# 一、Rotation Motion

1、$SO(3)\in\{R\in R^{3\times 3}\mid R^TR=I,\ \det R=1\}\longrightarrow R^{-1}=R^T$

2、∧运算：$\hat{a}=\begin{bmatrix}0&-a_3&a_2\\a_3&0&-a_1\\-a_2&a_1&0\end{bmatrix}$，

$\hat{a}b=a\times b$

$R\hat{w}R^T=(Rw)^\wedge \xrightarrow[\text{可推}]{+R^TR=I} R(v\times w)=(Rv)\times(Rw)$

3、$so(3)=\{S\in R^{3\times 3},\ S^T=-S\}$，skew-symmetric

$\hat{w}\in so(3)$

if $\|w\|=1$, $e^{\hat{w}\theta}=I+\hat{w}\sin\theta+\hat{w}^2(1-\cos\theta)$

if $\|w\|\neq 1$, $e^{\hat{w}\theta}=I+\frac{\hat{w}}{\|w\|}\cdot\sin(\|w\|\theta)+\frac{\hat{w}^2}{\|w\|^2}(1-\cos\|w\|\theta)$

$\hat{w}^2=ww^T-\|w\|^2I$

$\hat{w}^3=-\|w\|\hat{w}$

4、根据 $R\longrightarrow w、\theta$

$R=e^{\hat{w}\theta}=\begin{bmatrix}w_1^2v_\theta+c_\theta & w_1w_2v_\theta-w_3s_\theta & w_1w_3v_\theta+w_2s_\theta\\ w_1w_2v_\theta+w_3s_\theta & w_2^2v_\theta+c_\theta & w_2w_3v_\theta-w_1s_\theta\\ w_1w_3v_\theta-w_2s_\theta & w_2w_3v_\theta+w_1s_\theta & w_3^2v_\theta+c_\theta\end{bmatrix}$

① $tr(R)=3$ or $R=I$，$\theta、w=0$，$tr=r_{11}+r_{22}+r_{33}$

② $tr(R)=-1\longrightarrow\cos\theta=-1\longrightarrow\theta=\pm\pi$

③ $-1<tr(R)<3$，$\theta=\arccos\frac{tr(R)-1}{2}\longrightarrow w=\frac{1}{2s\theta}\begin{bmatrix}r_{32}-r_{23}\\r_{13}-r_{31}\\r_{21}-r_{12}\end{bmatrix}$

5、Eular angle

$R_x(\varphi)=e^{\hat{x}\varphi}=\begin{bmatrix}1&0&0\\0&\cos\varphi&-\sin\varphi\\0&\sin\varphi&\cos\varphi\end{bmatrix}$

$R_Y(\theta)=e^{\hat{y}\theta}=\begin{bmatrix}\cos\theta&0&\sin\theta\\0&1&0\\-\sin\theta&0&\cos\theta\end{bmatrix}$

$R_Z(\gamma)=e^{\hat{z}\gamma}=\begin{bmatrix}\cos\psi&-\sin\psi&0\\\sin\psi&\cos\psi&0\\0&0&1\end{bmatrix}$

① XYZ Fix angle

$R_{ab}(\gamma,\beta,\alpha)=R_z(\alpha)R_y(\beta)R_x(\gamma)$

$=\begin{bmatrix}c\alpha c\beta & c\alpha s\beta s\gamma-s\alpha c\gamma & c\alpha s\beta c\gamma+s\alpha s\gamma\\ s\alpha c\beta & s\alpha s\beta s\gamma+c\alpha c\gamma & s\alpha s\beta s\gamma-c\alpha s\gamma\\ -s\beta & c\beta s\gamma & c\beta c\gamma\end{bmatrix}$

$\beta=atan2(-r_{31},\sqrt{r_{32}^2+r_{33}^2})$

$\alpha=atan2(r_{21}/c\beta,\ r_{11}/c\beta)$

$\gamma=atan2(r_{32}/c\beta,\ r_{33}/c\beta)$

② XYZ Eular

$R_{ab}(\varphi,\theta,\psi)=R_x(\varphi)R_y(\theta)R_z(\psi)$

$=\begin{bmatrix}c\theta c\psi & -c\theta s\psi & s\theta\\ s\varphi s\theta s\psi+c\varphi s\psi & -s\varphi s\theta s\psi+c\varphi c\psi & -s\varphi c\theta\\ -c\varphi s\theta c\psi+s\varphi s\psi & c\varphi s\theta s\psi+s\varphi c\psi & c\varphi c\theta\end{bmatrix}$

③ ZYZ Eular

$R_{ab}=R_z(\alpha)R_y(\beta)R_z(\gamma)$

$=\begin{bmatrix}c\alpha c\beta c\gamma-s\alpha s\gamma & -c\alpha c\beta s\gamma-s\alpha c\gamma & c\alpha s\beta\\ s\alpha c\beta c\gamma+c\alpha s\gamma & -s\alpha c\beta s\gamma+c\alpha c\gamma & s\alpha s\beta\\ -s\beta s\gamma & s\beta s\gamma & c\beta\end{bmatrix}$

$\beta=atan2(\sqrt{r_{31}^2+r_{32}^2},\ r_{33})$

$\alpha=atan2(r_{23}/s\beta,\ r_{13}/s\beta)$

$\gamma=atan2(r_{32}/s\beta,\ -r_{31}/s\beta)$

# 二、Rigid Motion

1、$SE(3)=\left\{\begin{bmatrix}R&P\\0&1\end{bmatrix}\in R^{4\times 4}\ \middle|\ P\in R^3, R\in SO^3\right\}$

