Lecture 6

Roots: Bracketing Methods

Chapter Objectives

- Understanding what roots problems are and where they occur in engineering and science.
- Knowing how to determine a root graphically.
- Understanding the incremental search method and its shortcomings.
- Knowing how to solve a roots problem with the bisection method.
- Understanding false position.

Roots

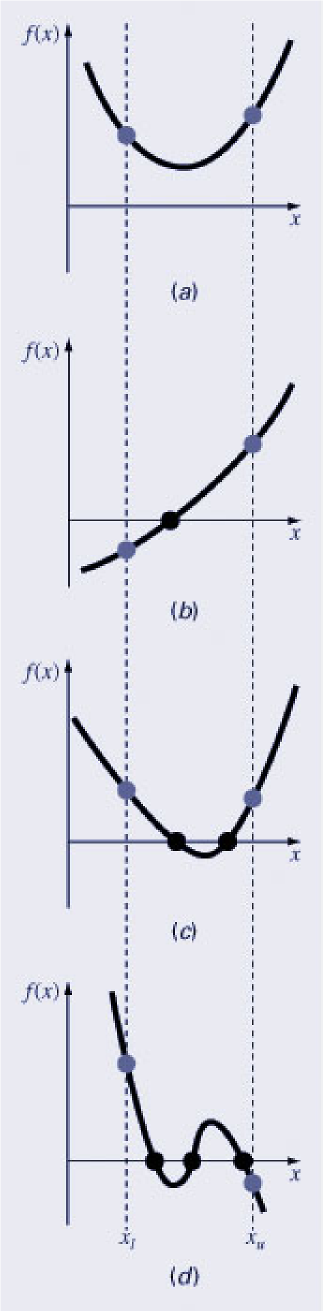
- “Roots” problems occur when some function *f* can be written in terms of one or more dependent variables *x*, where the solutions to $f(x)=0$ yields the solution to the problem.
- These problems often occur when a design problem presents an implicit equation for a required parameter.

Graphical Methods

- A simple method for obtaining the estimate of the root of the equation *f(x)*=0 is to make a plot of the function and observe where it crosses the *x*-axis.

- Graphing the function can also indicate where roots may be and where some root-finding methods may fail:

  a) Same sign, no roots
  b) Different sign, one root
  c) Same sign, two roots
  d) Different sign, three roots



A Mathematical Property

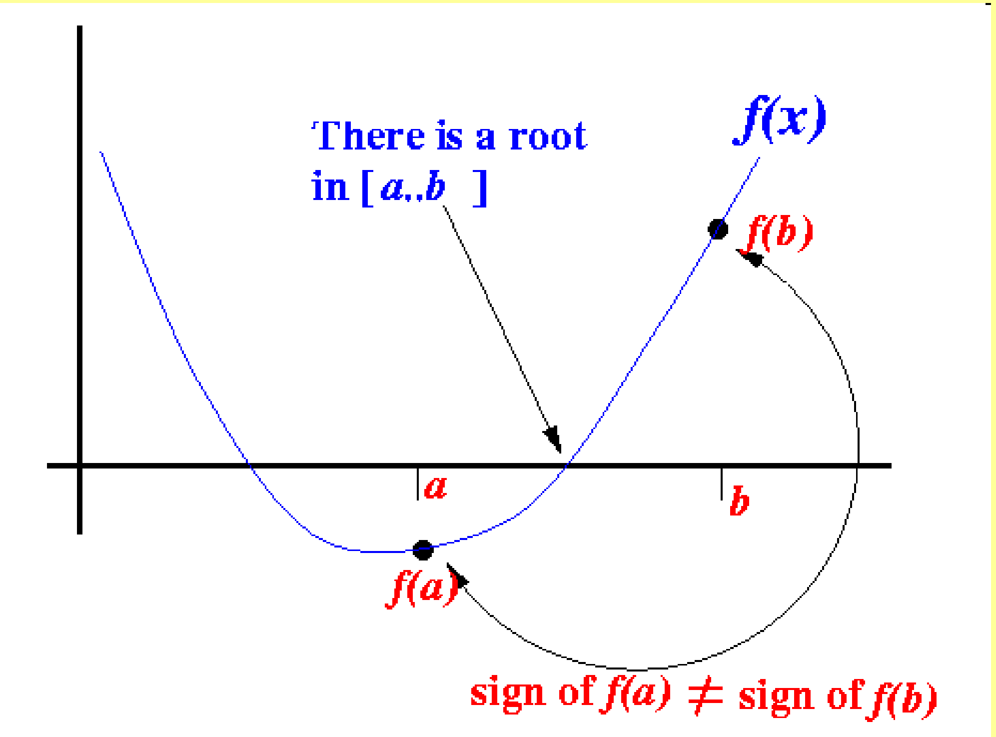
- Well-known Mathematical Property:

