CSI2110

Data Structures and Algorithms

Prof. WonSook Lee

Trying to overcome the limitations of Array based Stacks and Queues

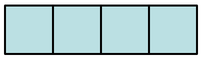
A Growable Array-Based Stack

Two strategies:

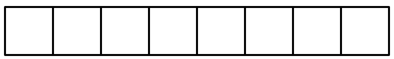
- **tight strategy**
- **growth strategy**

Growth Strategy

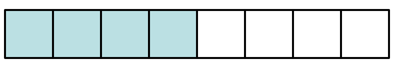
A full



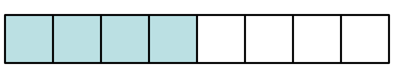
Create B

B 

Copy A into B

B 

Reassign reference A to the new array

A 

A Growable Array-Based Stack

Idea: when the array S is full, replace it with a larger one and continue processing push operations.

```
Algorithm push(obj):
    if size() = N then
        A ← new array of length f(N)
        for i ← 0 to N - 1
            A[i] ← S[i]
        S ← A
    t ← t + 1
    S[t] ← obj
```

- How large should the new array be?
  - **tight strategy** (add a constant): $f(N) = N + c$
  - **growth strategy** (double up): $f(N) = 2N$

Tight vs. Growth Strategies: a comparison

- To compare the two strategies, we use the following cost model:

OPERATION	RUN TIME
regular push operation: add one element	1
special push operation: create an array of size f(N), copy N elements, and add one element	f(N)+N+1

Tight Strategy (c=4)

$f(N) = N + c = N + 4$

- start with an array of size 0
- the cost of a special push is

$f(N)+N+1 = 2N + 5$

$= N + c + N + 1$

$= 2N + c + 1$ in general

push	phase	N	cost
1	1	0	5
2	1	4	1
3	1	4	1
4	1	4	1
5	2	4	13
6	2	8	1
7	2	8	1
8	2	8	1
9	3	8	21
10	3	12	1
11	3	12	1
12	3	12	1
13	4	12	29

$c = 4$

$5 = 2N + c + 1 = c+1$

1

1

1

$\Big\} \; c-1$

$13 = 2c + c + 1$

Now $N = c$

1

1

1

21

Now $N = 2c$

$= 2(2c) + c + 1$

1

1

1

push	phase	N	cost
1	1	0	5
2	1	4	1
3	1	4	1
4	1	4	1
5	2	4	13
6	2	8	1
7	2	8	1
8	2	8	1
9	3	8	21
10	3	12	1
11	3	12	1
12	3	12	1
13	4	12	29

Phase 1: $c+1 + (c-1) = 2c$

Phase 2: $2c + c + 1 + (c-1) = 4c$

Phase 3: $2(2c) + c + 1 + (c-1) = 6c$

Phase $i = 2ci$

Performance of the Tight Strategy

- We consider k phases, where $k = n/c$
- Each phase corresponds to a new array size
- The cost of phase i is $2ci$
- The total cost of n push operations is the total cost of k phases, with $k = n/c$:

  $$2c\,(1 + 2 + 3 + \ldots + k),$$

  which is $O(k^2)$ and $\mathbf{O(n^2)}$.

Growth Strategy

(double up): f(N) = 2N

- start with an array of size 0, then 1, 2, 4, 8, ...
- the cost of a special push is

$f(N)+N+1$ =

$2N + N + 1$ =

$3N + 1$

for $N>0$

push	phase	N	cost
1	0	0	1
2	1	1	4
3	2	2	7
4	2	4	1
5	3	4	13
6	3	8	1
7	3	8	1
8	3	8	1
9	4	8	25
10	4	16	1
11	4	16	1
12	4	16	1
...	...	...	...
16	4	16	1
17	5	16	49

Phase $i > 1$

Special

Normal

...

