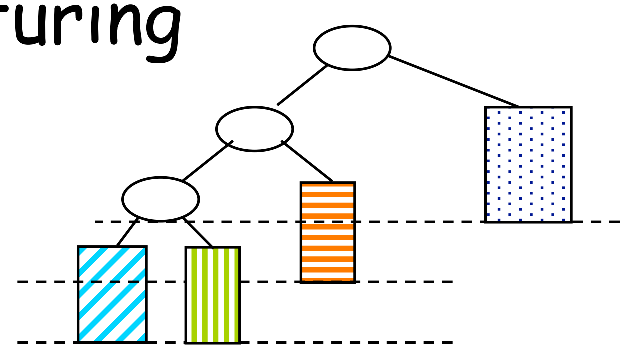
AVL Trees

Adel'son-Vel'skii and Landis

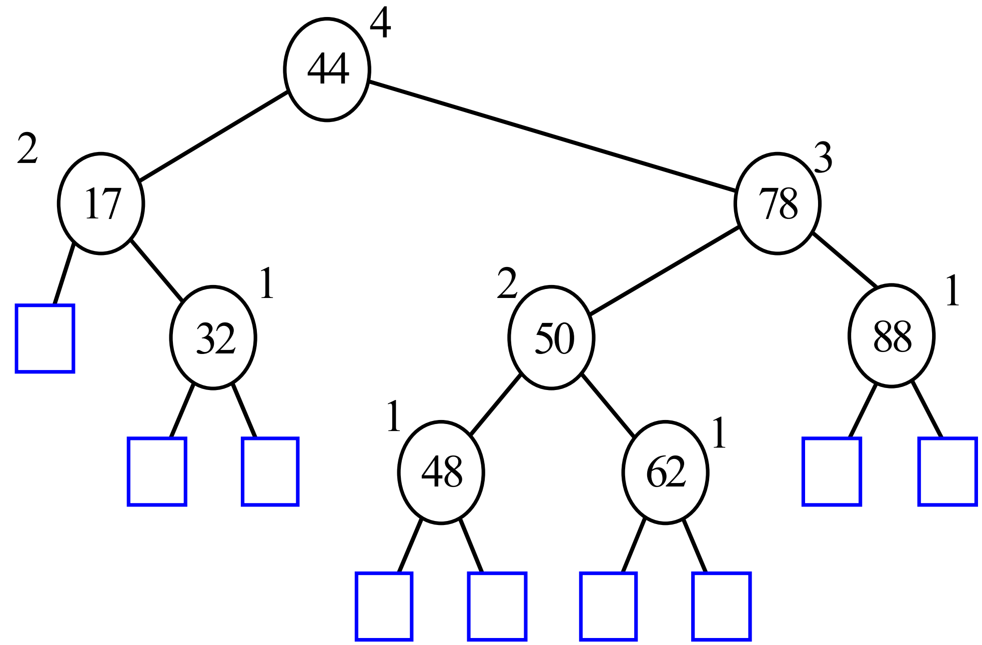
Data structure that implements MAP ADT

- Height of an AVL Tree
- Insertion and restructuring
- Removal and restructuring
- Costs



AVL Tree

- **AVL trees are balanced.**
- An AVL Tree is a ***binary search tree*** such that for every internal node v of T, the *heights of the children of v can differ by at most 1*.

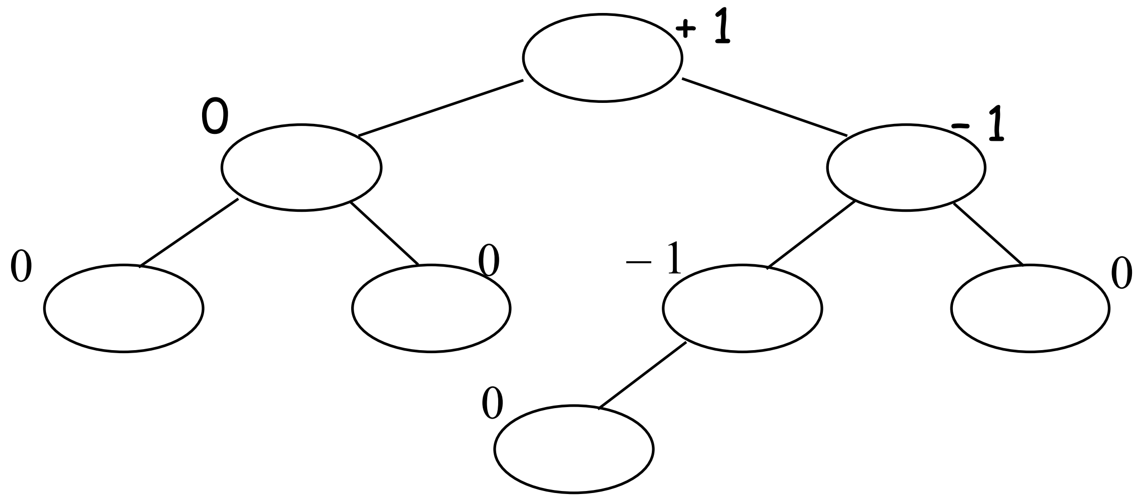


An example of an AVL tree where the heights are shown next to the nodes:

Balancing Factor

height(right s.a.) − height(left s.a.)

∈{-1, 0, 1} for AVL tree



Height of an AVL Tree

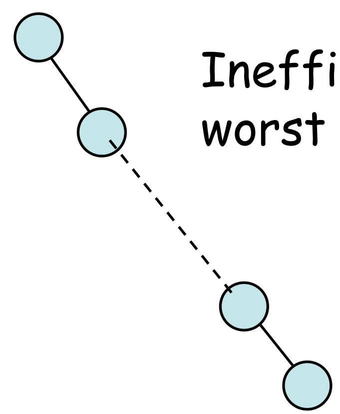
Note: "longest" possible heap with n nodes.

