WELFARE ECONOMICS AND EXISTENCE OF AN EQUILIBRIUM FOR A COMPETITIVE ECONOMY

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1. - The proof of the existence of an equilibrium for a competitive economy is given by Arrow and Debreu [1] and many others such as Gale [4], Kuhn [6], McKenzie [8], [9], and Nikaido [10]. In this note, we shall give another proof of the existence of an equilibrium, putting emphasis on the welfare aspect of the competitive equilibrium (1).

As is well known, an equilibrium point of an economic system under perfect competition is an efficient state in Pareto's sense in which we cannot make anyone better off without making someone worse off. In other words, it can be said that a competitive equilibrium is a maximum point of some properly defined social welfare function subject to the resource and technological constraints.

In the following, we shall show that a competitive equilibrium is a maximum point of a social welfare function which is a linear combination of utility functions of consumers, with the weights in the combination in inverse proportion to the marginal utilities of income. Then, the existence of an equilibrium is equivalent to the existence of a maximum point of this special welfare function. Therefore, we can prove the former by showing the latter.

2. - Let us construct our economic model, the existence of whose equilibrium we shall prove, as follows. Let there be m goods, n consumers, and r firms. Let x_i be a consumption vector (whose element is $x_{ij} \geq 0$), $\bar{x}_i$ be an initial holding vector (whose element is $\bar{x}_{ij} > 0$), and $U_i(x_i)$ be the utility (function) of the i^{th} consumer. Let y_k be a production vector of the k^{th} firm whose element $y_{kj} > 0$ (< 0) is the output (input) of the j^{th} good, and Y_k be the possible set of y_k, i. e., the set of y_k which satisfies the restriction on production $F_k(y_k) \geq 0$. Let P (whose element $P_j \geq 0$) be the price vector. For a non-free good, $P_j > 0$. Let λ_{ik} be the proportion of profit of the k^{th} firm distributed to the i^{th} consumer.

We define an equilibrium point under perfect competition:

Definition 1. The following are the conditions of an *equilibrium point* (x_i, y_k, P):

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a) Equalities of demand and supply for non-free goods:

$$\Sigma_i x_{ij} - \Sigma_k y_{kj} - \Sigma_i \bar{x}_{ij} \leq 0 \quad , \quad P_j(\Sigma_i x_{ij} - \Sigma_k y_{kj} - \Sigma_i \bar{x}_{ij}) = 0 \, , \quad \text{for } j = 1 , \ldots , m \, .$$

b) The equilibrium of consumers: x_i is a maximum point of $U_i(x_i)$ subject to

$$\Sigma_j P_j x_{ij} \leq \Sigma_j P_j \bar{x}_{ij} + \max[0 , \Sigma_k \lambda_{ki} \Sigma_j P_j y_{kj}] \equiv M_j \, , \quad \text{for } i = 1 , \ldots , n \, .$$

c) The equilibrium of firms: y_k is a maximum point of $\Sigma_j P_j y_{kj}$ subject to

$$F_k(y_k) \geq 0 \quad (y_k \in Y_k) \, , \quad \text{for } k = 1 , \ldots , r \, .$$

Next, we define a welfare maximum point as follows:

Definition 2. Consider the weighted sum of utility functions $\Sigma_i \alpha_i U_i(x_i)$ with weights $\alpha_i \geq 0$, $i = 1 , \ldots , n$, $\Sigma_i \alpha_i = 1$, as a social welfare function. We call a point (x_i , y_k), which maximizes it, subject to the condition of no excess of demand over supply, $\Sigma_i x_i \leq \Sigma_i \bar{x}_i + \Sigma_k y_k$, and production subject to the restriction on $F_k(y_k) \geq 0$, $k = 1 , \ldots , r$, a *welfare maximum point.*

3. - The assumptions on utility functions and production restrictions are as follows:

Assumption 1. $U_i(x_i)$ is continuous, increasing, and concave; more precisely, we can make it concave by a strictly positive monotone transformation. See Fenchel [3].

