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The Lift on a Flat Plate between Parallel Walls.

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1. *Introduction.*

Some time ago an interesting paper was published by T. Sasaki* in which was investigated the lift on a flat plate (and aerofoil) in an N.P.L. type of wind tunnel—a wind tunnel which is bounded by plane walls. The second part of the paper dealt with the case of a flat plate in an unbounded jet of air—which is equivalent to the Göttingen and Eiffel type of wind tunnel. In all cases the obstacle was of infinite span, so that the flow was two-dimensional. The flow was everywhere continuous and the problems resolved themselves into the discovery of suitable conformal transformations. This was done by the application of a method developed by H. Villat.† An independent check of the work on the Göttingen and Eiffel type of wind tunnel was recently obtained by von Karman‡ who developed an approximate theory, and whose results are in close agreement with those of Sasaki. In connection with the work on the flat plate and aerofoil between parallel walls Mr. Glauert pointed out that the results did not agree with those derived from an approximate theory. The difficulty was that the results of Sasaki indicated a loss of lift due to the presence of the walls, whereas a simple calculation made several years ago suggested a small increase of lift when the walls were far apart and the angle of incidence small. It was expected that the exact solution would give Glauert's result as a limiting value. As there was no such agreement, it was suggested that the problem might be re-investigated, and I am indebted to Mr. Glauert for this suggestion.

An initial check was applied by obtaining the limiting forms of some of the conformal transformations, and it was found that they did not agree with the well-known expressions for the continuous flow past a flat plate in an unbounded stream.

* "On the Effect of the Walls of a Wind Tunnel upon the Lift Coefficient of a Model," Report of the Aeronautical Research Institute, Tokyo,' No. 46 (1928).

† "Aperçus théoriques sur la Résistance des Fluides," 'Scientia,' No. 38 (1920).

‡ "Vorträge aus dem Gebiete der Aerodynamik und verwandter Gebiete," Aachen, p. 95 (1929). J. Springer.

In this paper we confine ourselves to a consideration of a flat plate between parallel walls, for aerofoils are usually used at small angles of incidence, and at these angles of incidence the corrections of lift for a flat plate and aerofoil are almost identical. Further, it is indicated how the problem of a plate in the neighbourhood of a wall can be solved by taking an appropriate limiting value of the results of the more general problem.

Section 2 gives a brief summary of results, and also a list of various formulæ scattered throughout the paper. Section 3 gives the method of Glauert's approximation, the results of which, it is seen, agree with limiting values obtained from the exact solution when the angle of incidence is small and the plate is in the neighbourhood of the centre of the stream. In later paragraphs the divergence from Sasaki's analysis is noted and a brief explanation is given of the reason why the original work is correct for the unbounded jet of air, but not for the bounded stream.

2. *Summary of Results.*

The problem of continuous flow past a flat plate of infinite span between plane parallel walls is solved. A circulation is superposed to make the velocity at the trailing edge finite and the effect of the walls on the lift coefficient is obtained. To a first approximation the results agree with those obtained by Glauert's theory. It is shown that the effect of the walls is to increase the lift coefficient, and curves and tables are given showing this increase for various values of the angle of attack and the ratio of chord of aerofoil to width of channel. It is also shown that the solution of the problem of a plate in the neighbourhood of a single plane wall can be obtained from the general result by a suitable limiting process.

A list of symbols, and of expressions that may be of practical value, are given below :—

l = length of plate, or chord of aerofoil.
D = distance between channel walls.
h = distance between mid-point of plate and the centre line of the channel.
α = angle of incidence of the plate.
κ = circulation round the plate.
ρ, U = density, velocity at infinity, of the fluid.
L = lift on aerofoil (= y component of resultant force on aerofoil = Y).
k_L = lift coefficient of flat plate in an unbounded stream (= (lift)/$\rho l U^2 = \pi \sin \alpha$).

Δk_L = increase of lift coefficient due to the channel walls.
k_m = moment coefficient.
k_{m0} = moment coefficient at zero lift.

$$\delta = \tfrac{1}{2}\pi - \alpha\,; \qquad h' = \tfrac{1}{2}\mathrm{D} - h.$$

I.—Results of Glauert's Approximate Theory.

For a monoplane between channel walls—

(*a*) When $h = 0$, l/D is small, α is small,

$$\Delta k_L = \frac{\pi^2}{24}\left(\frac{l}{\mathrm{D}}\right)^2 (k_L + 2k_{m0}).$$

(*b*) When $l/(\mathrm{D} - 2h)$ is small, α is small,

$$\Delta k_L = \frac{\pi^2}{16}\left(\frac{l}{\mathrm{D}}\right)^2 \left(\sec^2\frac{\pi h}{\mathrm{D}} - \tfrac{1}{3}\right)(k_L + 2k_{m0}).$$

II.—Results of General Theory.

For a monoplane between channel walls—

(*a*) The exact formulæ are given by equations (5.21 to 5.27), (6.2), (6.5).

(*b*) When $h = 0$, l/D is small,

$$\Delta k_L/k_L = \frac{\pi^2}{24}(1 + \sin^2\alpha)\left(\frac{l}{\mathrm{D}}\right)^2 - \frac{\pi^4}{15{,}360}(22 - 121\sin^2\alpha - 14\sin^4\alpha)\left(\frac{l}{\mathrm{D}}\right)^4,$$

$$= 0\cdot4112\,(1 + \sin^2\alpha)\left(\frac{l}{\mathrm{D}}\right)^2 - 0\cdot1395\,(1 - 5\cdot5\sin^4\alpha - 0\cdot6364\sin^4\alpha)\left(\frac{l}{\mathrm{D}}\right)^4.$$

(*c*) When l/D is small, $0 \leqslant h \leqslant \mathrm{D}/2$.

$$\Delta k_L/k_L = \left\{\pi\sin\alpha\tan\frac{\pi h}{\mathrm{D}}\right\}\frac{l}{\mathrm{D}} + \frac{\pi^2}{24}\sec^2\frac{\pi h}{\mathrm{D}}\left\{(1 + \sin^2\alpha) + \tfrac{1}{2}\sin^2\frac{\pi h}{\mathrm{D}}(1 + 22\sin^2\alpha)\right\}\left(\frac{l}{\mathrm{D}}\right)^2,$$

and if in addition α is small, we have

$$\Delta k_L/k_L = \frac{\pi^2}{16}\left\{\sec^2\frac{\pi h}{\mathrm{D}} - \tfrac{1}{3}\right\}\left(\frac{l}{\mathrm{D}}\right)^2 + \left\{\pi\tan\frac{\pi h}{\mathrm{D}}\cdot\frac{l}{\mathrm{D}}\right\}\alpha,$$

and if also h/D is small,

$$\Delta k_L/k_L = \frac{\pi^2}{24}\left(\frac{l}{\mathrm{D}}\right)^2.$$

(*d*) When α is small, $h = 0$, l/D any value,

$$\Delta k_{\mathrm{L}}/k_{\mathrm{L}} = \left[\tfrac{1}{4}\,\frac{\vartheta_2{}^4(0)}{\log \vartheta_3(0)/\vartheta_4(0)} - 1\right],$$

where

$$l/\mathrm{D} = \frac{2}{\pi}\log \vartheta_3(0)/\vartheta_4(0).$$

When l/D is small this becomes

$$\Delta k_{\mathrm{L}}/k_{\mathrm{L}} = \frac{\pi^2}{24}\left(\frac{l}{\mathrm{D}}\right)^2 - \frac{11\pi^4}{7680}\left(\frac{l}{\mathrm{D}}\right)^4 = 0\cdot4112\left(\frac{l}{\mathrm{D}}\right)^2 - 0\cdot1395\left(\frac{l}{\mathrm{D}}\right)^4.$$

3. *Glauert's Approximation.**

The basis of the approximation follows a line of argument published previously,† but it can be described briefly as follows. In the first place, the effect of the straight walls on the flow past any body can be accurately represented by a suitable series of images of the body, and the flow can then be determined without any further reference to the walls, for they will automatically emerge as stream lines of the system. Secondly, if these images are far away (that is l/D is small) the interference they exert on the body can be calculated with sufficient accuracy by replacing each image by a point vortex at its centre of pressure. The interference itself can be represented with sufficient accuracy by the downwash w and by the curvature of the stream lines dw/dx—that is, by a change of the effective incidence and of the effective camber of the aerofoil.

The above principles are stated in the reference given above, and the results are worked out in some detail for the mutual interference of the wings of a biplane. The method can be extended quite simply and the interference experienced by the aerofoil (assumed a straight line) can be represented by the velocity induced at its centre and by the curvature of the stream lines in its neighbourhood. In order to maintain the same lift the incidence must be reduced by

$$\Delta\alpha = 2\sum_{n=1}^{\infty}\frac{(-)^{n+1}}{4\pi}\left(\frac{l}{n\mathrm{D}}\right)^2(k_{\mathrm{L}} + 2k_{m0}) = \frac{\pi}{24}\left(\frac{l}{\mathrm{D}}\right)^2(k_{\mathrm{L}} + 2k_{m0}), \tag{1}$$

where k_{m0} is the value of the moment coefficient at zero lift. At the same angle of incidence therefore, the lift coefficient would be increased by

$$\Delta k_{\mathrm{L}} = \pi\,\Delta\alpha = \frac{\pi^2}{24}\left(\frac{l}{\mathrm{D}}\right)^2(k_{\mathrm{L}} + 2k_{m0}). \tag{2}$$

* I am indebted to Mr. Glauert for permission to reproduce the material of this section.