$g^{-1}=\begin{bmatrix}R^T&-R^TP\\0&1\end{bmatrix}$

2、$g_{ac}=g_{ab}g_{bc}=\begin{bmatrix}R_{ab}R_{bc} & R_{ab}p_{bc}+p_{ab}\\0&1\end{bmatrix}$

3、Twist：$\hat{\xi}\in se(3)$

$\hat{\xi}=\begin{bmatrix}\hat{w}&-w\times q\\0&0\end{bmatrix}$，纯移动时：$\hat{\xi}=\begin{bmatrix}0&v\\0&0\end{bmatrix}$

4、Twist coordinate：$R^6$

$\xi:(v,w)=\begin{bmatrix}v\\w\end{bmatrix}=\begin{bmatrix}-w\times q\\w\end{bmatrix}$

$\xi \underset{\vee}{\overset{\wedge}{\rightleftarrows}} \hat{\xi}$

5、exp map from $se(3)\longrightarrow SE(3)$

$\hat{\xi}\in se(3)、\theta\in R\longrightarrow e^{\hat{\xi}\theta}\in SE(3)$

① translation，$w=0$

$e^{\hat{\xi}\theta}=\begin{bmatrix}I&v\theta\\0&1\end{bmatrix}$，$v$ 为单位平移轴

② $w\neq 0$，$\|w\|=1$

$e^{\hat{\xi}\theta}=\begin{bmatrix}e^{\hat{w}\theta} & (I-e^{\hat{w}\theta})w\times v+ww^Tv\theta\\0&1\end{bmatrix}$

6、$p(\theta)=e^{\hat{\xi}\theta}p(0)$

$g_{ab}(\theta)=e^{\hat{\xi}\theta}g_{ab}(0)$

7、screw associated with a twist

$\xi\in(v,w)\in R^6$

① Pitch：$h=\begin{cases}\frac{w^Tv}{\|w\|^2}, & w\neq 0\\ \infty, & w=0\end{cases}$

② Axis：$l=\begin{cases}\frac{w\times v}{\|w\|^2}+\lambda w, & \lambda\in R, w\neq 0\\ 0+\lambda v, & \lambda\in R, w=0\end{cases}$

③ $M=\begin{cases}\|w\|\\\|v\|\end{cases}$

screw：$h\neq\infty$

$l=\{q+\lambda w\ \|w\|=1, \lambda\in R\}$

$M$

$\overrightarrow{\text{Twist}}$：$\theta=M$

$\hat{\xi}=\begin{bmatrix}\hat{w}&-\hat{w}q+hw\\0&0\end{bmatrix}$

# 三、Velocity

1、Rotational vel

$v_{qa}(t)=\dot{R}_{ab}R_{ab}^{-1}q_a(a)$

$R(t)R^T=I\longrightarrow\dot{R}R^T+R\dot{R}^T=0$

$\longrightarrow\dot{R}R^T=-(\dot{R}R^T)^T\in so(3)$

$\hat{w}_{ab}^s=\dot{R}_{ab}R_{ab}^{-1}\longrightarrow w_{ab}^s$

$\hat{w}_{ab}^b=R_{ab}^{-1}\dot{R}\longrightarrow w_{ab}^b$

$v_{qa}=w_{ab}^s\times q_a$

2、Rigid Body

$q_a(t)=g_{ab}(t)\cdot q_b$

$\dot{q}_a(t)=v_{qa}(t)=\dot{g}_{ab}(t)q_b$

$=\dot{g}_{ab}\cdot g_{ab}^{-1}\cdot g_{ab}\cdot q_b=\hat{V}_{ab}^s\cdot q_a$

$\hat{V}_{ab}^s=\dot{g}_{ab}\cdot g_{ab}^{-1}=\begin{bmatrix}\dot{R}_{ab}R_{ab}^T & -\dot{R}_{ab}R_{ab}^Tp_{ab}+\dot{p}_{ab}\\0&0\end{bmatrix}$

$=\begin{bmatrix}\hat{w}_{ab}^s & -w_{ab}^s\times p_{ab}+\dot{p}_{ab}\\0&0\end{bmatrix}$

$=\begin{bmatrix}\hat{w}_{ab}^s & v_{ab}^s\\0&0\end{bmatrix}\in se(3)$

$v_{qa}=\hat{V}_{ab}^s\cdot q_a=w_{ab}^s\times q_a+v_{ab}^s$

$\hat{V}_{ab}^b=g_{ab}^{-1}\dot{g}_{ab}=\begin{bmatrix}R_{ab}^T\dot{R}_{ab} & R_{ab}^T\dot{p}_{ab}\\0&0\end{bmatrix}$

$\overset{\triangle}{=}\begin{bmatrix}\hat{w}_{ab}^b & v_{ab}^b\\0&0\end{bmatrix}$

$v_{qb}=g_{ab}^{-1}v_{qa}=g_{ab}^{-1}\dot{g}_{ab}\cdot q_b$

$=\hat{V}_{ab}^b\cdot q_b$

3、Adg

$\hat{V}_{ab}^s=g_{ab}\hat{V}_{ab}^b g_{ab}^{-1}$

$V_{ab}^s=\begin{bmatrix}v_{ab}^s\\w_{ab}^s\end{bmatrix}=\begin{bmatrix}R_{ab} & \hat{p}_{ab}R_{ab}\\0&R_{ab}\end{bmatrix}V_{ab}^b$

