General rigid motion

1、properties

$\|g(p)-g(q)\| = \|p-q\|$

$v_1^T v_2 = g_*(v_1)^T g_*(v_2)$

$g_*(\vec{v}\times\vec{w}) = g_*(\vec{v})\times g_*(\vec{w})$

$$\|R_{ab}(p_b-p_a)\|^2 = (R_{ab}(p_b-p_a))^T R_{ab}(p_b-p_a) = (p_b-p_a)^T R_{ab}^T R_{ab}(p_b-p_a) = \|p_b-p_a\|^2$$

2、homogeneous representation

point $\begin{bmatrix} x\\ y\\ z\\ 1\end{bmatrix}$ vector $\begin{bmatrix} x\\ y\\ z\\ 0\end{bmatrix}$ $\begin{cases} v+v=v\\ v+p=p\\ p-p=v\\ p+p=N/A\end{cases}$

3. configuration

☆ g_{ab} 可看成 A 対 B 变换 也可看成 B 対 A 的位姿

$g=\begin{bmatrix} R & p\\ 0 & 1\end{bmatrix}_{4\times4}$ R: rotation, p: translation $g^{-1}=\begin{bmatrix} R^T & -R^Tp\\ 0 & 1\end{bmatrix}$

$g\in SE(3)$

$R\in SO(3)$

表示位姿，位 = p，姿 = R

$(G,\triangle)$ is a group if

1. △运算封闭
2. △存在 1 元
3. 任意元素可逆
4. △有结合律

4、Transform matrix $\bar{g}_{ac}=\bar{g}_{ab}\cdot\bar{g}_{bc}=\begin{bmatrix} R_{ab}R_{bc} & R_{ab}p_{bc}+p_{ab}\\ 0 & 1\end{bmatrix}$ composition rule

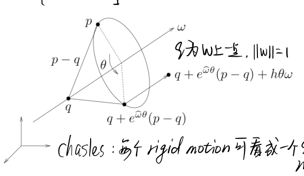
$\begin{cases}\text{显然易证 } \exp(\hat{\xi}\theta) \text{ 的形式} \in SE(3)\\ \text{在旧 } g \text{ 阵左乘得到新 } g,\ e^{\hat{\xi}\theta}\in SE(3)\\ \text{逆矩阵表示逆变换}\end{cases}$

$$e^{\hat{\xi}\theta}=\begin{bmatrix} R & (I-R)\omega\times v+\omega\omega^Tv\theta\\ 0 & 1\end{bmatrix}=\begin{bmatrix} R & (I-R)q+\omega h\theta\\ 0 & 1\end{bmatrix}$$

其中 $\hat{\xi}=\begin{bmatrix}\hat{\omega} & v\\ 0 & 0\end{bmatrix}$ 称为 twist

解 ω,θ,v: 1. $e^{\hat{\omega}\theta}=R$ 2. $A v = p$

$\xi=\begin{bmatrix} v=-\omega\times q+h\omega\\ \omega\end{bmatrix}$ 称为 twist coords



Chasles: 每个 rigid motion 可看成一个 screw motion

若无平移，令 $h=0$

若无旋转，令 $\xi=\begin{bmatrix} v\\ 0\end{bmatrix}$，$e^{\hat{\xi}\theta}=\begin{bmatrix} I & \theta v\\ 0 & 1\end{bmatrix}$

5. 只看 R，旋转矩阵

$\begin{cases} RR^T=R^TR=I & \Rightarrow R^T=R^{-1}\\ \det R = r_1^T(r_2\times r_3)=1\end{cases}$

复合旋转 $R_{ac}=R_{ab}R_{bc}$ $\begin{cases} g_a=R_{ab}g_b\\ g_b=R_{bc}g_c\end{cases}\Rightarrow g_a=R_{ac}g_c$ 顺序见 T fixed, euler

$R\in SO(3)$, $SO(n)=\{A\in\mathbb{R}^{n\times n}\mid RR^T=I, |R|=1\}$

$\hat{\omega}\in so(3)$, $so(n)=\{A\in\mathbb{R}^{n\times n}\mid A^T=-A\}$

$\vec{v}\times\vec{w}=\hat{v}w$, $\vec{v}\cdot\vec{w}=v^Tw$

$so(3) \xrightarrow{\text{onto}} SO(3)$; $\hat{\omega}$ / R; axis / rotation; $n \xrightarrow{e^{\hat{\omega}\theta}} 1$; surjective; $\theta+2k\pi$

$\omega=\begin{bmatrix}\omega_1\\ \omega_2\\ \omega_3\end{bmatrix}$ $\hat{\omega}=\begin{bmatrix} 0 & -\omega_3 & \omega_2\\ \omega_3 & 0 & -\omega_1\\ -\omega_2 & \omega_1 & 0\end{bmatrix}$

$\hat{\omega}^2=\omega\omega^T-\|\omega\|^2 I$

$\hat{\omega}^3=-\|\omega\|\hat{\omega}$

$R=e^{\hat{\omega}\theta}=I+\theta\hat{\omega}+\frac{\theta^2}{2!}\hat{\omega}^2+\cdots$

$=I+\frac{\hat{\omega}}{\|\omega\|}\sin(\|\omega\|\theta)+\frac{\hat{\omega}^2}{\|\omega\|^2}(1-\cos(\|\omega\|\theta))$

$=\begin{bmatrix} r_{11} & r_{12} & r_{13}\\ r_{21} & r_{22} & r_{23}\\ r_{31} & r_{32} & r_{33}\end{bmatrix}$

$$I+\frac{\hat{\omega}}{\|\omega\|}\left(\|\omega\|\theta-\frac{(\|\omega\|\theta)^3}{3!}+\frac{(\|\omega\|\theta)^5}{5!}-\cdots\right)+\frac{\hat{\omega}^2}{\|\omega\|^2}\left(\frac{(\|\omega\|\theta)^2}{2!}-\frac{(\|\omega\|\theta)^4}{4!}+\frac{(\|\omega\|\theta)^6}{6!}-\cdots\right)$$

由 mat 求 ω,θ 3种情况 见右边

$\begin{cases}\theta=\cos^{-1}\left(\frac{tr(R)-1}{2}\right)\\ \omega=\frac{1}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23}\\ r_{13}-r_{31}\\ r_{21}-r_{12}\end{bmatrix}_{3\times1}\\ \hat{\omega}\theta=\frac{\theta}{2\sin\theta}(R-R^T)\\ \omega\theta=\frac{\theta}{2\sin\theta}\begin{bmatrix} r_{32}-r_{23}\\ r_{13}-r_{31}\\ r_{21}-r_{12}\end{bmatrix}\Rightarrow \text{expo coords for R}\end{cases}$

