More efficient Sorting Algorithms

Recursive Sorts

Recursive sorts Divide the data roughly in half and are called Again on the smaller data sets. This is called the Divide-and-Conquer paradigm. We will see 2 recursive sorts:

- Merge Sort
- QuickSort

Divide-and-Conquer

- Divide-and-conquer paradigm:
  - Divide: divide one large problem into 2 smaller problems of the same type.
  - Recur: solve the 2 subproblems.
  - Conquer: combine the 2 solutions into a solution to the larger problem.
- The base case for the recursion are subproblems of manageable size, usually 0 or 1.

Merge-sort is a sorting algorithm based on the divide-and-conquer paradigm

Merge Sort

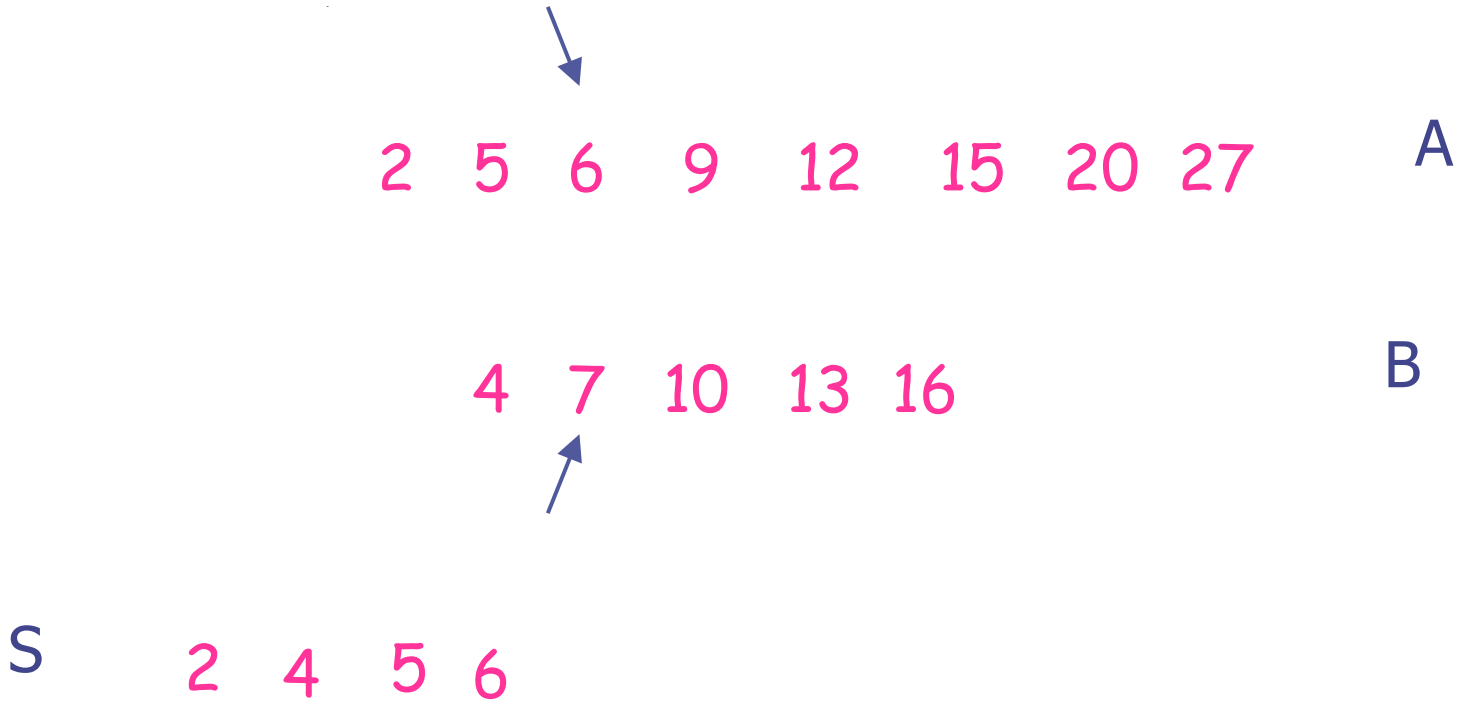
Merge-Sort

Merge-sort on an input sequence S with n elements consists of three steps:

- Divide: partition into 2 groups of about n/2 each
- Recur: recursively sort S_1 and S_2
- Conquer: merge S_1 and S_2 into a unique sorted sequence

Merging Two Sorted Sequences

- The conquer step merges the 2 sorted sequences A and B into one sorted sequence S
- ***How: Compare the lowest element of each of A and B and insert whichever is smaller.***
- Merging two sorted sequences, each with $n/2$ elements and implemented by means of a doubly linked list, takes $O(n)$ time


2 5 6 9 12 15 20 27
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4 7 10 13 16
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Merging Two Sorted Sequences

```
Algorithm merge(A, B)
    Input sorted sequences A and B
    Output sorted sequence of A ∪ B

    S ← empty sequence
    while !A.isEmpty() ∧ !B.isEmpty()
        if isLessThan(A.first().element(), B.first().element())
            S.insertLast(A.remove(A.first()))
        else
            S.insertLast(B.remove(B.first()))
    while !A.isEmpty()
            S.insertLast(A.remove(A.first()))
    while !B.isEmpty()
            S.insertLast(B.remove(B.first()))
    return S
```

Not In-Place

Merge-Sort

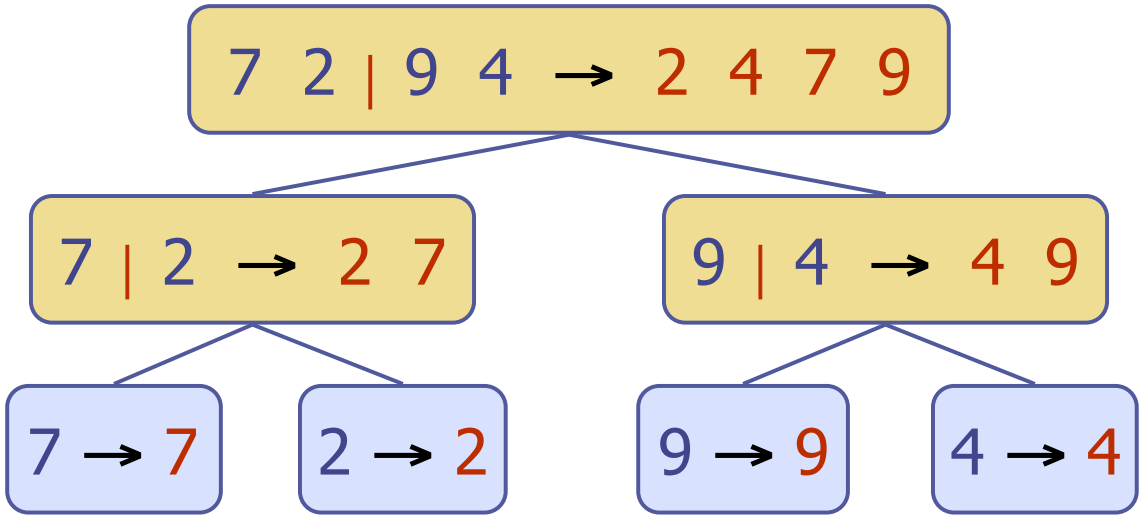
Algorithm *mergeSort*(*S*)
  Input sequence *S* with *n* elements,
  Output sequence *S* sorted
  if *S.size*() > 1
    $(S_1, S_2) \leftarrow$ *partition*(*S*, *n*/2)
    mergeSort(S_1)
    mergeSort(S_2)
    $S \leftarrow$ *merge*(S_1, S_2)

Not In-Place

Merge-Sort Tree

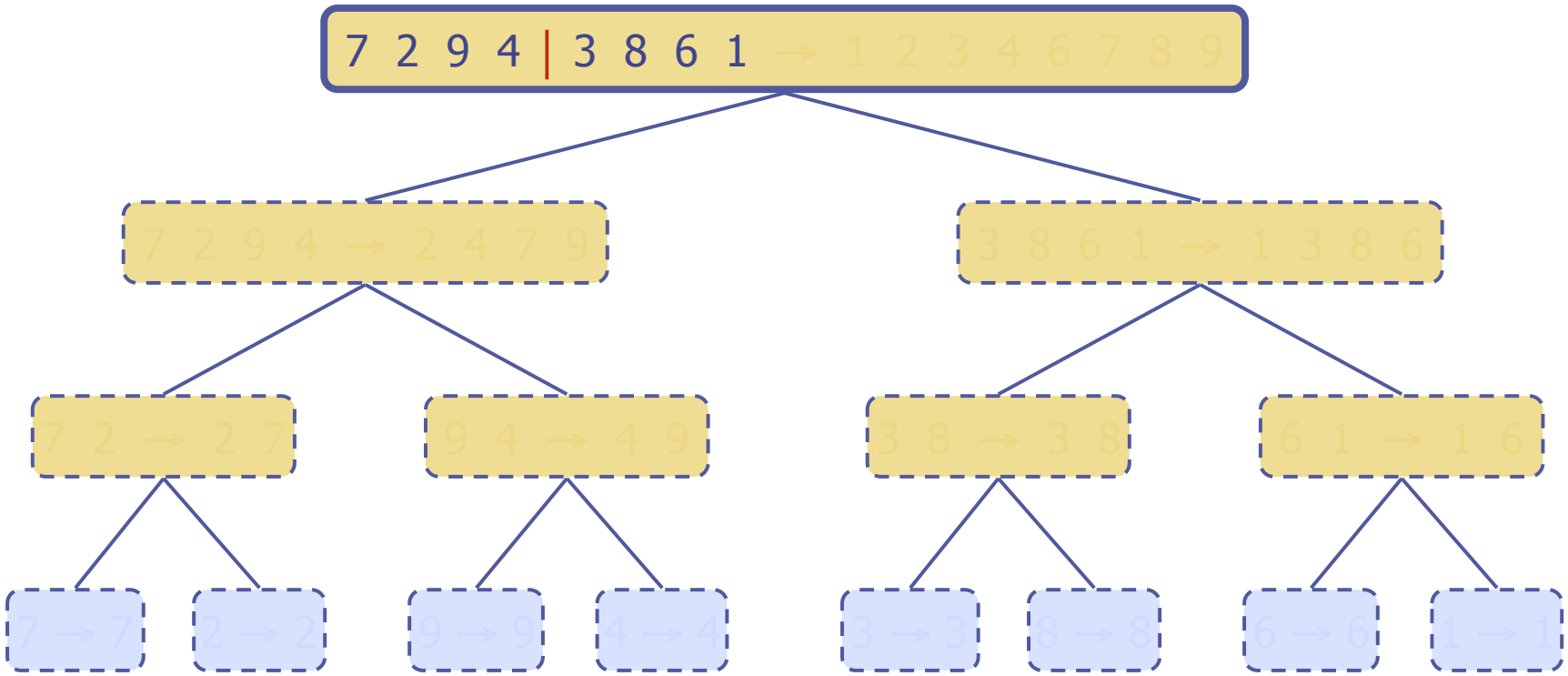
An execution of merge-sort is depicted by a binary tree

- each node represents a recursive call of merge-sort and stores
  - unsorted sequence before the execution and its partition
  - sorted sequence at the end of the execution
- the root is the initial call
- the children are calls on subsequences
- the leaves are calls on sequences of size 0 or 1



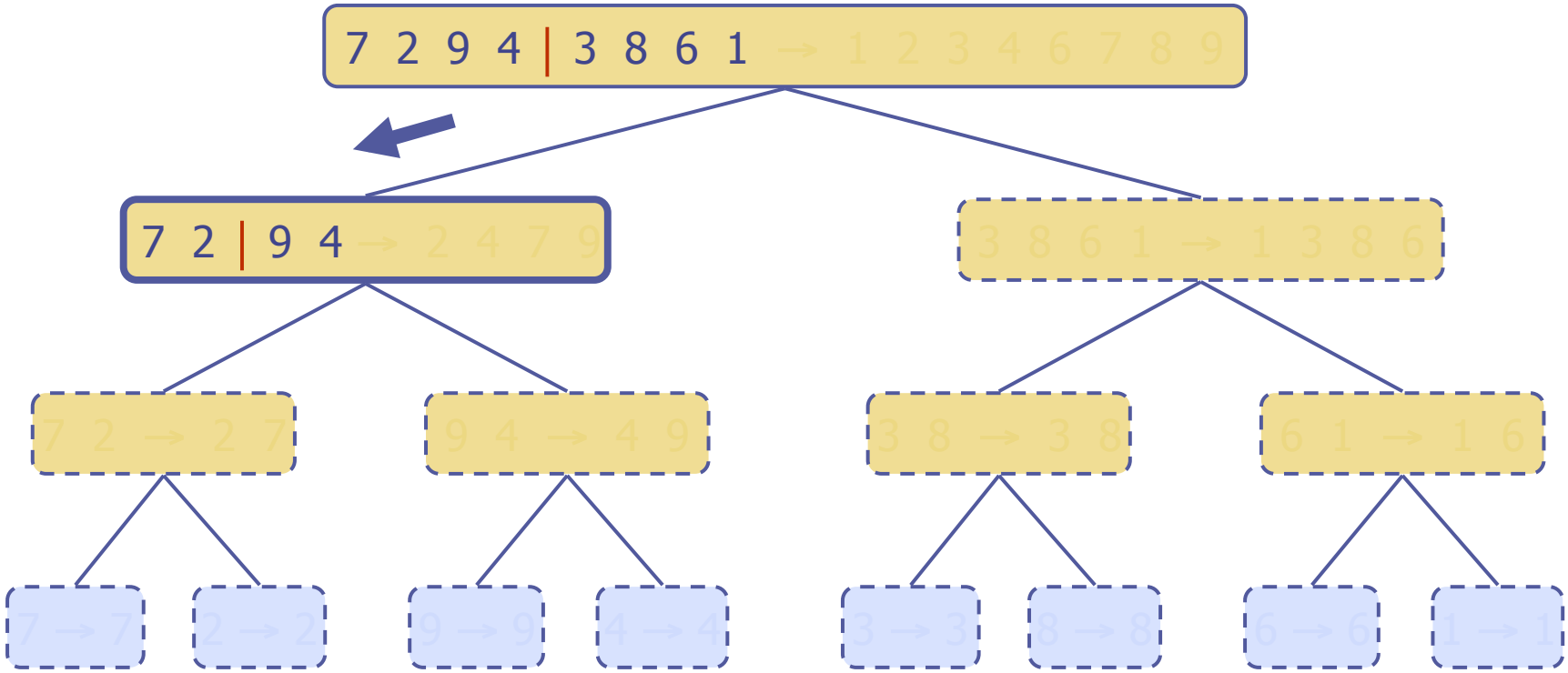
Execution Example

- Partition



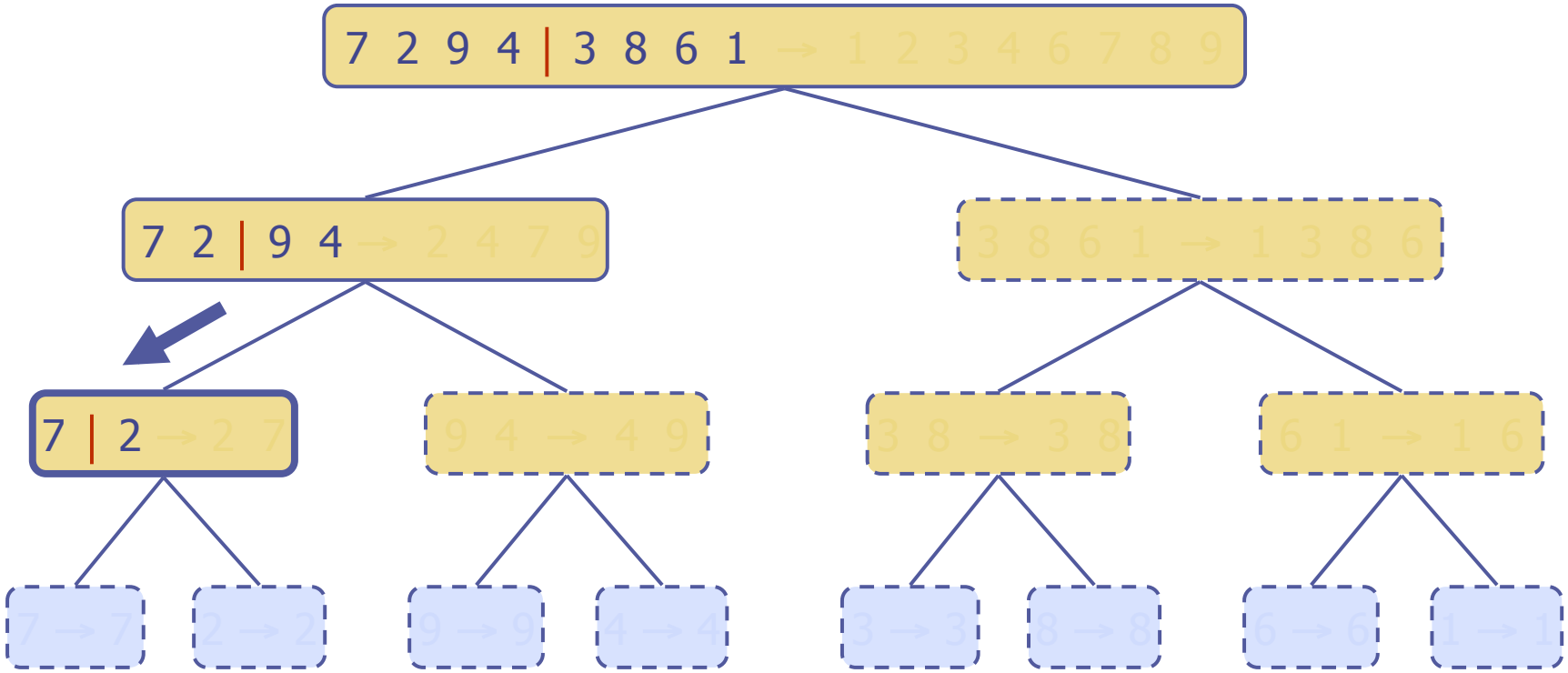
Execution Example (cont.)

- Recursive call, partition



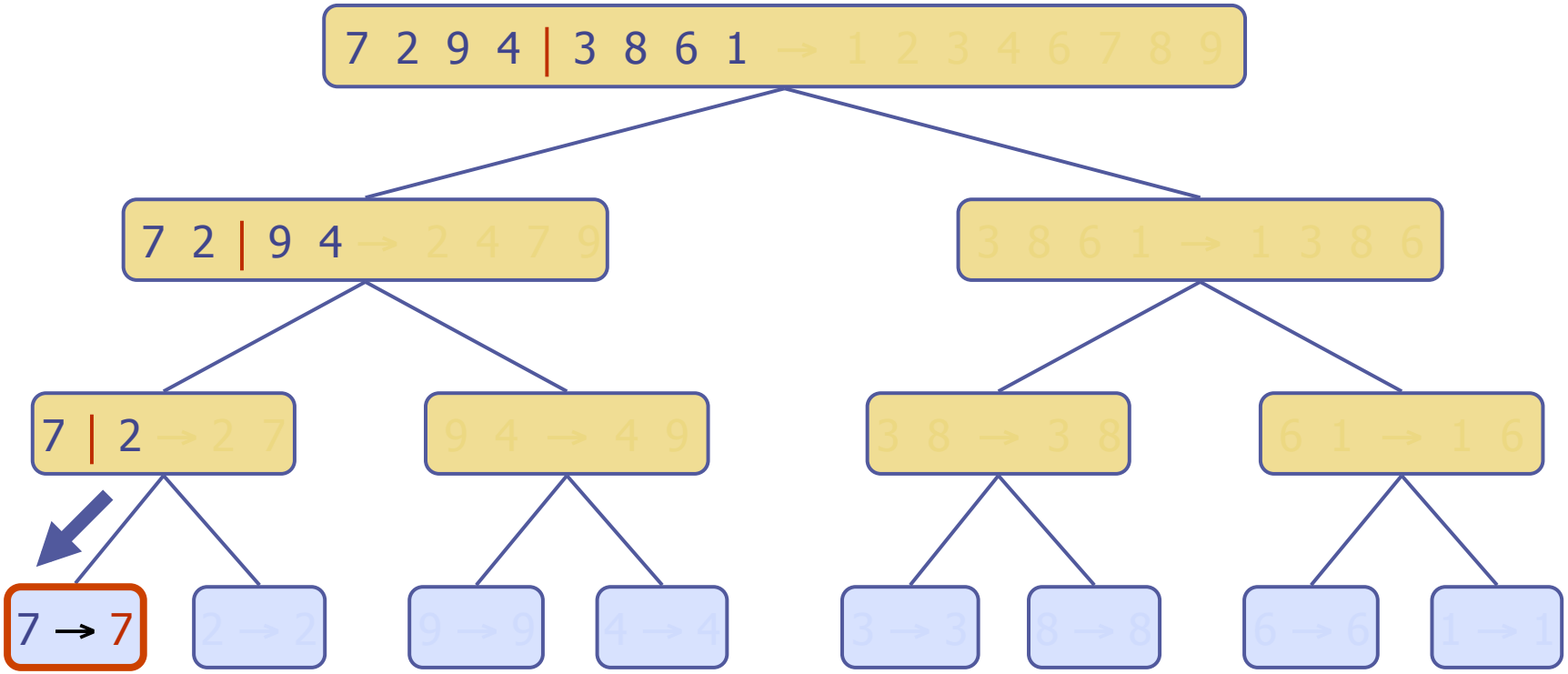
Execution Example (cont.)

- Recursive call, partition



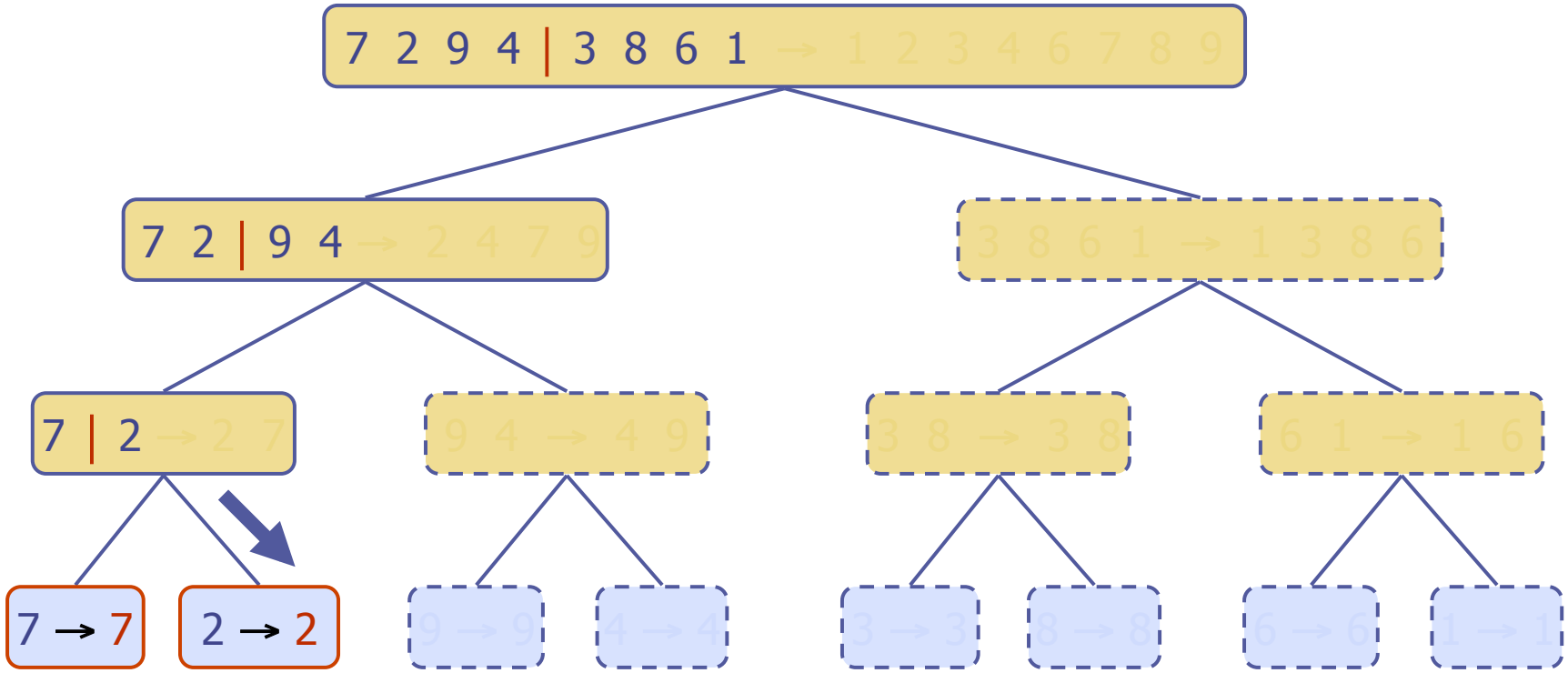
Execution Example (cont.)

- Recursive call, base case



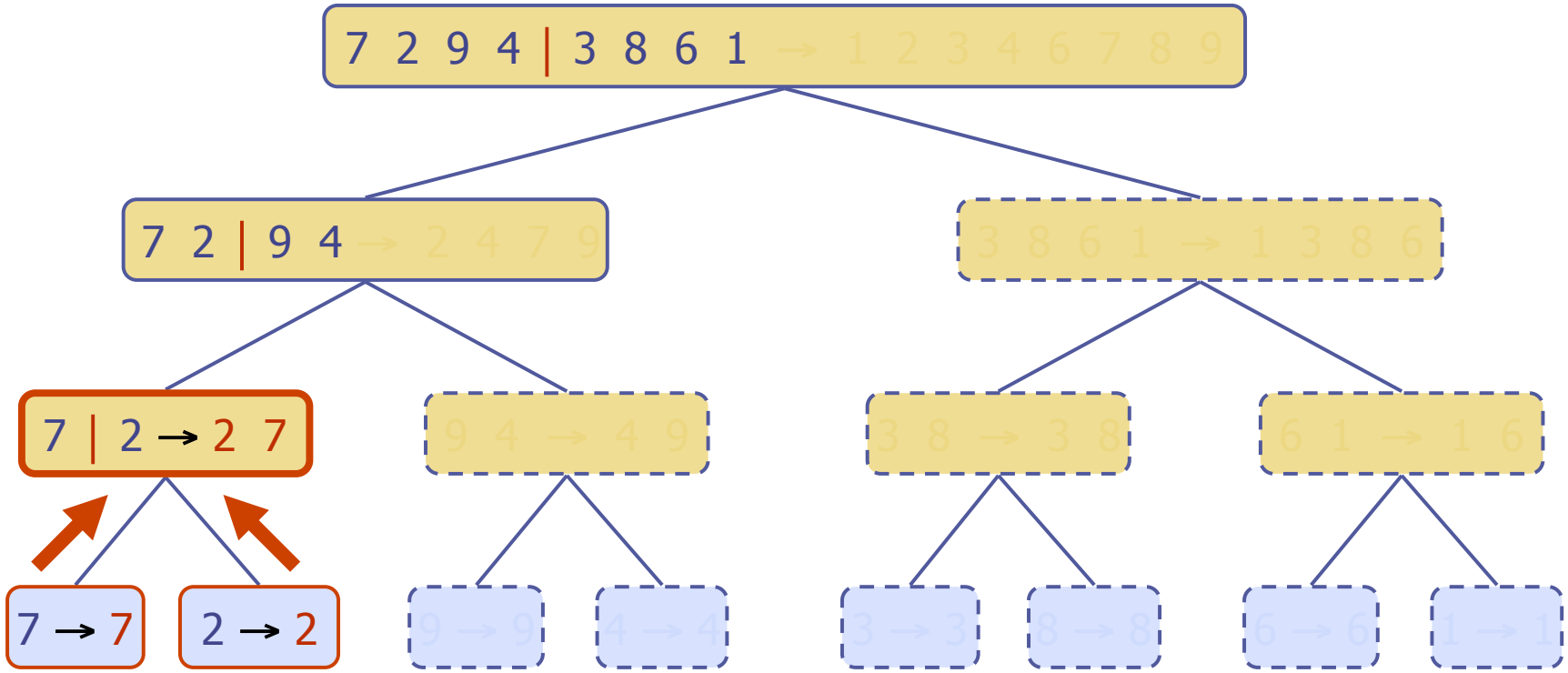
Execution Example (cont.)