† Glauert, 'Elements of Aerofoil and Airscrew Theory' (C.U.P. 1930), pp. 177–179.

This approximate method may also be applied to the case where the aerofoil is displaced from the centre of the channel. The condition of the simpler problem, that l/D must be small, is now replaced by the condition that $l/(\mathrm{D} - 2h)$ must be small, and the angle of incidence is small as before. We can again replace each image of the aerofoil by a vortex, and a summation of the effects of these image vortices gives us

$$\Delta k_{\mathrm{L}} = \frac{\pi^2}{16}\left(\sec^2 \frac{\pi h}{\mathrm{D}} - \tfrac{1}{3}\right)\left(\frac{l}{\mathrm{D}}\right)^2 (k_{\mathrm{L}} + 2k_{m0}). \tag{3}$$

Formulæ (2) and (3) are limiting forms of expressions that will be obtained later.

4. *The Conformal Transformations.*

The exact solution is obtained by means of the following transformations. Let $z = x + iy$ where (x, y) are rectangular co-ordinates in the plane of motion. The z plane is shown in fig. 1.

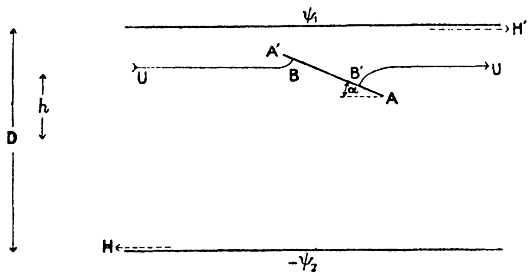

Fig. 1.—z-plane.

The velocity at infinity is U and the streaming function is $f = \phi + i\psi$, where ϕ is the velocity potential and ψ the stream function. The value of ψ on the upper and lower boundaries is ψ_1 and $-\psi_2$, so that $\mathrm{UD} = \psi_1 + \psi_2$. The f-plane is shown in fig. 2.

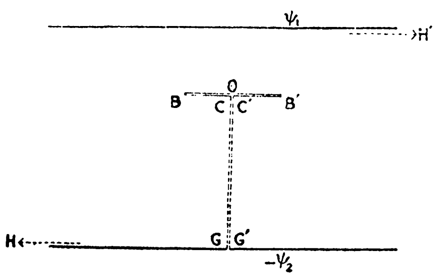

Fig. 2.—f-plane.

By making a cut along CGG′C′ as above, the f-plane can be transformed on to the upper half of a t-plane by the relation

$$\frac{df}{dt} = \mathrm{M}\frac{t^2 - b^2}{(t^2 - h^2)\sqrt{(t^2 - c^2)(t^2 - g^2)}}. \tag{1}$$

where b corresponds to B, $-b$ to B′, and so on. The constant M is so chosen that when t passes through h the ordinate changes suddenly by $\psi_1 + \psi_2$, and we therefore get

$$\mathrm{M} = \frac{\psi_1 + \psi_2}{\pi}\frac{2h\sqrt{(h^2 - c^2)(h^2 - g^2)}}{h^2 - b^2}. \tag{2}$$

The t-plane is shown in fig. 3.

The upper half of this t-plane is transformed conformally into a rectangle of sides $2\omega_1$ and ω_3/i in an s-plane by the relation

$$t^2 = \wp(s) - e_3. \tag{3}$$

The points $t = b, c, g, h$ are transformed into the points $s = \mu$, $\omega_1 + \omega_3$, ω_1, ν and the s-plane is shown in fig. 4.

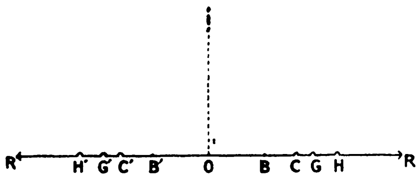

Fig. 3.—t-plane.

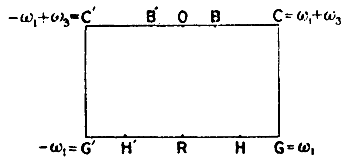

Fig. 4.—s-plane.

This gives us

$$\frac{df}{ds} = \frac{\psi_1 + \psi_2}{\pi}\frac{\wp'(\nu)}{\wp(\nu) - \wp(\mu)}\frac{\wp(s) - \wp(\mu)}{\wp(s) - \wp(\nu)}$$

$$= \frac{\psi_1 + \psi_2}{\pi}[\zeta(\mu + \nu) - \zeta(\mu - \nu) - \zeta(s + \nu) + \zeta(s - \nu)]. \tag{4}$$

This equation can be integrated immediately, and if we choose the constant of integration so that $f = 0$ at $s = \mu$, we get

$$f = \frac{\psi_1 + \psi_2}{\pi}\left\{[\zeta(\mu + \nu) - \zeta(\mu - \nu)](s - \mu) - \log\frac{\sigma(s + \nu)\,\sigma(\mu - \nu)}{\sigma(s - \nu)\,\sigma(\mu + \nu)}\right\}. \tag{5}$$

The fact that the circulation round the plate is zero gives a relation between the constants. (The circulation κ will be accounted for later by superposing a rotational flow.) Since the circulation is zero, f has a period $2\omega_1$, and we obtain the condition

$$\zeta(\mu+\nu)-\zeta(\mu-\nu)=2\eta_1\nu/\omega_1. \tag{6}$$

Another relation is obtained from the condition that f differs by $i\psi_2$ at $s=\omega_1$ and $s=\omega_1+\omega_3$, so that we get

$$f_C-f_G=i\psi_2=\frac{\psi_1+\psi_2}{\pi}\{[\zeta(\mu+\nu)-\zeta(\mu-\nu)]\,\omega_3-2\eta_3\nu\},$$

giving

$$\nu=\frac{\psi_2}{\psi_1+\psi_2}\,\omega_1. \tag{7}$$

The inside of the rectangle in the s-plane is transformed conformally into a ring region in a Z-plane bounded by two concentric circles of radii 1 and $q\left(=\exp.\left(i\pi\,\dfrac{\omega_3}{\omega_1}\right)<1\right)$, by the relation

$$s=\omega_1+\omega_3-\frac{\omega_1}{\iota\pi}\log Z. \tag{8}$$

The face of the inclined plate corresponds to the outer circle and those of the two parallel planes to the inner circle. The various points are transformed as follows :—

$$\left.\begin{array}{l} A=e^{\iota\theta_4};\quad B=e^{\iota\theta_1};\quad A'=e^{\iota\theta_3};\quad B'=e^{\iota(2\pi-\theta_1)}; \\ H=qe^{\iota\theta_2};\quad H'=qe^{\iota(2\pi-\theta_2)};\quad \mu=\omega_3+\omega_1\left(1-\dfrac{\theta_1}{\pi}\right);\quad \nu=\omega_1\left(1-\dfrac{\theta_2}{\pi}\right)\end{array}\right\}. \tag{9}$$

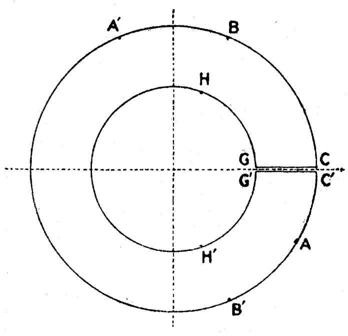

FIG. 5.—Z-plane.

We now connect the f and z-planes by the relation

$$\frac{df}{dz}=e^{-\iota\Omega}, \tag{10}$$

where

$$\Omega=i\log df/dz=\theta-i\log q_1,$$

θ being the direction of velocity and q_1 its absolute magnitude. If we can express the direction of the stream along the boundaries as a function of the central angle in the Z-plane, we can calculate the function Ω at any point in the ring region by Villat's formula (*loc. cit.*, p. 16),

$$\Omega(Z) = \frac{i\omega_1}{\pi^2}\int_0^{2\pi} \Phi(\theta)\, \zeta\left(\frac{\omega_1}{i\pi}\log Z - \frac{\omega_1}{\pi}\theta\right) d\theta$$

$$- \frac{i\omega_1}{\pi^2}\int_0^{2\pi} \Psi(\theta)\, \zeta_3\left(\frac{\omega_1}{i\pi}\log Z - \frac{\omega_1}{\pi}\theta\right) d\theta, \tag{11}$$

with the condition

$$\int_0^{2\pi} \Phi(\theta)\, d\theta = \int_0^{2\pi} \Psi(\theta)\, d\theta, \tag{12}$$

where $\Phi(\theta)$ is the angle which the direction of flow along the face of the inclined plate makes with the positive direction of the x-axis, expressed as a function of the central angle θ in the Z-plane; and $\Psi(\theta)$, the corresponding function for the two parallel boundaries, is zero, as the flow along them makes a zero angle with the positive direction of the x-axis. Hence the *real* part of Ω is defined for the boundaries, and we can use Villat's method to determine a function satisfying these conditions.

