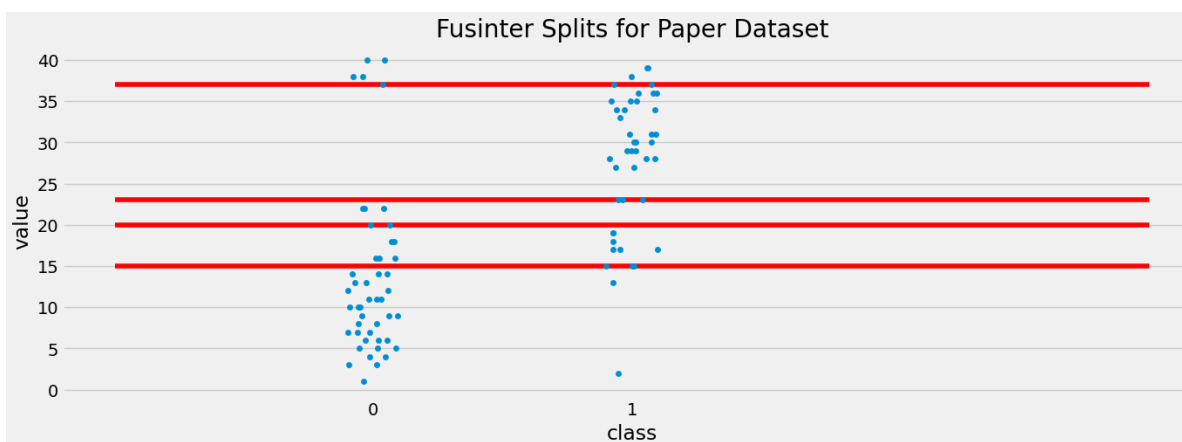


Figure 8: Splits with  $\lambda = 0.85$  and  $\alpha = 0.95$

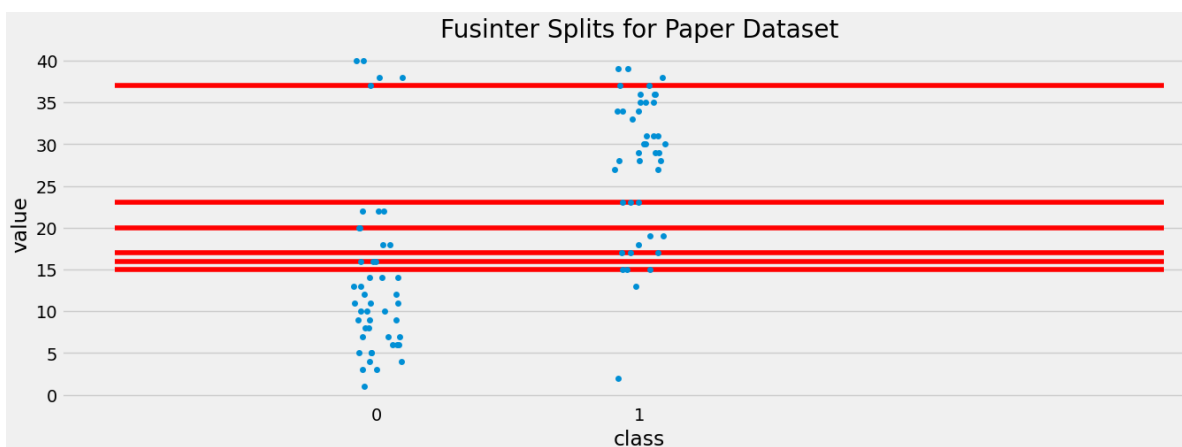


Figure 9: Splits with  $\lambda = 0.35$  and  $\alpha = 0.95$

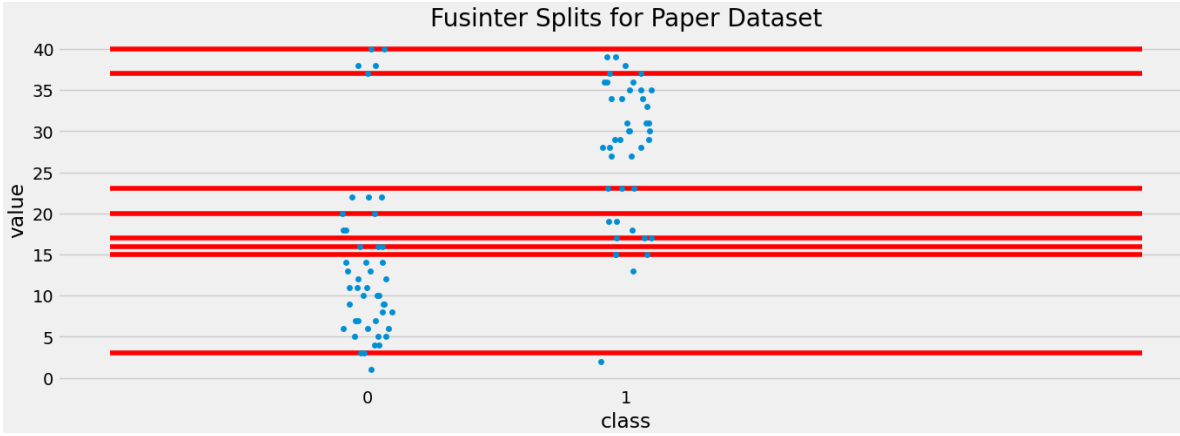


Figure 10: Splits with  $\lambda = 0.15$  and  $\alpha = 0.95$

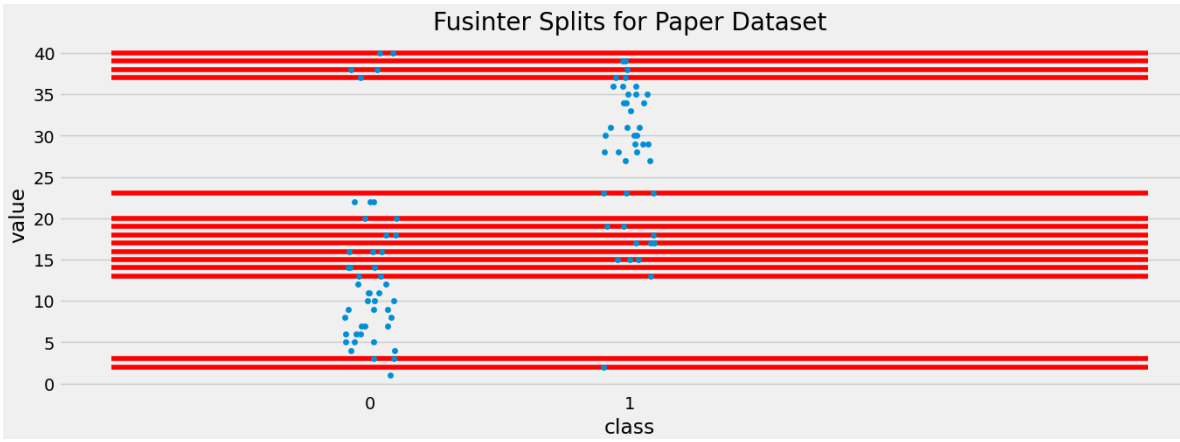


Figure 11: Splits with  $\lambda = 0.05$  and  $\alpha = 0.95$

We see that the lower the value of  $\lambda$  the less likely the algorithm is to remove initial splits. If it approaches zero all initial splits remain in the output.

The value of  $\lambda$  gives the maximal amount of possible splits and the value of  $\alpha$  then is responsible for removing splits from the possible ones. We see that when we set  $\lambda = 0.05$  and  $\alpha = 0.45$  the output is equal to  $\lambda = 0.1$  and  $\alpha = 0.95$