Always O(log n)

But cannot search efficiently

Note: "longest" possible binary tree with n nodes:

O(n)

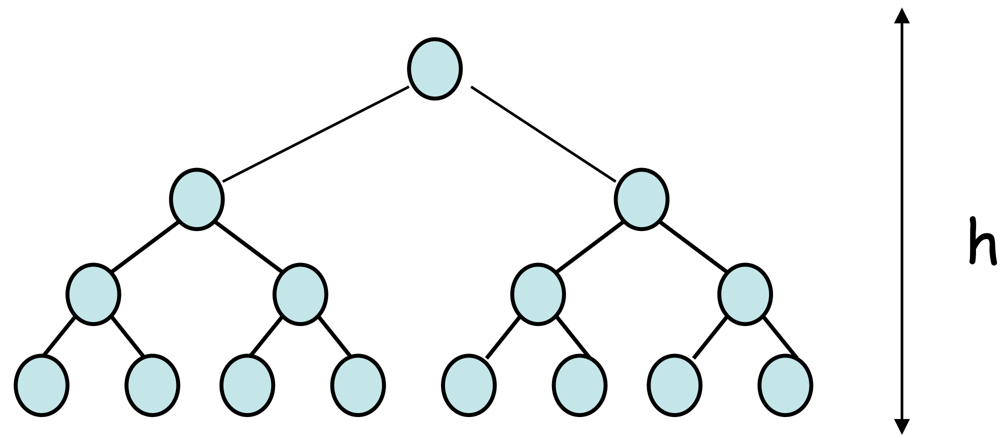


Inefficient to search in the worst case.

We'll now see that the *height* of an AVL tree T storing n keys is O(log n).

Note: AVL tree with the highest possible number of internal nodes for a given height h:

Perfect:



$n = 2^h$

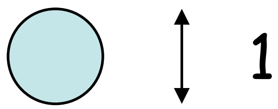
So $h = \log n$

To construct the "longest" possible AVL tree, we look for

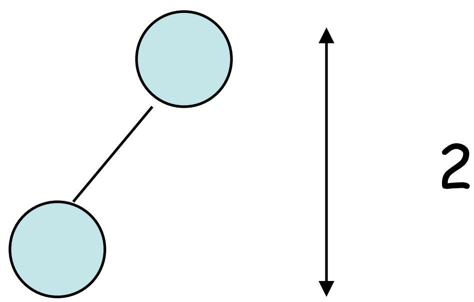
the *minimum number of nodes* of an *AVL* tree of height h.

$n(h)$

Easy to see that $n(1) = 1$ and $n(2) = 2$



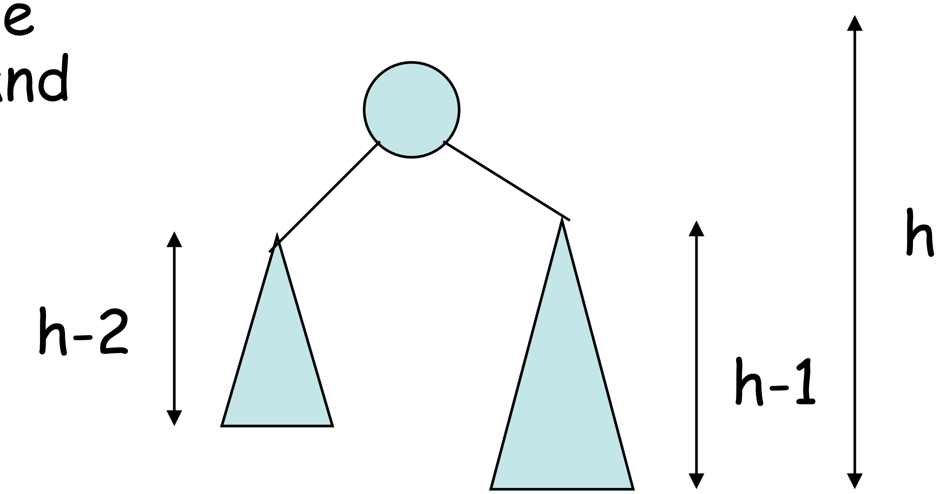
$n(1) = 1$



$n(2) = 2$

n(h): the *minimum number of internal nodes* of an AVL tree of height h.

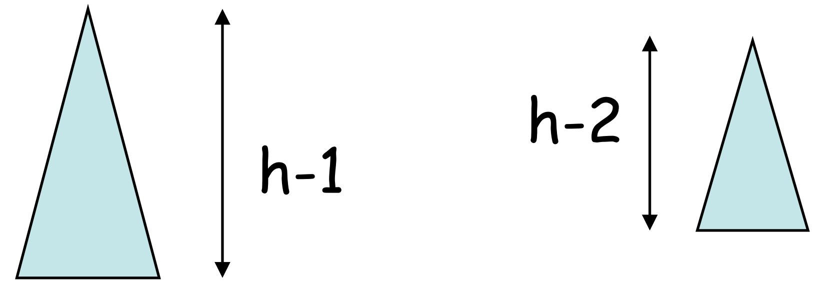
For n ≥ 3, an AVL tree of height h contains the root node, one AVL subtree of height h-1 and the other AVL subtree of height h-2.



$$n(h) = 1 + n(h-1) + n(h-2)$$

Height of an AVL Tree

$n(h) = 1 + \boxed{n(h-1)} + n(h-2)$



Clearly: $\boxed{n(h-1) > n(h-2)}$

so: $n(h) > 1 + n(h-2) + n(h-2)$

$n(h) > 1 + 2\,n(h-2) > 2\,n(h-2)$

$n(h) > 2n(h-2)$

for any h

Height of an AVL Tree

So, now we know: $n(h) > 2\ n(h-2)$

but then also: $n(h-2) > 2\ n(h-4)$

$n(h) > 4\ n(h-4)$

$n(h) > 4n(h-4)$

but then also: $n(h-4) > 2n(h-6)$

$n(h) > 8n(h-6)$

We can continue:

$n(h) > 2n(h-2)$

$n(h) > 4n(h-4)$

$n(h) > 8n(h-6)$

...

