Shortest Paths in Weighted Graphs

Madhavan Mukund

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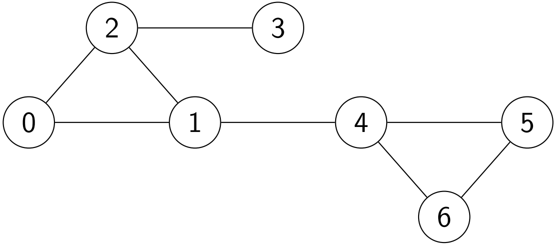
Programming, Data Structures and Algorithms using Python

Week 5

Weighted graphs

- Recall that BFS explores a graph level by level
- BFS computes shortest path, in terms of number of edges, to every reachable vertex



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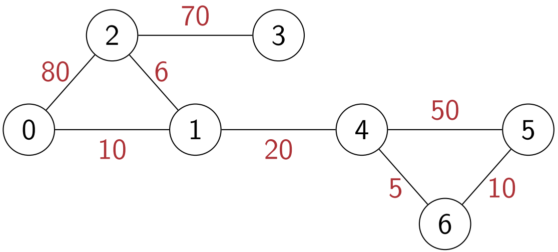
- May assign values to edges
  - Cost, time, distance, ...
  - Weighted graph
- $G = (V, E)$, $W : E \to \mathbb{R}$



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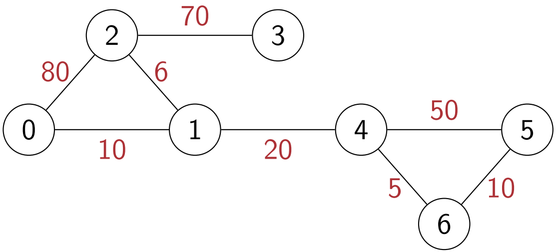
- Adjacency matrix

  Record weights along with edge information — weight is always 0 if no edge



	0	1	2	3	4	5	6
0	(0,0)	(1,10)	(1,80)	(0,0)	(0,0)	(0,0)	(0,0)
1	(1,10)	(0,0)	(1,6)	(0,0)	(1,20)	(0,0)	(0,0)
2	(1,80)	(1,6)	(0,0)	(1,70)	(0,0)	(0,0)	(0,0)
3	(0,0)	(0,0)	(1,70)	(0,0)	(0,0)	(0,0)	(0,0)
4	(0,0)	(1,20)	(0,0)	(0,0)	(0,0)	(1,50)	(1,5)
5	(0,0)	(0,0)	(0,0)	(0,0)	(1,50)	(0,0)	(1,10)
6	(0,0)	(0,0)	(0,0)	(0,0)	(1,5)	(1,10)	(0,0)

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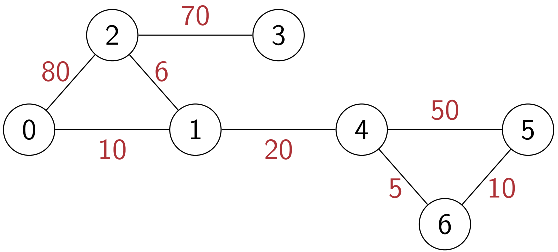

0	[(1,10),(2,80)]
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Shortest paths in weighted graphs

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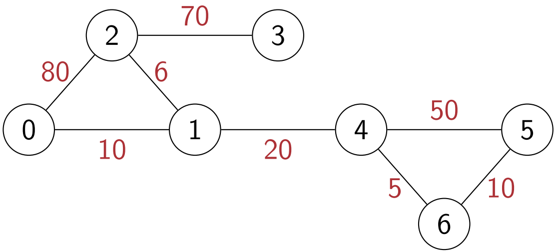

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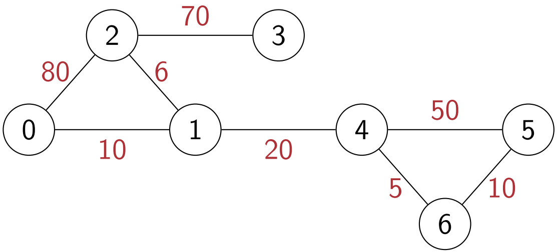
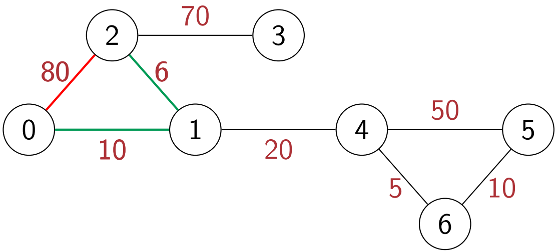


Shortest paths in weighted graphs

- BFS computes shortest path, in terms of number of edges, to every reachable vertex
- In a weighted graph, add up the weights along a path
- Weighted shortest path need not have minimum number of edges
  - Shortest path from 0 to 2 is via 1



Shortest path problems

Single source shortest paths

- Find shortest paths from a fixed vertex to every other vertex
- Transport finished product from factory (single source) to all retail outlets
- Courier company delivers items from distribution centre (single source) to addressees

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All pairs shortest paths

- Find shortest paths between every pair of vertices i and j
- Optimal airline, railway, road routes between cities

Negative edge weights

Negative edge weights

- Can negative edge weights be meaningful?
- Taxi driver trying to head home at the end of the day
  - Roads with few customers, drive empty (positive weight)
  - Roads with many customers, make profit (negative weight)
  - Find a route toward home that minimizes the cost

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Negative cycles

- A negative cycle is one whose weight is negative
  - Sum of the weights of edges that make up the cycle
- By repeatedly traversing a negative cycle, total cost keeps decreasing
- If a graph has a negative cycle, shortest paths are not defined
- Without negative cycles, we can compute shortest paths even if some weights are negative

Summary

- In a weighted graph, each edge has a cost
  - Entries in adjacency matrix capture edge weights
- Length of a path is the sum of the weights
  - Shortest path in a weighted graph need not be minimum in terms of number of edges
- Different shortest path problems
  - Single source — from one designated vertex to all others
  - All-pairs — between every pair of vertices
- Negative edge weights
  - Should not have negative cycles
  - Without negative cycles, shortest paths still well defined

Single Source Shortest Paths

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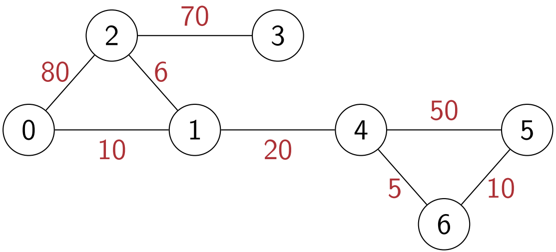

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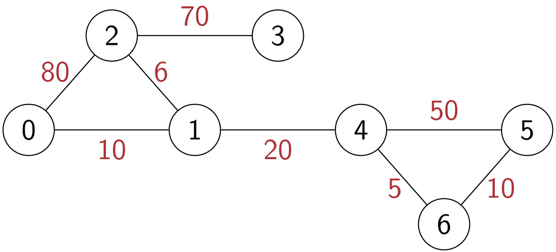
- Assume, for now, that edge weights are all non-negative