- If a function $f(x)$ is continuous on the interval $[a..b]$ and sign of $f(a) \neq$ sign of $f(b)$, then

    - There is a value $c \in [a..b]$ such that: $f(c) = 0$ I.e., there is a root c in the interval $[a..b]$

A Mathematical Property (cont.)

- Example:



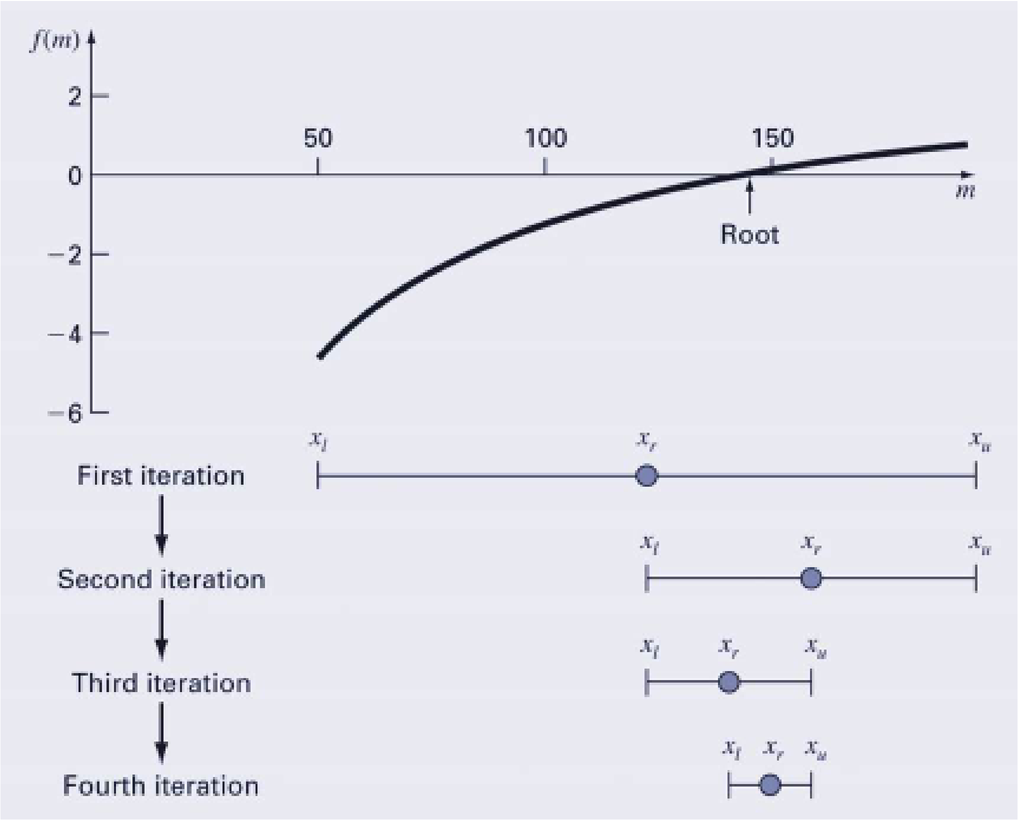
Bisection

The ***bisection method*** is a variation of the incremental search method in which the interval is always divided in half.

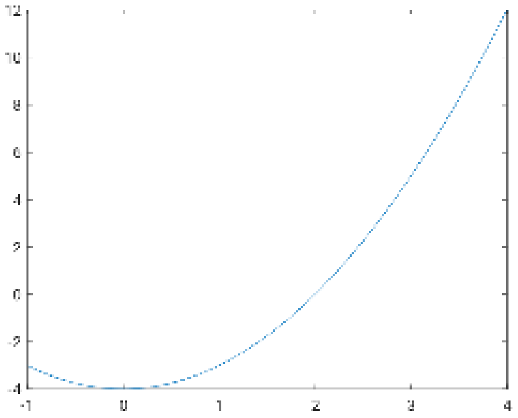
If a function changes sign over an interval, the function value at the midpoint is evaluated.

