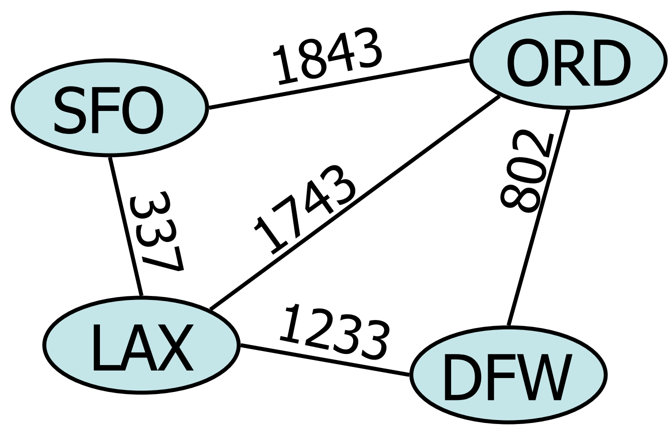
Graphs

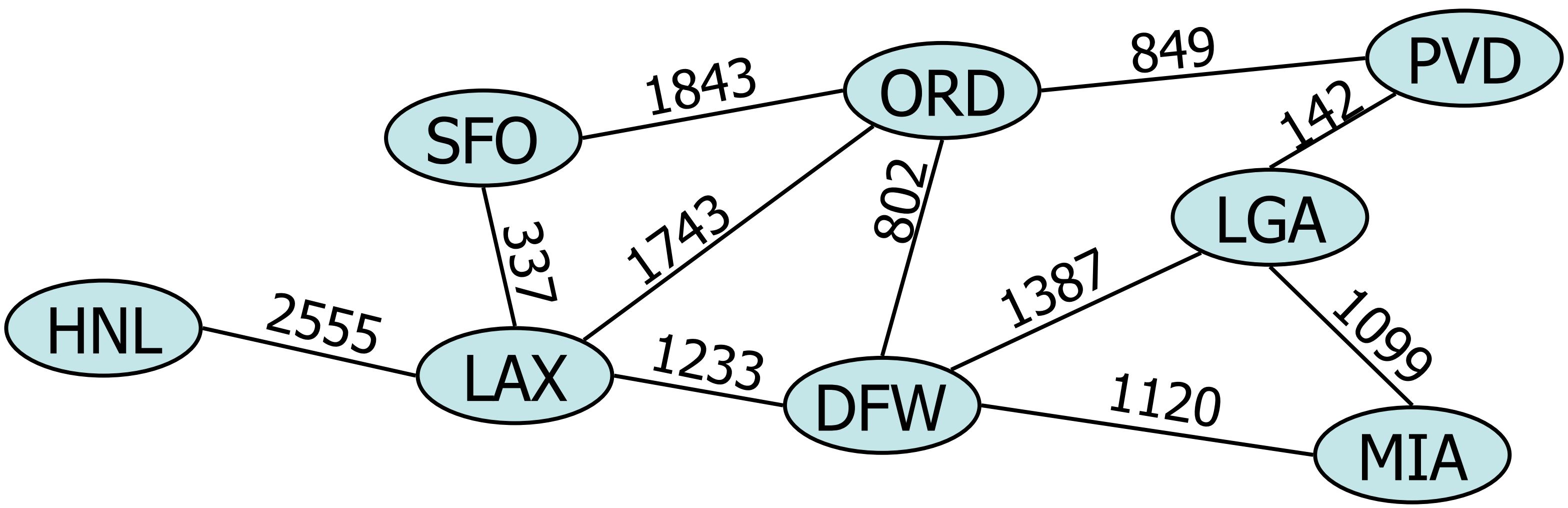


Outline and Reading

- Graphs
  - Definition
  - Applications
  - Terminology
  - Properties
  - ADT
- Data structures for graphs
  - Edge list structure
  - Adjacency list structure
  - Adjacency matrix structure

Graph

- A graph is a pair (***V***, ***E***), where
  - ***V*** is a set of nodes, called vertices
  - ***E*** is a collection of pairs of vertices, called edges
  - Vertices and edges are positions and store elements
- Example:
  - A vertex represents an airport and stores the three-letter airport code
  - An edge represents a flight route between two airports and stores the mileage of the route

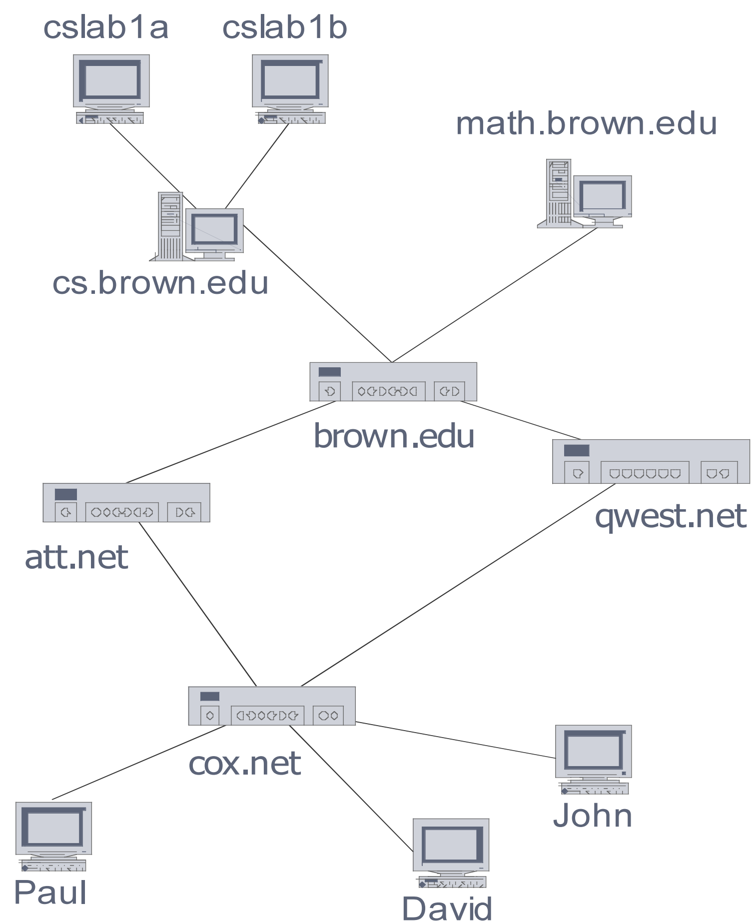


Edge Types

- Directed edge
  - ordered pair of vertices (***u***,***v***)
  - first vertex ***u*** is the origin
  - second vertex ***v*** is the destination
  - e.g., a flight
- Undirected edge
  - unordered pair of vertices (***u***,***v***)
  - e.g., a flight route
- Directed graph
  - all the edges are directed
  - e.g.,flight network
- Undirected graph
  - all the edges are undirected
  - e.g., route network

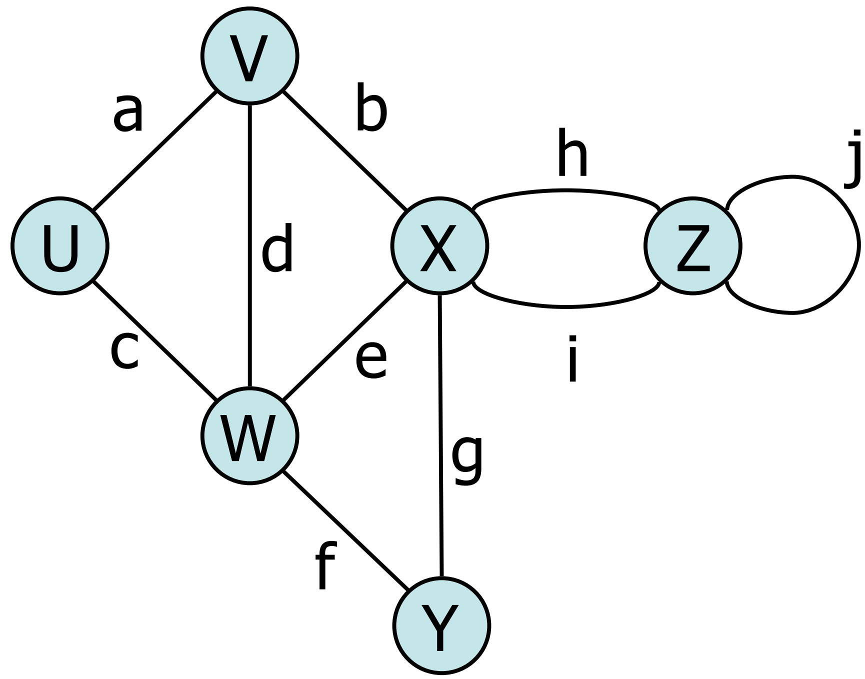
Applications

- Electronic circuits
  - Printed circuit board
  - Integrated circuit
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram



