A CURSED FUNCTION

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Abstract

On a misty autumn morning, I woke up and decided to create a cursed function. "What could it be?", I thought to myself. I had recently been trying to find the integral of $\sqrt{\tan(x)}$, which after a lot of trials and tribulations, I decided to put on hold before I learnt more integral calculus. Though I did look up the solution, and saw it contained a hyberbolic function. These hyperbolics had been playing on my mind, so I decided to create a function that combined two of them, along with sine, so it looks like three functions stacked in a trench coat. In addition to this, the denominator arises from the failed attempt to integrate $\sqrt{\tan(x)}$. I found myself having to use a u-substitution due to the presence of $1 + x^2$. I never substituted the u, so I decided to incorporate that binomial into my cursed function. And lo and behold, I present to you the following:

$$f(x) = \frac{\tanh\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right) + x^2}{\sqrt{1 + x^2}}.$$

Let's take a look

Though the function may seem cursed at a first glance, the reality is far less intimidating. It starts off like a nice little parabola, as we can in see in Figure 1.

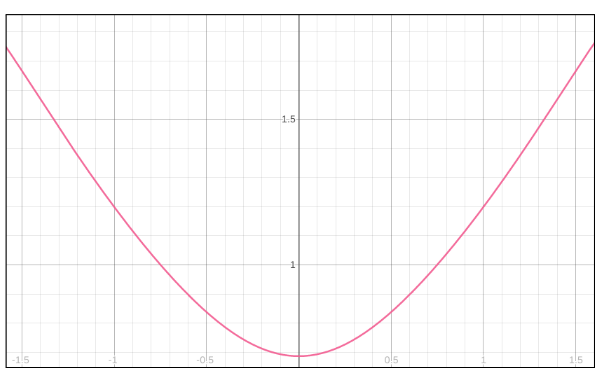

Figure 1: A nice little parabola [1].

However, at just over $x = 2$, we see that our parabola turns into a worm, shown in Figure 2 below.

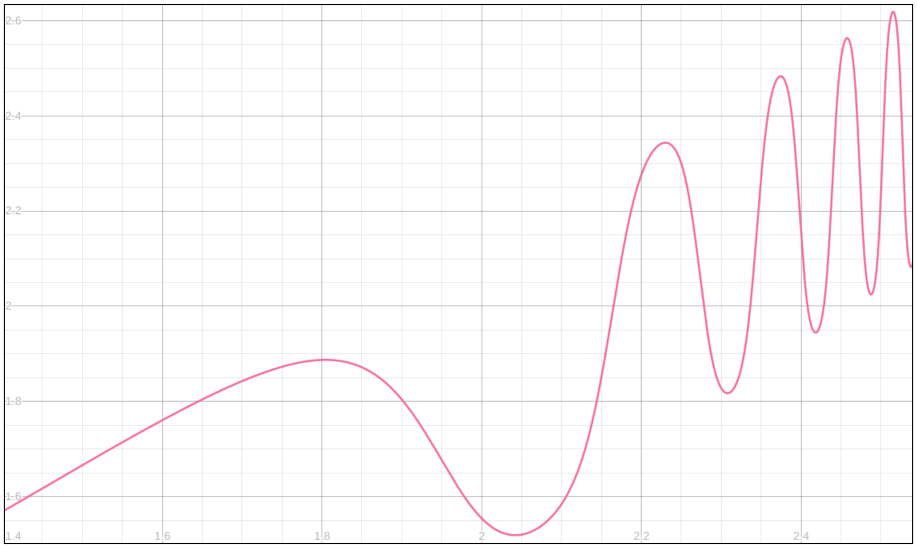

Figure 2: The oscillating worm begins [1].

Then without further ado, we see the worm wiggle off into infinity, with each wiggle becoming closer and closer as x increases in Figure 3.