Ω, however, is not unique, for if Ω_1 is a solution, then $\Omega_1 + i\mathrm{P}$, where P is a purely real constant, will also satisfy the boundary conditions. In fact, Ω is uniquely defined *except for a purely imaginary constant*. The effect of this constant is to multiply all the velocities in the flow by some real number—as can be seen from (10). This velocity must be chosen so as to make the velocity in the z-plane equal to U, at infinity. In the original* work, it was assumed that Ω was uniquely determined by Villat's method, and as a result errors were introduced. The multiplying factor is itself a function of l/D and α, so that the ordinates in the Sasaki curve of lift (fig. 14 of his paper) must be multiplied by a number which varies as we go along the curve, and it will be seen that this multiplying factor alters entirely the nature of the curve. The results of the investigation on a plate in a jet of air need no correcting factor, for a modified form of Villat's equation was used, the modified form being so determined that the imaginary part of Ω is zero over the free surface of the jet. This corresponds to systems in which the velocity is unity over the whole free surface, and therefore at infinity too.

The expression for Ω (Z) is valid over the whole ring region between the two

* Sasaki, *loc. cit.*

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circles, and on the circles themselves—because the singularities which occur on the boundaries correspond to singularities in the flow.

Condition (12) therefore is $\int_0^{2\pi} \Phi(\theta)\, d\theta = 0$, or

$$\int_{\theta_4}^{\theta_1} (-\tfrac{1}{2}\pi + \delta)\, d\theta + \int_{\theta_1}^{\theta_3} (\tfrac{1}{2}\pi + \delta)\, d\theta + \int_{\theta_3}^{2\pi-\theta_1} (-\tfrac{1}{2}\pi + \delta)\, d\theta + \int_{2\pi-\theta_1}^{2\pi+\theta_4} (\tfrac{1}{2}\pi + \delta)\, d\theta = 0,$$

or

$$\theta_3 + \theta_4 = \pi - 2\delta = 2\alpha. \tag{13}$$

We therefore find

$$\frac{dz}{ds} = \frac{dz}{df} \cdot \frac{df}{ds} = e^{i\Omega} \frac{df}{ds}$$

$$= -\,\mathrm{C}' \exp\left[\frac{2}{\pi}(\eta_1 s - \eta_3\omega_1)(\tfrac{1}{2}\pi + \delta) + 2\eta_3 s\right] \frac{\sigma(s - s_3)\,\sigma(s - s_4)}{\sigma(s + \nu)\,\sigma(s - \nu)}$$

$$= \mathrm{F}(s), \tag{14}$$

where

$$\left.\begin{aligned} s_3 &= \omega_3 + \omega_1 - \frac{\omega_1}{\pi}\theta_3 \\ s_4 &= \omega_3 + \omega_1 - \frac{\omega_1}{\pi}\theta_4 \end{aligned}\right\}, \tag{15}$$

and where C′ is the arbitrary multiplying factor mentioned above. If we test this function we find that it gives the correct slopes on the boundaries. The expression for dz/ds must now be integrated.

5. *Development of the Transformations.*

The addition formulæ for the periodic functions give

$$\left.\begin{aligned} \mathrm{F}(s + 2\omega_1) &= \mathrm{F}(s), \\ \mathrm{F}(s + 2\omega_3) &= \exp\left[2i(\tfrac{1}{2}\pi + \delta)\right]\mathrm{F}(s), \end{aligned}\right\}, \tag{1}$$

and hence F (s) is a "doubly periodic function of the second kind" with simple poles at $s = \pm\nu$. It can, therefore, be split up into simple elements by introducing a function A(s) where

$$\mathrm{A}(s) = -\frac{\sigma\left[s - \dfrac{2\omega_1}{\pi}\left(\dfrac{\pi}{2} + \delta\right)\right]}{\sigma(s)\,\sigma\left[\dfrac{2\omega_1}{\pi}\left(\dfrac{\pi}{2} + \delta\right)\right]} e^{\frac{2\eta_1}{\pi}\left(\frac{\pi}{2}+\delta\right)s}$$

$$= \frac{\vartheta_1'(0)}{2\omega_1\vartheta_2\left(\dfrac{\delta}{\pi}\right)} \frac{\vartheta_2\left[\dfrac{s}{2\omega_1} - \dfrac{\delta}{\pi}\right]}{\vartheta_1\left[\dfrac{s}{2\omega_1}\right]}, \tag{2}$$

and as a result of this the function $F(s)$ can be integrated. The law for the decomposition of $F(s)$ is

$$F(s) = C_{\nu} A(s-\nu) + C_{-\nu} A(s+\nu), \tag{3}$$

where C_{ν} and $C_{-\nu}$ are the residues of the poles at $+\nu$ and $-\nu$ respectively. We find

$$\left.\begin{aligned} C_{\nu} &= -C' \frac{\sigma^2(\omega_3)}{\vartheta_4{}^2(0)\,\sigma(2\nu)} \exp\left[\frac{\eta_1\omega_1}{2\pi^2}\left(\theta_3{}^2+\theta_4{}^2+4\pi\delta+\frac{2\pi^2\nu^2}{\omega_1{}^2}\right)\right] \\ &\qquad\qquad \times \vartheta_4\left(\frac{\theta_3-\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4-\theta_2}{2\pi}\right) \\ C_{-\nu} &= C' \frac{\sigma^2(\omega_3)}{\vartheta_4{}^2(0)\,\sigma(2\nu)} \exp\left[\frac{\eta_1\omega_1}{2\pi^2}\left(\theta_3{}^2+\theta_4{}^2+4\pi\delta+\frac{2\pi^2\nu^2}{\omega_1{}^2}\right)\right] \\ &\qquad\qquad \times \vartheta_4\left(\frac{\theta_3+\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4+\theta_2}{2\pi}\right) \end{aligned}\right\}. \tag{4}$$

From equation (4.14) we see that as s passes through ν (that is the point H in the z diagram) there is a discontinuity in z of $i\pi C_{\nu}$, and as s passes through $-\nu$ there is discontinuity of $i\pi C_{-\nu}$. From fig. 1 we see that these discontinuities must be equal to $+i$D and $-i$D respectively. Hence an essential condition for the consistency of the system is

$$\vartheta_4\left(\frac{\theta_3-\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4-\theta_2}{2\pi}\right) = \vartheta_4\left(\frac{\theta_3+\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4+\theta_2}{2\pi}\right). \tag{5}$$

When this is satisfied the equation

$$C_{\nu} = D/\pi \tag{6}$$

may be used as one which determines the hitherto arbitrary constant C', and we get

$$\frac{1}{C'} = -\frac{\pi}{D}\frac{\sigma^2(\omega_3)}{\vartheta_4{}^2(0)\,\sigma(2\nu)} \exp\left[\frac{\eta_1\omega_1}{2\pi^2}\left(\theta_3{}^2+\theta_4{}^2+4\pi\delta+\frac{2\pi^2\nu^2}{\omega_1{}^2}\right)\right] \times \vartheta_4\left(\frac{\theta_3-\theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4-\theta_2}{2\pi}\right), \tag{7}$$

or we may put quite simply

$$F(s) = \frac{D}{\pi}\left[A(s-\nu) - A(s+\nu)\right] \tag{8}$$

combined with condition (5). This ensures that as we go round the rectangle in the s-plane, we arrive at that point in the z-plane from which we started, and it also ensures that the velocity at infinity is U in all cases.

It should be noted that when the mid-point of the plate AA′ is on the centre line of the channel in fig. 1, then $\psi_1 = \psi_2$, and from (4.7) we get $\nu = \frac{1}{2}\omega_1$, and if this is inserted in (4.9) we get $\theta_2 = \frac{1}{2}\pi$. The insertion of this value for θ_2 in (5) gives the relation

$$\theta_3 - \theta_4 = \pi. \tag{9}$$

This relation is only true *if* the mid-point of the plate AA′ is on the centre line of the channel. Sasaki assumes without proof that this relation is true generally, but as the reasoning is not quite clear, condition (5) will be taken as the relation in the most general case.

Reverting to equation (8) we find that when a complex quantity p satisfies the relation

$$-\,\mathrm{R}\,(\tau/i) < 2\mathrm{R}\left(\left[\frac{p}{2\omega_1} - \frac{\tau}{2}\right]\Big/ i\right) < \mathrm{R}\,(\tau/i), \tag{10}$$

where R (z) means "the real part of z," the following expansion is valid for A (p)*

$$\mathrm{A}\,(p) = \frac{1}{2\omega_1}\,\frac{\vartheta_1{}'(0)\,\vartheta_3\left[\dfrac{p}{2\omega_1} - \dfrac{1+\tau}{2} - \dfrac{1}{\pi}\left(\dfrac{\pi}{2}+\delta\right)\right]}{\vartheta_3\left[\dfrac{p}{2\omega_1} - \dfrac{1+\tau}{2}\right]\vartheta_1\left[-\dfrac{1}{\pi}\left(\dfrac{\pi}{2}+\delta\right)\right]}\,e^{\iota(\frac{1}{2}\pi+\delta)}$$

$$= \frac{2\pi}{\omega_1}\,e^{i(\frac{1}{2}\pi+\delta)}\left[\cdot - \frac{1}{4\,\sin\left(\dfrac{\pi}{2}+\delta\right)}\right.$$

$$\left. + \sum_{n=1}^{\infty}(-)^n q^n\,\frac{\sin\pi\left[2n\left(\dfrac{p}{2\omega_1}-\dfrac{1+\tau}{2}\right)-\dfrac{1}{\pi}\left(\dfrac{\pi}{2}+\delta\right)\right]-q^{2n}\sin\pi\left[2n\left(\dfrac{p}{2\omega_1}-\dfrac{1+\tau}{2}\right)+\dfrac{1}{\pi}\left(\dfrac{\pi}{2}+\delta\right)\right]}{1-2q^{2n}\cos 2\left(\dfrac{\pi}{2}+\delta\right)+q^{4n}}\right]. \tag{11}$$