The location of the root is then determined as lying within the subinterval where the sign change occurs.

The absolute error is reduced by a factor of 2 for each iteration.



Bisection: Example



- $f(x) = x^2 - 4$
- Step 1: Determine the initial conditions:

$$x_l = 0 \rightarrow f(x_l) = -4$$
$$x_u = 3 \rightarrow f(x_u) = 5$$
$$f(x_l)f(x_u) < 0$$

- Step 2: Halve the bracket and determine the value of $f(x)$ at $x_r = 0.5(x_l + x_u)$

$$x_r = 0.5(x_l + x_u) = 1.5 \rightarrow f(x_r = 1.5) = -1.75$$

- Step 3: Based on the sign of $f(x_r)$, Replace x_r with x_l or x_u
  - If $f(x_r)\ f(x_u) > 0$, then the root lies between x_r and x_l, then x_l x_l and x_u x_r
  - If $f(x_r)\ f(x_l) > 0$, then the root lies between x_r and x_u, then $x_u = x_u$ and $x_l = x_r$

$$f(1.5)f(3) < 0 \rightarrow x_l = 1.5$$
$$\text{error} \ |f(1.5)| \ |1.75|$$

- Step 4: Go to Step 2 until error is sufficiently small

Bisection Example

Iteration					
1	0[-4]	3[+5]	1.5 [-1.75]	100	25%
2	1.5[-1.75]	3[+5]	2.25 [+1.06]	33.3%	12.5%
3	1.5[-]	2.25[+]	1.875[-]	20%	6.25%
4	1.875[-]	2.25[+]	2.06[+]	8.9%	3.125%
5	1.875[-]	2.06[+]			

$$\varepsilon_t = \frac{\text{true value} - \text{approximation}}{\text{true value}} 100\%$$

$$\varepsilon_a = \frac{\text{present approximation} - \text{previous approximation}}{\text{present approximation}} 100\%$$

Bisection Error

- The absolute error of the bisection method is solely dependent on the absolute error at the start of the process (the space between the two guesses) and the number of iterations:

$$E_a^n = \frac{\Delta x^0}{2^n}$$

- The required number of iterations to obtain a particular absolute error can be calculated based on the initial guesses:

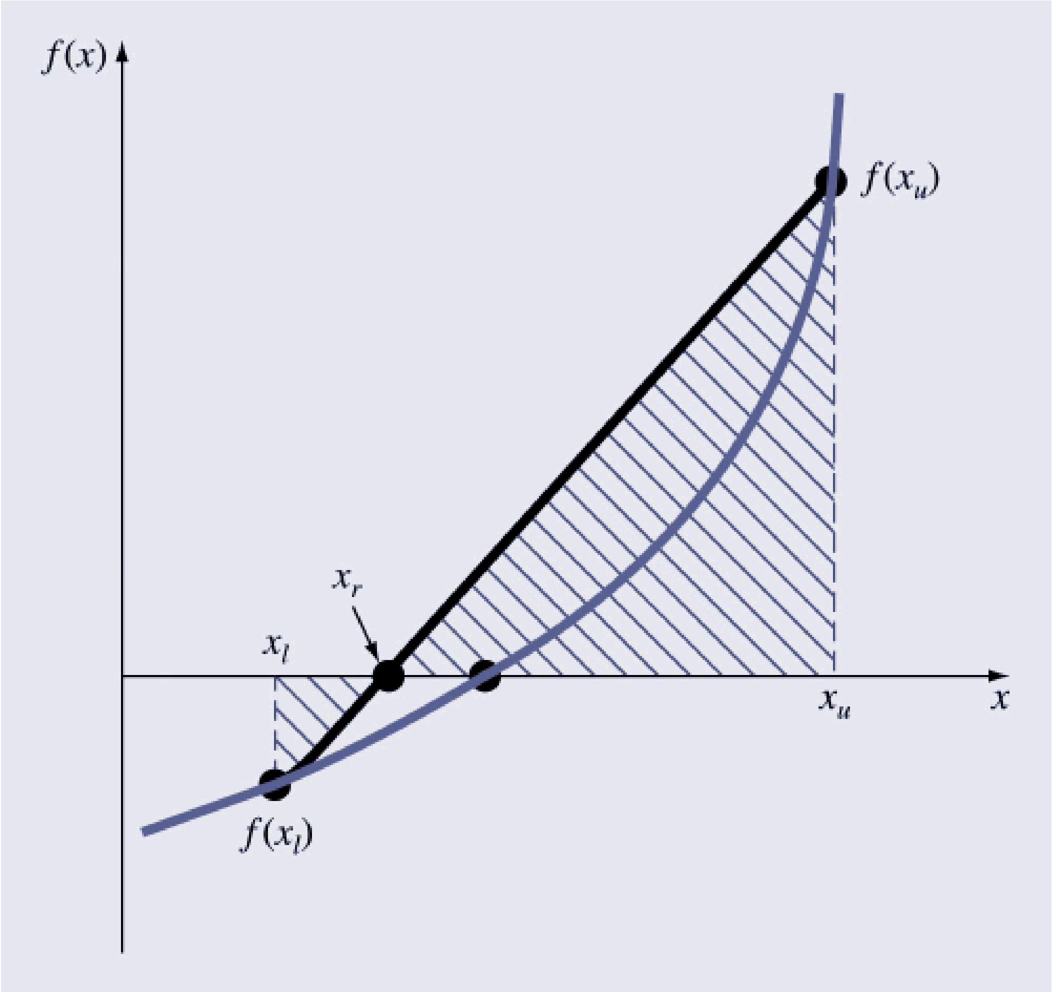
$$n = \log_2\left(\frac{\Delta x^0}{E_{a,d}}\right)$$