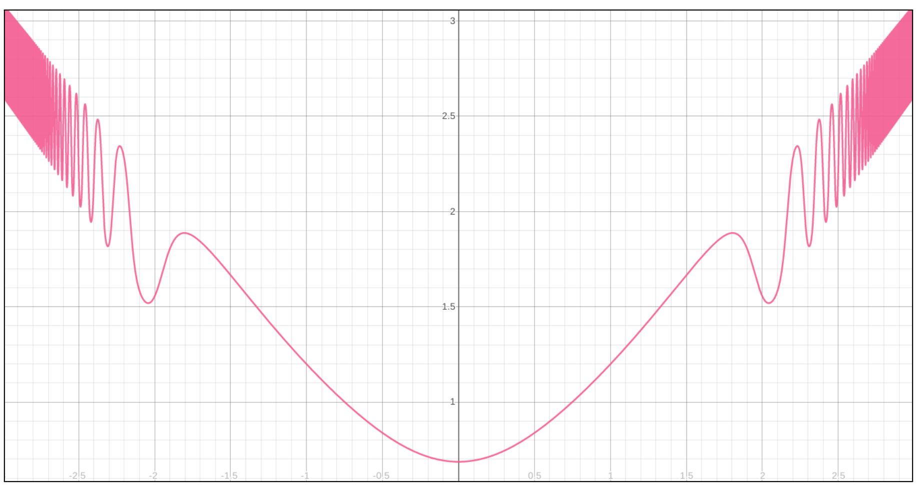

Figure 3: The cursed function in all its beauty [1].

My first thought upon analysing this function was, “Wow!” My second thought though, and the prompt for this piece of writing, was, “What are the points at which this parabola turns into a worm?” To answer such a question, we would need to know the stationary points, which are the solutions to $f'(x) = 0$. So in this piece of writing, we will focus on the first step of finding the derivative. Thus begins our endeavour.

Divide and conquer!

Our function is a fraction, so when thinking of how to solve this I was immediately drawn to the Quotient Rule. To make use of this rule we need to know the derivative of the numerator and the denominator, which I will label as $p(x)$ and $q(x)$ respectively, and then we can substitute those values into this equation:

$$f'(x) = \frac{q(x)p'(x) - p(x)q'(x)}{(q(x))^2}.$$

Sadly, our expressions for p and q are not so simple, meaning we need to break our function down a little bit further than just a ratio of two. So I have taken it upon myself to divide our cursed function $f(x)$ into seven functions that combine together as follows.

$$f(x) = \frac{p(x)}{q(x)} = \frac{h(j(k(l(x)))) + g(x)}{m(n(x))}.$$

Here is a list of the seven individual functions and their respective derivatives.

$$\begin{aligned}
g(x) &= x^2 & g'(x) &= 2x\\
h(x) &= \tanh(x) & h'(x) &= \operatorname{sech}^2(x)\\
j(x) &= \sin(x) & j'(x) &= \cos(x)\\
k(x) &= \cosh(x) & k'(x) &= \sinh(x)\\
l(x) &= x^3/4 & l'(x) &= 3x^2/4\\
m(x) &= \sqrt{x} & m'(x) &= (2\sqrt{x})^{-1}\\
n(x) &= 1 + x^2 & n'(x) &= 2x.
\end{aligned}$$

The derivatives for the trigonometric [2] and hyperbolic functions [3] were looked up in a highly reputable encyclopedia which is referenced at the end, whilst the derivatives of the polynomials were found using the Power Rule, which I describe as: *bring the power down in front, then take one from the power*. This power rule is also used for m, with the power in question being $\frac{1}{2}$. I find it is more intuitive to write square roots as powers of a half during calculations, though I like to leave them written with the square root symbol in the last step. The working for m' using the Power Rule is as follows:

$$m'(x) = (\sqrt{x})' = (x^{\frac{1}{2}})' = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}} = (2\sqrt{x})^{-1}.$$

With the above information, we can now consider the denominator of our function, $q(x) = m(n(x))$. We use the Chain Rule to obtain,

$$\begin{aligned}
q'(x) &= m'(n(x)) \cdot n'(x)\\
&= \frac{1}{2\sqrt{1+x^2}} \cdot 2x\\
&= \frac{x}{\sqrt{1+x^2}}.
\end{aligned}$$

Just like that, we have gathered half of what we need to use the Quotient Rule to find the derivative of the cursed function! But the real question is, “What are we going to do about the numerator?”