由角表示旋转阵 (Euler theorem：可以分解为三次旋转定理)

理解 Euler 表示 相对随动的: XYZ Euler angle $R_XR_YR_Z$ (右乘轴在动); XYZ fixed $R_ZR_YR_X$ (左乘轴不动)

//同理 ZYX Euler 同 RPY

$$R_x(\varphi):=e^{\hat{x}\varphi}=\begin{bmatrix} 1 & 0 & 0\\ 0 & \cos\varphi & -\sin\varphi\\ 0 & \sin\varphi & \cos\varphi\end{bmatrix}$$

$$R_y(\theta):=e^{\hat{y}\theta}=\begin{bmatrix}\cos\theta & 0 & \sin\theta\\ 0 & 1 & 0\\ -\sin\theta & 0 & \cos\theta\end{bmatrix}$$

$$R_z(\psi):=e^{\hat{z}\psi}=\begin{bmatrix}\cos\psi & -\sin\psi & 0\\ \sin\psi & \cos\psi & 0\\ 0 & 0 & 1\end{bmatrix}$$

由矩阵求角度 △ R 转 Z,Y,Z 欧拉

Y: $\beta=\operatorname{atan2}(\sqrt{r_{31}^2+r_{32}^2}, r_{33})$

Z₁: $\alpha=\operatorname{atan2}(r_{23}/s\beta, r_{13}/s\beta)$ 奇异点

Z₂: $\gamma=\operatorname{atan2}(r_{32}/s\beta, -r_{31}/s\beta)$

若奇异令 $\alpha=0$，$\gamma=\operatorname{atan2}(r_{21},r_{22})$

△ R 转 XYZ fixed or ZYX Eu

奇异 $\alpha=0$，$\gamma=\operatorname{atan2}(r_{12},r_{22})$ 与 β 同号

Y: $\beta=\operatorname{atan2}(-r_{31},\sqrt{r_{11}^2+r_{21}^2})$

Z: $\alpha=\operatorname{atan2}(r_{21}/c_\beta, r_{11}/c_\beta)$

X: $\gamma=\operatorname{atan2}(r_{32}/c_\beta, r_{33}/c_\beta)$

Velocity

1. 只看 R

R(t): rotation "trajectory" relative to fixed frame

spatial angular vel $:=\omega^s$（与 ω 同向向量）

$\hat{\omega}^s=\dot{R}R^{-1}$

body angular vel $:=\omega^b$

$\hat{\omega}^b=R^{-1}\dot{R}$

转换 $\hat{\omega}^b=R^{-1}\hat{\omega}^sR$, $\omega^b=R^{-1}\omega^s$

vel of point

$v_q^s=\hat{\omega}^sq_a=\hat{\omega}^sRq_b$

$v_q^b=\hat{\omega}^bq_b$

2. 看 g

spatial: $V^s=\begin{bmatrix} v^s\\ \omega^s\end{bmatrix}=\begin{bmatrix} -\dot{R}R^Tp+\dot{p}\\ (\dot{R}R^T)^\vee\end{bmatrix}$

$\hat{V}^s=\dot{g}g^{-1}=\begin{bmatrix}\dot{R}R^T & -\dot{R}R^Tp+\dot{p}\\ 0 & 0\end{bmatrix}$

注意是 homo 坐标

$V_q^s=\dot{g}g_b=\dot{g}g^{-1}g_a$

$=\hat{V}^sg_a=\hat{\omega}^sg_a+v^s$

body: $V^b=\begin{bmatrix} v^b\\ \omega^b\end{bmatrix}=\begin{bmatrix} R^T\dot{p}\\ (R^T\dot{R})^\vee\end{bmatrix}$

$\hat{V}^b=g^{-1}\dot{g}=\begin{bmatrix} R^T\dot{R} & R^T\dot{p}\\ 0 & 0\end{bmatrix}$

$V_q^b=g^{-1}V_q^s=g^{-1}\dot{g}g_b$

$=\hat{V}^bg_b=\hat{\omega}^bg_b+v^b$

转换: $V^s=\underbrace{\begin{bmatrix} R & \hat{p}R\\ 0 & R\end{bmatrix}}_{Ad(g)}V^b$ adjoint transformation associated with g

广义上讲 Ad(g) 是表示对 se 坐标变换

body $\xrightarrow{Ad_g}$ spatial

$Ad_g^{-1}=Ad_{g^{-1}}=\begin{bmatrix} R^T & -R^T\hat{p}\\ 0 & R^T\end{bmatrix}$

$V_{ab}^s=-V_{ba}^s$

$V_{ab}^b=-Ad_{g_{ba}}V_{ba}^b$

且 $(Ad(g)\xi)^\wedge=g\hat{\xi}g^{-1}$

3. 看 $g(\theta)=e^{\hat{\xi}\theta}g(0)$

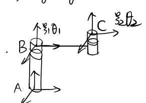
$\hat{V}^s=\hat{\xi}\dot{\theta}$, $V^s=\xi\dot{\theta}$

$\hat{V}^b=(Ad_{g(0)}^{-1}\xi)^\wedge\dot{\theta}$, $V^b=Ad_{g(0)}^{-1}\xi\dot{\theta}$

坐标变换 $\begin{cases} V_{ac}^s=V_{ab}^s+Ad_{g_{ab}}V_{bc}^s\\ V_{ac}^b=V_{bc}^b+Ad_{g_{bc}}^{-1}V_{ab}^b\end{cases}$

下标对应变换，叉对应 screw

e.g.



$\begin{cases} g_{ab}(\theta_1)=e^{\hat{\xi}_1\theta_1}g_{ab}(0)\\ V_{ab}^s=\xi_1\dot{\theta}_1\end{cases}$

$\begin{cases} g_{bc}(\theta_2)=e^{\hat{\xi}_2\theta_2}g_{bc}(0)\\ V_{bc}^s=\xi_2\dot{\theta}_2\end{cases}$

于是 V_{ac}^s 可求

frame, which is apparent from equation (2.53). Rather, $v_{ab}^s(t)$ is the velocity of a (possibly imaginary) point on the rigid body which is traveling through the origin of the spatial frame at time t. That is, if one stands at the origin of the spatial frame and measures the velocity of a point attached to the rigid body and traveling through the origin at time t, this is what one sees.

The interpretation of the body velocity is straightforward: v_{ab}^b is the velocity of the origin of the body coordinate frame relative to the spatial frame, as viewed in the current body frame. ω_{ab}^b is the angular velocity of the coordinate frame, also as viewed in the current body frame. Note that the body velocity is *not* the velocity of the body relative to the body frame; this latter quantity is always zero.