In this expression for A(p) put $p = (s - \nu)$ and $p = (s + \nu)$ consecutively and subtract. We then get

$$\mathrm{F}\,(s) = \frac{\mathrm{D}}{\pi}\cdot\frac{2\pi}{\omega_1}\,e^{i(\frac{1}{2}\pi+\delta)}\sum_{n=1}^{\infty}\frac{i\,(-)^{n+1}q^n\sin\dfrac{n\nu\pi}{\omega_1}}{1+2q^{2n}\cos 2\delta+q^{4n}}$$

$$\times\,[\mathrm{Z}^n\,(e^{\iota\delta}+q^{2n}e^{-\iota\delta}) - \mathrm{Z}^{-n}\,(e^{-\iota\delta}+q^{2n}e^{\iota\delta})]. \tag{12}$$

* See Tannery and Molk, vol. 4, p. 104.

and where s has been transformed to Z by means of equation (4.8), for the sake of convenience. We now have

$$z = \int \frac{dz}{ds} \cdot ds = \int \mathrm{F}(s) \cdot \frac{ds}{d\mathrm{Z}} \cdot d\mathrm{Z} = -\frac{\omega_1}{i\pi} \int \mathrm{F}(s) \cdot \frac{d\mathrm{Z}}{\mathrm{Z}}$$

$$= -\frac{2\mathrm{D}}{\pi} e^{i(\frac{1}{2}\pi+\delta)} \sum_{n=1}^{\infty} \frac{(-)^{n+1} q^n \sin \dfrac{n\nu\pi}{\omega_1}}{n(1 + 2q^{2n} \cos 2\delta + q^{4n}}$$
$$\times [\mathrm{Z}^n (e^{i\delta} + q^{2n} e^{-i\delta}) + \mathrm{Z}^{-n} (e^{-i\delta} + q^{2n} e^{i\delta})]. \quad (13)$$

To this expression must be added an arbitrary constant, which can, however, be neglected.

We know that

$$z_{\mathrm{A}'} - z_{\mathrm{A}} = l\, e^{i(\frac{1}{2}\pi+\delta)},$$

and if we put $\mathrm{Z} = e^{i\theta_3}$, and $e^{i\theta_4}$ in equation (13), and then subtract, we get

$$l = \frac{8\mathrm{D}}{\pi} \sum_{n=1}^{\infty} \frac{q^n \sin n\theta_2 \sin \dfrac{n}{2}(\theta_3 - \theta_4)}{n(1 - 2q^{2n} \cos 2\alpha + q^{4n})} [\cos (n-1)\alpha - q^{2n} \cos (n+1)\alpha], \quad (14)$$

where

$$\alpha = \tfrac{1}{2}\pi - \delta.$$

From (13) it is not immediately evident that we get the boundary walls of the channel by putting $\mathrm{Z} = qe^{i\theta}$. This is due to the expansion adopted for $\mathrm{A}(p)$. If, however, the terms involving Z are grouped rather differently, that is, if we adopt an alternative expansion for $\mathrm{A}(p)$ the above property becomes evident, as follows :—

We note that

$$\mathrm{A}(p) = \frac{2\pi}{\omega_1} \left\{ \frac{\vartheta_1'(0)}{4\pi} \frac{\vartheta_1\left[\dfrac{p}{2\omega_1} + \dfrac{\alpha}{\pi}\right]}{\vartheta_1\left[\dfrac{\alpha}{\pi}\right] \vartheta_1\left[\dfrac{p}{2\omega_1}\right]} \right\}, \quad (15)$$

and from Tannery and Molk, vol. 4, p. 102, this can be expanded as

$$\mathrm{A}(p) = \frac{2\pi}{\omega_1} \left\{ \tfrac{1}{4} \cot \frac{\pi p}{2\omega_1} + \tfrac{1}{4} \cot \alpha \right.$$
$$\left. + \sum_{n=1}^{\infty} \frac{q^{2n} \sin 2\pi \left(\dfrac{np}{2\omega_1} + \dfrac{\alpha}{\pi}\right) - q^{4n} \sin \dfrac{n\pi p}{\omega_1}}{1 - 2q^{2n} \cos 2\alpha + q^{4n}} \right\}, \quad (16)$$

since p, which is going to be put equal to $(s - \nu)$ and $(s + \nu)$, satisfies the relation

$$-\mathrm{R}(\tau/i) < \mathrm{R}(p/i) < \mathrm{R}(\tau/i).$$

Applying this expansion to (8) we get

$$\mathrm{F}(s) = \frac{2\mathrm{D}}{\omega_1}\left[\frac{\mathrm{Z}q\sin\theta_2}{\mathrm{Z}^2 - 2\mathrm{Z}q\cos\theta_2 + q^2} + \sum_{n=1}^{\infty}\frac{q^{2n}\sin n\theta_2}{1 - 2q^{2n}\cos 2\alpha + q^{4n}}\left\{\left(\frac{\mathrm{Z}}{q}\right)^n(e^{-2\iota\alpha} - q^{2n}) + \left(\frac{q}{\mathrm{Z}}\right)^n(e^{2\iota\alpha} - q^{2n})\right\}\right]. \tag{17}$$

Inserting this in the equation

$$z = -\frac{\omega_1}{i\pi}\int \mathrm{F}(s)\,\frac{d\mathrm{Z}}{\mathrm{Z}},$$

we get

$$z = i\,\frac{2\mathrm{D}}{\pi}\left[\frac{1}{2i}\log\frac{\mathrm{Z} - qe^{\iota\theta_2}}{\mathrm{Z} - qe^{-\iota\theta_2}} + \sum_{n=1}^{\infty}\frac{q^{2n}\sin n\theta_2}{n(1 - 2q^{2n}\cos 2\alpha + q^{4n})}\left\{\left(\frac{\mathrm{Z}}{q}\right)^n(e^{-2\iota\alpha} - q^{2n}) - \left(\frac{q}{\mathrm{Z}}\right)^n(e^{2i\alpha} - q^{2n})\right\}\right]. \tag{18}$$

From this equation we wish to show that when $z = qe^{i\theta}$, the real part of z/i is independent of θ. It is at once evident that the expression within the summation sign is the difference of two conjugate complexes, and therefore a pure imaginary—which we can ignore. It can also be shown quite simply that when

$$(2\pi - \theta_2) > \theta > \theta_2, \quad \log\frac{e^{\iota\theta} - e^{\iota\theta_2}}{e^{\iota\theta} - e^{-\iota\theta_2}} = \log\left|\frac{\sin\frac{1}{2}(\theta - \theta_2)}{\sin\frac{1}{2}(\theta + \theta_2)}\right| + i\theta_2,$$

and when

$$\theta_2 > \theta > -\theta_2, \quad \log\frac{e^{\iota\theta} - e^{\iota\theta_2}}{e^{\iota\theta} - e^{-\iota\theta_2}} = \log\left|\frac{\sin\frac{1}{2}(\theta - \theta_2)}{\sin\frac{1}{2}(\theta + \theta_2)}\right| + i(\theta_2 - \pi),$$

so that the first of these ranges corresponds to

$$\left.\begin{aligned} y &= \frac{\mathrm{D}}{\pi}\theta_2, \\ \text{and the second to}\quad y &= \frac{\mathrm{D}}{\pi}\theta_2 - \mathrm{D} \end{aligned}\right\}. \tag{19}$$

By means of (13) we see that the co-ordinates of the mid-point of AA′ are given by

$$z_m = \tfrac{1}{2}(z_{\mathrm{A}} + z_{\mathrm{A}'})$$
$$= \frac{4\mathrm{D}}{\pi} e^{\iota(\frac{1}{2}\pi+\delta)} \sum_{n=1}^{\infty} \frac{q^n \sin n\theta_2 \cos \frac{1}{2}n(\theta_3 - \theta_4)}{n(1 - 2q^{2n}\cos 2\alpha + q^{4n})} \times [\sin(n-1)\alpha - q^{2n}\sin(n+1)\alpha].$$

Hence h, the distance of the mid-point of the plate from the centre line of the channel, is given by

$$h = \frac{\mathrm{D}}{\pi}(\tfrac{1}{2}\pi - \theta_2) + \frac{4\mathrm{D}}{\pi}\sin\alpha \sum_{n=1}^{\infty} \frac{q^n \sin n\theta_2 \cos \frac{1}{2}n(\theta_3 - \theta_4)}{n(1 - 2q^{2n}\cos 2\alpha + q^{4n})} \times [\sin(n-1)\alpha - q^{2n}\sin(n+1)\alpha]. \tag{20}$$

Equations (14) and (20) give the ratios l/D and h/D in terms of the inclination α and the other quantities.