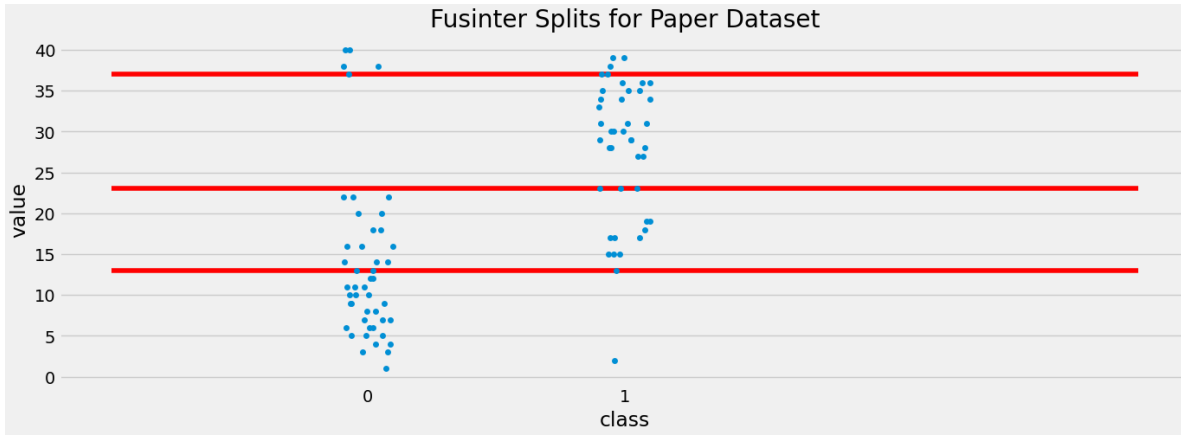


Figure 12: Splits with  $\lambda = 0.05$  and  $\alpha = 0.45$

### 3.4 Iris Dataset

The application of the algorithm to the petal length feature of the iris dataset [iri]:

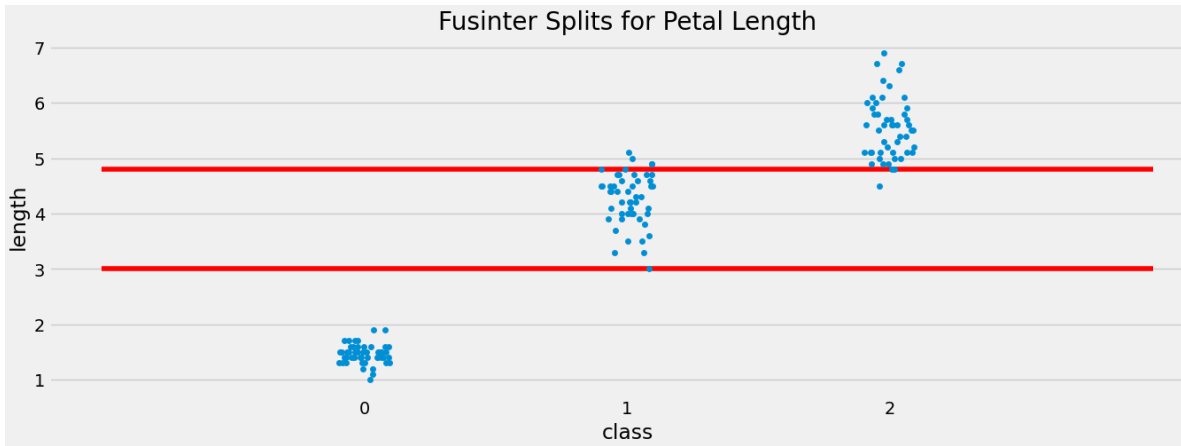


Figure 13: Splits with  $\lambda = 1$  and  $\alpha = 0.95$

Heuristic  
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formula

### 3.5 Duplicated Data

As it is stated in the paper, the algorithm should yield zero splits, if there is no meaningful discretization possible. [ZRR98, p. 316 (Step 8)] We put that to the test by generating a normal distributed set of datapoints and duplicated it. The first set is completely labeled 1 the second one is completely labeled 2.