Useful Proof. Reference

- If $\omega=0$, then $\hat{\xi}^2=\hat{\xi}^3=\cdots=0$, $e^{\hat{\xi}\theta}=\begin{bmatrix} I & v\theta\\ 0 & 1\end{bmatrix}\in SE(3)$
- If ω is not 0, assume $\|\omega\|=1$.

Define:

$$g_0=\begin{bmatrix} I & \omega\times v\\ 0 & 1\end{bmatrix},\ \hat{\xi}'=g_0^{-1}\cdot\hat{\xi}\cdot g_0=\begin{bmatrix}\hat{\omega} & h\omega\\ 0 & 0\end{bmatrix}$$

where $h=\omega^T\cdot v$.

$$e^{\hat{\xi}\theta}=e^{g_0\cdot\hat{\xi}'\cdot g_0^{-1}}=g_0\cdot e^{\hat{\xi}'\theta}\cdot g_0^{-1}$$

and as

$$\hat{\xi}'^2=\begin{bmatrix}\hat{\omega}^2 & 0\\ 0 & 0\end{bmatrix},\ \hat{\xi}'^3=\begin{bmatrix}\hat{\omega}^3 & 0\\ 0 & 0\end{bmatrix}$$

we have

$$e^{\hat{\xi}'\theta}=\begin{bmatrix} e^{\hat{\omega}\theta} & h\omega\theta\\ 0 & 1\end{bmatrix}\Rightarrow e^{\hat{\xi}\theta}=\begin{bmatrix} e^{\hat{\omega}\theta} & (I-e^{\hat{\omega}\theta})\hat{\omega}v+\omega\omega^Tv\theta\\ 0 & 1\end{bmatrix}$$

$\xi=(v,\omega)\in\mathbb{R}^6$

- Pitch: $h=\begin{cases}\frac{\omega^Tv}{\|\omega\|^2}, & \text{if } \omega\neq0\\ \infty, & \text{if } \omega=0\end{cases}$
- Axis: $l=\begin{cases}\frac{\omega\times v}{\|\omega\|^2}+\lambda\omega, & \lambda\in\mathbb{R}, \text{if } \omega\neq0\\ 0+\lambda v & \lambda\in\mathbb{R}, \text{if } \omega=0\end{cases}$
- Magnitude: $M=\begin{cases}\|\omega\|, & \text{if } \omega\neq0\\ \|v\|, & \text{if } \omega=0\end{cases}$

Special cases:

- $h=\infty$, Pure translation (prismatic joint)
- $h=0$, Pure rotation (revolute joint)

$tr(R)=r_{11}+r_{22}+r_{33}=1+2\cos\theta=\sum_{i=1}^3\lambda_i$

where λ_i is the eigenvalue of R, $i=1,2,3$

Case 1: $tr(R)=3$ or $R=I$, $\theta=0\Rightarrow\omega\theta=0$

Case 2: $-1<tr(R)<3$, $\theta=\arccos\frac{tr(R)-1}{2}\Rightarrow\omega=\frac{1}{2s_\theta}\begin{bmatrix} r_{32}-r_{23}\\ r_{13}-r_{31}\\ r_{21}-r_{12}\end{bmatrix}$

Case 3: $tr(R)=-1\Rightarrow\cos\theta=-1\Rightarrow\theta=\pm\pi$

Wrench. Work

wrench: $F_b=\begin{bmatrix} f_b\\ \tau_b\end{bmatrix}$ 对 B 原点加的力、矩，坐标相对 B

work: $W=\int_{t_1}^{t_2}V_{ab}^b\cdot F_b\,dt$

等效 wrench: generate same work for all rigid motion

力的传递 $F_c=\mathrm{Ad}_{g_{bc}}^T\cdot F_b$

Forward Kinematics

1. 列 ω，列向量；轴上点，列向量
2. $\xi=\begin{bmatrix}-\omega\times q\\ \omega\end{bmatrix}$ 或 $\begin{bmatrix} v\\ 0\end{bmatrix}$ ← 要算
3. $e^{\hat{\xi}\theta}=\begin{bmatrix} R & (I-R)q+\omega h\theta\\ 0 & 1\end{bmatrix}$ 或 $=\begin{bmatrix} I & \theta v\\ 0 & 1\end{bmatrix}$
4. 选 S.T. 列式 $g_i=e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_{100}\theta_{100}}g_0$

雅可比. 速度与力的传递. 奇异点

末端速度 $\hat{V}^s=\dot{g}g^{-1}=J^s(\theta)[\dot{\theta}_1\dot{\theta}_2\cdots\dot{\theta}_n]^T$

$J^s=[\xi_1, \xi_2', \cdots \xi_n']$，$\xi_i'=Ad(e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_{i-1}\theta_{i-1}})\xi_i$

$J_{st}^s(\theta)=Ad(g_{st}(\theta))J_{st}^b(\theta)$

$V_{st}^b=J_{st}^b(\theta)\cdot\dot{\theta}$

$J_{st}^b(\theta)=[\xi_1^\dagger\ldots\xi_{n-1}^\dagger,\xi_n^\dagger]$

$\xi_i^\dagger=\mathrm{Ad}^{-1}_{e^{\hat{\xi}_{i+1}\theta_{i+1}}\ldots e^{\hat{\xi}_n\theta_n}g_{st}(0)}\xi_i$

奇异点(情况) 为 J^s drop rank 不可逆

此时 $V_{st}^s=J_{st}^s(\theta)\dot{\theta}=0$

与机构有关 (对列向量有抑制，但维令为 0) 或相关于

反求 Given $g(t)$, how to find $\theta(t)$?

1) $\hat{V}_{st}^s=\dot{g}(t)g^{-1}(t)$

2) $\begin{cases}\dot{\theta}(t)=(J_{st}^s(\theta))^{-1}V_{st}^s(t)\\ \theta(0)=\theta_0\end{cases}\Rightarrow\theta(t)$

末端施加在轴上的力 相对末端坐标系的力 $F_t\Rightarrow\tau=(J_{st}^b)^TF_t$

相对 Spatial 坐标系力 $F_s\Rightarrow\tau=(J_{st}^s)^TF_s$

(1. 2 轴共线 2. 3 轴共面 3. 4 轴相交)

$w_1=\pm w_2$, $w_1\times(q_1-q_2)=0$, $w_2\times(q_1-q_2)=0$ 同向相减

$w_1=\pm w_2=\pm w_3$ 列相减 $w_3\times(q_2-q_1)$ 与法向同向

$w_i\times(q_i-\text{一定点})=0$ 左 $\begin{bmatrix} 0 & 0 & 0 & 0\\ w_1 & w_2 & w_3 & w_4\end{bmatrix}$