- Recursive call, base case



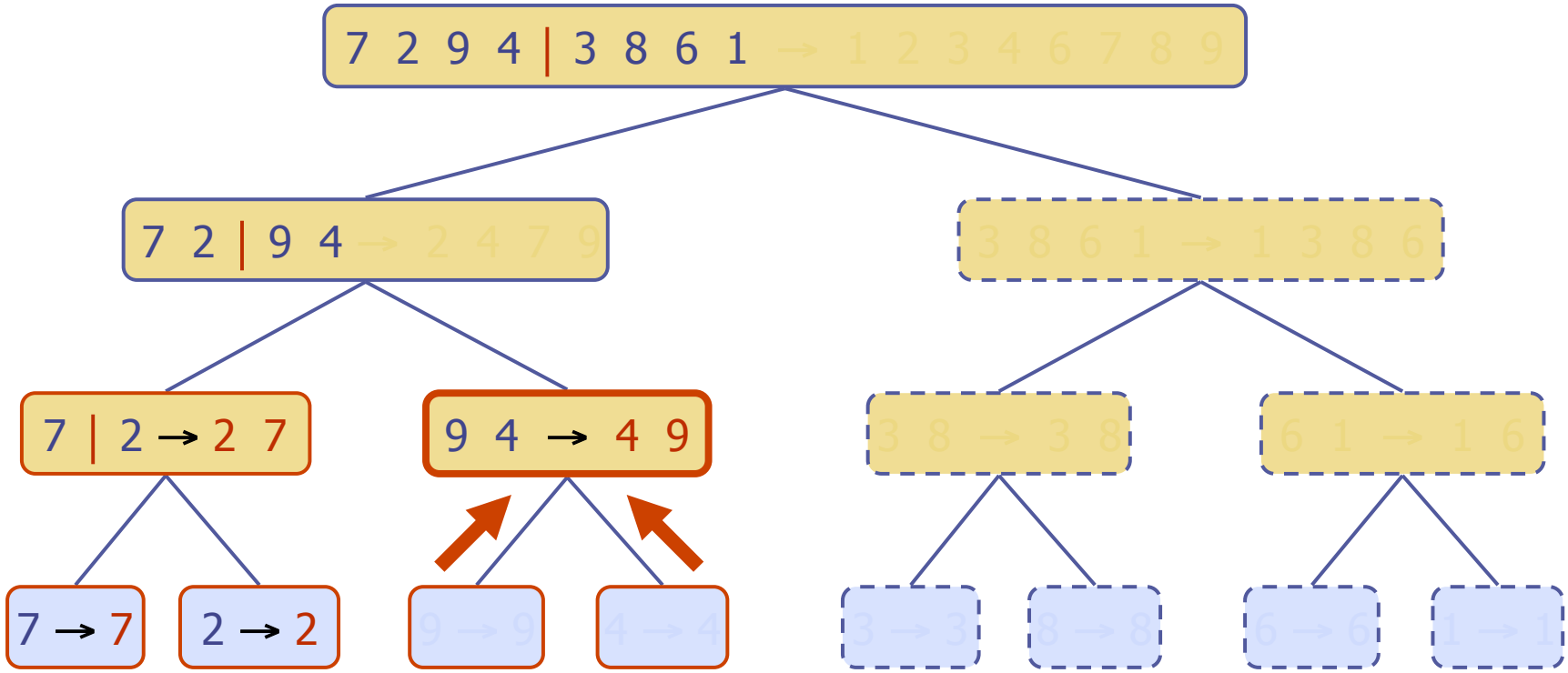
Execution Example (cont.)

- Merge



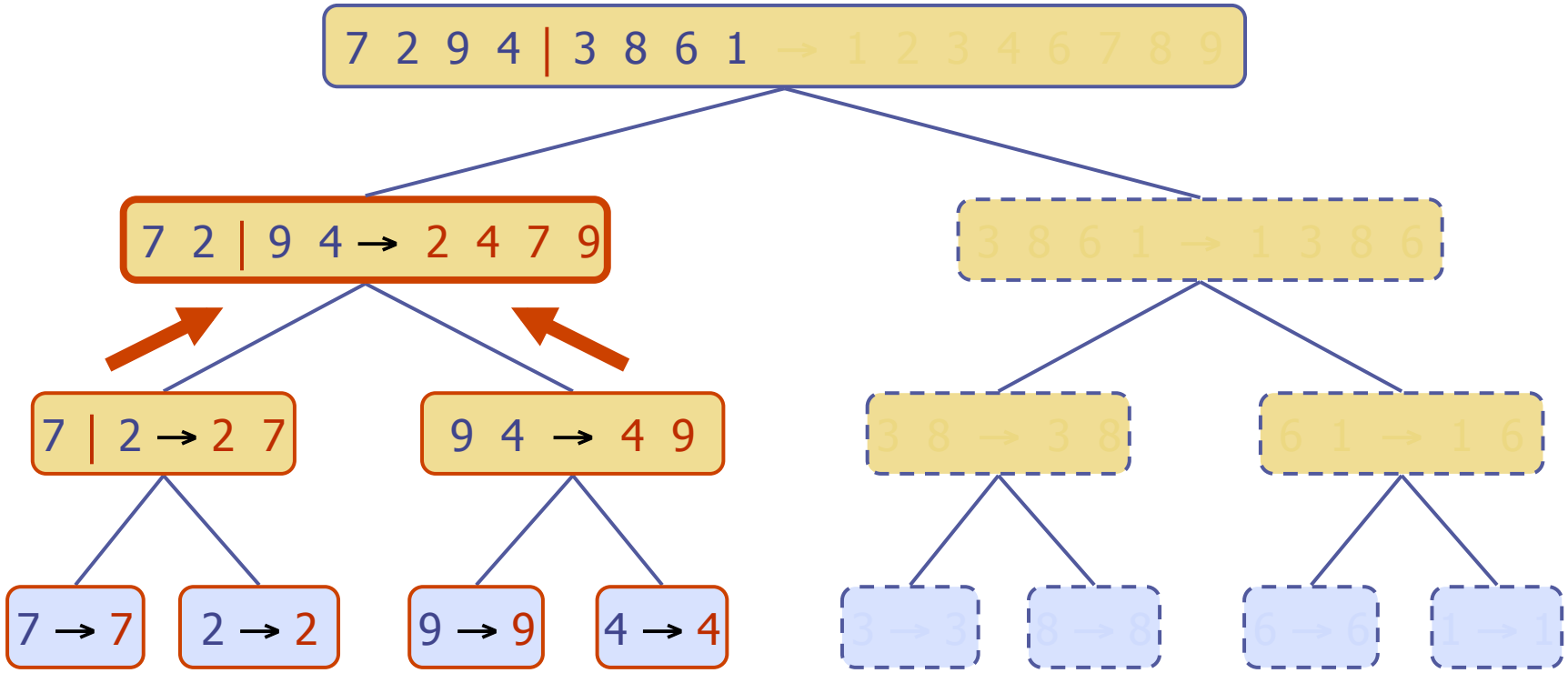
Execution Example (cont.)

- Recursive call, …, base case, merge



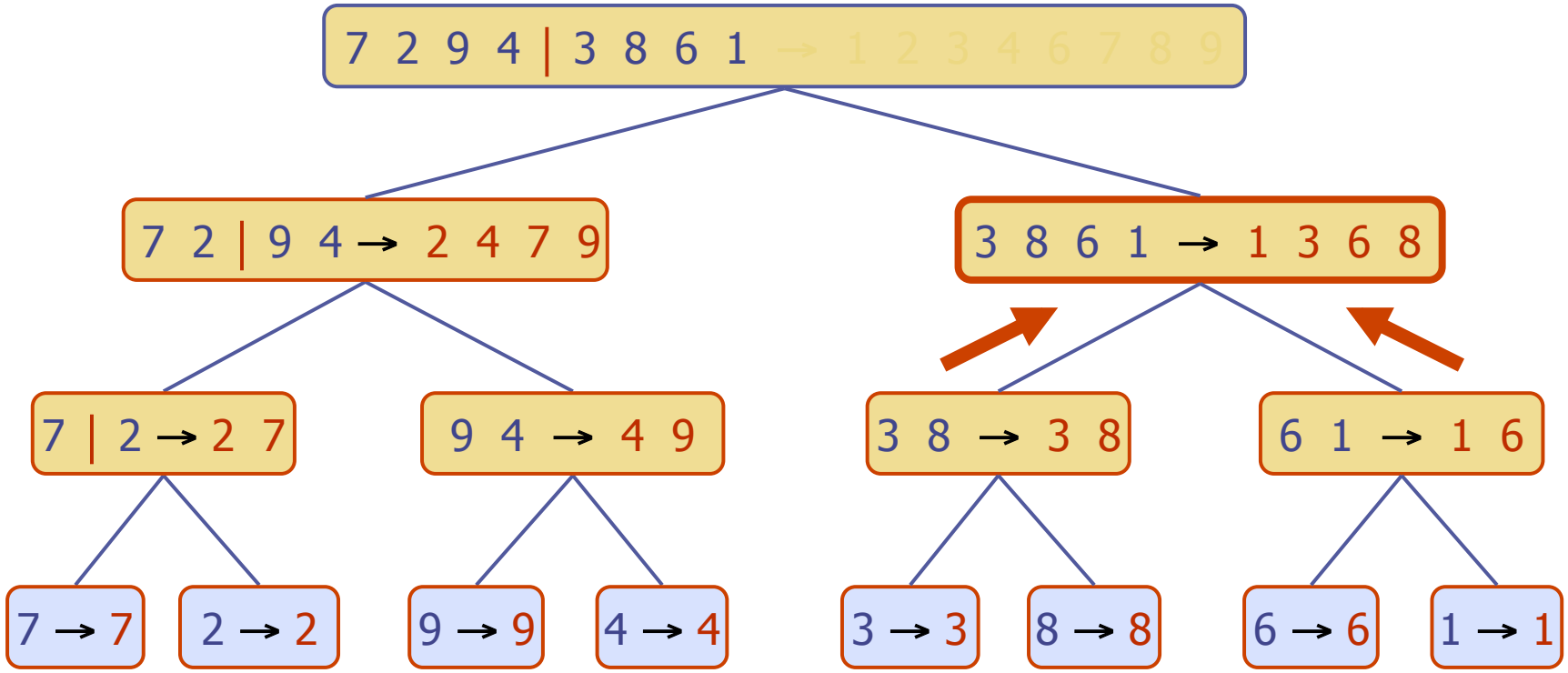
Execution Example (cont.)

- Merge



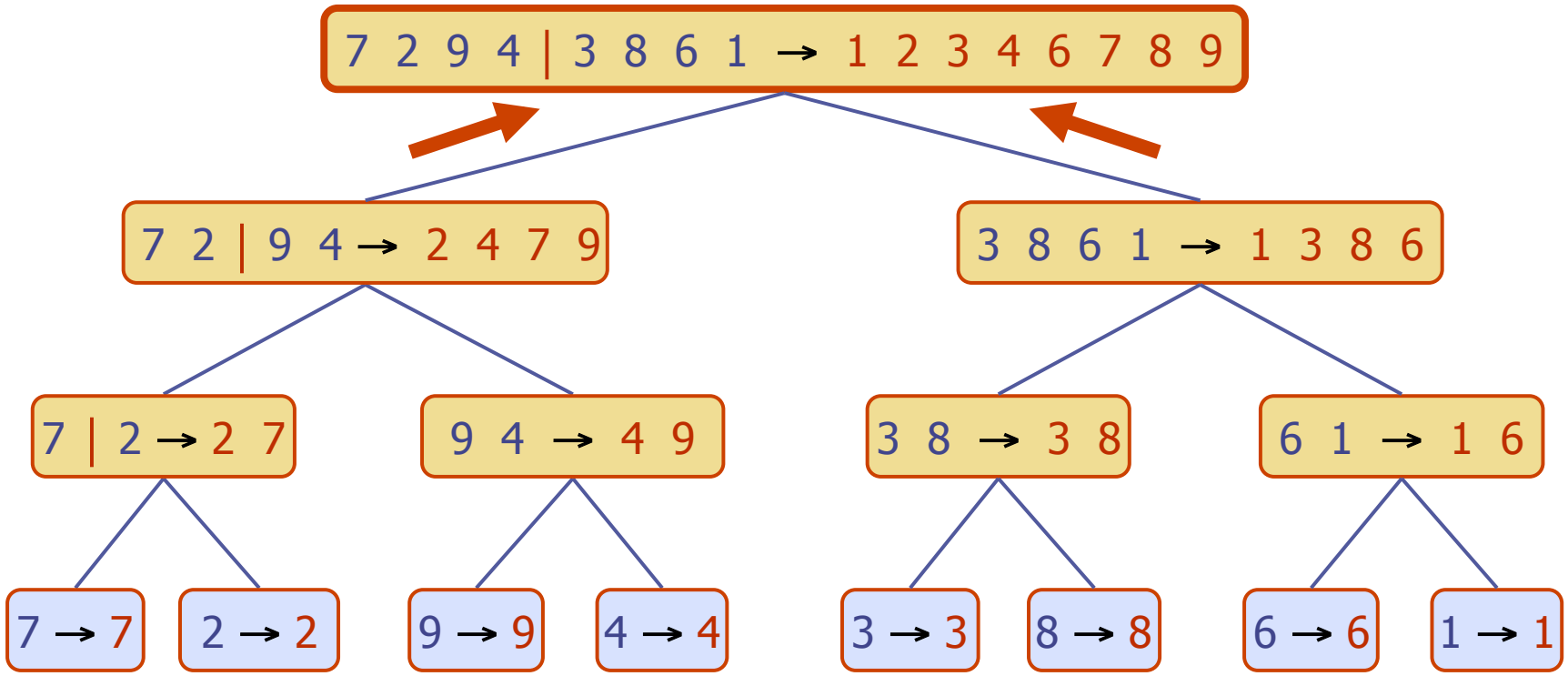
Execution Example (cont.)

- Recursive call, …, merge, merge



Execution Example (cont.)