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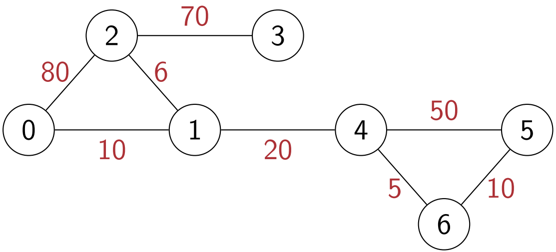

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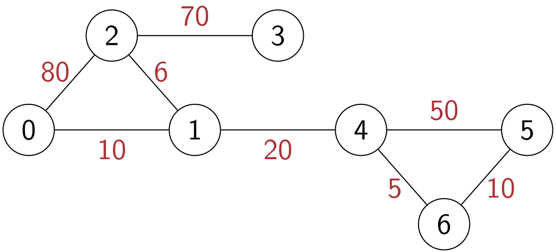

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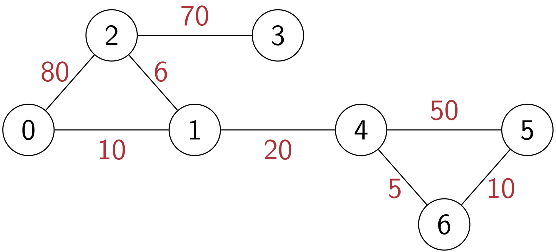

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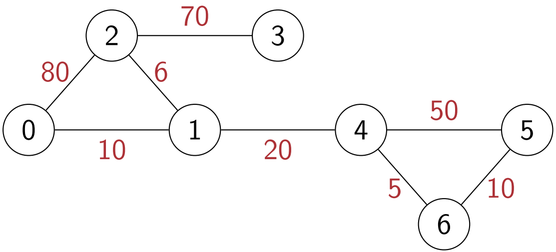

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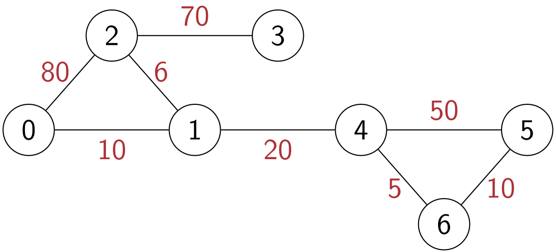

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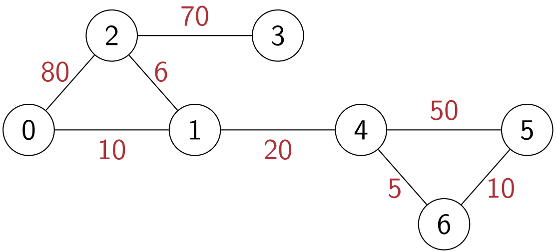

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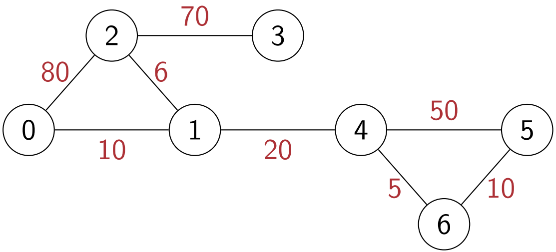

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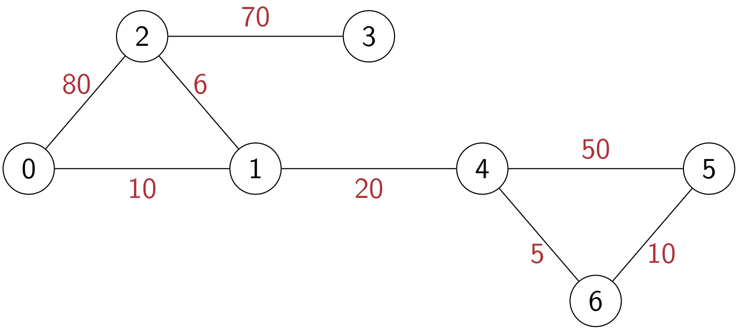

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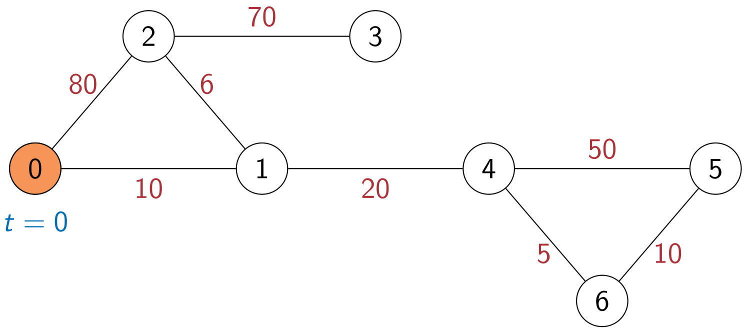

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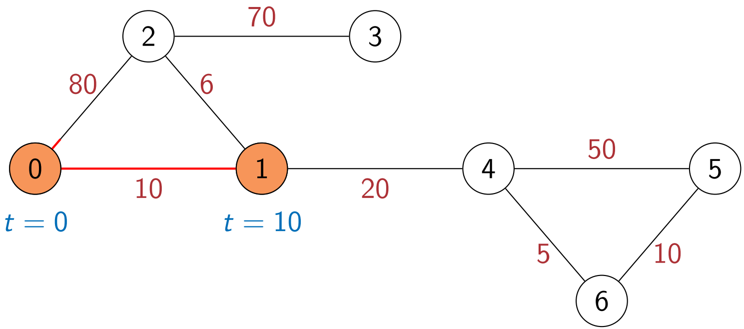

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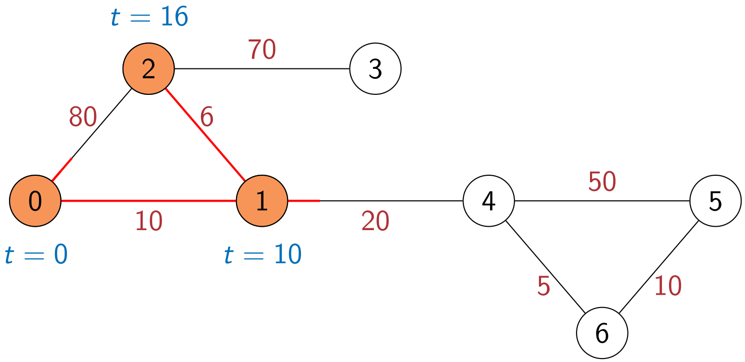

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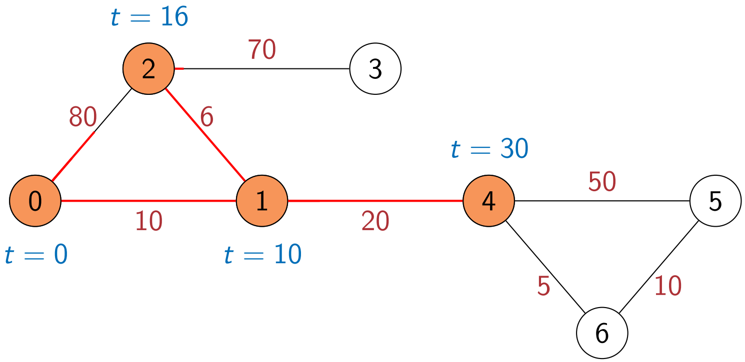