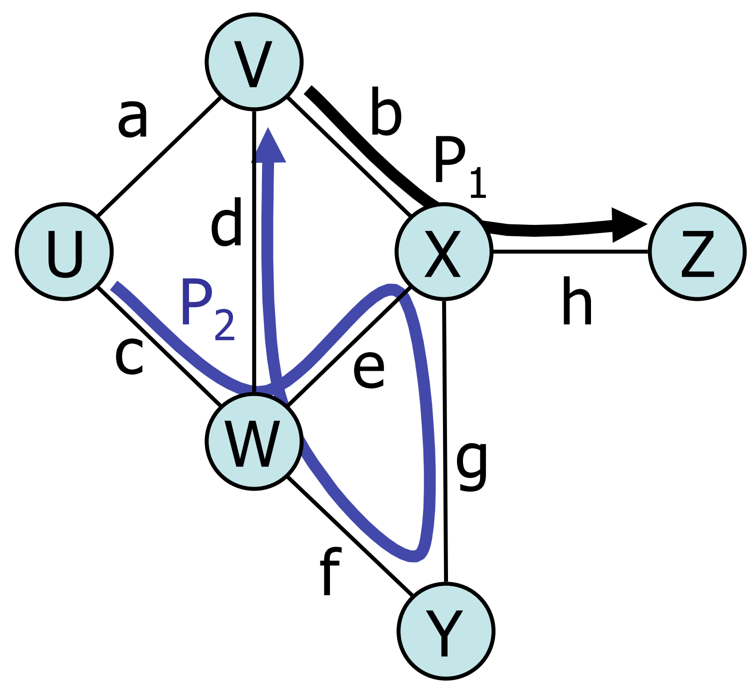
Terminology

- End vertices (or endpoints) of an edge
  - U and V are the endpoints of a
- Edges incident on a vertex
  - a, d, and b are incident on V
- Adjacent vertices
  - U and V are adjacent
- Degree of a vertex
  - X has degree 5
- Parallel edges
  - h and i are parallel edges
- Self-loop
  - j is a self-loop



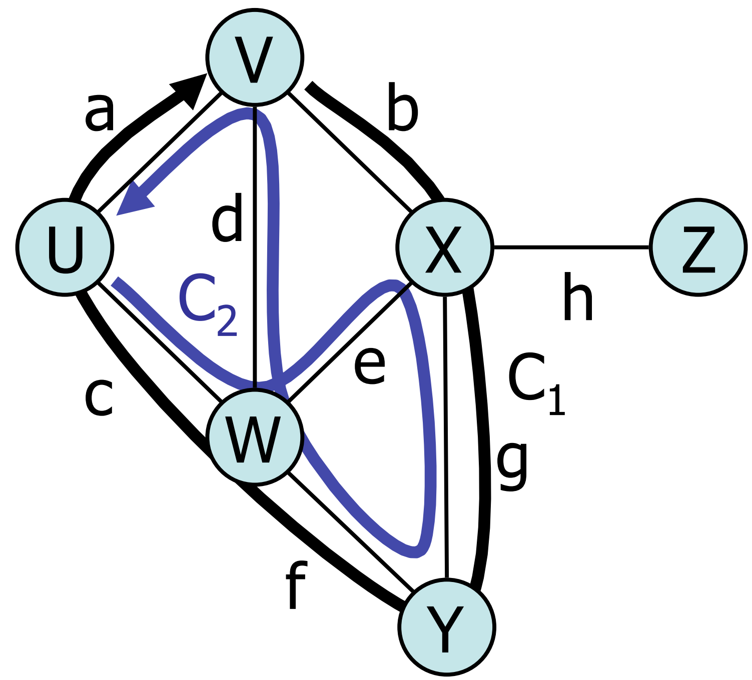
Terminology (cont.)

- Path
  - sequence of alternating vertices and edges
  - begins with a vertex
  - ends with a vertex
  - each edge is preceded and followed by its endpoints
- Simple path
  - path such that all its vertices and edges are distinct
- Examples
  - P_1=(V,b,X,h,Z) is a simple path
  - P_2=(U,c,W,e,X,g,Y,f,W,d,V) is a path that is not simple



Terminology (cont.)

- Cycle
  - circular sequence of alternating vertices and edges
  - each edge is preceded and followed by its endpoints
- Simple cycle
  - cycle such that all its vertices and edges are distinct
- Examples
  - C_1=(V,b,X,g,Y,f,W,c,U,a,↵) is a simple cycle
  - C_2=(U,c,W,e,X,g,Y,f,W,d,V,a,↵) is a cycle that is not simple



Properties

Property 1

$\sum_v \deg(v) = 2m$

Proof: each endpoint is counted twice

Property 2

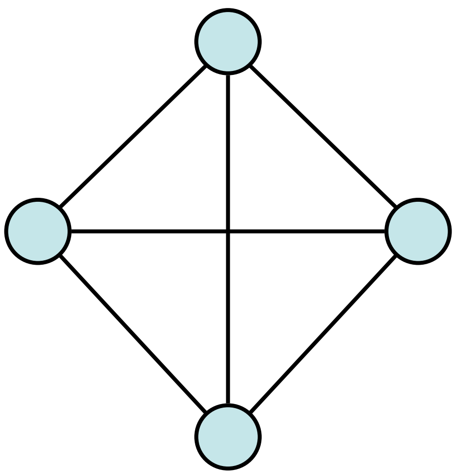
In an undirected graph with no self-loops and no multiple edges

$m \leq n\,(n - 1)/2$

Proof: each vertex has degree at most $(n - 1)$

Notation

n	number of vertices
m	number of edges
$\deg(v)$	degree of vertex v

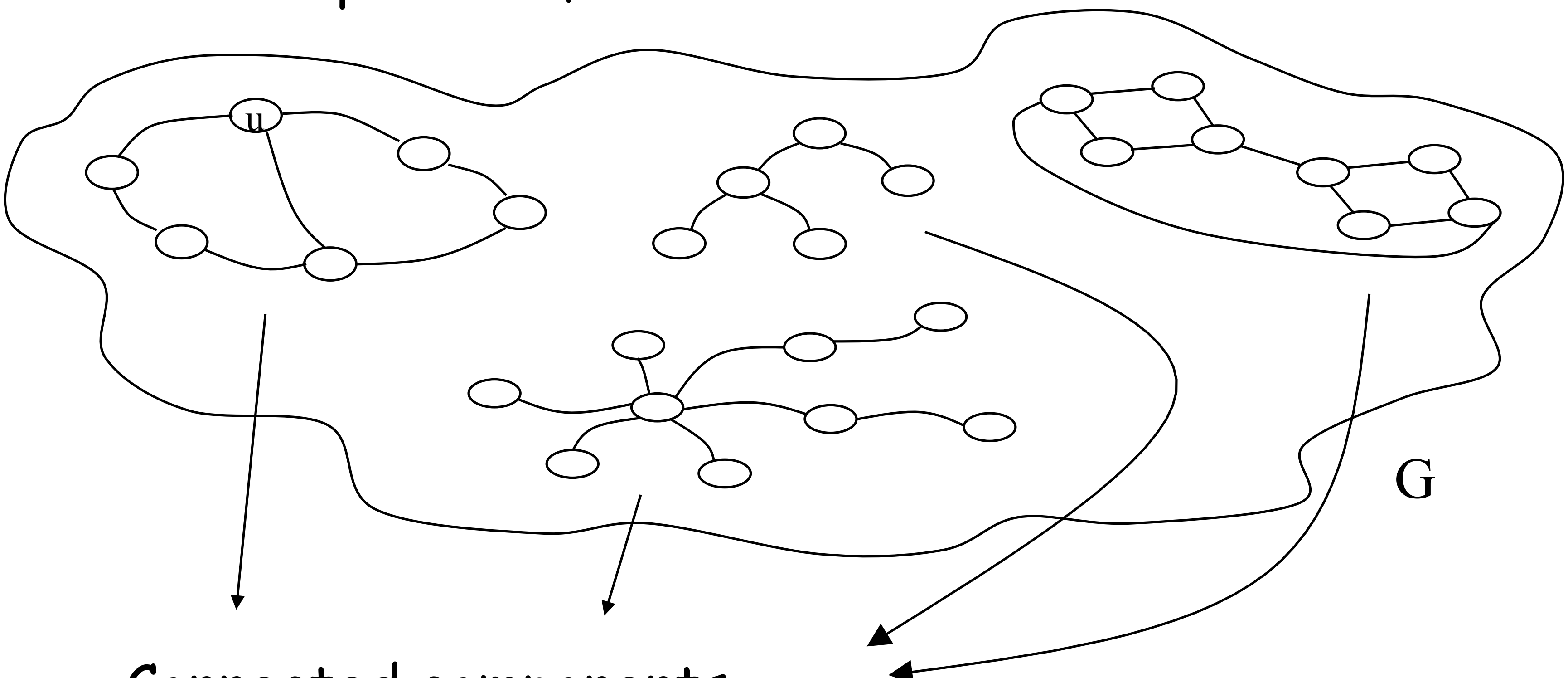


Example

- $n = 4$
- $m = 6$
- $\deg(v) = 3$

Connected Graphs

A (non-directed) graph is connected if there exists a path $\forall$ u, v $\in$ V.