- Merge

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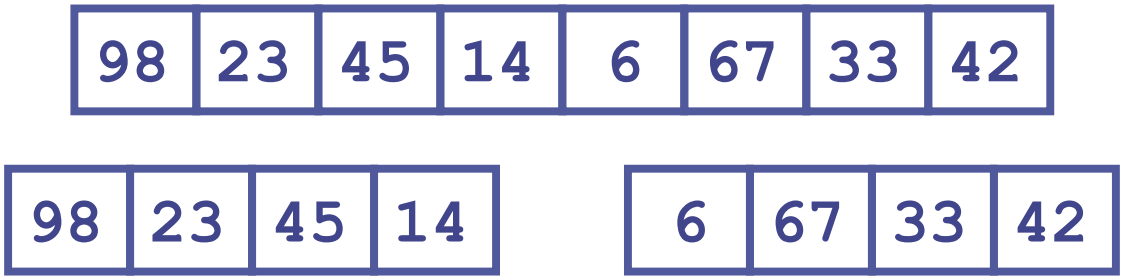
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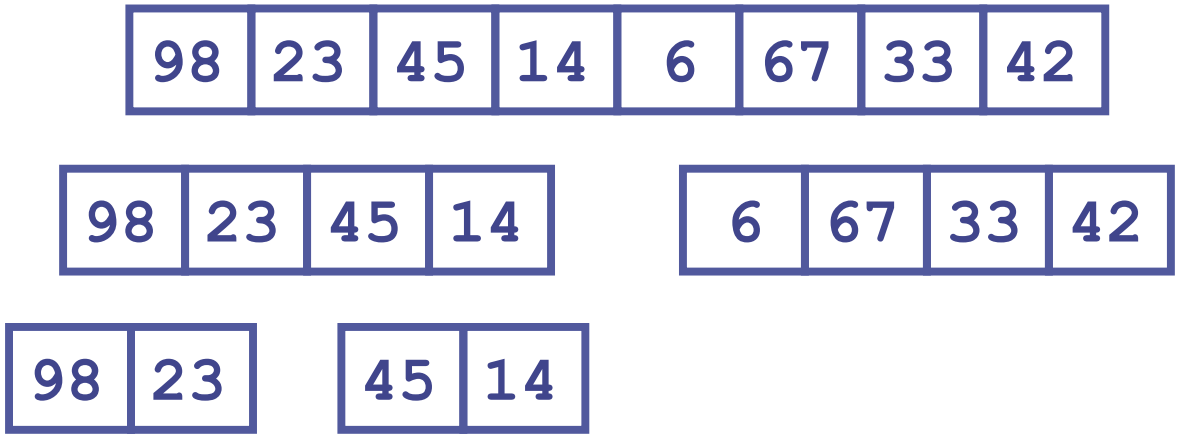
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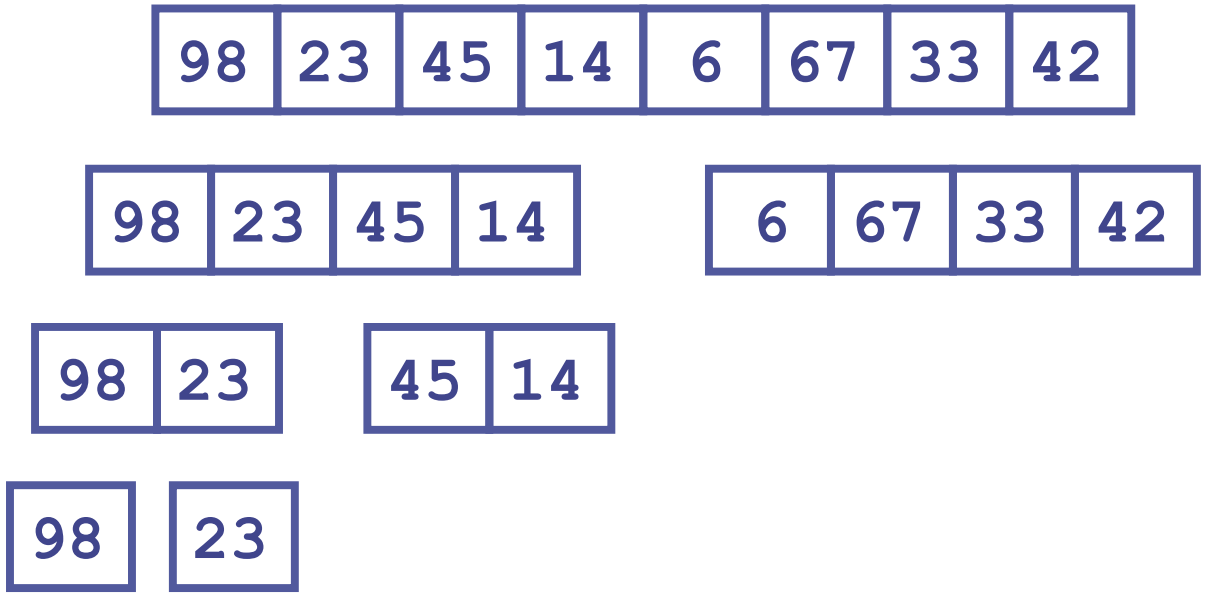
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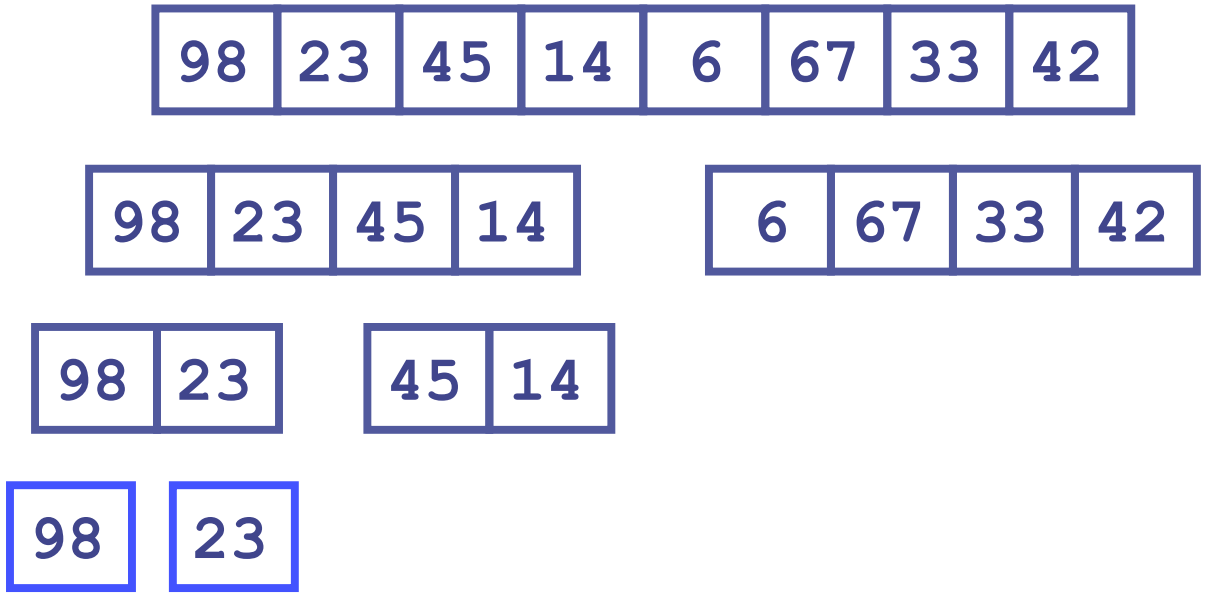
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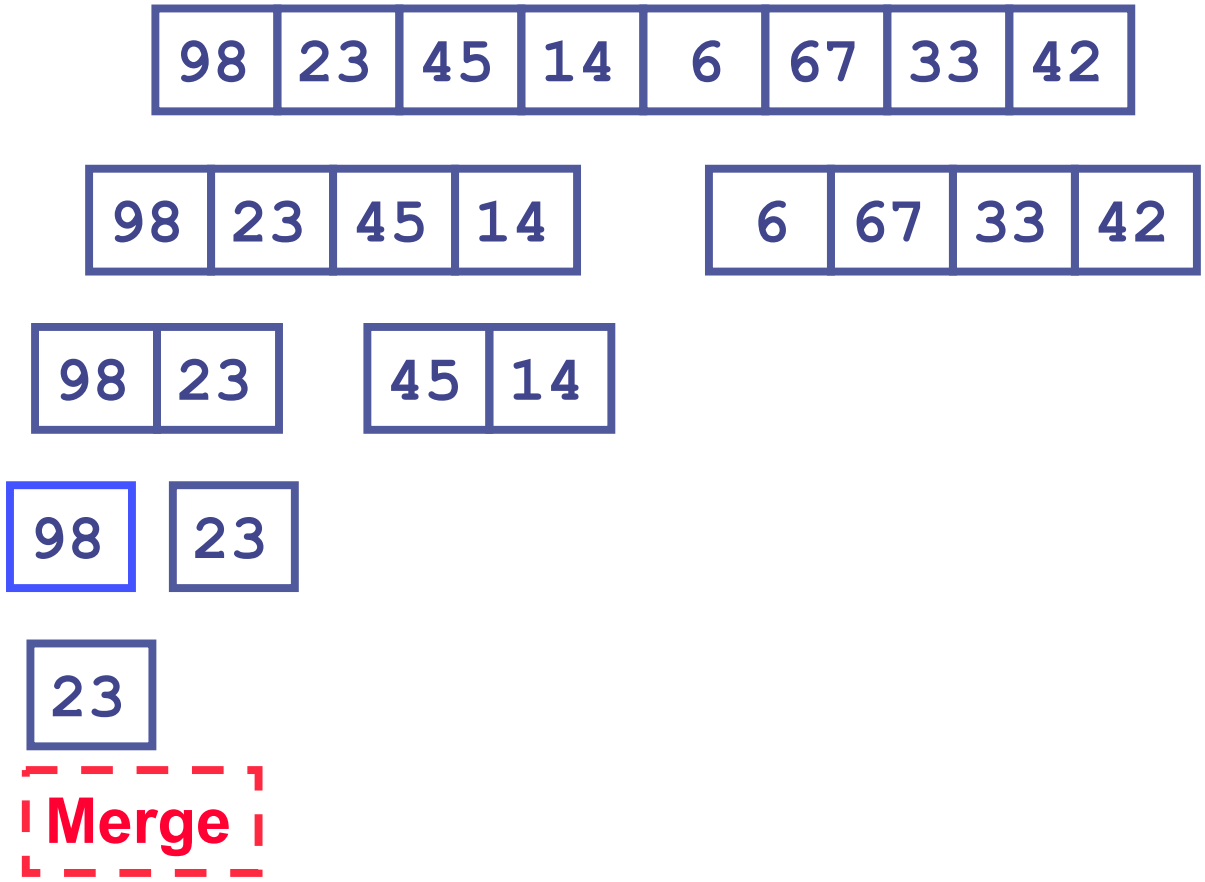
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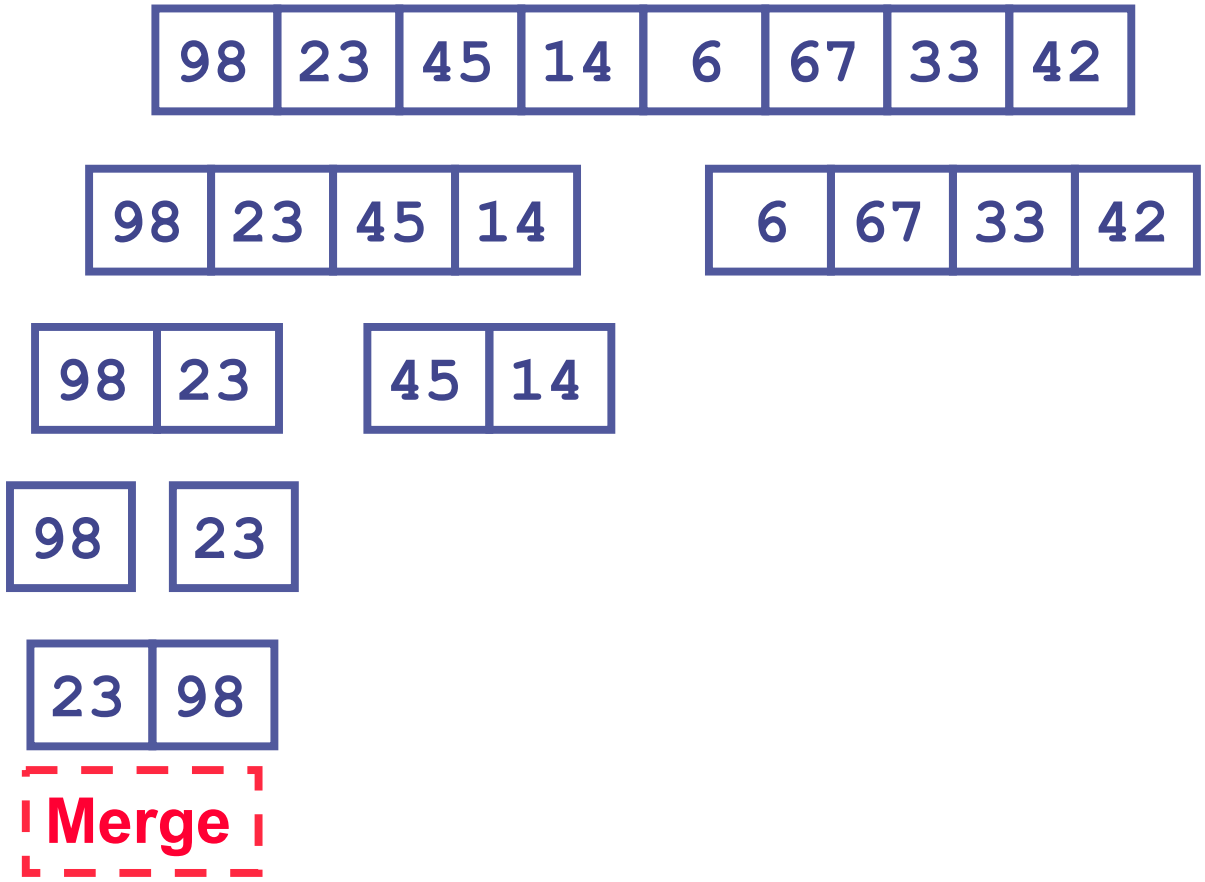
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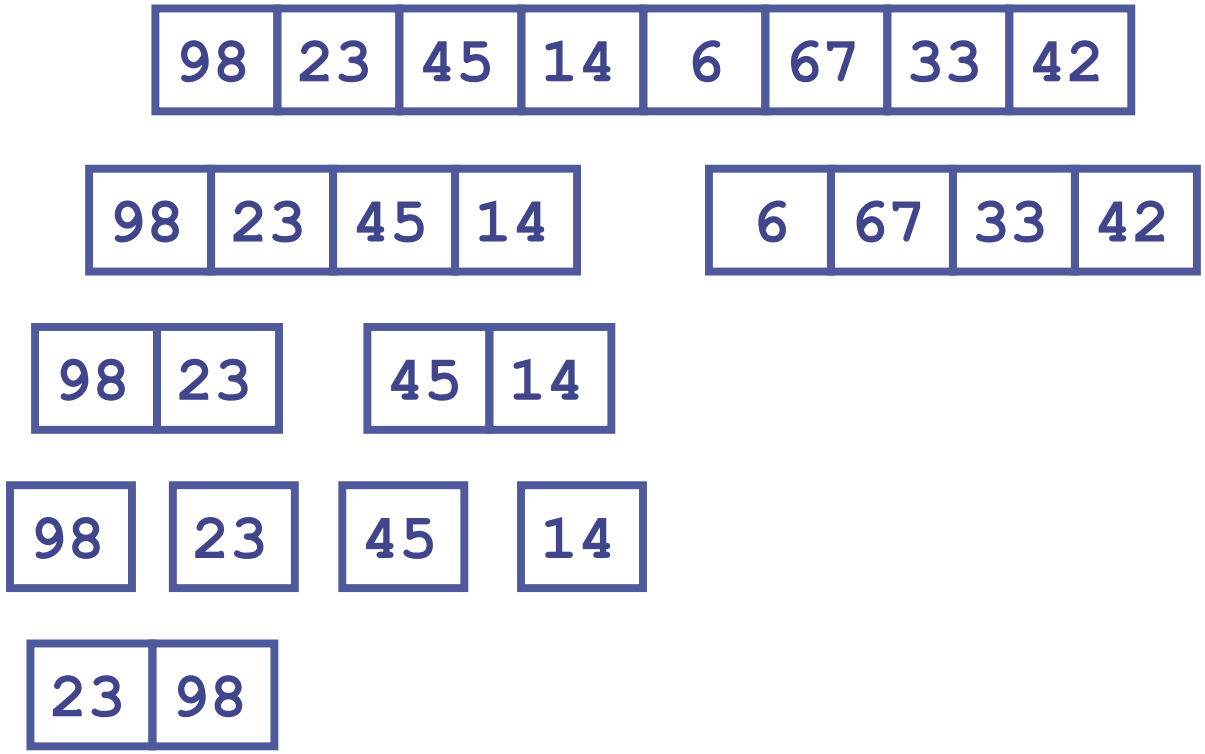
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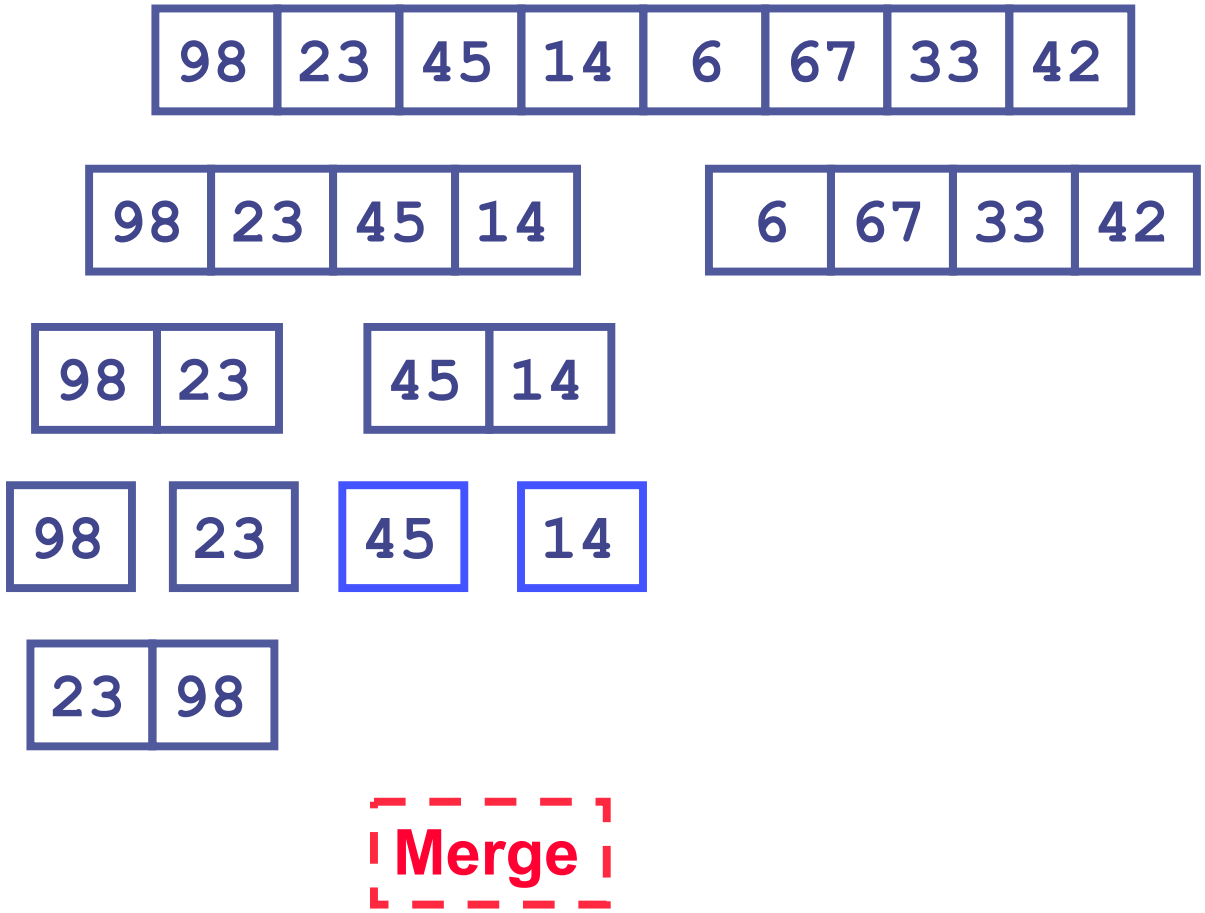
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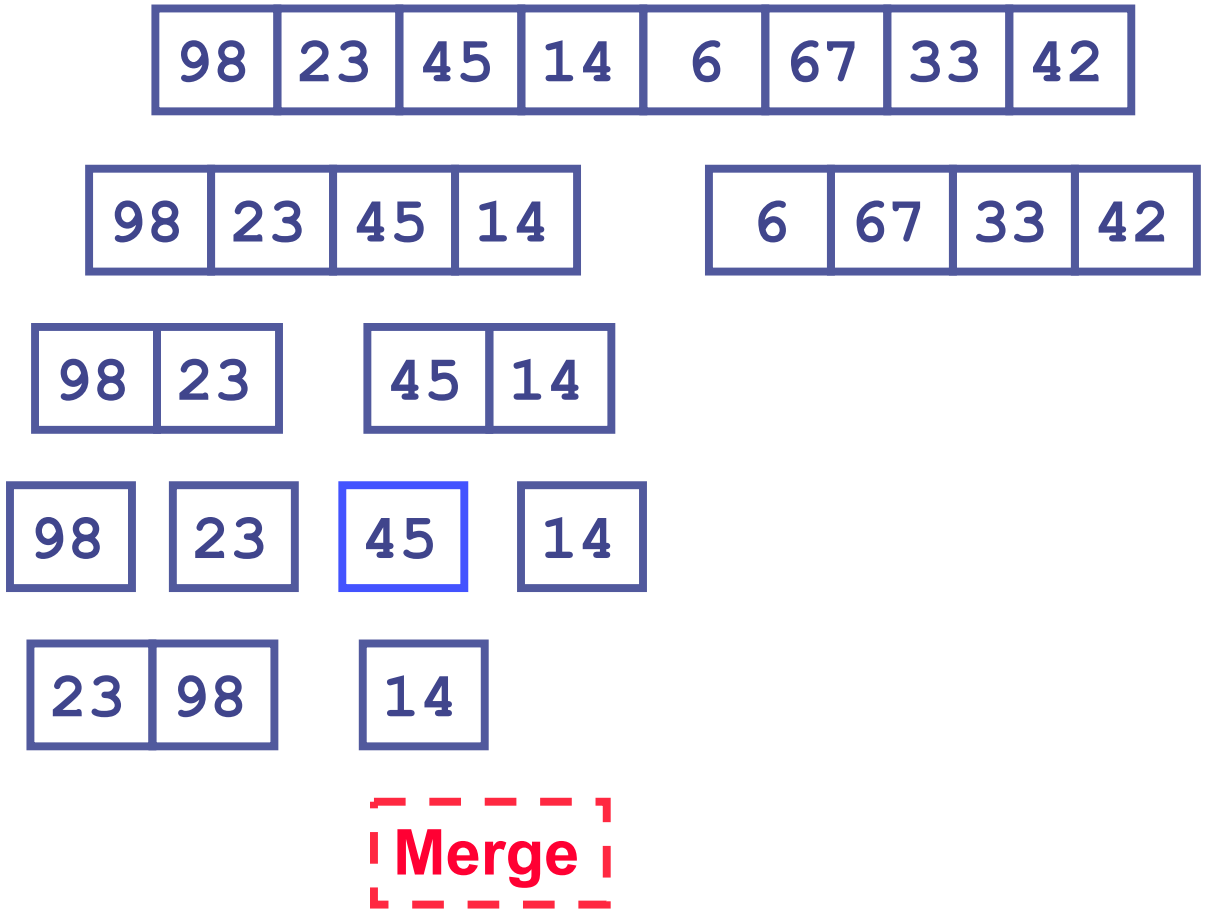
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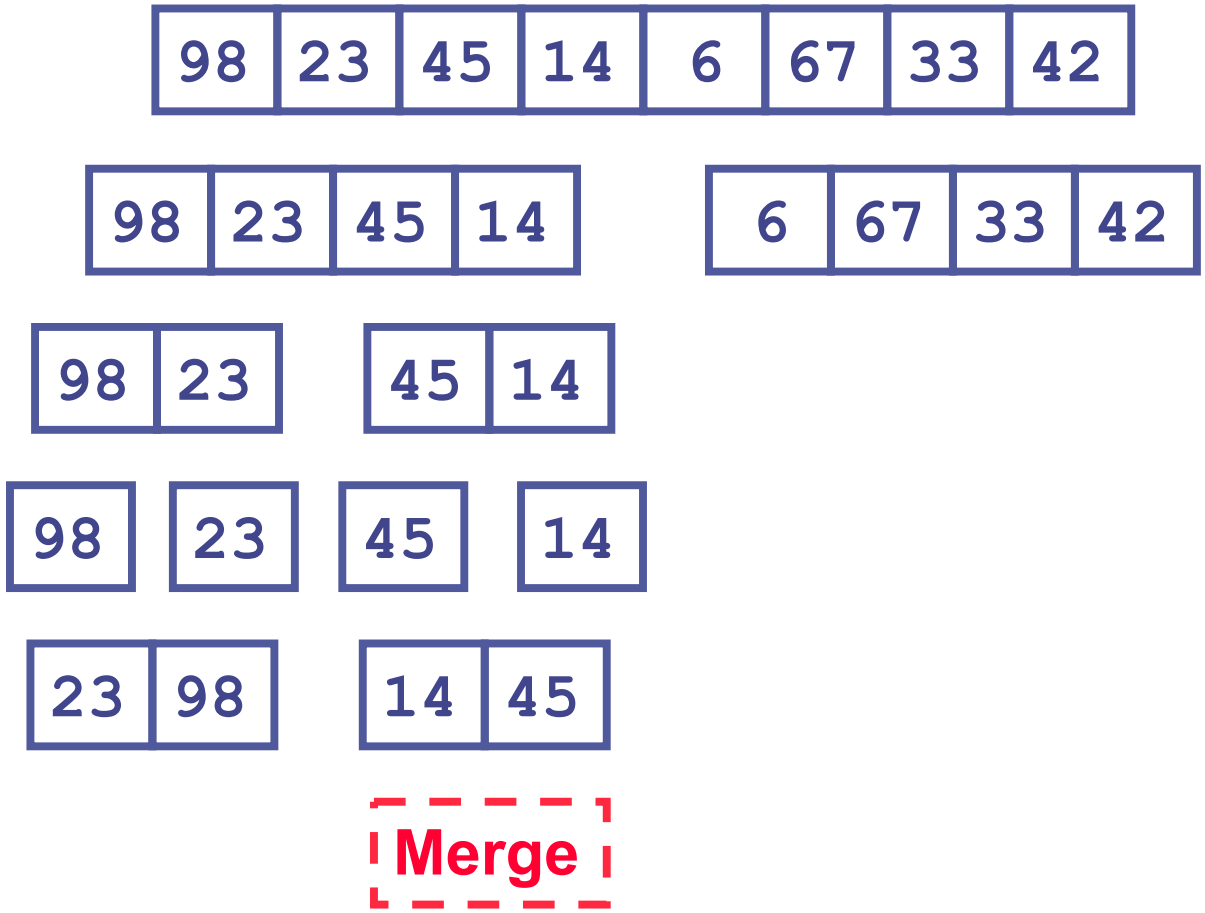
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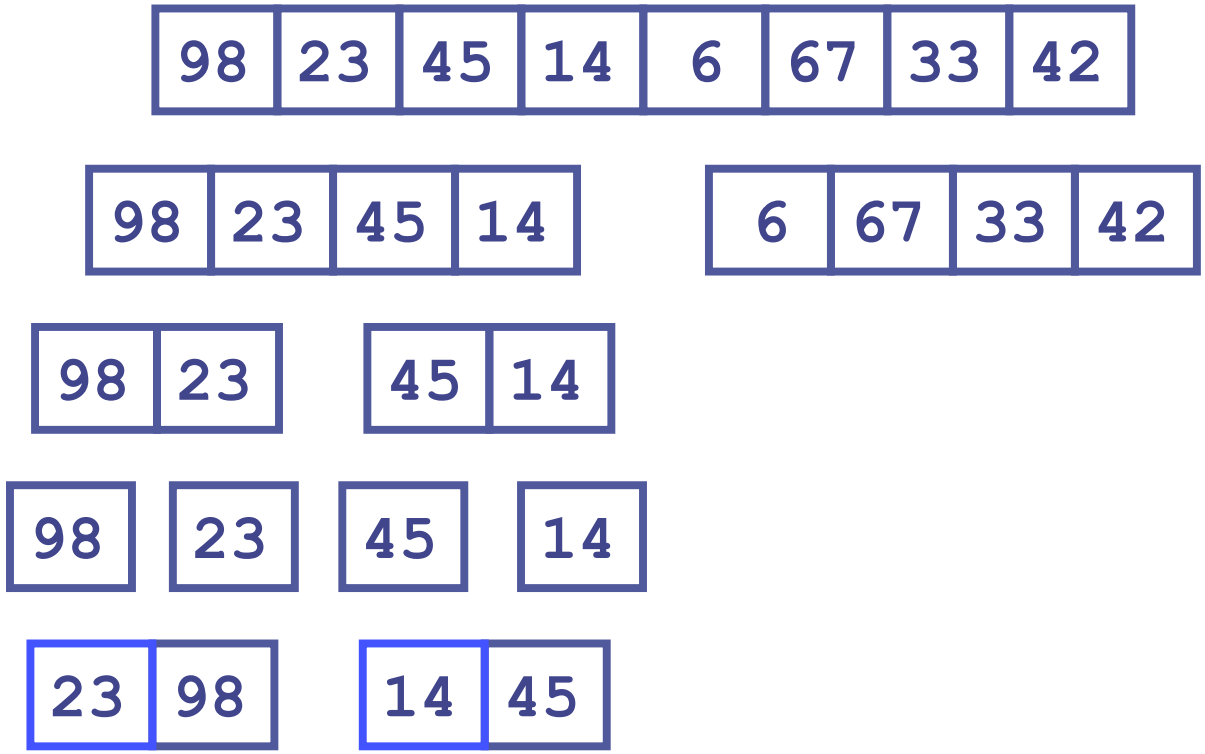
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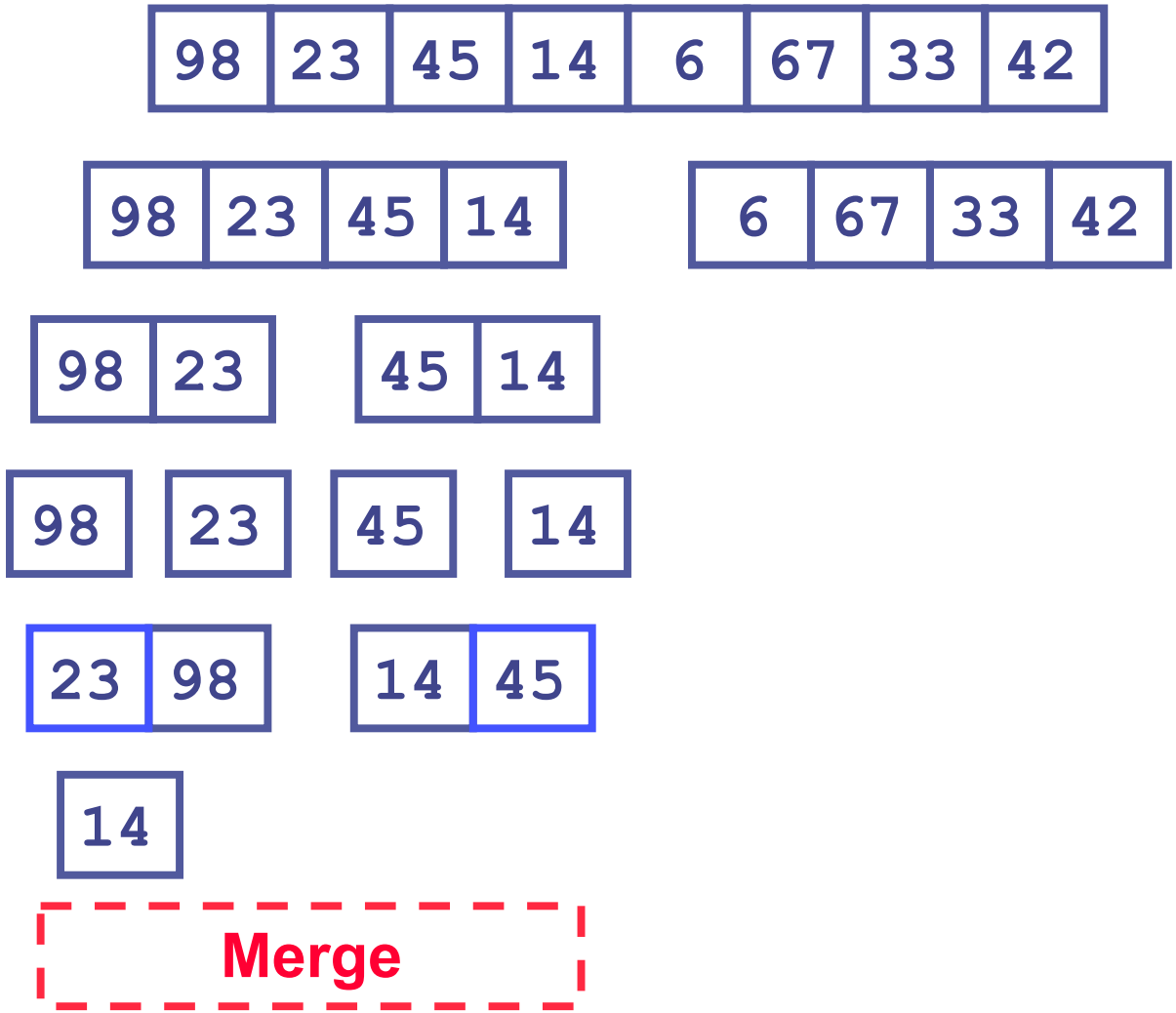
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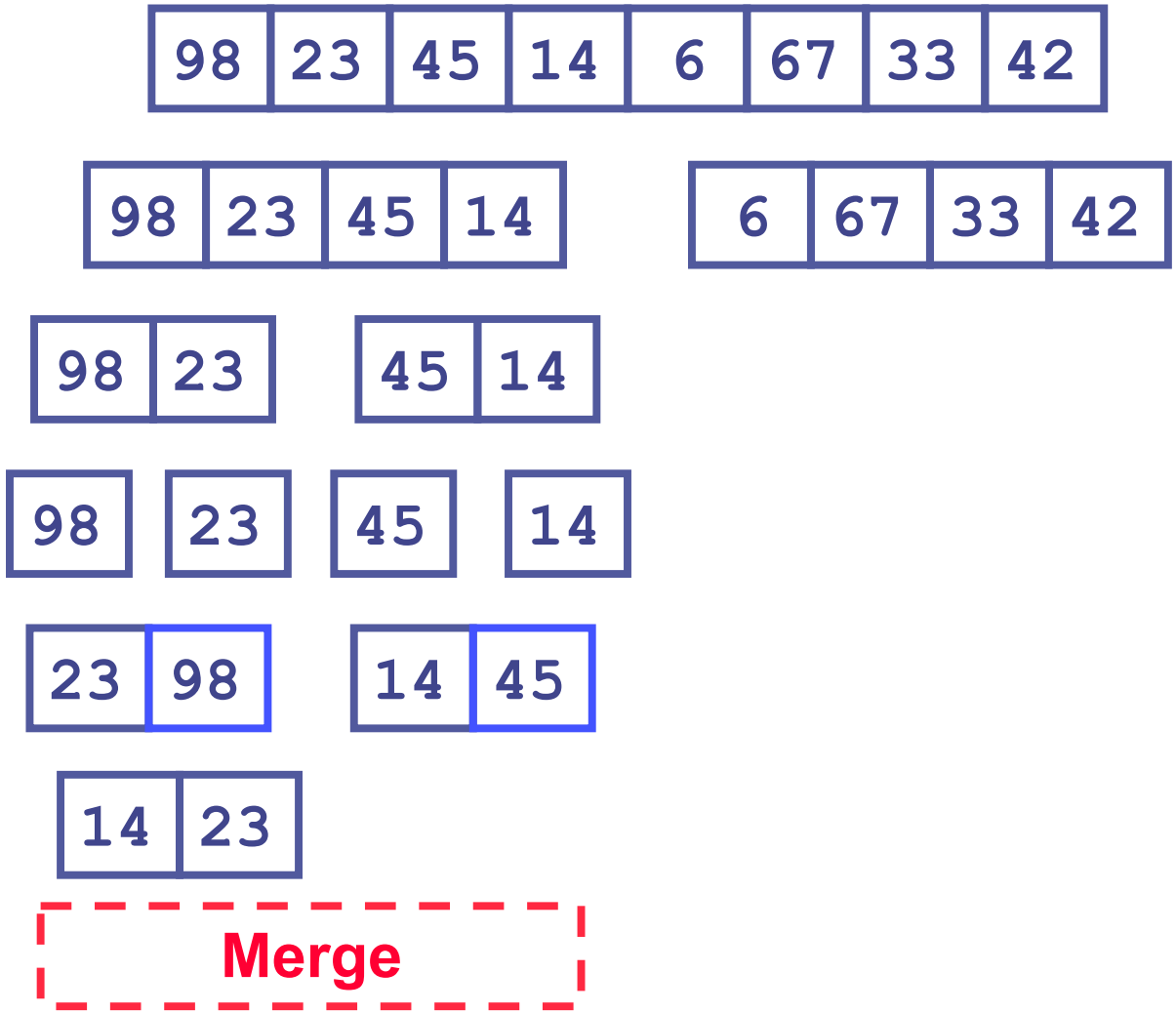
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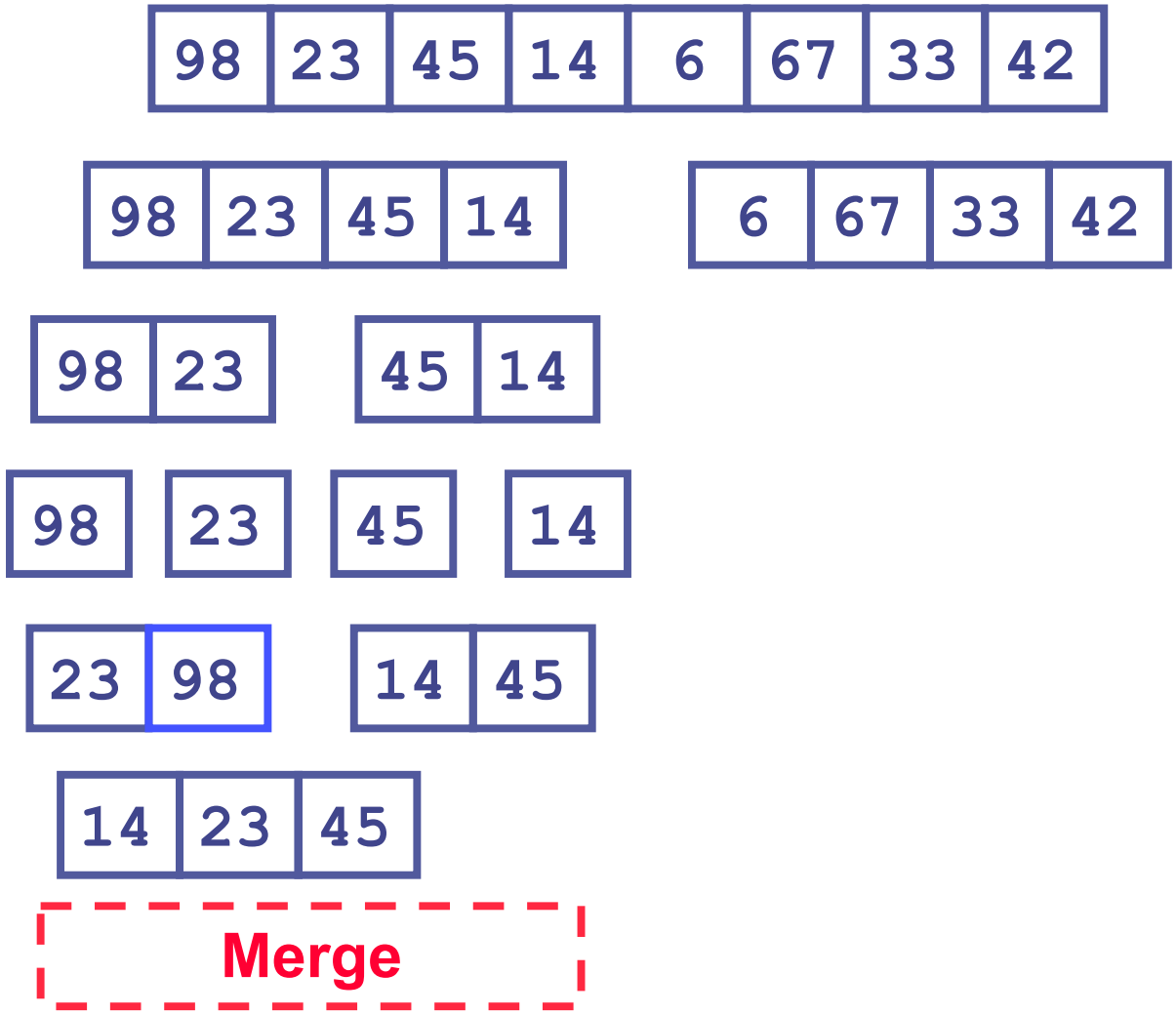