$n(h) > 2^i n(h-2i)$

$n(h) > 2^i n(\underline{h-2i})$

$h-2i = 2$

for $i = h/2 - 1$

And we know that

$n(1) = 1$

$n(2) = 2$

We substitute $h/2 - 1$

$n(h) > 2^{h/2 - 1} n(2)$

$n(h) > 2^{h/2}$

$\log n(h) > \log 2^{h/2}$

$\log n(h) > h/2$

$h < 2 \log n(h)$

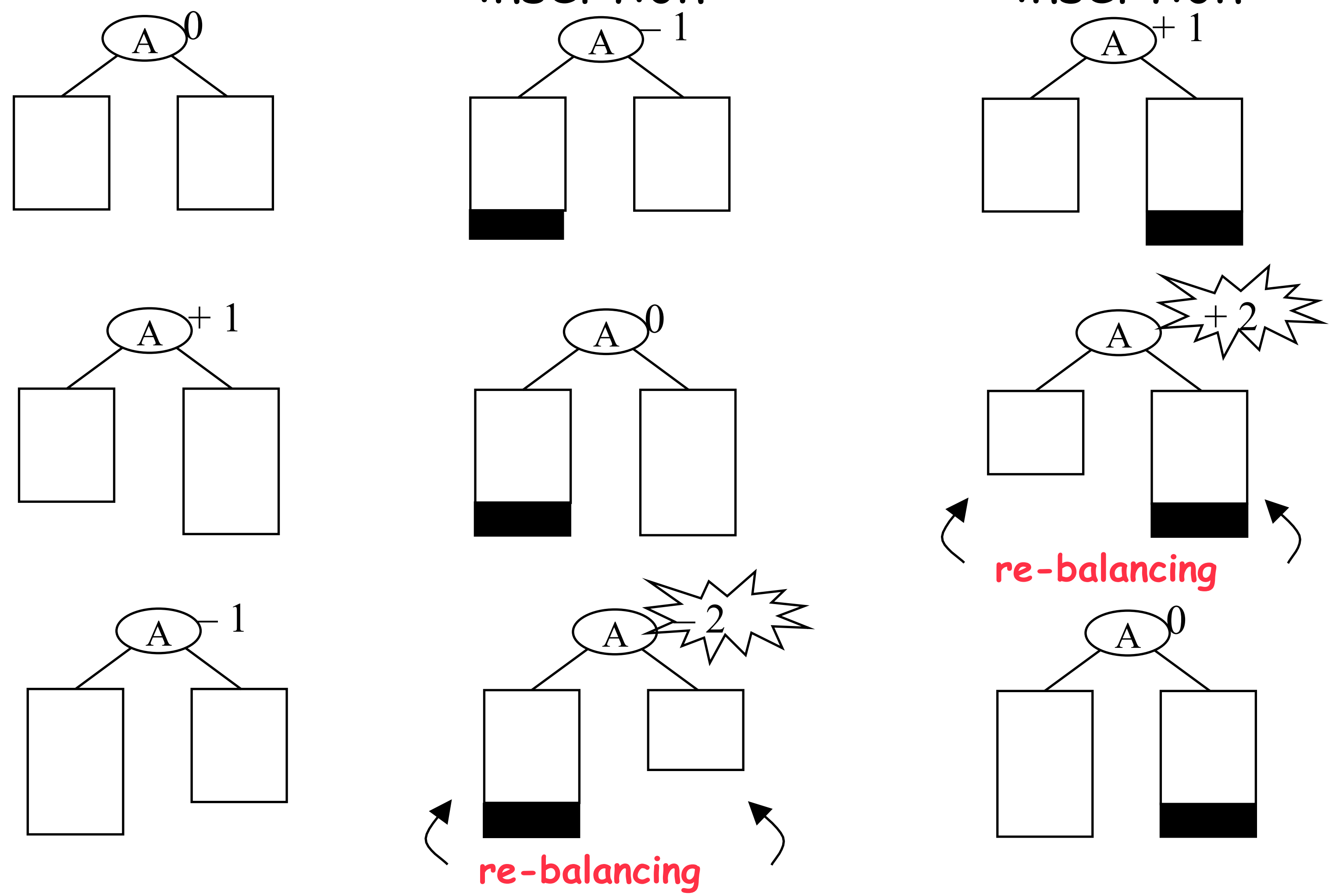
which means that h is **$O(\log n)$**

Insertion

- A binary search tree T is called ***balanced*** if for every node v, the height of v's children differ by at most one.
- Inserting a node into an AVL tree involves performing an expandExternal(w) on T, which changes the heights of some of the nodes in T.
- If an insertion causes T to become unbalanced we have to rebalance...

Insertion

Before | After left insertion | After right insertion



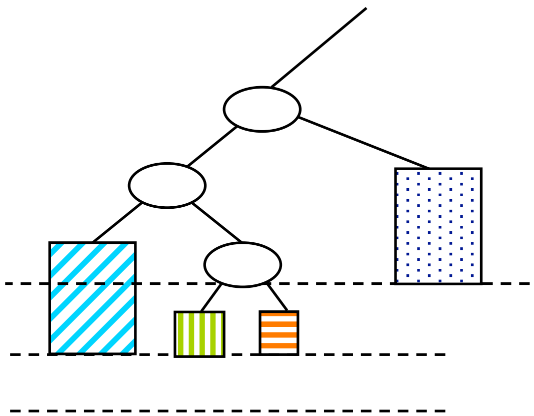
Rebalancing after insertion

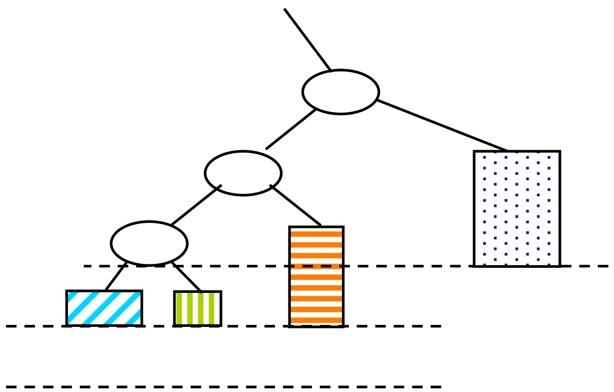
We are going to identify 3 nodes which form a grandparent, parent, child triplet and the 4 subtrees attached to them. We will rearrange these elements to create a new balanced tree.

Rebalancing

Step 1: Trace the path back from the point of insertion to the first node whose grandparent is unbalanced. Label this node x, its parent y, and grandparent z.

Examples .....





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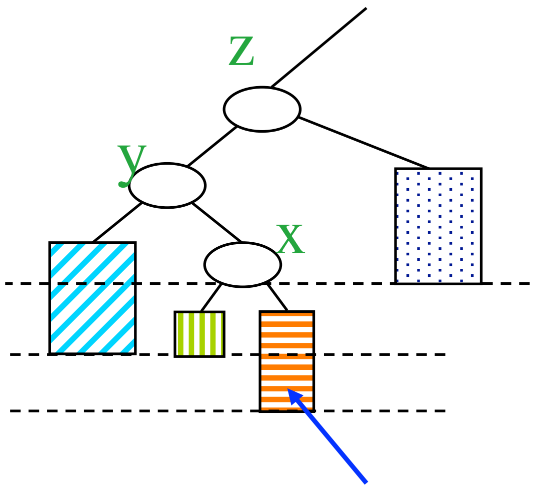
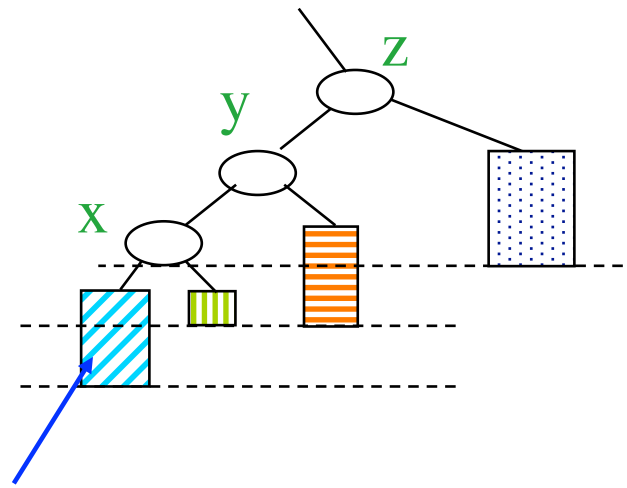
Examples …..

