Section 2.5 - Euler's Method for Systems

1. **Euler's Method for Autonomous Systems**
   Given the system
$$\frac{dx}{dt} = f(x, y)$$
$$\frac{dy}{dt} = g(x, y),$$
   the initial condition (x_0, y_0), and step size Δt, Euler's method approximates a solution (x, y) by
$$x_{k+1} = x_k + f(x_k, y_k)\Delta t$$
$$y_{k+1} = y_k + g(x_k, y_k)\Delta t.$$

2. **Euler's Method in Vector Notation:**
   Consider the system
$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(\mathbf{Y}),$$
   where $\mathbf{Y} = (x, y)$, $\dfrac{d\mathbf{Y}}{dt} = \left(\dfrac{dx}{dt}, \dfrac{dy}{dt}\right)$, and $\mathbf{F}(\mathbf{Y}) = (f(x, y), g(x, y))$. If we are given the initial condition $\mathbf{Y}_0 = (x_0, y_0)$, then Euler's method approximates a solution (x, y) by
$$(x_{k+1}, y_{k+1}) = (x_k, y_k) + \Delta t\mathbf{F}(x_k, y_k).$$

3. Example: (Problem 2, p. 202)
   For the system
$$\frac{dx}{dt} = 2x$$
$$\frac{dy}{dt} = y,$$
   we claim that the curve $\mathbf{Y}(t) = (e^{2t}, 3e^t)$ is a solution. Its initial position is $\mathbf{Y}(0) = (1, 3)$.

   (a) Check that $\mathbf{Y}(t) = (e^{2t}, 3e^t)$ is a solution.

   (b) Use Euler's method with step size $\Delta t = 0.5$ to approximate this solution, and check how close the approximate solution is to the real solution when $t = 2$, $t = 4$, and $t = 6$.

   (c) Use Euler's method with step size $\Delta t = 0.1$ to approximate this solution, and check how close the approximate solution is to the real solution when $t = 2$, $t = 4$, and $t = 6$.

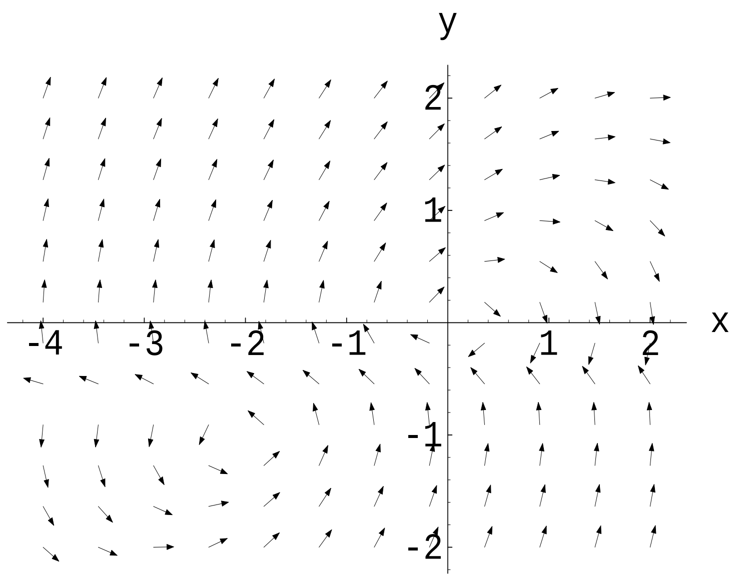
   (d) How and why do the Euler approximations differ from the solution?

4. Example: (Problem 6, p. 204) Answer the following concerning the system

$$\frac{dx}{dt} = y + y^2$$

$$\frac{dy}{dt} = -\frac{x}{2} + \frac{y}{5} - xy + \frac{6y^2}{5},$$

the initial condition $(x_0, y_0) = (-0.5, 0)$, step size $\Delta t = 0.25$, and $n = 7$.



(a) Use EulersMethodForSystems to calculate the approximate solution given by Euler's method for the given system with the given initial condition and step size for n steps.

(b) Plot your approximate solution on the direction field. Make sure that your approximate solution is consistent with the direction field.

(c) Using HPGSystemSolver, obtain a more detailed sketch of the phase portrait for the system.

Section 2.6 - Existence and Uniqueness Theorem for Systems

1. **Existence and Uniqueness Theorem:**
Let

$$\frac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y})$$

be a system of differential equations. Suppose that t_0 is an initial time and $\mathbf{Y}_0$ is an initial value. Suppose also that the function $\mathbf{F}$ is continuously differentiable. Then there is an $\epsilon > 0$ and a function $\mathbf{Y}(t)$ defined for $t_0 - \epsilon < t < t_0 + \epsilon$, such that $\mathbf{Y}(t)$ satisfies the initial-value problem $\dfrac{d\mathbf{Y}}{dt} = \mathbf{F}(t, \mathbf{Y})$ and $\mathbf{Y}(t_0) = \mathbf{Y}_0$. Moreover, for t in this interval, this solution is unique.
(Two different solutions can't start at the same place at the same time.)

2. In this chapter, we're only dealing with autonomous differential equations. Since the vector field $\mathbf{F}(\mathbf{Y})$ doesn't change with time, two different solutions that start at the point $\mathbf{Y}_0$ at different times correspond to the same solution curve. Hence distinct solution curves $\mathbf{Y}_1(t)$ and $\mathbf{Y}_2(t)$ cannot cross.