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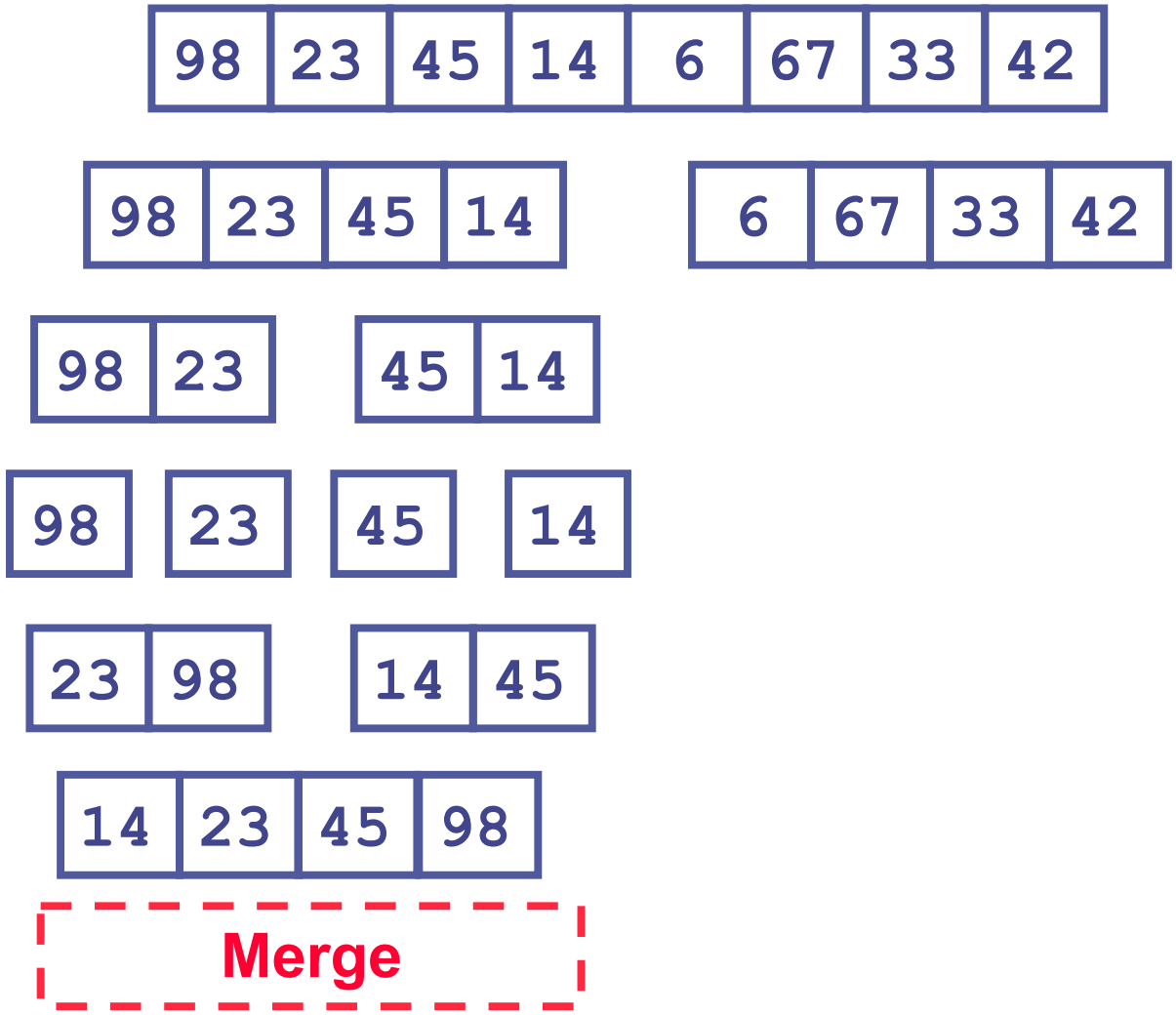
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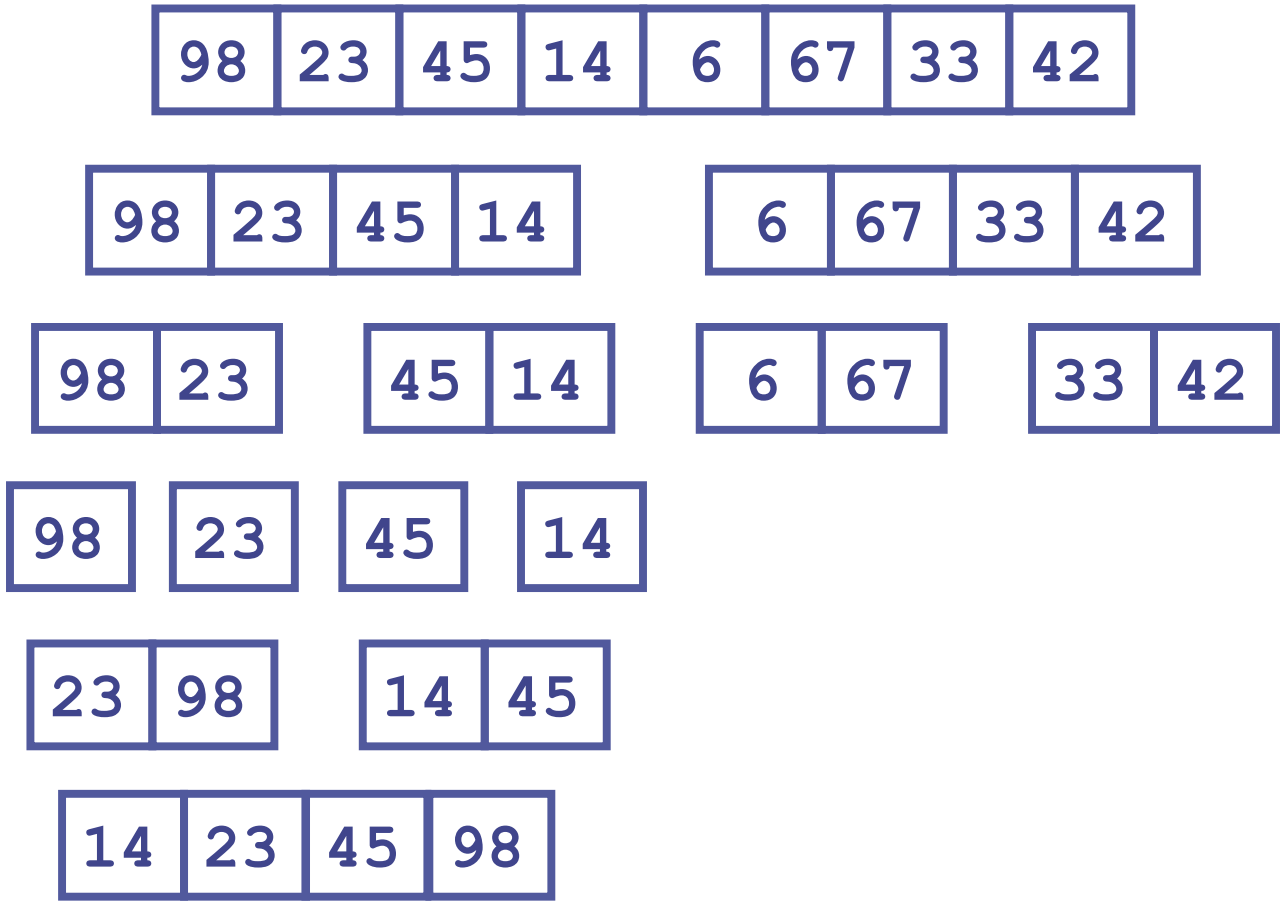
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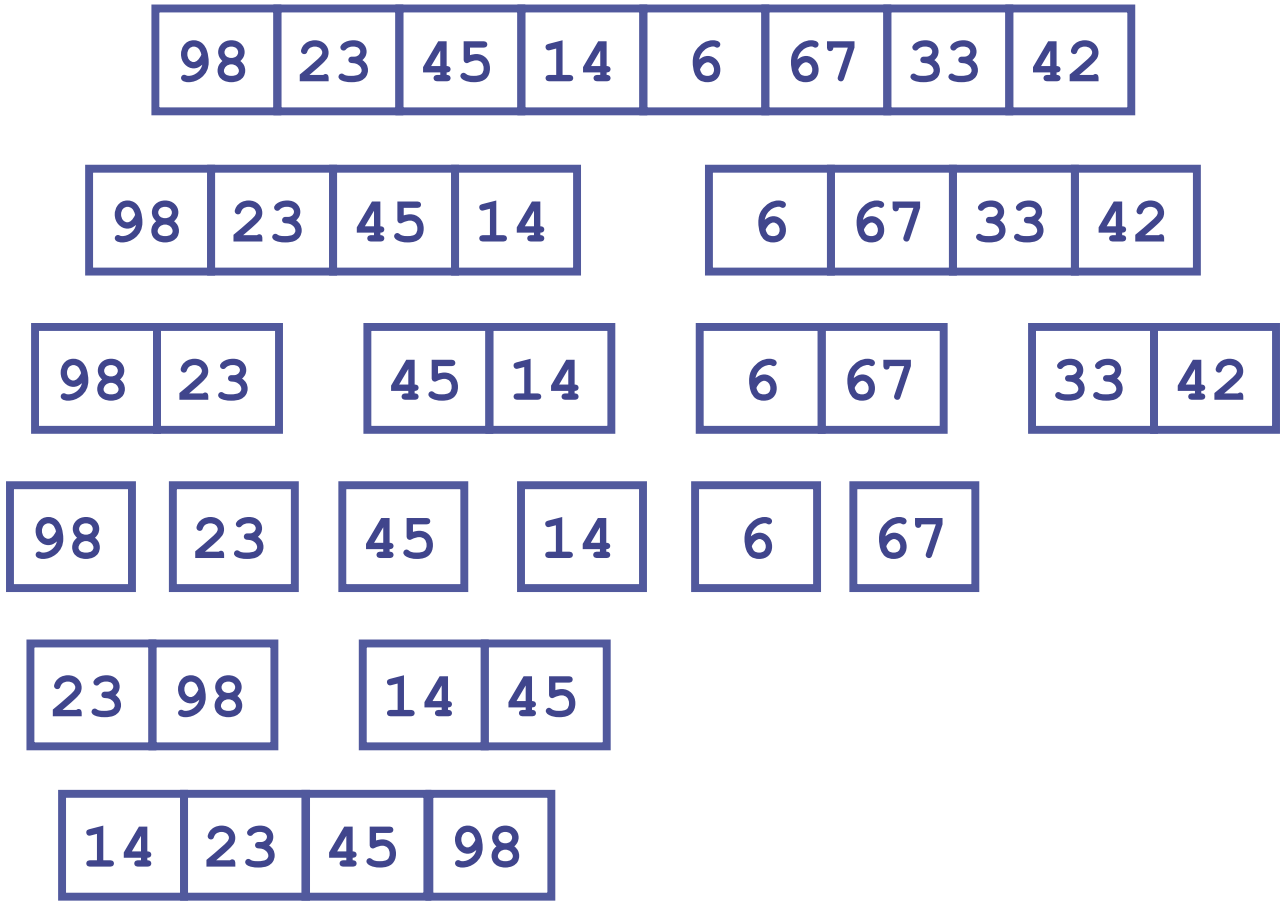
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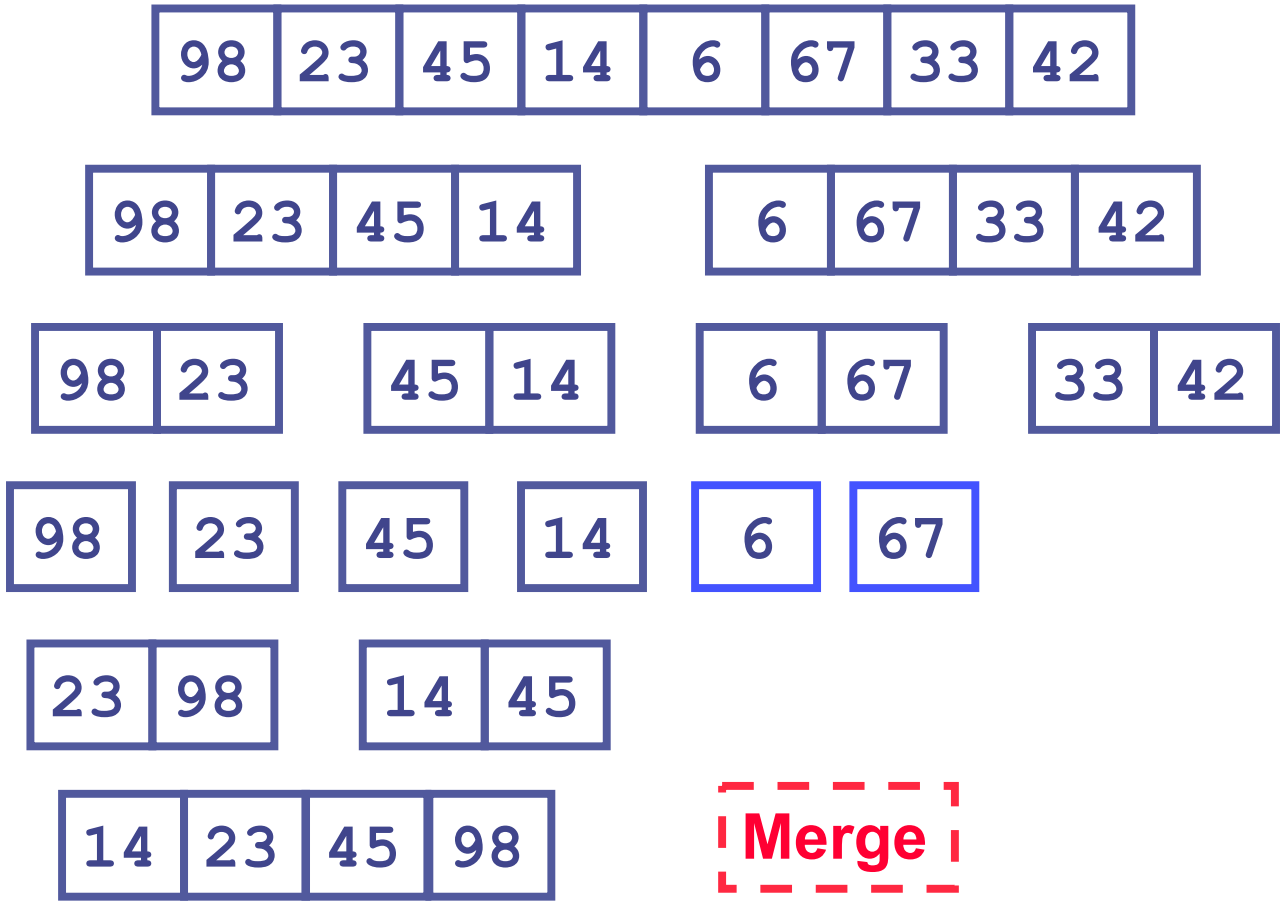
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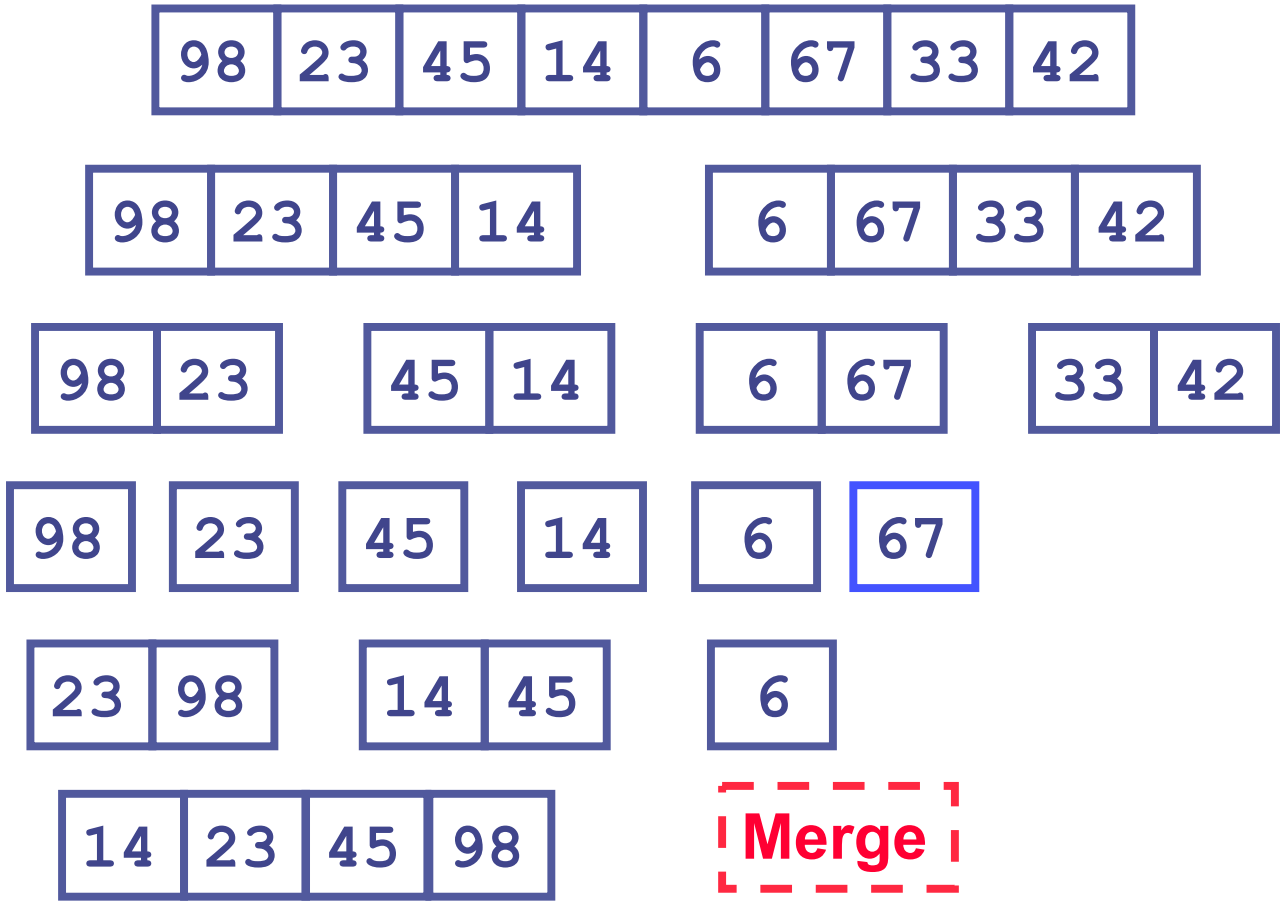
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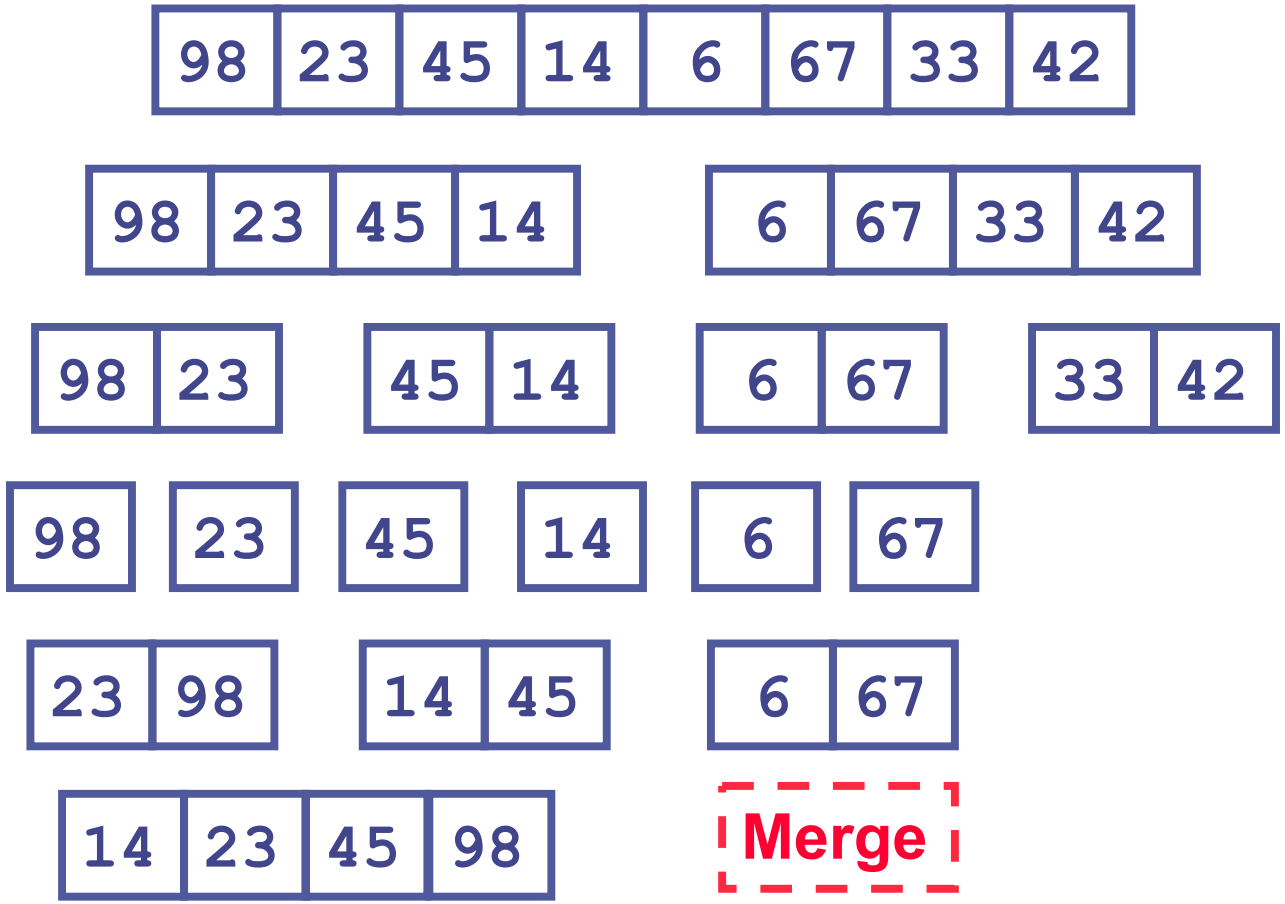
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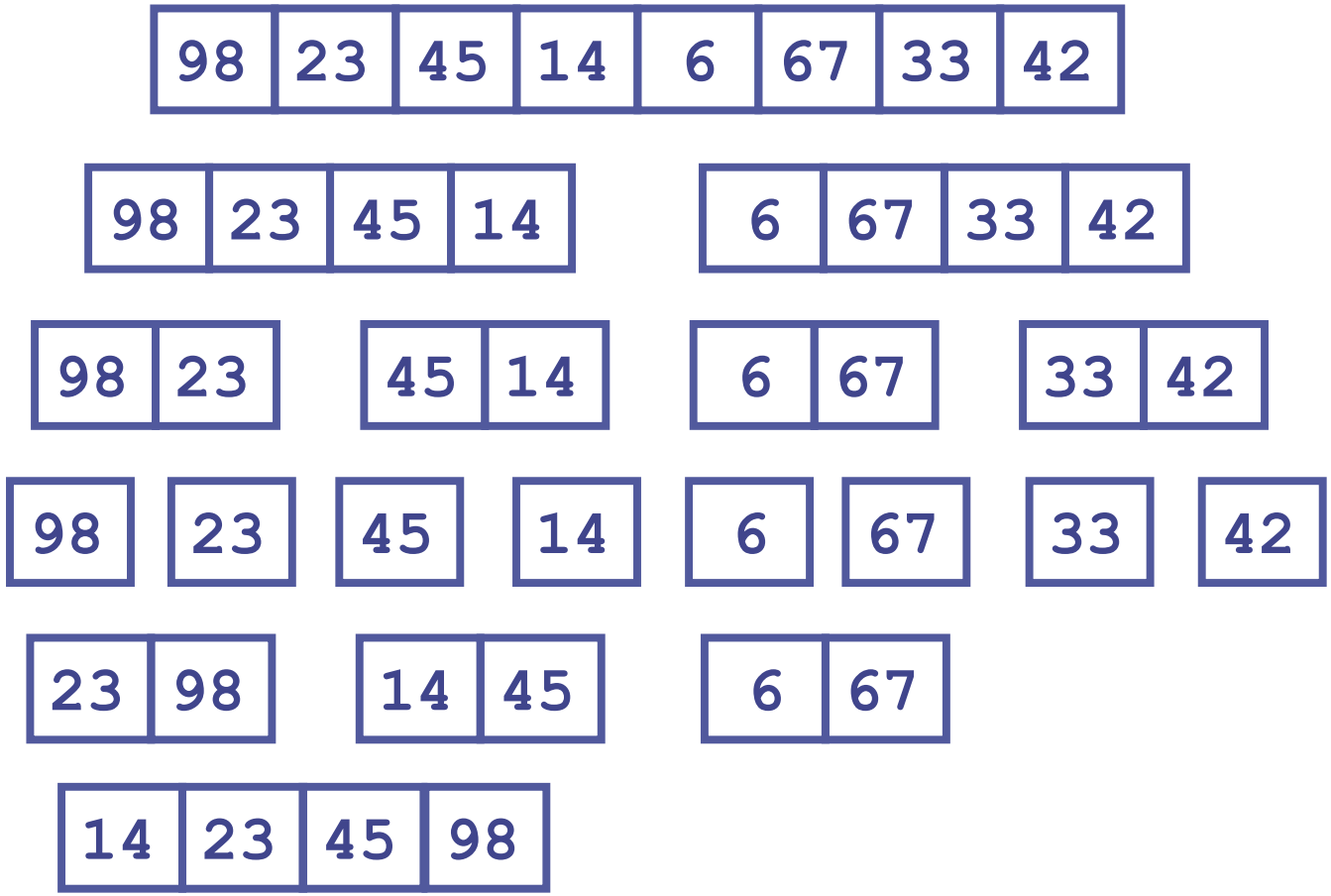
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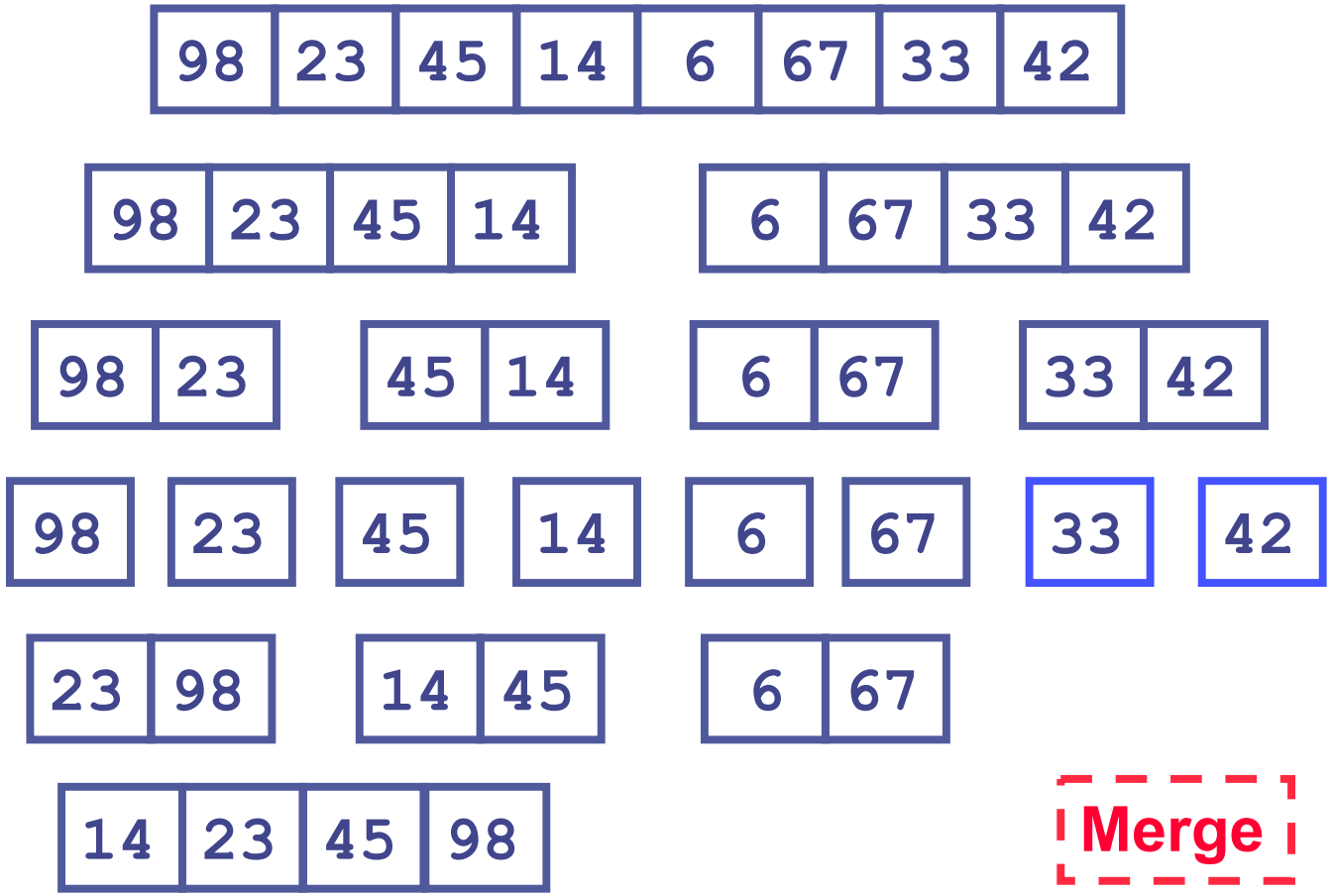
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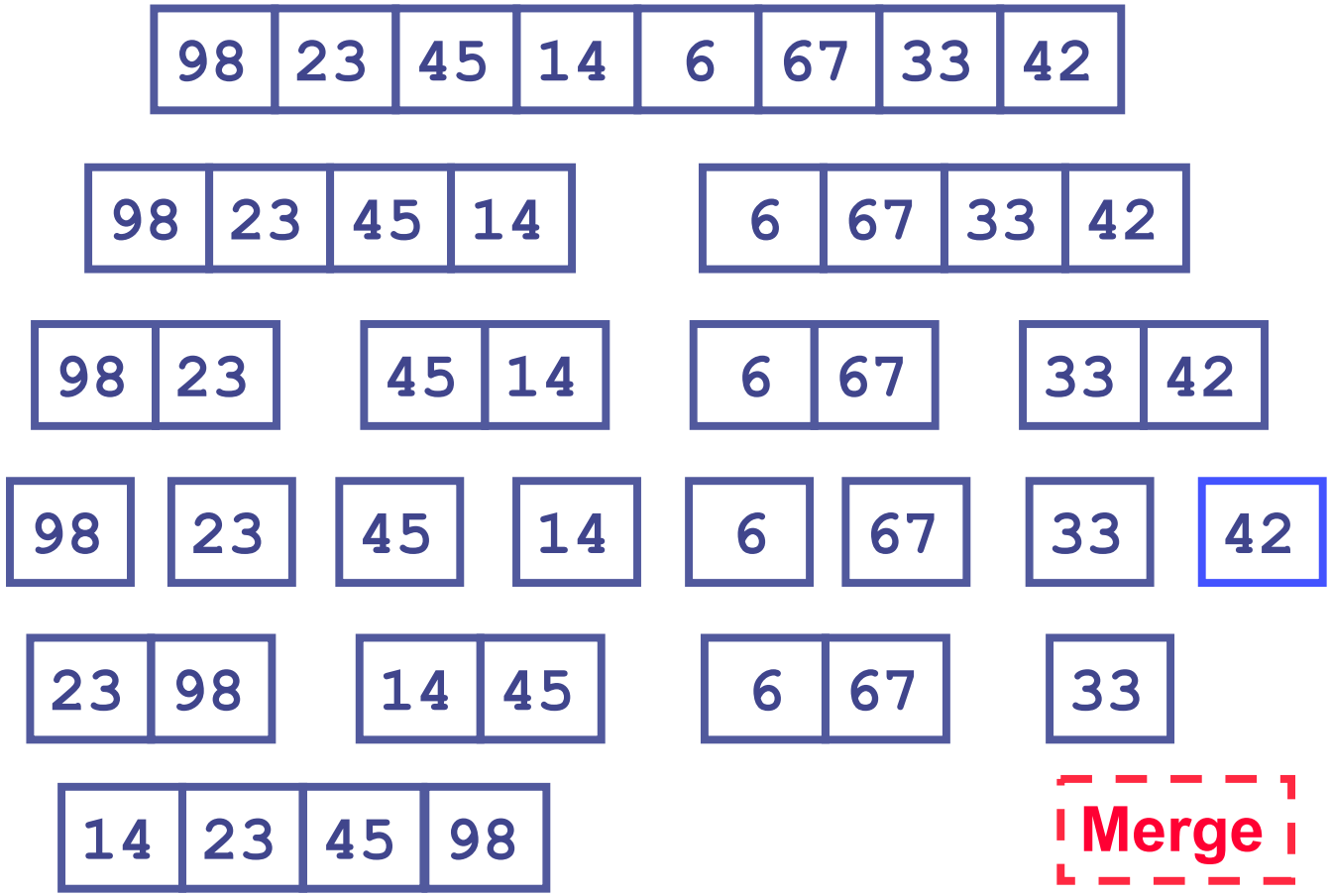
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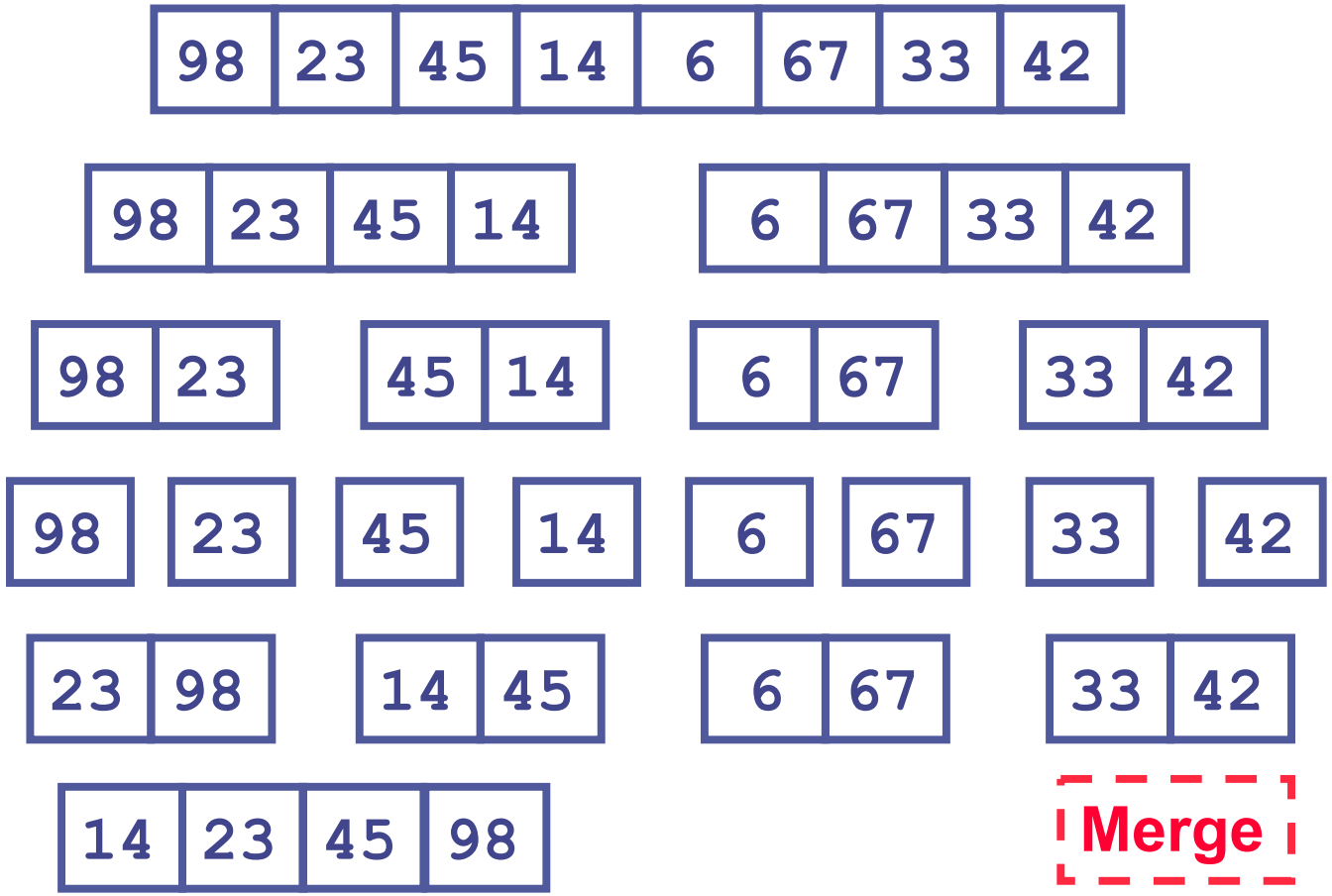
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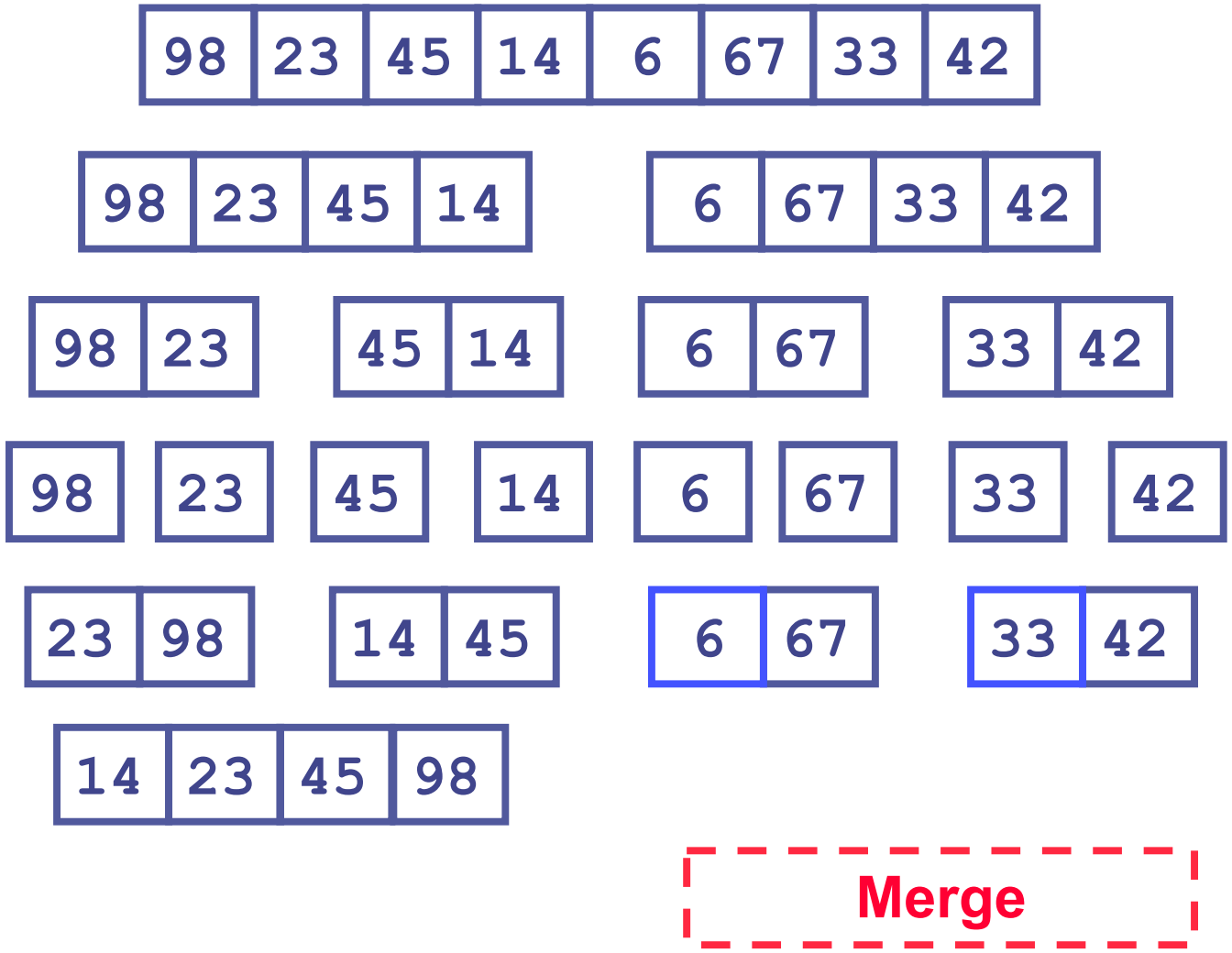
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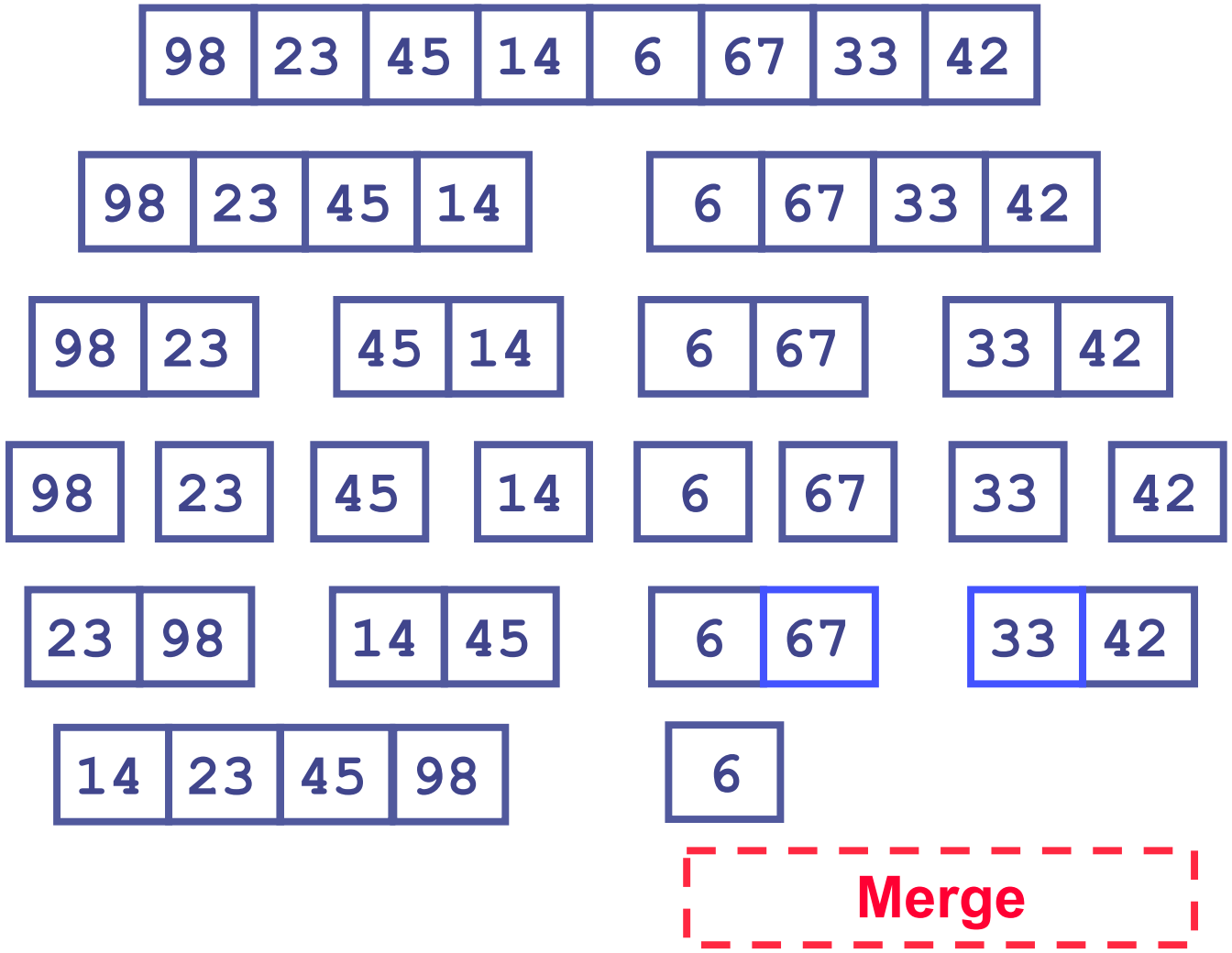
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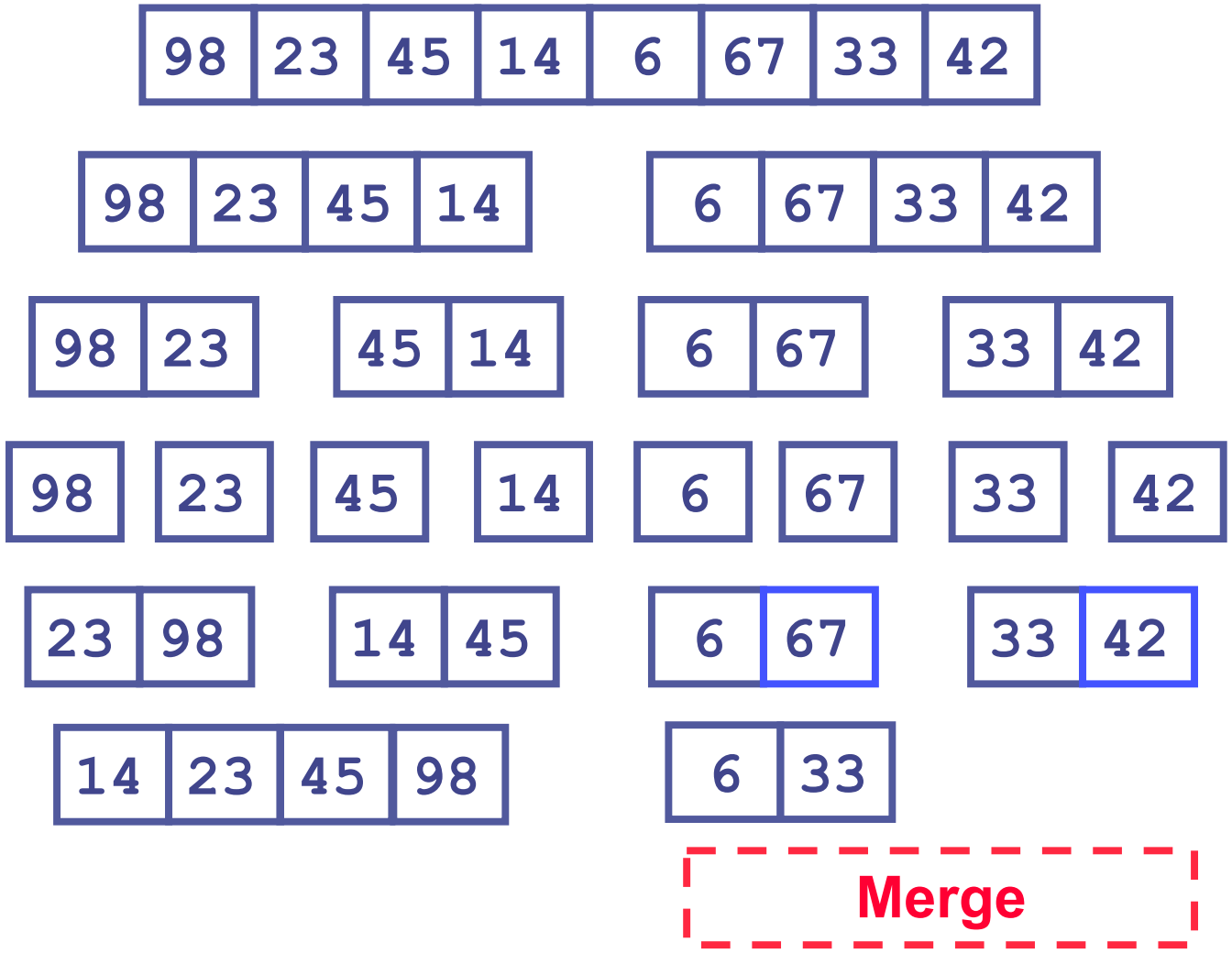