False Position

- The *false position* method is another bracketing method.
- It determines the next guess not by splitting the bracket in half but by connecting the endpoints with a straight line and determining the location of the intercept of the straight line (x_r).
- The value of x_r then replaces whichever of the two initial guesses yields a function value with the same sign as $f(x_r)$.

False Position Illustration

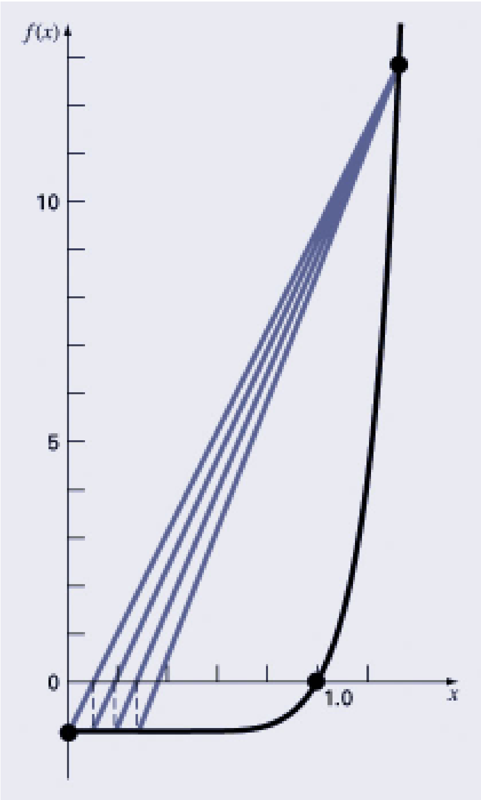
$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)}$$



Bisection vs. False Position

- Bisection does not take into account the shape of the function; this can be good or bad depending on the function!
- Bad:

$$f(x) = x^{10} - 1$$

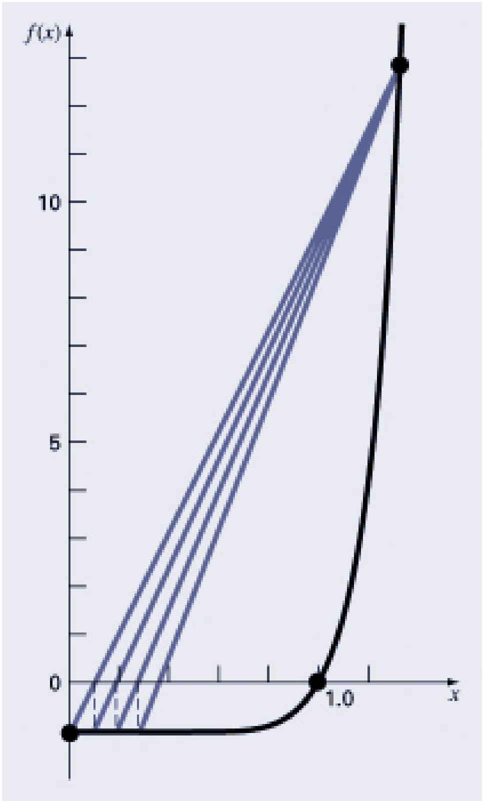


Bisection

Iteration	x_l	x_u	x_r	ε_a (%)	ε_t (%)
1	0	1.3	0.65	100.0	35
2	0.65	1.3	0.975	33.3	2.5
3	0.975	1.3	1.1375	14.3	13.8
4	0.975	1.1375	1.05625	7.7	5.6
5	0.975	1.05625	1.015625	4.0	1.6