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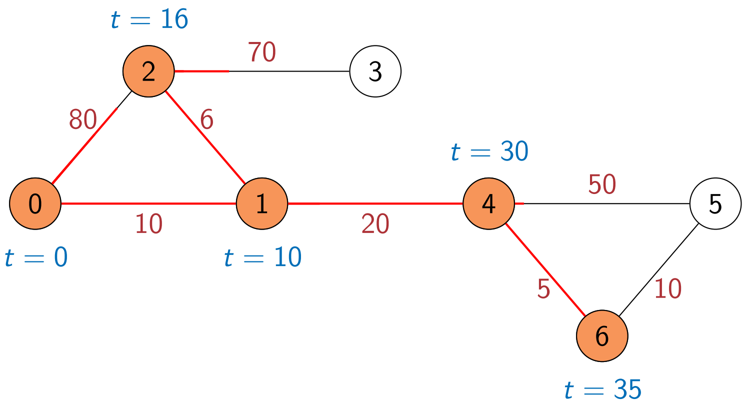

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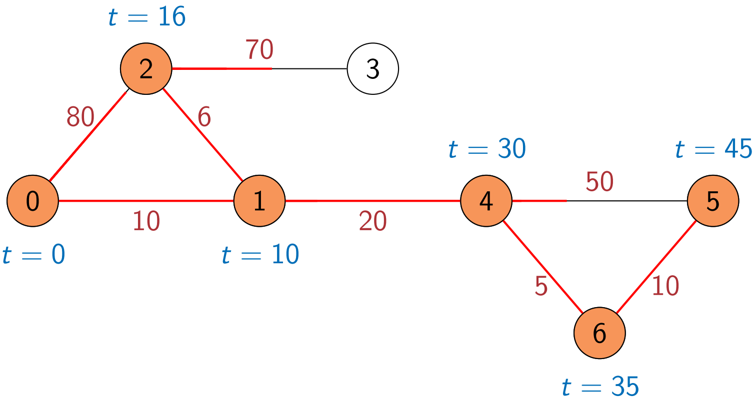

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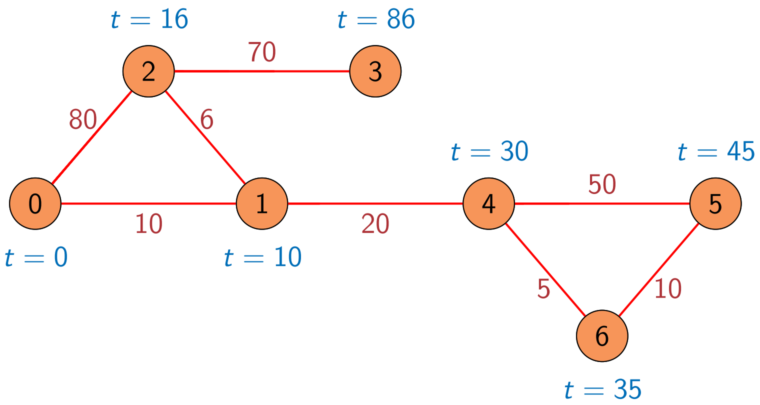
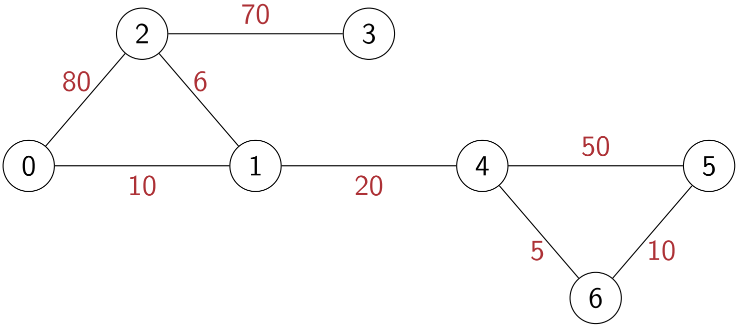


Single source shortest paths

- Compute expected burn time for each vertex
- Each time a new vertex burns, update the expected burn times of its neighbours



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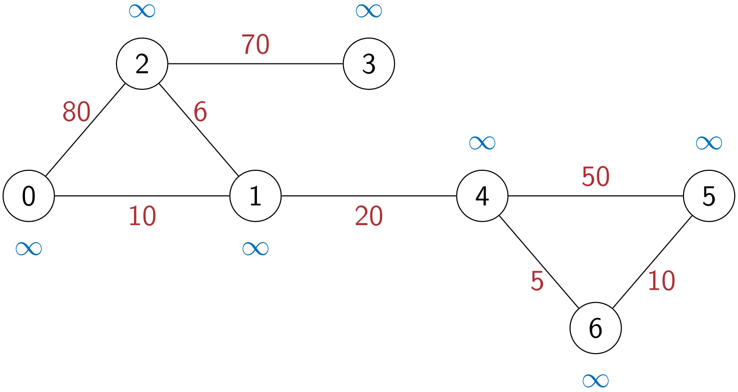

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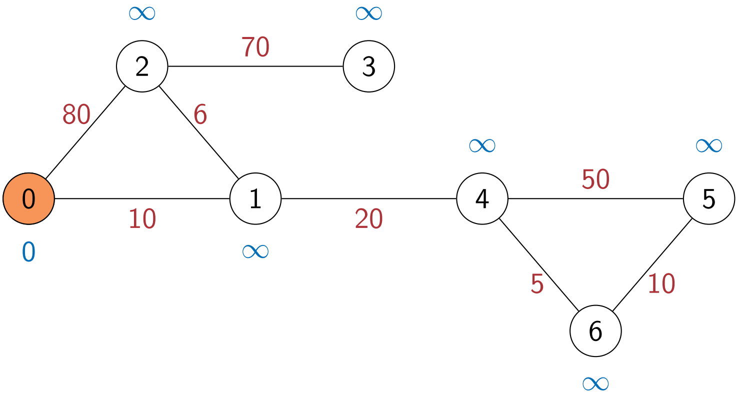

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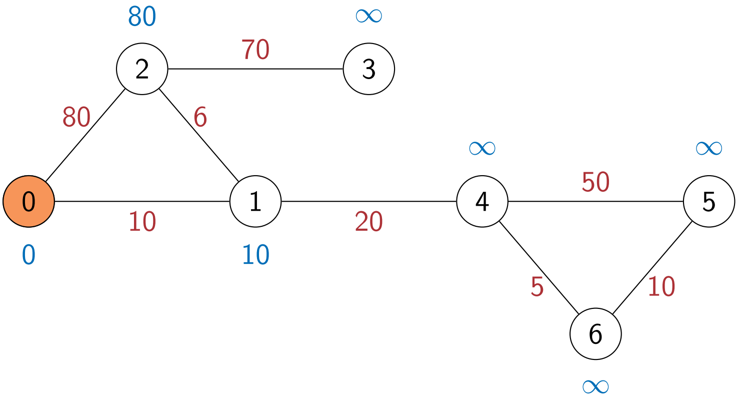