Connected components

Main Methods of the Graph ADT

- Vertices and edges
  - are positions
  - store elements
- Accessor methods
  - aVertex()
  - incidentEdges(v)
  - endVertices(e)
  - isDirected(e)
  - origin(e)
  - destination(e)
  - opposite(v, e)
  - areAdjacent(v, w)
- Update methods
  - insertVertex(o)
  - insertEdge(v, w, o)
  - insertDirectedEdge(v, w, o)
  - removeVertex(v)
  - removeEdge(e)
- Generic methods
  - numVertices()
  - numEdges()
  - vertices()
  - edges()

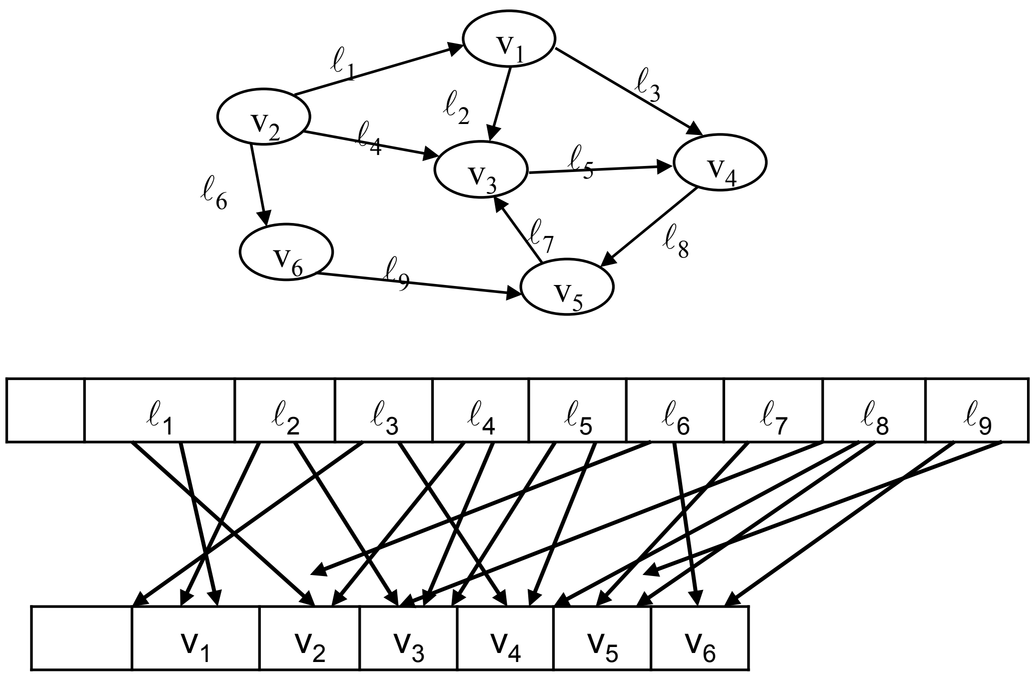
There could be other methods .....

Representations

Edge List
Adjacency List
Adjacency Matrix
Incidence Matrix

n = number of nodes
m = number of edges

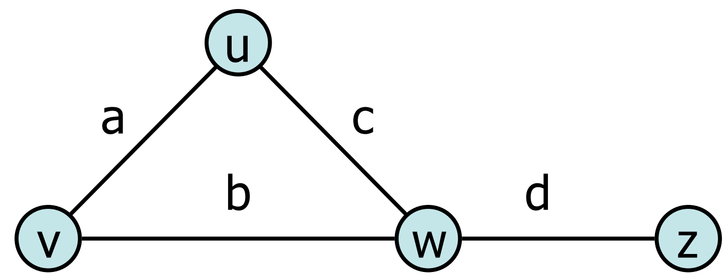
Edge List Structure (example)

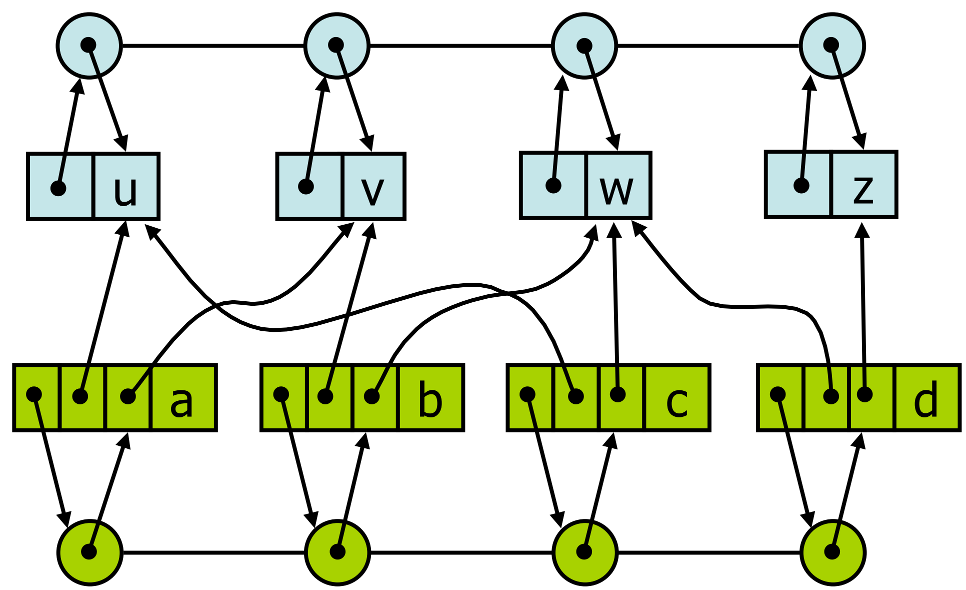


Space: $\boxed{n + m}$

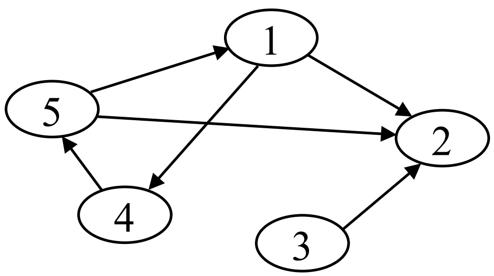
Edge List Structure

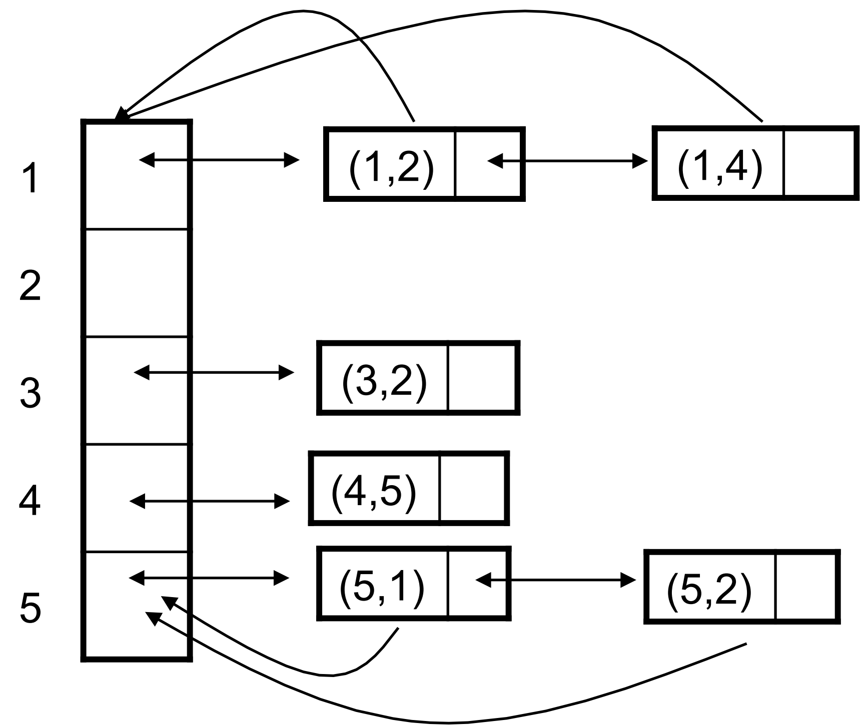
- Vertex object
  - element
  - reference to position in vertex sequence
- Edge object
  - element
  - origin vertex object
  - destination vertex object
  - reference to position in edge sequence
- Vertex sequence
  - sequence of vertex objects
- Edge sequence
  - sequence of edge objects