$\dot{\theta}\neq0$

*平移轴与旋转轴一般不相关，平移轴间 V

Inverse Kinematics. 已知 g_d, g_{st} 求 θ

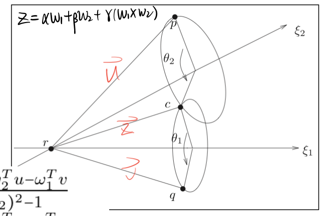
subproblem 1. $e^{\hat{\xi}\theta}p=q$ $u'=(I-\omega\omega^T)u$, $v'=(I-\omega\omega^T)v$

$\theta=\operatorname{atan2}(\omega^T(u'\times v'),u'^Tv')$

$u=p-r$, $v=q-r$. r 为轴上一点

subproblem 2 $e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}p=q$

设中间点 c $\Rightarrow\begin{cases} e^{\hat{\xi}_2\theta_2}p=c\\ e^{\hat{\xi}_1\theta_1}p=c\end{cases}$ 化为 1



$$\omega_2^Tu=\alpha\omega_2^T\omega_1+\beta \quad \omega_1^Tv=\alpha+\beta\omega_1^T\omega_2 \Rightarrow \begin{cases}\alpha=\frac{(\omega_1^T\omega_2)\omega_2^Tu-\omega_1^Tv}{(\omega_1^T\omega_2)^2-1}\\ \beta=\frac{(\omega_1^T\omega_2)\omega_1^Tv-\omega_2^Tu}{(\omega_1^T\omega_2)^2-1}\end{cases}$$

$$\|z\|^2=\|u\|^2\Rightarrow\gamma^2=\frac{\|u\|^2-\alpha^2-\beta^2-2\alpha\beta\omega_1^T\omega_2}{\|\omega_1\times\omega_2\|^2}$$

subproblem 3 $\|q-e^{\hat{\xi}\theta}p\|=\delta$

Define: $u=p-r$, $v=q-r$, $\|v-e^{\hat{\omega}\theta}u\|^2=\delta^2$

$u'=u-\omega\omega^Tu$

$v'=v-\omega\omega^Tv$

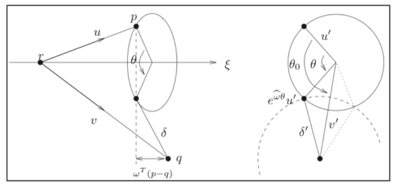


Figure 3.7

$$\Rightarrow u=u'+\omega\omega^Tu, v=v'+\omega\omega^Tv, \delta'^2=\delta^2-|\omega^T(p-q)|^2$$

$$\|v'-e^{\hat{\omega}\theta}u'\|^2=\delta^2-|\omega^T(p-q)|^2=\delta'^2,$$

$$\theta_0=\operatorname{atan2}(\omega^T(u'\times v'),u'^Tv'),$$

$$\phi=\theta_0-\theta\Rightarrow\|u'\|^2+\|v'\|^2-2\|u'\|\cdot\|v'\|\cos\phi=\delta'^2,$$

$$\theta=\theta_0\pm\cos^{-1}\frac{\|u'\|+\|v'\|-\delta'^2}{2\|u'\|\cdot\|v'\|} \quad (*)$$

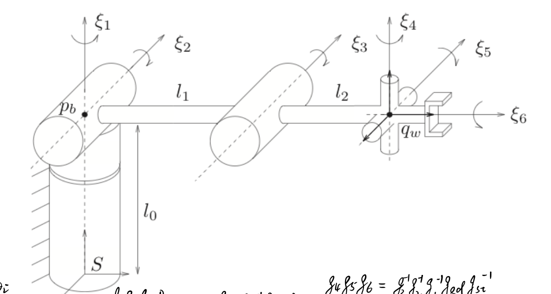
化简：右乘轴上的点左乘 $e^{\hat{\xi}\theta}$ 因为没作用

减去一个点距 $g_1g_2g_3p-q=g_1g_2g_3p-g_1g_2q=g_1g_2(g_3p-q)$

$\|g_1g_2(g_3p-q)\|=\|g_3p-q\|$（模不变）

几何关系很好使用

elbow 例



$g_i=e^{\hat{\xi}_i\theta_i}$

$g_1g_2g_3g_4g_5g_6g_{st}=g_{ed}$

$g_1g_2g_3g_4g_5g_6=g_{ed}g_{st}^{-1}$

$g_1g_2g_3p_w=g_{ed}g_{st}^{-1}p_w$

$g_1g_2g_3p_w-p_b=g_{ed}g_{st}^{-1}p_w-p_b$

$=g_1g_2(g_3p_w-p_b)$

sub3 → θ₃

$g_1g_2(g_3p_w)=g_{ed}g_{st}^{-1}p_w$

sub2 → θ₁, θ₂

$g_4g_5g_6=g_3^{-1}g_2^{-1}g_1^{-1}g_{ed}g_{st}^{-1}$

$g_4g_5p_t=g_3^{-1}g_2^{-1}g_1^{-1}g_{ed}g_{st}^{-1}p_t$

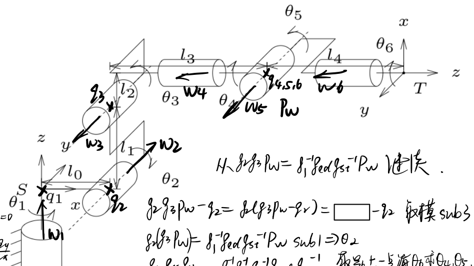
sub2 ⇒ θ₄, θ₅

$g_6p_0=g_5^{-1}g_4^{-1}g_3^{-1}g_2^{-1}g_1^{-1}g_{ed}g_{st}^{-1}p_0$ (原点)

sub1 ⇒ θ₆

IRB 4400 例

1. 列轴、取点
2. 写出 g_{st}
3. 列出运动学式子，求解



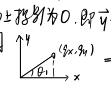
$g_1g_2g_3g_4g_5g_6=g_{ed}g_{st}^{-1}$

$g_1g_2g_3p_w=g_{ed}g_{st}^{-1}p_w$

$g_2g_3p_w=g_1^{-1}g_{ed}g_{st}^{-1}p_w\triangleq g_1^{-1}q$ ☆ 几何

在特轴上投影为 0，即 $\vec{y}\cdot g_1^{-1}\vec{q}=y^Tg_1^{-1}q=0$

或看图



$\Rightarrow\theta_1=\operatorname{atan}\frac{q_y}{q_x}$

从 $g_2g_3p_w=g_1^{-1}g_{ed}g_{st}^{-1}p_w$ 进行

$g_2(g_3p_w-q_2)=g_2(g_3p_w-q_r)=$ ▭ $-q_2$ 取模 sub3 ⇒ θ₃

$g_2(g_3p_w)=g_1^{-1}g_{ed}g_{st}^{-1}p_w$ sub1 ⇒ θ₂

$g_4g_5g_6=g_3^{-1}g_2^{-1}g_1^{-1}g_{ed}g_{st}^{-1}$，取 ξ_6 上一点消 θ₆ 求 θ₄, θ₅，再求 θ₆