At this point it is convenient to collect all those relations between the parameters that are going to be of use in the future work. First come four relations between θ_1, θ_2, θ_3, θ_4, which suffice to determine these quantities uniquely. They are

I. $$\theta_3 + \theta_4 = \pi - 2\delta = 2\alpha; \tag{21}$$

II. $$\nu = \frac{\psi_2}{\psi_1 + \psi_2}\omega_1, \quad \text{or} \quad \frac{\theta_2}{\pi} = \frac{\psi_1}{\psi_1 + \psi_2}; \tag{22}$$

III. $$\zeta(\mu + \nu) - \zeta(\mu - \nu) = \frac{2\eta_1\nu}{\omega_1},$$

or

$$\frac{\vartheta_4'\left(\dfrac{\theta_1 + \theta_2}{2\pi}\right)}{\vartheta_4\left(\dfrac{\theta_1 + \theta_2}{2\pi}\right)} - \frac{\vartheta_4'\left(\dfrac{\theta_1 + \theta_2}{2\pi}\right)}{\vartheta_4\left(\dfrac{\theta_1 - \theta_2}{2\pi}\right)} = 0,$$

or

$$8\pi \sum_{n=1}^{\infty} \frac{q^n}{1 - q^{2n}} \cos n\theta_1 \sin n\theta_2 = 0; \tag{23}$$

IV. $$\vartheta_4\left(\frac{\theta_3 - \theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4 - \theta_2}{2\pi}\right) = \vartheta_4\left(\frac{\theta_3 + \theta_2}{2\pi}\right)\vartheta_4\left(\frac{\theta_4 + \theta_2}{2\pi}\right). \tag{24}$$

Then come the following three relations

$$\psi_1 + \psi_2 = \mathrm{UD}; \tag{25}$$

$$l/\mathrm{D} = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{q^n \sin n\theta_2 \sin \frac{1}{2}n(\theta_3 - \theta_4)}{n(1 - 2q^{2n} \cos 2\alpha + q^{4n})} [\cos(n-1)\alpha - q^{2n}\cos(n+1)\alpha]; \quad (26)$$

$$h/\mathrm{D} = \frac{1}{\pi}[\tfrac{1}{2}\pi - \theta_2] + \frac{4\sin\alpha}{\pi} \sum_{n=1}^{\infty} \frac{q^n \sin n\theta_2 \cos \frac{1}{2}n(\theta_3 - \theta_4)}{n(1 - 2q^{2n}\cos 2\alpha + q^{4n})} \times [\sin(n-1)\alpha - q^{2n}\sin(n+1)\alpha]. \quad (27)$$

6. *Circulation and Lift.*

The flow obtained by the f-function used above, namely,

$$f = \frac{\psi_1 + \psi_2}{\pi}\left\{[\zeta(\mu+\nu) - \zeta(\mu-\nu)](s-\mu) - \log\frac{\sigma(s+\nu)\,\sigma(\mu-\nu)}{\sigma(s-\nu)\,\sigma(\mu+\nu)}\right\}$$

$$= \frac{\psi_1 + \psi_2}{\pi}\left\{\frac{2\eta_1\nu}{\omega_1}(s-\mu) - \log\frac{\sigma(s+\nu)\,\sigma(\mu-\nu)}{\sigma(s-\nu)\,\sigma(\mu+\nu)}\right\}$$

$$= \frac{\mathrm{UD}}{\pi}\log\frac{\vartheta_1\left(\dfrac{s-\nu}{2\omega_1}\right)\vartheta_1\left(\dfrac{\mu+\nu}{2\omega_1}\right)}{\vartheta_1\left(\dfrac{s+\nu}{2\omega_1}\right)\vartheta_1\left(\dfrac{\mu-\nu}{2\omega_1}\right)}$$

is irrotational and satisfies the boundary conditions of the problem. It also makes the velocity at the trailing edge A (fig. 1) infinite, and the stream line does not leave the edge "smoothly." In order to avoid this we superimpose a rotational flow of circulation κ, the value of κ being determined by the condition that the velocity at A is finite. The corresponding z diagram is shown in fig. 6.

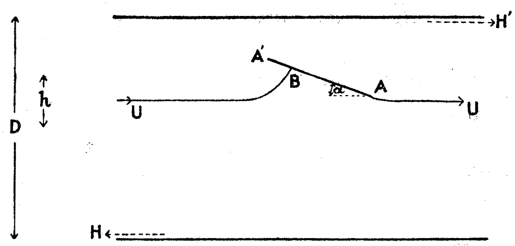

Fig. 6.—z-plane.

If f_1 is the superposed streaming function, the final streaming function w is equal to $f + f_1$. Now

$$\frac{dw}{dz} = \frac{dw}{d\mathrm{Z}}\bigg/\frac{dz}{ds}\cdot\frac{ds}{d\mathrm{Z}} = -i\,\frac{\pi}{\omega_1}\,\mathrm{Z}\,\frac{dw}{d\mathrm{Z}}\bigg/\frac{dz}{ds}.$$

At the trailing edge $s = s_4$, and dz/ds is zero. Hence dw/dZ must be zero at this point—that is, dw/dZ must be zero at $Z = e^{\iota\theta_4}$. But

$$w = f + f_1 = f - \frac{i\kappa}{2\pi}\log Z, \tag{1}$$

so that the boundaries in the Z-plane (fig. 5) are still stream lines of the system. Hence the circulation is given by the equation

$$\left(\frac{dw}{dZ}\right)_A = 0 = -\frac{\omega_1}{\iota\pi}\frac{1}{Z_4}\frac{UD}{2\pi\omega_1}\left\{\frac{\vartheta_1'\left(\dfrac{s-\nu}{2\omega_1}\right)}{\vartheta_1\left(\dfrac{s-\nu}{2\omega_1}\right)} - \frac{\vartheta_1'\left(\dfrac{s+\nu}{2\omega_1}\right)}{\vartheta_1\left(\dfrac{s+\nu}{2\omega_1}\right)}\right\}_{s_4} - \frac{i\kappa}{2\pi}\frac{1}{Z_4},$$

and so

$$\kappa = \frac{UD}{\pi}\left\{\frac{\vartheta_4'\left(\dfrac{\theta_4+\theta_2}{2\pi}\right)}{\vartheta_4\left(\dfrac{\theta_4+\theta_2}{2\pi}\right)} - \frac{\vartheta_4'\left(\dfrac{\theta_4-\theta_2}{2\pi}\right)}{\vartheta_4\left(\dfrac{\theta_4-\theta_2}{2\pi}\right)}\right\},$$

$$= 8UD\sum_{n=1}^{\infty}\frac{q^n}{1-q^{2n}}\cos n\theta_4 \sin n\theta_2. \tag{2}$$

The components of force on the plate are given by the well-known formula

$$X - iY = \tfrac{1}{2}i\rho\int\left(\frac{dw}{dz}\right)^2 . dz.$$

For convenience in the evaluation of this integral we transform the integrand in such a way that the integration takes place in the Z-plane.

$$\int\left(\frac{dw}{dz}\right)^2 dz = \int_C\left(\frac{dw}{ds}\right)^2 . \frac{ds}{dz} . \frac{ds}{dZ} . dZ = -\frac{\omega_1}{i\pi}\int_C\left(\frac{dw}{ds}\right)^2\left(\frac{ds}{dz}\right) . \frac{dZ}{Z}, \tag{3}$$

where the contour of integration in the Z-plane is any closed line surrounding the inner circle. But

$$\left(\frac{dw}{ds}\right)^2 = \left(\frac{df}{ds} + \frac{df_1}{ds}\right)^2 = \left(\frac{df}{ds}\right)^2 - \frac{\kappa}{\omega_1}\frac{df}{ds} + \frac{\kappa^2}{4\omega_1^2}. \tag{4}$$

If we put $\left(\dfrac{dw}{ds}\right)^2 . \left(\dfrac{ds}{dz}\right) = G(s)$, we find that

$$G(s + 2\omega_1) = G(s), \qquad G(s + 2\omega_3) = \exp\left[-2i\left(\tfrac{1}{2}\pi + \delta\right)\right] G(s),$$

so that $G(s)$ is a doubly periodic function of the second kind and can be split up into simple elements. The expression $(dw/ds)^2$ has double poles at the points $s = \pm\nu$, and (ds/dz) has, as shown by equation (4.14), simple zeros at $\pm\nu$ and simple poles at s_3 and s_4. Hence $G(s)$ has simple poles at $\pm\nu$, s_3 and s_4. The law of decomposition is

$$G(s) = R_\nu B(s-\nu) + R_{-\nu} B(s+\nu) + R_{s_3} B(s-s_3) + R_{s_4} B(s-s_4),$$

where R_ν, $R_{-\nu}$, etc., are the residues of $G(s)$ at ν, $-\nu$, etc., respectively, and where $B(s)$ is a doubly periodic function of the second kind with a simple pole at $s=0$, and such that

$$B(s+2\omega_1) = B(s), \qquad B(s+2\omega_3) = e^{-2i(\frac{1}{2}\pi+\delta)} B(s).$$

We note immediately that

$$\begin{aligned} B(s) &= -A(-s), \\ &= \frac{\sigma\left[s+\frac{2\omega_1}{\pi}(\frac{1}{2}\pi+\delta)\right]}{\sigma[s]\,\sigma\left[\frac{2\omega_1}{\pi}(\frac{1}{2}\pi+\delta)\right]} e^{-\frac{2\eta_1}{\pi}(\frac{1}{2}\pi+\delta)s}, \\ &= \frac{2\pi}{\omega_1} e^{\iota(\frac{1}{2}\pi+\delta)} \frac{1}{4\sin(\frac{1}{2}\pi+\delta)} + \text{positive and negative integral powers of } Z. \end{aligned}$$