Normal

$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} 2^{i-1}$

push	phase	N	cost
1	0	0	1
2	1	1	4
3	2	2	7
4	2	4	1
5	3	4	13
6	3	8	1
7	3	8	1
8	3	8	1
9	4	8	25
10	4	16	1
11	4	16	1
12	4	16	1
...	...	...	...
16	4	16	1
17	5	16	49

Cost of special operations: $3N + 1$

Phase 0: 1 $\quad 0 + 1$

Phase 1: 4 $\quad 3 \cdot 1 + 1$

Phase 2: 7 $\quad 3 \cdot 2 + 1$

Phase 3: 13 $\quad 3 \cdot 4 + 1$

Phase i: $3 \cdot 2^{i-1} + 1$

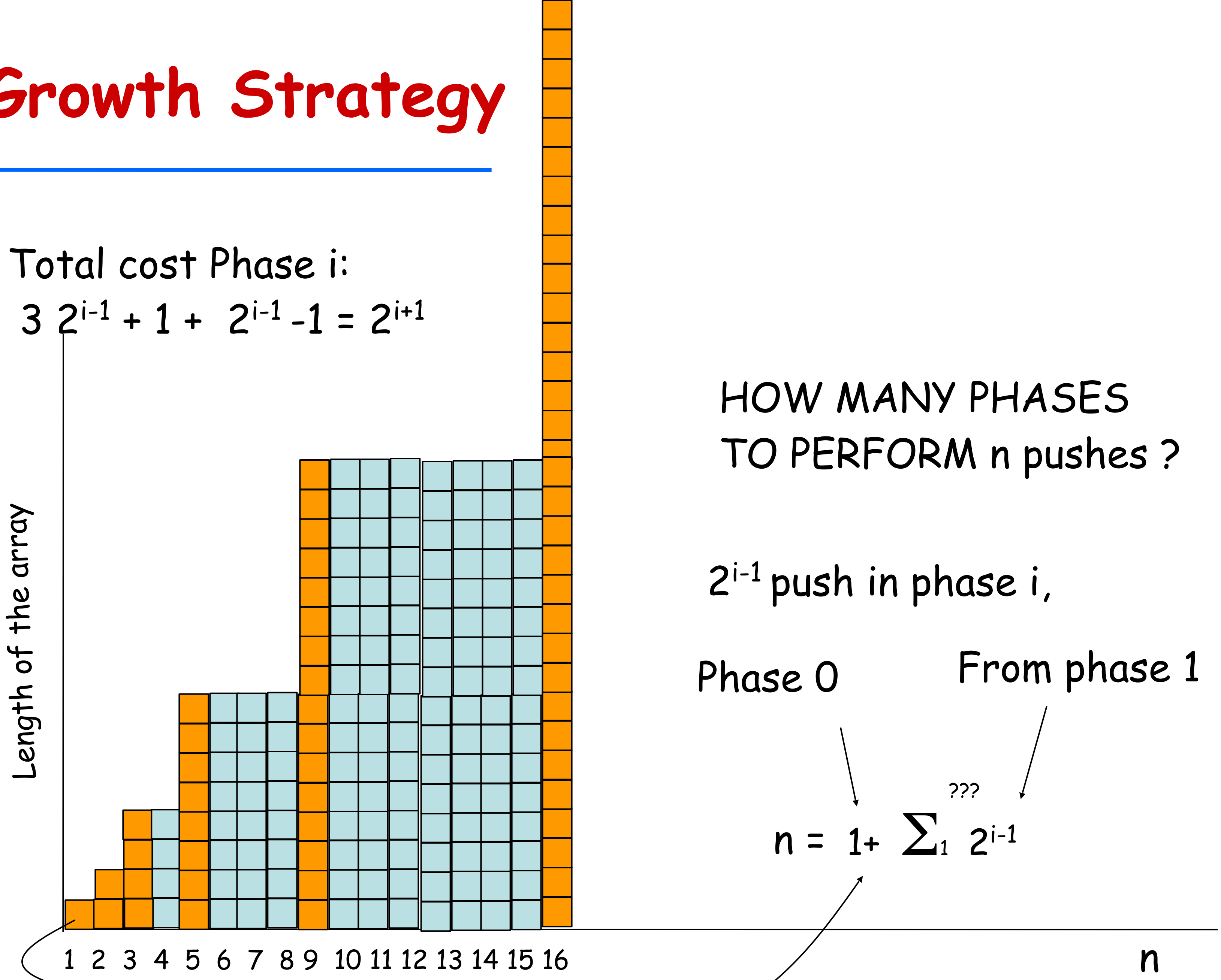
TOT Phase i: $2^{i-1} + 3 \cdot 2^{i-1} + 1$