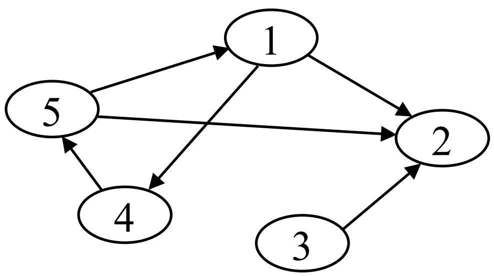
Adjacency List (example)

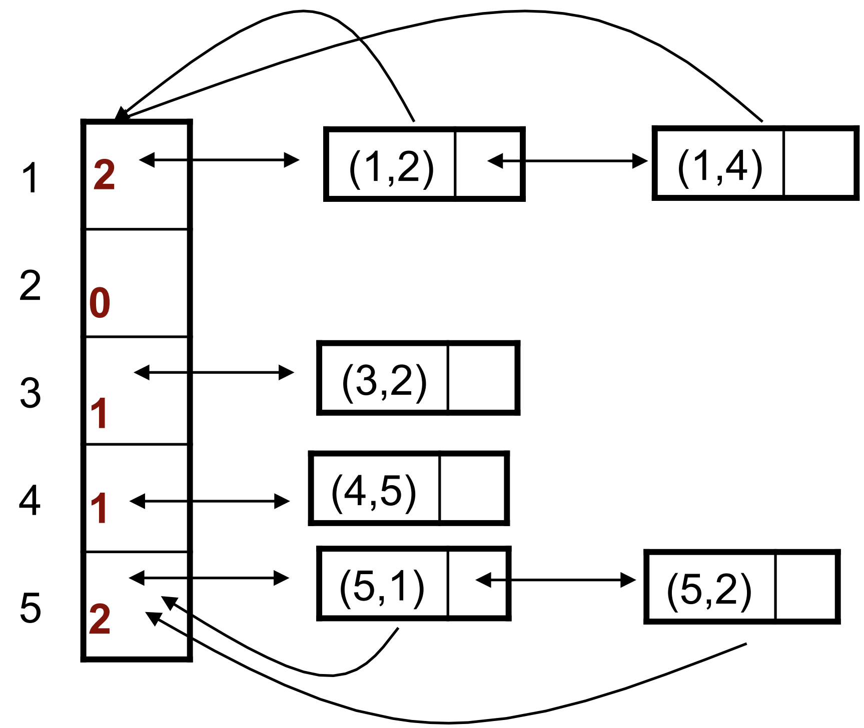




Edges augmented by link to node

Adjacency List (example)





Often, the node out-degree is kept at the node.

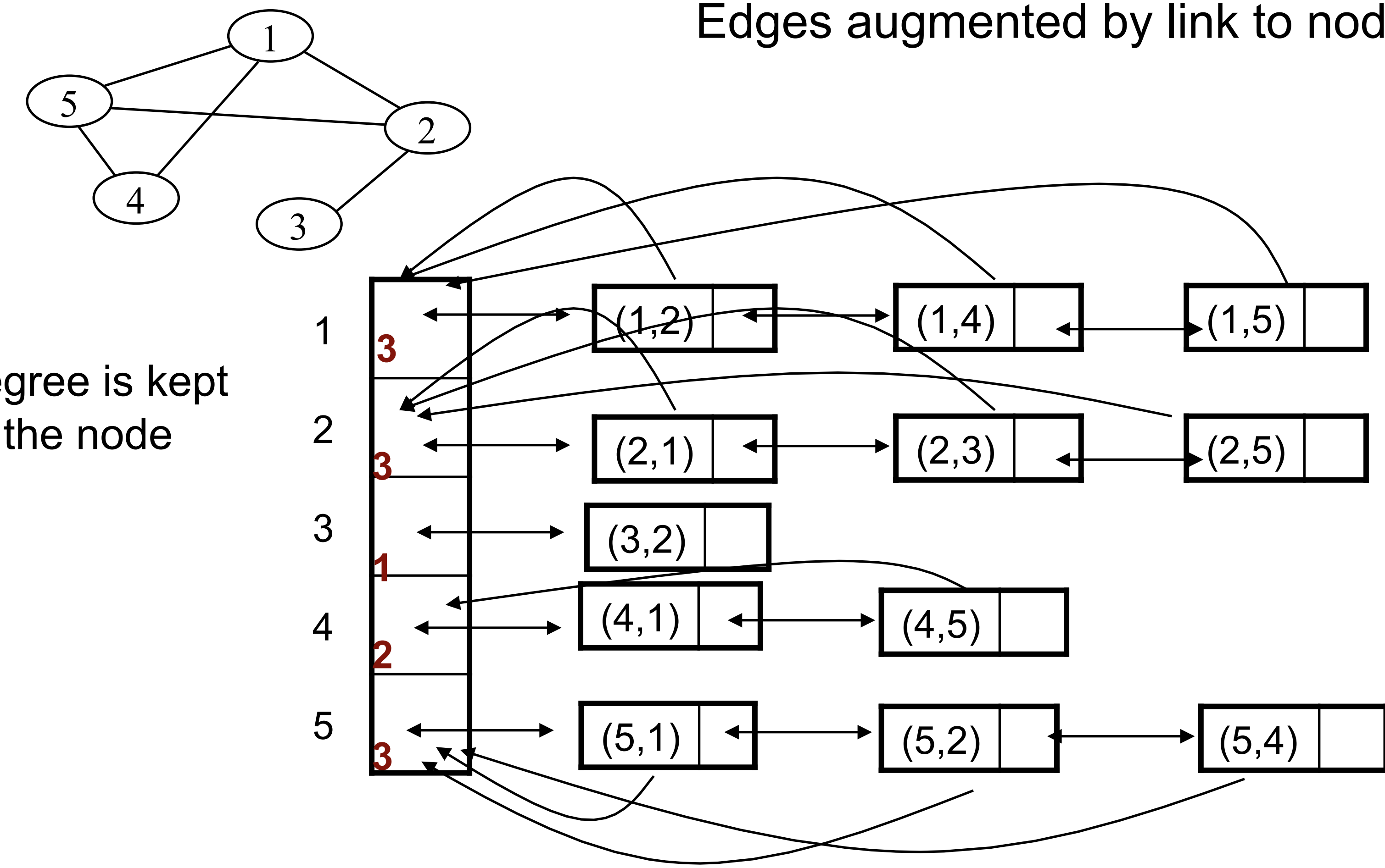
Edges augmented by link to node

Adjacency List

(another example -with undirected edges)