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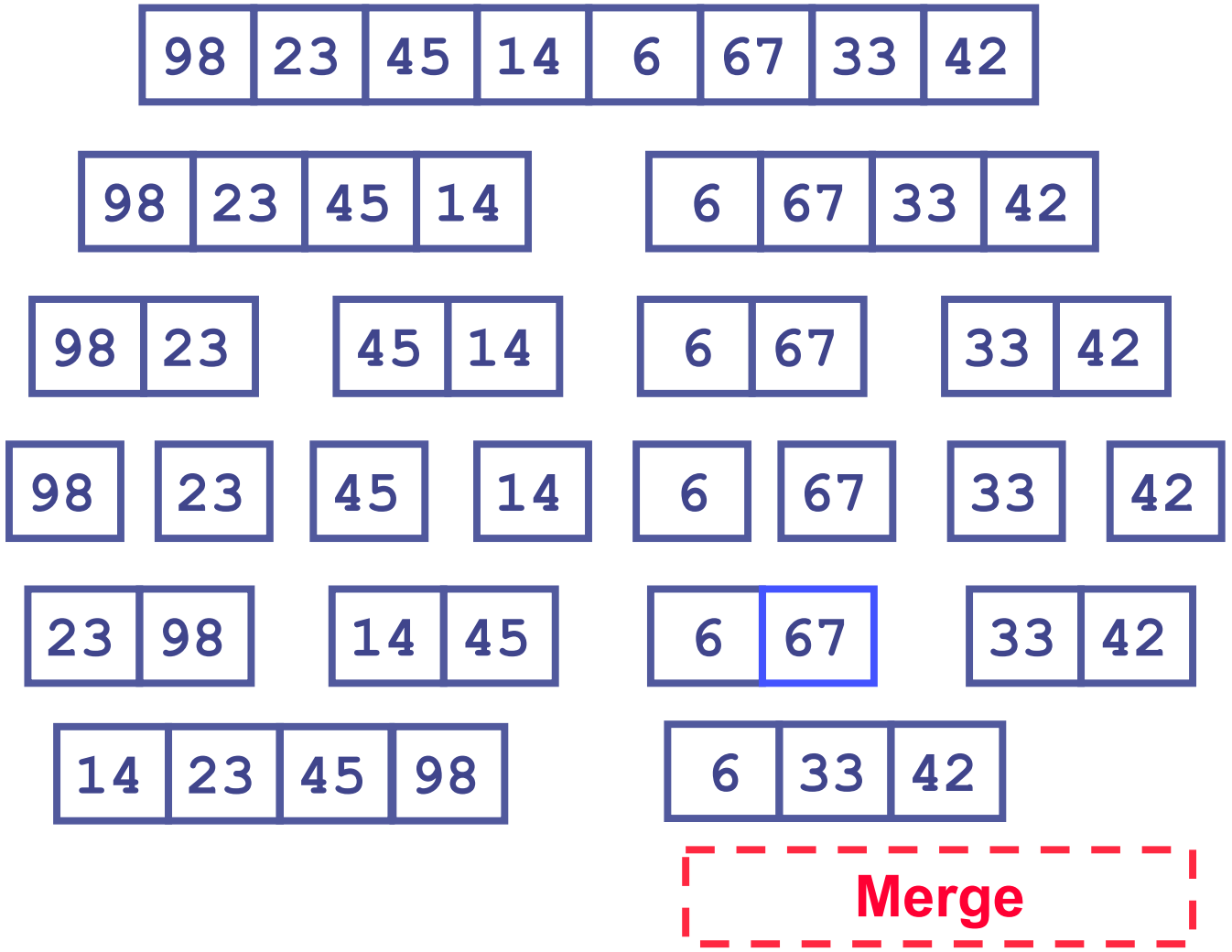
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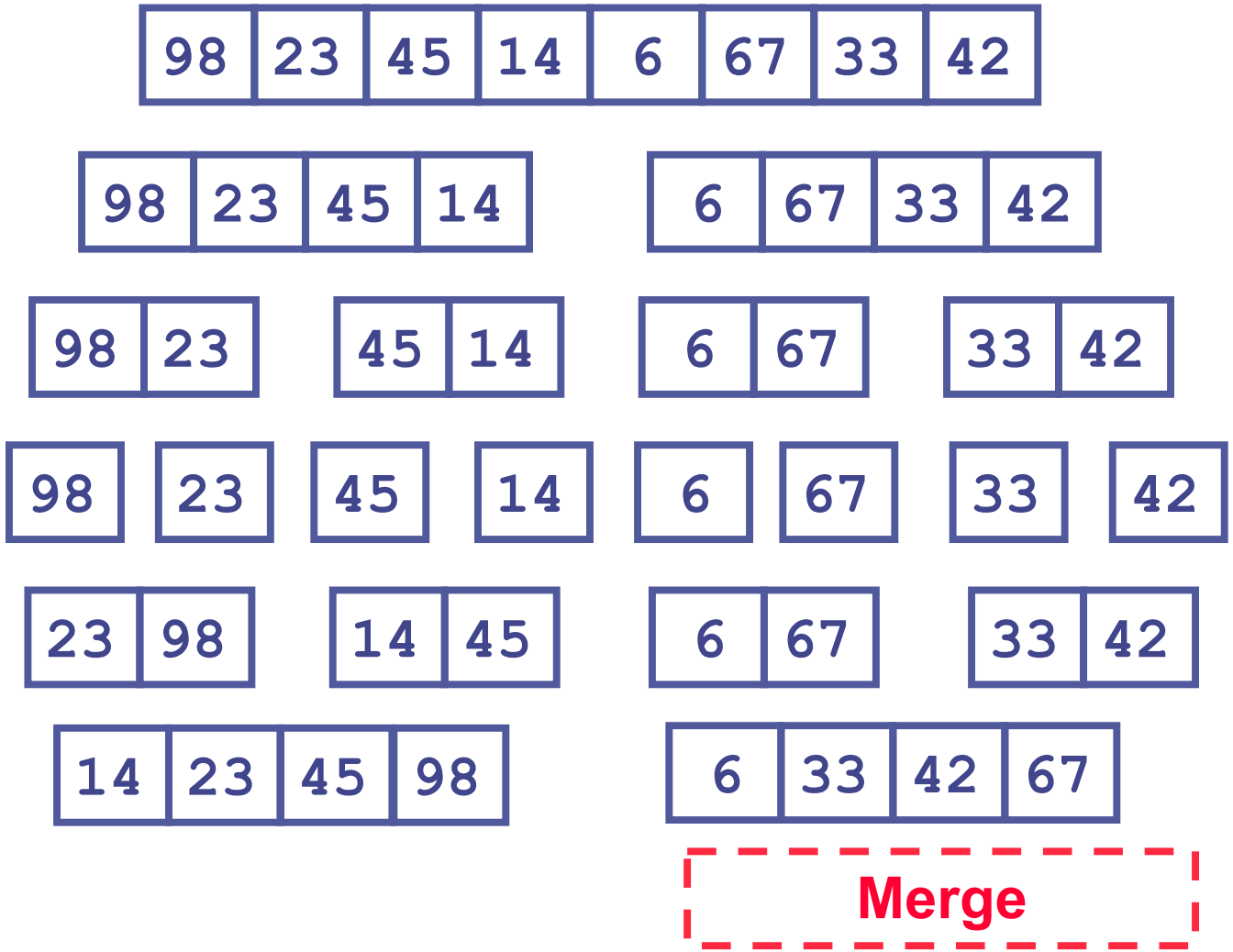
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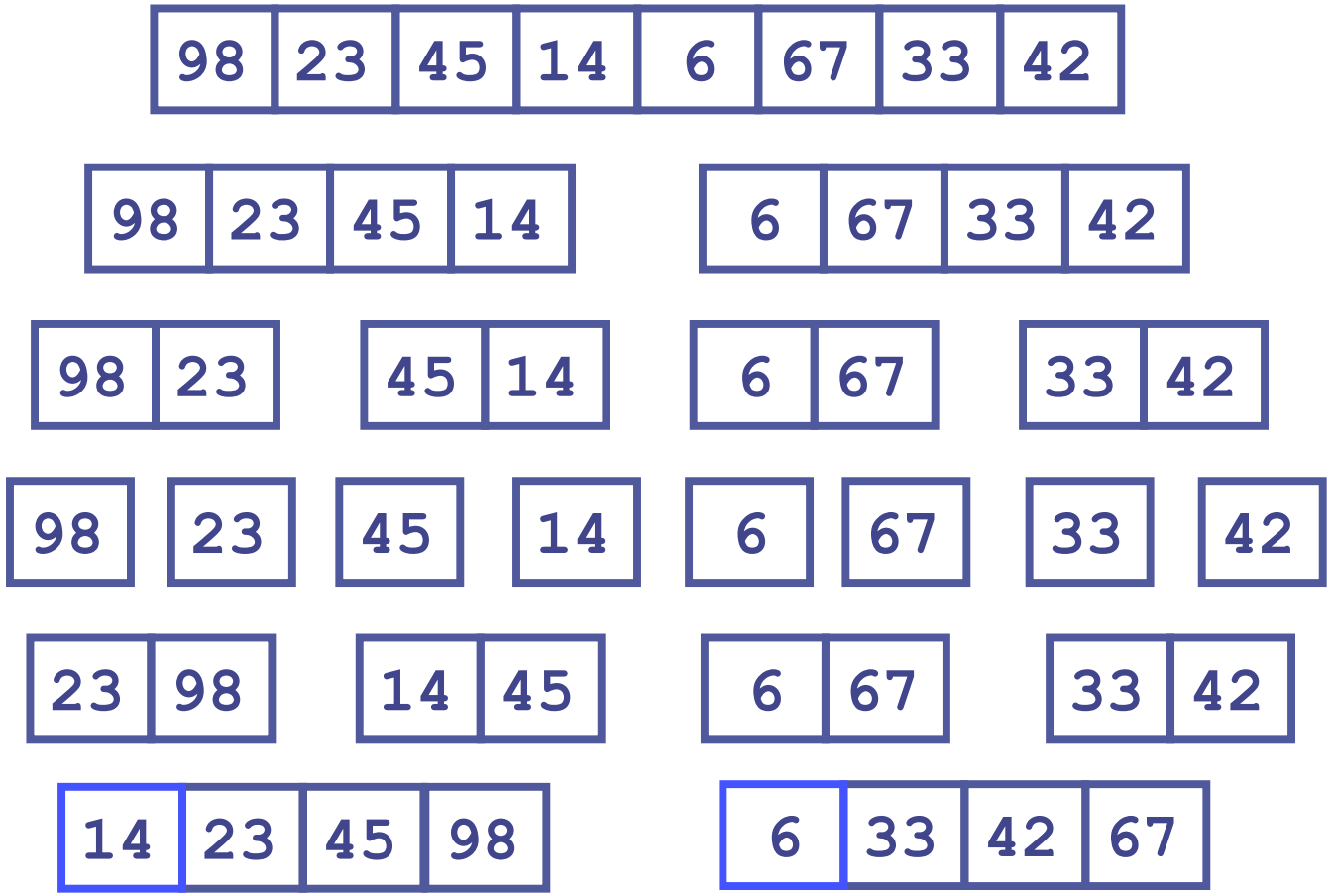
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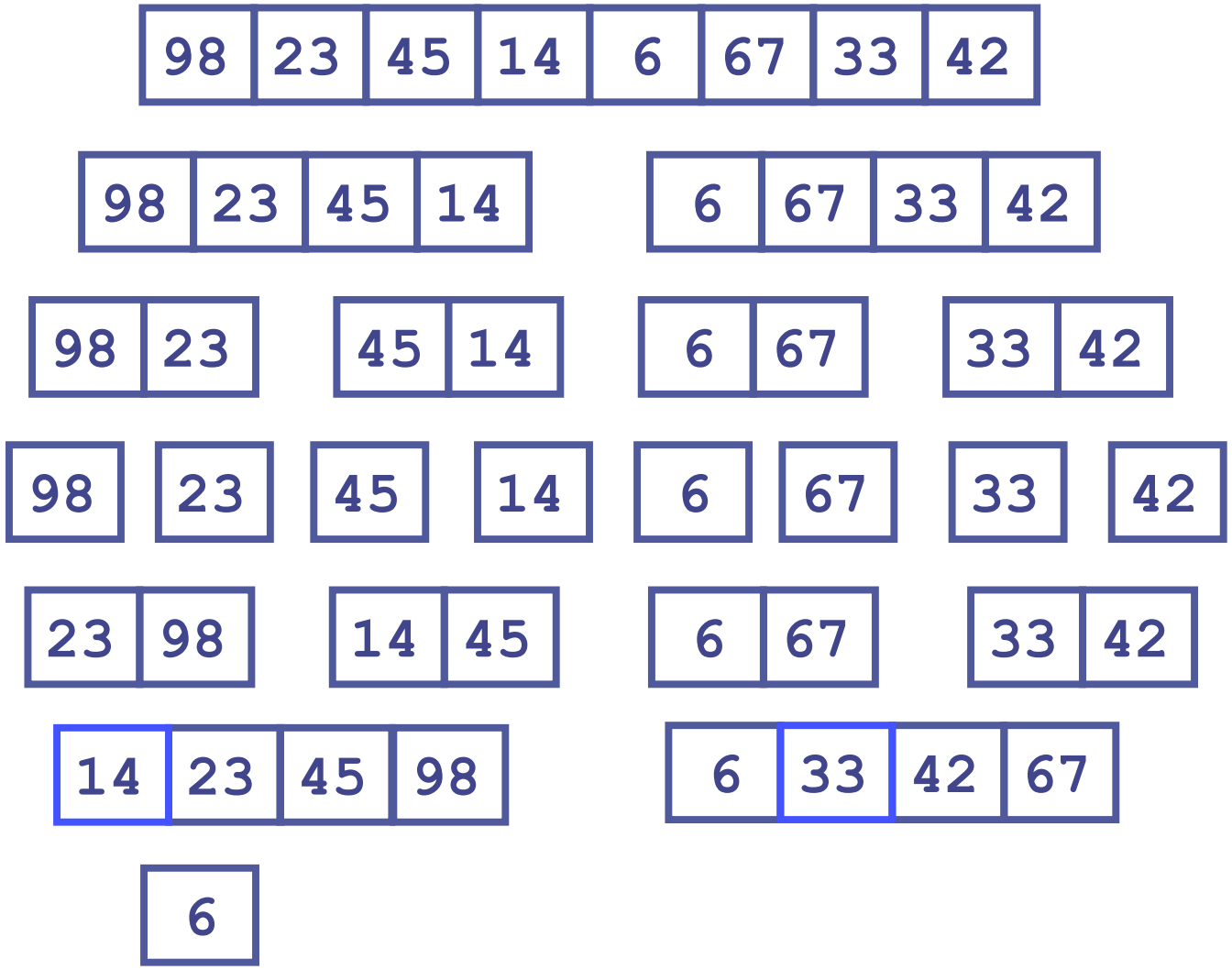
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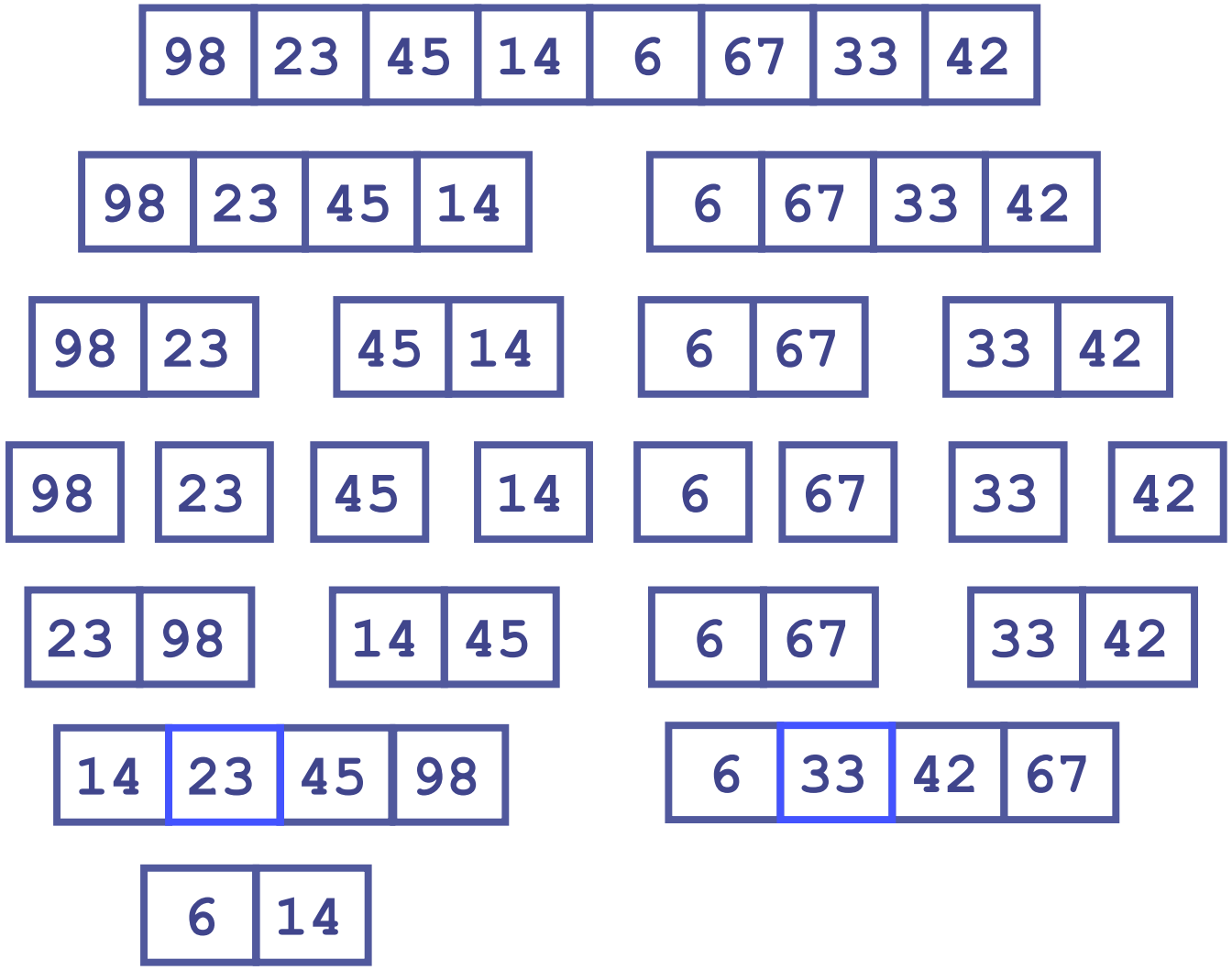
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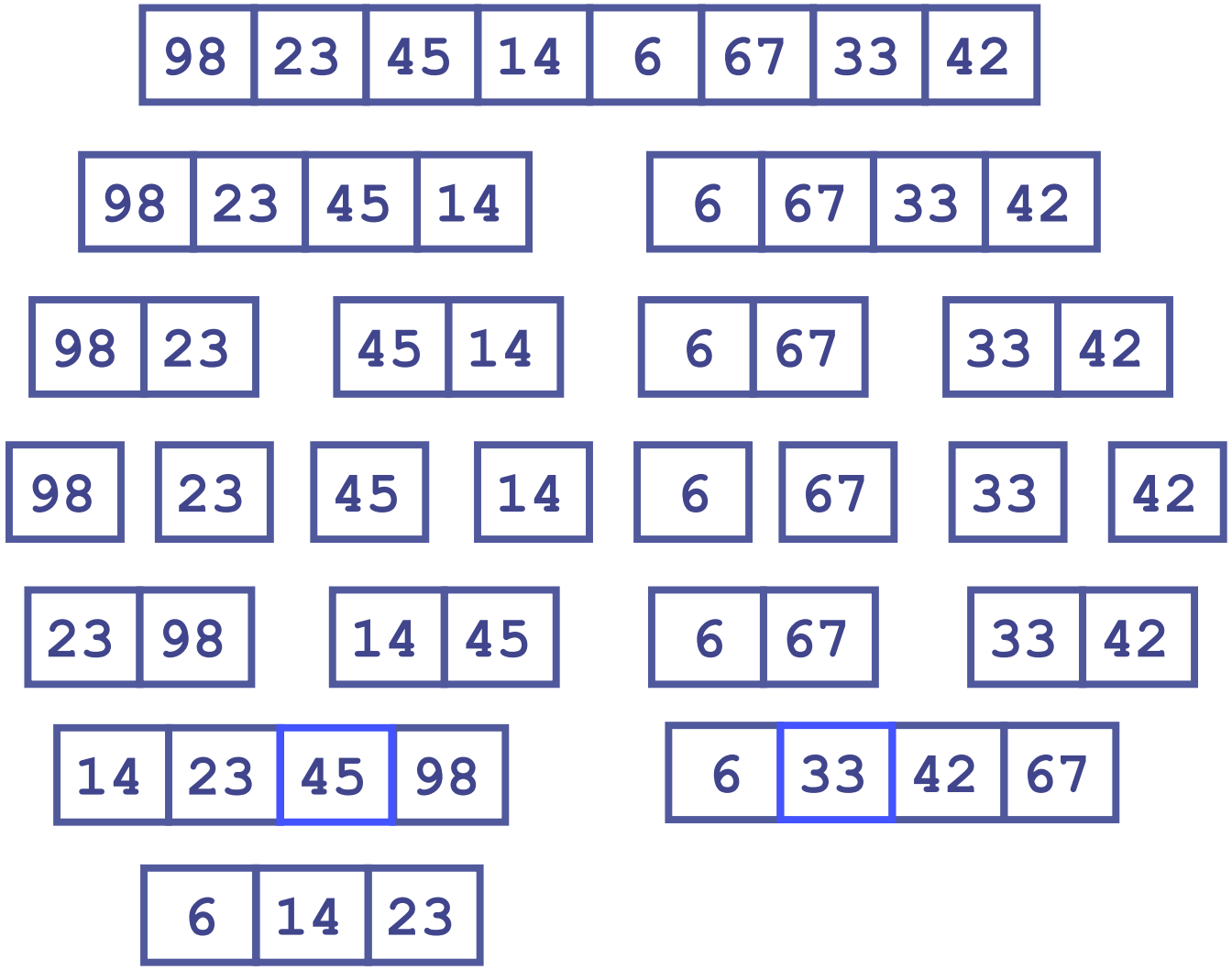
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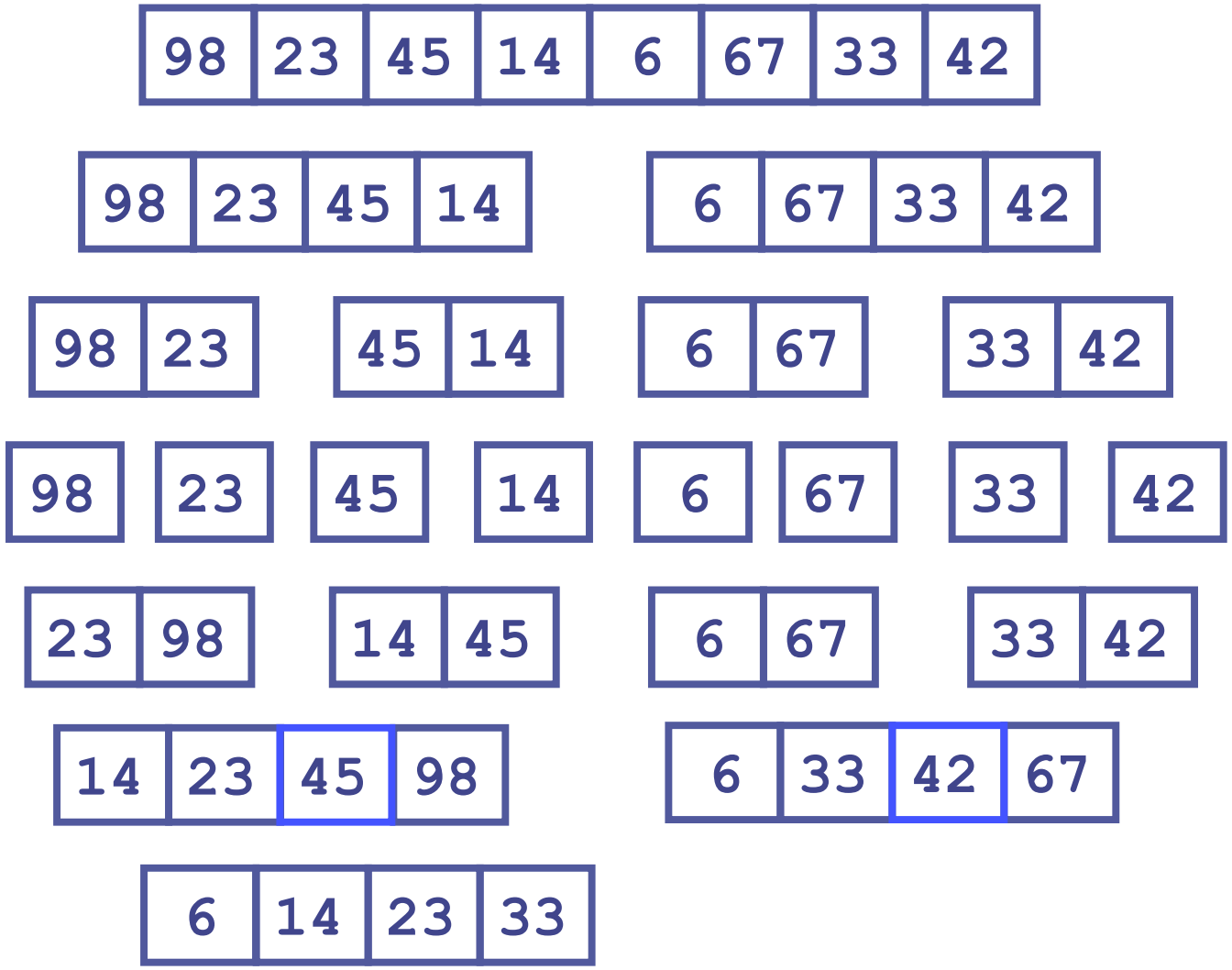
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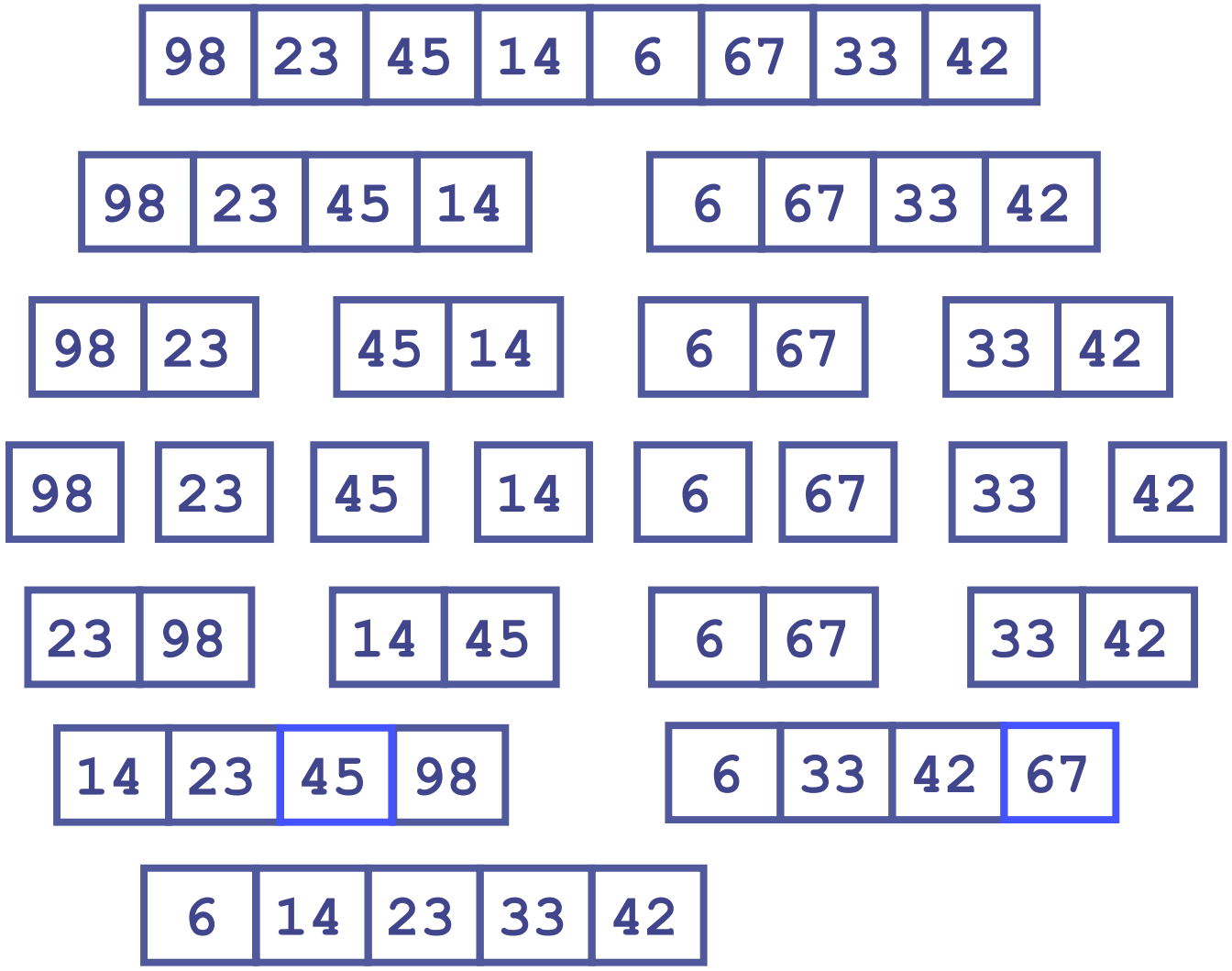
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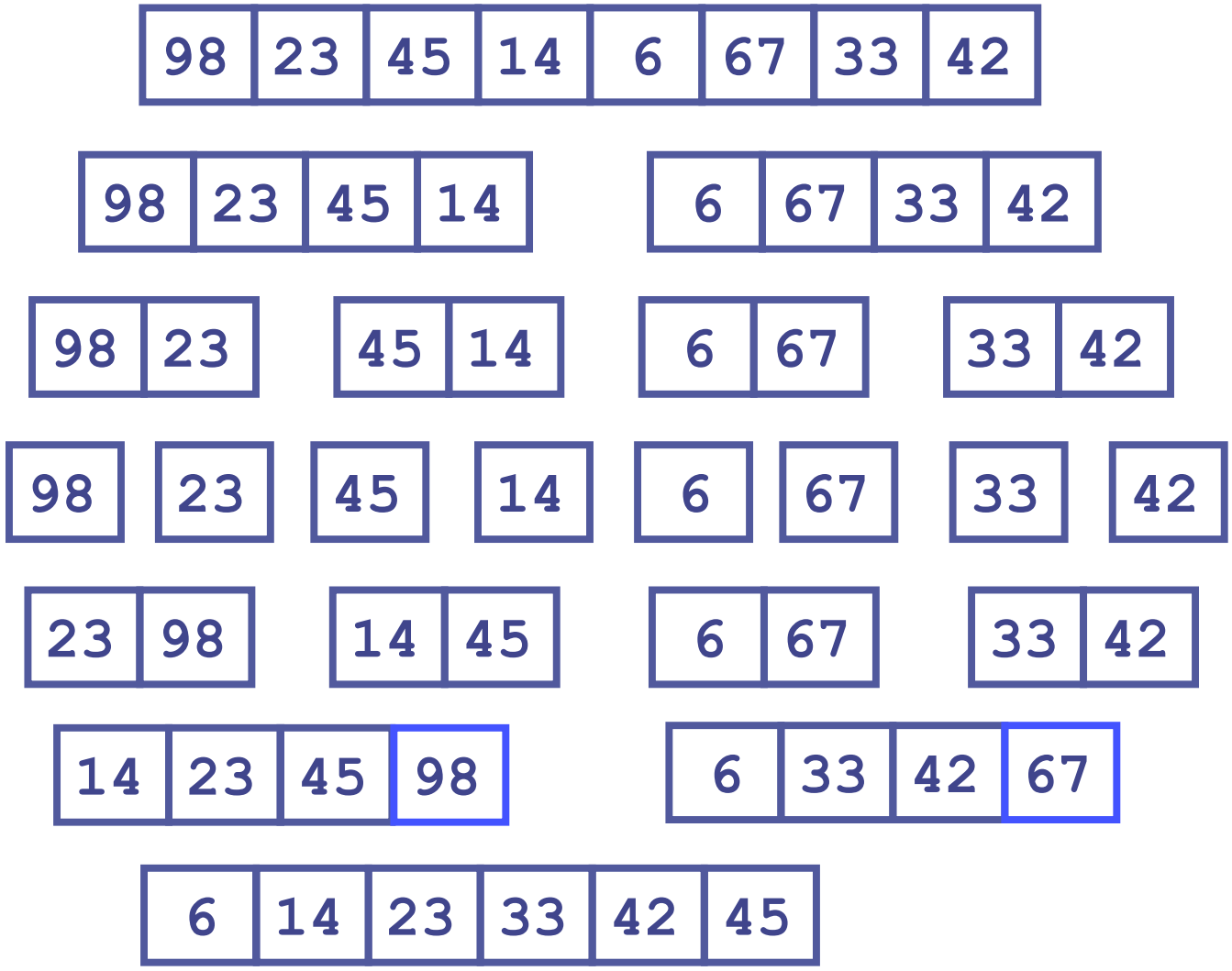
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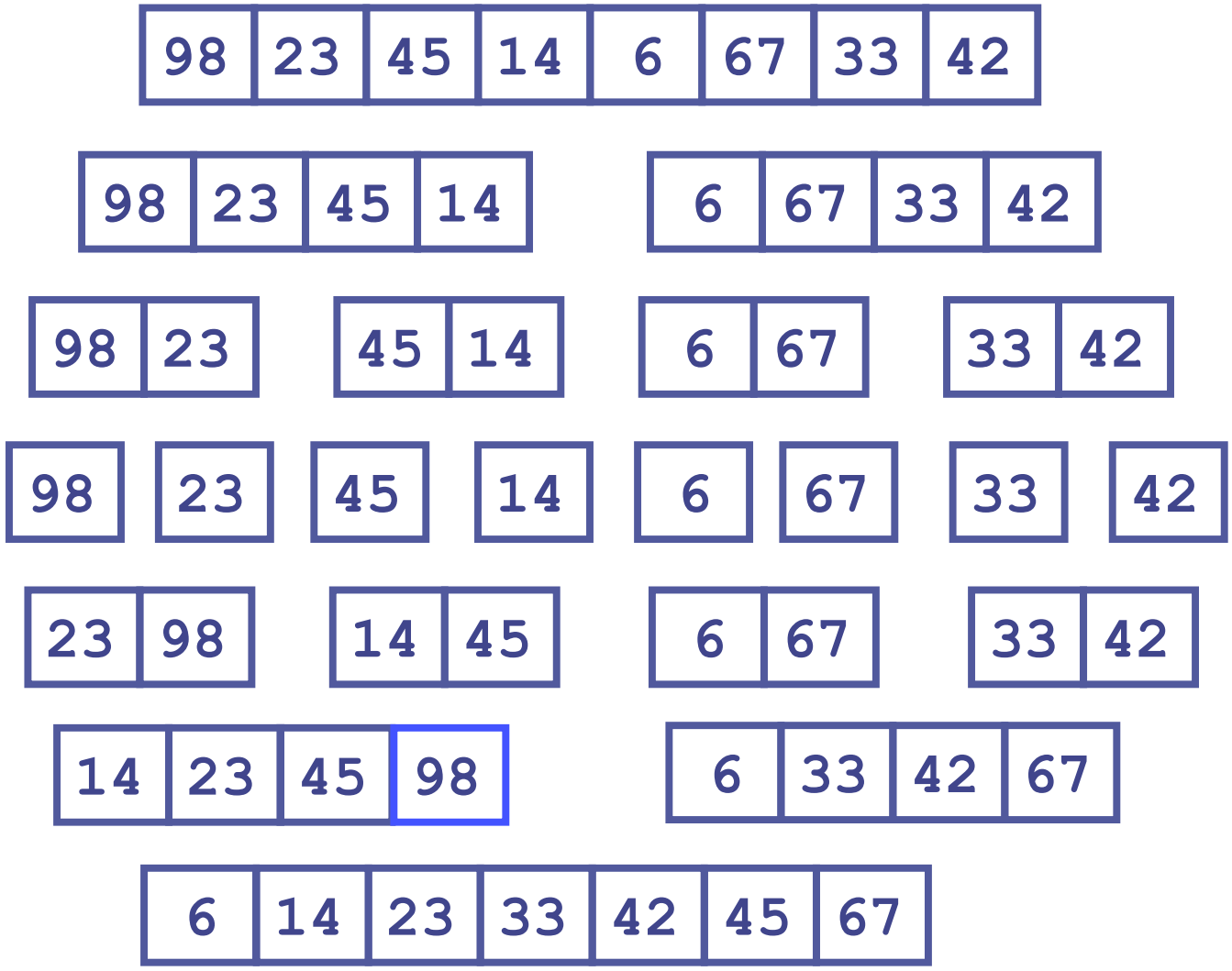
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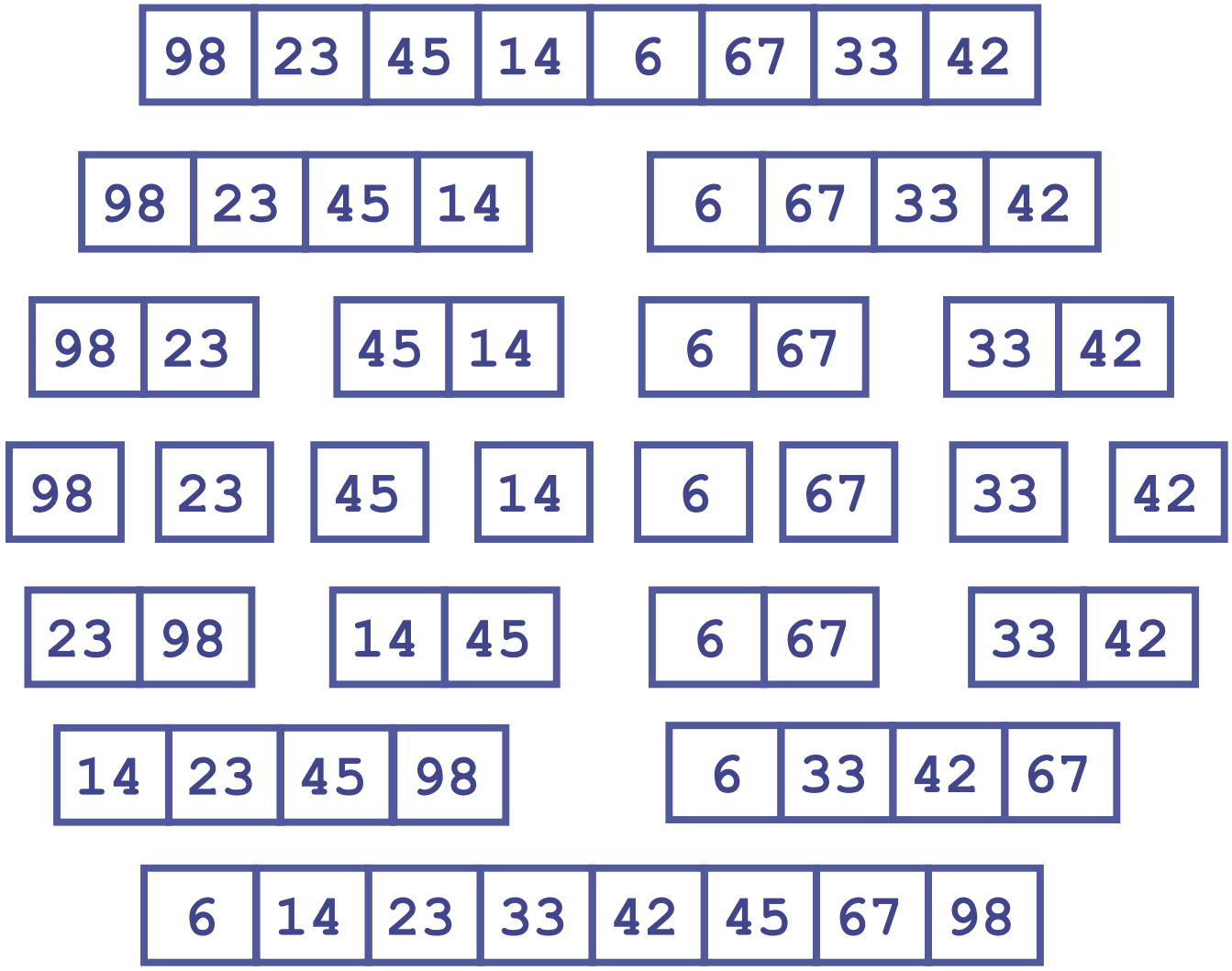
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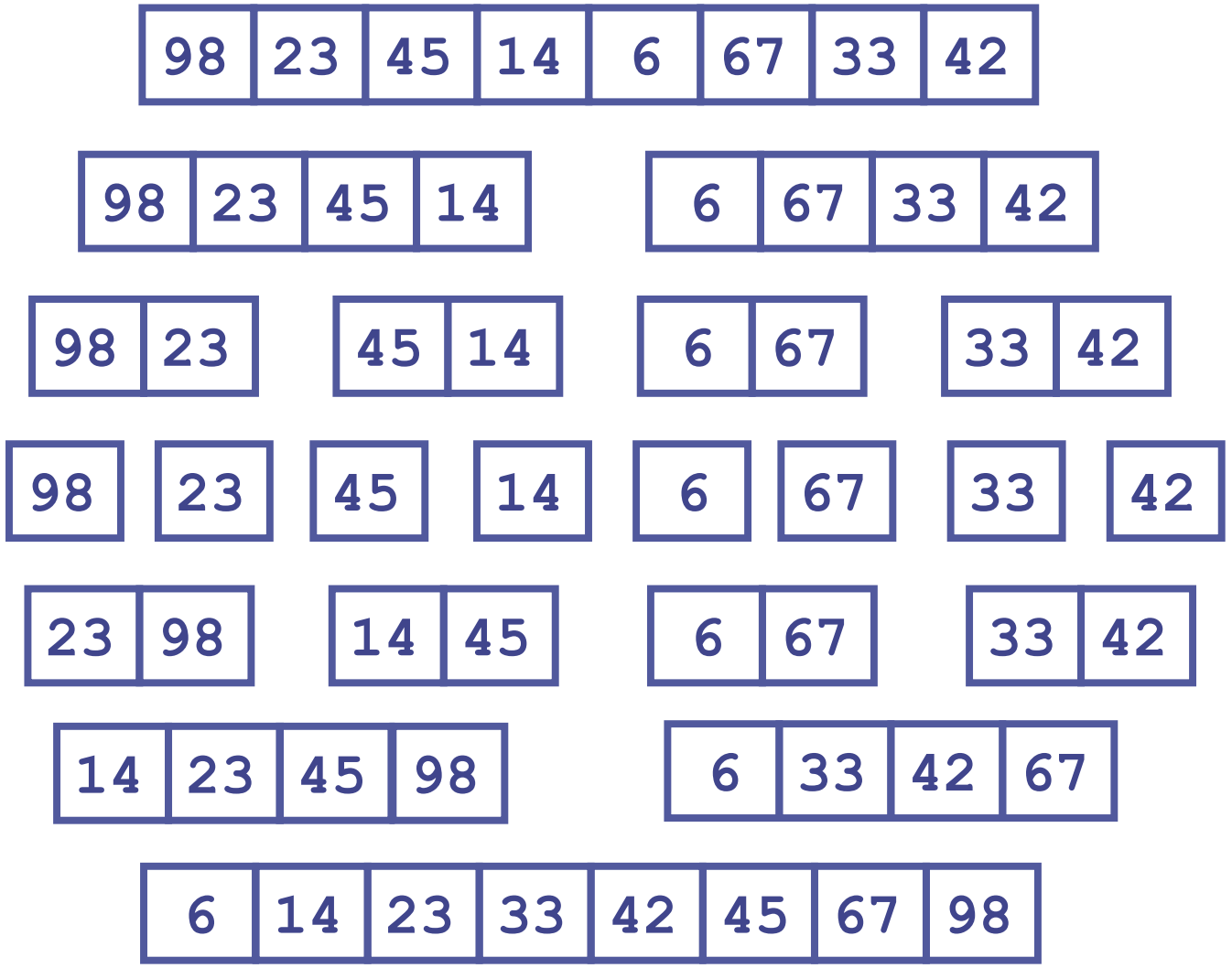
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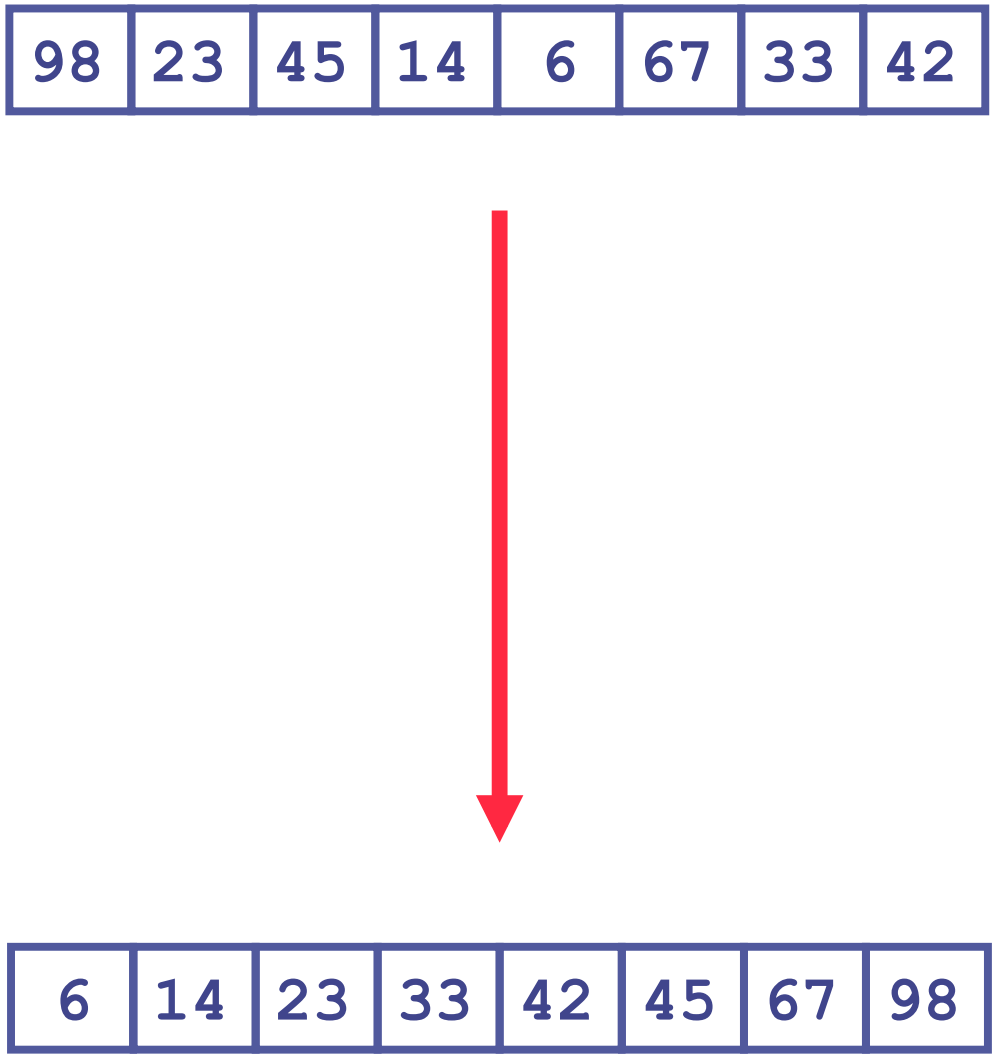
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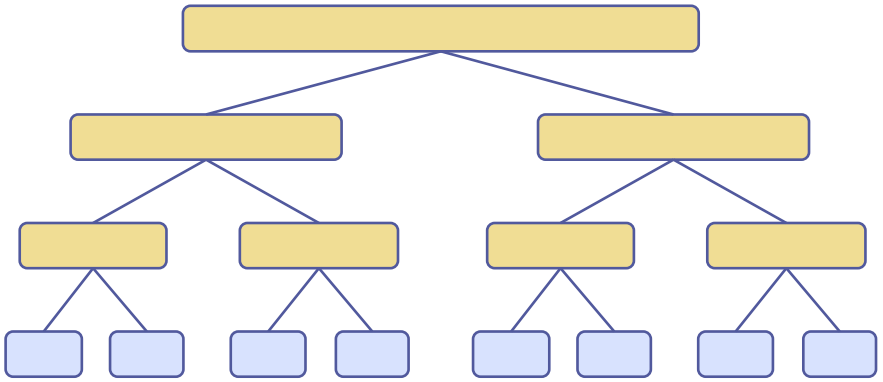
Analysis of Merge-Sort