$= 2^{i+1}$

Growth Strategy

Total cost Phase i:

$3\ 2^{i-1} + 1 + 2^{i-1} - 1 = 2^{i+1}$



HOW MANY PHASES
TO PERFORM n pushes ?

2^{i-1} push in phase i,

Phase 0

From phase 1

$$n = 1 + \sum_{1}^{???} 2^{i-1}$$

Performance of the Growth Strategy

- We consider k phases, where $k = \log n$
- Each phase corresponds to a new array size
- The cost of phase i is 2^{i+1}
- The total cost of n push operations is the total cost of k phases, with $k = \log n$
- $2 + 4 + 8 + \ldots + 2^{\log n + 1} = 2\,(1 + 2 + \ldots + 2^{\log n})$

  $= 2((2^{\log n + 1} - 1)/(2-1)) = 2\,(2(n) - 1) = 4n - 2$

- $S = \sum_{i=0}^{m} r^i = 1 + r + r^2 + \ldots + r^m = (r^{m+1}-1)/(r-1)$
  - $b = a^{\log_a b} \to b = 2^{\log b} \to n = 2^{\log n}$
- The growth strategy wins! **O(n)**

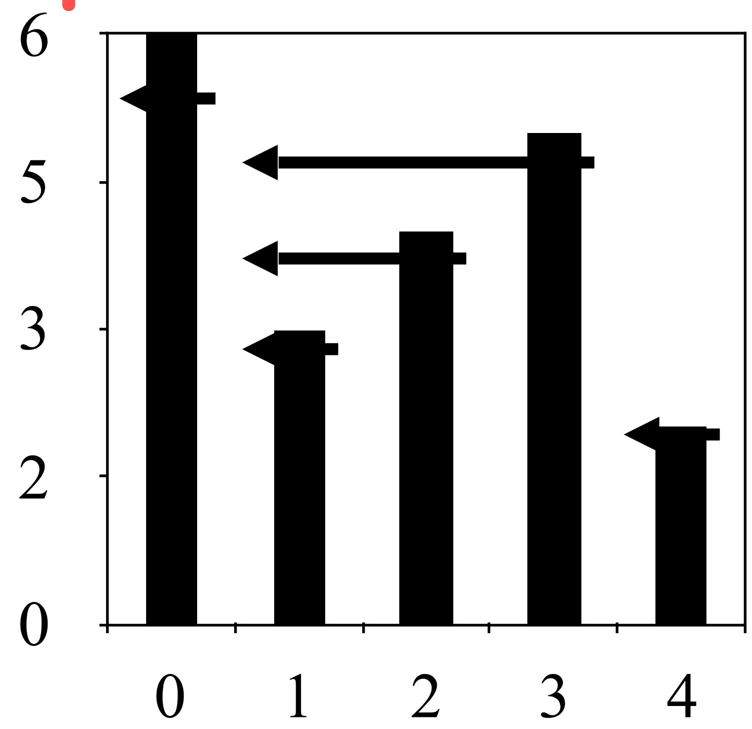
Another Example of the use of a Stack

Computing Spans

Given a series of n daily price quotes for a stock, we call the span of the stock's price on a certain day the maximum number of consecutive days up to the current day that the price of the stock has been less than or equal to its price on that day.