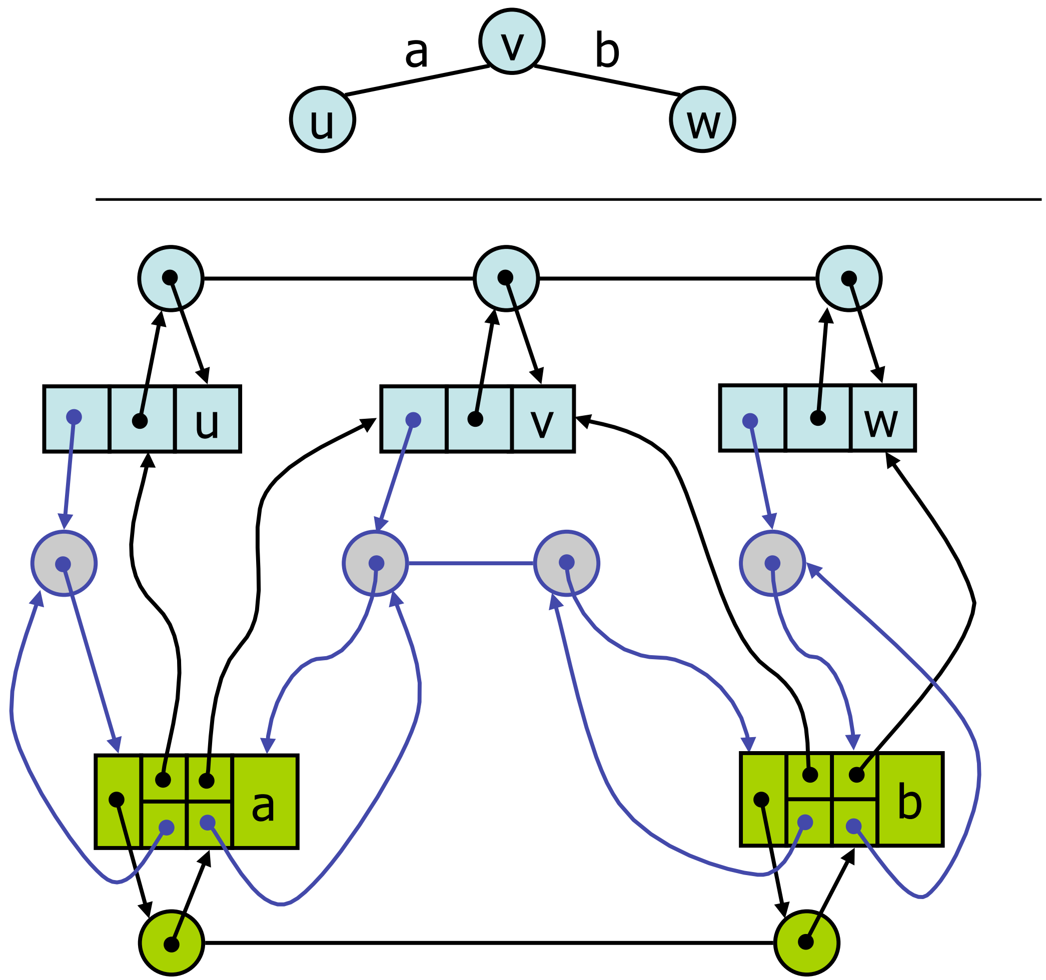
Edges augmented by link to node

degree is kept at the node

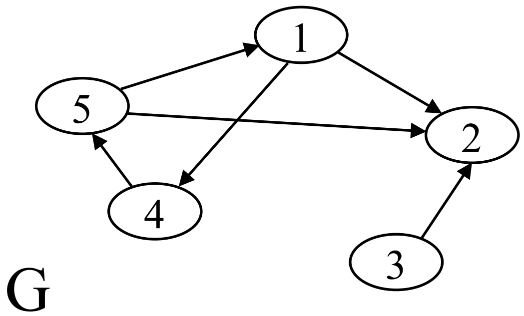


Adjacency List Structure

- Edge list structure
- Incidence sequence for each vertex
  - sequence of references to edge objects of incident edges
- Augmented edge objects
  - references to associated positions in incidence sequences of end vertices

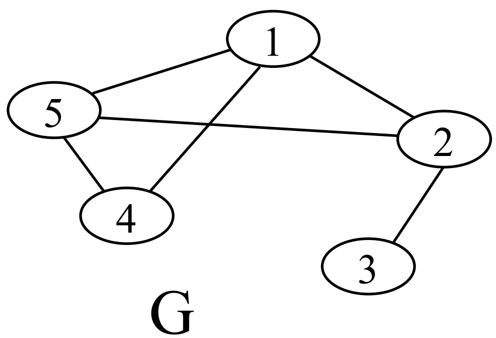


Adjacency Matrix (examples)



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

If G is not-directed



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

symmetric matrix

Adjacency Matrix (observation)

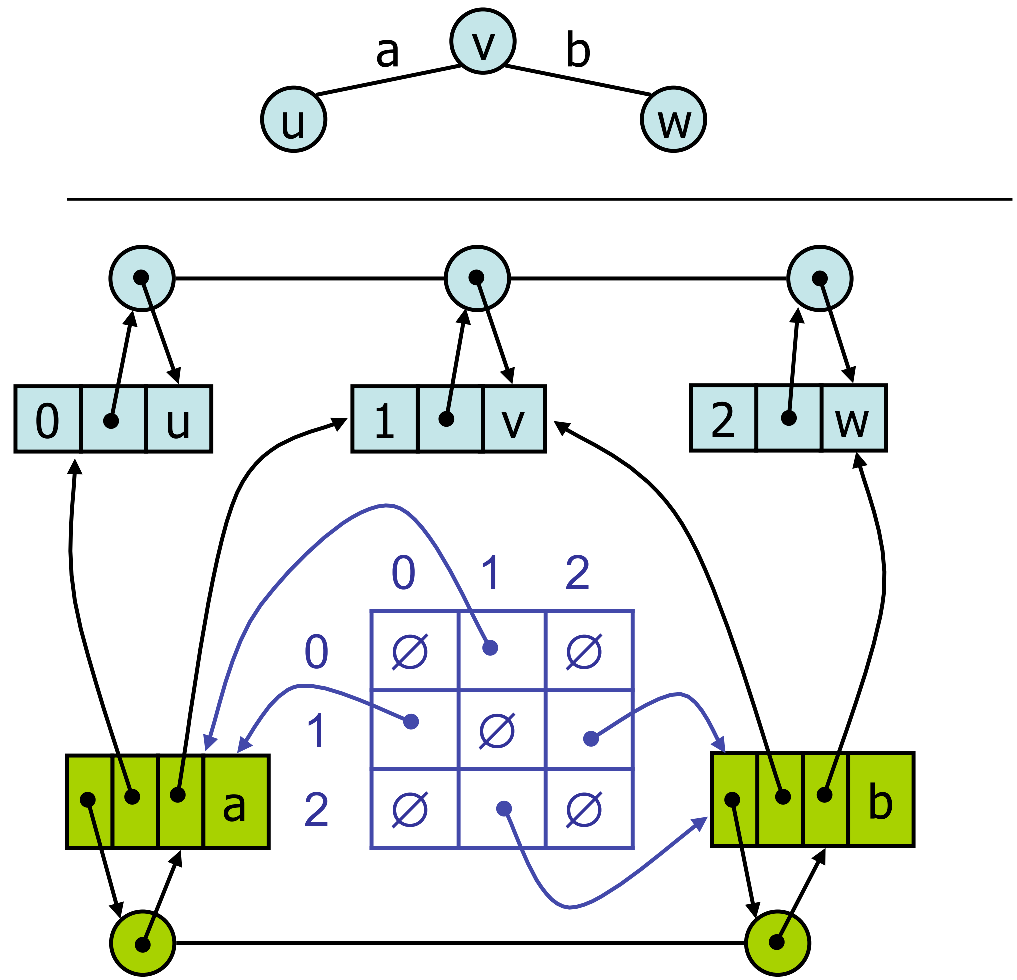
Space: $\boxed{n \times n}$

Lots of waste space if the matrix is SPARSE ...

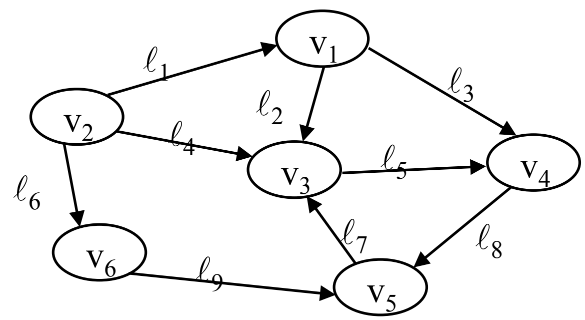
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Adjacency Matrix Structure

- Edge list structure
- Augmented vertex objects
  - Integer key (index) associated with vertex
- 2D-array adjacency array
  - Reference to edge object for adjacent vertices
  - Null for non nonadjacent vertices



Incidence Matrix (1)



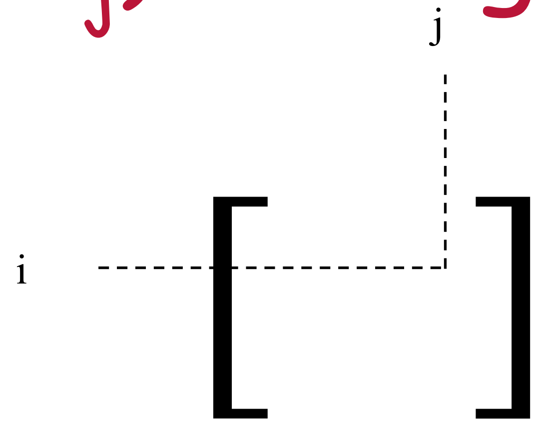
	ℓ_1	ℓ_2	ℓ_3	ℓ_4	ℓ_5	ℓ_6	ℓ_7	ℓ_8	ℓ_9
v_1	-1	1	1	0	0	0	0	0	0
v_2	1	0	0	1	0	1	0	0	0
v_3	0	-1	0	-1	1	0	-1	0	0
v_4	0	0	-1	0	-1	0	0	1	0
v_5	0	0	0	0	0	0	1	-1	-1
v_6	0	0	0	0	0	-1	0	0	1

Space: $\boxed{n \times m}$

Example in non-directed graphs

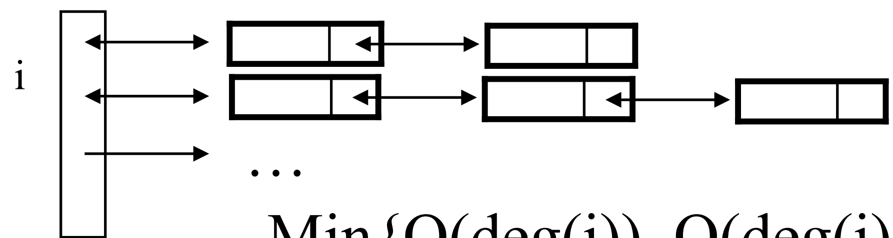
Is (v_i, v_j) an edge?

