Question 1

Let x be irrational and let r be rational and non-zero. Then

$$r = \frac{p}{q}, \qquad \text{where } p, q \in \mathbb{Z}, \qquad \text{and } p \neq 0, q \neq 0.$$

Towards a contradiction, assume $r + x$ is rational, that is

$$r + x = \frac{n}{m}, \qquad \text{where } n, m \in \mathbb{Z}, \qquad \text{and } m \neq 0.$$

Substituting in our expression for r, we have

$$\frac{p}{q} + x = \frac{n}{m}.$$

Solving for x, we obtain

$$x = \frac{nq - pm}{qm}.$$

As $nq - pm$ and qm are integers, it follows that x is rational, which is a contradiction. Thus $r + x$ must be irrational.

Define x and r in the same way, and towards a contradiction, assume rx is rational, that is

$$rx = \frac{s}{t} \qquad \text{where } s, t \in \mathbb{Z}, \qquad \text{and } t \neq 0.$$

Moreover,

$$\left(\frac{p}{q}\right)x = \frac{s}{t}.$$

Solving for x gives

$$x = \frac{sq}{pt}.$$

Since sq and pt are integers, it follows that x is rational, which is a contradiction. Thus rx must also be irrational. ■

Question 2

The number $\sqrt{12}$ is equal to $\sqrt{4 \cdot 3}$, which can further be simplified to $2\sqrt{3}$. Therefore, to show $\sqrt{12}$ is irrational, we will prove $\sqrt{3}$ is irrational.

Suppose $\sqrt{3}$ is rational, then

$$\sqrt{3} = p/q,$$

where p and q are nonzero integers that are comprime. Squaring both sides and making p the subject gives

$$p^2 = 3q^2. \tag{1}$$

Hence p^2 is divisible by 3, and by the Fundamental Theorem of Arithmetic, it follows that p is divisible by 3. We can now rewrite p as

$$p = 3k, \tag{2}$$

for some integer k. Substituting (2) into (1) yields

$$\begin{aligned} (3k)^2 &= 3q^2 \\ 9k^2 &= 3q^2 \\ 3k^2 &= q^2. \end{aligned}$$

As k^2 is an integer, it follows that q^2 and subsequently q are divisible by 3. This is a contradiction, as p and q were assumed to be coprime. Thus we conclude $\sqrt{3}$ is irrational.

Using the result from Question 1, we know that the product of a rational and irrational number is also irrational. Therefore $\sqrt{12}$ is irrational. ■