Computing Spans

- Stack as an auxiliary data structure in an algorithm
- Given an an array **X**, the span **S[i]** of **X[i]** is the maximum number of consecutive elements **X[j]** immediately preceding **X[i]** and such that **X[j]** ≤ **X[i]**
- Spans have applications to financial analysis
  - E.g., stock at 52-week high



X	6	3	4	5	2
S	1	1	2	3	1

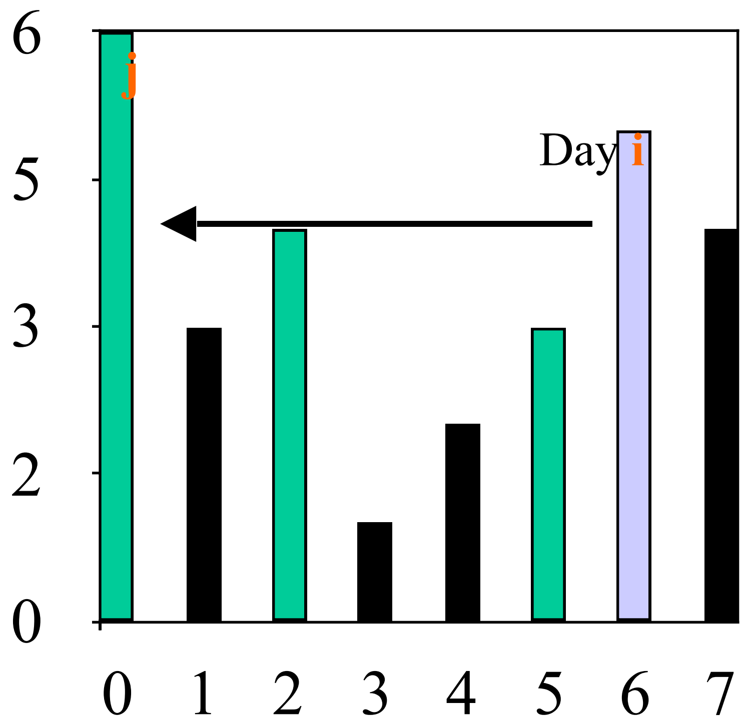
Quadratic Algorithm

	#
Algorithm ***spans1***(X, n)	
Input array X of n integers	
Output array S of spans of X	#
$S \leftarrow$ new array of n integers	n
for $i \leftarrow 0$ **to** $n - 1$ **do**	n
$s \leftarrow 1$	n
while $s \leq i \wedge X[i - s] \leq X[i]$	$1 + 2 + \ldots + (n - 1)$
$s \leftarrow s + 1$	$1 + 2 + \ldots + (n - 1)$
$S[i] \leftarrow s$	n
return S	1

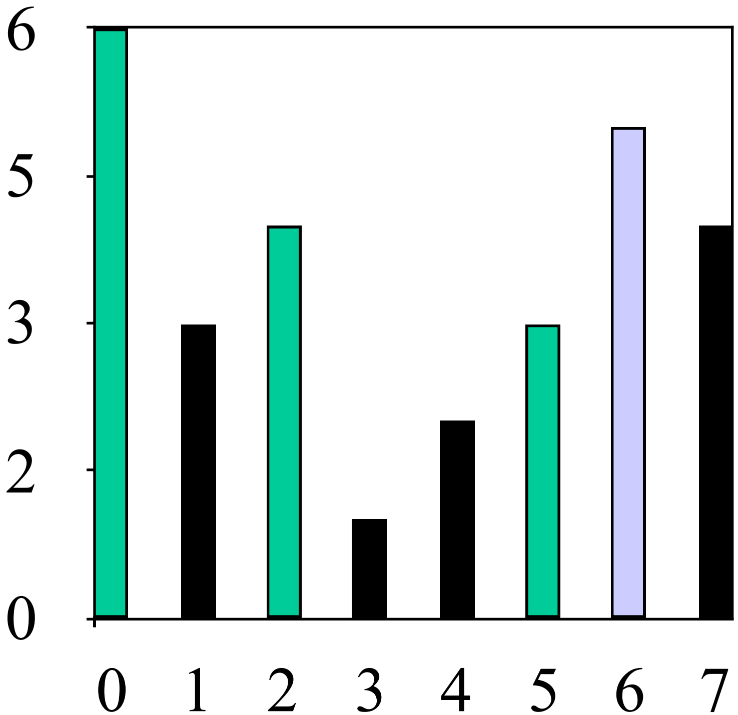
- Algorithm ***spans1*** runs in $O(n^2)$ time