Now,

$$R_\nu = \left(\frac{\psi_1+\psi_2}{\pi}\right)^2 \frac{1}{C_\nu} = \frac{D}{\pi} U^2,$$

$$R_{-\nu} = \left(\frac{\psi_1+\psi_2}{\pi}\right)^2 \frac{1}{C_{-\nu}} = -\frac{D}{\pi} U^2,$$

$$R_{s_4} = \left[\left(\frac{\kappa}{2\omega_1}\right)^2 - \frac{\kappa}{\omega_1}\left(\frac{\kappa}{2\omega_1}\right) + \frac{\kappa^2}{4\omega_1^2}\right] \times [\text{residue of } ds/dz \text{ at } s=s_4] = 0,$$

since $(df/ds)_{s_4} = \dfrac{\kappa}{2\omega_1}$. If also we make use of the property that $(df/ds)_{s_3} = -\dfrac{\kappa}{2\omega_1}$ we get

$$\begin{aligned} R_{s_3} &= \left[\left(\frac{\kappa}{2\omega_1}\right)^2 - \frac{\kappa}{\omega_1}\left(-\frac{\kappa}{2\omega_1}\right) + \frac{\kappa^2}{4\omega_1^2}\right] \times [\text{residue of } ds/dz \text{ at } s=s_3] \\ &= \frac{\kappa^2}{\omega_1^2} \times -\frac{1}{C'} \exp\left[-\frac{2}{\pi}(\eta_1 s_3 - \eta_3\omega_1)(\tfrac{1}{2}\pi+\delta) + 2\eta_3 s_3\right] \frac{\sigma(s_3+\nu)\,\sigma(s_3-\nu)}{\sigma(s_3-s_4)}. \end{aligned}$$

We note also from (3) that only the term independent of Z in $G(s)$ makes a contribution to the value of the integral, and the integral itself is equal to $2\pi i$ times this constant term. We see at once that this constant is

$$\frac{2\pi}{\omega_1} e^{\iota(\frac{1}{2}\pi+\delta)} R_{s_3}/4\sin(\tfrac{1}{2}\pi+\delta).$$

After some reduction this becomes

$$X - iY = -i\,\frac{2\rho\pi^2\kappa^2}{D\sin(\frac{1}{2}\pi+\delta)}\,\frac{\vartheta_4\left(\dfrac{\theta_3+\theta_2}{2\pi}\right)\vartheta_4\left(\dfrac{\theta_4-\theta_2}{2\pi}\right)\left[\vartheta_4\left(\dfrac{\theta_3-\theta_2}{2\pi}\right)\right]^2}{\vartheta_1\left(\dfrac{\theta_2}{\pi}\right)\vartheta_1\left(\dfrac{\theta_3-\theta_4}{2\pi}\right)[\vartheta_1'(0)]^2}.$$

It therefore follows that $X = 0$, and that

$$Y = \frac{2\rho\pi^2\kappa^2}{D\sin\alpha}\,\frac{\left[\vartheta_4\left(\dfrac{\theta_3-\theta_2}{2\pi}\right)\right]^2\vartheta_4\left(\dfrac{\theta_3+\theta_2}{2\pi}\right)\vartheta_4\left(\dfrac{\theta_4-\theta_2}{2\pi}\right)}{\left[\vartheta_1'(0)\right]^2\vartheta_1\left(\dfrac{\theta_2}{2\pi}\right)\vartheta_1\left(\dfrac{\theta_3-\theta_4}{2\pi}\right)}. \qquad (5)$$

7. *The Lift Coefficient when l/D is Small.*

The value of the coefficient of lift $k_L + \Delta k_L$ $(= Y/l\rho U^2)$ assumes a simple form in terms of l/D and α when the channel is wide and the mid-point of the plate is on the centre line of the channel. As a preliminary the limiting values, as $D \to \infty$, of several of the expressions used will be obtained, and it will be seen that they all reduce to the forms which are well known for the flow past a flat plate inclined to a stream of infinite width.

Equation (5.24), which is a necessary relation between θ_2, θ_3 and θ_4, gives on expansion as far as q^4,

$$0 = q\sin\theta_2(\sin\theta_3+\sin\theta_4) - q^2\sin(\theta_3+\theta_4)\sin 2\theta_2 - q^4\sin 2\theta_2(\sin 2\theta_3+\sin 2\theta_4). \qquad (1)$$

Taking only the first term of this expression as a first approximation, we get $\theta_3 - \theta_4 = \pi$. (This relation is exactly true if $\theta_2 = \pi/2$.) If now h/D is small enough to be neglected, we get from (5.27), $\theta_2 = \pi/2$ as an approximation of the same order, and $l/D = 8q/\pi$. The transformations for f (4.5) and z (5.13) become, as $q \to 0$,

$$f \to i\,\frac{2UDq}{\pi}\left[Z - \frac{1}{Z}\right],$$

$$z \to -\frac{2D}{\pi}\,e^{\iota(\frac{1}{2}\pi+\delta)}\,q\,[Ze^{\iota\delta} + Z^{-1}e^{-\iota\delta}].$$

The expressions transform the flow from the z-plane into the interior of a unit circle in the Z-plane. It is more usual to consider the flow as external to the unit circle—hence we here introduce the auxiliary transformations

$$Z = -i/\eta, \qquad \tfrac{1}{2}\pi + \delta = \beta,$$

and we get

$$f = \frac{Ul}{4}\left[\eta + \frac{1}{\eta}\right],$$

$$z = \frac{l}{4} e^{\iota\beta}[\eta e^{-\iota\beta} + \eta^{-1} e^{\iota\beta}],$$

$$= \frac{l}{4} e^{\iota\beta}\left[\cos\beta\left(\eta + \frac{1}{\eta}\right) - i\sin\beta\left(\eta - \frac{1}{\eta}\right)\right].$$

Hence the unit circle in the η-plane is transformed into a plate of length l at an angle α in the z-plane, and the velocity in the infinite part of the z-plane is U.

From (6.2) we see that in the limit

$$\kappa_0 \to 8\mathrm{UD}q\cos\delta = \pi l \sin\alpha, \tag{2}$$

which is the usual form, and the lift Y becomes

$$\mathrm{Y}_0 \to \frac{\rho\kappa_0^2}{8\mathrm{D}q}\frac{1}{\sin\alpha} = \rho\pi l\sin\alpha \mathrm{U}^2. \tag{3}$$

If now we wish to make a further approximation, still keeping the mid-point of the plate on the centre line of the channel, we still have

$$\theta_2 = \pi/2, \quad \theta_3 = \pi - \delta, \quad \theta_4 = -\delta,$$

and we get

$$\frac{l}{\mathrm{D}} = \frac{8q}{\pi}[1 + \tfrac{4}{3}q^2\cos 2\alpha + \tfrac{6}{5}q^4\cos 4\alpha], \tag{4}$$

$$\frac{\kappa}{\mathrm{UD}} = 8q\sin\alpha\,[1 + 4(1 - \sin^2\alpha)q^2 + 2(3 - 10\sin^2\alpha + 8\sin^4\alpha)q^4], \tag{5}$$

$$\begin{aligned}\frac{\Delta k_{\mathrm{L}}}{k_{\mathrm{L}}} &= \tfrac{8}{3}(1 + \sin^2\alpha)q^2 + \tfrac{4}{45}(14 + 283\sin^2\alpha - 118\sin^4\alpha)q^4 \\ &= \frac{\pi^2}{24}(1 + \sin^2\alpha)\left(\frac{l}{\mathrm{D}}\right)^2 - \frac{\pi^4}{15{,}360}(22 - 121\sin^2\alpha - 14\sin^4\alpha)\left(\frac{l}{\mathrm{D}}\right)^4 \\ &= 0\cdot4112(1 + \sin^2\alpha)\left(\frac{l}{\mathrm{D}}\right)^2 - 0\cdot1395(1 - 5\cdot5\sin^2\alpha - 0\cdot6364)\left(\frac{l}{\mathrm{D}}\right)^4.\end{aligned} \tag{6}$$

At small angles of incidence, we have as a first approximation

$$\frac{\Delta k_{\mathrm{L}}}{k_{\mathrm{L}}} = \frac{\pi^2}{24}\left(\frac{l}{\mathrm{D}}\right)^2,$$

which is Glauert's result.