False Position

Iteration	x_l	x_u	x_r	ε_a (%)	ε_t (%)
1	0	1.3	0.09430		90.6
2	0.09430	1.3	0.18176	48.1	81.8
3	0.18176	1.3	0.26287	30.9	73.7
4	0.26287	1.3	0.33811	22.3	66.2
5	0.33811	1.3	0.40788	17.1	59.2



$$f(x) = x^{10} - 1$$

False Position: Example

- $f(x) = x^2 - 4$
- Step 1: Determine the initial conditions:

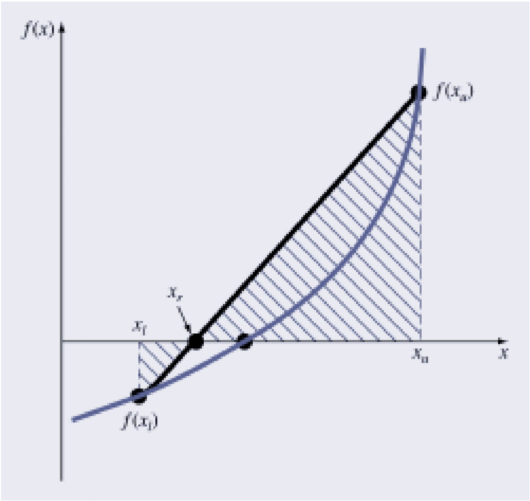
$$x_l = 0 \rightarrow f(x_l) = -4$$
$$x_u = 3 \rightarrow f(x_u) = 5$$
$$f(x_l)f(x_u) < 0$$

- Step 2: Estimate the roots

$$x_r = x_u - \frac{f(x_u)(x_l - x_u)}{f(x_l) - f(x_u)} \rightarrow x_r = 3 - \frac{5(0-3)}{5-(-4)} = 1.3333$$

$$error = |f(1.33)| = |\,2.2222\,|$$
$$x_l = x_r$$

- Step 4: Go to Step 2 until error is sufficiently small



False Position Example

Iteration					
1	0[-4]	3[+5]	1.33 [-2.22]	100	33.5
2	1.33[-2.22]	3[+5]	1.8462 [-0.5917]	27.7	8
3	1.8462 [-0.5917]	3[+5]	1.9683[-0.1260]	6.2	1.6
4	1.9683[-0.1260]	3[+5]	1.9936[-0.0255]	1.2	0.32
5	1.9936[-0.0255]	3[+5]	1.9987[-0.0051]	0.2	0.0065

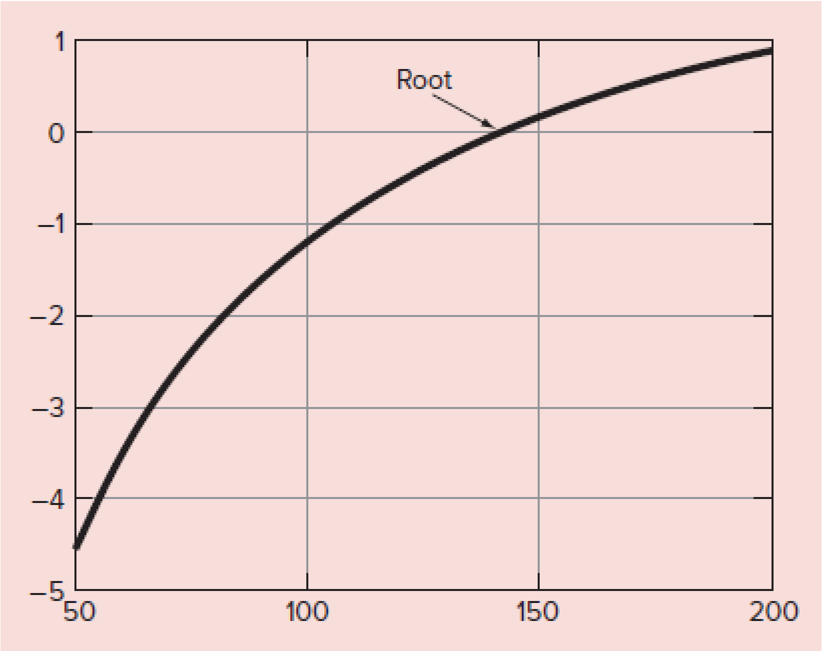
$$\varepsilon_t = \frac{\text{true value} - \text{approximation}}{\text{true value}} 100\%$$

