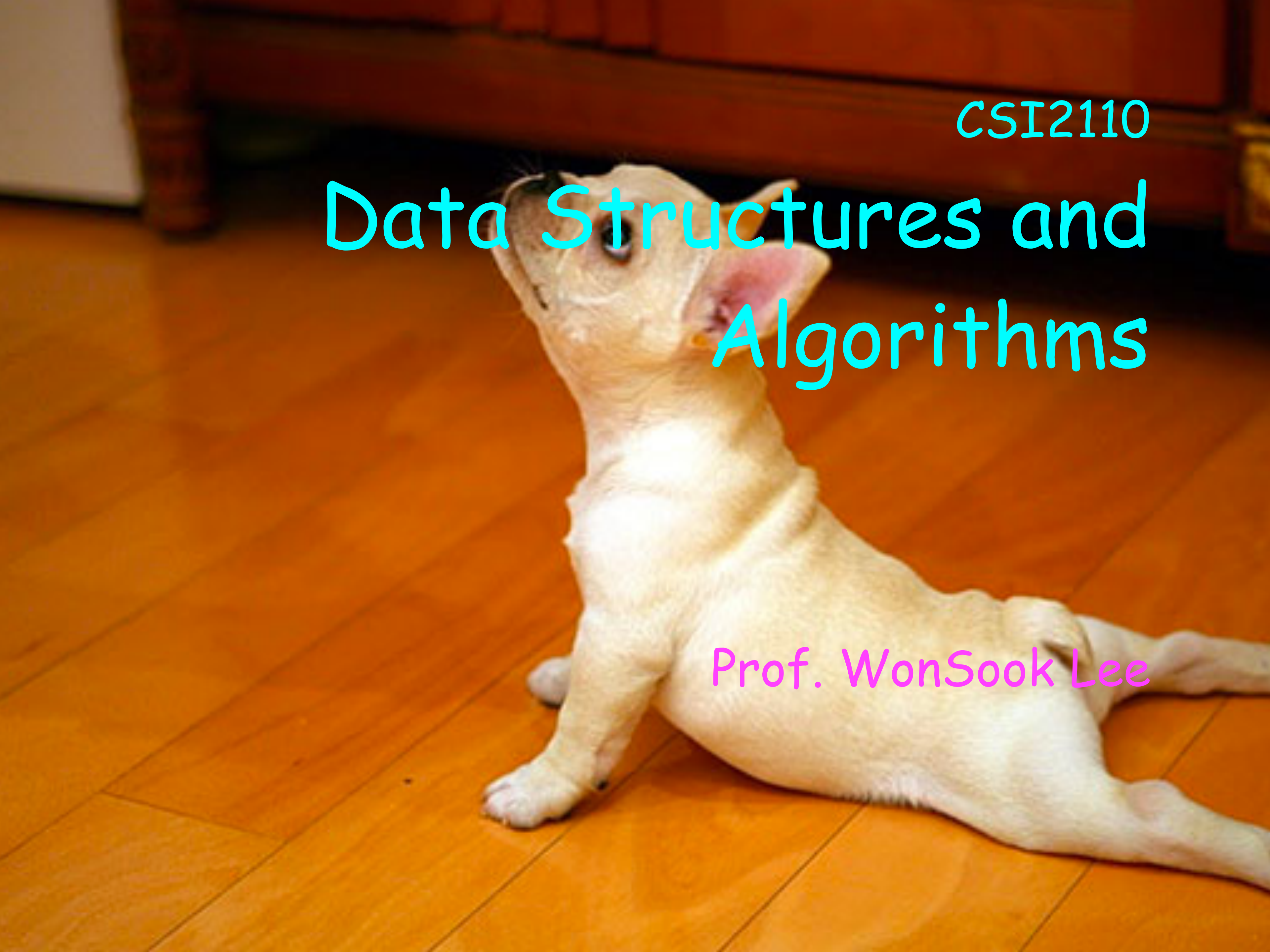


CSI2110

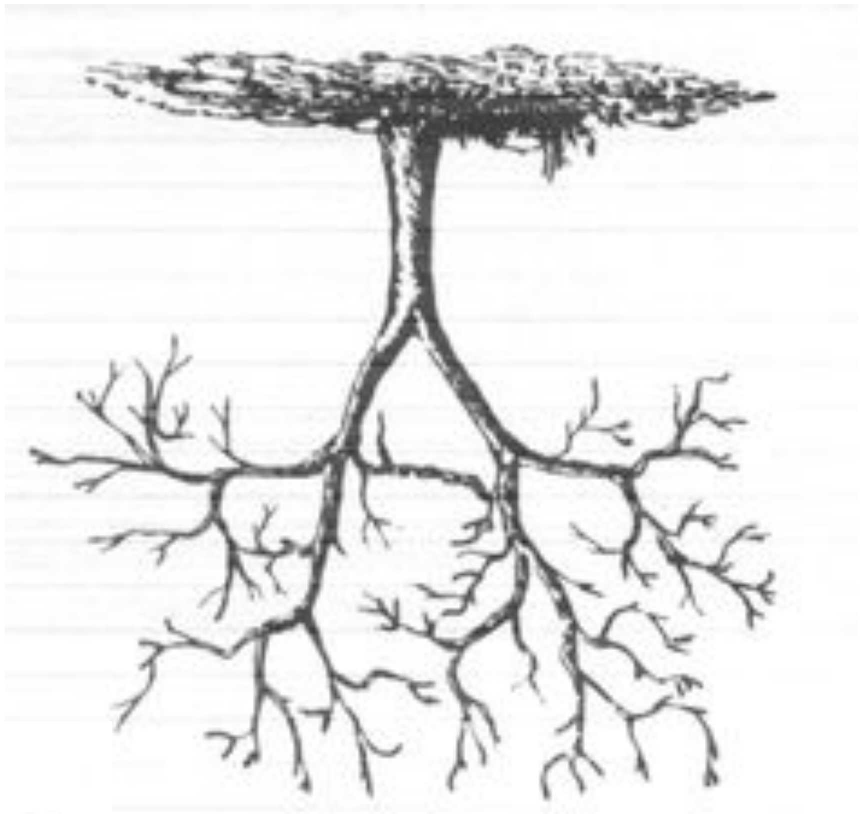
Data Structures and Algorithms

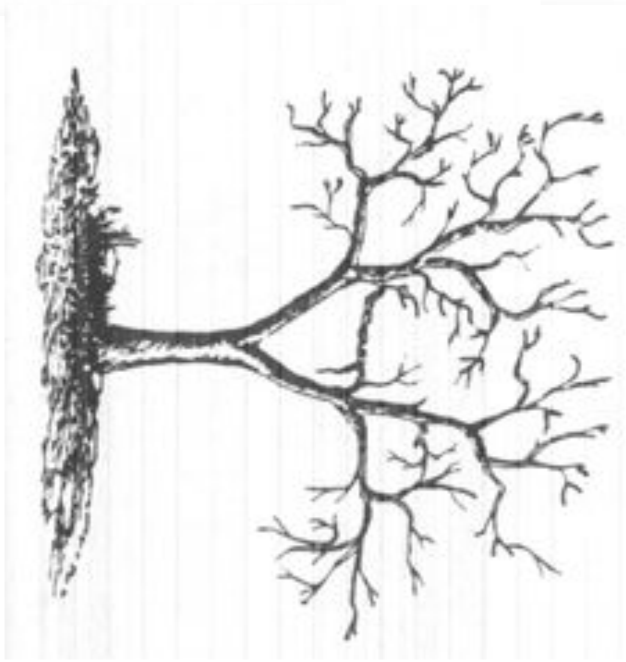
Prof. WonSook Lee

Trees

- Trees
- Binary Trees
- Properties of Binary Trees
- Traversals of Trees
- Data Structures for Trees

a Tree

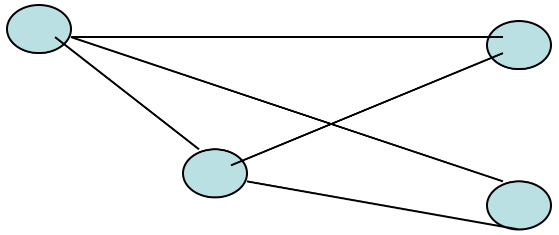




Trees

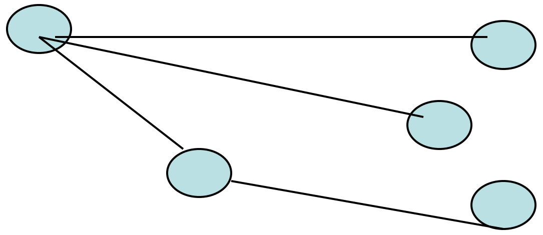
A graph $G = (V,E)$ consists of an set V of VERTICES

and a set E of edges, with $E = \{(u,v): u,v \in V, u \neq v\}$



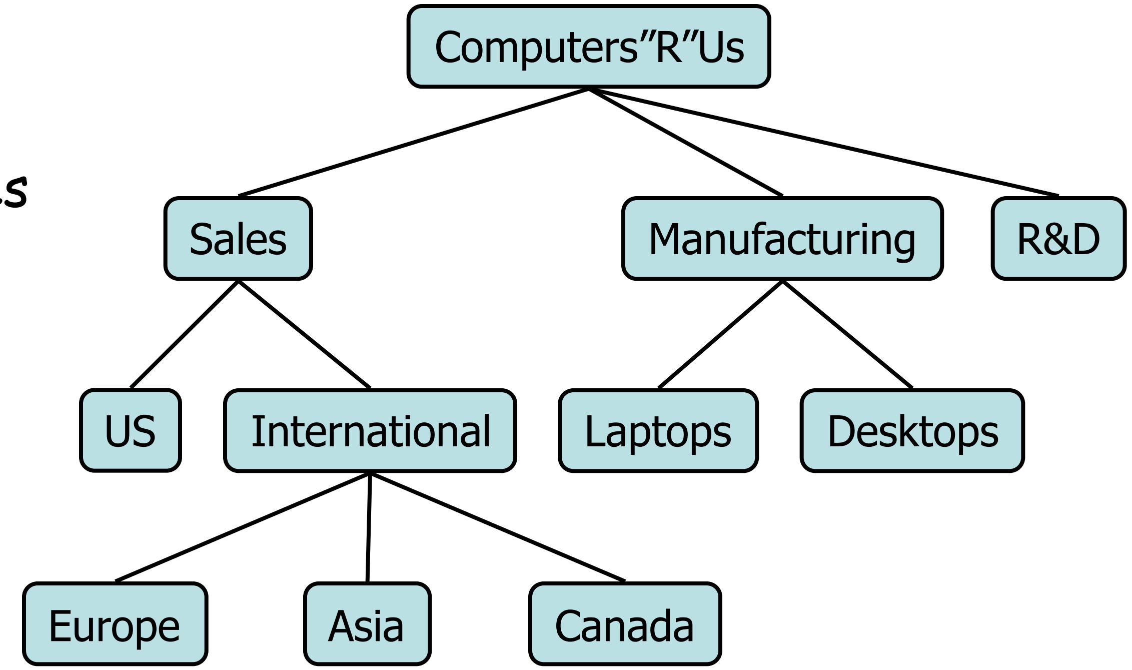
A tree is a connected graph with no cycles.

→ $\exists$ a path between each pair of vertices.

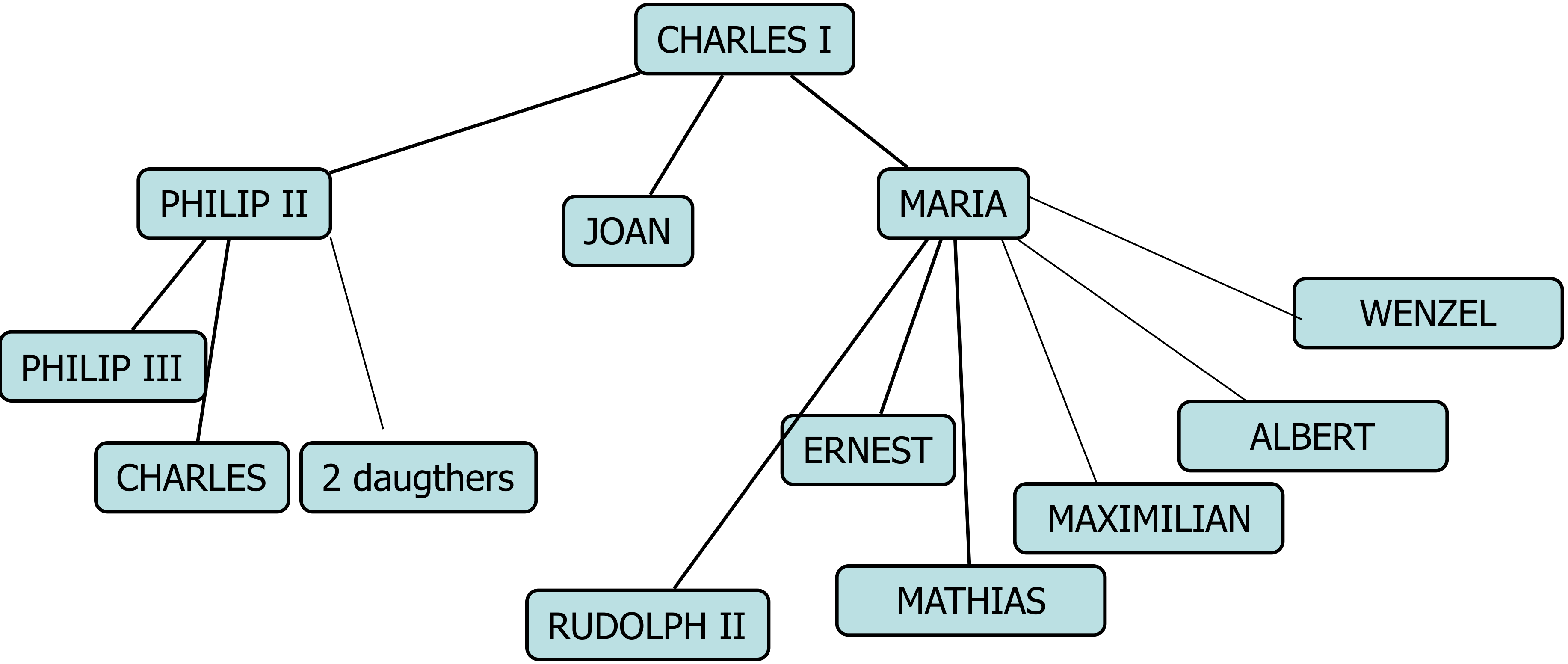


What is a Tree

- Abstract model of a hierarchical structure
- A tree consists of nodes with a parent-child relation
- Applications:
  - Organization charts
  - File systems
  - Programming environments



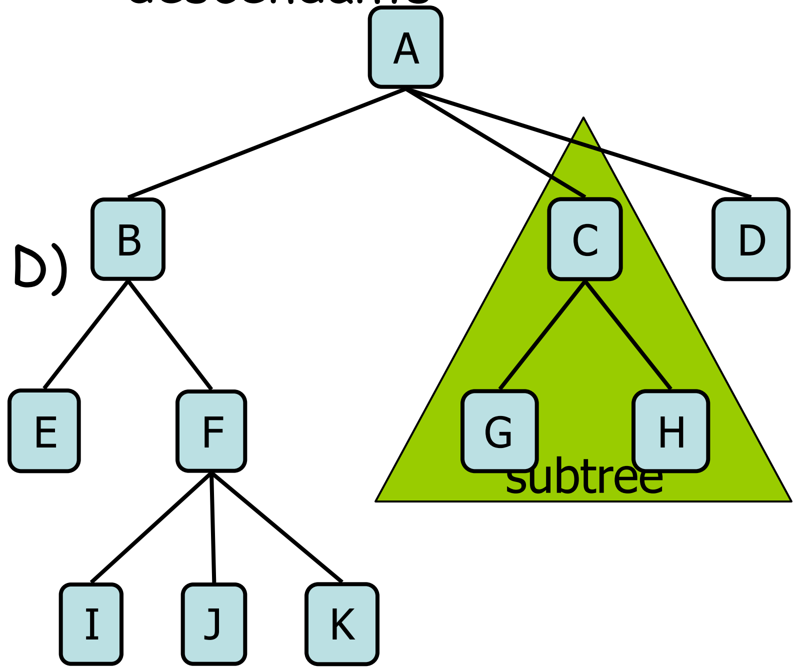
Example: Genealogical Tree



Hasburg Family

Tree Terminology

- Root: node without parent (A)
- Internal node: node with at least one child (A, B, C, F)
- External node (a.k.a. leaf): node without children (E, I, J, K, G, H, D)
- Ancestors of a node: parent, grandparent, grand-grandparent, etc.
- Subtree: tree consisting of a node and its descendants

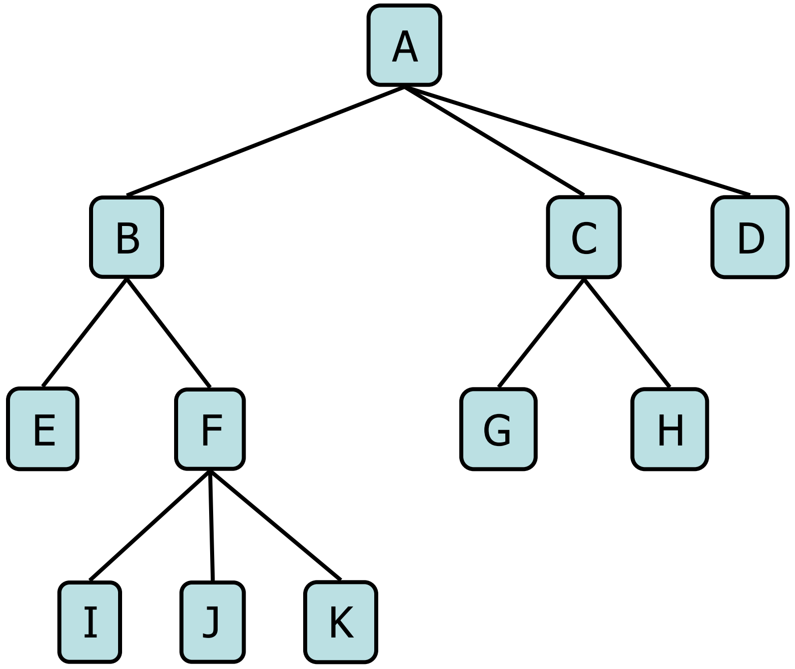


- Descendant of a node: child, grandchild, grand-grandchild, etc.