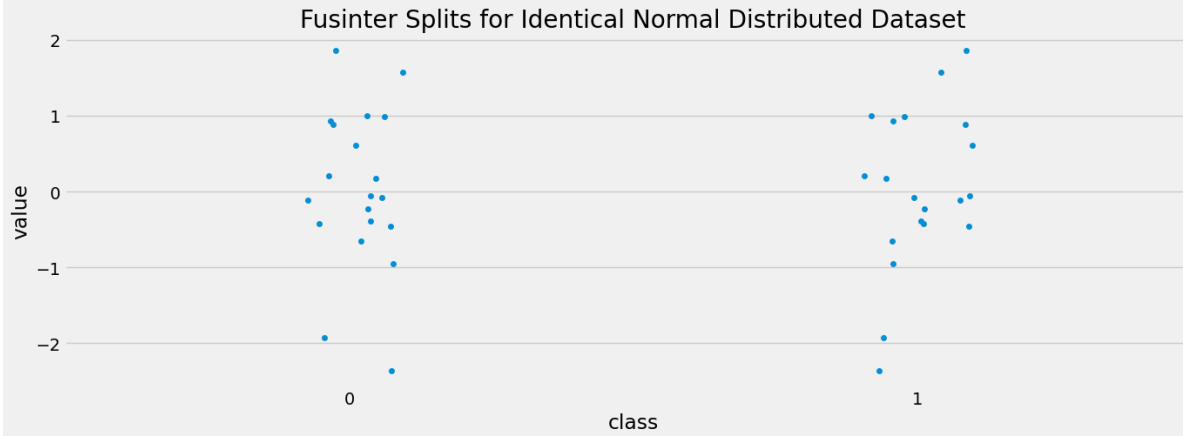


Figure 14: Duplicated Normal Distribution yields no splits for  $\lambda \neq 0$

This is the expected output.

## 4 Phase 2

### 4.1 Improvement Idea V1

The major problem of the naive implementation lies withing the `apply()` method of the `FUSINTERDiscretizer` class.

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**Algorithm 1** Apply Method of FUSINTERDiscretizer Version 1

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**Require:** *splits*: list of initial splits

**Require:** *table*: initial table

**Require:** *H*: Function to compute the entropy of a given table

**while** *splits*  $\neq \emptyset$  **do**

$max\_idx = \arg \max_i H(table_i)$   $\triangleright table_i$  is the original table merged at index  $i$

**if**  $H(table) - H(table_{max\_idx}) \geq 0$  **then**

$table = table_{max\_idx}$

**delete** *splits*[ $max\_idx$ ]

**else**

**break**

**end if**

**end while**

**return** *splits*

---

Here we compute the split values for all possible splits for each pass of the algorithm. If we have a look at the actual computation of these values we see that many steps are redundant. The entropy functions on page 2 are defined over two sums: The inner one computes weighted values for each column (depending on  $\lambda, \alpha$  and the number of elements in the table  $n$  as well as the number of rows in the table  $k$ ). The outer one just adds up those values.

The only operation applied on the tables is merging two adjacent columns. This neither alters the amount of rows in the table  $k$  nor does it alter the amount of elements in the table  $n$ .

If we create a class that stores the values  $n, k$  we can calculate the entropy value for each column separately and store them too. To compute  $H(table) - H(table_i)$  we simply add up those columns

except for the columns  $i$  and  $i + 1$  and then add the entropy value of the column created by merging columns  $i$  and  $i + 1$ .

This allows a new implementation of the `apply()` method:

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**Algorithm 2** Apply Method of FUSINTERDiscretizer Version 2.1

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**Require:** *splits*: list of initial splits

**Require:** *table*: initial table

**Require:** *MVC*: Class `MergedValueComputer`

```

while splits  $\neq \emptyset$  do
    mvc = MVC(table)
    max_val = max mvc.compute_split_values()
    max_idx = arg max mvc.compute_split_values()
    if max_val  $\geq 0$  then
        table = tablemax_idx  $\triangleright$  tablei is the original table merged at index i
        delete splits[max_idx]
    else
        break
    end if
end while

return splits

```

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The code can be found [here](#) (FUSINTERDiscretizer) and [here](#) (MergeValueComputer)

#### 4.1.1 MergeValueComputer V1

The `MergeValueComputer` is a class that takes a table created by a `TableManager` together with values for  $\alpha$  and  $\lambda$  as input. Then it computes a list of values corresponding to the inner loop of weighted entropy functions as they are described on page 2.

E.g.: For the following input table

|   |   |   |
|---|---|---|
| 5 | 0 | 2 |
| 3 | 1 | 0 |

we get the following entropy values per column

|         |         |         |
|---------|---------|---------|
| 0.68333 | 0.17931 | 0.19013 |
|---------|---------|---------|

which gives the table entropy of

|         |
|---------|
| 1.05277 |
|---------|

now for computing the split value for the first two columns we only need to take the entropy value, subtract the first two stored values from it and add the entropy value of the merged column  $[5, 4]$  with the original parameters  $n, k, \lambda, \alpha$ .