Dynamics

1. Lagrangian Equation

$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}-\frac{\partial L}{\partial q_i}=\tau_i$ 有i个方程

$L=T-V$ = 动能 − 势能

q_i 为第i个广义坐标轴(参数)

τ_i 为在第i个坐标轴上的外力(不算G)

代进去算可得到 $A(Q)\ddot{Q}+B(Q,\dot{Q})+C(Q)=\tau$ 形式的

2. 刚体的动能计算(单体)

刚体属性：体积V，密度$\rho(r)$

质量 $m=\int_V \rho(r)dV$

空间质心位置 $\bar{r}=\frac{1}{m}\int_V \rho(r)\, r\, dV=[\bar{x}\ \bar{y}\ \bar{z}]^T$

体坐标质心坐标为0

$T=\frac{1}{2}\int_V \rho(r)\|\dot{p}+\dot{R}r\|^2 dV$

$=\frac{1}{2}m\|\dot{p}\|^2+\frac{1}{2}(w^b)^T \mathbb{I} w^b$

$=\frac{1}{2}(\mathbb{V}^b)^T \mathbb{M}^b \mathbb{V}^b$

w^b: angular body vel

$\mathbb{V}^b$: body vel vector

$\mathbb{I}$: inertia tensor

$\mathbb{M}$: generalized inertia

其中 $\mathbb{M}=\begin{bmatrix} mI_{3\times3} & 0 \\ 0 & \mathbb{I}_{3\times3}\end{bmatrix}_{6\times6}$

而 $\mathbb{I}=-\int_V \rho(r)\hat{r}^2 dV=\begin{bmatrix}\mathbb{I}_{xx} & \mathbb{I}_{xy} & \mathbb{I}_{xz}\\ \mathbb{I}_{yx} & \mathbb{I}_{yy} & \mathbb{I}_{yz}\\ \mathbb{I}_{zx} & \mathbb{I}_{zy} & \mathbb{I}_{zz}\end{bmatrix}$

同下标项 $\mathbb{I}_{xx}=\int_V \rho(r)(y^2+z^2)dV$ (去掉下标^2)

异下标项 $\mathbb{I}_{xy}=-\int_V \rho(r)(xy)dV$ (对于下标)

3. 刚体上固连2个系，2相对1的config为g_0

做一个rigid motion $1\xrightarrow{g_1}1'$，$2\xrightarrow{g_2}2'$

$1'\to2'$ 还是 g_0。

此时 $\mathbb{M}_2^b=Ad(g_0)^T\mathbb{M}_1^b Ad(g_0)$

4. 多体的L方程(open chain robot)

总动能 $T(\theta,\dot{\theta})=\sum_{i=1}^{n}\frac{1}{2}(\mathbb{V}_{i}^b)^T\mathbb{M}_i^b\mathbb{V}_i^b$

$\mathbb{V}_i^b$ 为 link body vel $=J_{i}^b\dot{\theta}$

$T=\frac{1}{2}\dot{\theta}^T\boxed{\sum J_i^{bT}\mathbb{M}_i^b J_i^b}\dot{\theta}$

$=\frac{1}{2}\dot{\theta}^T\boxed{M(\theta)}\dot{\theta}$ M为对称正定阵

$=\frac{1}{2}\sum_{i,j=1}^{n}M_{ij}\dot{\theta}_i\dot{\theta}_j$ ij为i行j列元素

总势能 $V(\theta)=\sum_{i=1}^{n}m_i g h_i(\theta)$

拉格朗日方程 $L=T-V$

$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}_i}-\frac{\partial L}{\partial\theta_i}=\tau_i$

$=\sum_{j=1}^{n}M_{ij}\ddot{\theta}_j+\frac{\partial V}{\partial\theta_i}$

$+\sum_{j,k=1}^{n}\left(\frac{\partial M_{ij}}{\partial\theta_k}\dot{\theta}_j\dot{\theta}_k-\frac{1}{2}\frac{\partial M_{kj}}{\partial\theta_i}\dot{\theta}_k\dot{\theta}_j\right)$

$=\sum_{j=1}^{n}M_{ij}\ddot{\theta}_j+\sum_{j,k=1}^{n}\Gamma_{ijk}\dot{\theta}_j\dot{\theta}_k+\frac{\partial V}{\partial\theta_i}$

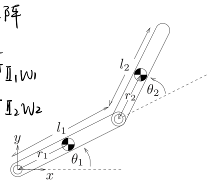
$N_i(\theta,\dot{\theta})=\frac{\partial V}{\partial\theta_i}-\beta\dot{\theta}_i$

$C_{ij}(\theta,\dot{\theta})=\sum_{k=1}^{n}\Gamma_{ijk}\dot{\theta}_k=\frac{1}{2}\sum_{k=1}^{n}\left(\frac{\partial M_{ij}}{\partial\theta_k}+\frac{\partial M_{ik}}{\partial\theta_j}-\frac{\partial M_{kj}}{\partial\theta_i}\right)\dot{\theta}_k$

$M(\theta)$ 上面有，$M(\theta)\ddot{\theta}+C(\theta,\dot{\theta})\dot{\theta}+N(\theta,\dot{\theta})=\tau$

$\dot{M}-2C$ 为反对称阵

例如：设坐标为θ_1,θ_2



$T=\frac{1}{2}m_1|V_1|^2+\frac{1}{2}w_1^T\mathbb{I}_1 w_1$

$+\frac{1}{2}m_2|V_2|^2+\frac{1}{2}w_2^T\mathbb{I}_2 w_2$

$=\frac{1}{2}m_1(\dot{x}_1^2+\dot{y}_1^2)$

$+\frac{1}{2}\mathbb{I}_{z1}\dot{\theta}_1^2$

$+\frac{1}{2}m_2(\dot{x}_2^2+\dot{y}_2^2)+\frac{1}{2}\mathbb{I}_{z2}(\dot{\theta}_1+\dot{\theta}_2)^2$

$x_1=r_1\cos\theta_1$，$y_1=r_1\sin\theta_1$，$x_2=l_1\cos\theta_1+r_2\cos(\theta_1+\theta_2)$

$y_2=l_1\sin\theta_1+r_2\sin(\theta_1+\theta_2)$

$\begin{bmatrix}\alpha+2\beta C_2 & \delta+\beta C_2\\ \delta+\beta C_2 & \delta\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}+\begin{bmatrix}-\beta S_2\dot{\theta}_2 & -\beta S_2(\dot{\theta}_1+\dot{\theta}_2)\\ \beta S_2\dot{\theta}_1 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}=\tau$