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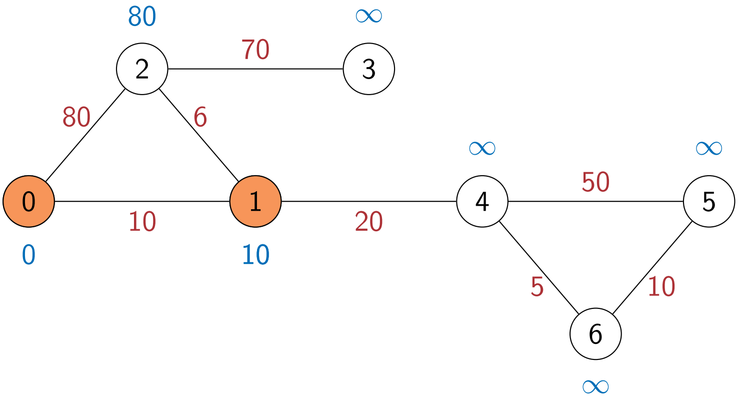

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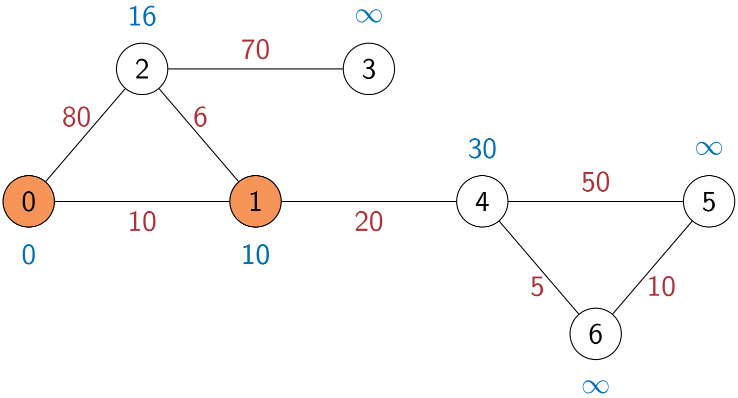

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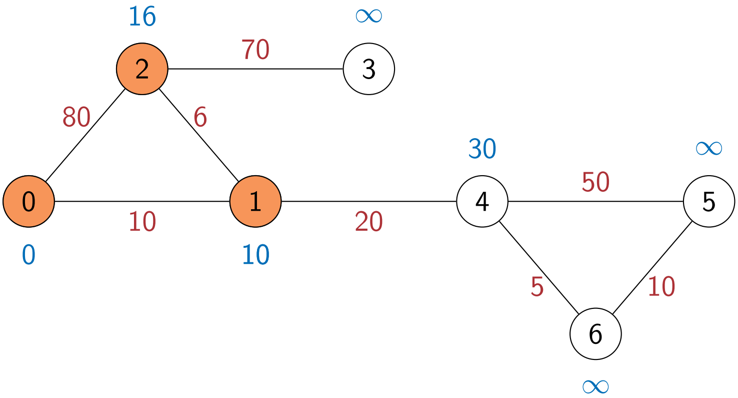

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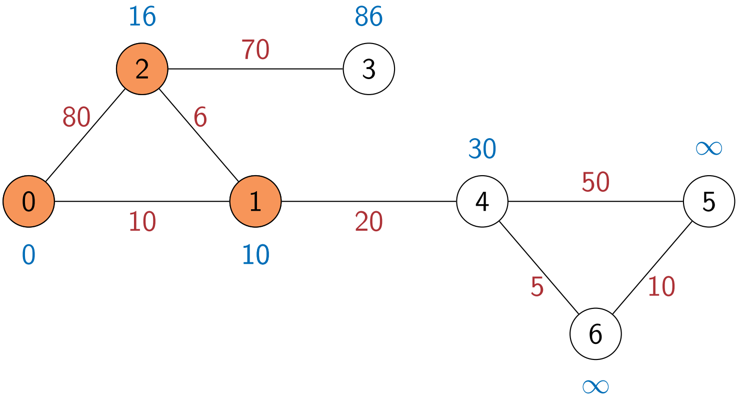

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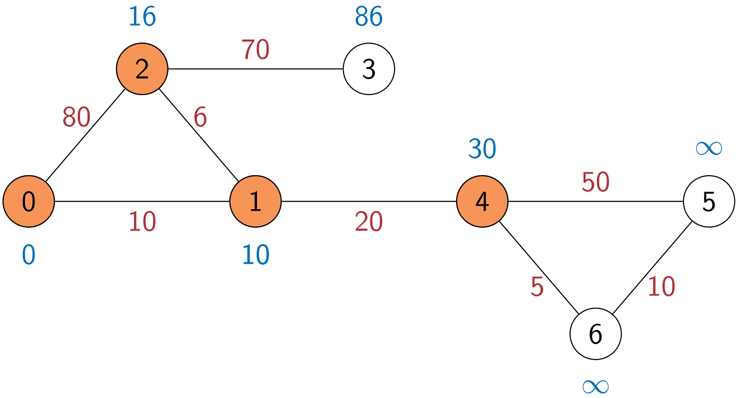

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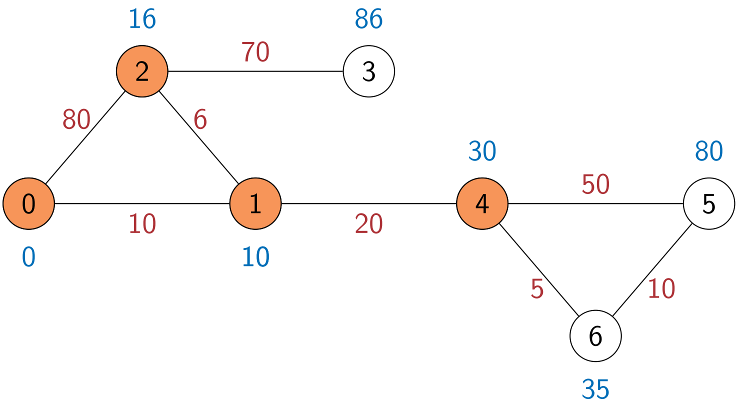

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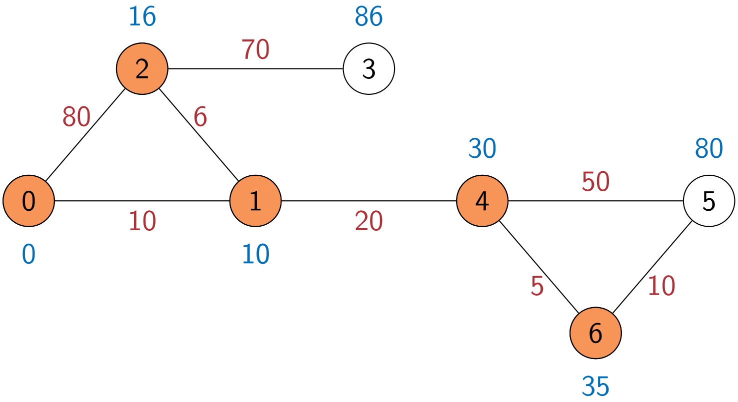