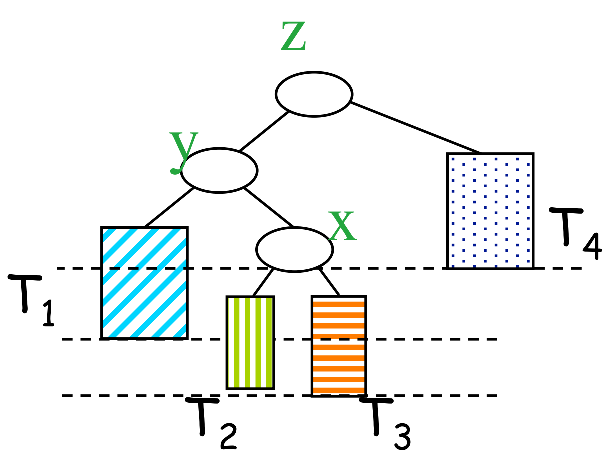


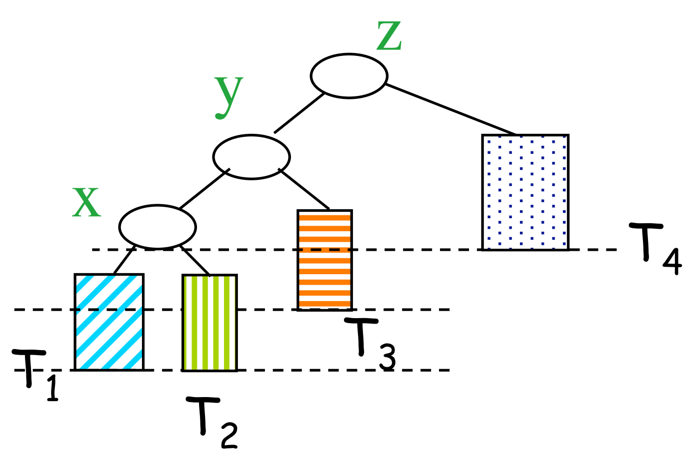


Rebalancing

Step 2: These nodes will have 4 subtrees connected to them. Label them T_1, T_2, T_3, T_4 from left to right.

Examples .....

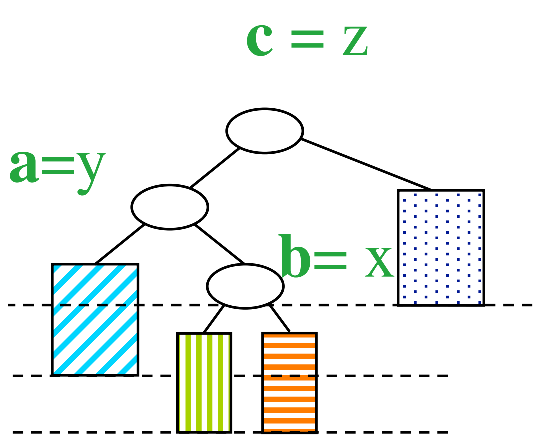




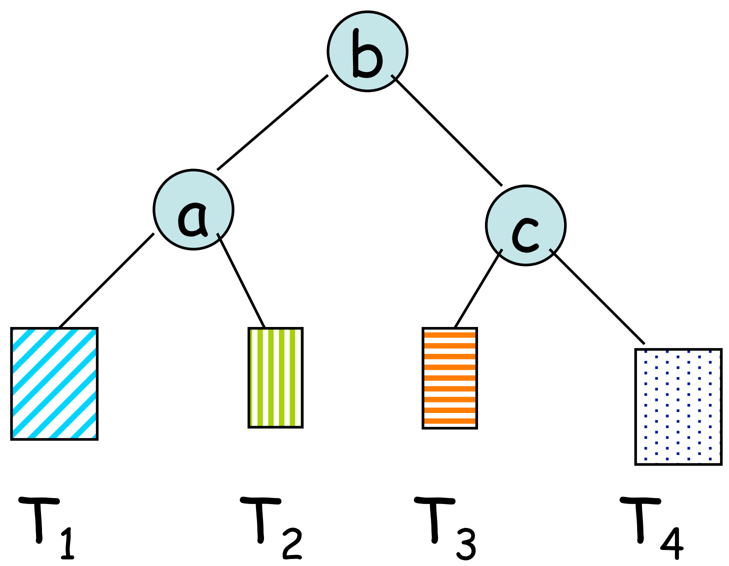
Rebalancing

Step 3: Rename x, y, z to a, b, c according to their inorder traversal i.e. if y,x,z is the relative order of those nodes following the inorder traversal then label y 'a', x 'b' and z 'c'.