When life gives you lemons, apply the chain rule thrice

The numerator is the sum of two terms, so the derivative is going to be found by using the Sum Rule: *find the derivative of each term in the expression, then add them together*. Luckily we have already found the derivative of the second term x^2, as we noted earlier this was $2x$, so all we are left with is finding the derivative of the first term,

$$h(j(k(l(x)))) = \tanh\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right).$$

To find this we are going to use the Backwards Substitution Chain Rule Extravaganza method. All will become clear about how this works in due course. We begin by breaking up our function to first look at $k(l(x))$.

$$k(l(x)) = \cosh\left(\frac{x^3}{4}\right)$$

We are going to rewrite this as an *algebraic expression* with a and b like so,

$$b = \cosh a.$$

Beautiful, isn't it? Now that we have our first *algebraic expression*, we are going to make another one! This time for $j(k(l(x)))$. Here is the part of the function in question:

$$j(k(l(x))) = \sin\left(\cosh\left(\frac{x^3}{4}\right)\right).$$

Instead of writing out our *algebraic expression* in full, we substitute b to obtain,

$$c = \sin b.$$

Finally, we write an *algebraic expression* for all of $h(j(k(l(x))))$ by substituting c as follows,

$$d = \tanh c.$$

Now that we have our four *algebraic expressions*, we take their individual derivatives. For b, c, and d, we apply the Chain Rule, leaving us ready to substitute in the next step.

$$\begin{aligned}
a &= x^3/4 & a' &= 3x^2/4\\
b &= \cosh a & b' &= \sinh(a)\cdot a'\\
c &= \sin b & c' &= \cos(b)\cdot b'\\
d &= \tanh c & d' &= \operatorname{sech}^2(c)\cdot c'.
\end{aligned}$$

On the previous page, we completed the Chain Rule part of the Backwards Substitution Chain Rule Extravaganza method. It is now the time for the Backwards Substitution part. Grab your popcorn, make note of the results at the bottom of the previous page, and let's go!

Beginning with our first *algebraic expression*, we substitute a and a' into b' to obtain

$$b' = \sinh\left(\frac{x^3}{4}\right) \cdot \frac{3}{4}x^2,$$

rearranging ever so slightly to give

$$b' = \frac{3}{4}x^2 \sinh\left(\frac{x^3}{4}\right).$$

Repeating the process with c', we substitute the above b', along with b to produce the following

$$c' = \cos\left(\cosh a\right) \cdot \frac{3}{4}x^2 \sinh\left(\frac{x^3}{4}\right).$$

Upon rearranging and substituting the expression for a into $\cosh(a)$, we have

$$c' = \frac{3}{4}x^2 \sinh\left(\frac{x^3}{4}\right)\cos\left(\cosh\left(\frac{x^3}{4}\right)\right).$$

Now we commence with our final substitution. First we substitute c' and c into d' to obtain

$$d' = \operatorname{sech}^2\left(\sin b\right) \cdot \frac{3}{4}x^2 \sinh\left(\frac{x^3}{4}\right)\cos\left(\cosh\left(\frac{x^3}{4}\right)\right).$$

From above we know that $b = \cosh a = \cosh\dfrac{x^3}{4}$. Substituting this into d' and bringing our $\dfrac{3}{4}x^2$ term to the front, we arrive at

$$d' = \frac{3}{4}x^2 \sinh\left(\frac{x^3}{4}\right)\cos\left(\cosh\left(\frac{x^3}{4}\right)\right)\operatorname{sech}^2\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right).$$

Henceforth,

$$(h(j(k(l(x)))))' = \frac{3}{4}x^2 \sinh\left(\frac{x^3}{4}\right)\cos\left(\cosh\left(\frac{x^3}{4}\right)\right)\operatorname{sech}^2\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right).$$