Roughly speaking, this assumption implies that, among utility functions which satisfy the same indifference map, there is a utility function with non-increasing marginal utility.

Assumption 2. $F_k(y_k)$ is continuous and concave, and $F_k(y^*_k) > 0$ for some y^*_k such that $\Sigma_k y^*_k < \Sigma_i \bar{x}_i$ [2]. Furthermore, the sets Y_k and their vector sum Y satisfy the following conditions: $0 \in Y_k$, $Y \frown B = 0$ (B is a closed positive orthant), $Y \frown (-Y) = 0$.

The concavity of F_k implies non-increasing returns. The conditions on Y_k and Y are explained in Arrow and Debreu [1], p. 276.

We get from Assumption 2 and the conditions of no excess of demand over supply in Definition 2, or equalities of demand and supply for non-free goods in Definition 1, the following lemma:

Lemma 1. The domain of x_i and y_k can be restricted as $x_i \in \Gamma_i$, $y_k \in \Gamma_k$, Γ_i , Γ_k being suitably large convex, compact sets, without causing any change in the definitions of a welfare maximum point causing any change in the definitions of a welfare maximum point and an equilibrium point (3).

Other lemmas we shall use in this paper are:

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(2) This condition is needed for the application of the Kuhn-Tucker Theorem. See Lemma 2 below.

(3) Arrow and Debreu [1], pp. 276, 277, 279.

Lemma 2. (Kuhn-Tucker Theorem) Let $f(x)$ and $g(x) = \{g_1(x), \ldots, g_n(x)\}$ be concave in $x \geq 0$ and $g(x)$ satisfy Slater's condition that there is a vector x^0 such that $x^0 \geq 0$ and $g(x^0) > 0$. Then $\bar{x}$ maximizes $f(x)$ subject to the restrictions that $x \geq 0$ and $g(x) \geq 0$ if, and only if, there is a vector $\bar{u}$ such that $(\bar{x}, \bar{u})$ is a non-negative saddle point of the Lagrangian $\varphi(x, u) = f(x) + u \cdot g(x)$, i. e., $\varphi(x, \bar{u}) \leq \varphi(\bar{x}, \bar{u}) \leq \varphi(\bar{x}, u)$ for all $x \geq 0$ and $u \geq 0$. See Kuhn and Tucker [7] and Arrow, Hurwicz and Uzawa [2], pp. 32-37.

Lemma 3. (Kakutani's Fixed Point Theorem) Let K be a compact convex set in n dimensional Euclidian space R^n and $f(x)$ be a point-to-set, upper semi-continuous mapping from K into K, whose image is non-void and convex. Then, there is a fixed point $\hat{x}$ such that $\hat{x} = f(\hat{x})$. See Kakutani [5] and Nikaido [10].

4. - We are now in the position to state the following theorems on a welfare maximum point.

THEOREM 1. *For any set of weights α_i, there is a welfare maximum point under Assumptions 1 and 2.*

Proof. From Assumption 1 and Definition 2 the social welfare function is continuous. From Lemma 1 the domain is compact. As a continuous function on the compact domain the social welfare function has a maximum.

THEOREM 2. *A welfare maximum point is a saddle point of*

$$\varphi(x_i, y_k, P_j, \mu_k) \equiv \Sigma_i \alpha_i U_i(x_i) - \Sigma_j P_j (\Sigma_i x_{ij} - \Sigma_k y_{kj} - \Sigma_i \bar{x}_{ij}) + \Sigma_k \mu_k F_k(y_k)$$

where $x_i \geq 0$, y_k are maximizing variables and $P_j \geq 0$, $\mu_k \geq 0$, are minimizing variables. The necessary and sufficient condition for it is as follows [1]:

$$\alpha_i U^{-0}_{ix_{ij}} - P_j \geq 0, \ \alpha_i U^{+0}_{ix_{ij}} - P_j \leq 0, \text{ for } i = 1, \ldots, n, \ j = 1, \ldots, m.$$