We may also consider the value of the lift-coefficient when l/D is small, and when the centre of the plate is at an appreciable distance from the middle of the channel—that is, when squares and higher powers of h/D cannot be neglected. In mounting aerofoils in wind tunnels, however, it is not usual to put the aerofoil at a great distance from the middle; that is the ratio h/D is usually small and quite frequently it is small in relation to l/D. When l/D is small, however, an approximate formula can be obtained without restricting the value of the ratio h/D. We put $\theta_2 = \frac{1}{2}\pi - \lambda$, and in the final result neglect powers of q greater than the second. To this accuracy the equations become

$$\theta_3 + \theta_4 = 2\alpha$$

$$\theta_3 - \theta_4 = \pi - 4q(1-2q^2)\cos\alpha\sin\lambda.$$

Also

$$\left.\begin{aligned} \pi l/D &= 8q\cos\lambda\,[1 + (\tfrac{4}{3}\cos 2\alpha\cos^2\lambda + 2\cos^2\alpha\sin^2\lambda)\,q^2], \\ \pi h/D &= [\lambda - 2\sin^2\alpha\sin^2\lambda\,.\,q^2] \\ \kappa/UD &= 8q\sin\alpha\cos\lambda\,[1 + 2q\sin\lambda\sin\alpha \\ &\qquad + 2q^2(2\cos^2\alpha + \sin^2\lambda + \sin^2\lambda\sin^2\alpha)] \end{aligned}\right\}, \tag{7}$$

and

$$\frac{k_L + \Delta k_L}{k_L} = \frac{1}{\pi\sin\alpha}\,\frac{Y}{\rho l U^2} = 1 + 8q\sin\lambda\sin\alpha + \tfrac{8}{3}q^2\,[(1+\sin^2\alpha) + \tfrac{1}{2}\sin^2\lambda\,(1 + 22\sin^2\alpha)],$$

and hence

$$\Delta k_L/k_L = \left\{\pi\sin\alpha\tan\frac{\pi h}{D}\right\}\frac{l}{D} + \frac{\pi^2}{24}\sec^2\frac{\pi h}{D}\left\{(1+\sin^2\alpha) + \tfrac{1}{2}\sin^2\frac{\pi h}{D}(1 + 22\sin^2\alpha)\right\}\left(\frac{l}{D}\right)^2. \tag{8}$$

At small angles of incidence this becomes

$$\frac{\Delta k_L}{k_L} = \frac{\pi^2}{16}\left(\sec^2\frac{\pi h}{D} - \tfrac{1}{3}\right)\left(\frac{l}{D}\right)^2 + \frac{\pi l}{D}\tan\frac{\pi h}{D}\,.\,\alpha, \tag{9}$$

and if in addition h/D is small, and of the same order as l/D, we get the original result

$$\Delta k_L/k_L = \frac{\pi^2}{24}\left(\frac{l}{D}\right)^2.$$

Hence when l/D is small, and the aerofoil is mounted at a small angle of incidence, the value of $\Delta k_L/k_L$ remains unaltered within the required limits of accuracy, when the displacement of the aerofoil from the centre of the channel is of the same order of magnitude as the length.

8. *The Lift Coefficient at Small Angles of Incidence.*

In experimental work aerofoils are usually used at small angles of incidence, and in the range considered $k_L \propto \alpha$. If now in our results we make the assumption that α is small, and that powers of α higher than the second can be neglected, the result for $\Delta k_L/k_L$ assumes a rather simple form suitable for computation. The following results are accurate to within 1 or 2 per cent. in the range $0 < |\alpha| < 8°$ (approx.), that is $|k_L| \leqslant 0\cdot45$ (approx.). The previous section showed that at small values of α, small values of h/D can be neglected, so that we shall assume h/D to be zero.

We know that

$$\kappa = \frac{UD}{\pi}\left\{\frac{\vartheta_4'\left(\dfrac{\theta_4+\theta_2}{2\pi}\right)}{\vartheta_4\left(\dfrac{\theta_4+\theta_2}{2\pi}\right)} - \frac{\vartheta_4'\left(\dfrac{\theta_4-\theta_2}{2\pi}\right)}{\vartheta_4\left(\dfrac{\theta_4-\theta_2}{2\pi}\right)}\right\},$$

so that when $\theta_2 = \pi/2$

$$\kappa = \frac{UD}{\pi}\left\{\frac{\vartheta_4'\left(\dfrac{\alpha}{2\pi}\right)}{\vartheta_4\left(\dfrac{\alpha}{2\pi}\right)} - \frac{\vartheta_3'\left(\dfrac{\alpha}{2\pi}\right)}{\vartheta_3\left(\dfrac{\alpha}{2\pi}\right)}\right\}$$

and when α is small

$$\kappa = \frac{UD}{\pi}\left\{\frac{\vartheta_4''(0)}{\vartheta_4(0)} - \frac{\vartheta_3''(0)}{\vartheta_3(0)}\right\}\frac{\alpha}{2\pi} = [\tfrac{1}{2}UD\vartheta_2^4(0)]\,\alpha. \tag{1}$$

Under these assumptions

$$\frac{l}{D} = \frac{8}{\pi}\sum_{n=1}^{\infty}\frac{q^{2n-1}}{(2n-1)(1-q^{4n-2})} = \frac{2}{\pi}\log[\vartheta_4(\tfrac{1}{2})/\vartheta_4(0)]$$

$$= \frac{2}{\pi}\log[\vartheta_3(0)/\vartheta_4(0)] = \frac{1}{\pi}\log(1/k'). \tag{2}$$

Hence

$$k' = \exp_0(-\pi l/D),$$

and if l/D is given, the corresponding value of k', and therefore q, can be obtained from tables.

If now we make use of the fact that

$$\vartheta_1'(0) = 2\pi\,\vartheta_2(0)\,\vartheta_3(0)\,\vartheta_4(0),$$

and of the fact that $\kappa \propto \alpha$, we get

$$\frac{Y}{\rho U^2} = \frac{D}{2}\vartheta_2^4(0)\,.\,\alpha,$$

so that

$$k_L + \Delta k_L = \frac{\pi\alpha}{4} \vartheta_2{}^4 (0) \Big/ \log [\vartheta_3 (0)/\vartheta_4 (0)],$$

and therefore

$$\frac{\Delta k_L}{k_L} = \left\{\tfrac{1}{4} \frac{\vartheta_2{}^4 (0)}{\log [\vartheta_3 (0)/\vartheta_4 (0)]} - 1\right\}, \tag{3}$$

where

$$\frac{l}{D} = \frac{2}{\pi} \log \vartheta_3 (0)/\vartheta_4 (0).$$

We see that

$$\frac{l}{D} \to \frac{8q}{\pi}\left(1 + \tfrac{4}{3}q^2 + \frac{6}{5} q^4 + \dots\right)$$

and

$$\vartheta_2{}^4 (0) \to 16q (1 + 4q^2 + 6q^4 + 8q^6 \dots)$$

giving

$$\frac{\Delta k_L}{k_L} \to \frac{8q^2}{3} + \frac{56q^4}{45} = \frac{\pi^2}{24}\left(\frac{l}{D}\right)^2 - \frac{11\pi^4}{7680}\left(\frac{l}{D}\right)^4 = 0\cdot4112\left(\frac{l}{D}\right)^2 - 0\cdot1395\left(\frac{l}{D}\right)^4. \tag{4}$$

Table I gives the various coefficients that can be used in the range $-8° < \alpha < 8°$ with an accuracy of within 1 or 2 per cent.—the limits of the range being only approximate. [The constants are tabulated against "irregular" values of q only because these values of q correspond to certain values of the ϑ-functions that are already tabulated in Hayashi, 'Bessel-Theta Functions, etc.' (1930), Julius Springer.]

Table I.—Valid in the range $|\alpha| < 8°$ (approx.), *i.e.*, $k_L < 0\cdot45$ (approx.) The last column gives the value of the approximate expression

$$\frac{\Delta k_L}{k_L} = \frac{\pi^2}{24}\left(\frac{l}{D}\right)^2 - \frac{11\pi^4}{7680}\left(\frac{l}{D}\right)^4 = 0\cdot4112\left(\frac{l}{D}\right)^2 - 0\cdot1395\left(\frac{l}{D}\right)^4.$$

q.	l/D.	κ/UD.	$\Delta k_L/k_L$.	$\Delta k_L/k_L$ (approx.).
0·006308	0·0161	$\alpha\times$0·0505	0·0001	0·0001
0·01441	0·0367	0·1154	0·0005	0·0005
0·02642	0·0673	0·2120	0·0019	0·0019
0·04321	0·1104	0·3483	0·0050	0·0050
0·06613	0·1694	0·5383	0·0117	0·0117
0·1023	0·2643	0·8537	0·0281	0·0280
0·1425	0·3730	1·2358	0·0547	0·0545
0·2068	0·5581	1·9577	0·1166	0·1146
0·2943	0·8432	3·2896	0·2418	0·2219
0·3608	1·0994	4·7443	0·3736	0·2933

Table II. $k_L = 0 \cdot 5$ (*i.e.*, $\alpha = 9° \, 9'$). The last column gives the value of the approximate expression

$$\frac{\Delta k_L}{k_L} = 0 \cdot 4112 \, (1 + \sin^2 \alpha) \left(\frac{l}{D}\right)^2 - 0 \cdot 1395 \, (1 - 5 \cdot 5 \sin^2 \alpha - 0 \cdot 6364 \sin^4 \alpha) \left(\frac{l}{D}\right)^4,$$

$$= 0 \cdot 4216 \left(\frac{l}{D}\right)^2 - 0 \cdot 1200 \left(\frac{l}{D}\right)^4.$$

q.	l/D.	κ/UD.	Y/ρU²D.	$\Delta k_L/k_L$.	$\Delta k_L/k_L$ (approx.)
0·005	0·0127	0·0064	0·0064	0·0000	0·0000
0·050	0·1277	0·0642	0·0642	0·0066	0·0069
0·075	0·1923	0·0976	0·0976	0·0149	0·0154
0·150	0·3930	0·2082	0·2083	0·0598	0·0622
0·200	0·5359	0·2965	0·2977	0·1110	0·1122
0·300	0·8572	0·5348	0·5410	0·2622	0·2450

Tables III and IV give the results obtained from the exact and approximate solutions when $k_L = 0 \cdot 75$ and $k_L = 1 \cdot 0$.