Tree Terminology

Distance between two nodes: number of "edges" between them

- Depth of a node: number of ancestors (= distance from the root)
- Height of a tree: maximum depth of any node (3)



ADTs for Trees

- generic container methods
  - size(), isEmpty(), elements()
- positional container methods
  - positions(), swapElements(p,q), replaceElement(p,e)
- query methods
  - isRoot(p), isInternal(p), isExternal(p)
- accessor methods
  - root(), parent(p), children(p)
- update methods
  - application specific

Computing the depth of a node

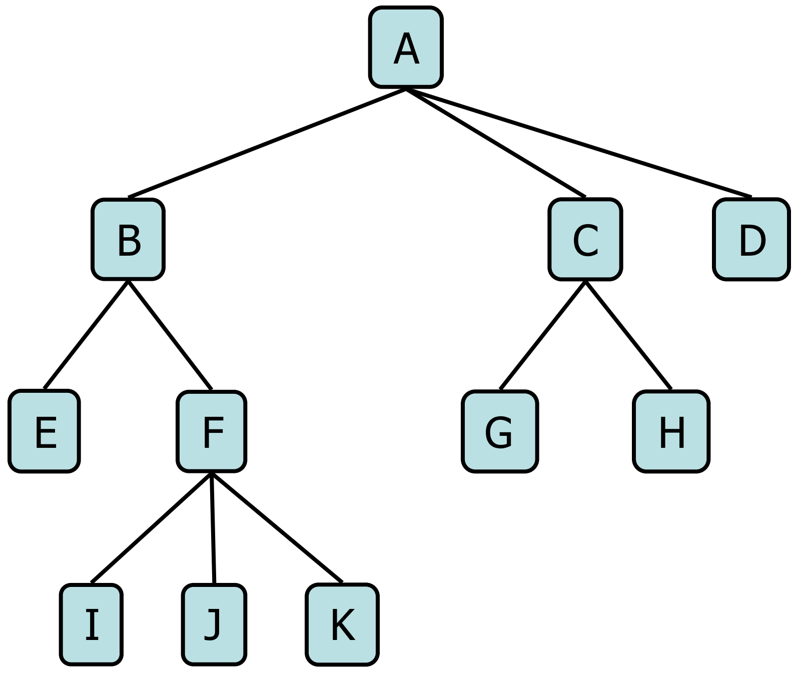
If v is the root the depth is 0
If v is an internal node the depth is 1 + the depth of its parent

```
Algorithm depth(T,v)
if T.isRoot(v) then
    return 0
else
return 1 + depth(T, T.parent(v))
```

Complexity ?

Now, Traversing a tree!

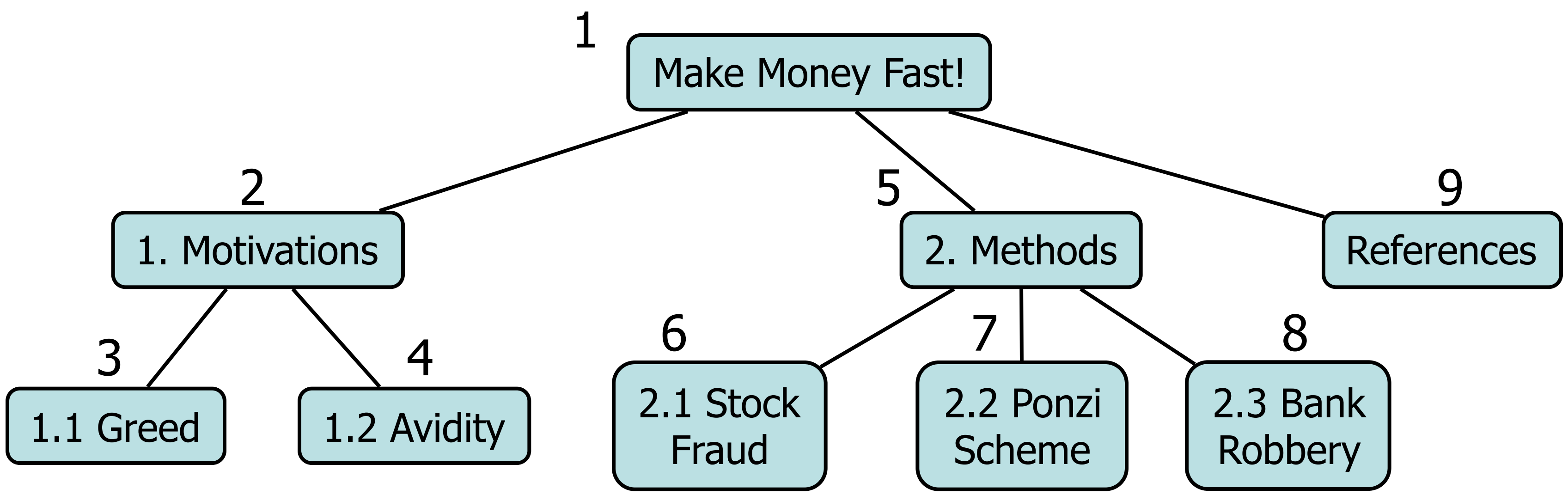
How to visit all the nodes in a tree?



Traversing Trees
Preorder Traversal

- A traversal visits the nodes of a tree in a systematic manner
- In a preorder traversal, a **node is visited before its descendants**
- Application: print a structured document

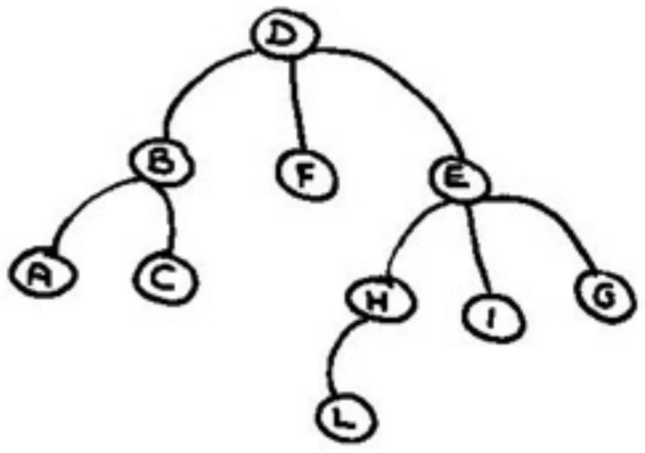
```
Algorithm preOrder(v)
    visit(v)
    for each child w of v
        preorder (w)
```



Traversing Trees

Preorder Traversal

```
Algorithm preOrder(v)
    visit(v)
    for each child w of v
        preorder (w)
```



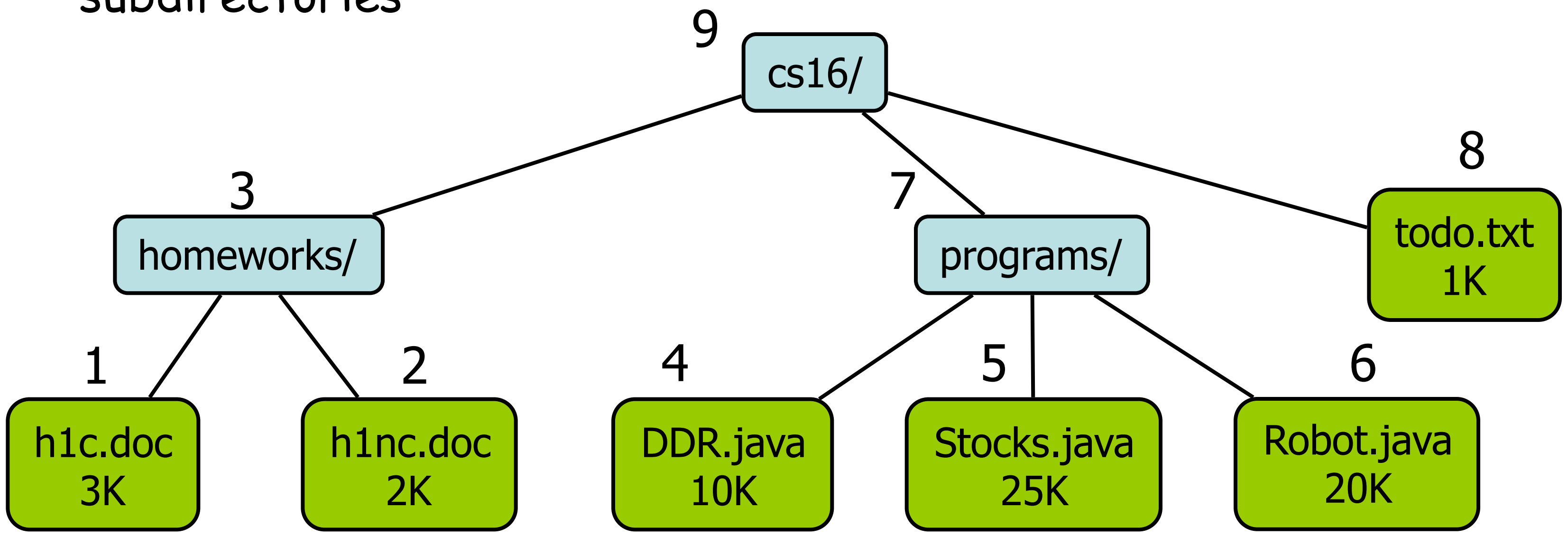
D B A C F E H L I G

Traversing Trees

Postorder Traversal

- In a postorder traversal, a **node is visited after its descendants**
- Application: compute space used by files in a directory and its subdirectories

```
Algorithm postOrder(v)
    for each child w of v
        postOrder (w)
    visit(v)
```

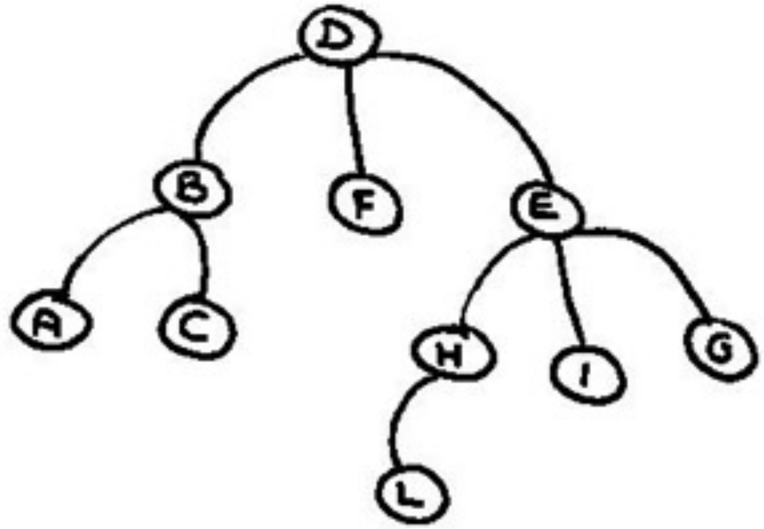


Traversing Trees

Postorder Traversal

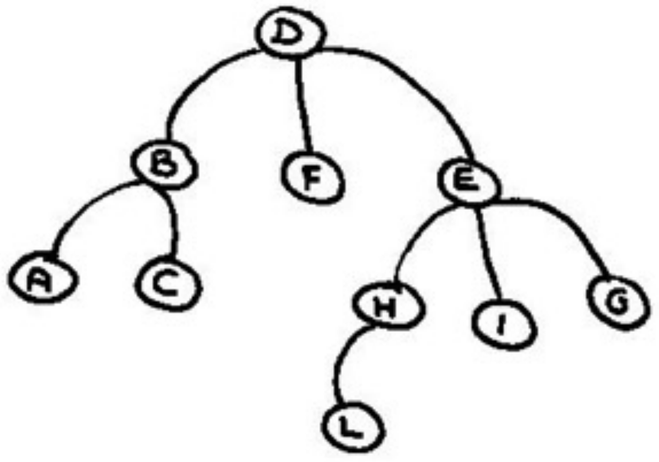
```
Algorithm postOrder(v)
    for each child w of v do
    recursively perform postOrder(w)
            "visit" node v
```

A C B F L H I G E D



Traversing Trees

Inorder Traversal of a tree (Depth-first)



Let d(x) be the number
of sub-trees of node x.