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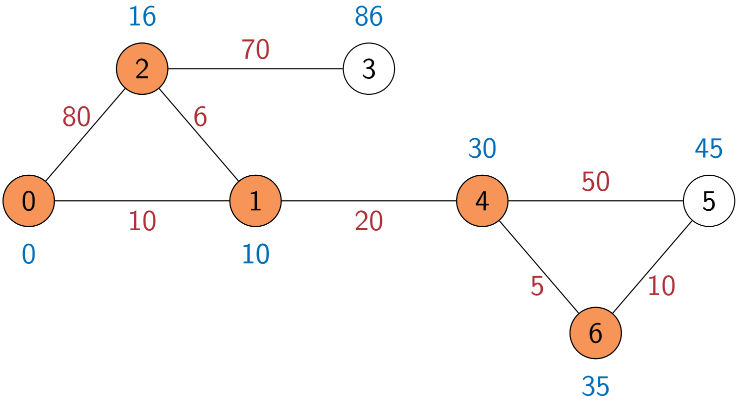

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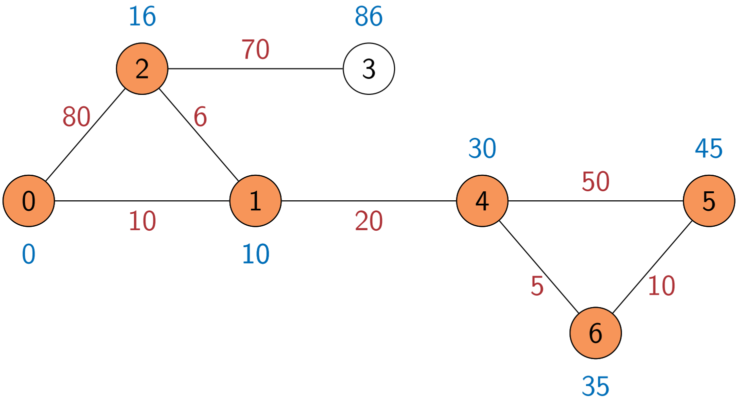

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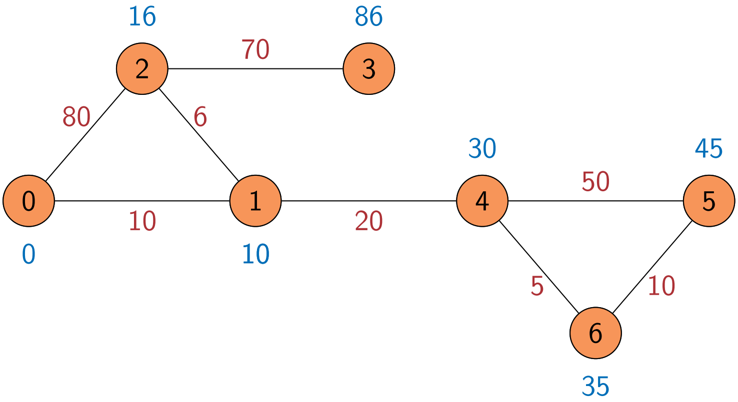
- Algorithm due to Edsger W Dikjstra



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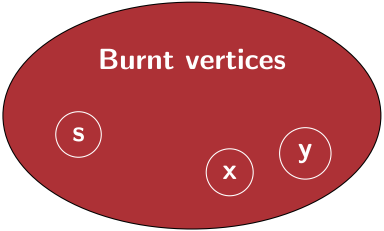

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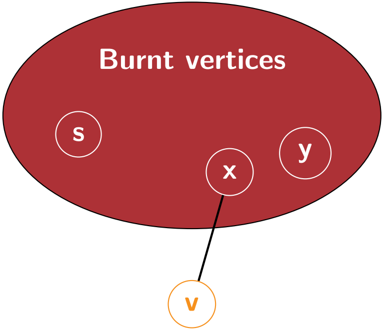

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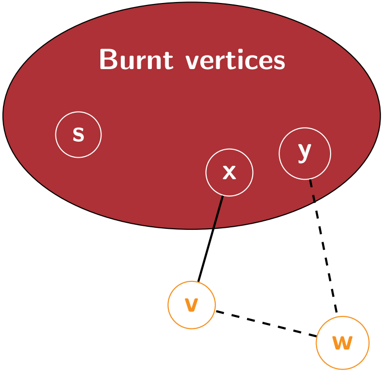
- Cannot find a shorter path later from **y** to **v** via **w**
  - Burn time of **w** $\geq$ burn time of **v**
  - Edge from **w** to **v** has weight ≥ 0



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  - Burn time of **w** $\geq$ burn time of **v**
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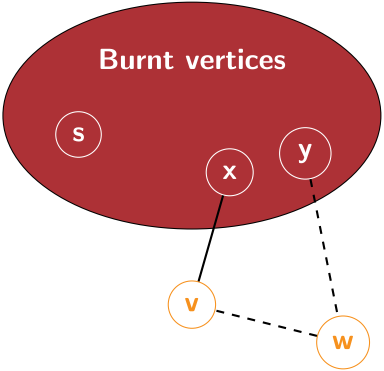
- This argument breaks down if edge (**w**,**v**) can have negative weight
  - Can't use Dijkstra's algorithm with negative edge weights



Implementation

- Maintain two dictionaries with vertices as keys
  - `visited`, initially `False` for all `v` (burnt vertices)
  - `distance`, initially `infinity` for all `v` (expected burn time)

```
def dijkstra(WMat,s):
  (rows,cols,x) = WMat.shape
  infinity = np.max(WMat)*rows+1
  (visited,distance) = ({},{})
  for v in range(rows):
    (visited[v],distance[v]) = (False,infinity)
  distance[s] = 0
  for u in range(rows):
    nextd = min([distance[v] for v in range(rows)
                    if not visited[v]])
    nextvlist = [v for v in range(rows)
                    if (not visited[v]) and
                       distance[v] == nextd]
    if nextvlist == []:
      break
    nextv = min(nextvlist)
    visited[nextv] = True
    for v in range(cols):
      if WMat[nextv,v,0] == 1 and (not visited[v]):
        distance[v] = min(distance[v],distance[nextv]
                                      +WMat[nextv,v,1])
  return(distance)
```

Implementation

- Maintain two dictionaries with vertices as keys
  - `visited`, initially `False` for all `v` (burnt vertices)
  - `distance`, initially `infinity` for all `v` (expected burn time)
- Set `distance[s]` to `0`

```
def dijkstra(WMat,s):
  (rows,cols,x) = WMat.shape
  infinity = np.max(WMat)*rows+1
  (visited,distance) = ({},{})
  for v in range(rows):
    (visited[v],distance[v]) = (False,infinity)
  distance[s] = 0
  for u in range(rows):
    nextd = min([distance[v] for v in range(rows)
                    if not visited[v]])
    nextvlist = [v for v in range(rows)
                    if (not visited[v]) and
                       distance[v] == nextd]
    if nextvlist == []:
      break
    nextv = min(nextvlist)
    visited[nextv] = True
    for v in range(cols):
      if WMat[nextv,v,0] == 1 and (not visited[v]):
        distance[v] = min(distance[v],distance[nextv]
                                      +WMat[nextv,v,1])
  return(distance)
```

Implementation

- Maintain two dictionaries with vertices as keys
  - `visited`, initially `False` for all `v` (burnt vertices)
  - `distance`, initially `infinity` for all `v` (expected burn time)
- Set `distance[s]` to `0`
- Repeat, until all reachable vertices are visited
  - Find unvisited vertex `nextv` with minimum distance
  - Set `visited[nextv]` to `True`
  - Recompute `distance[v]` for every neighbour `v` of `nextv`