$$P_j + \mu_k F^{-0}_{ky_{kj}} \geq 0, \ P_j + \mu_k F^{+0}_{ky_{kj}} \leq 0, \text{ for } k = 1, \ldots, r, \ j = 1, \ldots, m.$$

$$\Sigma_i x_{ij} - \Sigma_k y_{kj} - \Sigma_i \bar{x}_{ij} \leq 0, \ P_j(\Sigma_i x_{ij} - \Sigma_k y_{kj} - \Sigma_i \bar{x}_{ij}) = 0, \quad \text{for } j = 1, \ldots, m.$$

$$\mu_k F_k(y_k) = 0 \quad , \quad F_k(y_k) \geq 0, \quad \text{for } k = 1, \ldots, r.$$

Proof. By putting $x_i = 0$, the assumption $F_k(y^*_k) > 0$, $\Sigma_k y^*_k < \Sigma \bar{x}_i$, guarantees the satisfaction of Slater's condition in

Lemma 2. Then we can apply the Kuhn-Tucker Theorem. The second half of the theorem follows from the definition of the saddle point.

(4) Here $U^{-0}_{ix_{ij}}$ stands for the left-hand derivative of U_i with respect to x_{ij}.

5. - Next, we shall prove the following theorems on a competitive equilibrium point.

THEOREM 3. *Conditions* b), c) *of an equilibrium point in Definition* 1 *can be written respectively in the following form:*

b') x_i *is a saddle point of*

$$\varphi_i(x_i, \delta_i) \equiv U_i(x_i) - \delta_i(\Sigma_j P_j x_{ij} - M_i)$$

where $x_i \geq 0$ *are maximizing variables and* $\delta_i \geq 0$ *are minimizing variables. The necessary and sufficient condition for it is:*

$$u_{xi_{ij}}^{-0} + \delta_i P_j \geq 0, \quad u_{ix_{ij}}^{+0} - \delta P_j \leq 0, \qquad \text{for } j = 1, \dots, m$$
$$\Sigma_j P_j x_{ij} - M_i = 0$$

c') y_k *is a saddle point of*

$$\varphi_k(y_k, \mu_k) \equiv \Sigma_j P_j y_{kj} - \mu_k F_k(y_k)$$

where y_k *are maximizing variables and* $\mu_k \geq 0$ *are minimizing variables. The necessary and sufficient condition for it is:*

$$P_j - \mu_k F_{ky_{kj}}^{-0} \geq 0, \quad P_j - \mu_k F_{ky_{kj}}^{+0} \leq 0, \quad \text{for } j = 1, \dots, m,$$
$$F_k(y_k) \geq 0.$$

Proof. *b*) $\rightarrow$ *b'*). $\bar{x}_i > 0$ implies $M_i > 0$. Then, putting $x_i = 0$, Slater's condition in Lemma 2 can be satisfied and the Kuhn-Tucker Theorem can be applied.

c) $\rightarrow$ *c'*). The assumption $F_k(y^*_k) > 0$ implies Slater's condition and the Kuhn-Tucker Theorem can be applied.

THEOREM 4. *At any welfare maximum point, the conditions* a), c), *of an equilibrium in Definition* 1 *are satisfied.*

Proof. Compare Definitions 1 and 2 and use Theorem, 2 and the second half of Theorem 3.

6. - From Theorem 4 we know that if condition *b*) of an equilibrium is satisfied at a welfare maximum point for some set of weights α_i, then it is an equilibrium point. We have to seek such a set of weights. For this, we construct the following mapping:

a) For any point $\alpha = (\alpha_1, \dots, \alpha_n)$ on the $n - 1$ dimensional simplex S^{n-1} we get a welfare maximum point $(x^o_i, y^o_k, P^o_j, \mu^o_k)$ and, by $P'^o_j = P^o_j / \Sigma_j P^o_j$, we have (x^o_i, y^o_k, P'^o), where $P'^o \in S^{m-1}$.