Example

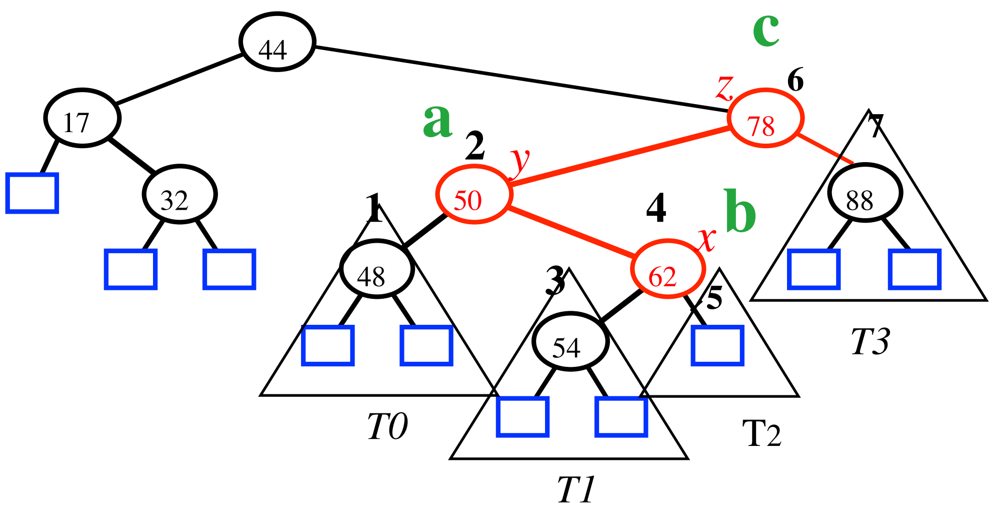


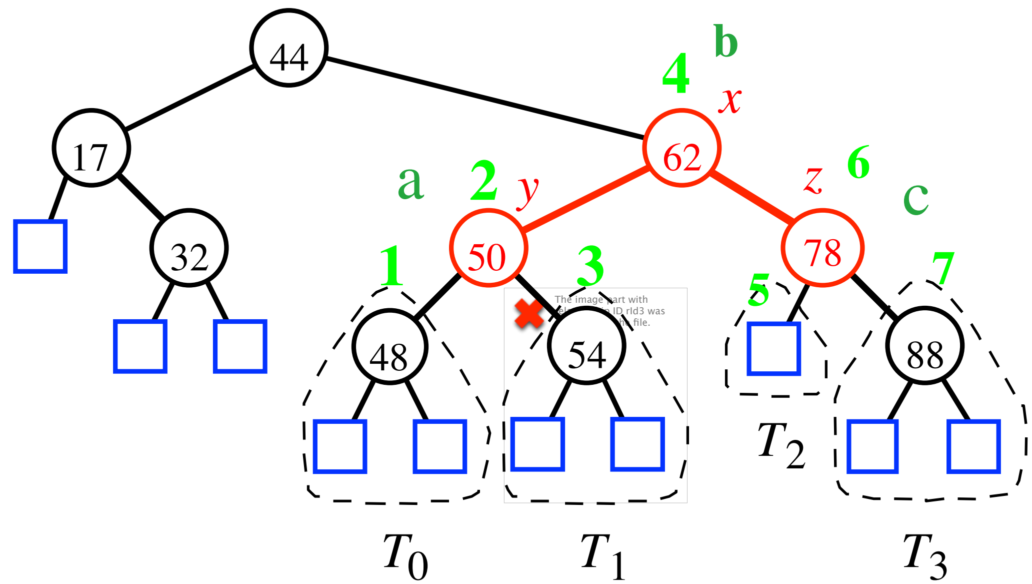
Step 4: Replace the tree rooted at z with the following tree:



Rebalance done!

Example: after inserting 54





Does this really work?

We need to see that the new tree is :

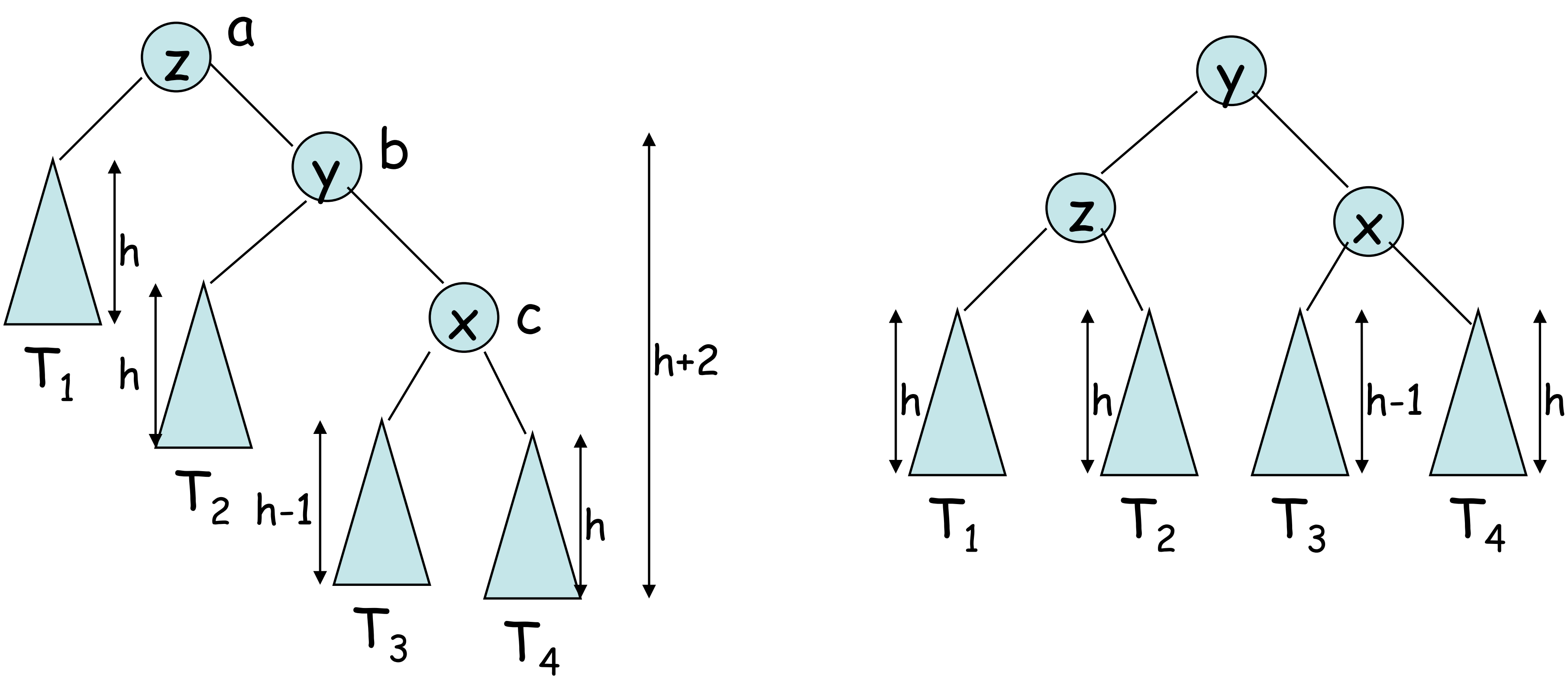
a) A Binary search tree - the inorder traversal of our new tree should be the same as that of the old tree

Inorder traversal: by definition is T1 a T2 b T3 c T4

b) Balanced: have we fixed the problem?