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Complexity

- Setting `infinity` takes $O(n^2)$ time

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Complexity

- Setting `infinity` takes $O(n^2)$ time
- Main loop runs `n` times
  - Each iteration visits one more vertex
  - $O(n)$ to find next vertex to visit
  - $O(n)$ to update `distance[v]` for neighbours

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- Overall $O(n^2)$

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Complexity

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- Main loop runs `n` times
  - Each iteration visits one more vertex
  - $O(n)$ to find next vertex to visit
  - $O(n)$ to update `distance[v]` for neighbours
- Overall $O(n^2)$
- If we use an adjacency list
  - Setting `infinity` and updating distances both $O(m)$, amortised
  - $O(n)$ bottleneck remains to find next vertex to visit
  - Better data structure? Later …

```
def dijkstralist(WList,s):
  infinity = 1 + len(WList.keys())*
                max([d for u in WList.keys()
                       for (v,d) in WList[u]])
  (visited,distance) = ({},{})
  for v in WList.keys():
    (visited[v],distance[v]) = (False,infinity)
  distance[s] = 0
  for u in WList.keys():
    nextd = min([distance[v] for v in WList.keys()
                     if not visited[v]])
    nextvlist = [v for v in WList.keys()
                     if (not visited[v]) and
                        distance[v] == nextd]
    if nextvlist == []:
      break
    nextv = min(nextvlist)
    visited[nextv] = True
    for (v,d) in WList[nextv]:
      if not visited[v]:
        distance[v] = min(distance[v],distance[nextv]+d)
  return(distance)
```

Summary

- Dijkstra's algorithm computes single source shortest paths
- Use a greedy strategy to identify vertices to visit
  - Next vertex to visit is based on shortest distance computed so far
  - Need to prove that such a strategy is correct
  - Correctness requires edge weights to be non-negative
- Complexity is $O(n^2)$
  - Even with adjacency lists
  - Bottleneck is identifying unvisited vertex with minimum distance
  - Need a better data structure to identify and remove minimum (or maximum) from a collection

Single Source Shortest Paths with Negative Weights

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Programming, Data Structures and Algorithms using Python

Week 5

Dijkstra's algorithm

- Recall the burning pipeline analogy

Dijkstra's algorithm

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- We keep track of the following
  - The vertices that have been burnt
  - The expected burn time of vertices

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  - The vertices that have been burnt
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- Initially
  - No vertex is burnt
  - Expected burn time of source vertex is 0
  - Expected burn time of rest is ∞

Initialization (assume source vertex 0)

- $B(i) = \text{False}$, for $0 \leq i < n$
  - $UB = \{k \mid B(k) = \text{False}\}$
- $EBT(i) = \begin{cases} 0, & \text{if } i = 0 \\ \infty, & \text{otherwise} \end{cases}$

Dijkstra's algorithm

- Recall the burning pipeline analogy
- We keep track of the following
  - The vertices that have been burnt
  - The expected burn time of vertices
- Initially
  - No vertex is burnt
  - Expected burn time of source vertex is 0
  - Expected burn time of rest is ∞
- While there are vertices yet to burn
  - Pick unburnt vertex with minimum expected burn time, mark it as burnt
  - Update the expected burn time of its neighbours

Initialization (assume source vertex 0)

- $B(i) = \text{False}$, for $0 \leq i < n$
  - $UB = \{k \mid B(k) = \text{False}\}$
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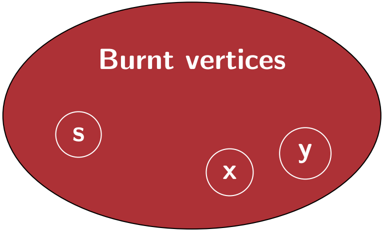
Update, if $UB \neq \emptyset$

- Let $j \in UB$ such that $EBT(j) \leq EBT(k)$ for all $k \in UB$
- Update $B(j) = \text{True}$, $UB = UB \setminus \{j\}$
- For each $(j,k) \in E$ such that $k \in UB$, $EBT(k) = \min(\ EBT(k),\ EBT(j) + W(j,k)\ )$

Correctness requires non-negative edge weights

- Each new shortest path we discover extends an earlier one
- By induction, assume we have found shortest paths to all vertices already burnt



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- Each new shortest path we discover extends an earlier one
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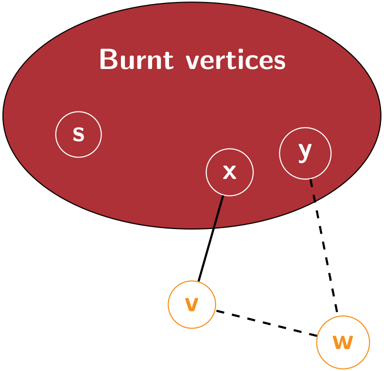
- Next vertex to burn is **v**, via **x**
- Cannot find a shorter path later from **y** to **v** via **w**
  - Burn time of **w** $\geq$ burn time of **v**
  - Edge from **w** to **v** has weight ≥ 0



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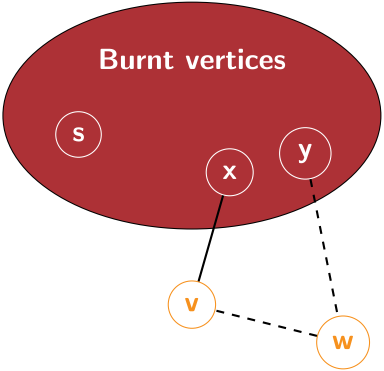
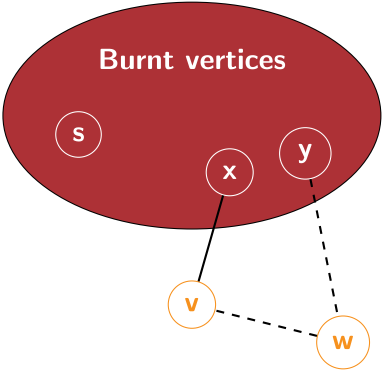
- This argument breaks down if edge (**w**,**v**) can have negative weight



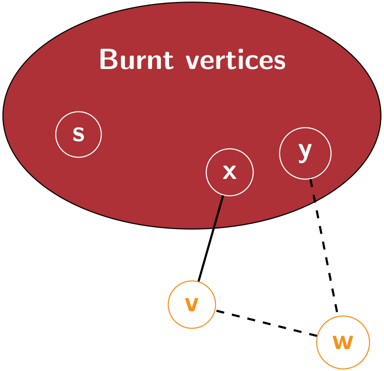
Extending to negative edge weights

- The difficulty with negative edge weights is that we stop updating the burn time once a vertex is burnt



Extending to negative edge weights

- The difficulty with negative edge weights is that we stop updating the burn time once a vertex is burnt
- What if we allow updates even after a vertex is burnt?