- The height h of the merge-sort tree is $O(\log n)$
  - at each recursive call we divide in half the sequence,
- The overall amount or work done at the nodes of depth i is $O(n)$
  - we partition and merge 2^i sequences of size $n/2^i$
  - we make 2^{i+1} recursive calls
- Thus, the total running time of merge-sort is $O(n \log n)$

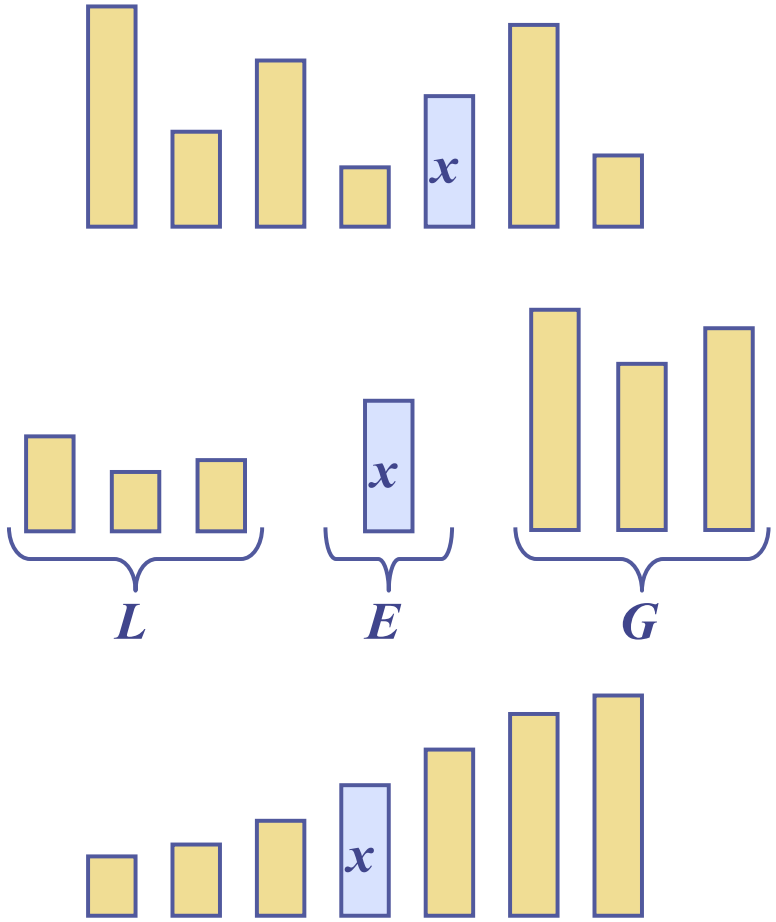
depth	#seqs	size
0	1	n
1	2	$n/2$
i	2^i	$n/2^i$
…	…	…