Computing Spans with a Stack

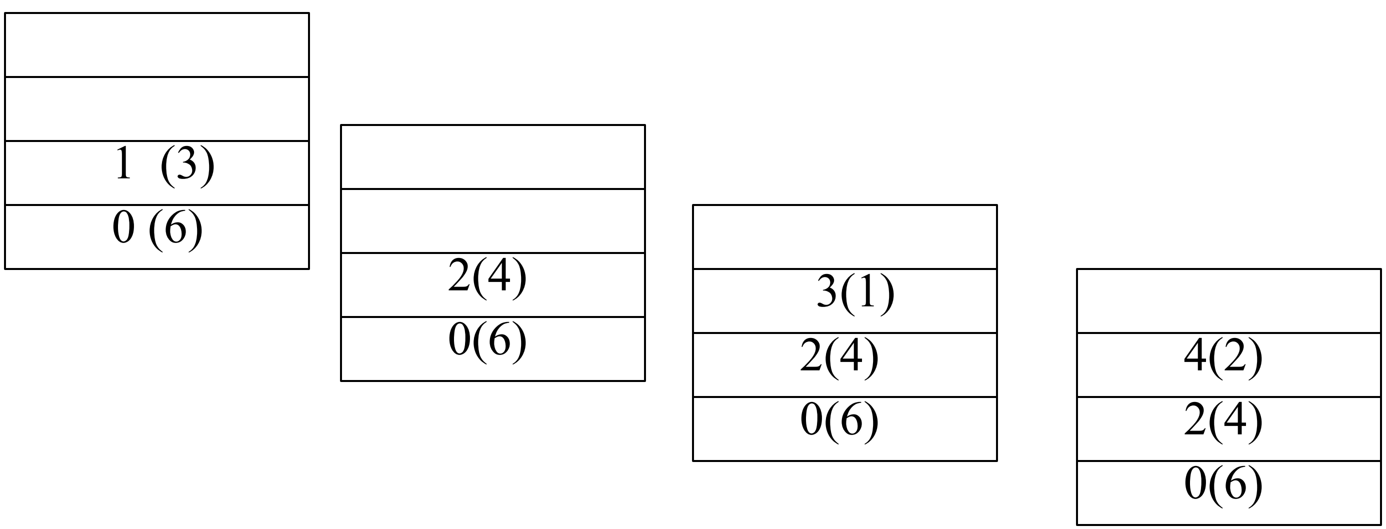
Closest higher element preceding i



$$S_i = i - j$$



Index:	0, 1, 2, 3, 4, 5, 6, 7
X :	6, 3, 4, 1, 2, 3, 5, 4
S :	1, 1, 2, 1, 2, 3, 6, 1



- We keep in a stack the indices of the elements visible when "looking back"

- We scan the array from left to right
  - Let **i** be the current index

  - We pop indices from the stack until we find index **j** such that

    $$X[j] > X[i]$$

  - We set $S[i] \leftarrow i - j$

  - We push **i** onto the stack

Linear Algorithm

- Each index of the array
  - Is pushed into the stack exactly one
  - Is popped from the stack at most once
- The statements in the while-loop are executed at most ***n*** times
- Algorithm ***spans2*** runs in ***O***(***n***) time

```
Algorithm computeSpans2(P):
Input: an n-element array P of numbers
    representing stock prices
Output: an n-element array S of numbers such
    that S[i] is the span of the stock on day i
    Let D be an empty stack
    for i ←0 to n - 1 do k ← 0
      done ← false
      while not(D.isEmpty() or done) do
        if P[i] ≥ P[D.top()] then
          D.pop()
        else done ← true
      if D.isEmpty() then h ← -1
      else h ← D.top()
      S[i] ← i - h
      D.push(i)
    return S
```

Ex1.