$Adg=\begin{bmatrix}R&\hat{P}R\\0&0\end{bmatrix}\in R^{6\times 6}$

twist b $\xrightarrow{Adg_{ab}}$ twist a

$Adg^{-1}=(Adg)^{-1}$

$g\hat{\xi}g^{-1}\longrightarrow Adg\xi$

4、V of screw

$\frac{d}{dt}e^{\hat{\xi}\theta(t)}=\dot{\theta}(t)\cdot e^{\hat{\xi}\theta(t)}\hat{\xi}$

$\hat{V}_{ab}^s=\dot{g}_{ab}g_{ab}^{-1}=\hat{\xi}\dot{\theta}\longrightarrow V_{ab}^s=\xi\dot{\theta}$

$\hat{V}_{ab}^b=g_{ab}^{-1}\cdot\dot{g}_{ab}=g_{ab}^{-1}(0)\hat{\xi}\dot{\theta}g_{ab}(0)$

$=(Adg_{ab(0)}^{-1}\xi)^\wedge\dot{\theta}\longrightarrow V_{ab}^b=Adg_{ab(0)}^{-1}\xi\dot{\theta}$

5、transformation of V

$V_{ac}^s=V_{ab}^s+Adg_{ab}V_{bc}^s$

$V_{ac}^b=Adg_{bc}^{-1}V_{ab}^b+V_{bc}^b$

# 四、Forward kinematics

1、$g_{st}(\theta)=e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_n\theta_n}g_{st}(0)$

$\xi_i=\begin{bmatrix}-w_i\times q_i\\w_i\end{bmatrix}$ or $\xi_i=\begin{bmatrix}v_i\\0\end{bmatrix}\longrightarrow e^{\hat{\xi}_i\theta_i}=\begin{bmatrix}I&v_i\theta_i\\0&1\end{bmatrix}$

2、Workspace

① $W=\{g_{st}(\theta)\mid\forall\theta\in Q\}\subset SE(3)$

② $W_R=\{p(\theta)\mid\forall\theta\in Q\}\subset R^3$

③ $W_D=\{p\in R^3\mid\forall R\in SO(3),\exists\theta, g_{st}(\theta)=(p,R)\}$

# 五、inverse kinematic

1、subproblem 1

$e^{\hat{\xi}\theta}p=q$

$u'=u-ww^Tu$

$v'=v-ww^Tv$

$\begin{cases}w^Tu=w^Tv\\\|u\|^2=\|v\|^2\end{cases}$

$u'\times v'=w\sin\theta\|u'\|\|v'\|$

$u'\cdot v'=\cos\theta\|u'\|\|v'\|$

$\longrightarrow\theta=atan2(w^T(u'\times v'),\ u'^Tv')$

2、subproblem2

$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p=q\longrightarrow e^{\hat{\xi}_2\theta_2}p=c=e^{-\hat{\xi}_1\theta_1}q$

$\begin{cases}w_2^Tu=w_2^Tz\\w_1^Tv=w_1^Tz\end{cases}$，$\|u\|=\|v\|=\|z\|$

$z=\alpha w_1+\beta w_2+\gamma(w_1\times w_2)$

$\longrightarrow\|z\|=\alpha^2+\beta^2+2\alpha\beta w_1^Tw_2+\gamma^2\|w_1\times w_2\|^2$

$\begin{cases}w_2^Tu=\alpha w_2^Tw_1+\beta\\w_1^Tv=\beta w_1^Tw_2+\alpha\end{cases}\longrightarrow\begin{cases}\alpha=\frac{(w_1^Tw_2)w_2^Tu-w_1^Tv}{(w_1^Tw_2)^2-1}\\\beta=\frac{(w_1^Tw_2)w_1^Tv-w_2^Tu}{(w_1^Tw_2)^2-1}\end{cases}$

$\|z\|^2=\|u\|^2=\frac{\|u\|^2-\alpha^2-\beta^2-2\alpha\beta w_1^Tw_2}{\|w_1\times w_2\|^2}$

3、subproblem3：$\|q-e^{\hat{\xi}\theta}p\|=\delta$

$\|v-e^{\hat{\xi}\theta}u\|^2=\delta^2$

$\delta'^2=\delta^2-|w^T(p-q)|^2$

$\delta^2=\|v-e^{\hat{\xi}\theta}u\|$

$=\|v'+ww^Tv-e^{\hat{\xi}\theta}(u'+ww^Tu)\|^2$

$=\|v'-e^{\hat{w}\theta}+\underbrace{ww^T(v-u)}_{ww^T(q-p)}\|^2$

$\|v'-e^{\hat{w}\theta}u'\|^2=\delta^2-\|w^T(p-q)\|^2=\delta'^2$

$\theta_0=atan2(w^T(u'\times v'),\ u'^Tv')$

$\phi=\theta_0-\theta\longrightarrow\|u'\|^2+\|v'\|^2-2\|u'\|\cdot\|v'\|\cos\phi=\delta'^2$

$\theta=\theta_0\pm\cos^{-1}\frac{\|u'\|+\|v'\|-\delta'^2}{2\|u'\|\|v'\|}$

# 六、Jacobian

1、$J_{st}^s(\theta)$

$\hat{V}_{st}^s=\dot{g}_{st}(\theta)g_{st}^{-1}(\theta)=\sum_{i=1}^n\left(\frac{\partial g_{st}}{\partial\theta_i}\cdot\dot{\theta}_i\right)\cdot g_{st}^{-1}$

$\longrightarrow V_{st}^s=\sum_{i=1}^n\left(\frac{\partial g_{st}}{\partial\theta_i}\cdot g_{st}\right)^\vee\theta_i$

$\longrightarrow\quad=\left[\left(\frac{\partial g_{st}}{\partial\theta_1}g_{st}^{-1}(\theta)\right)^\vee\cdots\right]\begin{bmatrix}\dot{\theta}_1\\\vdots\end{bmatrix}$