Start: x = root

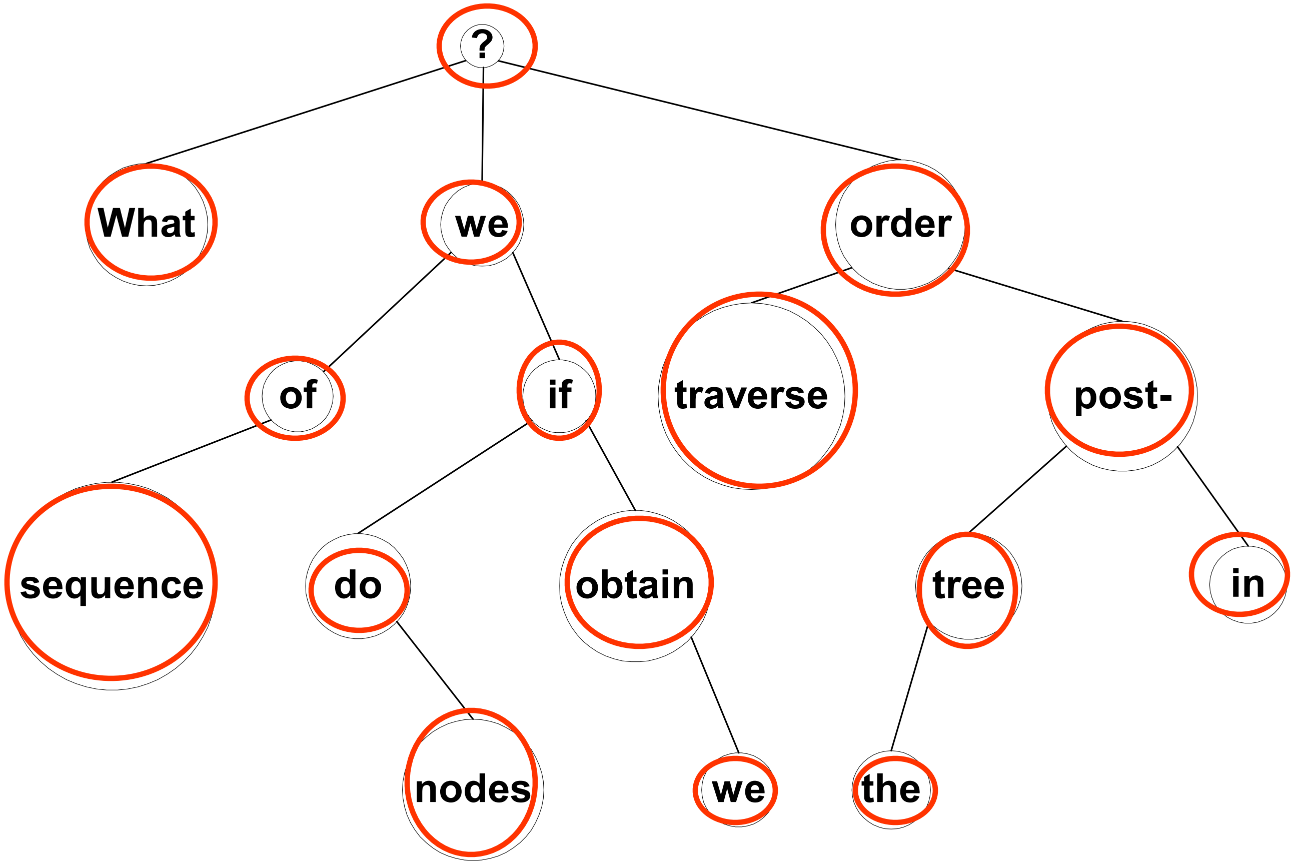
IN-ORDER VISIT

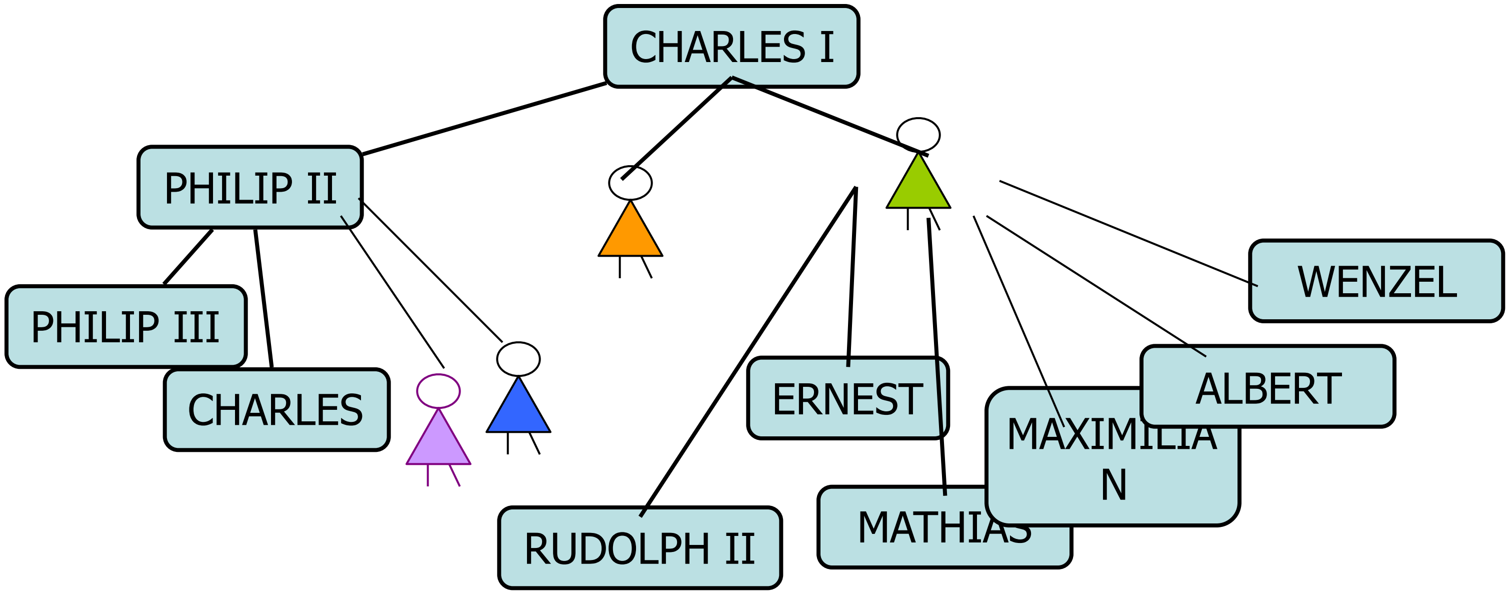
1. Visit the first sub-tree (inorder)
2. Visit the root
3. Visit the second sub-tree (inorder)

? ?

d(x)+1. Visit the d(x)th sub-tree (inorder)

A B C D F L H E I G


?
What
we
order
of
if
traverse
post-
sequence
do
obtain
tree
in
nodes
we
the

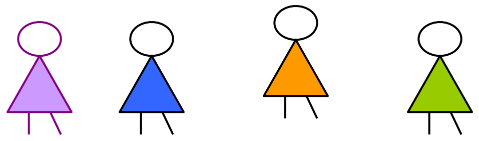


When Charles dies, Philip II becomes King.
If Philip II dies as well ….

Charles I,

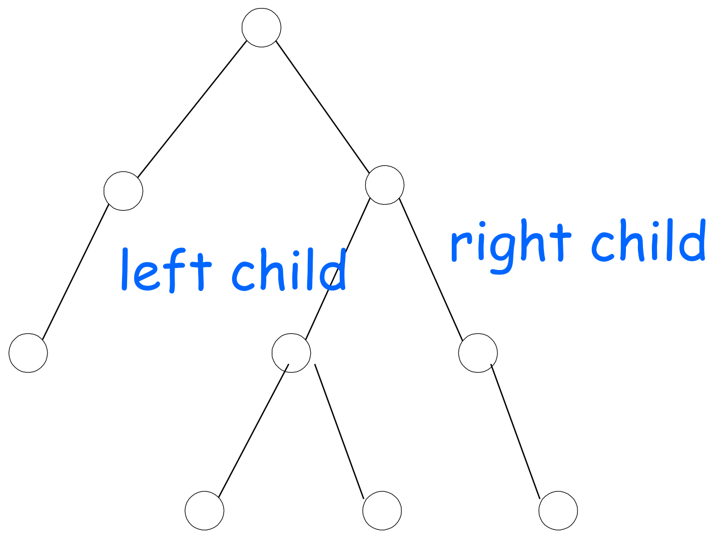
Philip II,

Philip III, Charles,



Rudolph II, Ernest, Mathias, Max, Albert, Wenzel,

Binary Trees



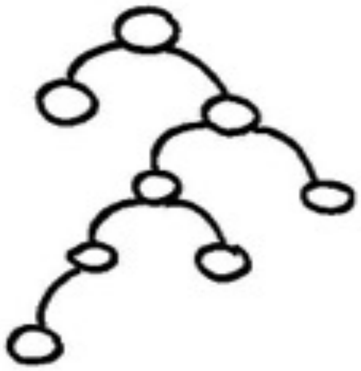
Children are ordered

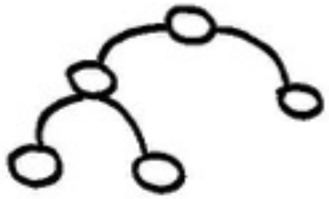
Each node has at most two children:

[0, 1, or 2]

“Full” Binary Trees (or “Proper”)

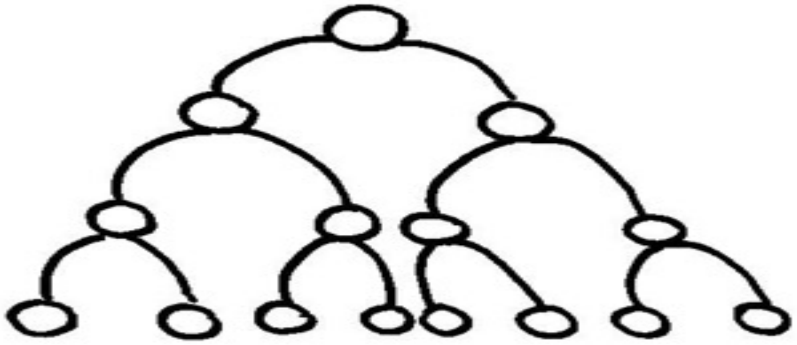
Each node: $\left\{ \begin{array}{l} \text{is a leaf, or} \\ \text{has two children} \end{array} \right.$





Perfect Binary Trees

Full binary trees with all leaves at the same level:

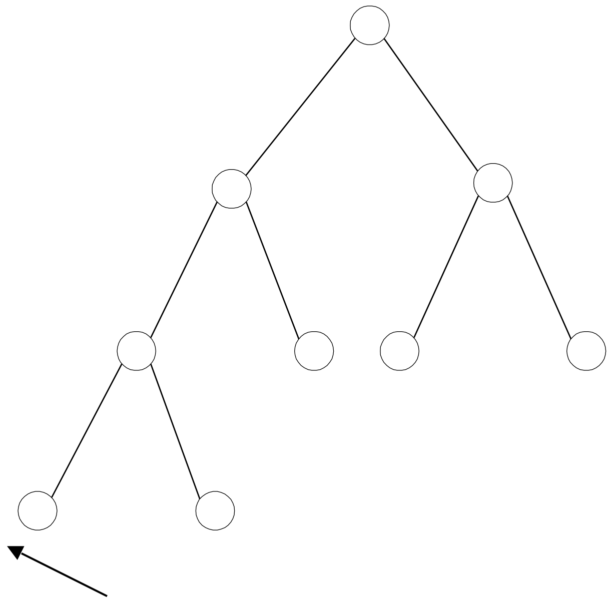


Complete Binary Trees

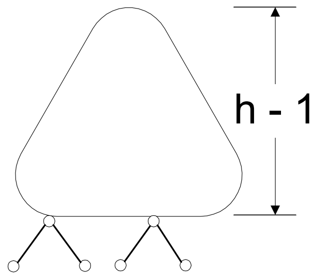
of depth h = Perfect trees of depth (h-1)

+

one or more leaves at level h.



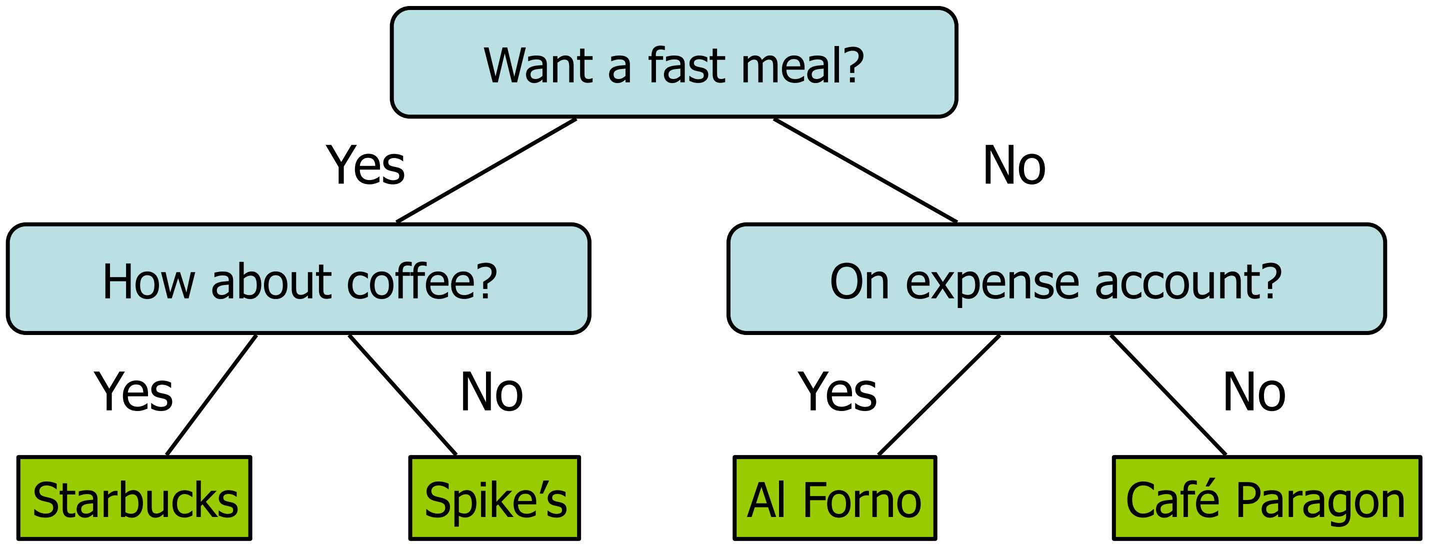
Leaves go at the left



Examples of Binary Trees

Decision Tree

- Binary tree associated with a decision process
  - internal nodes: questions with yes/no answer
  - external nodes: decisions
- Example: dining decision

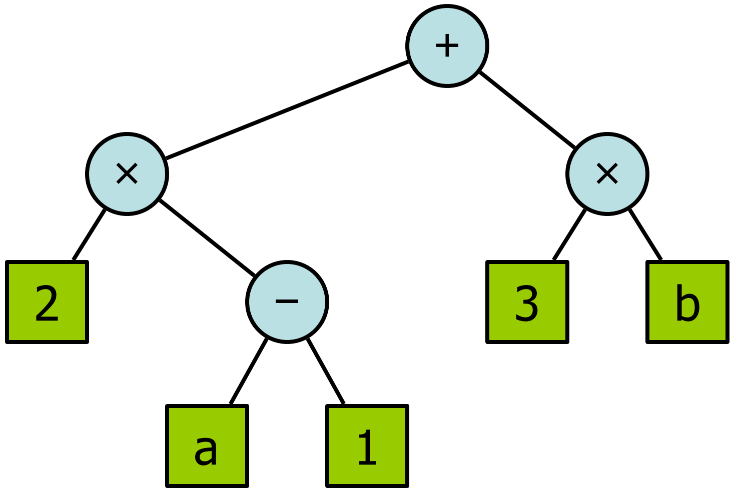


Examples of Binary Trees

Arithmetic Expression Tree

- Binary tree associated with an arithmetic expression
  - internal nodes: operators
  - external nodes: operands

Example: arithmetic expression tree for the expression $(2 \times (a - 1) + (3 \times b))$

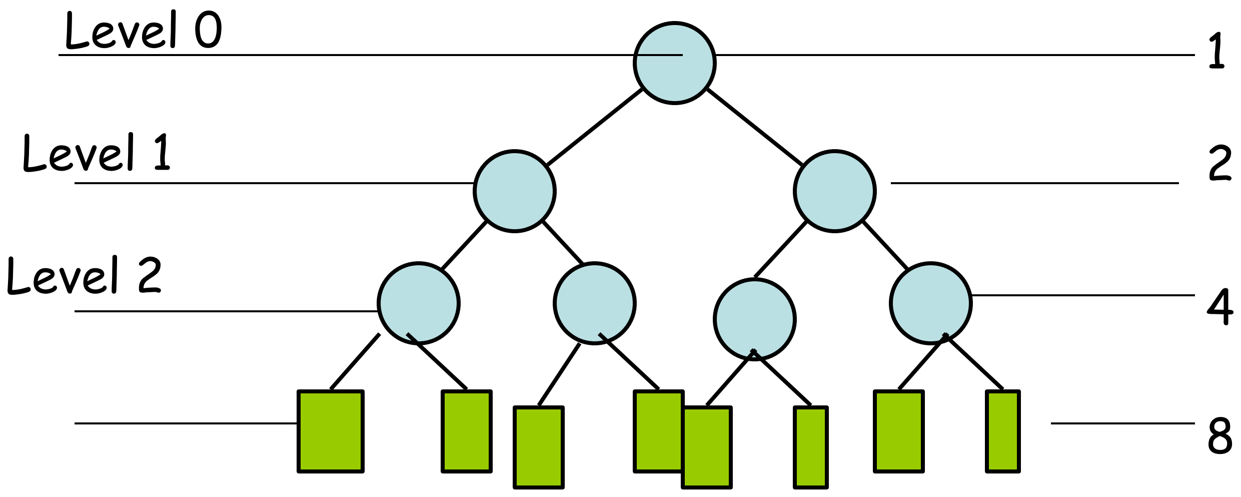


Properties of Binary Trees

- Notation

n	# of nodes	**e**	# of leaves
i	# of internal nodes	**h**	height

Maximum number of nodes at each level ?