Tada! Just like that, we have found the derivative of the second term in our numerator. Moreover, applying the Sum Rule gives the derivative of the entire numerator as

$$p'(x) = \frac{3}{4}x^2 \sinh\left(\frac{x^3}{4}\right)\cos\left(\cosh\left(\frac{x^3}{4}\right)\right)\operatorname{sech}^2\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right) + 2x.$$

Marvelous! Truly a calculus extravaganza. ♥

Picking up the pieces

After using the Backwards Substitution Chain Rule Extravaganza method, we have acquired all the information we need to apply the Quotient Rule to find the derivative of the cursed function. Each term is summarised below.

$$
\begin{aligned}
p(x) &= \tanh\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right) + x^2, \\
p'(x) &= \frac{3}{4}x^2 \sinh\left(\frac{x^3}{4}\right)\cos\left(\cosh\left(\frac{x^3}{4}\right)\right)\operatorname{sech}^2\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right) + 2x, \\
q(x) &= \sqrt{1+x^2}, \\
q'(x) &= \frac{x}{\sqrt{1+x^2}}.
\end{aligned}
$$

At this point, I have used far too many letters of the alphabet, so now we will make use of the beloved halloweenmath package for LaTeX and adopt a little pumpkin and a little bat to hide the trigonometric and hyperbolic functions for the time being. Thus, we simplify $p(x)$ and $p'(x)$ to,

$$
p(x) = \pumpkin + x^2, \qquad\qquad p'(x) = \frac{3}{4}x^2\mathbat + 2x.
$$

The reason for this is because we only need the x terms and the constants in order to simplify our expression. Trig will remain trig, hyperbolics will remain hyperbolics, so we don't need to write them out each time and only need to concern ourselves with the other parts of the equation.

We also simplify q and q' to

$$
q(x) = \left(1+x^2\right)^{\frac{1}{2}} \qquad\qquad q'(x) = x\left(1+x^2\right)^{-\frac{1}{2}}.
$$

As mentioned earlier, working with indices is more intuitive when it comes to manipulating algebra.

And the following page is just that! I will briefly explain each line for the benefit of the reader. ♥

$$
\begin{aligned}
f'(x) &= \frac{q(x)p'(x) - p(x)q'(x)}{(q(x))^2} \\
&= \frac{\left(1+x^2\right)^{\frac{1}{2}}\left(\frac{3}{4}x^2\text{🦇} + 2x\right) - (\text{🎃} + x^2)\left(x\left(1+x^2\right)^{-\frac{1}{2}}\right)}{1+x^2} && \text{(1) Substituting in each term} \\
&= \frac{\left(1+x^2\right)^{-\frac{1}{2}}\left[(1+x^2)\left(\frac{3}{4}x^2\text{🦇} + 2x\right) - (\text{🎃} + x^2)(x)\right]}{1+x^2} && \text{(2) Factoring out } (1+x^2)^{-\frac{1}{2}} \\
&= \frac{(1+x^2)\left(\frac{3}{4}x^2\text{🦇} + 2x\right) - (\text{🎃} + x^2)(x)}{(1+x^2)^{\frac{3}{2}}} && \text{(3) Cancelling the } (1+x^2) \text{ terms} \\
&= \frac{\frac{3}{4}x^2\text{🦇} + 2x + \frac{3}{4}x^4\text{🦇} + 2x^3 - \text{🎃}x - x^3}{(1+x^2)^{\frac{3}{2}}} && \text{(4) Expanding all the brackets} \\
&= \frac{x\left[\frac{3}{4}x\text{🦇} + 2 + \frac{3}{4}x^3\text{🦇} + x^2 - \text{🎃}\right]}{(1+x^2)^{\frac{3}{2}}} && \text{(5) Factoring out } x \\
&= \frac{x\left[3x\text{🦇} + 8 + 3x^3\text{🦇} + 4x^2 - 4\text{🎃}\right]}{4\,(1+x^2)^{\frac{3}{2}}} && \text{(6) Factoring out } 1/4 \\
&= \frac{x\left[3x(1+x^2)\text{🦇} + 8 + 4x^2 - 4\text{🎃}\right]}{4\,(1+x^2)^{\frac{3}{2}}} && \text{(7) Factoring out } 3x \text{ and } \text{🦇} \\
&= \frac{x\left[3x(1+x^2)\text{🦇} + 4(2 + x^2 - \text{🎃})\right]}{4\,(1+x^2)^{\frac{3}{2}}} && \text{(8) Factoring out 4.}
\end{aligned}
$$