$=J_{st}^s(\theta)\theta_i\quad(J_{st}^s(\theta)\in R^{6\times n})$

$J_{st}^s(\theta)$ 第 i 列：$\xi_i'$

$\xi_i'=\left(\frac{\partial g_{st}}{\partial\theta_i}\cdot g_{st}^{-1}\right)^\vee$

$=Ad(e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_{i-1}\theta_{i-1}})\xi_i=\begin{bmatrix}-w_i'\times q_i'\\w_i'\end{bmatrix}$

$w_i'=e^{\hat{w}_1\theta_1}\cdots e^{\hat{w}_{i-1}\theta_{i-1}}w_i$

$q_i'=e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_{i-1}\theta_{i-1}}q_i$

2、$J_{st}^b(\theta)$

$J_{st}^b(\theta)=[\xi_1^+\ \ \xi_2^+\cdots\xi_n^+]$

$\xi_i^+=Ad^{-1}(e^{\hat{\xi}_i\theta_i}\cdots e^{\hat{\xi}_n\theta_n}g_{st}(0))\xi_i$

$J_{st}^s(\theta)=Adg_{st}(\theta)\cdot J_{st}^b(\theta)$

3、有平动轴时 $\xi_i'$ 的计算

$\xi_i=\begin{bmatrix}v_i'\\0\end{bmatrix}$，$v_i'=e^{\hat{w}_1\theta_1}\cdots e^{\hat{w}_{i-1}\theta_{i-1}}v$

# 七、singularities

1、Definition：$\exists\dot{\theta}\neq 0$, s.t. $V_{st}^s=J_{st}^s(\theta)\cdot\dot{\theta}=0$

2、example for 6R

① two collinear revolute joints

$\longrightarrow w_1=\pm w_2$

$w_i\times(q_1-q_2)=0$

$J(\theta)=\begin{bmatrix}-w_1\times q_1 & -w_2\times\overset{\to 0}{(q_2-q_1)} & \cdots\\w_1&0&\cdots\end{bmatrix}$

② three parallell coplanar revolute joint axes

$\longrightarrow w_i=\pm w_j$，$\exists n$，$n^Tw_i=0$，$n^T(q_i-q_j)=0$

③ four intersecting revolute joints axes with intersection q

$\longrightarrow w_i\times(q_i-q)=0$

以 q 为原点 $\longrightarrow J(\theta)=\begin{bmatrix}0&0&0&0&\cdots\\w_1&w_2&w_3&w_4&\cdots\end{bmatrix}$

# 八、Lagrange's Equations

1、拉格朗日函数

$L(q,\dot{q})=T(q,\dot{q})-V(q)$

$T=\frac{1}{2}\sum m_i\|\dot{r}_i^2\|$

2、拉格朗日方程

$\frac{d}{dt}\cdot\frac{\partial L}{\partial\dot{q}_i}-\frac{\partial L}{\partial q_i}=\tau_i$

3、动力学方程（见题）

# 九、kinetic energy of a rigid body

1、$m=\int_V\rho(r)dV$

$\bar{r}=\frac{1}{m}\int_V\rho(r)r\,dV$

2、inertia tensor

$I=\begin{bmatrix}I_{xx}&I_{xy}&I_{xz}\\I_{yx}&I_{yy}&I_{yz}\\I_{zx}&I_{zy}&I_{zz}\end{bmatrix}=-\int_V\rho(r)\hat{r}^2dV$

$I_{xx}=\int_V\rho(y^2+z^2)dxdydz$

$I_{xy}=-\int_V\rho(r)(xy)dxdydz$

3、动能

$T=\frac{1}{2}\int_V\rho(r)\|\dot{p}+\dot{R}r\|dv$

$=\frac{1}{2}m\|\dot{p}\|^2+\frac{1}{2}\int_V\rho(r)\|\dot{R}r\|^2dV$

$\hookrightarrow\frac{1}{2}w^TIw$

w: body angular vel

$w^b=(R_{ab}^{-1}\dot{R}_{ab})^\vee$

$T=\frac{1}{2}(V^b)^T\begin{bmatrix}mI&0\\0&\mathcal{I}\end{bmatrix}V^b$

$M_b=\begin{bmatrix}mI&0\\0&\mathcal{I}\end{bmatrix}$

# 十、Dynamics of Open-chain

目标：$M(\theta)\ddot{\theta}+C(\theta,\dot{\theta})\dot{\theta}+N(\theta,\dot{\theta})=\tau$

① $M(\theta)=\sum_{i=1}^n J_i^T(\theta)M_iJ(\theta)$

$M_i$ 为广义惯性矩阵，$M_i=\begin{bmatrix}mI&&&\\&I_{xi}&&\\&&I_{yi}&\\&&&I_{zi}\end{bmatrix}$

$J_i=[\xi_1^+\ \cdots\ \xi_i^+, 0, 0, \cdots 0]$

$\xi_j^+=Ad^{-1}(e^{\hat{\xi}_j\theta_j}\cdots e^{\hat{\xi}_i\theta_i})\xi_j$，$j\leq i$

② $C(\theta,\dot{\theta})$

$C_{ij}(\theta,\dot{\theta})=\sum_{k=1}^n T_{ijk}\dot{\theta}_k$

$=\frac{1}{2}\sum_{k=1}^n\left(\frac{\partial M_{ij}}{\partial\theta_k}+\frac{\partial M_{ik}}{\partial\theta_j}-\frac{\partial M_{kj}}{\partial\theta_i}\right)\cdot\dot{\theta}_k$