Give a big-Oh characterization, in terms of *n*, of the running time of the following method. Show your analysis!

```
public void Ex(int n)
     int a ;
     for (int i = 0 ; i < n*n ; i++)
               for (int j = 0; j <= i; j++)
                         a = i
}
```

Give a big-Oh characterization, in terms of *n*, of the running time of the following method. Show your analysis!

```
public void Ex(int n)
      int a ;
      for (int i = 0 ; i < n*n ; i++)
                for (int j = 0; j <= i; j++)
                          a = i
}
```

the inner loop j goes from 1 to i

the outer loop i goes from 1 to $n*n=n^2$

the total for the two loops will be $1+2+3 + \ldots + n^2$

which equals $n^2(n^2+1)/2 = (n^4+ n^2)/2 < (n^4+ n^4)/2 = n^4 = O(n^4)$

The Ex method runs in $O(n^4)$ time.

Ex2.

Prove that 2^{n+1} is $O(2^n)$ using definition of O.

Prove that 2^{n+1} is $O(2^n)$ using definition of O.

By the definition of big-Oh, we need to find a real constant $c > 0$ and an integer constant $n_0 \geq 1$ such that $2^{n+1} \leq c(2^n)$ for $n \geq n_0$. One possible solution is choosing $c = 2$ and $n_0 = 1$.

Ex3.

In big-Oh notation, what is the running time of the following code fragment:

```
public void EX5(int n)
{
int a=0;
for (int i=0; i< n; i++)
    for(int j=0; j<=i; j++)
        for( int k=0; k<=n; k++)
            a+=i;
}
```

Show your analysis!

```
public void EX5(int n){
int a=0;
for (int i=0; i< n; i++)
for(int j=0; j<=i; j++)
for( int k=0; k<=n; k++)
a+=i; }
```

Solution:

Initializing the integer variable "a" at the beginning of the code fragment takes O (1) time.

There are three nested loops in this code fragment. The inner most loop, controlled by counter k adds O(n) operations every time it is executed. But how often is it executed?

Consider the outer 2 loops. When i is 0, the j loop executes once; when i is 1, j executes twice, and so on giving:
$1 + 2 + 3 + ... + n = n(n+1)/2$ or $O(n^2)$.
Since the k loop within the j loop has O(n), we get $O(n^* n^2) = O(n^3)$.

Ex4.

Describe how to implement the stack ADT using two queues. That is: write pseudocode algorithms which implement the push() and pop() methods of the stack using the methods of the queue. What are the running times of your push () and pop () algorithms?

Describe how to implement the stack ADT using two queues. That is: write pseudocode algorithms which implement the push() and pop() methods of the stack using the methods of the queue. What are the running times of your push () and pop () algorithms?

Assume that we have two queues named ***Q1*** and ***Q2***.
Q1 will have store the stack elements assuming the front element is the top of the stack. ***Q2*** will be used as temporary storage when pushing new element.

When we want to pop an element, we can just perform dequeue on ***Q1*** (Thus we are taking the top of the stack). While when we want to push an element ***o***, we should dequeue all the elements from ***Q1*** and enqueue them in ***Q2***, then enqueue the element ***o*** in ***Q1*** and then dequeue all the elements from ***Q2*** and enqueue them in ***Q1***. The code follows:

Pseudocode algorithms which implement the push() and pop() methods of the stack using the methods of the queue.

Algorithm Pop()
If Q1.isEmpty() then
        ERROR
Return Q1.dequeue()

Algorithm push(object o)
While ! Q1.isEmpty
      Q2.enqueue(Q1.dequeue())

Q1.enqueue(o)

While ! Q2.isEmpty
      Q1.enqueue(Q2.dequeue())

Pop will be O(1)
Push will be O(n)