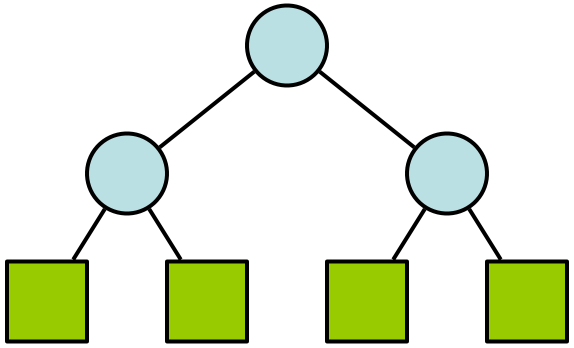
level i ------- 2^i

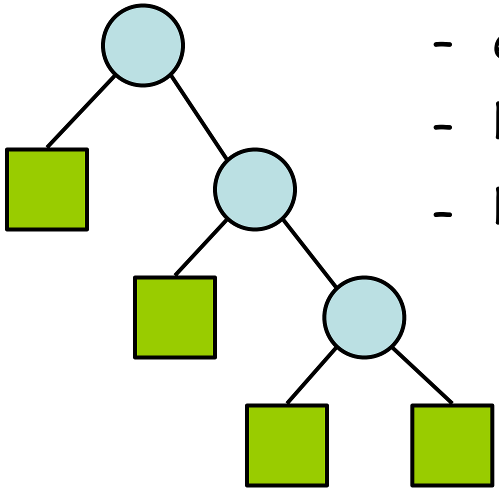
Properties of Full Binary Trees

- Notation

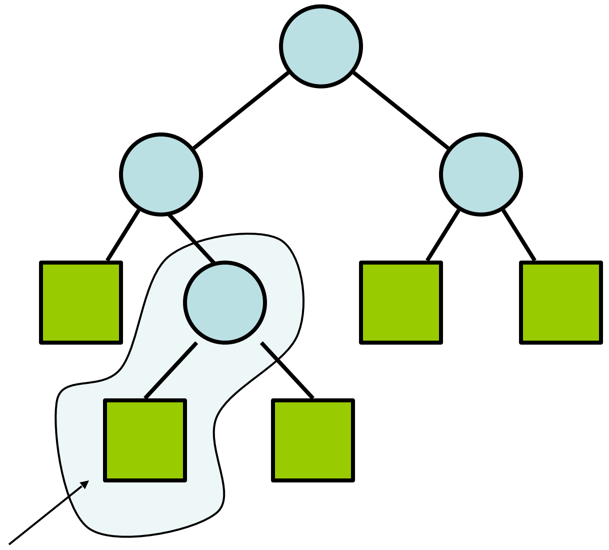
n	number of nodes
e	number of leaves
i	number of internal nodes
h	height

- Properties:
  - $e = i + 1$
  - $n = 2e - 1$
  - $h \leq i$
  - $h \leq (n - 1)/2$
  - $e \leq 2^h$
  - $h \geq \log_2 e$
  - $h \geq \log_2 (n + 1) - 1$

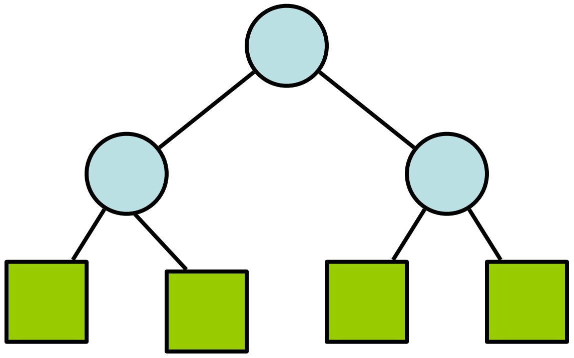


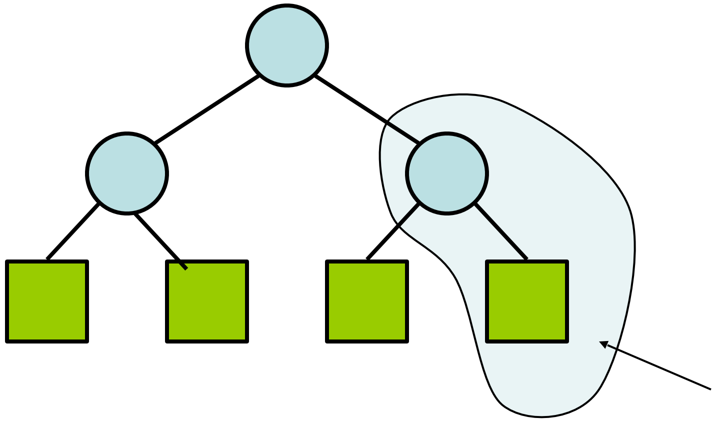


$e = i + 1$

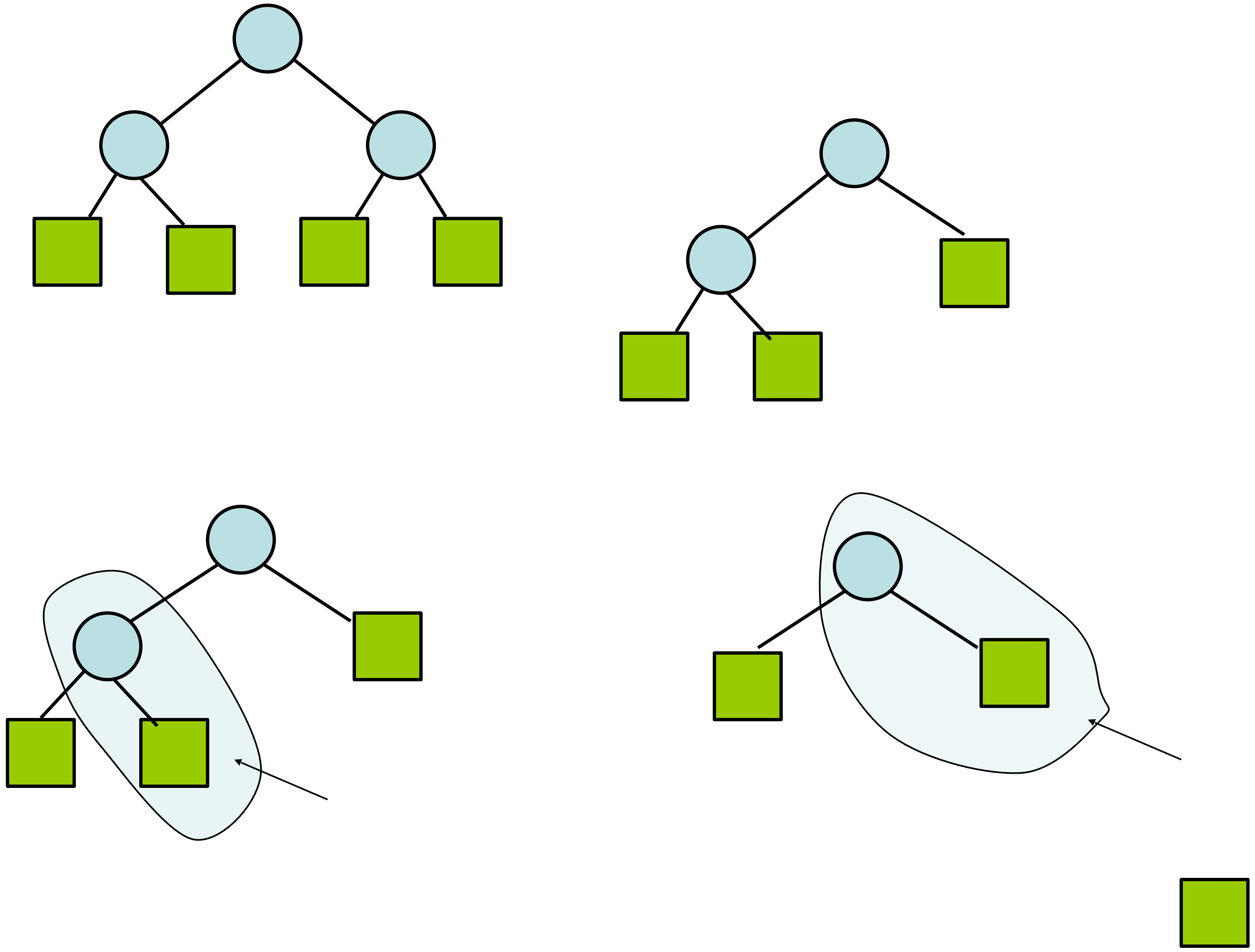


$e = i + 1$





$e = i + 1$



$$\boxed{n = 2e - 1}$$

$n = i + e$

$e = i + 1$ (just proved)

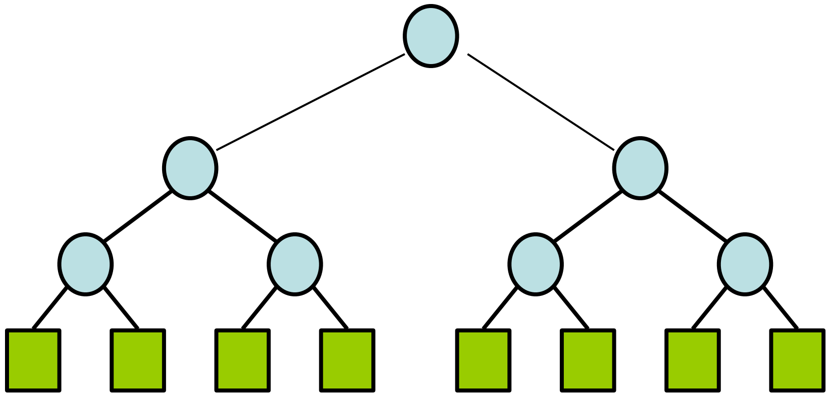
$i = e - 1$

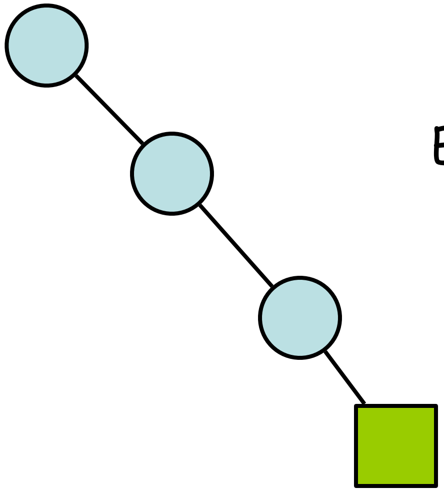
$n = 2e - 1$

$\boxed{h \leq i}$

(h = max num of ancestors)

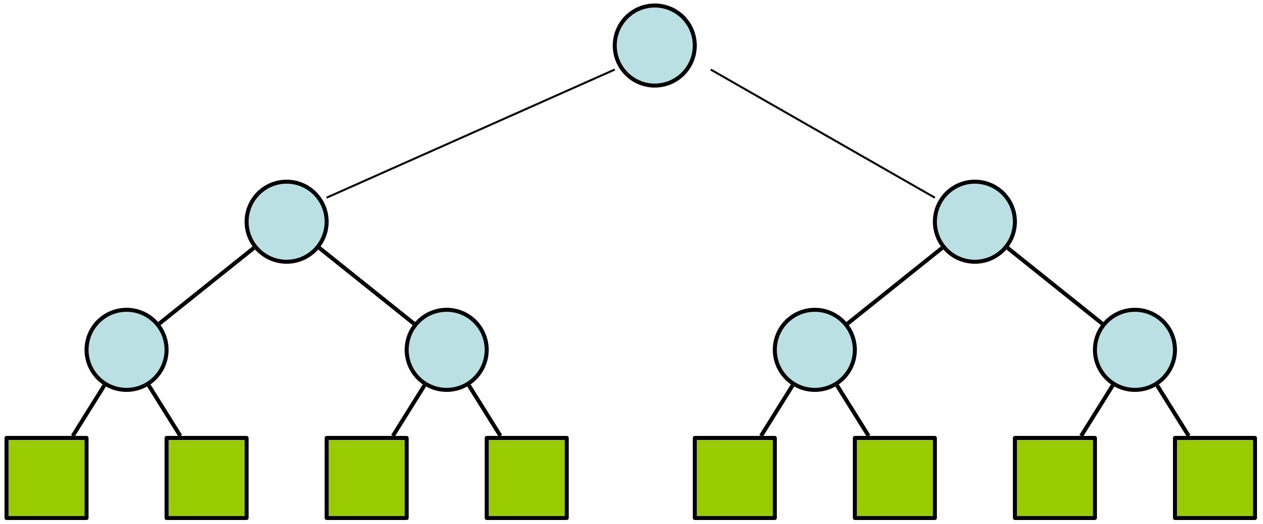
There must be at least one internal node for each level (except the last) !





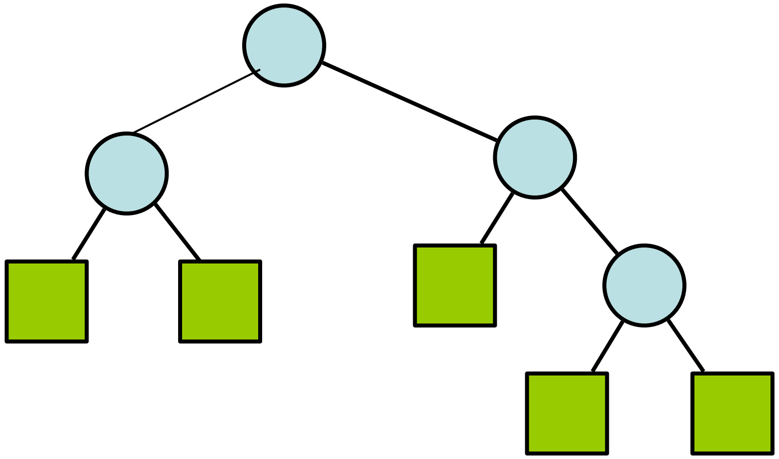
$$\boxed{e \leq 2^h}$$

level i ------- max num of nodes is 2^i



h = 3

2^3 leaves
if all at last
level h



otherwise less

Since $e \leq 2^h$

$\log_2 e \leq \log_2 2^h$

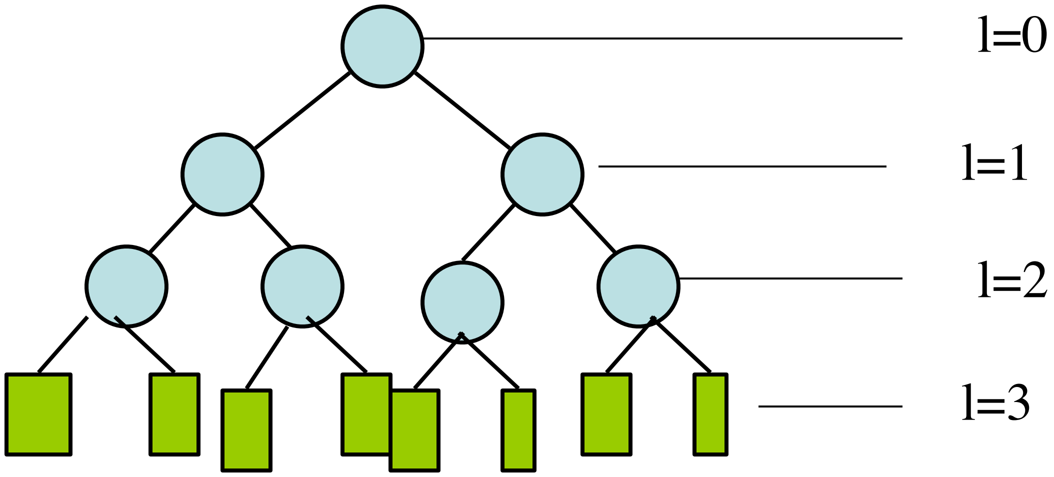
$\log_2 e \leq h$

$$\boxed{h \geq \log_2 e}$$

In Perfect Binary Trees...

with height h there are 2^{h+1} -1 nodes

$n = 2^{h+1} - 1$



At each level there are 2^l nodes, so the tree has:

$$\sum_{l=0}^{h} 2^l = 1 + 2 + 4 + \cdots + 2^h = 2^{h+1} - 1$$

As a consequence:

In Binary trees:

obviously $n \leq 2^{h+1} - 1$

$n \leq 2^{h+1}-1$

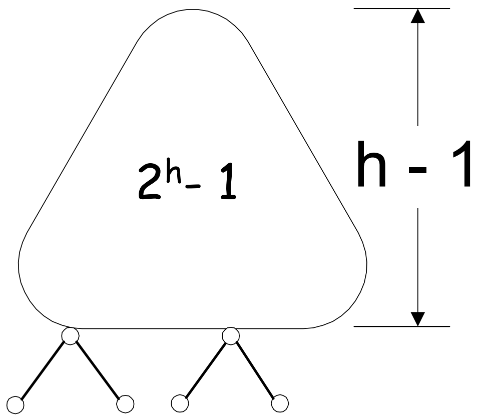
$n+1 \leq 2^{h+1}$

$\log(n+1) \leq h+1$

$$\boxed{h \geq \log(n+1) - 1}$$

In Complete Binary Trees …

with height h $2^h \leq n \leq 2^{h+1} - 1$



From previous observation: $n \leq 2^{h+1} - 1$

A complete binary tree is a perfect binary tree of height h-1 plus some more leaves …

$n \geq 2^h$

$$n \geq 2^h$$

It follows that:

Height of a complete binary tree with n nodes:

$$\lfloor \log n \rfloor$$

ADTs for Binary Trees

- accessor methods
  - -leftChild(p), rightChild(p), sibling(p)
- update methods
  - -expandExternal(p), removeAboveExternal(p)

other application specific methods

Traversing Binary Trees

Pre-, post-, in- (order)

- Refer to the place of the parent relative to the children
- pre is before: parent, child, child
- post is after: child, child, parent
- in is in between: child, parent, child

Traversing Binary Trees

Preorder, Postorder,

```
Algorithm preOrder(T,v)
        visit(v)
         if v is internal:
           preOrder (T,T.LeftChild(v))
           preOrder (T,T.RightChild(v))
```

```
Algorithm postOrder(T,v)
         if v is internal:
         postOrder (T,T.LeftChild(v))
         postOrder(T,T.RightChild(v))
       visit(v)
```