求 $T_{ijk}$ 就行

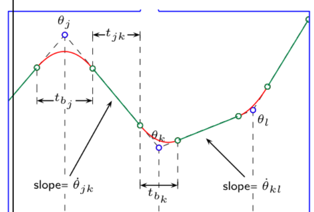
③ $V=\sum_{i=1}^n m_igh_i(\theta_i)$

$N_i(\theta,\dot{\theta})=\frac{\partial V}{\partial\theta_i}$

# 八、LFPB

give $t_0$、$\theta_0$、$t_f$、$\theta_f$ $\ddot{\theta}_b$，via $\{\theta_i\}$ at $\{t_i\}$

① 对 via point



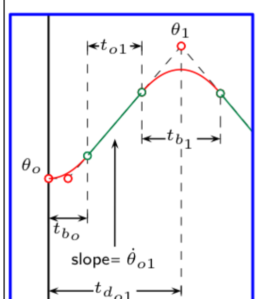

$\dot{\theta}_{jk}=\frac{\theta_k-\theta_j}{t_{djk}}$

$\ddot{\theta}_k=sgn(\dot{\theta}_{kl}-\dot{\theta}_{jk})|\ddot{\theta}_b|$

$t_{bk}=\frac{\dot{\theta}_{kl}-\dot{\theta}_{jk}}{\ddot{\theta}_k}$

$t_{jk}=t_{djk}-\frac{1}{2}t_{bj}-\frac{1}{2}t_{bk}$

② First Segment




$\ddot{\theta}_0=sgn(\theta_1-\theta_0)|\ddot{\theta}_b|$

$t_{b0}=t_{d01}-\sqrt{t_{d01}^2-\frac{2(\theta_1-\theta_0)}{\ddot{\theta}_0}}$

$\dot{\theta}_{01}=\frac{\theta_1-\theta_0}{t_{d01}-\frac{1}{2}t_{b0}}$

$t_{01}=t_{d01}-t_{b0}-\frac{1}{2}t_{b1}$

③ Last Segment

$\ddot{\theta}_f=sgn(\theta_m-\theta_f)|\ddot{\theta}_b|$

$t_{bf}=t_{dmf}-\sqrt{t_{dmf}^2+\frac{2(\theta_f-\theta_m)}{\ddot{\theta}_f}}$

$\dot{\theta}_{mf}=\frac{\theta_f-\theta_m}{t_{dmf}-\frac{1}{2}t_{bf}}$

$t_{mf}=t_{dmf}-t_{bf}-\frac{1}{2}t_{bm}$

# Exercise

## 一、Rotation Motion

1、proof：$R\hat{\omega}R^T=(R\omega)^\wedge$

设 $R=(X_{ab}, Y_{ab}, Z_{ab})$

$\omega=(a, b, c)^T$

$\therefore R\omega = aX_{ab}+bY_{ab}+cZ_{ab}$

$(R\omega)\times R$

$=[-bZ_{ab}+cY_{ab},\ aZ_{ab}-cX_{ab},\ -aY_{ab}+bX_{ab}]$

$R^T(R\omega)\times R = \begin{bmatrix} X_{ab}^T \\ Y_{ab}^T \\ Z_{ab}^T \end{bmatrix}$

$=\begin{bmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{bmatrix}=\hat{\omega}$

$\longrightarrow R^T(R\omega)^\wedge R=\hat{\omega}$

$\longrightarrow (R\omega)^\wedge = R\hat{\omega}R^T$

2、向量空间证明

①加法、数乘封闭

② $(a+b)+c = a+(b+c)$

③零元，$a+0=a$

④ $1, a1=a$

⑤ $k、l\in R$，$(kl)a=k(la)=l(ka)$

$(k+l)a=ka+la$

$k(a+b)=ka+kb$

3、给定 $g=\begin{bmatrix} R & P \\ 0 & 1 \end{bmatrix}$，求 $\xi=\begin{bmatrix} v \\ \omega \end{bmatrix}$

设 $A=(I-e^{\hat{\omega}\theta})\hat{\omega}+\omega\omega^T\theta$

$v=A^{-1}p$，$\theta$ 为弧度

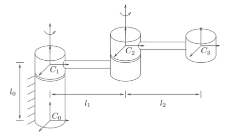
1. Figure 1 shows a two degree of freedom manipulator. Let $l_0, l_1, l_2$ be the link length parameters and $\theta_1, \theta_2$ the joint angle variables of link 1 and link 2, respectively.

(a) Express the position and orientation of frame $C_3$ relative to frame $C_0$ in terms of the joint angle variables and the link parameters.

(b) Compute the spatial velocity of $C_3$ relative to $C_0$ as functions of the joint angles and the joint rates.

(c) Compute the body velocity of $C_3$ relative to $C_0$ as functions of the joint angles and the joint rates.

(d) *Optional:* Find the spatial velocity of the origin of $C_3$ and use this to check your answer for parts (b) and (c).