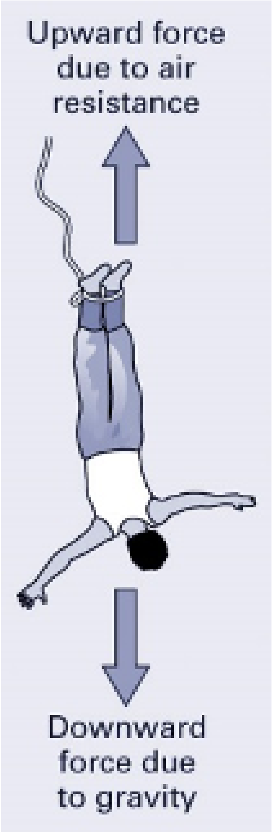
$$\varepsilon_a = \frac{\text{present approximation} - \text{previous approximation}}{\text{present approximation}} 100\%$$

False Position and Bisection Example

Use the bisection and false position approach to determine the mass of the bungee jumper with a drag coefficient of 0.25 kg/m to have a velocity of 36 m/s after 4 s of free fall. Note: The acceleration of gravity is 9.81 m/s^2.

$$v(t) = \sqrt{\frac{gm}{c_d}}\tanh\left(\sqrt{\frac{gc_d}{m}}t\right)$$

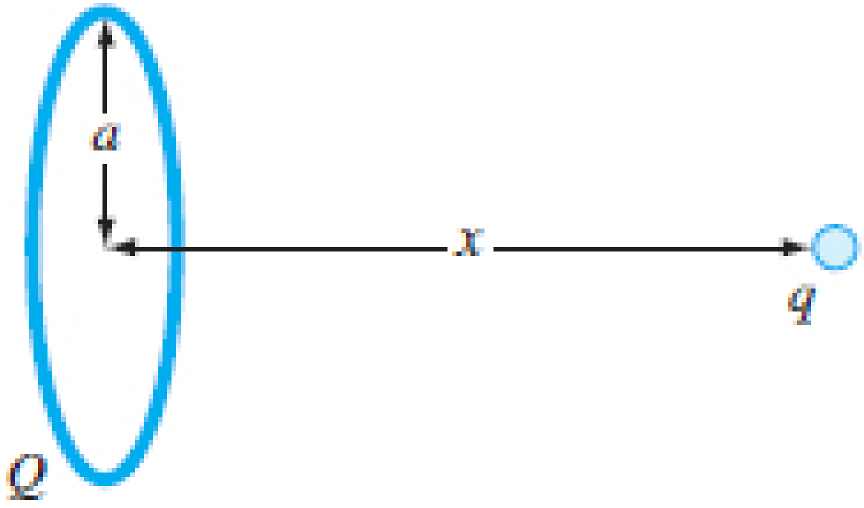




Problem 5.19

A total charge Q is uniformly distributed around a ring-shaped conductor with radius a. A charge q is located at a distance x from the center of the ring. The forced exerted on the charge by the ring is given by,

$$F = \frac{1}{4\pi e_0} \frac{qQx}{(x^2 + a^2)^{1.5}}$$



Where $e_0 = 8.9 \times 10^{-12}\ C^2/(Nm^2)$. Find the distance x where the force is 1.25 N if $q = Q = 2 \times 10^{-5}\ C$ for a ring with radius of $a = 0.85\ m$.

Hint: You need to find the root of the following function,

$$F(x) = F - \frac{1}{4\pi e_0} \frac{qQx}{(x^2 + a^2)^{1.5}}$$

to have an idea about the location of the roots, you can plot the variation of *F(x)* vs. *x* between *x*=0 and *x*=3 with an increment of 0.1, i.e., x = 0:0.1:3

```
#init parameters
#Define function
# plot function

  while (________)

        ~~~~~~~~

  end
```