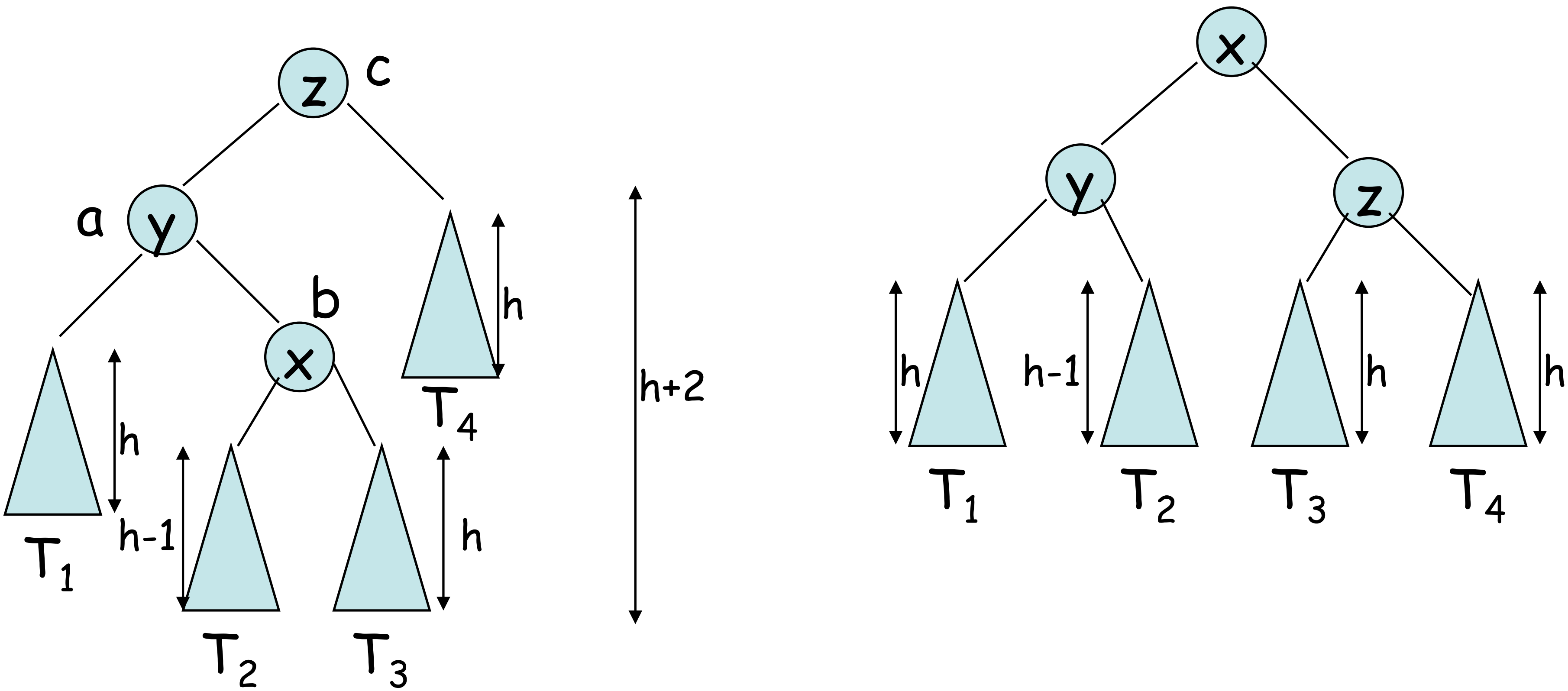
We consider 2 types of examples

Example 1



Inorder: T1 z T2 y T3 x T4

Example 2



Inorder: T1 y T2 x T3 z T4

An Observation…

Notice that in both cases, the new tree rooted at b has the same height that the old tree rooted at z had before insertion.

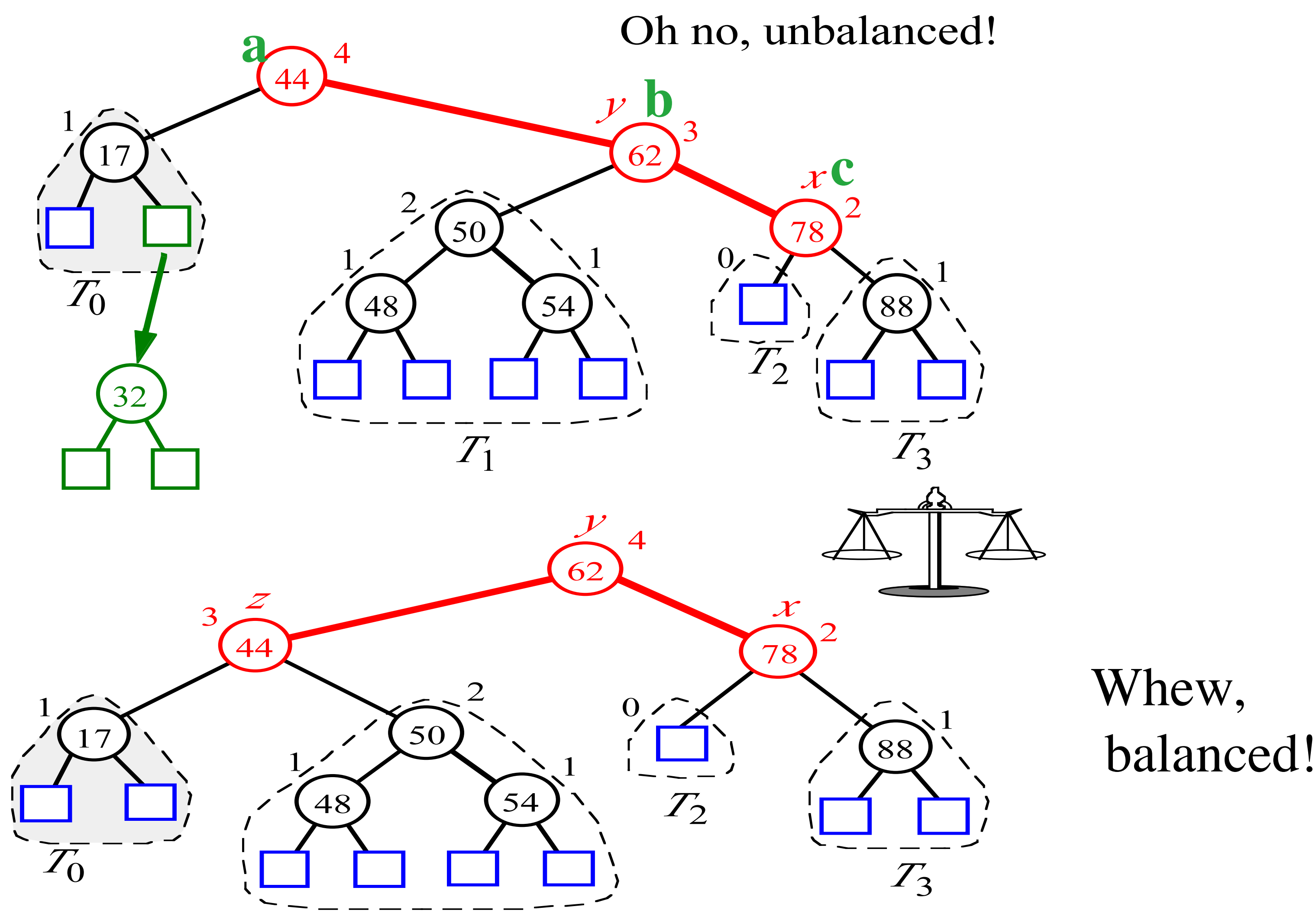
So.. once we have done one rebalancing act, we are done.