$\alpha=\mathbb{I}_{z1}+\mathbb{I}_{z2}+m_1r_1^2+m_2(l_1^2+r_2^2)$

$\beta=m_2l_1r_2$　$\delta=\mathbb{I}_{z2}+m_2r_2^2$

5. 牛二的完全体：牛顿欧拉方程

$\mathbb{M}\begin{bmatrix}\dot{v}^b\\ \dot{w}^b\end{bmatrix}+\begin{bmatrix}w^b\times m v^b\\ w^b\times\mathbb{I}w^b\end{bmatrix}=F^b$

F^b 为相对体坐标的wrench

类似 $F=ma$

Path, trajectory

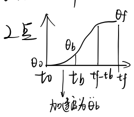
Via points 是 path planning 的结果

再由 trajectory planner 生成轨迹 $\theta(t)$

workspace $\xrightarrow{\text{逆解}}$ joint space − 般joint空间

1. Linear + parabolic (θ为2次)

2点

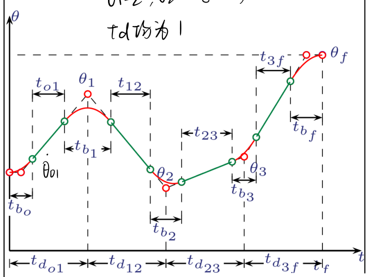


$\begin{cases}\theta_0+\frac{1}{2}\ddot{\theta}_b(t-t_0)^2\\ \theta_0+\ddot{\theta}_b t_b(t-t_0-\frac{t_b}{2})\\ \theta_f-\frac{1}{2}\ddot{\theta}_b(t_f-t)^2\end{cases}$

via point $\theta_0=1$，$\theta_f=2.5$，$\ddot{\theta}_b=4$

$\theta_1=2$，$\theta_2=0.8$，$\theta_3=1.2$

t_d 均为1



I. $\theta_1>\theta_0$，故 $\ddot{\theta}_0=\ddot{\theta}_b=4$，$t_{b0}=t_{d01}-\sqrt{t_{d01}^2-\frac{2(\theta_1-\theta_0)}{\ddot{\theta}_0}}=0.29$

$\dot{\theta}_{01}=\frac{\theta_1-\theta_0}{t_{d01}-\frac{1}{2}t_{b0}}=1.17$

II. via 1，$\dot{\theta}_{12}=\frac{\theta_2-\theta_1}{t_{d12}}=-1.2$，$\dot{\theta}_{12}<\dot{\theta}_{01}$ 故 $\ddot{\theta}_1=-\ddot{\theta}_b=-4$

$t_{b1}=\frac{\dot{\theta}_{12}-\dot{\theta}_{01}}{\ddot{\theta}_1}=0.59$，$t_{01}=t_{d01}-t_{b0}-\frac{1}{2}t_{b1}=0.41$

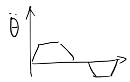
III via 2，同上操作

IV via 3，最后一个via pt，先算$\ddot{\theta}_f$，$\theta_3<\theta_f$ 故 $\ddot{\theta}_f=-4$

$t_{bf}=t_{d3f}-\sqrt{t_{d3f}^2-\frac{2(\theta_f-\theta_3)}{\ddot{\theta}_f}}$

$\dot{\theta}_{3f}=\frac{\theta_f-\theta_3}{t_{d3f}-\frac{1}{2}t_{bf}}$，然后有了$\ddot{\theta}_3$，求$t_{b3}$、$t_{23}$、$t_{3f}$

2. double S，把梯形速度的线性段LFPB

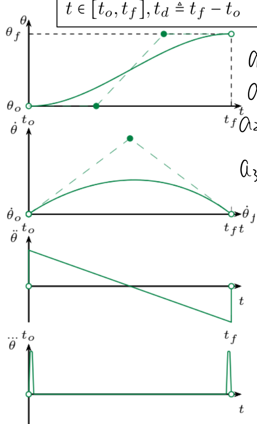


可在LFPB基础上改

3. Cubic(3)

2点 $\theta(t)=a_0+a_1(t-t_0)+a_2(t-t_0)^2+a_3(t-t_0)^3$,
$t\in[t_0,t_f], t_d\triangleq t_f-t_0$

根据边界解方程



$a_0=\theta_0$　$h=\triangle\theta$

$a_1=\dot{\theta}_0$

$a_2=\frac{1}{t_d^2}(3h-t_d(2\dot{\theta}_0+\dot{\theta}_f))$

$a_3=\frac{1}{t_d^3}(-2h+t_d(\dot{\theta}_0+\dot{\theta}_f))$

via points 问题：

可以分段单独计算

若via pts的速度未给

则根据$\dot{\theta}_k$算$\dot{\theta}_k$值

且连续

4. Quintic(5) $\ddot{\theta}$一定连续

$\theta(t)=\sum_{i=0}^{5}a_i(t-t_0)^i, t\in[t_0,t_f]$

给定初始 $\theta,\dot{\theta},\ddot{\theta}$，则

$a_0=\theta_0$，$a_1=\dot{\theta}_0$，$a_2=\frac{1}{2}\ddot{\theta}_0$

$a_3=\frac{1}{2t_d^3}(20h-(8\dot{\theta}_f+12\dot{\theta}_0)t_d-(3\ddot{\theta}_0-\ddot{\theta}_f)t_d^2)$

$a_4=\frac{1}{2t_d^4}(-30h-(14\dot{\theta}_f+16\dot{\theta}_0)t_d-(3\ddot{\theta}_0-2\ddot{\theta}_f)t_d^2)$

$a_5=\frac{1}{2t_d^5}(12h-(6\dot{\theta}_f+6\dot{\theta}_0)t_d-(\ddot{\theta}_f-\ddot{\theta}_0)t_d^2)$

Control

设 $e=\theta$期望$-\theta$实际，求施加的Joint torque τ，使 $e=\dot{\theta}=\ddot{\theta}=0$

1. Computed torque 开环控制 $\tau=M\ddot{\theta}+C\dot{\theta}+N$

2. PD. $\tau=k_v\dot{e}+k_p e$

3. Augmented PD (加了前馈) $\tau=$ 计算τ + PD

例题

1. 证明 $R\hat{w}R^T=(Rw)^\wedge$

设元素，乘开，用$R^T=R^{-1}$即可得

2. $(Rv)\times(Rw)=(Rv)^\wedge Rw$ 代入1

3. Vector space：加法交换结合

0元逆元1元，乘分配结合

4. 2维旋转情况 $\hat{w}=\begin{bmatrix}0 & -w\\ w & 0\end{bmatrix}\in so(2)$

$e^{\hat{w}\theta}=\begin{bmatrix}\cos w\theta & -\sin w\theta\\ \sin w\theta & \cos w\theta\end{bmatrix}$ 可用Taylor证

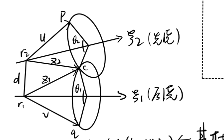
5. 角速度可分阶段算(例题为如此算)

以段 $\mathbb{V}^s=\begin{bmatrix}v^s\\ w^s\end{bmatrix}=\begin{bmatrix}-\dot{R}R^Tp+\dot{p}\\ (\dot{R}R^T)^\vee\end{bmatrix}$

令故 $\mathbb{V}_{ac}^s=\mathbb{V}_{ab}^s+Ad_{g_{ab}}\mathbb{V}_{bc}^s$

$\mathbb{V}^b$ 同理

6.