Table III. $k_L = 0 \cdot 75$ (*i.e.*, $\alpha = 13° \, 49'$). The last column gives the value of the approximate expression

$$\frac{\Delta k_L}{k_L} = 0 \cdot 4112 \, (1 + \sin^2 \alpha) \left(\frac{l}{D}\right)^2 - 0 \cdot 1395 \, (1 - 5 \cdot 5 \sin^2 \alpha - 0 \cdot 6364 \sin^4 \alpha) \left(\frac{l}{D}\right)^4$$

$$= 0 \cdot 4346 \left(\frac{l}{D}\right)^2 - 0 \cdot 0955 \left(\frac{l}{D}\right)^4.$$

q.	l/D.	κ/UD.	Y/ρU²D.	$\Delta k_L/k_L$.	$\Delta k_L/k_L$ (approx.).
0·005	0·0127	0·0095	0·0095	0·0000	0·0000
0·050	0·1277	0·0964	0·0965	0·0068	0·0071
0·075	0·1923	0·1463	0·1464	0·0154	0·0160
0·150	0·3922	0·3116	0·3127	0·0623	0·0646
0·200	0·5339	0·4426	0·4470	0·1163	0·1161
0·300	0·8495	0·7922	0·8132	0·2763	0·2640

Table IV. $k_L = 1 \cdot 0$ (*i.e.*, $\alpha = 18° \; 34'$). The last column gives the value of the approximate expression

$$\frac{\Delta k_L}{k_L} = 0 \cdot 4112\,(1 + \sin^2\alpha)\left(\frac{l}{D}\right)^2 - 0 \cdot 1395\,(1 - 5 \cdot 5 \sin^2\alpha - 0 \cdot 6364 \sin^4\alpha)\left(\frac{l}{D}\right)^4$$

$$= 0 \cdot 4529\left(\frac{l}{D}\right)^2 - 0 \cdot 0609\left(\frac{l}{D}\right)^4.$$

q.	l/D.	κ/UD.	Y/ρU²D.	$\Delta k_L/k_L$.	$\Delta k_L/k_L$ (approx.).
0·005	0·0127	0·0127	0·0127	0·0000	0·0000
0·050	0·1277	0·1285	0·1286	0·0070	0·0074
0·075	0·1921	0·1948	0·1952	0·0160	0·0166
0·150	0·3912	0·4136	0·4168	0·0654	0·0679
0·200	0·5312	0·5859	0·5965	0·1230	0·1230
0·300	0·8388	1·0390	1·0893	0·2986	0·2884

Table V gives the values of $\Delta k_L / k_L$ for various values of l/D and α. The numerical results are obtained from the curves of fig. 6 and from the approximate expression for $\Delta k_L / k_L$, and have an error of $\pm 0 \cdot 002$ when $l/D \geqslant 0 \cdot 7$. This accuracy is well within that required in experimental work, though it has yet to be proved that the theory is applicable in actual practice when l/D is large or when α is large.

Table V.—Values of $\Delta k_L / k_L$ are tabulated against l/D and k_L, and the top row gives the approximate value of $\Delta k_L / k_L$ obtained from the Glauert approximation.

$\frac{\pi^2}{24}\left(\frac{l}{D}\right)^2$	0·001	0·004	0·009	0·016	0·037	0·066	0·103	0·202	0·333
k_L \\ l/D	0·05	0·10	0·15	0·20	0·30	0·40	0·50	0·70	0·90
$k_L \leqslant 0 \cdot 4$ (approx.)	0·001	0·004	0·009	0·016	0·036	0·060	0·094	0·175	0·270
0·50	0·001	0·004	0·009	0·016	0·037	0·062	0·098	0·185	0·285
0·75	0·001	0·004	0·009	0·017	0·038	0·065	0·103	0·198	0·307
1·00	0·001	0·004	0·009	0·017	0·040	0·069	0·109	0·215	0·338

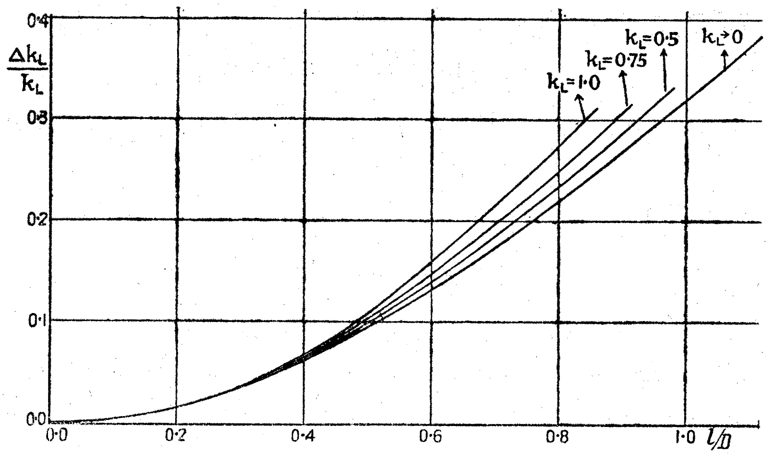

FIG. 7.

9. *A Flat Plate in the Neighbourhood of a Plane Wall.*

Sasaki (*loc. cit.*) investigated the problem of a flat plate in the vicinity of a plane wall *ab initio*, the method and analysis being similar to that of the more general problem. There is, however, no necessity to go through the investigation again, for the results already at our disposal can be made to give the required information as follows: The width of the channel D is allowed to increase indefinitely, and if $h'[=(D/2-h)]$ be the distance of the plate from the upper wall, then $h'/D=0$ is zero when $\theta_2=0$. When D is very big h'/D does not differ greatly from zero and θ_2 is therefore very small. Hence in our approximation $D\rightarrow\infty$ and $\theta_2\rightarrow 0$. The two quantities must approach their limiting values in such a way, however, that $D\theta_2\,(=\gamma)$ remains finite, and the equations for the boundaries become, from (5.19),

$$y=\gamma/\pi, \qquad y=-\infty.$$

Subject to these conditions we find that (5.21) and (5.22) become

$$\left.\begin{aligned}&\theta_3+\theta_4=2\alpha,\\ &\frac{\vartheta_4'\,(\theta_3/2\pi)}{\vartheta_4\,(\theta_3/2\pi)}+\frac{\vartheta_4'\,(\theta_4/2\pi)}{\vartheta_4\,(\theta_4/2\pi)}=0\end{aligned}\right\}. \tag{1}$$

These equations suffice to determine θ_3 and θ_4. The equations for l and h' become

$$\left.\begin{aligned}l&=\frac{8\gamma}{\pi}\sum_{n=1}^{\infty}\frac{q^n\sin\frac{1}{2}n\,(\theta_3-\theta_4)}{(1-2q^{2n}\cos 2\alpha+q^{4n})}[\cos\,(n-1)\,\alpha-q^{2n}\cos\,(n+1)\,\alpha],\\ h'&=\frac{\gamma}{\pi}\Big\{1-4\sin\alpha\sum_{n=1}^{\infty}\frac{q^n\cos\frac{1}{2}n\,(\theta_3-\theta_4)}{1-2q^{2n}\cos 2\alpha+q^{4n}}\\ &\qquad\times[\sin\,(n-1)\,\alpha-q^{2n}\sin\,(n+1)\,\alpha]\Big\}\end{aligned}\right\}. \tag{2}$$

The circulation becomes

$$\kappa = \frac{U\gamma}{\pi^2}\left[\frac{\vartheta_4''(\theta_4/2\pi)}{\vartheta_4(\theta_4/2\pi)} - \left(\frac{\vartheta_4'(\theta_4/2\pi)}{\vartheta_4(\theta_4/2\pi)}\right)^2\right]$$

$$= 8U\gamma \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} \cos n\theta_4, \tag{3}$$

and the lift Y is

$$Y = \frac{2\rho\pi^3\kappa^2}{\gamma \sin\alpha} \frac{[\vartheta_4(\theta_3/2\pi)]^3}{[\vartheta_1'(0)]^3} \frac{\vartheta_4(\theta_4/2\pi)}{\vartheta_1((\theta_3-\theta_4)/2\pi)}. \tag{4}$$

The ratio h'/l, which does not involve γ, and α are sufficient to determine the system. The ratios κ/l and $Y/\rho l U^2$ do not involve γ, and can be evaluated in terms of the other parameters, and the problem may be considered as solved.

Absorption Spectra in Relation to the Colour of Solutions of Iodine Monochloride.

By A. E. Gillam and R. A. Morton, Liverpool University.

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In a previous investigation* the absorption spectra of solutions in carbon tetrachloride of chlorine, bromine and iodine and some interhalogen compounds were studied. The solvent was selected on account of its high transparency to wave-lengths longer than 260 $\mu\mu$ and its stability towards the various solutes used.

The inertness of the medium facilitated a general survey of the absorptive properties of solutions of the halogens under the simplest conditions. Preliminary studies using other solvents soon showed, however, that the interest of the subject was far from exhausted; in this communication the properties of one of the interhalogen compounds, namely, iodine monochloride, are described in greater detail.

Iodine Monochloride.

Commercial "pure" iodine monochloride, m.p. 25·1° C., was subjected to fractional distillation. The liquid product obtained supercools largely,

* Gillam and Morton, 'Proc. Roy. Soc.,' A, vol. 124, p. 604 (1929).