CHAPTER 5

Lagrangian Duality

The present chapter pursues two goals. First, we take a (rather preliminary) look at nonlinear optimization problems by investigating to what extent fundamental concepts carry over from linear to general optimization problems and what kind of difficulties arise in the general context. Doing so, our second goal is to motivate much of the theory of nonlinear problems that are treated in more detail in subsequent chapters. The main points we want to make now are Lagrangian relaxation as a bounding technique for integer programs (*cf.* Chapter 9) and the optimality conditions (which we derive rather independently from the rest of the chapter in Section 5.4) that motivate many of the algorithmic approaches to non-linear problems.

5.1. Lagrangian Relaxation

A *nonlinear (constrained) optimization problem* is a problem of the type

$$\max\ f(\mathbf{x}) \qquad \text{s.t.} \quad g_j(\mathbf{x}) \le 0\,, \qquad j = 1, \ldots, m,$$

with *objective function* $f : \mathbb{R}^n \to \mathbb{R}$ and *constraint functions* $g_j : \mathbb{R}^n \to \mathbb{R}$. In terms of the vector-valued function $g(\mathbf{x}) = (g_1(\mathbf{x}), \ldots, g_m(\mathbf{x}))^T$, this problem can be stated more compactly as

$$\max\ f(\mathbf{x}) \quad \text{s.t.} \quad g(\mathbf{x}) \le \mathbf{0}\,. \tag{5.1}$$

REMARK. Usually, one assumes f and g to be (at least) continuous. In what follows, whenever a gradient ∇f or Jacobian ∇g are used, we implicitly assume that f and g are continuously differentiable.

The use of “max” resp. “min” is standard notation in nonlinear optimization although “sup” and “inf” would be more precise. For example,

$$\min\ x_1 \qquad \text{s.t.} \qquad x_1x_2 \ge 1, \quad x_1 \ge 0$$

has “minimum value” 0, but optimal solutions do not exist.

EX. 5.1. *Show that* $\overline{\mathbf{x}} = (1, 1)$ *(with* $f(\overline{\mathbf{x}}) = 6$*) is an optimal solution for*

$$\max\ f(\mathbf{x}) = 4(x_1 + x_2) - (x_1^2 + x_2^2) \quad \text{s.t.} \quad g(\mathbf{x}) = x_1x_2 - 1 \ \le\ 0\,.$$

Clearly, a linear program, maximizing a linear objective $f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$ under *linear constraints* $g(\mathbf{x}) = \mathbf{A}\mathbf{x} - \mathbf{b} \ \le\ \mathbf{0}$ is a special case of (5.1).

As in linear programming, we associate with the *primal problem* (5.1) a *dual* problem that wants to minimize certain upper bounds on the primal maximum

U. Faigle et al., *Algorithmic Principles of Mathematical Programming*

value. As in the linear case, we obtain such bounds from non-negative combinations of the constraints. Consider any $\mathbf{y} = (y_1, \ldots, y_m)^T \geq \mathbf{0}$. Then every *primal feasible* $\mathbf{x}$, *i.e.*, any $\mathbf{x} \in \mathbb{R}^n$ with $g(\mathbf{x}) \leq \mathbf{0}$, necessarily satisfies the "derived" inequality

$$\sum_{j=1}^{m} y_j g_j(\mathbf{x}) = \mathbf{y}^T g(\mathbf{x}) \leq 0 . \tag{5.2}$$

So each $\mathbf{y} \geq \mathbf{0}$ gives rise to an upper bound $L(\mathbf{y})$:

$$\max_{g(\mathbf{x}) \leq \mathbf{0}} f(\mathbf{x}) \leq \max_{\mathbf{x}} f(\mathbf{x}) - \mathbf{y}^T g(\mathbf{x}) = L(\mathbf{y}). \tag{5.3}$$

The (unconstrained) maximization problem defining the upper bound

$$L(\mathbf{y}) = \max_{\mathbf{x}} f(\mathbf{x}) - \mathbf{y}^T g(\mathbf{x})$$

is called the *Lagrangian relaxation* of (5.1) with *Lagrangian multipliers* $y_j \geq 0$ (which play the role of the dual variables in linear programming). So the Lagrangian relaxation is obtained by "moving the constraints into the objective function". We also say that we *relax* or *dualize* the constraints $g_j(\mathbf{x}) \leq 0$ with multipliers $y_j \geq 0$.

The problem of determining the best possible upper bound $L(\mathbf{y})$ is the *Lagrangian dual problem*

$$\min_{\mathbf{y} \geq \mathbf{0}} L(\mathbf{y}) = \min_{\mathbf{y} \geq \mathbf{0}} \max_{\mathbf{x}} f(\mathbf{x}) - \mathbf{y}^T g(\mathbf{x}). \tag{5.4}$$

We immediately observe the following relation between the primal problem (5.1) and its dual (5.4).

THEOREM 5.1. (Weak Duality)

$$\max_{g(\mathbf{x}) \leq \mathbf{0}} f(\mathbf{x}) \leq \min_{\mathbf{y} \geq \mathbf{0}} L(\mathbf{y}) .$$

Consequently, if equality is attained with the primal feasible $\overline{\mathbf{x}}$ and the (dual feasible) $\overline{\mathbf{y}} \geq \mathbf{0}$, then $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ are optimal primal resp. dual solutions. In this case $\overline{\mathbf{x}}$ and $\overline{\mathbf{y}}$ are necessarily complementary *in the sense that* $\overline{\mathbf{y}}^T g(\overline{\mathbf{x}}) = 0$.

Proof. In view of (5.2), we have the inequality

$$f(\overline{\mathbf{x}}) \leq f(\overline{\mathbf{x}}) - \overline{\mathbf{y}}^T g(\overline{\mathbf{x}}) \leq \max_{\mathbf{x}} f(\mathbf{x}) - \overline{\mathbf{y}}^T g(\mathbf{x}) = L(\overline{\mathbf{y}}) .$$

Equality can only hold if $\overline{\mathbf{y}}^T g(\overline{\mathbf{x}}) = 0$.

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REMARK. As in the linear (programming) case, an equality constraint $g_j(\mathbf{x}) = 0$ is formally equivalent to two opposite inequalities and corresponds to a *sign-unrestricted* multiplier $y_j \in \mathbb{R}$ in the Lagrangian dual.