b) By *a*) and Lemma 1, it can be considered that all (x_i, y_k, P) are contained in a convex compact set $K = \Pi_i \Gamma_i \times \Pi_k \Gamma_k \times S^{m-1}$. We can take a positive number A such that.

$$\Sigma_i |M_i - \Sigma_j P_j x_{ij}| < A \quad \text{for any} \quad (x_i, y_k, P) \in K .$$

For any $\alpha \in S^{n-1}$, $(x_i, y_k, P) \in K$, by

$$\alpha'_i = \max\left[0\ ,\ \alpha_i + \frac{M_i - \Sigma_j P_j x_{ij}}{A}\right], \qquad \alpha''_i = \alpha'_i \big| \Sigma_i \alpha'_i\ ,$$

we get $\alpha'' = (\alpha''_1\ , \ldots, \alpha''_n) \in S^{n-1}$.

c) Combining *a*) and *b*), we have a mapping from a convex compact set into itself

$$S^{n-1} \times K \in (\alpha\ ,\ x_i\ ,\ y_k\ ,\ P) \rightarrow (a'',\ x^o_i\ ,\ y^o_k\ ,\ P'^o) \in S^{n-1} \times K\ .$$

7\. - We shall prove that this mapping has a fixed point and that the corresponding welfare maximum point is an equilibrium point and so establish the following existence theorem for an equilibrium:

THEOREM 5. *Under Assumptions* 1 *and* 2, *there is an equilibrium point.*

Proof. A. The mapping has a fixed point.

1\) The point-to-set mapping $\alpha \in S^{n-1} \rightarrow (x^o_i\ ,\ y^o_k\ ,\ P^o_j)$ is the mapping from α to the saddle points of,

$$\varphi(x_i\ ,\ y_k\ ,\ P_j\ ,\ \mu_k) \equiv \Sigma_i \alpha_i U_i(x_i) - \Sigma_j P_j (\Sigma_i x_{ij} - \Sigma_k y_{kj} - \Sigma_i \bar{x}_{ij}) + \\ + \Sigma_k \mu_k F_k(y_k)$$

in Theorem 2. From Theorem 1, its image is non-void. It is convex because its elements are saddle points of a convex-concave function. This mapping is upper semi-continuous because if we have converging sequences $\alpha^n \rightarrow \alpha^o$, $(x^{on}_i\ ,\ y^{on}_k\ ,\ P^{on}_j\ ,\ \mu^{on}_k) \rightarrow (x^{oo}_i\ ,\ y^{oo}_k\ ,\ P^{oo}_i\ ,\ \mu^{oo}_k)$, and for each n, $(x^{on}_i\ ,\ y^{on}_k\ ,\ P^{on}_i\ ,\ \mu^{on}_k)$ is a saddle point of φ corresponding to α^n, then $(x^{oo}_i\ ,\ y^{oo}_k\ ,\ P^{oo}_j\ ,\ \mu^{oo}_k)$ is a saddle point of φ corresponding to α^o. The normalizing mapping $P^o \rightarrow P'^o$ preserves upper semi-continuity and convexity.

2\) $(\alpha\ ,\ x_i\ ,\ y_k\ ,\ P) \rightarrow \alpha''$ is a point-to-point mapping and continuous.

3\) From 1, 2, the mapping $(\alpha\ ,\ x_i\ ,\ y_k\ ,\ P) \rightarrow (\alpha'',\ x^o_i\ ,\ y^o_k\ ,\ P'^o)$ is an upper semi-continuous mapping from convex compact set $S^{n-1} \times K$ into itself whose image is non-void and convex. Therefore, there is a fixed point $(\bar{\bar{\alpha}}\ ,\ \bar{\bar{x}}_i\ ,\ \bar{\bar{y}}_k\ ,\ \bar{\bar{P}})$, from Kakutani's Fixed Point Theorem in Lemma 3.