Removal

- We can easily see that performing a removeAboveExternal(w) can cause T to become unbalanced.
- Let *z* be the first unbalanced node encountered while travelling up the tree from w. Also, let y be **the child of z with the larger height, and let x be the child of y with the larger height.**
- We can perform operation restructure(x) to restore balance at the subtree rooted at z.
- As this restructuring may upset the balance of another node higher in the tree, we must continue checking for balance until the root of T is reached

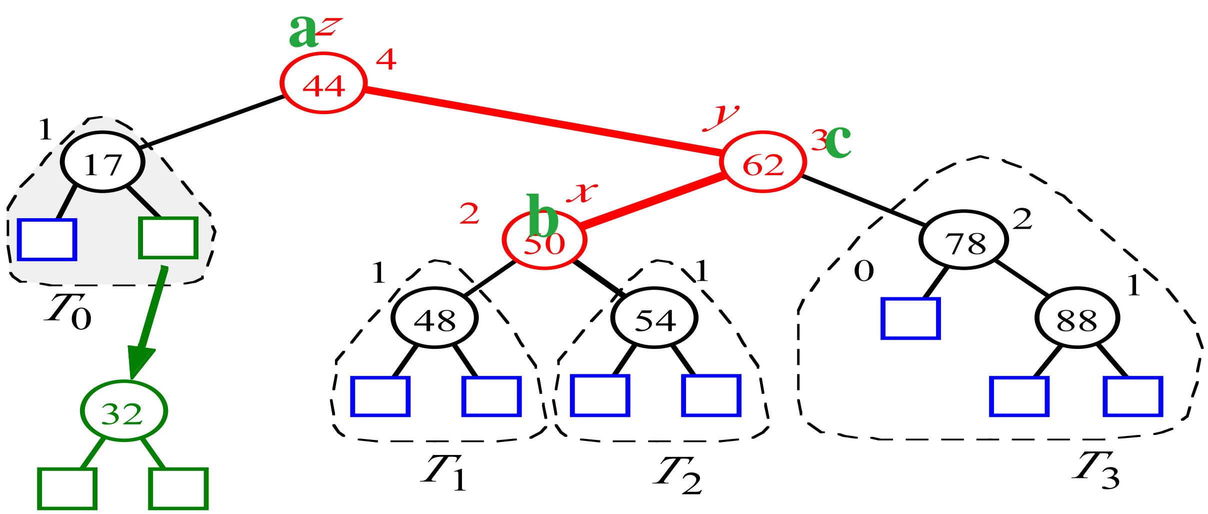
Removal (contd.)

the choice of x is not unique !!!

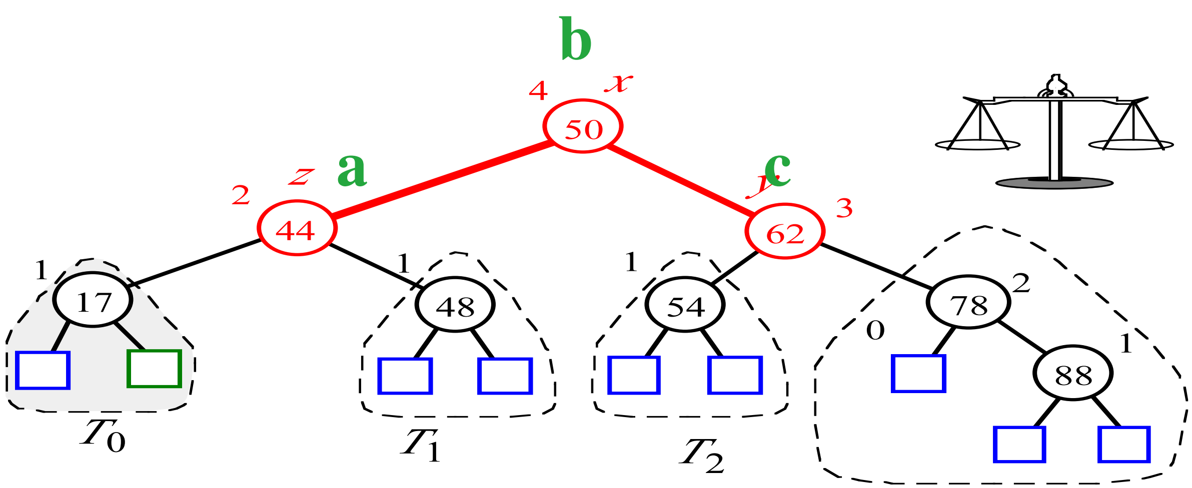


Removal (contd.)

- we could choose a different x:



Oh no, unbalanced!



Whew, balanced!

COMPLEXITY

Searching: findElement(k):

Inserting: insertItem(k, o):

Removing: removeElement(k):

O(log n)

Some implementation details are very important:

The trinode restructure is accomplished using the rotation operation:

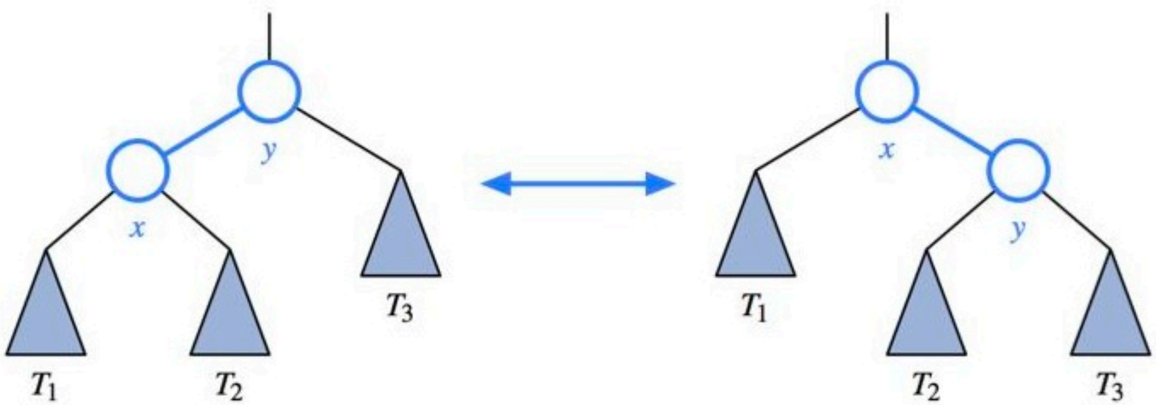


Figure 11.8: A rotation operation in a binary search tree. A rotation can be performed to transform the left formation into the right, or the right formation into the left. Note that all keys in subtree T_1 have keys less than that of position x, all keys in subtree T_2 have keys that are between those of positions x and y, and all keys in subtree T_3 have keys that are greater than that of position y.