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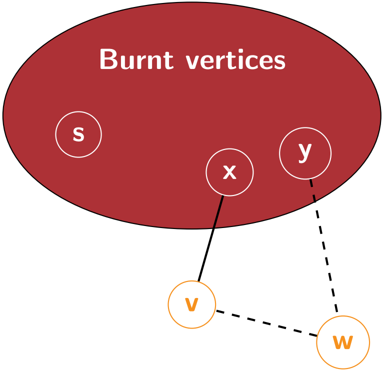
- Recall, negative edge weights are allowed, but no negative cycles



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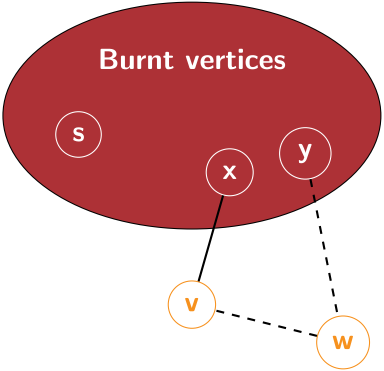
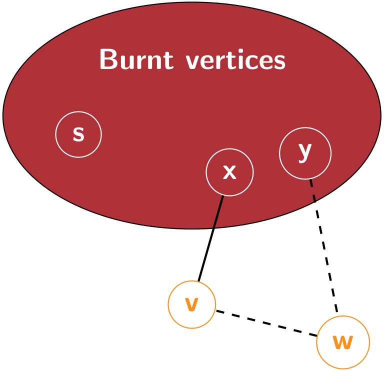


Extending to negative edge weights

- The difficulty with negative edge weights is that we stop updating the burn time once a vertex is burnt
- What if we allow updates even after a vertex is burnt?
- Recall, negative edge weights are allowed, but no negative cycles
- Going around a cycle can only add to the length
- Shortest route to every vertex is a path, no loops



Extending to negative edge weights

- Suppose minimum weight path from 0 to k is

  $$0 \xrightarrow{w_1} j_1 \xrightarrow{w_2} j_2 \xrightarrow{w_3} \cdots \xrightarrow{w_{\ell-1}} j_{\ell-1} \xrightarrow{w_\ell} k$$

  - Need not be minimum in terms of number of edges

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- Every prefix of this path must itself be a minimum weight path
  - $0 \xrightarrow{w_1} j_1$
  - $0 \xrightarrow{w_1} j_1 \xrightarrow{w_2} j_2$
  - ...
  - $0 \xrightarrow{w_1} j_1 \xrightarrow{w_2} j_2 \xrightarrow{w_3} \cdots \xrightarrow{w_{\ell-1}} j_{\ell-1}$

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- Repeatedly update shortest distance to each vertex based on shortest distance to its neighbours
  - Update cannot push this distance below actual shortest distance

- After ℓ updates, all shortest paths using $\leq \ell$ edges have stabilized
  - Minimum weight path to any node has at most $n-1$ edges
  - After $n-1$ updates, all shortest paths have stabilized

Bellman-Ford Algorithm

Initialization (source vertex 0)

- $D(j)$: minimum distance known so far to vertex j
- $D(j) = \begin{cases} 0, & \text{if } j = 0 \\ \infty, & \text{otherwise} \end{cases}$

Bellman-Ford Algorithm

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Repeat $n-1$ times

- For each vertex $j \in \{0, 1, \ldots, n-1\}$, for each edge $(j, k) \in E$,
  $D(k) = \min(\ D(k),$
  $\qquad D(j) + W(j, k)\ )$

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Works for directed and undirected graphs

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def bellmanford(WMat,s):
  (rows,cols,x) = WMat.shape
  infinity = np.max(WMat)*rows+1
  distance = {}
  for v in range(rows):
    distance[v] = infinity

  distance[s] = 0

  for i in range(rows):
    for u in range(rows):
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        if WMat[u,v,0] == 1:
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  return(distance)
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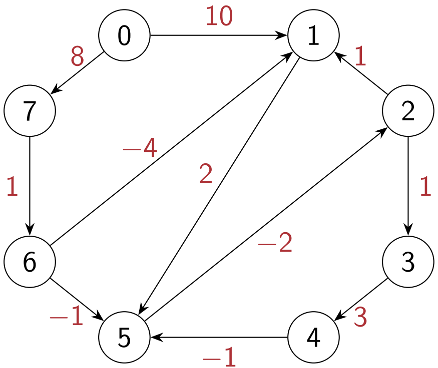
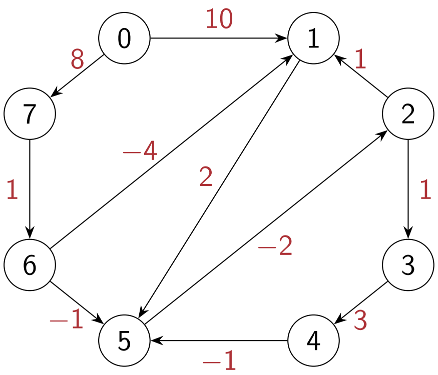


Works for directed and undirected graphs

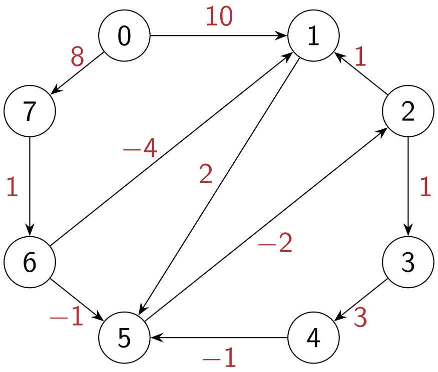
Bellman-Ford Algorithm

v	$D(v)$							
0								
1								
2								
3								
4								
5								
6								
7								



Bellman-Ford Algorithm

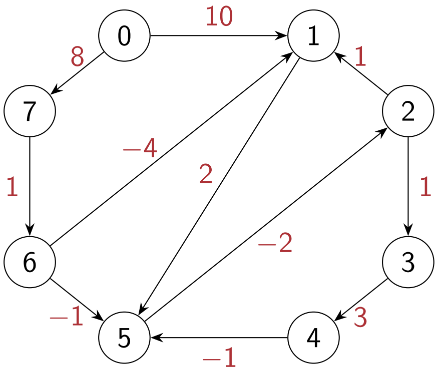
v	$D(v)$							
0	0							
1	∞							
2	∞							
3	∞							
4	∞							
5	∞							
6	∞							
7	∞							



- Initialize $D(0) = 0$

Bellman-Ford Algorithm

v	$D(v)$							
0	0	0						
1	∞	10						
2	∞	∞						
3	∞	∞						
4	∞	∞						
5	∞	∞						
6	∞	∞						
7	∞	8						



- Initialize $D(0) = 0$
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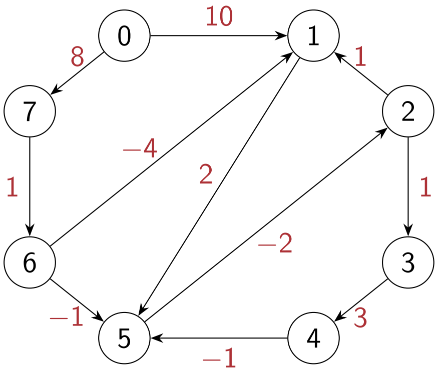
$$D(k) = \min(\ D(k),\ D(j) + W(j,k)\ )$$

Bellman-Ford Algorithm

v	$D(v)$							
0	0	0	0					
1	∞	10	10					
2	∞	∞	∞					
3	∞	∞	∞					
4	∞	∞	∞					
5	∞	∞	12					
6	∞	∞	9					
7	∞	8	8					



