# *Triangle Numbers*

Today we shall prove that for all natural numbers,

$$P(n) : \sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

Alternatively, this states that the triangle numbers are given by the expression $n(n+1)/2$.

First we consider $P(1)$. We have $\Sigma_{i=1}^{1} i = 1$, and $\frac{1}{2}(1(1+1)) = 1$, as required. Now let $P(k)$ be the statement

$$P(k) : \sum_{i=1}^{k} i = \frac{k(k+1)}{2},$$

and assume $P(k)$ is true. We seek to show that the statement $P(k+1)$ is true, that is $\Sigma_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2}$.

$$\begin{aligned} \sum_{i=1}^{k+1} i &= \sum_{i=1}^{k} i + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) && \text{(by } P(k)\text{)} \\ &= \frac{k^2 + k + 2k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2}, \end{aligned}$$

as required! Henceforth, by mathematical induction, $P(n)$ is true for all natural numbers. ■

*but...*

We have proved that the sum of the first $n$ natural numbers is $n(n+1)/2$, but how did we even know to check that *that* particular expression is what they sum to in the first place?

To answer this question, we make note of the fact that addition of natural numbers is commutative.

Let's define the sequence $S_n$ as

$$S_n = 1 + 2 + 3 + ... + (n-2) + (n-1) + n.$$

The first term is 1, so we are going to rewrite the sequence by basing all the subsequent terms off the first term.

$$S_n = 1 + (1+1) + (1+2) + ... + (1+(n-3)) + (1+(n-2)) + (1+(n-1)).$$

Now if you recall the fact we made note of, it doesn't matter what order we write our terms, so let's write them backwards for the fun of it!

$$S_n = (1+(n-1)) + (1+(n-2)) + (1+(n-3)) + ... + (1+2) + (1+1) + 1.$$

What if... we added our forwards $S_n$ and our backwards $S_n$ together? Since they are actually both the same, the sum is $2S_n$ and

$$\begin{aligned} 2S_n &= [1 + (1+(n-1))] + [(1+1) + (1+(n-2))] + [(1+2) + (1+(n-3))] + ... \\ &\qquad + [(1+(n-3)) + (1+2)] + [(1+(n-2)) + (1+1)] + [(1+(n-1)) + 1] \\ &= (n+1) + (n+1) + (n+1) + ... + (n+1) + (n+1) + (n+1) \\ &= n(n+1). \end{aligned}$$

Ultimately, we have $S_n = n(n+1)/2$, tada! So that's where that expression comes from.