(a) $g_{03}=g_{01}g_{12}g_{23}$

or $g_{03}=e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}\cdot g_{st}(0)$

$g_{12}=\begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b) $V_{ab}^s=\begin{bmatrix} v_{ab}^s \\ \omega_{ab}^s \end{bmatrix}=\begin{bmatrix} -\dot{R}_{ab}R_{ab}^T P_{ab}+\dot{P}_{ab} \\ (\dot{R}_{ab}R_{ab}^T)^\vee \end{bmatrix}$

$V_{ab}^b=\begin{bmatrix} v_{ab}^b \\ \omega_{ab}^b \end{bmatrix}=\begin{bmatrix} R_{ab}^T\dot{P}_{ab} \\ (R_{ab}^T\dot{R}_{ab})^\vee \end{bmatrix}$

$V_{ac}^s=V_{ab}^s+Ad_{g_{ab}}V_{bc}^s$

由 $g_{01}\longrightarrow V_{01}^s=[0,0,0,0,0,\dot{\theta}_1]$

由 $g_{12}\longrightarrow V_{12}^s=[l_1\dot{\theta}_2,0,0,0,0,\dot{\theta}_2]$

由 $g_{23}\longrightarrow V_{23}^s=0$

$V_{03}^s=V_{02}^s+Ad_{g_{02}}V_{23}^s$

$=V_{02}^s=V_{01}^s+Ad_{g_{01}}V_{12}^s$

$=V_{01}^s+\begin{bmatrix} R_{01} & \hat{P}_{01}R_{01} \\ 0 & R_{01} \end{bmatrix}V_{12}^s$

$=[l_1\dot{\theta}_2\cos\theta_1,\ l_1\dot{\theta}_2\sin\theta_1, 0,0,0,\dot{\theta}_1+\dot{\theta}_2]$

(c) $V_{ac}^b=Ad_{g_{bc}^{-1}}V_{ab}^b+V_{bc}^b$

$V_{02}^b=Ad_{g_{12}^{-1}}V_{01}^b+V_{12}^b$

$V_{03}^b=Ad_{g_{23}^{-1}}V_{02}^b+V_{23}^b$

$V_{01}^b=[0,0,0,0,0,\dot{\theta}_1]$

$V_{12}^b=[0,0,0,0,0,\dot{\theta}_2]$

$V_{23}^b=0$

$V_{02}^b=Ad_{g_{12}^{-1}}V_{01}^b+V_{12}^b$

$V_{03}^b=Ad_{g_{23}^{-1}}V_{02}^b+V_{23}^b$

$V_{qa}=g_{ab}V_{qb}$

$V_{qb}=g_{ab}^{-1}V_{qa}$

## 二、正运动学题目

① $\omega$、$q$ 列出来

② 求 $\xi_i$

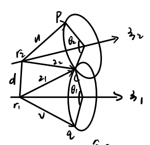
③平动轴的 $e^{\hat{\xi}\theta}$

$\begin{bmatrix} I & v\theta \\ 0 & 1 \end{bmatrix}$

## 三、逆运动学

1. *Subproblem* $2'$*: Rotation about two non-intersecting axes*

Solve Subproblem 2 when the two axes $\xi_1$ and $\xi_2$ do not intersect and write the Matlab program function.




① $e^{\hat{\xi}_2\theta_2}p=c=e^{-\hat{\xi}_1\theta_1}q$

② $e^{\hat{\xi}_2\theta_2}(p-r_2)=e^{\hat{\xi}_2\theta_2}u=c-r_2=z_2$

③ $e^{-\hat{\xi}_1\theta_1}(q-r_1)=e^{-\hat{\xi}_1\theta_1}v=c-r_1=z_1$

$d=r_2-r_1=z_1-z_2$

$\longrightarrow \omega_1^Tv=\omega_1^Tz_1,\ \|v\|^2=\|z_1\|^2$

$\omega_2^Tu=\omega_2^Tz_2,\ \|u\|^2=\|z_2\|^2$

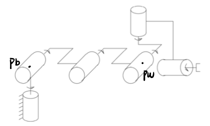
$z_2=\alpha\omega_1+\beta\omega_2+\gamma(\omega_1\times\omega_2)$

$z_1=\alpha\omega_1+\beta\omega_2+\gamma(\omega_1\times\omega_2)+d$

④ $\begin{cases} \|u\|^2=\alpha^2+\beta^2+2\alpha\beta\,\omega_1^T\omega_2+\gamma^2\|\omega_1\times\omega_2\|^2 \\ \omega_1^Tv=\alpha+\beta\,\omega_1^T\omega_2+\omega_1^Td \\ \omega_2^Tu=\beta+\alpha\,\omega_1^T\omega_2 \end{cases}$

得 $\begin{cases} \alpha=\dfrac{\omega_1^T\omega_2\omega_2^Tu-\omega_1^T(v-d)}{(\omega_1^T\omega_2)^2-1} \\ \beta=\dfrac{\omega_1^T\omega_2\omega_1^T(v-d)-\omega_2^Tu}{(\omega_1^T\omega_2)^2-1} \\ \gamma^2=\dfrac{\|u\|^2-\alpha^2-\beta^2-2\alpha\beta\omega_1^T\omega_2}{\|\omega_1\times\omega_2\|^2} \end{cases}$

2、Elow 逆运动学




① solve $\theta_3$

$e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_6\theta_6}=g_d g_{st}^{-1}(0)=g_1$

$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}p_w=g_1p_w$

$\rightarrow e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}(e^{\hat{\xi}_3\theta_3}p_w-p_b)=g_1p_w-p_b$

$\rightarrow\|e^{\hat{\xi}_3\theta_3}p_w-p_b\|=\|g_1p_w-p_b\|$

② solve $\theta_1$、$\theta_2$

$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}(e^{\hat{\xi}_3\theta_3}p_w)=g_1p_w$

sub 2

③ $\theta_4$、$\theta_5$

$e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}e^{\hat{\xi}_6\theta_6}=e^{-\hat{\xi}_3\theta_3}e^{-\hat{\xi}_2\theta_2}e^{-\hat{\xi}_1\theta_1}g_dg_{st}^{-1}(0)$