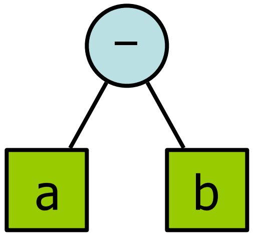
Traversing Binary Trees

Inorder

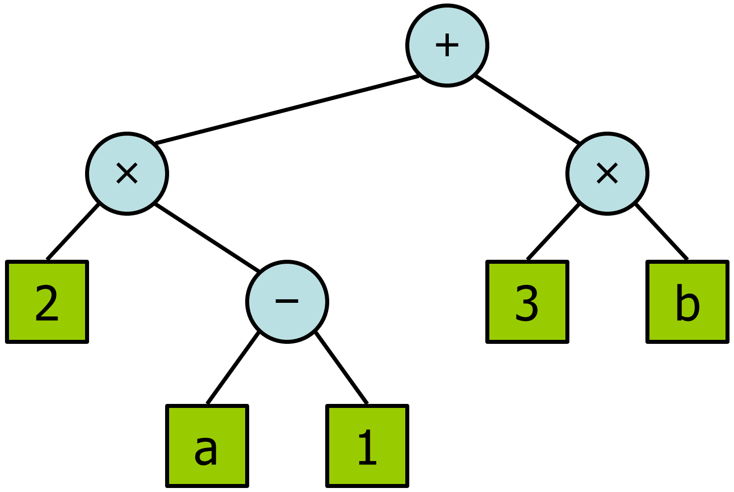
(Depth-first)

```
Algorithm inOrder(T,v)
        if v is internal:
        inOrder (T,T.LeftChild(v))
    visit(v)
    if v is internal:
     inOrder(T,T.RightChild(v))
```

Arithmetic Expressions



Inorder: a - b
Postorder: a b -
Preorder - a b



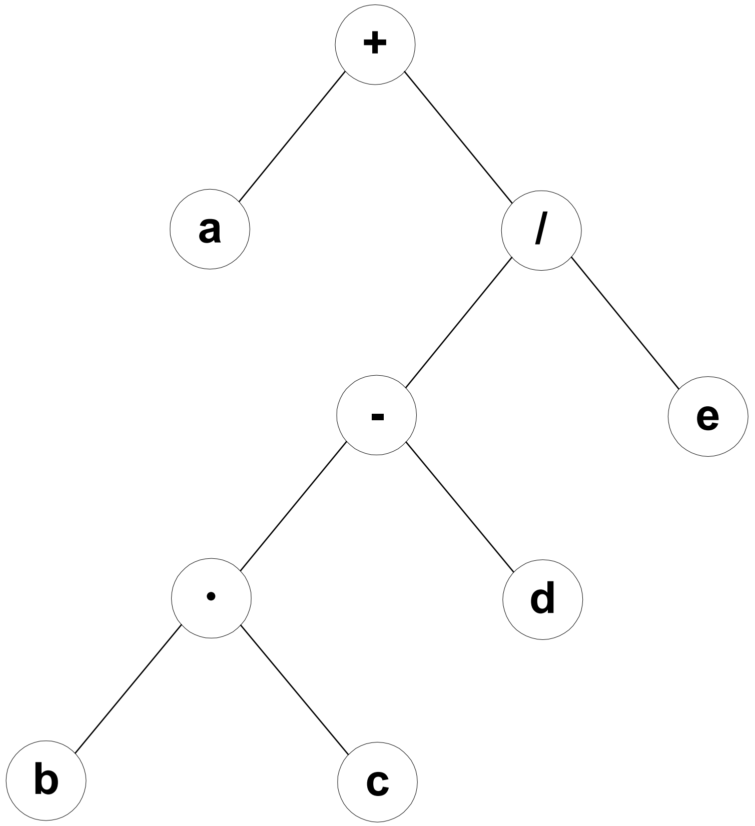
Inorder:

2 × a – 1 + 3 × b

Postorder:

2 a 1 - × 3 b × +

a + (b • c – d)/e



PRE-ORDER:

+ a / – • b c d e

POST-ORDER:

a b c • d – e / +

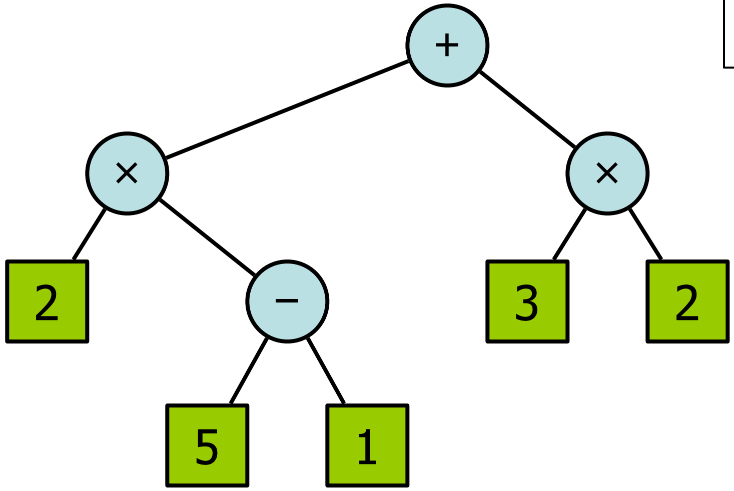
IN-ORDER:

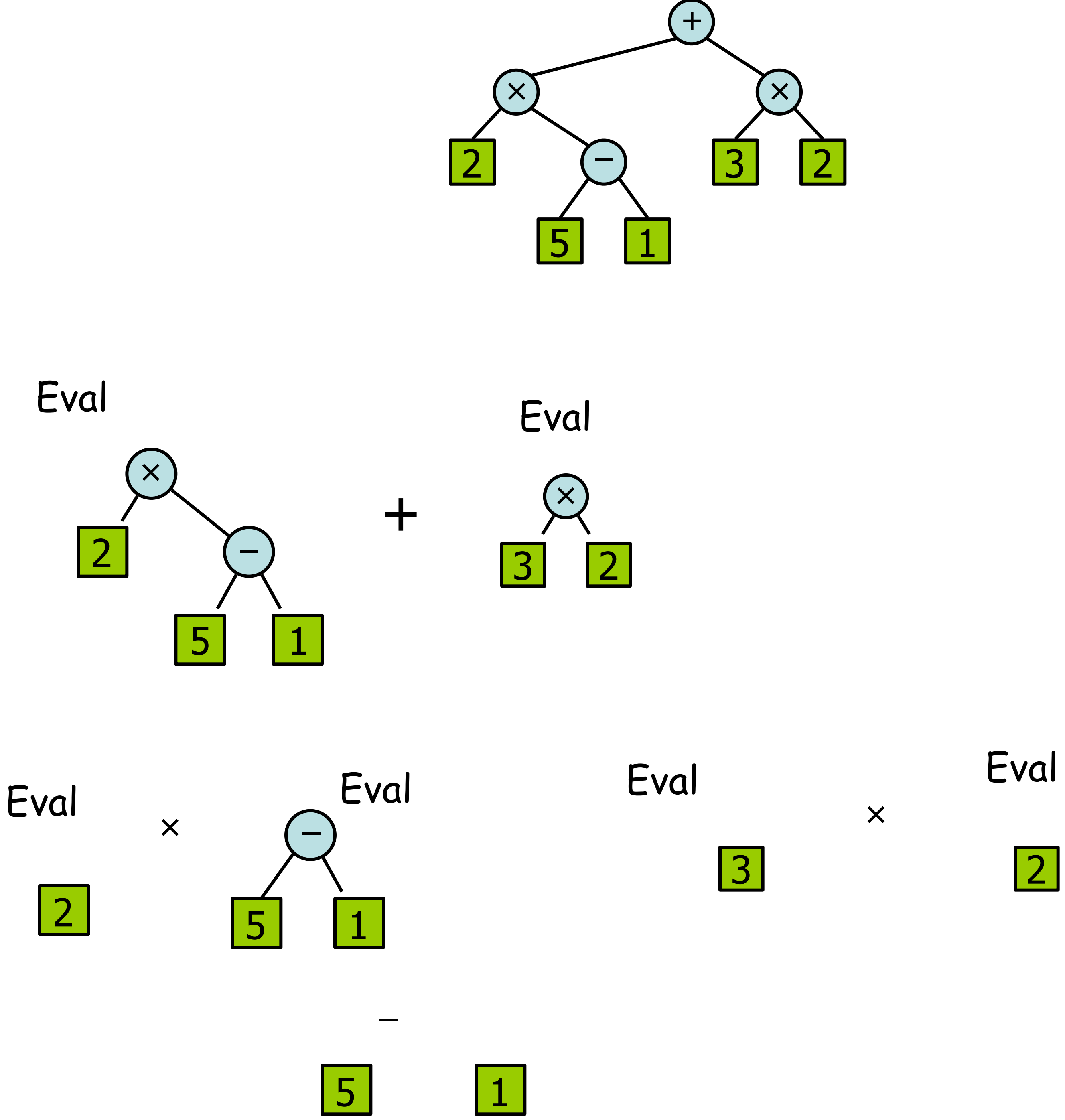
a + b • c – d / e

Evaluate Arithmetic Expressions

- Specialization of a **postorder** traversal
  - recursive method returning the value of a subtree
  - when visiting an internal node, combine the values of the subtrees

```
Algorithm evalExpr(v)
    if isExternal (v)
        return v.element ()
    else
        x ← evalExpr(leftChild (v))
        y ← evalExpr(rightChild (v))
        ◊ ← operator stored at v
        return x ◊ y
```

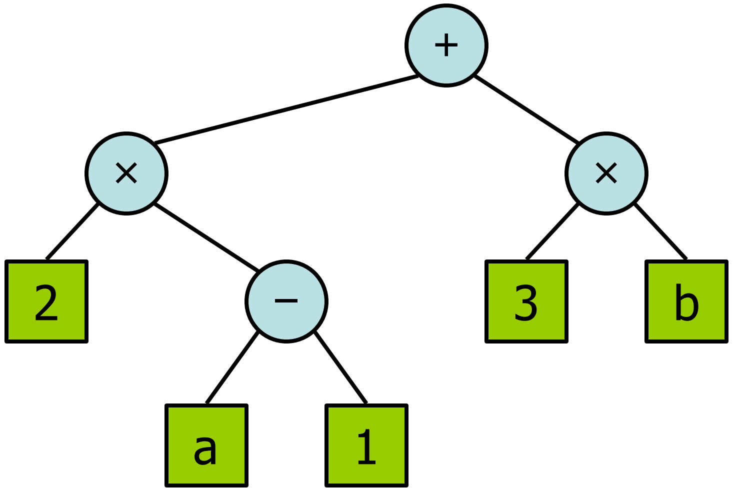



Eval
Eval
Eval
Eval
Eval
Eval

Print Arithmetic Expressions

- Specialization of an **inorder** traversal
  - print operand or operator when visiting node
  - print “(“ before traversing left subtree
  - print “)“ after traversing right subtree

```
Algorithm printExpression(v)
    if isInternal (v)
        print(“(”)
        inOrder (leftChild (v))
    print(v.element ())
    if isInternal (v)
        inOrder (rightChild (v))
        print (“)”)
```



$2 \times a - 1 + 3 \times b$

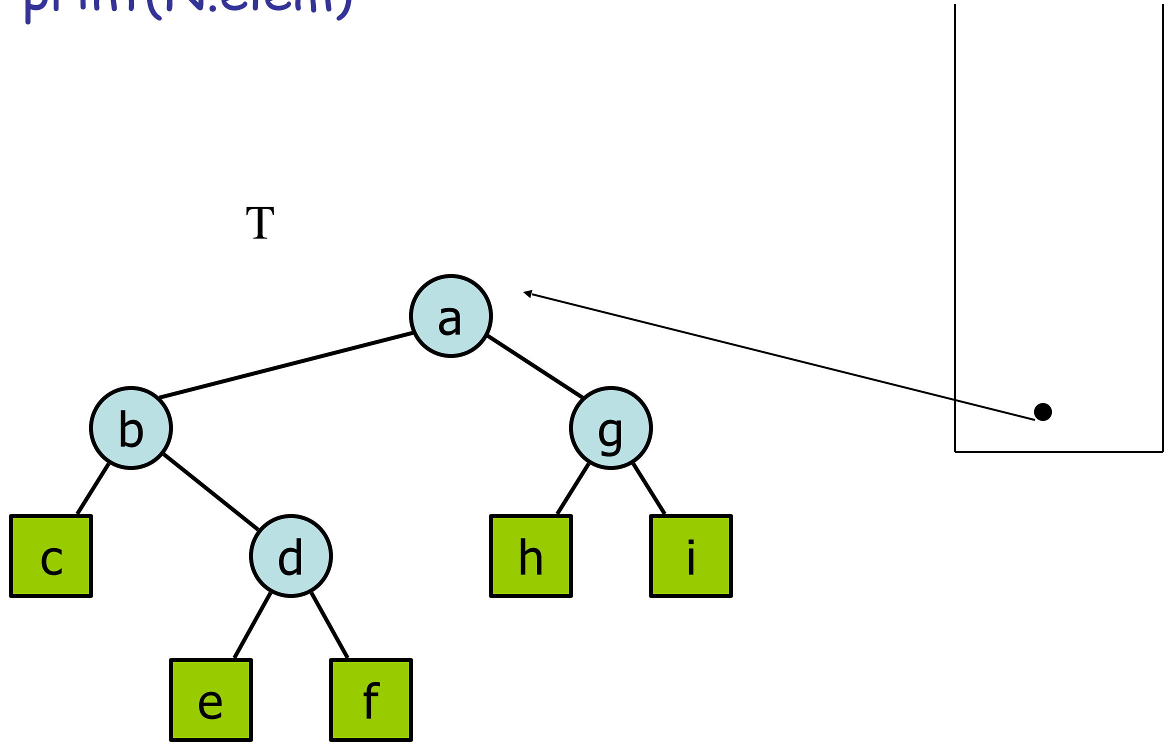
$((2 \times (a - 1)) + (3 \times b))$

Algorithm preOrderTraversalwithStack(T)

```
    Stack S
    TreeNode N

    S.push(T)   // push the reference to T in the empty stack
    While (not S.empty())
       N = S.pop()
       if (N != null) {
                  print(N.elem)           // print information
                  S.push(N.rightChild) // push the reference to
                                          the right child
                  S.push(N.leftChild) // push the reference to
                                          the left child
                   }
```

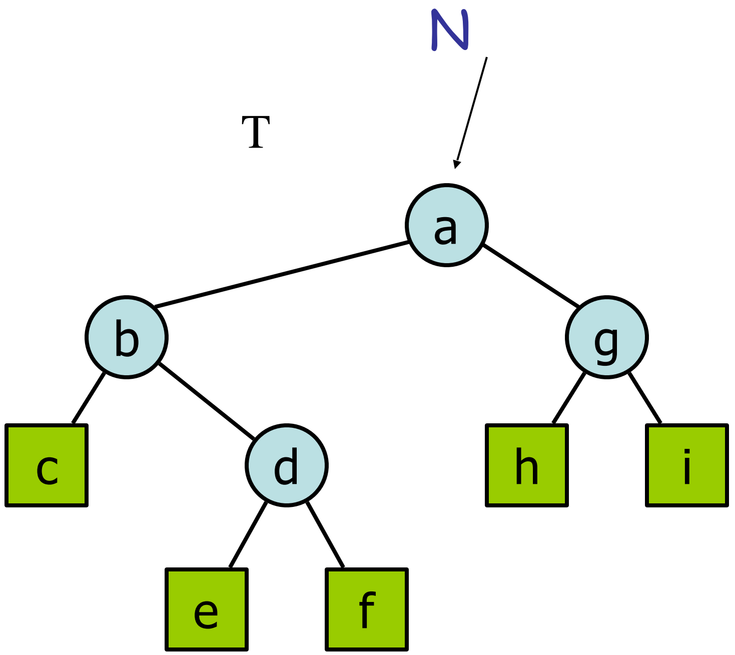
Algorithm preOrderTraversalwithStack(T)

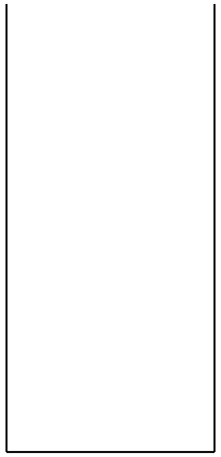
```
S.push(T)   // push the reference to T in the empty stack
N = S.pop()
  print(N.elem)
```



Algorithm preOrderTraversalwithStack(T)

```
S.push(T)   // push the reference to T in the empty stack
N = S.pop()
  print(N.elem)
```