This approach saves us from computing the split value for each possible merged table. The only new value that has to be computed is the entropy of the new column created by merging.

## 4.2 Improvement Idea V2

The described approach reduced the runtime, but it also requires the computation of the entropy values of all possible merges in each iteration of the while loop. We can reduce this even further by observing that if the merge takes place at index  $i$ , only possible merge values at  $i - 1$  and  $i + 1$  are affected. So most of the entropy values stay the same over one iteration.

For this I implemented a new version of the `MergeValueComputer` that stores the entropies of the given table and then computes the values  $\Delta_i = H(table) - H(table_i)$  by the method used in V1. Then after a merge the new table is given to the `update()` method to remove the delta corresponding to the merge index and also update the neighboring  $\Delta$ 's of the merge index.

For the paper dataset we compute the input table with the following entries.

|   |   |    |   |   |
|---|---|----|---|---|
| 1 | 0 | 26 | 2 | 3 |
| 0 | 1 | 0  | 1 | 0 |

The `MergeValueComputer` computes the following column entropies and stores them:

|         |         |         |         |         |
|---------|---------|---------|---------|---------|
| 0.12566 | 0.12566 | 0.16533 | 0.11472 | 0.09385 |
|---------|---------|---------|---------|---------|

The deltas following from this are:

|         |         |          |         |
|---------|---------|----------|---------|
| 0.14543 | 0.01415 | -0.00304 | 0.05589 |
|---------|---------|----------|---------|

The `FUSINTER` algorithm forces us to remove the maximum of the deltas. In this case it is the first one. This gives a new table:

|   |    |   |   |
|---|----|---|---|
| 1 | 26 | 2 | 3 |
| 1 | 0  | 1 | 0 |

When given to the `update()` method of the new `MergeValueComputer` it updates the entropies as follows:

|         |         |         |         |
|---------|---------|---------|---------|
| 0.12566 | 0.16533 | 0.11472 | 0.09385 |
|---------|---------|---------|---------|

The deltas are computed as:

|         |          |         |
|---------|----------|---------|
| 0.18511 | -0.00304 | 0.05589 |
|---------|----------|---------|

In this example only the second element has to be updated, all other entropies and deltas stay the same. If the maximum delta lies between two other values, the predecessor and successor have to be recalculated. If it lies at the end, only the predecessor needs to be recomputed.

For this to work we also need to keep the instance of the `MergeValueComputer` in the `FUSINTER-Discretizer`'s while loop.

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**Algorithm 3** Apply Method of FUSINTERDiscretizer Version 2.2

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**Require:** *splits*: list of initial splits

**Require:** *table*: initial table

**Require:** *MVC*: Class MergedValueComputer

*mvc* = *MVC*(*table*)

**while** *splits*  $\neq \emptyset$  **do**

*max\_val* =  $\max mvc.compute\_split\_values()$

*max\_idx* =  $\arg \max mvc.compute\_split\_values()$

**if** *max\_val*  $\geq 0$  **then**

*table* = *table*<sub>*max\_idx*</sub>  $\triangleright$  *table*<sub>*i*</sub> is the original table merged at index *i*

**delete** *splits*[*max\_idx*]

*mvc.update*(*table*, *max\_idx*)

**else**

**break**

**end if**

**end while**

**return** *splits*

---

This approach makes it possible to keep the already computed deltas and only recompute the ones that will change according to the table merge.

The code can be found [here](#) (FUSINTERDiscretizer) and [here](#) (MergeValueComputer)

## 5 Phase 3

## 6 Benchmarks

## References

- [iri] [https://scikit-learn.org/stable/datasets/toy\\_dataset.html#iris-dataset](https://scikit-learn.org/stable/datasets/toy_dataset.html#iris-dataset). Accessed: 04.11.2023.
- [ZRR98] D. A. Zighed, S. Rabaséda, and R. Rakotomalala. Fusinter: A method for discretization of continuous attributes. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, 06(03):307–326, 1998.