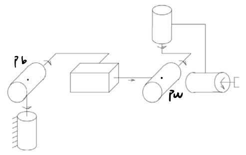
$=g_2$

$e^{\hat{\xi}_4\theta_4}e^{\hat{\xi}_5\theta_5}p_b=g_2p_b\longrightarrow$ sub 2

④ sub 1

3、Stanford

平移轴求解

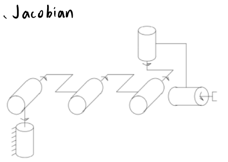



$e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}p_w=g_1p_w$

$\longrightarrow\|e^{\hat{\xi}_3\theta_3}p_w-p_b\|=\|g_1p_w-p_b\|=\delta$

$\longrightarrow\theta_3=\delta-l_1$

## 四、Jacobian



1、Spatial：

①定义 $\omega_1\sim\omega_6$，$q_1\sim q_6$

②求 $\omega'$，$q'\longrightarrow\xi'=\begin{bmatrix} -\omega'\times q' \\ \omega' \end{bmatrix}$

$\omega_1'=\omega$，$q_1'=q_1$，$\xi_1'=\xi_1$

$\omega_2'=R_{\omega_1}(\theta_1)\omega_2$，$q_2'=q_2$，

$\omega_3'=R_{\omega_1}(\theta_1)\omega_3$，$q_3'=e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}q_3$，

$\omega_4'=R_{\omega_1}(\theta_1)\omega_4$，$q_4'=e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}q_4$

$\omega_5'=R_{\omega_1}(\theta_1)\cdots R_{\omega_4}(\theta_4)\omega_5$，$q_5'=q_4'$

$\omega_6'=R_{\omega_1}(\theta_1)\cdots R_{\omega_5}(\theta_5)\omega_6$，$q_6'=q_5'$

$J^s(\theta)=[\xi_1'\ \xi_2'\ \xi_3'\ \xi_4'\ \xi_5'\ \xi_6']$

2、Body

$J^b(\theta)=Ad_{g_{st}(\theta)}^{-1}J^s(\theta)$

$=[\xi_1^\dagger\ \cdots\ \xi_6^\dagger]$

$\xi_i^\dagger=Ad^{-1}_{(e^{\hat{\xi}_i\theta_i}\cdots e^{\hat{\xi}_n\theta_n}g_{st}(0))}\xi_i$

把每一个的公式列出来

3、有平移轴情况

$v_i'=R_1(\theta_1)\cdots R_{i-1}(\theta_{i-1})v_i$，$\xi_i'=\begin{bmatrix} v_i' \\ 0 \end{bmatrix}$

## 五、Singularity

1、三种情况

2、ZYZ Eular angle 奇异证明：$\alpha=0$

$\omega_1'=\omega_1=\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$，

$\omega_2'=R_{\omega_1}(\alpha)\omega_2=\begin{bmatrix} -s\alpha \\ c\alpha \\ 0 \end{bmatrix}$

$\omega_3'=R_{\omega_1}(\alpha)R_{\omega_2}(\beta)\omega_3=\begin{bmatrix} c\alpha s\beta \\ s\alpha s\beta \\ c\beta \end{bmatrix}$

$J^s=[\omega_1'\ \omega_2'\ \omega_3']$

$\operatorname{rank}(J)<3\longrightarrow\dfrac{-\sin\alpha}{c\alpha s\beta}=\dfrac{\cos\alpha}{s\alpha s\beta}$

$\longrightarrow\sin\beta=0$，$\beta=k\pi$，$k\in Z$

3、ZYX 奇异

$\omega_1'=[0\ 0\ 1]^T$

$\omega_2'=[-s\alpha\ c\alpha\ 0]^T$

$\omega_3'=[c\alpha c\beta\ c\alpha s\beta\ -s\beta]^T$

$J^s=[\omega_1'\ \omega_2'\ \omega_3']$

rank < 3：$\dfrac{-\sin\alpha}{c\alpha c\beta}=\dfrac{c\alpha}{c\alpha c\beta}$

$\cos(\alpha-\beta)=0$，$\alpha-\beta=\frac{\pi}{2}+k\pi$

4、4个共面旋转轴

设轴在 $x$-$y$ 平面上

$\omega_i=[x_i\ y_i\ 0]^T$

$q_i=[q_{ix}\ q_{iy}\ 0]^T$

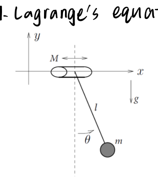
$-\omega_i\times q_i=[0\ 0\ y_iq_{ix}-x_iq_{iy}]$

$J=\begin{bmatrix} 0 & & 0 & & \\ 0 & & 0 & & \\ y_1q_{1x}-x_1q_{1y} & \cdots & y_4q_{4x}-x_4q_{4y} & \xi_5' & \xi_6' \\ x_1 & & x_4 & & \\ y_1 & & y_4 & & \\ 0 & & 0 & & \end{bmatrix}$

每列都是3维向量

## 五、Dynamic

1、Lagrange's equation




①广义坐标的选取：

每一个能动的自由度即为广义坐标、

$x$、$\theta$

②求 $T=\frac{1}{2}\sum m_i\|\dot{r}_i\|^2$

$r_m(x,\theta)=\begin{bmatrix} x+l\sin\theta \\ -l\cos\theta \end{bmatrix}$

$\dot{r}_m=\begin{bmatrix} \dot{x}+l\dot{\theta}\cos\theta \\ l\dot{\theta}\sin\theta \end{bmatrix}$

$r_M=\begin{bmatrix} x \\ 0 \end{bmatrix}$，$\dot{r}_M=\begin{bmatrix} \dot{x} \\ 0 \end{bmatrix}$