Trinode restructuring using rotation operation:

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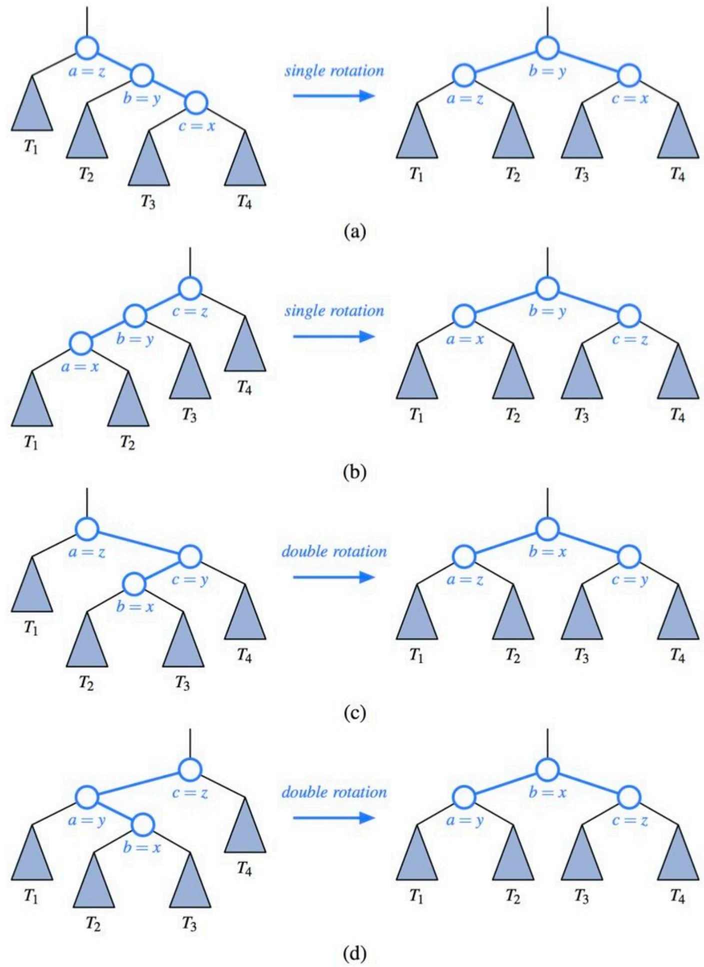

Figure 11.9: Schematic illustration of a trinode restructuring operation: (a and b) require a single rotation; (c and d) require a double rotation.

Rotate:

Restructure:

```
/** Relinks a parent node with its oriented child node. */
private void relink(Node<Entry<K,V>> parent, Node<Entry<K,V>> child,
                    boolean makeLeftChild) {
  child.setParent(parent);
  if (makeLeftChild)
    parent.setLeft(child);
  else
    parent.setRight(child);
}
/** Rotates Position p above its parent. */
public void rotate(Position<Entry<K,V>> p) {
  Node<Entry<K,V>> x = validate(p);
  Node<Entry<K,V>> y = x.getParent();         // we assume this exists
  Node<Entry<K,V>> z = y.getParent();         // grandparent (possibly null)
  if (z == null) {
    root = x;                                 // x becomes root of the tree
    x.setParent(null);
  } else
    relink(z, x, y == z.getLeft());           // x becomes direct child of z
  // now rotate x and y, including transfer of middle subtree
  if (x == y.getLeft()) {
    relink(y, x.getRight(), true);            // x's right child becomes y's left
    relink(x, y, false);                      // y becomes x's right child
  } else {
    relink(y, x.getLeft(), false);            // x's left child becomes y's right
    relink(x, y, true);                       // y becomes left child of x
  }
}
/** Performs a trinode restructuring of Position x with its parent/grandparent. */
public Position<Entry<K,V>> restructure(Position<Entry<K,V>> x) {
  Position<Entry<K,V>> y = parent(x);
  Position<Entry<K,V>> z = parent(y);
  if ((x == right(y)) == (y == right(z))) {   // matching alignments
    rotate(y);                                // single rotation (of y)
    return y;                                 // y is new subtree root
  } else {                                    // opposite alignments
    rotate(x);                                // double rotation (of x)
    rotate(x);
    return x;                                 // x is new subtree root
  }
}
}
```

Code Fragment 11.10: The BalanceableBinaryTree class, which is nested within the TreeMap class definition (continued from Code Fragment 11.9).

Rebalancing operation for AVL insertions and deletions:

```java
  /**
   * Utility used to rebalance after an insert or removal operation. This traverses the
   * path upward from p, performing a trinode restructuring when imbalance is found,
   * continuing until balance is restored.
   */
  protected void rebalance(Position<Entry<K,V>> p) {
    int oldHeight, newHeight;
    do {
      oldHeight = height(p);                      // not yet recalculated if internal
      if (!isBalanced(p)) {                       // imbalance detected
        // perform trinode restructuring, setting p to resulting root,
        // and recompute new local heights after the restructuring
        p = restructure(tallerChild(tallerChild(p)));
        recomputeHeight(left(p));
        recomputeHeight(right(p));
      }
      recomputeHeight(p);
      newHeight = height(p);
      p = parent(p);
    } while (oldHeight != newHeight && p != null);
  }
  /** Overrides the TreeMap rebalancing hook that is called after an insertion. */
  protected void rebalanceInsert(Position<Entry<K,V>> p) {
    rebalance(p);
  }
  /** Overrides the TreeMap rebalancing hook that is called after a deletion. */
  protected void rebalanceDelete(Position<Entry<K,V>> p) {
    if (!isRoot(p))
      rebalance(parent(p));
  }
}
```

At this point in the class I discuss how AVL trees are implemented in the 6th edition of the textbook by Goodrich, Tamassia and Goldwasser.

Please, refer to pages:

466-470 class TreeMap<K,V>
Note methods: put(K key, V value), remove(K key)

475-478 class BalancedBinaryTree<K,V>
Note: hooks for rebalancing present in TreeMap,
Methods: rotate, restructure

486-487 class AVLTreeMap<K,V>
Note methods: rebalanceInsert, rebalanceDelete, rebalance