Adjacency Matrix:



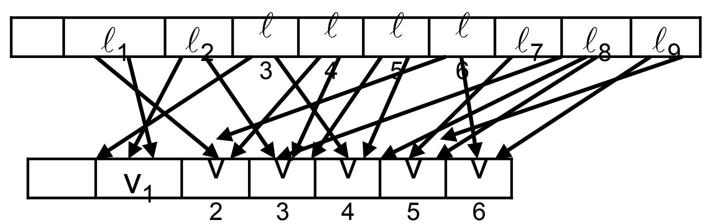
O(1)

Adjacency List:



Min{O(deg(i)), O(deg(j))}

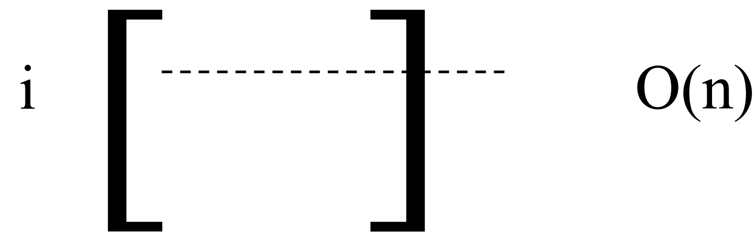
Edge List:



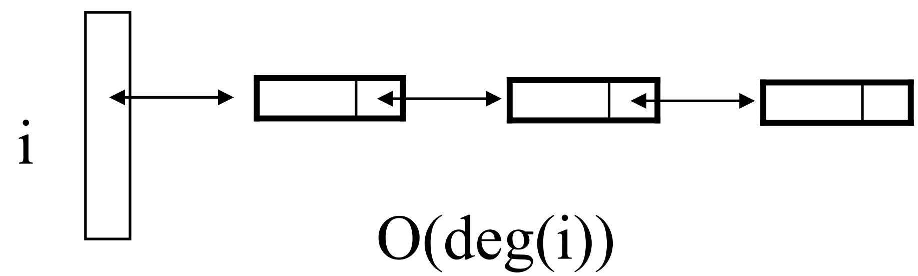
O(m)

Which nodes are adjacent to v_i?

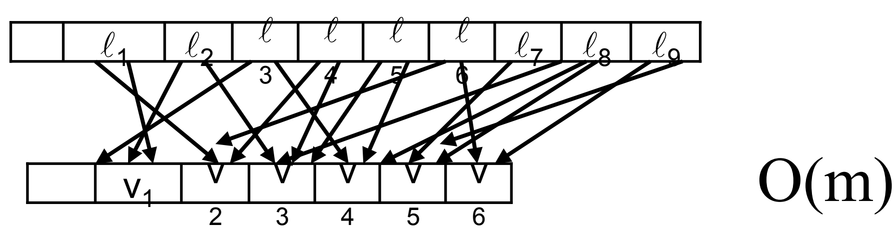
Adjacency Matrix:



Adjacency List:

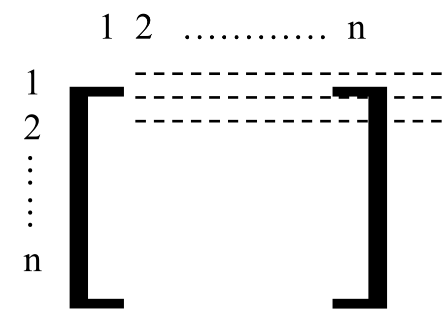


Edge List:



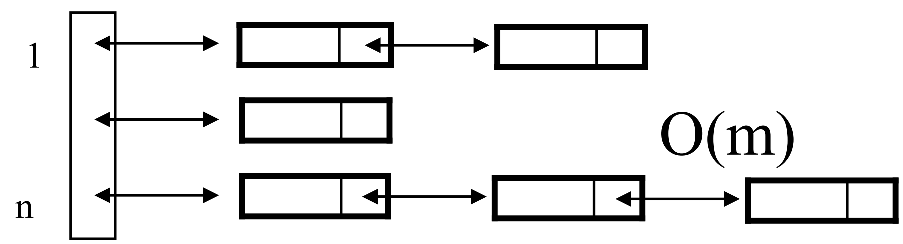
Mark all Edges

Adjacency Matrix:



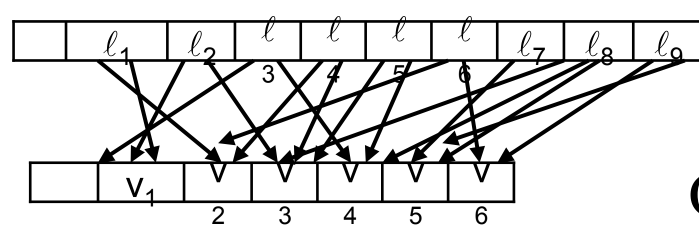
$O(n^2)$

Adjacency List:



$O(m)$

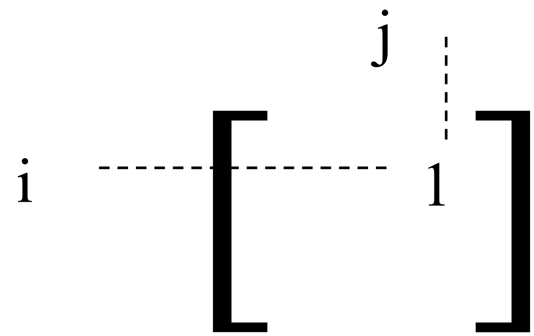
Edge List:



$O(m)$

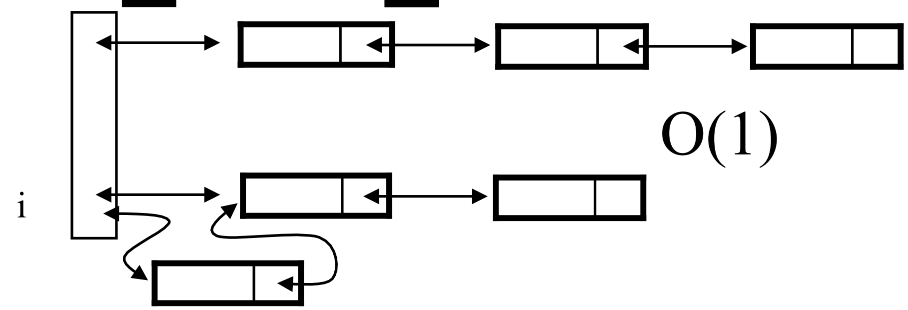
Add an Edge (v_i, v_j)

Adjacency Matrix:



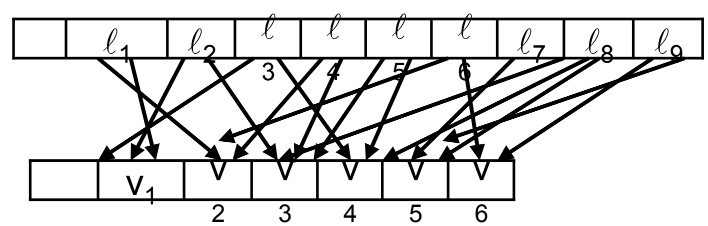
O(1)

Adjacency List (linked):



O(1)

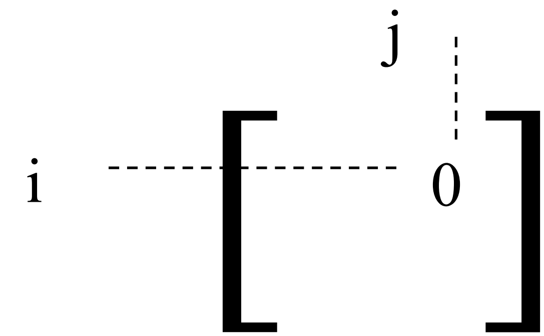
Edge List:



O(1)

Remove a given Edge $e=(v_i, v_j)$

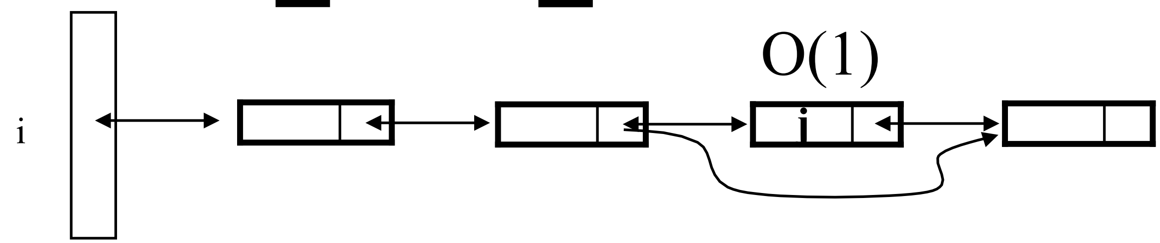
Adjacency Matrix:



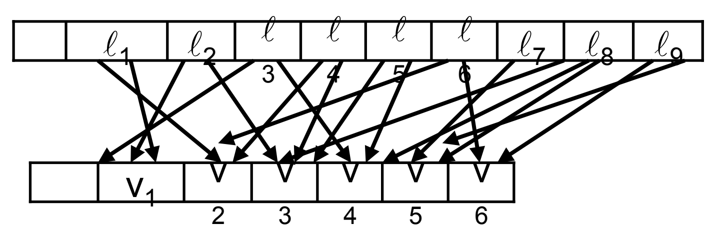
O(1)

Adjacency List :

O(1)



Edge List:



O(1)

	Adjacency Matrix	Adjacency List
Is (v_i, v_j) an edge?	O(1)	O(deg(i))
Which nodes are adjacent to v_i?	O(n)	O(deg(i))
Mark all edges	$O(n^2)$	O(m)
Add edge (v_i, v_j)	O(1)	O(1)
Remove edge (v_i, v_j)	O(1)	O(1)

O(deg(i)) = OUT-degree of node vi

G is directed

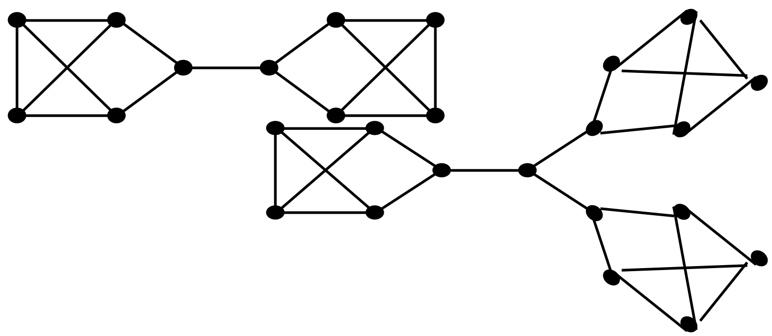
What are the predecessors of v_i?

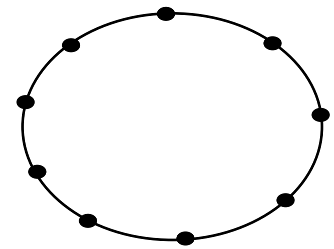
Performance

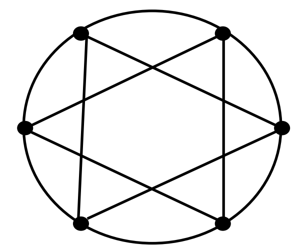
• n vertices • m edges • no parallel edges • no self-loops	Edge List	Adjacency List	Adjacency Matrix
Space	$n + m$	$n + m$	n^2
incidentEdges(v)	m	$\deg(v)$	n
areAdjacent (v, w)	m	$\min(\deg(v), \deg(w))$	1
insertVertex(o)	1	1	n^2
insertEdge(v, w, o)	1	1	1
removeVertex(v)	m	$\deg(v)$	n^2
removeEdge(e)	1	1	1