$T=\frac{1}{2}M\dot{x}^2+\frac{1}{2}m\dot{r}^T\dot{r}$

③势能 $V$

$V=-mgl\sin\theta$

④拉格朗日函数

$L=T-V$

$=\frac{1}{2}M\dot{x}+\frac{1}{2}m[(\dot{x}+l\dot{\theta}\cos\theta)^2+(l\dot{\theta}\sin\theta)^2]+mgl\cos\theta$

$=\frac{1}{2}(M+m)\dot{x}^2+m\dot{x}l\dot{\theta}\cos\theta+\frac{1}{2}ml^2\dot{\theta}^2+mgl\cos\theta$

⑤拉格朗日方程

$\frac{d}{dt}\cdot\frac{\partial L}{\partial\dot{x}}=(M+m)\ddot{x}+ml\ddot{\theta}\cos\theta-ml\dot{\theta}^2\sin\theta$

$\frac{\partial L}{\partial x}=0$

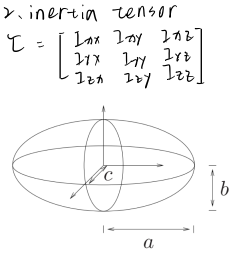
$\frac{d}{dt}\cdot\frac{\partial L}{\partial\dot{\theta}}=ml\ddot{x}\cos\theta-m\dot{x}l\dot{\theta}\sin\theta+ml^2\ddot{\theta}$

$\frac{\partial L}{\partial\theta}=-m\dot{x}l\dot{\theta}\sin\theta-mgl\sin\theta$

$\Longrightarrow\begin{bmatrix} M+m & ml\cos\theta \\ ml\cos\theta & ml^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix}+\begin{bmatrix} -ml\dot{\theta}^2\sin\theta \\ 0 \end{bmatrix}+\begin{bmatrix} 0 \\ mgl\sin\theta \end{bmatrix}=0$

2、inertia tensor

$\mathcal{I}=\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$




$I_{xx}=\int_V\rho(y^2+z^2)\,dx\,dy\,dz$

$=\rho\int_0^{2\pi}d\theta\int_0^{2\pi}d\varphi\int_0^1 a^2(a^2s^2\varphi s^2\theta+b^2c^2\varphi)abcr^2s\varphi dr$

$=\frac{abc\rho}{5}\left[\frac{4\pi c^2}{3}+\frac{4\pi b^2}{3}\right]$

$=\frac{4\pi abc\rho(b^2+c^2)}{15}$

## 六、速度规划

1、LFPB

1. Calculate $\dot{\theta}_{12}, \dot{\theta}_{23}, t_1, t_2,$ and $t_3$ for a two-segment linear function with parabolic blends (LFPB). For this joint, $\theta_1=5.0^\circ, \theta_2=15.0^\circ, \theta_3=40.0^\circ$. Assume that $t_{d12}=t_{d23}=1.0$ seconds and that the default acceleration to use during blends is 80 degrees/second$^2$. Use Matlab to sketch plots of position, velocity, and acceleration of $\theta$. The derivation, Matlab code, and result graphs are required.

① First Segment

$\ddot{\theta}_1=\operatorname{sgn}(\theta_2-\theta_1)|\ddot{\theta}_b|=80^\circ/s^2$

$t_{b1}=t_{d12}-\sqrt{t_{d12}^2-\dfrac{2(\theta_2-\theta_1)}{\ddot{\theta}_1}}\approx0.1345$

$\dot{\theta}_{12}=\dfrac{\theta_2-\theta_1}{t_{d12}-\frac{1}{2}t_{b1}}\approx10.718^\circ/s$

$t_{12}=t_{d12}-t_{b1}-\frac{1}{2}t_{b2}$

② Last Segment

$\ddot{\theta}_3=\operatorname{sgn}(\theta_2-\theta_3)|\ddot{\theta}_b|=80^\circ/s^2$

$t_{b3}=t_{d23}-\sqrt{t_{d23}+\dfrac{2(\theta_3-\theta_1)}{\ddot{\theta}_3}}\approx0.3885$

$\dot{\theta}_{23}=\dfrac{\theta_3-\theta_2}{t_{d23}-\frac{1}{2}t_{b3}}\approx31.01^\circ/s$

$t_{23}=t_{d23}-t_{b3}-\frac{1}{2}t_{b2}$

③ Via point

$\ddot{\theta}_2=\operatorname{sgn}(\dot{\theta}_{23}-\dot{\theta}_{12})\cdot|\ddot{\theta}_b|=80^\circ/s$

$t_{b2}=\dfrac{\dot{\theta}_{23}-\dot{\theta}_{12}}{\ddot{\theta}_2}\approx0.254$

$\longrightarrow t_{12}\approx0.7395$

$t_{23}\approx0.4855$

2. Use Matlab to sketch graphs of position, velocity, and acceleration for a two-segment spline where each segment is a cubic. Sketch them for a joint where $\theta_0=5.0^\circ$ for the initial point, $\theta_v=15.0^\circ$ is a via point, and $\theta_g=-10.0^\circ$ is the goal point. Assume that each segment has a duration of 2.0 seconds and that the velocity at the via point is to be 0.0 degrees/second. The derivation, Matlab code, and result graphs are required.

设 $\theta_1(t)=a_0+a_1t+a_2t^2+a_3t^3$

$\theta_2(t)=b_0+b_1t+b_2t^2+b_3t^3$

由 $\begin{cases} \theta_1(0)=a_0=\theta_0=5 \\ \dot{\theta}_1(0)=a_1=0 \\ \theta_1(2)=a_0+2a_1+4a_2+8a_3=\theta_v=15^\circ \\ \dot{\theta}_1(2)=a_1+4a_2+12a_3=0 \end{cases}$

$\theta_2(t)$ 同理