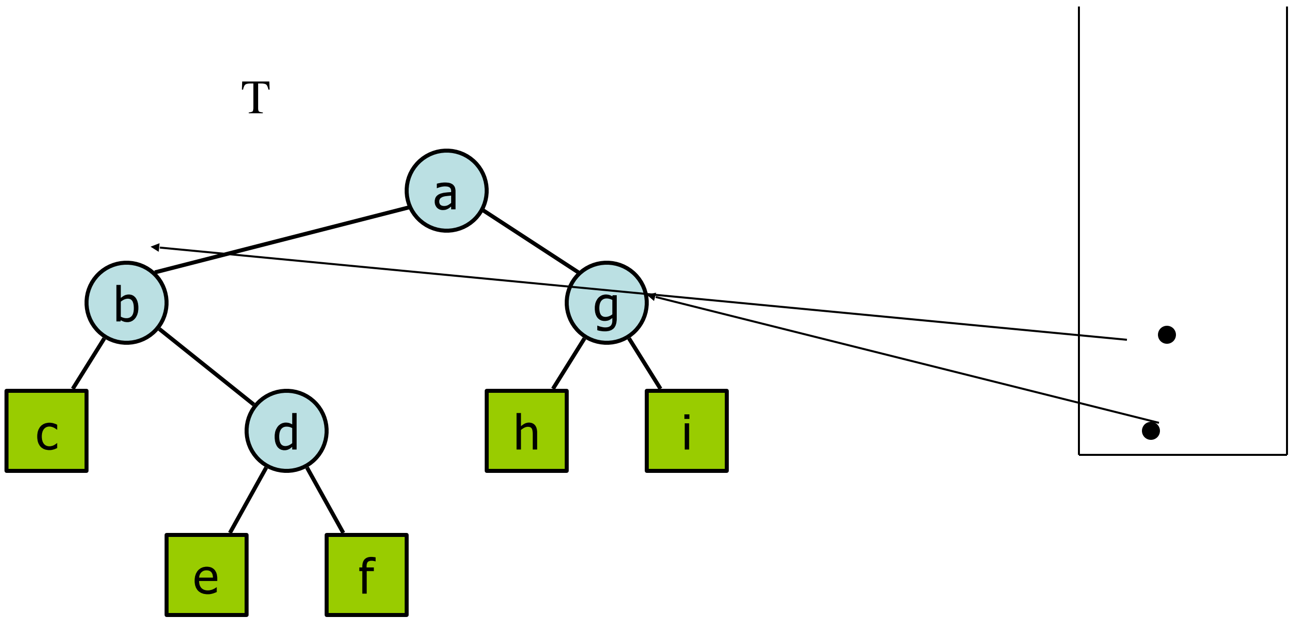




a

Algorithm preOrderTraversalwithStack(T)

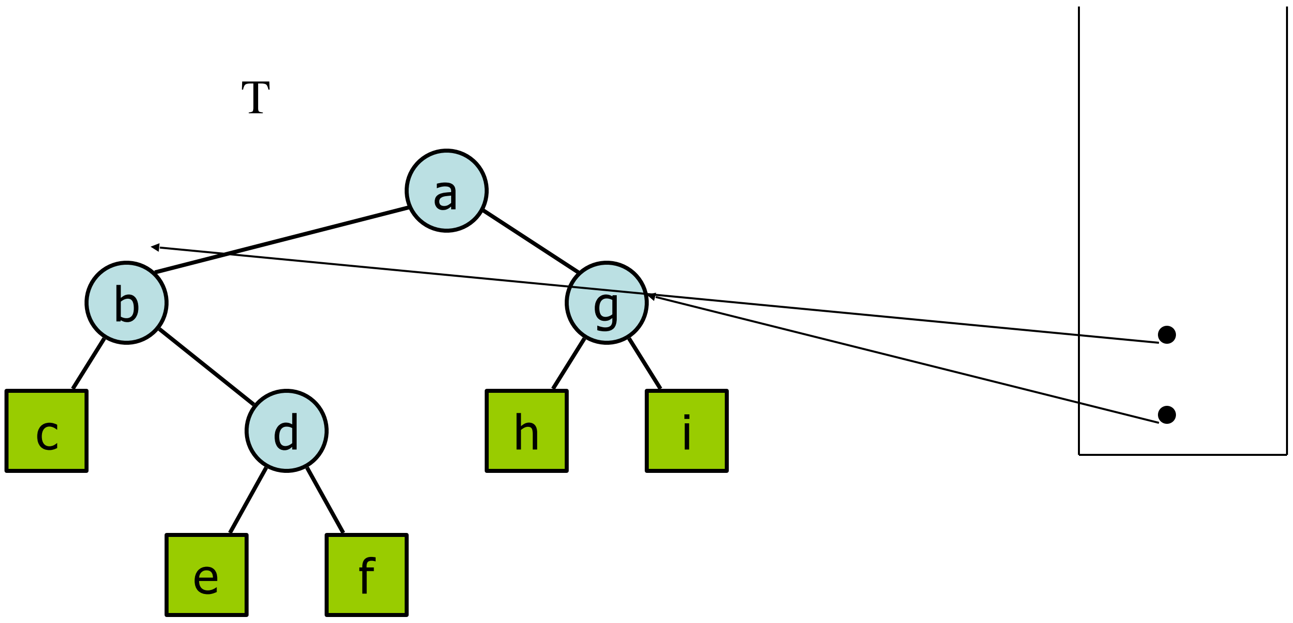
```
S.push(N.rightChild) // push the reference to
                                  the right child

S.push(N.leftChild) // push the reference to
                                  the left child
```



a

Algorithm preOrderTraversalwithStack(T)

N = S.pop()

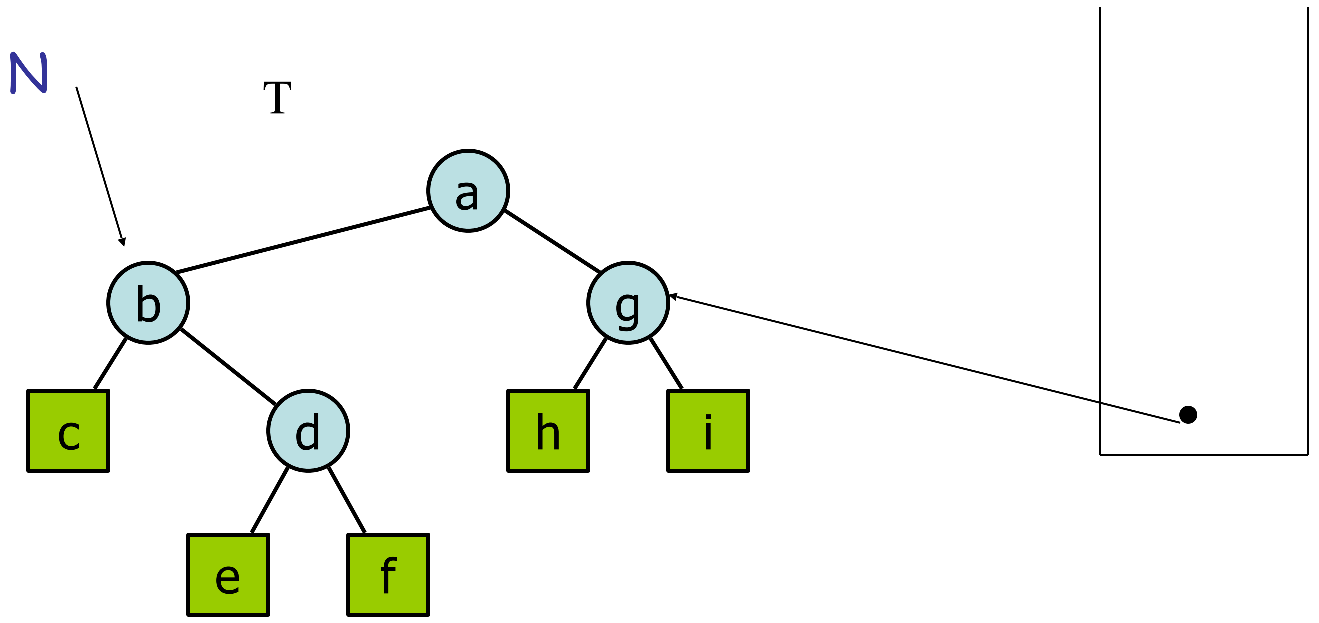


a

Algorithm preOrderTraversalwithStack(T)

N = S.pop()

print(N.elem)

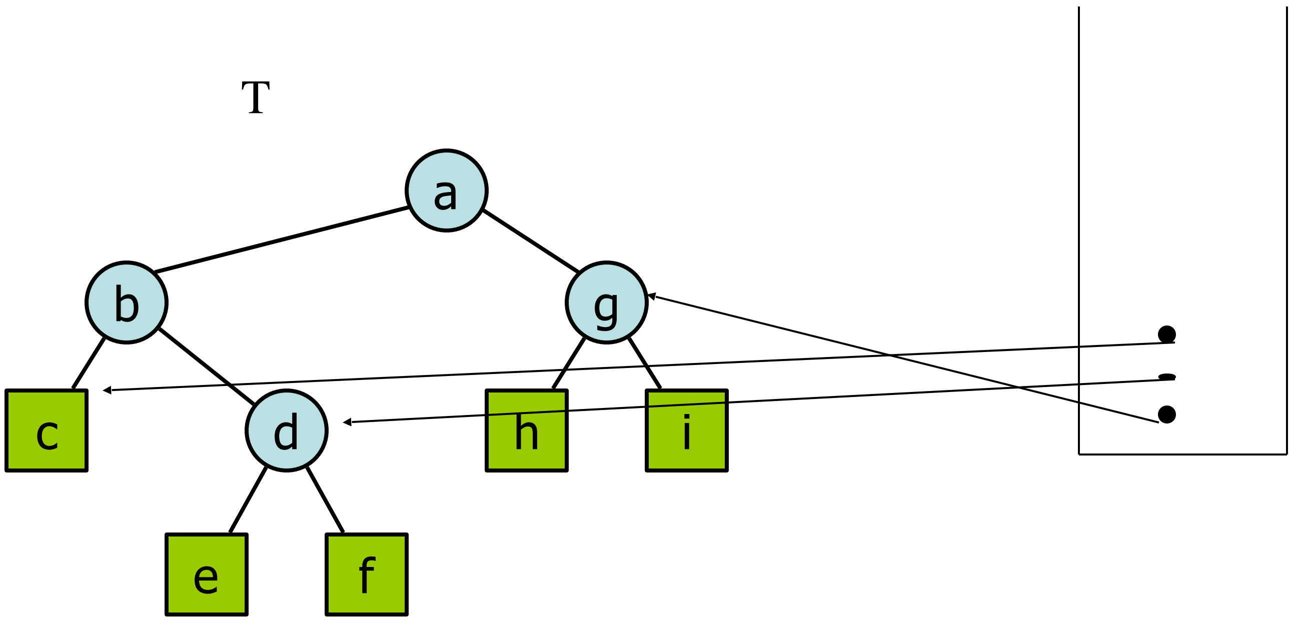


a b

Algorithm preOrderTraversalwithStack(T)

S.push(N.rightChild)

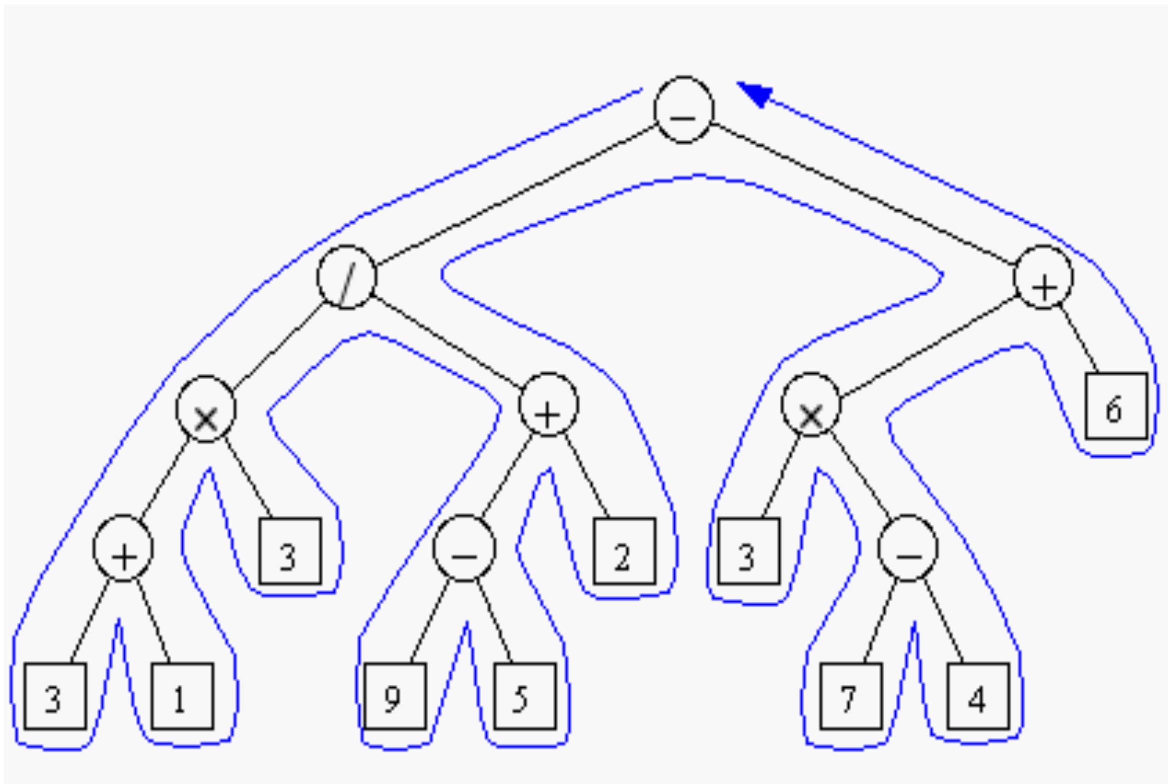
S.push(N.leftChild)



a b

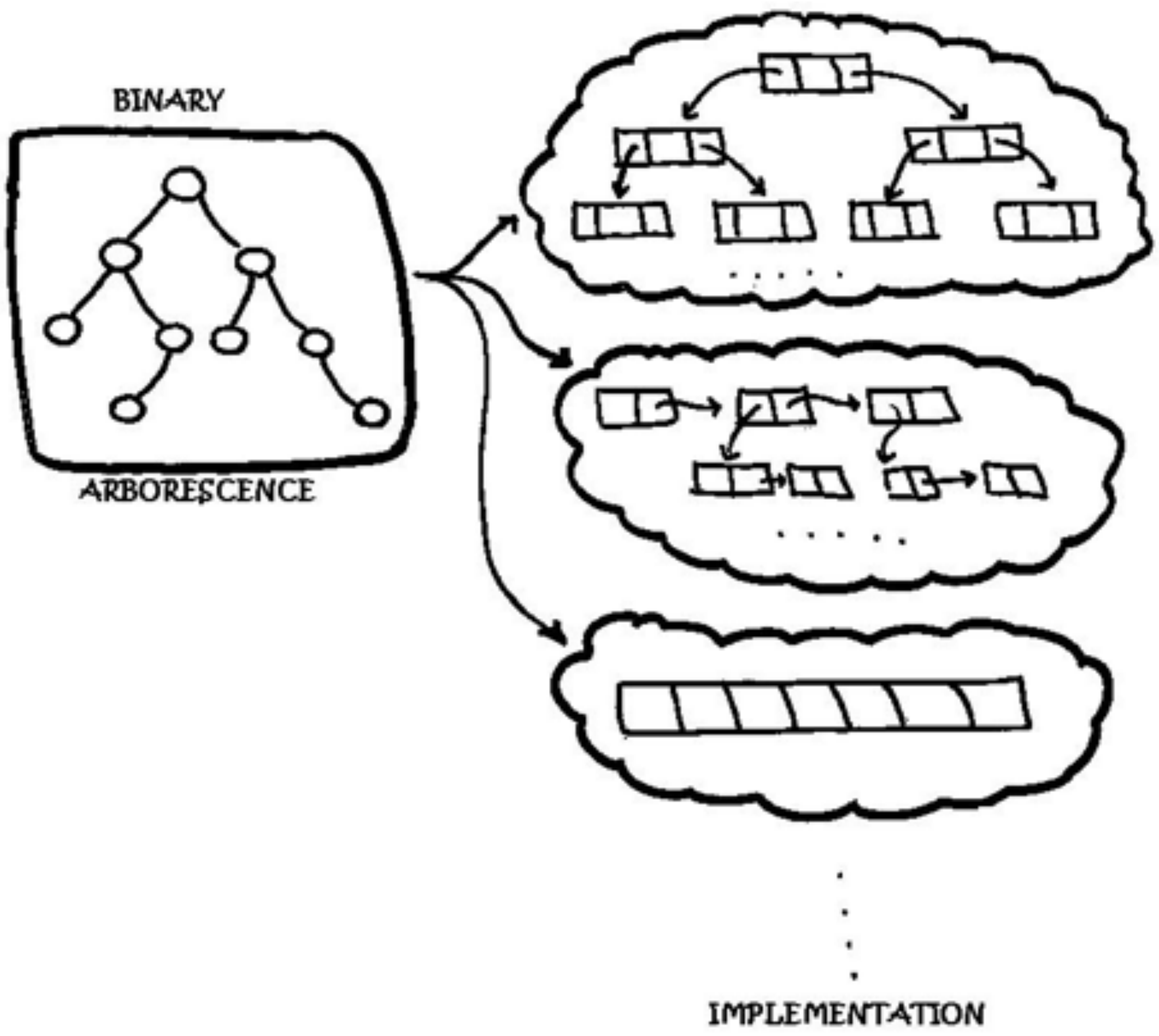
Euler Tour Traversal

- generic traversal of a binary tree
- the preorder, inorder, and postorder traversals are special cases of the Euler tour traversal
- "walk around" the tree and visit each node three times:
  - on the left
  - from below
  - on the right