$z_2=\alpha w_1+\beta w_2+\gamma(w_1\times w_2)$ ⇐ 基本操作

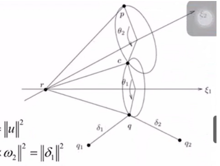
$\begin{cases}\alpha=\frac{w_1^Tw_2w_2^Tu-w_1^T(v-d)}{(w_1^Tw_2)^2-1}\\ \beta=\frac{w_1^Tw_2w_1^T(v-d)-w_2^Tu}{(w_1^Tw_2)^2-1}\\ \gamma^2=\frac{\|u\|^2-\alpha^2-\beta^2-2\alpha\beta w_1^Tw_2}{\|w_1\times w_2\|^2}\end{cases}$

考虑一般情况，r,q_1,q_2不共线：

$\omega_1=q_1-r, \omega_2=q_2-r, u=p-r$

$z=q-r=\alpha\omega_1+\beta\omega_2+\gamma(\omega_1\times\omega_2)$

根据距离关系，列出等式：



$$\begin{cases}\|z\|^2=\alpha^2+\beta^2+2\alpha\beta\omega_1^T\omega_2+\gamma^2\|\omega_1\times\omega_2\|^2=\|u\|^2\\ \|z-\omega_1\|^2=(\alpha-1)^2+\beta^2+2(\alpha-1)\beta\omega_1^T\omega_2+\gamma^2\|\omega_1\times\omega_2\|^2=\|\delta_1\|^2\\ \|z-\omega_2\|^2=\alpha^2+(\beta-1)^2+2\alpha(\beta-1)\omega_1^T\omega_2+\gamma^2\|\omega_1\times\omega_2\|^2=\|\delta_2\|^2\end{cases}$$

解方程组：

$$\begin{cases}\alpha=\frac{(\|u\|^2-\delta_2^2+1)\omega_1^T\omega_2-\|u\|^2+\delta_1^2-1}{2[(\omega_1^T\omega_2)^2-1]}\\ \beta=\frac{(\|u\|^2-\delta_1^2+1)\omega_1^T\omega_2-\|u\|^2+\delta_2^2-1}{2[(\omega_1^T\omega_2)^2-1]}\\ \gamma^2=\frac{\|u\|^2-\alpha^2-\beta^2-2\alpha\beta\omega_1^T\omega_2}{\|\omega_1\times\omega_2\|^2}\end{cases}$$

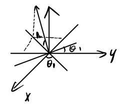
7. J^s的计算，先按对应关系列轴和点

$\xi_i'=\begin{bmatrix}-w_i'\times q_i'\\ w_i'\end{bmatrix}$　$w_i'=e^{\hat{w}_1\theta_1}\cdots e^{\hat{w}_{i-1}\theta_{i-1}}w_i$

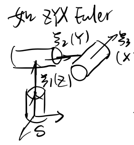
$q_i'=e^{\hat{\xi}_1\theta_1}\cdots e^{\hat{\xi}_{i-1}\theta_{i-1}}q_i$

J^b 用公式算 $\xi_i^+=$ 上面那个 $=Ad(g_{st}e^{-\hat{\xi}_n\theta_n}\cdots e^{-\hat{\xi}_i\theta_i})\xi_i$

8. 用J^s求旋转的奇异



如 ZYX Euler



$w_1'=\begin{bmatrix}0\\0\\1\end{bmatrix}$　$w_2'=\begin{bmatrix}-\sin\theta_1\\ \cos\theta_1\\ 0\end{bmatrix}$

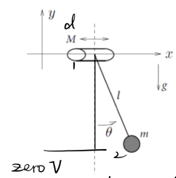
$w_3'=\begin{bmatrix}-\cos\theta_2\cos\theta_1\\ -\cos\theta_2\sin\theta_1\\ \sin\theta_2\end{bmatrix}$　当 $\sin\theta_2=\pm1$，$\cos\theta_2=0$ 时奇异，1和3列相关

$J^s=\begin{bmatrix}0 & -\sin\theta_1 & -\cos\theta_2\cos\theta_1\\ 0 & \cos\theta_1 & -\cos\theta_2\sin\theta_1\\ 1 & 0 & \sin\theta_2\end{bmatrix}$

9. 4个轴由共面的奇异，找一个系使 $w_i=[x_i\ y_i\ 0]^T$ $q_i=[q_{ix}\ q_{iy}\ 0]^T$，$-w_i\times q_i=\begin{bmatrix}0\\0\\y_iq_{ix}-x_iq_{iy}\end{bmatrix}$

于是J^s里可找到4列如下q_{iy}相关

10.



设坐标轴为d,θ，表示1和2位置 $r_1=\begin{bmatrix}d\\0\end{bmatrix}$　$r_2=\begin{bmatrix}d+l\sin\theta\\ -l\cos\theta\end{bmatrix}$

$T=\frac{1}{2}m\left((l\dot{\theta}\sin\theta)^2+(\dot{d}+l\dot{\theta}\cos\theta)^2\right)+\frac{1}{2}M\dot{d}^2$

$V=mgl(1-\cos\theta)+Mgl$

$L=T-V=\frac{1}{2}m(\dot{d}^2+l^2\dot{\theta}^2+2\dot{d}l\dot{\theta}\cos\theta)+\frac{1}{2}M\dot{d}^2-mgl(1-\cos\theta)-Mgl$

$L'_{\dot{d}}=m\dot{d}+ml\dot{\theta}\cos\theta+M\dot{d}$，$\frac{dL'_{\dot{d}}}{dt}=m\ddot{d}+ml(\ddot{\theta}\cos\theta-\dot{\theta}^2\sin\theta)+M\ddot{d}$

$L'_{\dot{\theta}}=ml^2\dot{\theta}+m\dot{d}l\cos\theta$，$\frac{dL'_{\dot{\theta}}}{dt}=ml^2\ddot{\theta}+ml(\ddot{d}\cos\theta-\dot{\theta}\dot{d}\sin\theta)$