B. The fixed point $(\bar{\bar{\alpha}}\ ,\ \bar{\bar{x}}_i\ ,\ \bar{\bar{y}}_k\ ,\ \bar{\bar{P}})$ is an equilibrium point. The point $(\bar{\bar{x}}_i\ ,\ \bar{\bar{y}}_k\ ,\ \bar{\bar{P}})$ is the welfare maximum point corresponding to $\bar{\bar{\alpha}}$. Therefore, to demonstrate that it is also an equilibrium point, it is sufficient to show that condition *b*) is satisfied. In order to see this we first note that $\bar{\bar{M}} = \Sigma \bar{P} \bar{x} + \Sigma \lambda\ \Sigma \bar{\bar{P}} \bar{\bar{y}} =$

this, we first note that $M_i = \Sigma_j P_j x_{ij} + \Sigma_k \lambda_{ik} \Sigma_j P_j y_{kj} -$ $= \Sigma_j \overline{\overline{P}}_j \overline{\overline{x}}_{ij}$ (5). This is because $\overline{M}_i - \Sigma_j \overline{\overline{P}}_j \overline{\overline{x}}_{ij}$, $i = 1, \ldots, n$, must be of equal sign by the construct on of the mapping and from Theorem 2,

(5) Here it must be noted all $\Sigma_k \lambda_{ki} \Sigma_j P_j y_{kj}$ obtained from P^o and y^o_k are non negative.

$$\Sigma_i \left(\Sigma_j \bar{\bar{P}}_j \bar{\bar{x}}_{ij} - \bar{\bar{M}}_i \right) = \Sigma_j \bar{\bar{P}}_j \left(\Sigma_i \bar{\bar{x}}_{ij} - \Sigma_k \bar{\bar{y}}_{kj} - \Sigma_i \bar{x}_{ij} \right) = 0 .$$

From Theorem 2, $(\bar{\bar{x}}_i , \bar{\bar{y}}_k , \bar{\bar{P}})$ satisfies the following conditions,

$$\bar{\bar{\alpha}}_i U^{+0}_{ix_{ij}} - \bar{\bar{P}}_j \leq 0 , \quad \bar{\bar{\alpha}}_i U^{-0}_{ix_{ij}} - \bar{\bar{P}}_j \geq 0 , \qquad \text{for } \begin{matrix} j = 1 , \dots , m . \\ i = 1 , \dots , n \end{matrix}$$

Because of the assumption of $M_i > 0$ we are sure $\bar{\bar{\alpha}}_i > 0$. Therefore we get

$$U^{+0}_{ix_{ij}} - \frac{1}{\bar{\bar{\alpha}}_i} \bar{P}_j \leq 0 , \quad U^{-0}_{ix_{ij}} - \frac{1}{\bar{\bar{\alpha}}_i} \bar{P} \geq 0 , \qquad \text{for } \begin{matrix} j = 1 , \dots , m . \\ i = 1 , \dots , n \end{matrix}$$

Replacing $\frac{1}{\bar{\bar{\alpha}}_i}$ by $\bar{\bar{\delta}}_i$, we get

$$U^{+0}_{ix_{ij}} (\bar{\bar{x}}_i) - \bar{\bar{\delta}}_i \bar{\bar{P}}_j \leq 0 , \quad U^{-0}_{ix_{ij}} (\bar{\bar{x}}_i) - \bar{\bar{\delta}}_i \bar{\bar{P}}_j \geq {}^{1} , \qquad \text{for } \begin{matrix} j = 1 , \dots , m . \\ i = 1 , \dots , n \end{matrix}$$

These, together with $\bar{\bar{M}}_i = \Sigma_j \bar{\bar{P}}_j \bar{\bar{x}}_{ij}$, $i = 1 , \dots , n$, are the necessary and sufficient conditions of *b*) as is shown in Theorem 3.

It is well known that δ_i is the so called marginal utility of income of the i^{th} consumer. Thus, we proved the existence of an equilibrium for a competitive economy as a welfare maximum point, with the weight of a consumer being in inverse relation to the equilibrium marginal utility if income.

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