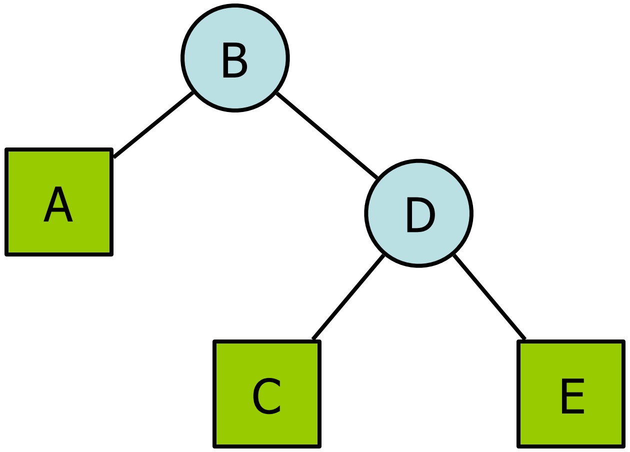
```
Algorithm eulerTour(T,v)

      visit v                                   (from the left)
    if v is internal:
      eulerTour (T,T.LeftChild(v))
    visit v                              (from below)
    if v is internal:
      eulerTour(T,T.RightChild(v))
    visit v                              (from the right)
```

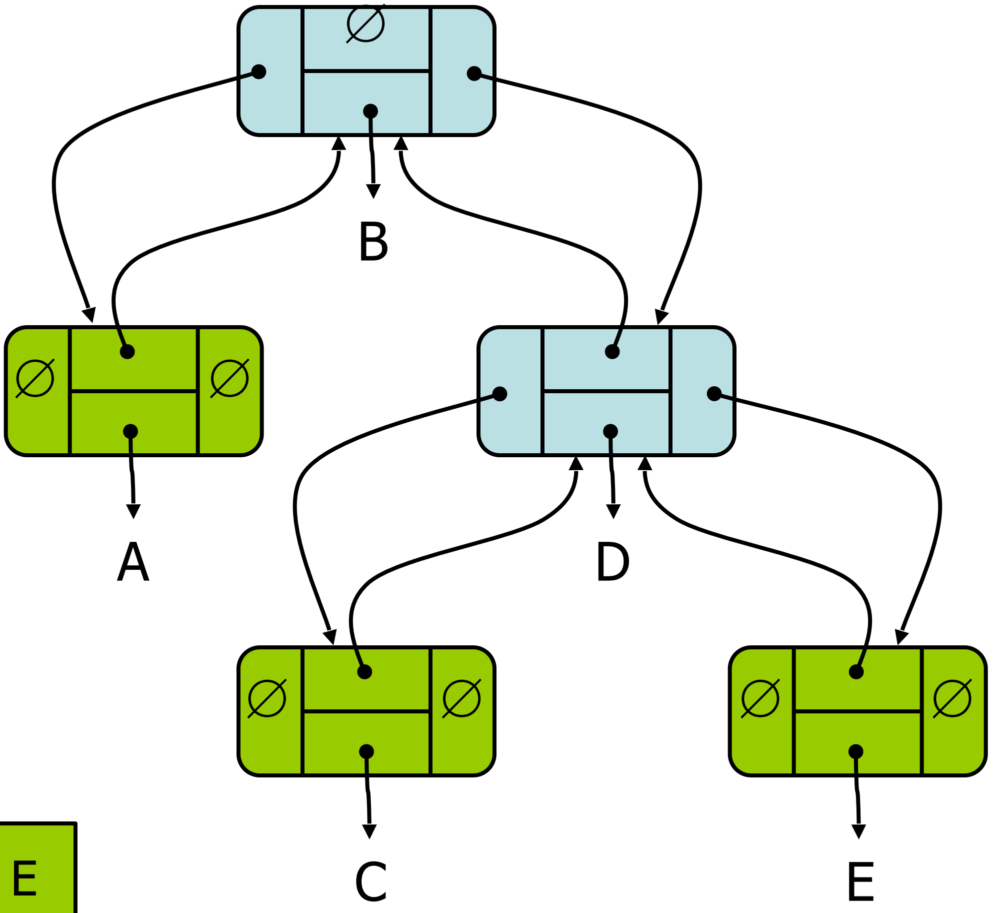
Implementations of Binary trees….



Implementing Binary Trees with a Linked Structure

- A node is represented by an object storing
  - Element
  - Parent node
  - Left child node
  - Right child node
- Node objects implement the Position ADT



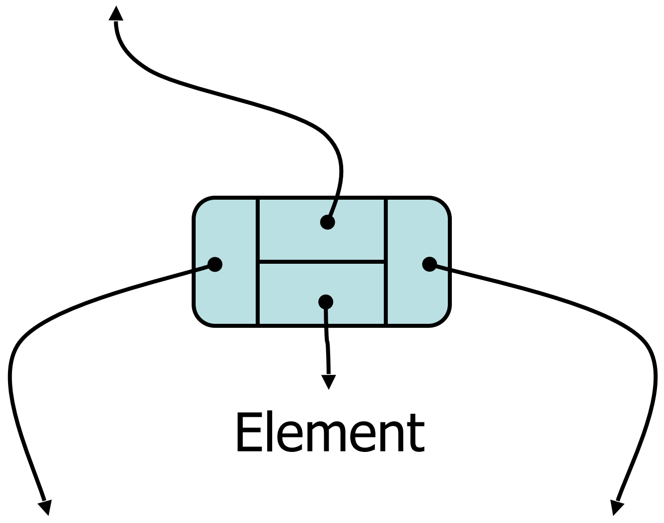


leftChild(p), rightChild(p), sibling(p):

Input: Position   Output: Position

swapElements(p,q)	Input: 2 Positions	Output: None
replaceElement(p,e)	Input: Position and an object	Output: Object
isRoot(p)	Input: Position	Output: Boolean
isInternal(p)	Input: Position	Output: Boolean
isExternal(p)	Input: Position	Output: Boolean

```
BTNode
        Object   Element
        BTNode left, right, parent
```



```
leftChild(v)   return v.left

rightChild(v) return v.right

sibling(v)
    p ← parent(v)
    q ← leftChild(p)
    if (v = q) return rightChild(p)
      else return q
```

```
replaceElement(v,obj)
    temp ← v.element
    v.element ← obj
    return temp
```

```
swapElements(v,w)
    temp ← w.element
    w.element ← v.element
    v.element ← temp
```

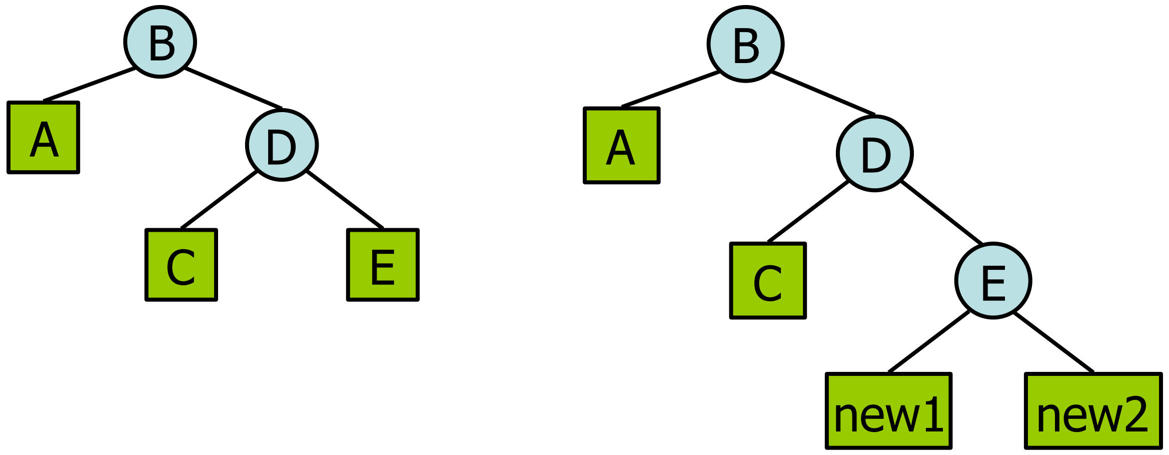
leftChild(p), rightChild(p), sibling(p):

swapElements(p,q),
replaceElement(p,e)
isRoot(p), isInternal(p),
isExternal(p)

They all have
complexity O(1)

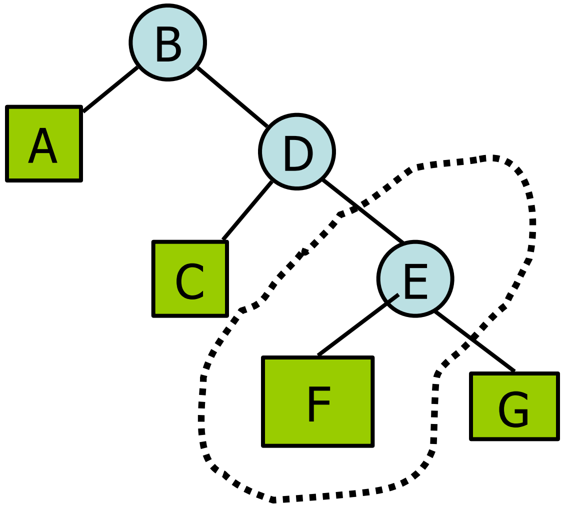
Other interesting methods for the ADT Binary Tree:

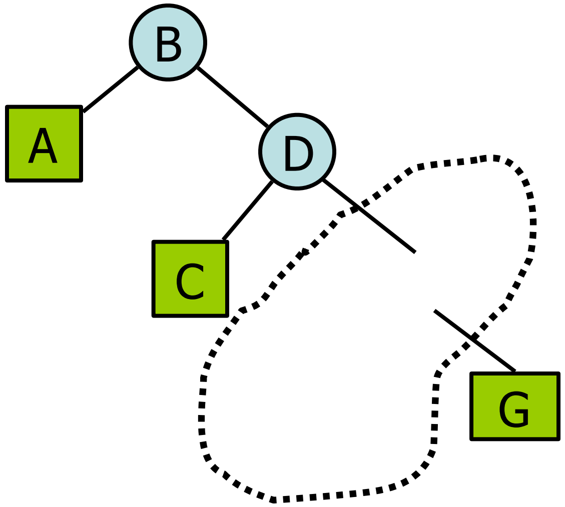
expandExternal(v): Transform v from an external node into an internal node by creating two new children

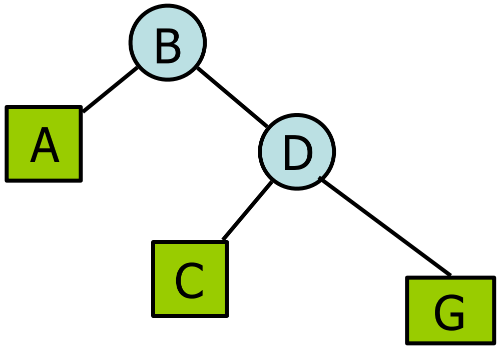


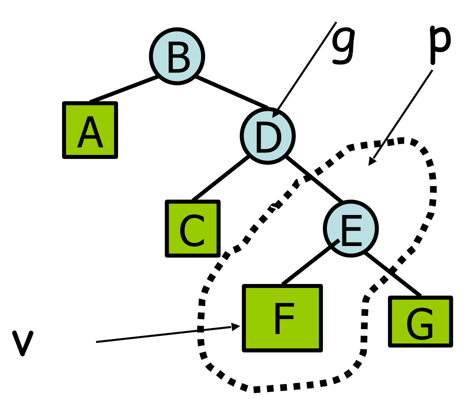
```
expandExternal(v):
new1 and new 2 are the new nodes
if isExternal(v)
    v.left ← new1
    v.right ← new2
    size ← size +2
```

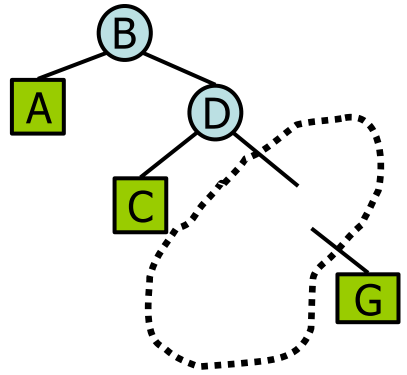
removeAboveExternal(v):

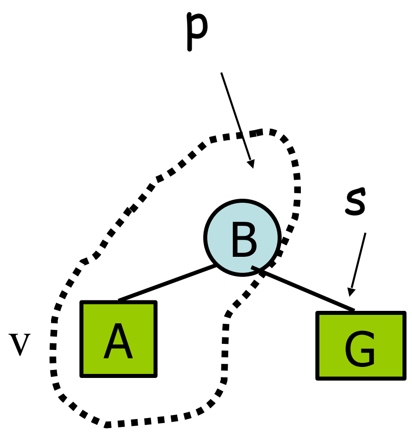






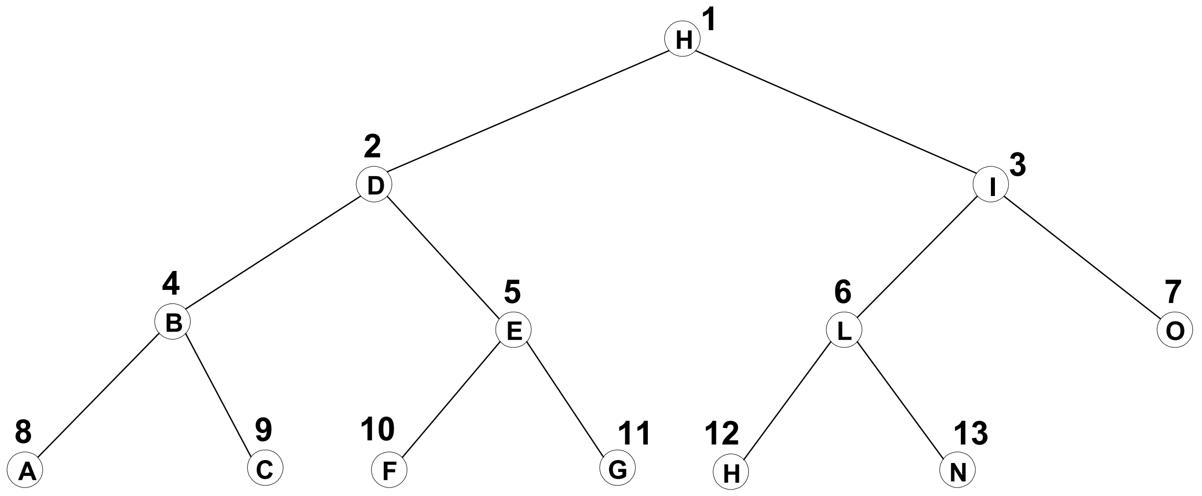


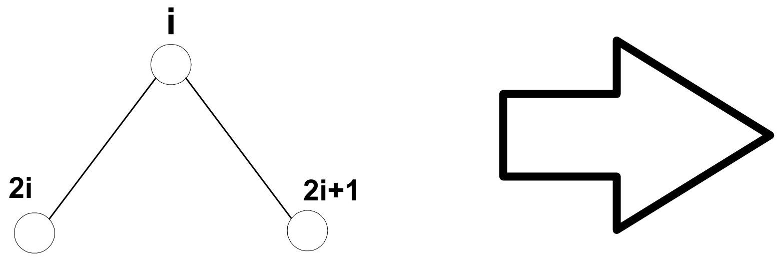




```
removeAboveExternal(v):
if isExternal(v)
    { p ← parent(v)
      s ← sibling(v)
      if isRoot(p)   s.parent   ← null and root ← s
      else
        { g ← parent(p)
          if p is leftChild(g)     g.left ← s
               else g.right ← s
          s.parent ← g
          }
    size ← size -2 }
```

Implementing Complete Binary Trees with Vectors (Array-based)



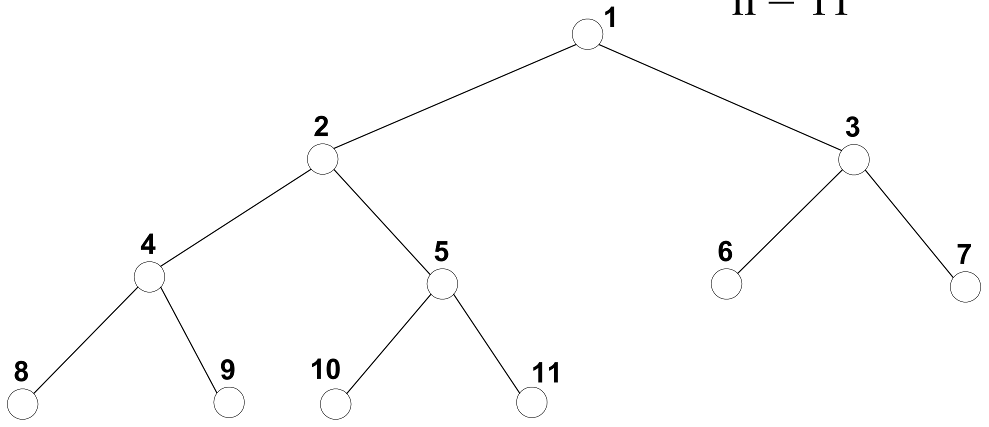


1	2	3	4	5	6	7	8	9	10	11	12	13
H	D	I	B	E	L	O	A	C	F	G	H	N

leftChild(p), rightChild(p), sibling(p):

swapElements(p,q),
replaceElement(p,e)
isRoot(p), isInternal(p),
isExternal(p)

n = 11



They all have complexity O(1)