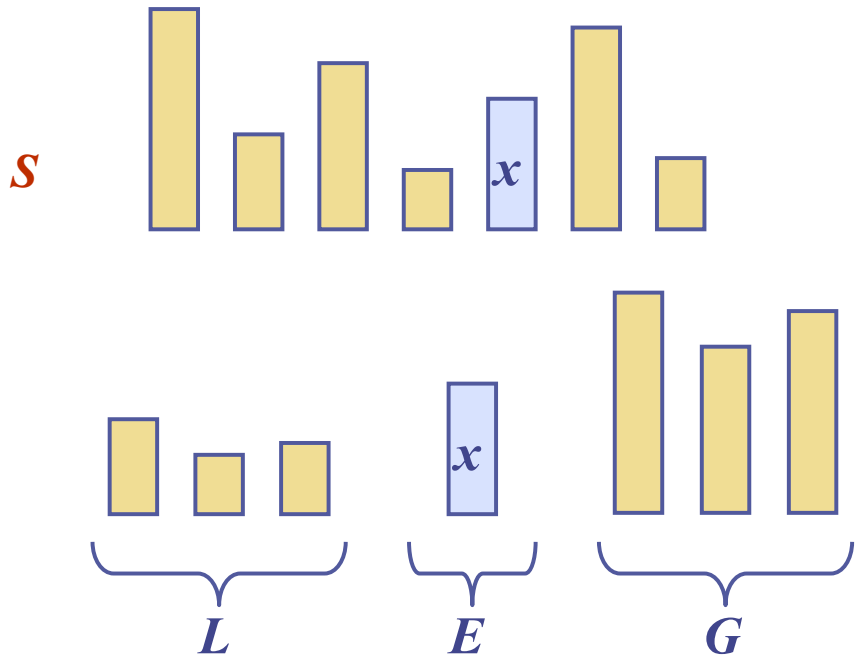
Quick-Sort

Quick-Sort

- Quick-sort is a sorting algorithm based on the divide-and-conquer paradigm:
  - Divide: pick an element x (called pivot) and partition S into
    - L elements less than x
    - E elements equal x
    - G elements greater than x
  - Recur: sort L and G
  - Conquer: join L, E and G



Not In-Place



***QuickSort*(*S*)**

```
i ← PIVOT
x ← S.elemAtRank(i)
(L,G) ← Partition(S,x)
QuickSort(L)
QuickSort(G)
```

In this example the PIVOT is chosen randomly, but we could decide always to choose the first element of the array, or the last.

Partition

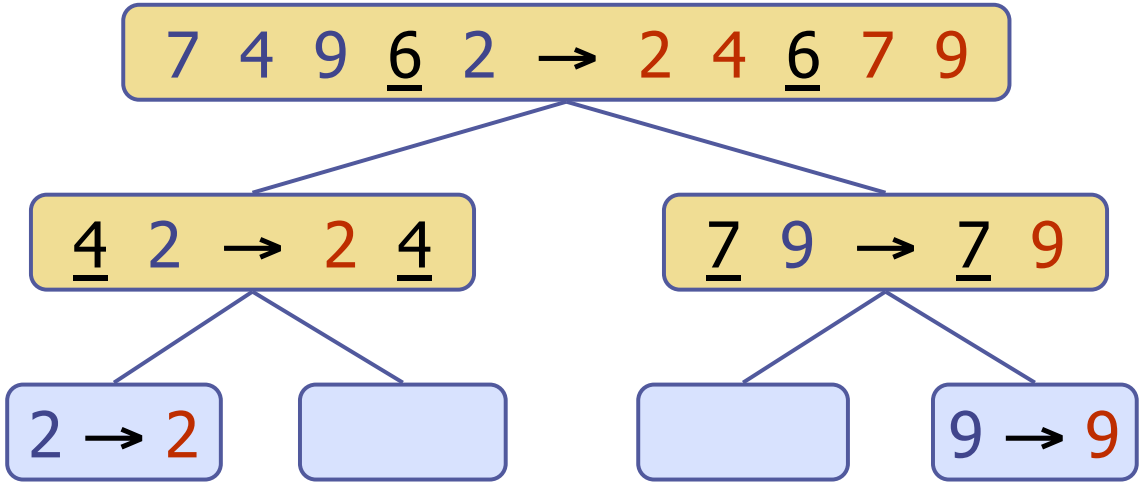
Not in-place

- We partition an input sequence as follows:
  - We remove, in turn, each element y from S and
  - We insert y into L, E or G, depending on the result of the comparison with the pivot x
- Each insertion and removal is at the beginning or at the end of a sequence, and hence takes $O(1)$ time
- Thus, the partition step of quick-sort takes $O(n)$ time

```
Algorithm partition(S, p)
    Input sequence S, position p of pivot
    Output subsequences L, E, G of the
        elements of S less than, equal to,
        or greater than the pivot, resp.
    L, E, G ← empty sequences
    x ← S.remove(p)
    while !S.isEmpty()
        y ← S.remove(S.first())
        if y < x
            L.insertLast(y)
        else if y = x
            E.insertLast(y)
        else { y > x }
            G.insertLast(y)
    return L, E, G
```

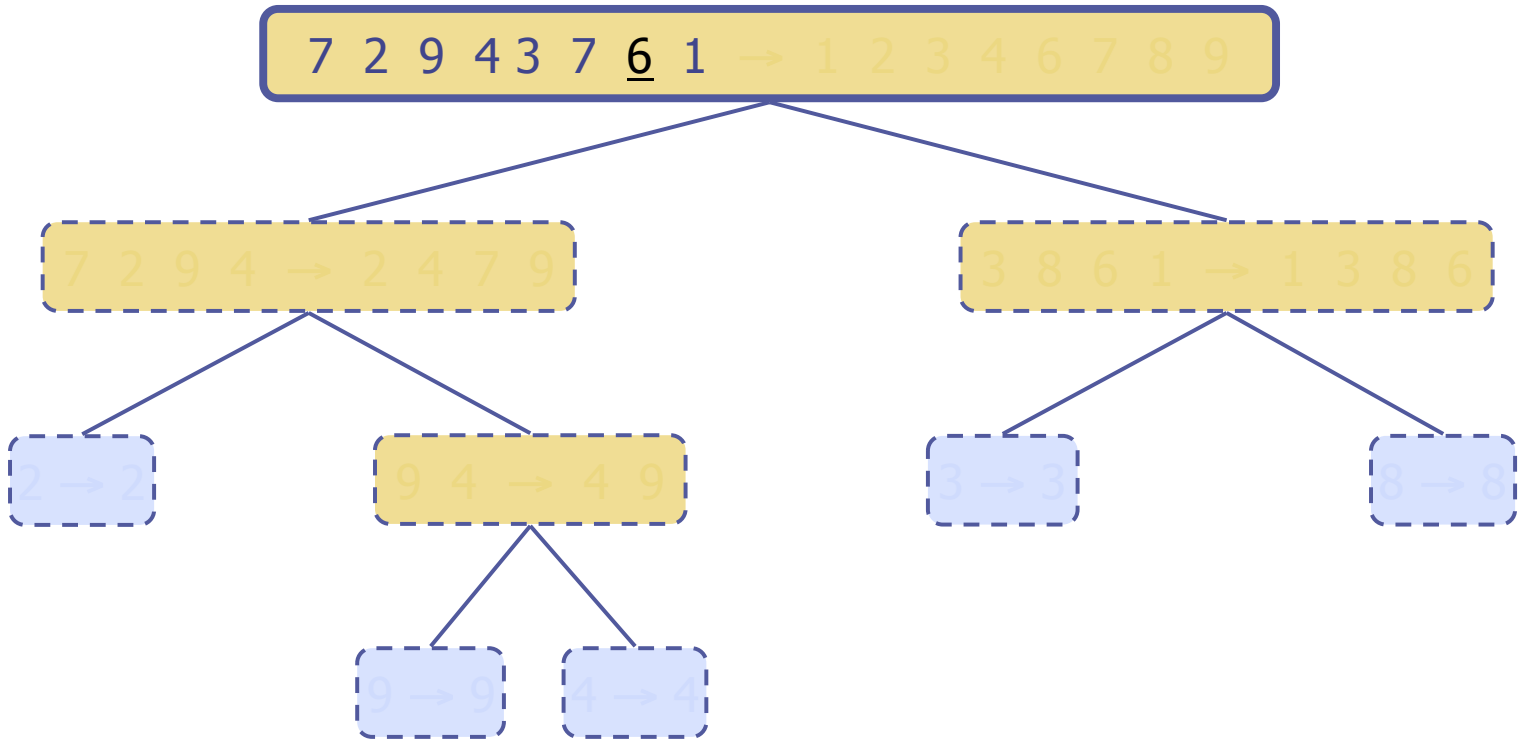
Quick-Sort Tree

- An execution of quick-sort is depicted by a binary tree
  - Each node represents a recursive call of quick-sort and stores
    - Unsorted sequence before the execution and its pivot
    - Sorted sequence at the end of the execution
  - The root is the initial call
  - The leaves are calls on subsequences of size 0 or 1



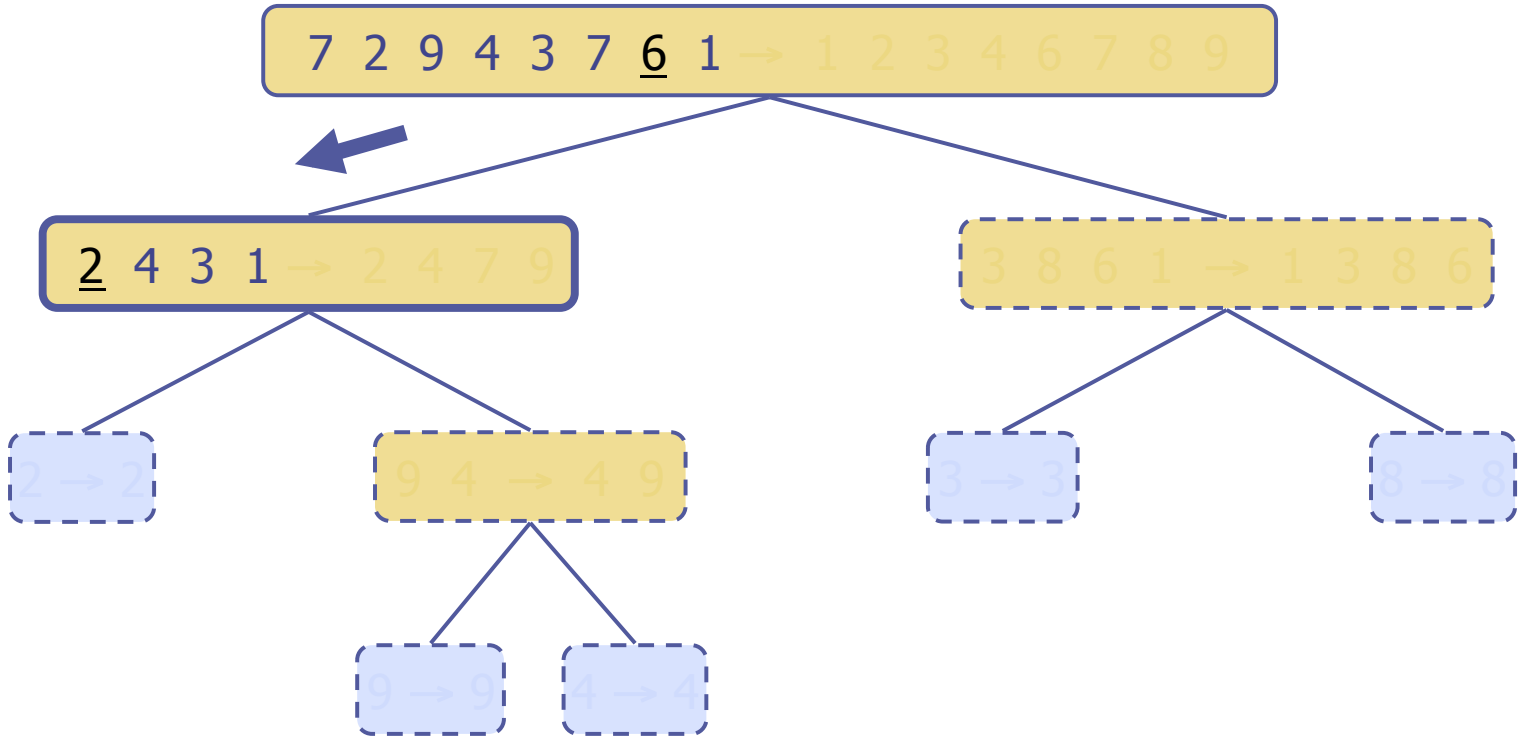
Execution Example

- Pivot selection



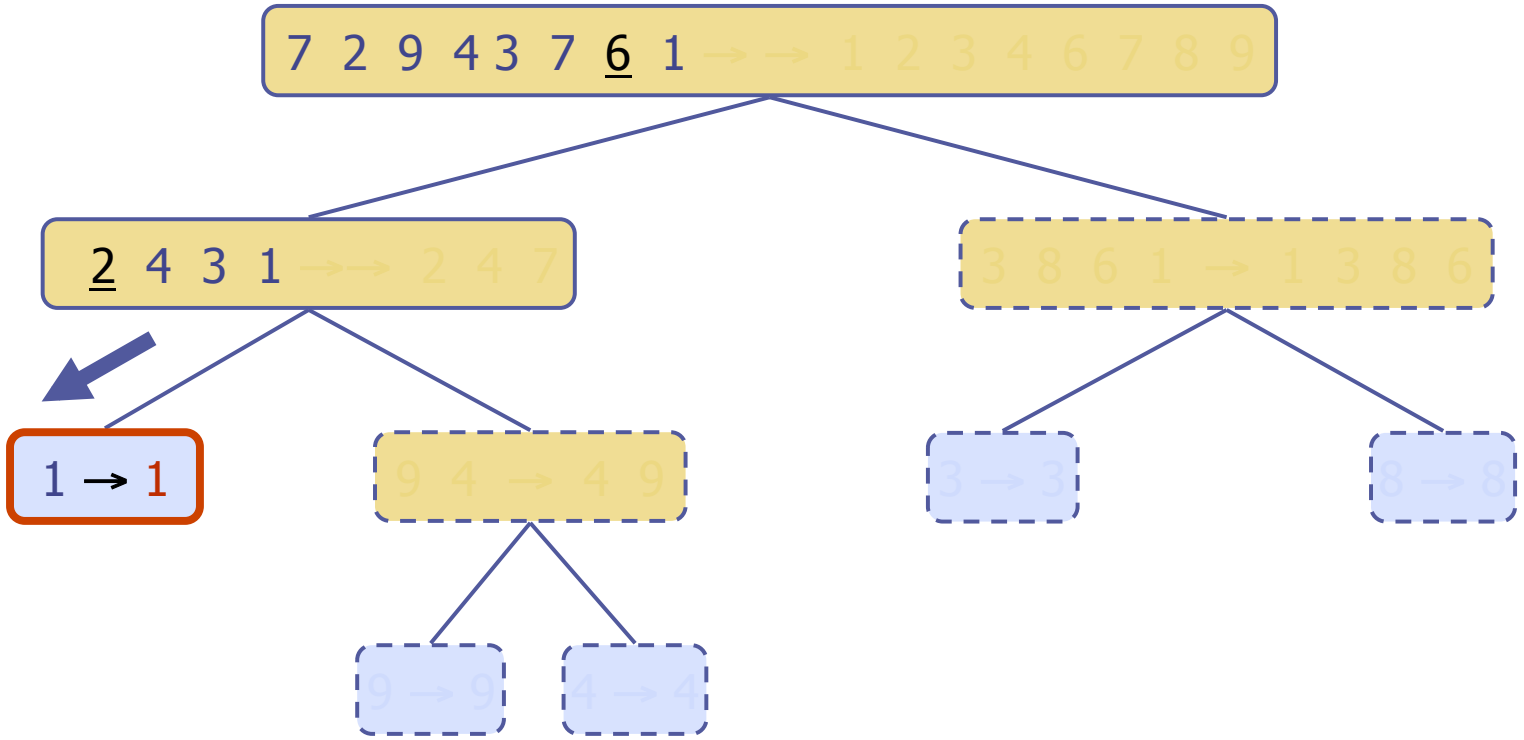
Execution Example (cont.)