$L'_d=0$

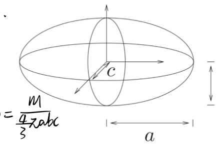
$L'_\theta=-m\dot{d}l\dot{\theta}\sin\theta-mgl\sin\theta$

$\begin{cases}\frac{dL'_{\dot{d}}}{dt}-L'_d=\tau_d\\ \frac{dL'_{\dot{\theta}}}{dt}-L'_\theta=\tau_\theta\end{cases}$

整理哦 $=-ml(\dot{d}\dot{\theta}\sin\theta+g\sin\theta)$

$\begin{bmatrix}m+M & ml\cos\theta\\ ml\cos\theta & ml^2\end{bmatrix}\begin{bmatrix}\ddot{d}\\ \ddot{\theta}\end{bmatrix}+\begin{bmatrix}-ml\sin\theta\dot{\theta}^2\\ 0\end{bmatrix}+\begin{bmatrix}0\\ mlg\sin\theta\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

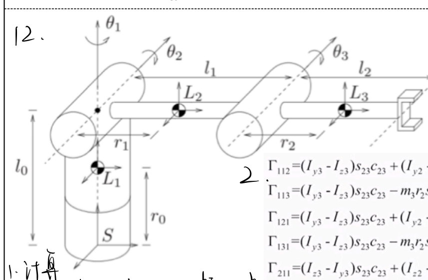
11.



$\rho=\frac{m}{\frac{4}{3}\pi abc}$

$\mathbb{I}=\begin{bmatrix}\frac{4}{15}\pi abc\rho(a^2+b^2) & 0 & 0\\ 0 & \cdots(c^2+b^2) & 0\\ 0 & 0 & \cdots(a^2+c^2)\end{bmatrix}$

12.



设 $M_i=\begin{bmatrix}m_i & & & & & \\ & m_i & & & & \\ & & m_i & & & \\ & & & \mathbb{I}_{xi} & & \\ & & & & \mathbb{I}_{yi} & \\ & & & & & \mathbb{I}_{zi}\end{bmatrix}$

1. 计算 $M(\theta)=J_1^{bT}M_1J_1^b+\cdots+J_3^{bT}M_3J_3^b$

所以要先算好各J^b

$J_1^b=\begin{bmatrix}0 & | & | \\ 0 & | & | \\ 0 & 0 & 0 \\ 0 & 0 & 0\\ 0 & 0 & 0\\ 1 & 0 & 0\end{bmatrix}$

$J_2^b=\begin{bmatrix}-r_1C_2 & 0 & | \\ 0 & 0 & | \\ 0 & -r_1 & | \\ -S_2 & 0 & 0 \\ 0 & -1 & 0 \\ C_2 & 0 & 0\end{bmatrix}$　$J_3^b=\begin{bmatrix}-r_2C_{23}-l_1C_2 & 0 & 0\\ 0 & l_1S_3 & 0\\ 0 & -r_2-l_1C_3 & -r_2\\ -S_{23} & 0 & 0\\ 0 & -1 & -1\\ C_{23} & 0 & 0\end{bmatrix}$

$M_{11}=I_{z1}+m_2r_1^2c_2^2+I_{z3}c_{23}^2+m_3(r_2c_{23}+l_1c_2)^2+I_{y3}s_{23}^2+\mathbb{I}_{z2}C_2^2+\mathbb{I}_{y2}S_2^2$

$M_{12}=0, M_{13}=0, M_{21}=0$

$M_{22}=I_{x2}+I_{x3}+m_3(l_1s_3)^2+m_2r_1^2+m_3(r_2+l_1c_3)^2$

$M_{23}=I_{x3}+m_3r_2(r_2+l_1c_3)$

$M_{31}=0$

$M_{32}=I_{x3}+m_3r_2(r_2+l_1c_3)$

$M_{33}=I_{x3}+m_3r_2^2$

2.

$\Gamma_{112}=(I_{y3}-I_{z3})s_{23}c_{23}+(I_{y2}-I_{z2})s_2c_2-m_2r_1^2c_2s_2-m_3(r_2s_{23}+l_1s_2)(r_2c_{23}+l_1c_2)$

$\Gamma_{113}=(I_{y3}-I_{z3})s_{23}c_{23}-m_3r_2s_{23}(r_2c_{23}+l_1c_2)$

$\Gamma_{121}=(I_{y3}-I_{z3})s_{23}c_{23}+(I_{y2}-I_{z2})s_2c_2-m_2r_1^2c_2s_2-m_3(r_2s_{23}+l_1s_2)(r_2c_{23}+l_1c_2)$

$\Gamma_{131}=(I_{y3}-I_{z3})s_{23}c_{23}-m_3r_2s_{23}(r_2c_{23}+l_1c_2)$

$\Gamma_{211}=(I_{z3}-I_{y3})s_{23}c_{23}+(I_{z2}-I_{y2})s_2c_2+m_2r_1^2c_2s_2+m_3(r_2s_{23}+l_1s_2)(r_2c_{23}+l_1c_2)$

$\Gamma_{223}=-l_1m_3r_2s_3$

$\Gamma_{232}=-l_1m_3r_2s_3$

$\Gamma_{233}=-l_1m_3r_2s_3$

$\Gamma_{311}=(I_{z3}-I_{y3})s_{23}c_{23}+m_3r_2s_{23}(r_2c_{23}+l_1c_2)$

$\Gamma_{322}=l_1m_3r_2s_3$

Other $\Gamma_{ijk}=0$

3. $h_1(\theta)=r_0$

$g_{sl_2}(\theta)=e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}g_{sl_2}(0)$

$h_2(\theta)=g_{sl_2}(\theta)(3,4)=l_0-r_1s_2$

$g_{sl_3}(\theta)=e^{\hat{\xi}_1\theta_1}e^{\hat{\xi}_2\theta_2}e^{\hat{\xi}_3\theta_3}g_{sl_3}(0)$

$h_3(\theta)=g_{sl_3}(\theta)(3,4)=l_0-r_2s_{23}-l_1s_2$

$V(\theta)=m_1gh_1(\theta)+m_2gh_2(\theta)+m_3gh_3(\theta)$

$=m_1gr_0+m_2g(l_0-r_1s_2)+m_3g(l_0-r_2s_{23}-l_1s_2)$

$N(\theta,\dot{\theta})=\frac{\partial V}{\partial\theta}=\begin{bmatrix}0\\ -(m_2gr_1+m_3gl_1)c_2-m_3gr_2c_{23}\\ -m_3gr_2c_{23}\end{bmatrix}$

4. $M(\theta)\ddot{\theta}+C(\theta,\dot{\theta})\dot{\theta}+N(\theta,\dot{\theta})=\tau$