Left child of T[i]	T[2i]	if	$2i \leq n$
Right child of T[i]	T[2i+1]	if	$2i + 1 \leq n$
Parent of T[i]	T[i div 2]	if	$i > 1$
The Root	T[1]	if	$T \neq 0$
Leaf? T[i]	TRUE	if	$2i > n$

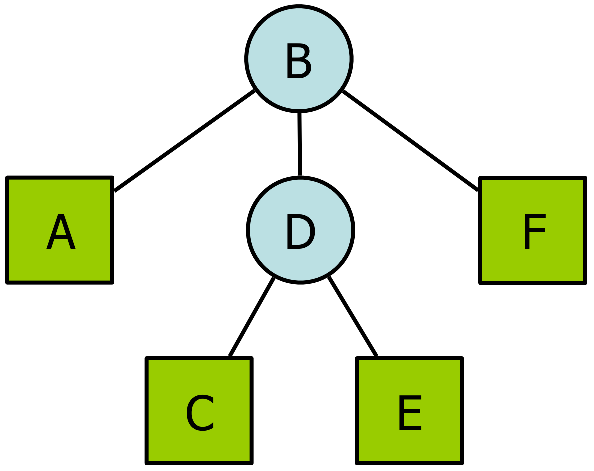
leftChild(p), rightChild(p), sibling(p):

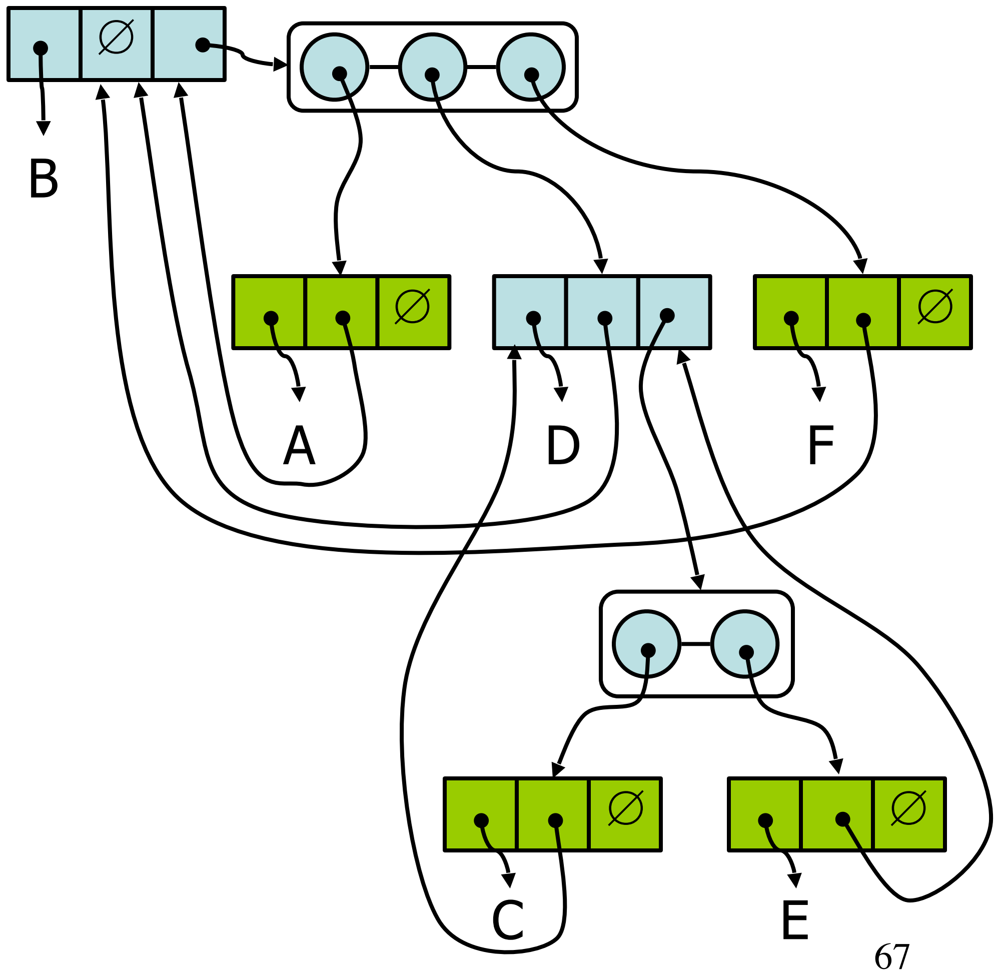
swapElements(p,q),
replaceElement(p,e)
isRoot(p), isInternal(p),
isExternal(p)

They all have complexity O(1)

Implementing General Trees with a Linked Structure

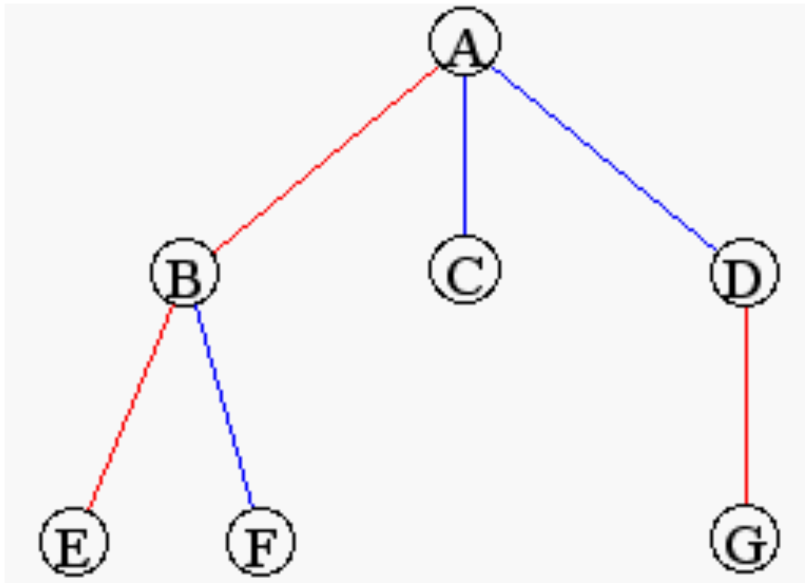
- A node is represented by an object storing
  - Element
  - Parent node
  - Sequence of children nodes
- Node objects implement the Position ADT



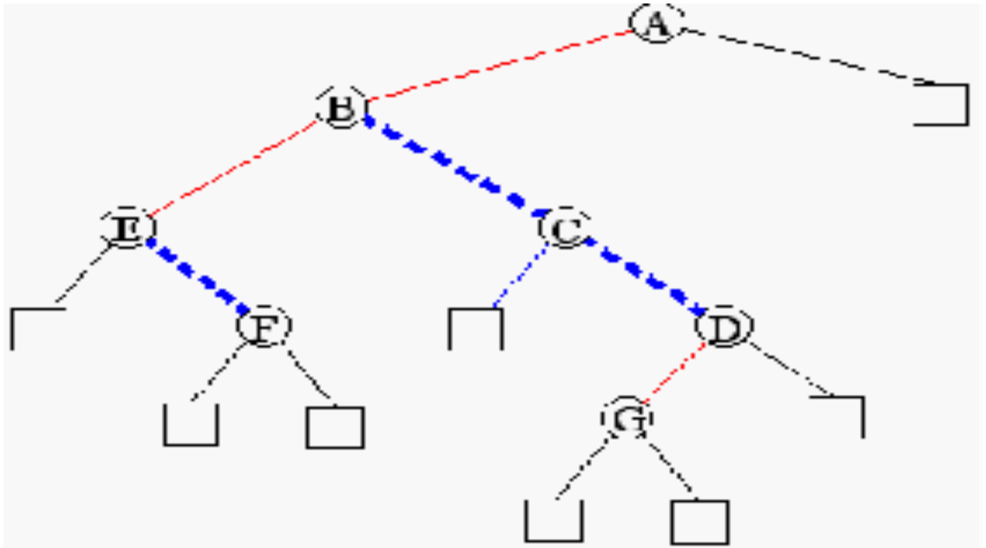


Representing General Trees

tree T



binary tree T' representing T

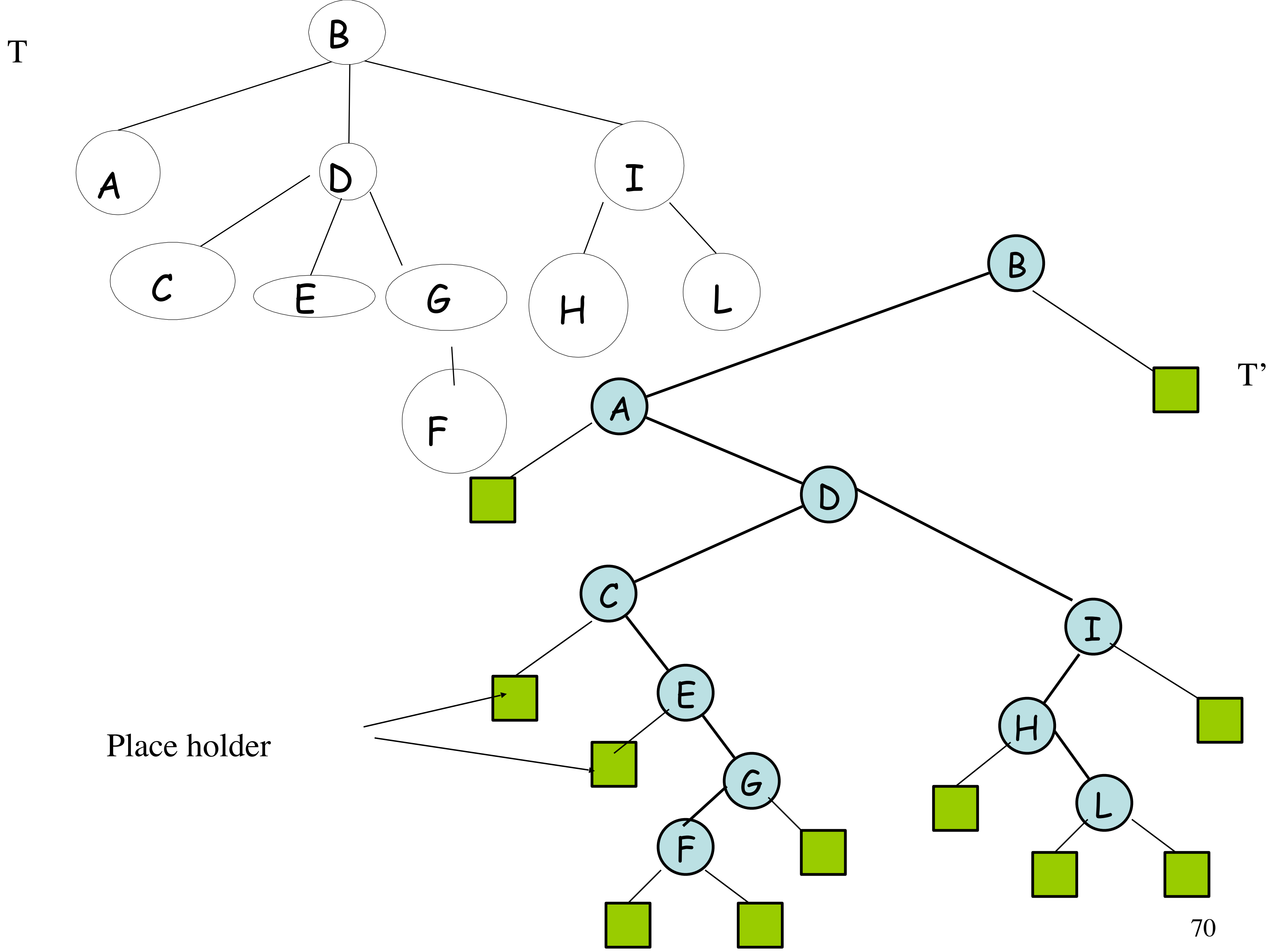


RULES

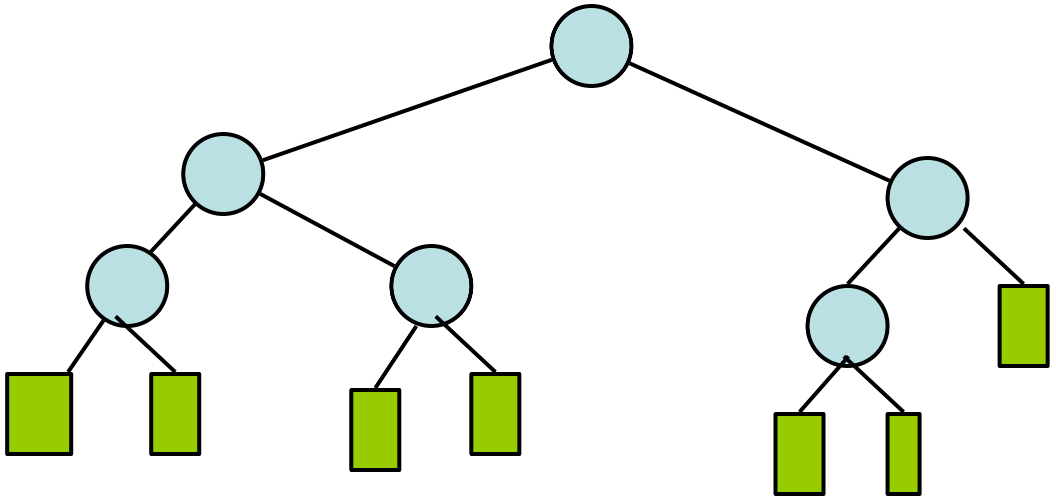
u in T u' in T'

first child of u in T is left child of u' in T'

first sibling of u in T is right child of u' in T'


T
B
A
D
I
C
E
G
H
L
F
T'
B
A
D
C
I
E
H
G
L
F
Place holder

children are "completed" with "fake" nodes

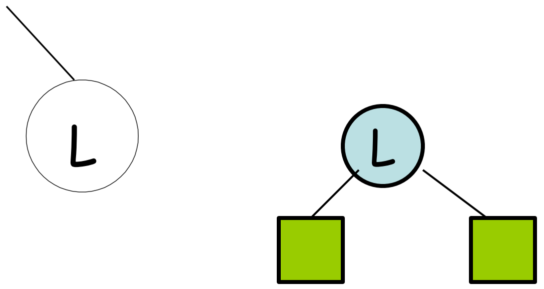


The green squared nodes are the dummy nodes.

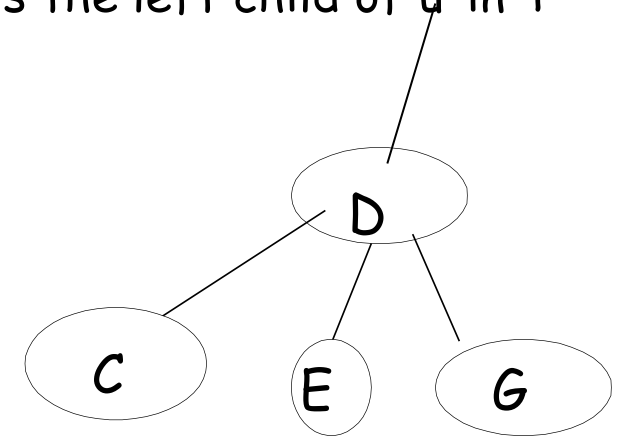
In this way ALL the original nodes are internal.
The leaves are the fake green nodes.

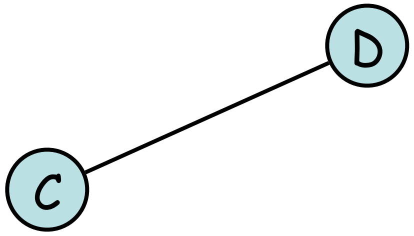
RULE:
to u in T corresponds u' in T'

if u is a leaf in T and has no siblings,
then the children of u' are leaves

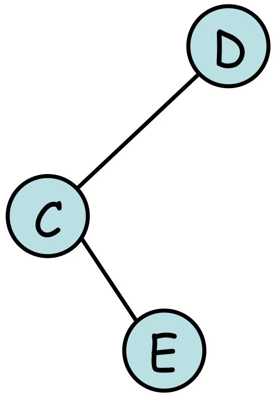


If u is internal in T and v is its first child
then v' is the left child of u' in T'





If v has a sibling w immediately following it,
w' is the right child of v' in T'



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I NEED A SMARTER HUMAN.