Special Graphs

Regular Graphs

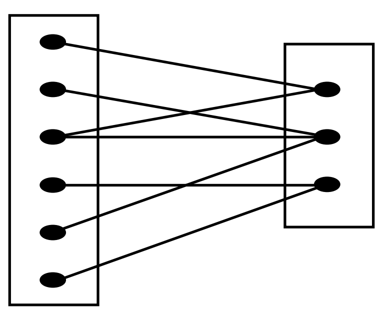




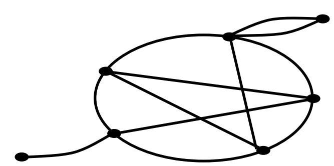


$\forall v_i, v_j \in V \qquad d(v_i) = d(v_j)$

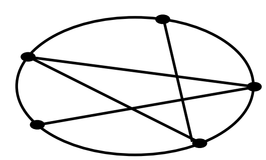
Bipartite Graphs

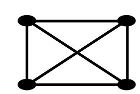


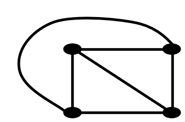
Planar Graphs



Cannot have







$$n - 1 \leq m \leq \frac{n(n-1)}{2}$$

$$1 \leq deg(i) \leq n - 1$$

degree

connected, non-directed

$n = | V |$ $m = | E |$

$$n - 1 \leq m \leq n(n - 1)$$

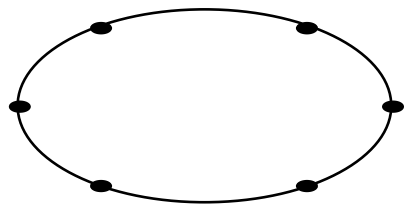
$$1 \leq deg(i) \leq n - 1$$

OUT-degree

connected, directed

Some Regular Graphs

— Ring —



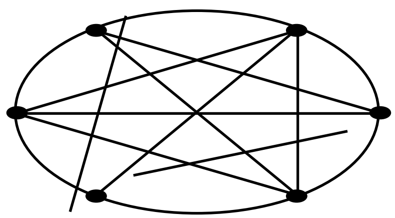
$m = n$

$d_i = 2 \quad \forall i$

Tree

$m = O(n)$

— Complete Graph —



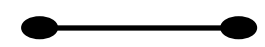
$m = \frac{n(n-1)}{2}$

$d_i = n - 1 \quad \forall i$

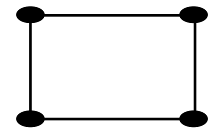
$m = O(n^2)$

— Hypercube —

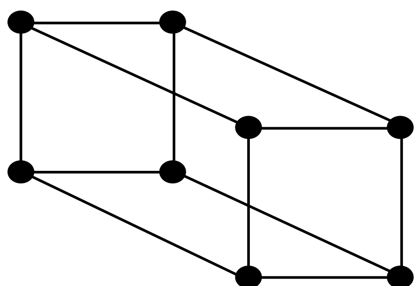
dim 0



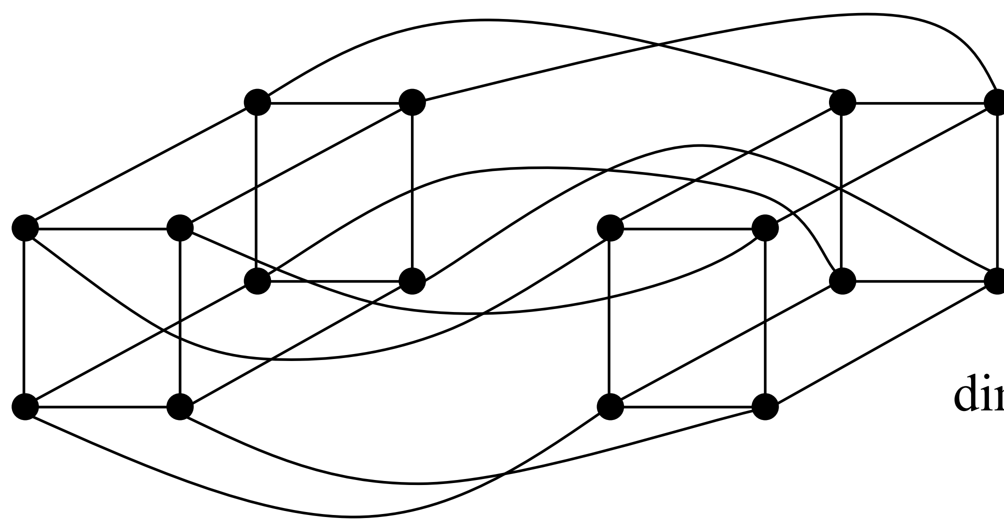
dim 1



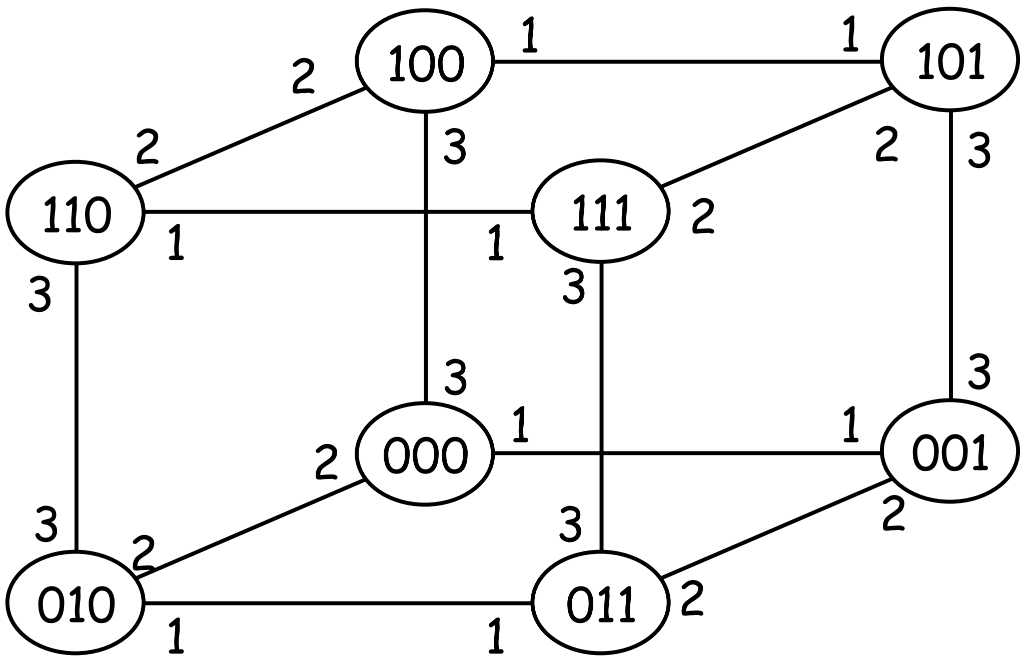
dim 2



dim 3



dim 4


100
101
110
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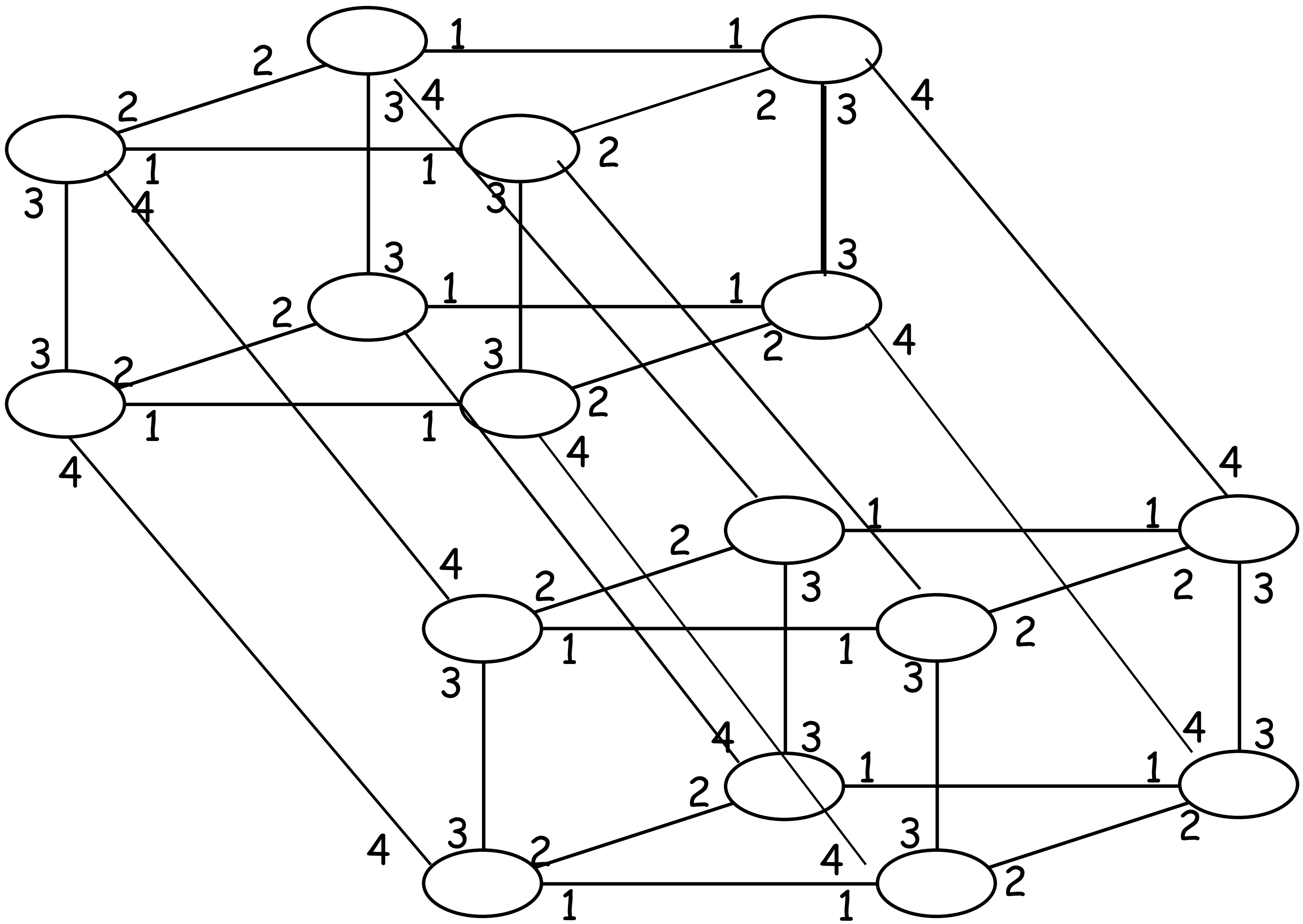
000
001
010
011
1
2
3



Hypercube

	n	m
h_0	1	0
h_1	2	1
h_2	4	1x2 + 2 = 4
h_3	8	4x2 + 4 = 12
h_4	16	12x2 + 8 = 32

$m_i = i \bullet 2^{i-1}$

h_i:

$n_0 = 1$
$n_i = 2\, n_{i-1}$ $\longrightarrow$ $n_i = 2^i$

$$\begin{cases} n_0 = 1 \\ n_i = 2\, n_{i-1} \end{cases} \longrightarrow n_i = 2^i$$

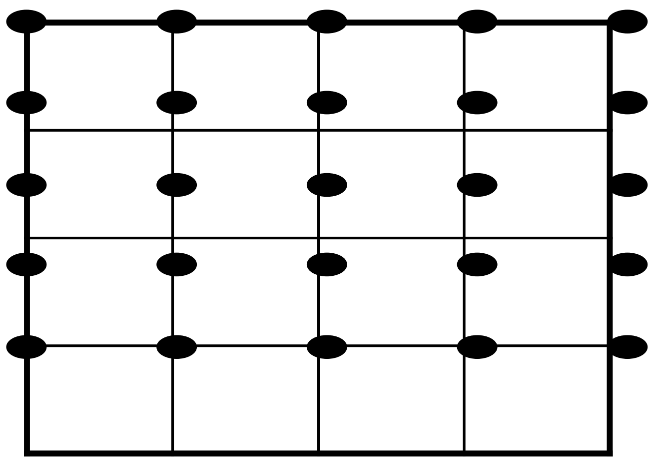
$$m_i = i \bullet 2^{i-1}$$

$$m = \frac{n \log n}{2}$$

$$\text{degree} = \log n$$

$$m = O(n \log n)$$

— Grid —



Not regular

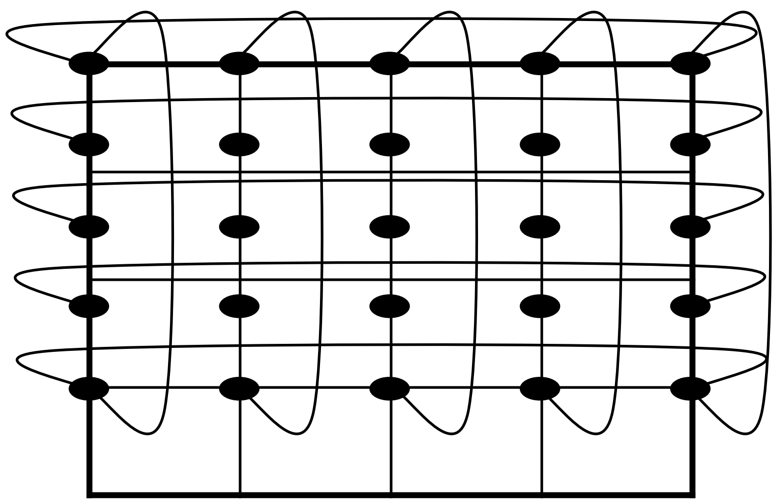
$m = O(n)$

$deg(i) = 4 \quad v_i = internal$

$deg(i) = 3 \quad v_i = border$

$deg(i) = 2 \quad v_i = corner$

— Torus —



$m = O(n)$

$deg(i) = 4 \quad \forall i$