The trigonometric-hyperbolic nonsense we hid earlier was,

$$\text{🎃} = \tanh\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right), \qquad \text{🦇} = \sinh\left(\frac{x^3}{4}\right)\cos\left(\cosh\left(\frac{x^3}{4}\right)\right)\operatorname{sech}^2\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right).$$

Though if we immediately substitute these values back into our equation, we will end up with a derivative which this page is too small to contain. In this case, we must split our fraction into two, before finally substituting our values for 🎃 and 🦇 in the last line.

$$\begin{aligned}
f'(x) &= \frac{x\left[3x(1+x^2)\text{🦇} + 4(2+x^2-\text{🎃})\right]}{4\left(1+x^2\right)^{\frac{3}{2}}} \\
&= \frac{x}{4(1+x^2)^{\frac{3}{2}}}\left[4(2+x^2-\text{🎃}) + 3x(1+x^2)\text{🦇}\right] \\
&= \frac{x}{4(1+x^2)^{\frac{3}{2}}}\left[4\left(2+x^2-\tanh\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right)\right)\right. \\
&\qquad \left. + 3x(1+x^2)\sinh\left(\frac{x^3}{4}\right)\cos\left(\cosh\left(\frac{x^3}{4}\right)\right)\operatorname{sech}^2\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right)\right].
\end{aligned}$$

And there you have it!

The cursed function and its derivative:

$$f(x) = \frac{\tanh\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right) + x^2}{\sqrt{1+x^2}},$$

$$\begin{aligned}
f'(x) &= \frac{x}{4(1+x^2)^{\frac{3}{2}}}\left[4\left(2+x^2-\tanh\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right)\right)\right. \\
&\qquad \left. + 3x(1+x^2)\sinh\left(\frac{x^3}{4}\right)\cos\left(\cosh\left(\frac{x^3}{4}\right)\right)\operatorname{sech}^2\left(\sin\left(\cosh\left(\frac{x^3}{4}\right)\right)\right)\right].
\end{aligned}$$

Glorious! ♥

Further adventures await

My initial query of, "What are the points at which this parabola turns into a worm?" can now be answered. We have our derivative, and we can find the stationary points by solving the equation, $f'(x) = 0$. By inspection, we see that $x = 0$ is a solution to this equation, and substituting this value back into our function gives,

$$f(0) = \frac{\tanh\left(\sin\left(\cosh\left(\frac{0^3}{4}\right)\right)\right) + 0^2}{\sqrt{1 + 0^2}} = \tanh(\sin(1)) \approx 0.67.$$

This value is the point where the function intersects the y-axis. Further calculations will be needed to find the other solutions to $f'(x) = 0$, a task which I will save for another day. And of course, there is the ever looming question, "What is the indefinite integral? Can I eloquently explain its solution in a similar manner?"

Acknowledgements

I would like to thank my friend and algebraist for inspiring me to create this function. I hope that one day, they can look at an analysis in awe too, and their jaded view of the subject becomes rose tinted instead. ♥

References

[1] Graphics created using Desmos:
https://www.desmos.com/calculator/s5llcwb360

[2] For derivatives of trigonometric functions:
https://en.wikipedia.org/wiki/Differentiation_of_trigonometric_functions

[3] For derivatives of hyperbolic functions:
https://en.wikipedia.org/wiki/Hyperbolic_functions#Derivatives