- Partition, recursive call, pivot selection



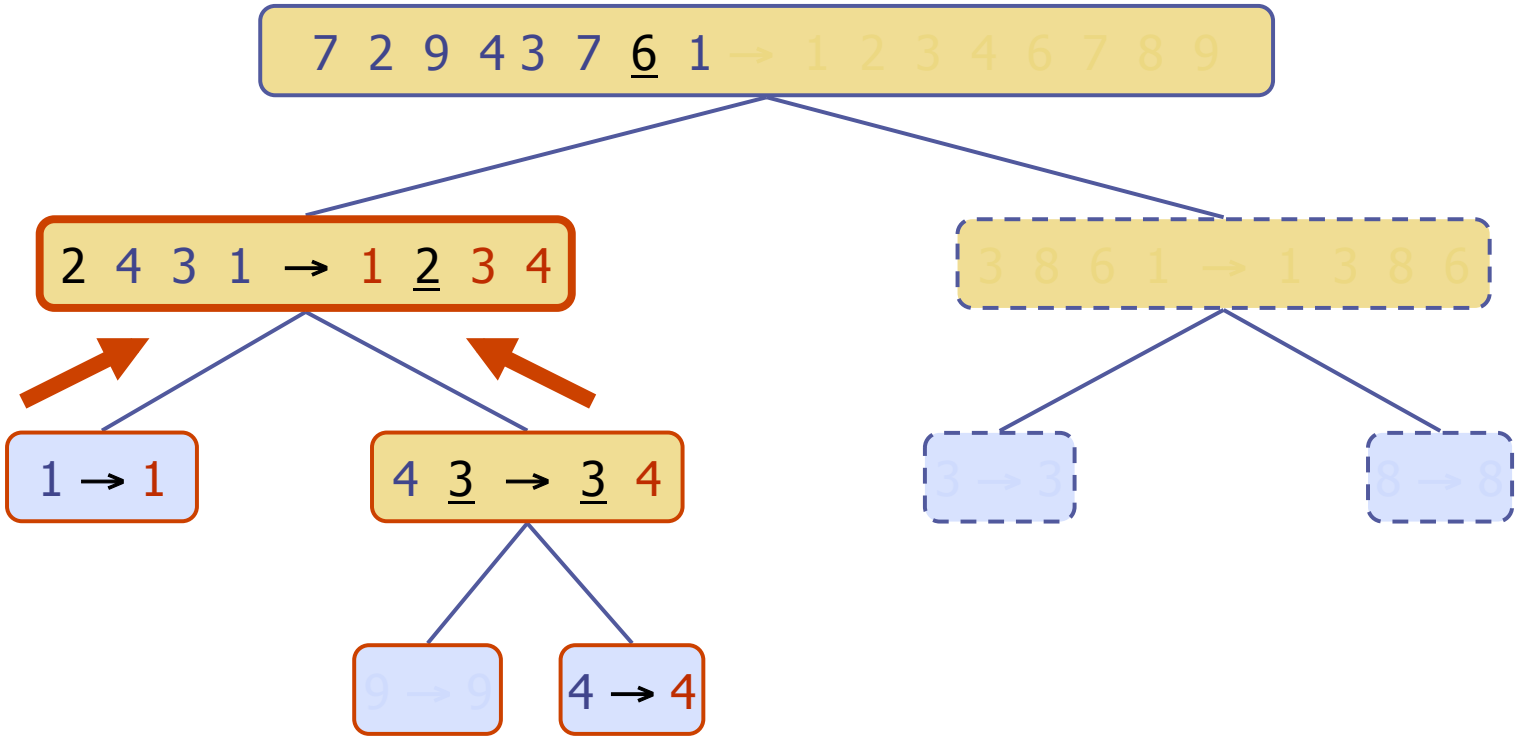
Execution Example (cont.)

- Partition, recursive call, base case



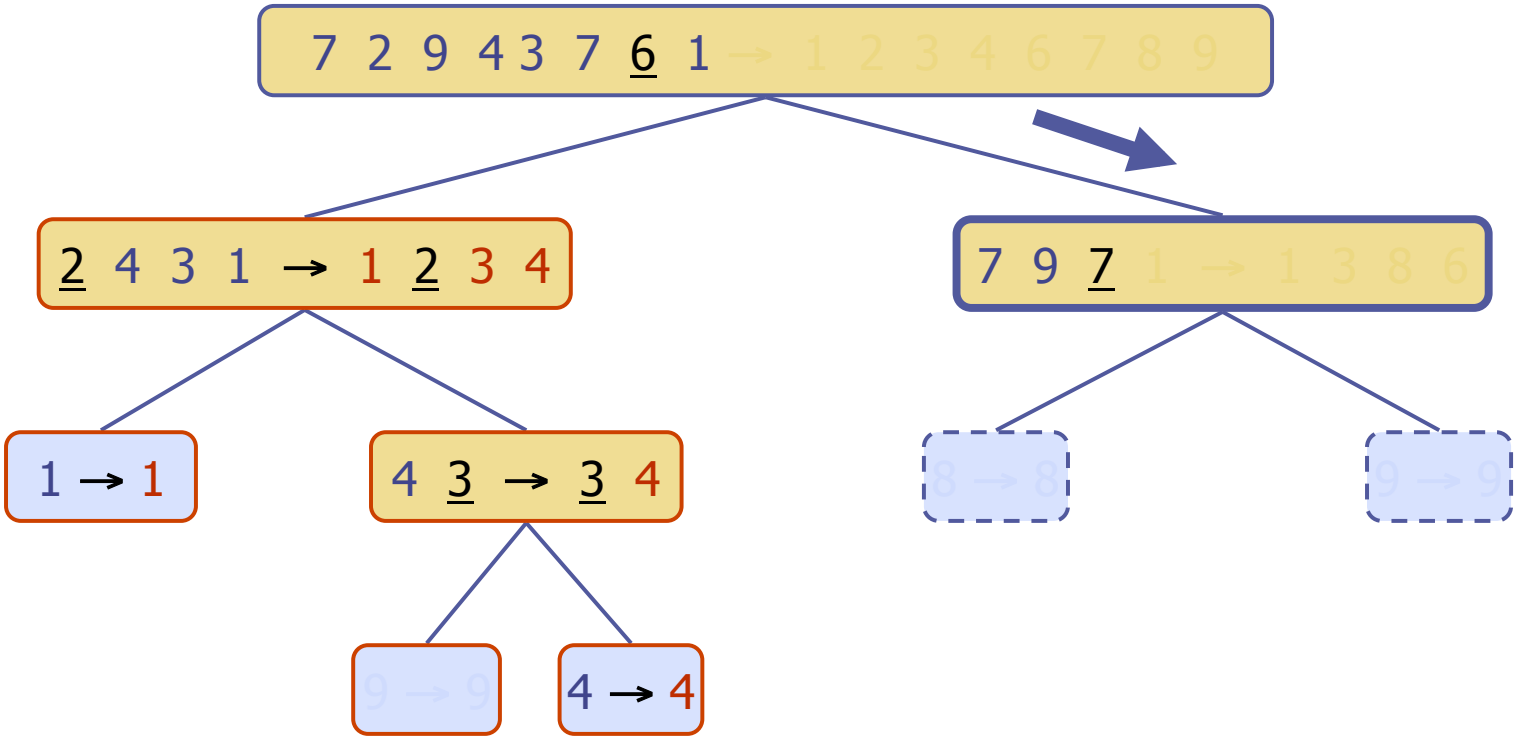
Execution Example (cont.)

- Recursive call, …, base case, join



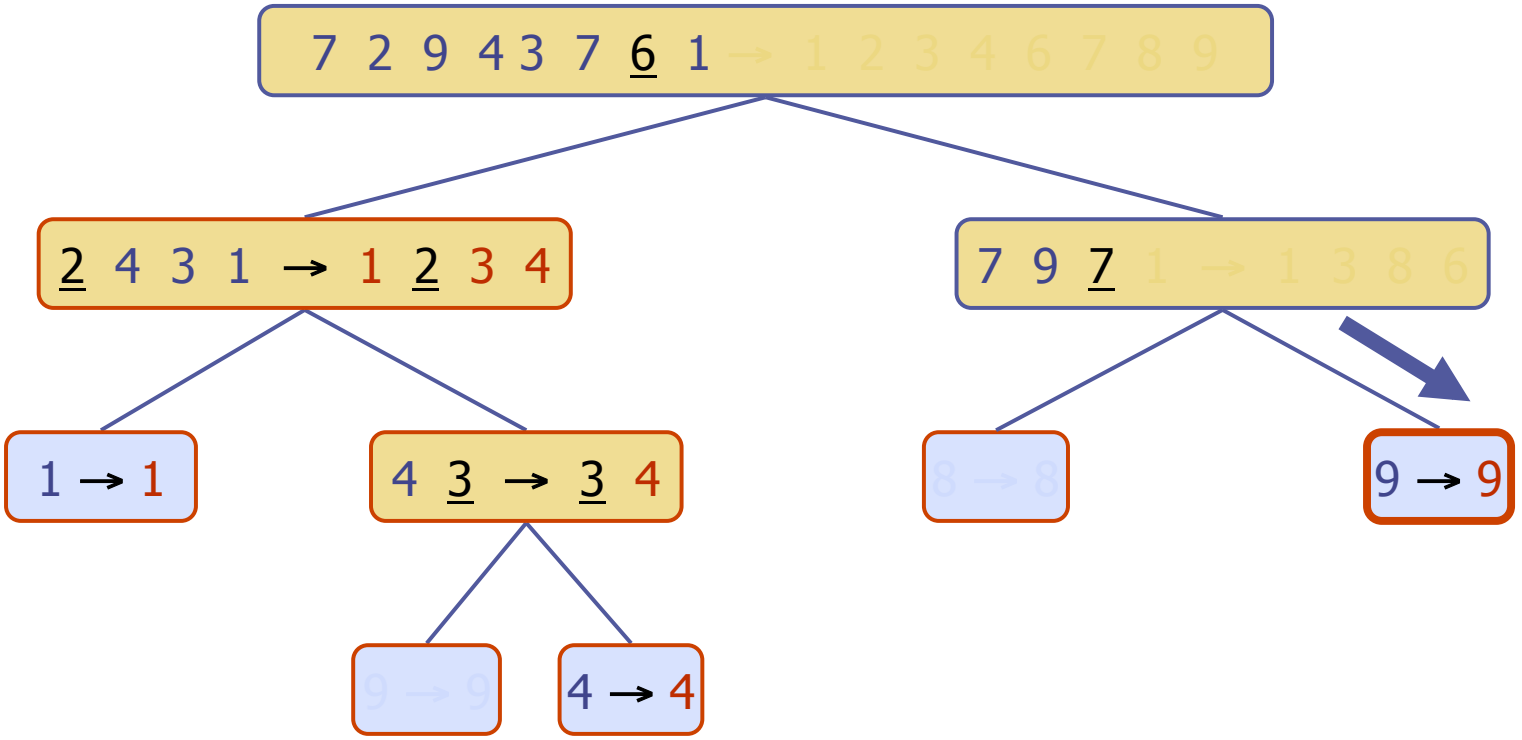
Execution Example (cont.)

- Recursive call, pivot selection



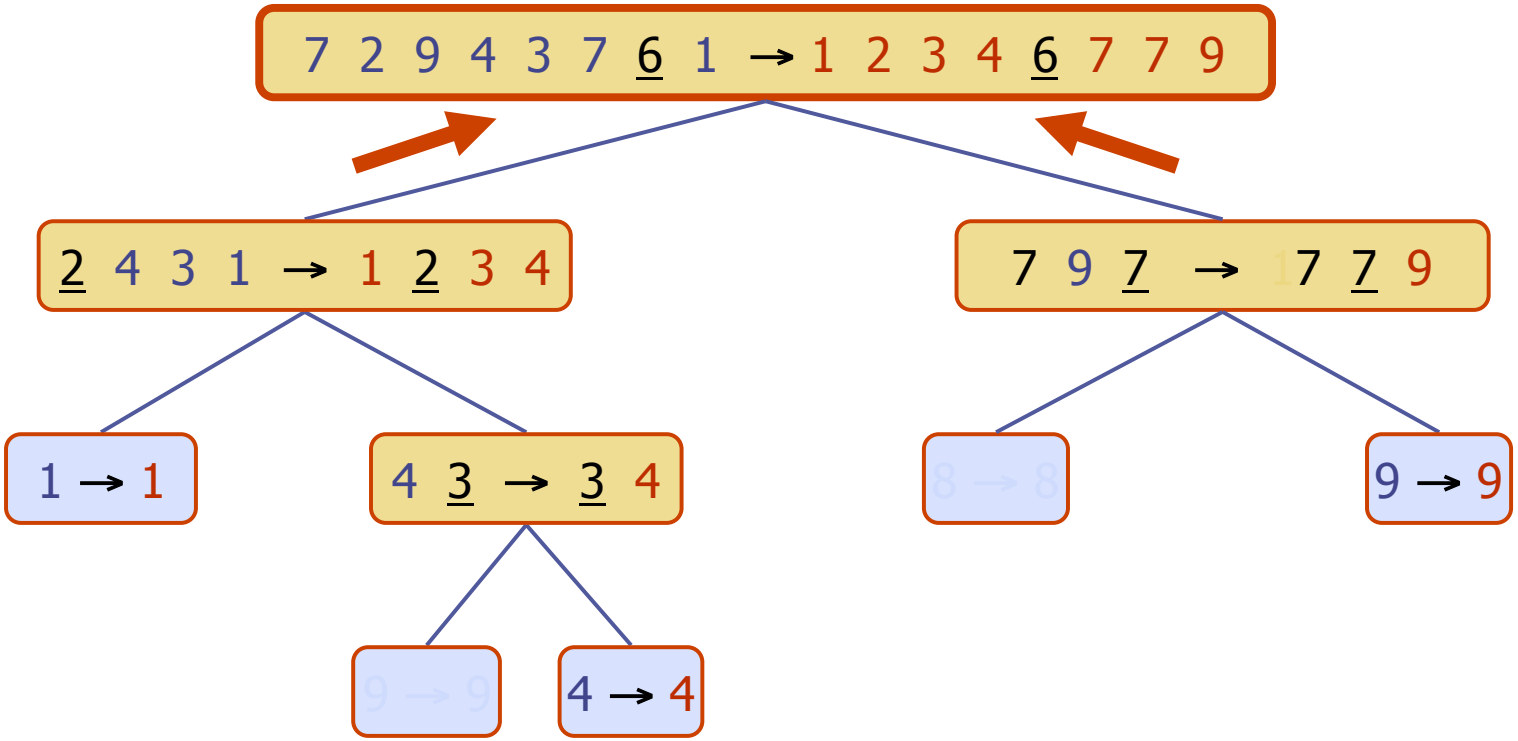
Execution Example (cont.)

- Partition, …, recursive call, base case



Execution Example (cont.)

- Join, join



In-Place Quick-Sort

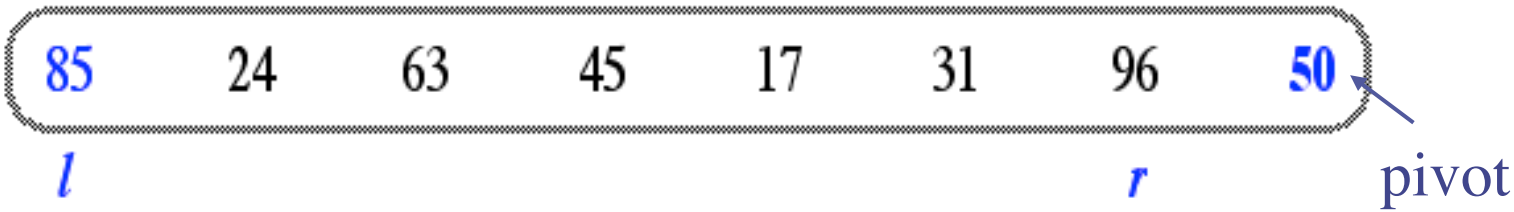
In the partition step, we use replace operations to rearrange the elements of the input sequence such that

- the elements less than the pivot have rank less than h
- the elements equal to the pivot have rank between h and k
- the elements greater than the pivot have rank greater than k

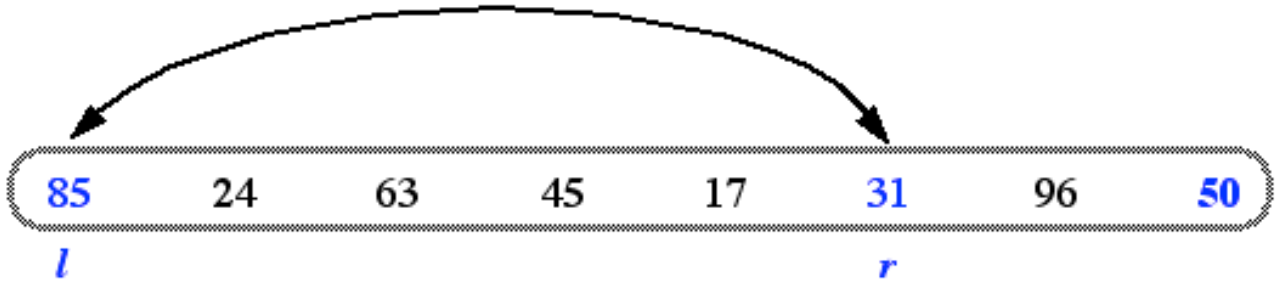
- The recursive calls consider
  - elements with rank less than h
  - elements with rank greater than k

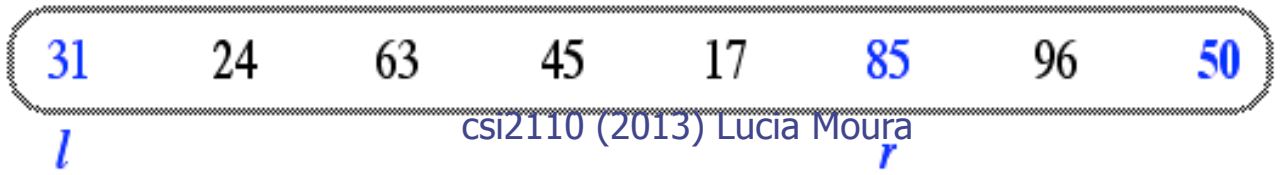
In-Place Quick-Sort

Divide step: l scans the sequence from the left, and r from the right.

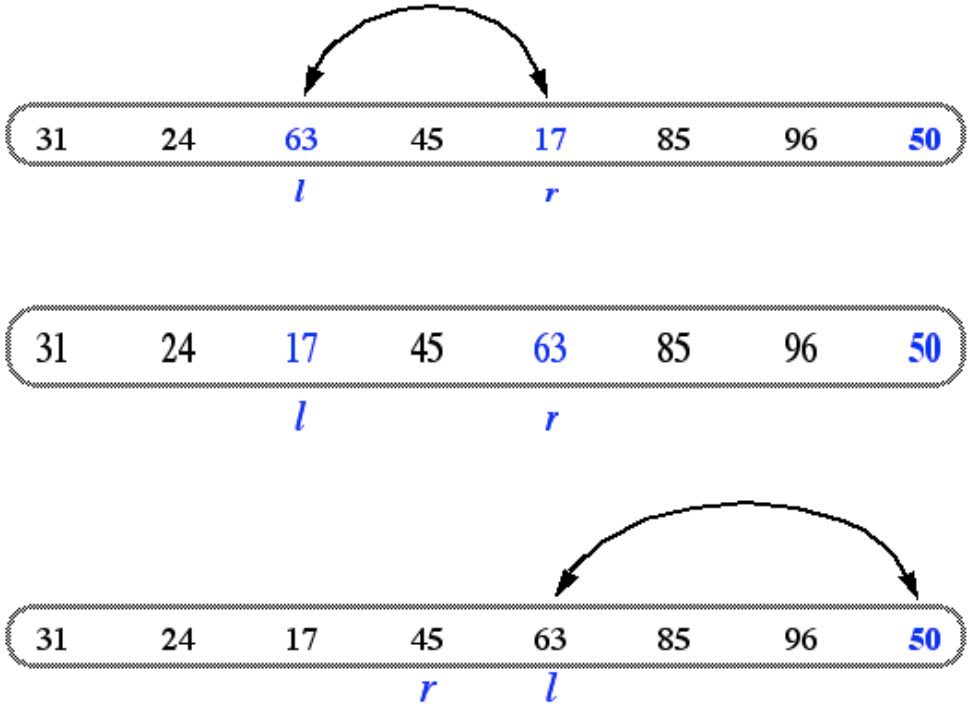


A swap is performed when l is at an element larger than the pivot and r is at one smaller than the pivot.

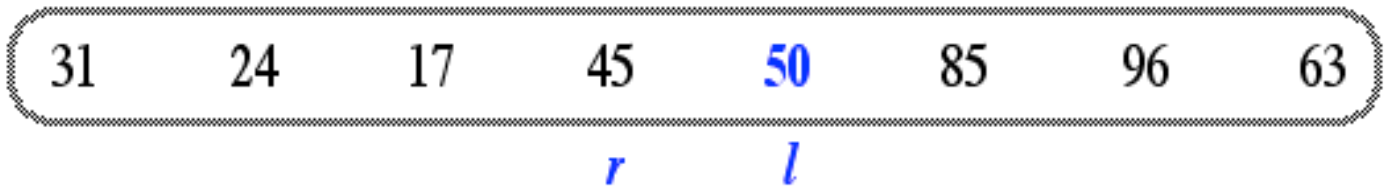




In Place Quick Sort (contd.)



A final swap with the pivot completes the divide step



In-Place Quick-Sort

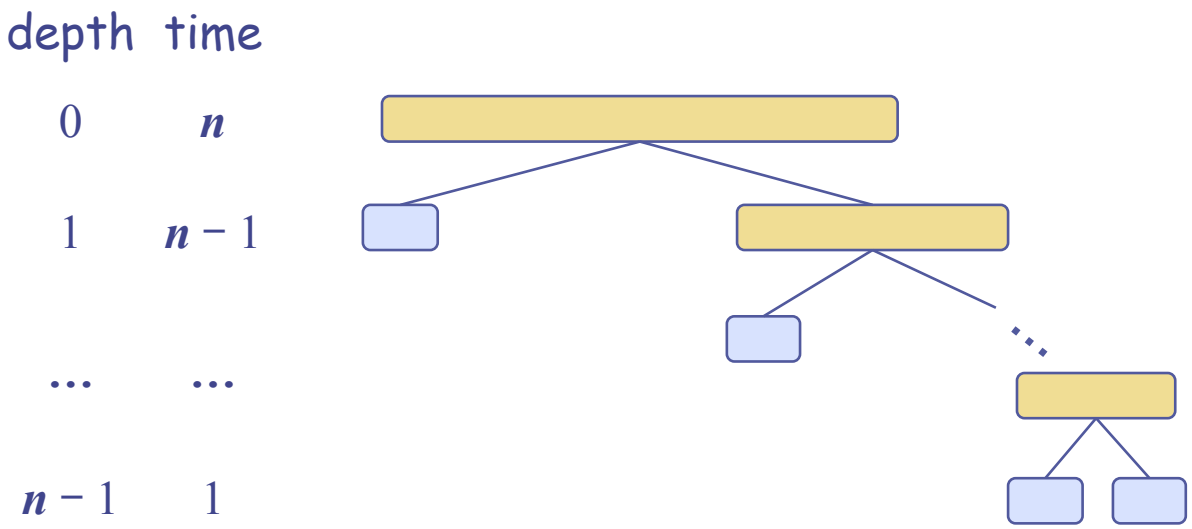
Algorithm *inPlaceQuickSort*(*S*, *l*, *r*)
  Input sequence *S*, ranks *l* and *r*
  Output sequence *S* with the elements of rank between *l* and *r* rearranged in increasing order
  if $l \geq r$
    return
  $i \leftarrow$ a random integer between *l* and *r*
  $x \leftarrow S.elemAtRank(i)$
  $(h, k) \leftarrow inPlacePartition(x)$
  inPlaceQuickSort(*S*, *l*, $h - 1$)
  inPlaceQuickSort(*S*, $k + 1$, *r*)

Worst-case Running Time

- The worst case for quick-sort occurs when the pivot is the unique minimum or maximum element
- One of L and G has size $n - 1$ and the other has size 0
- The running time is proportional to the sum

$$n + (n - 1) + \ldots + 2 + 1$$

- Thus, the worst-case running time of quick-sort is $O(n^2)$

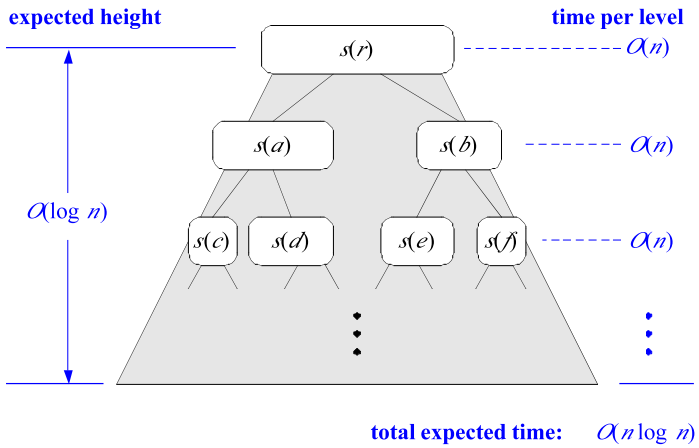


Expected Running Time

Consider a recursive call of quick-sort on a sequence of size s

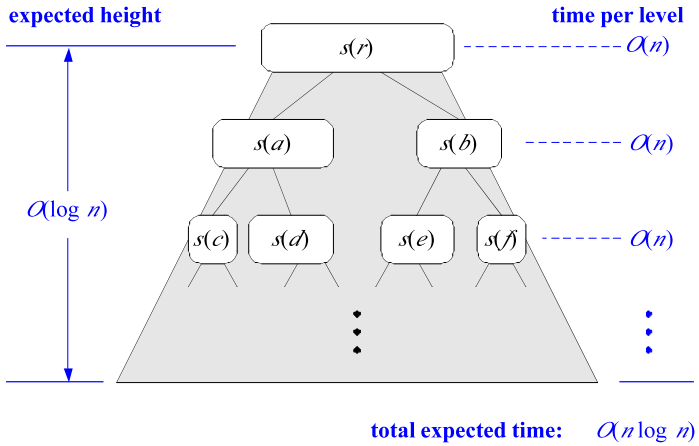
- Good call: the sizes of L and G are each less than $3s/4$
- Bad call: one of L and G has size greater than $3s/4$

- A call is good with probability $1/2$ *(for an element, the expected number of calls until a good call is 2)*
- Hence, for a node of depth i, we expect that
  - $i/2$ ancestor nodes are associated with good calls
  - the expected size of the input sequence for the current call is at most $(3/4)^{i/2}n$



Expected Running Time

- Thus, we have
  - For a node of depth $2\log_{4/3}n$, the expected size of the input sequence is one ((3/4)^((2log$_{4/3}$n)/2) n = 1)
  - The expected height of the quick-sort tree is $O(\log n)$
- The overall amount or work done at the nodes of the same depth of the quick-sort tree is $O(n)$
- Thus, the expected running time of quick-sort is $O(n \log n)$



Algorithm	Time	Notes
selection-sort	$O(n^2)$ w.c. and av.	◆ in-place ◆ slow (good for small inputs)
insertion-sort	$O(n^2)$ w.c. and av.	◆ in-place ◆ slow (good for small inputs)
quick-sort	$O(n^2)$ *w.c.* $O(n \log n)$ average	◆ in-place, randomized ◆ fastest (good for large inputs)
heap-sort	$O(n \log n)$ w.c. and av.	◆ in-place ◆ fast (good for large inputs)
merge-sort	$O(n \log n)$ w.c. and av.	◆ sequential data access ◆ fast (good for huge inputs)