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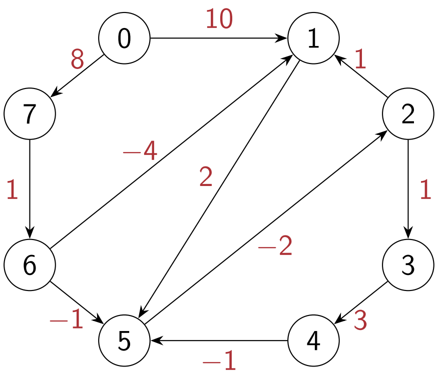
Bellman-Ford Algorithm

v	$D(v)$							
0	0	0	0	0				
1	∞	10	10	5				
2	∞	∞	∞	10				
3	∞	∞	∞	∞				
4	∞	∞	∞	∞				
5	∞	∞	12	8				
6	∞	∞	9	9				
7	∞	8	8	8				



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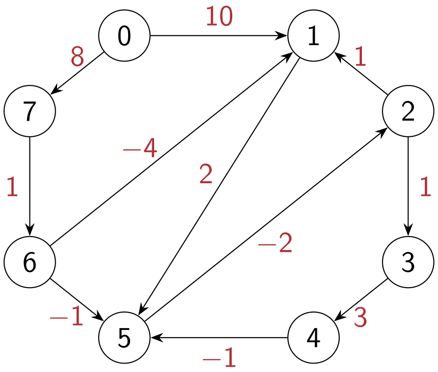
v	$D(v)$							
0	0	0	0	0	0			
1	∞	10	10	5	5			
2	∞	∞	∞	10	6			
3	∞	∞	∞	∞	11			
4	∞	∞	∞	∞	∞			
5	∞	∞	12	8	7			
6	∞	∞	9	9	9			
7	∞	8	8	8	8			



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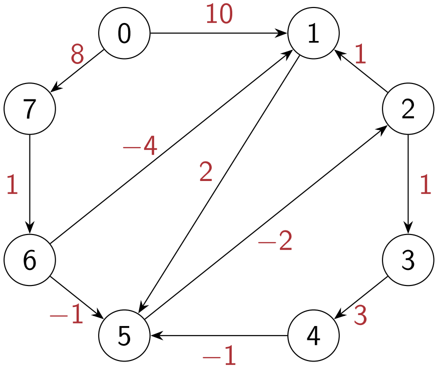
Bellman-Ford Algorithm

v	$D(v)$							
0	0	0	0	0	0	0		
1	∞	10	10	5	5	5		
2	∞	∞	∞	10	6	5		
3	∞	∞	∞	∞	11	7		
4	∞	∞	∞	∞	∞	14		
5	∞	∞	12	8	7	7		
6	∞	∞	9	9	9	9		
7	∞	8	8	8	8	8		

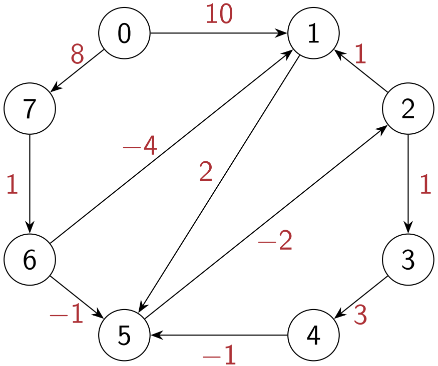


- Initialize $D(0) = 0$
- For each $(j, k) \in E$, update

$$D(k) = \min(\ D(k),\ D(j) + W(j, k)\ )$$

Bellman-Ford Algorithm

v	$D(v)$							
0	0	0	0	0	0	0	0	
1	∞	10	10	5	5	5	5	
2	∞	∞	∞	10	6	5	5	
3	∞	∞	∞	∞	11	7	6	
4	∞	∞	∞	∞	∞	14	10	
5	∞	∞	12	8	7	7	7	
6	∞	∞	9	9	9	9	9	
7	∞	8	8	8	8	8	8	



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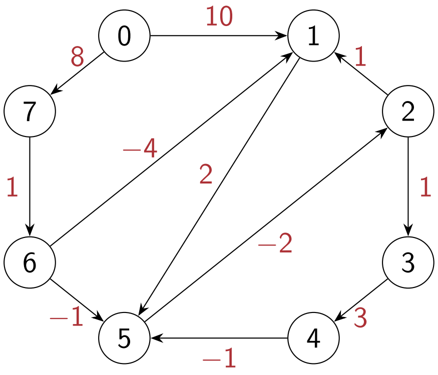
Bellman-Ford Algorithm

v	$D(v)$							
0	0	0	0	0	0	0	0	0
1	∞	10	10	5	5	5	5	5
2	∞	∞	∞	10	6	5	5	5
3	∞	∞	∞	∞	11	7	6	6
4	∞	∞	∞	∞	∞	14	10	9
5	∞	∞	12	8	7	7	7	7
6	∞	∞	9	9	9	9	9	9
7	∞	8	8	8	8	8	8	8



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Bellman-Ford Algorithm

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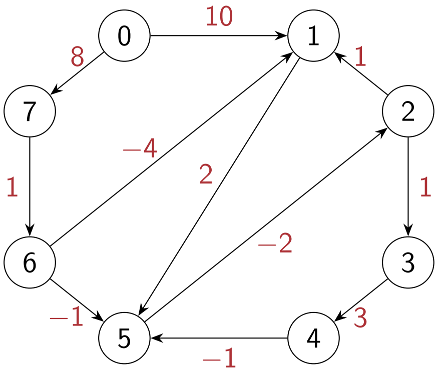
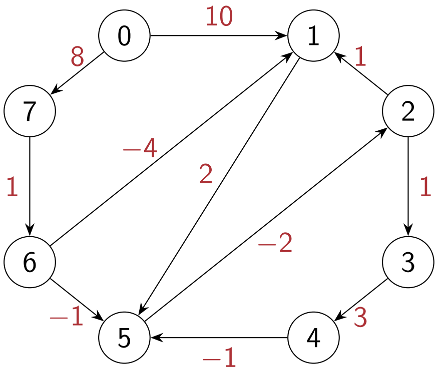
0	0	0	0	0	0	0	0	0
1	∞	10	10	5	5	5	5	5
2	∞	∞	∞	10	6	5	5	5
3	∞	∞	∞	∞	11	7	6	6
4	∞	∞	∞	∞	∞	14	10	9
5	∞	∞	12	8	7	7	7	7
6	∞	∞	9	9	9	9	9	9
7	∞	8	8	8	8	8	8	8



- What if there was a negative cycle?

Bellman-Ford Algorithm

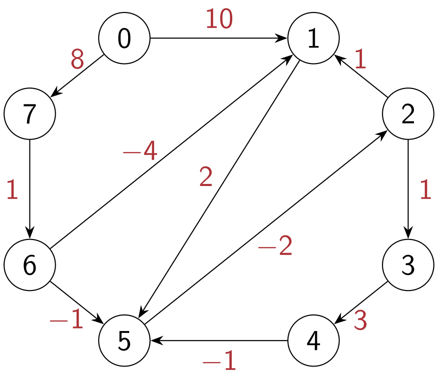
v	$D(v)$							
0	0	0	0	0	0	0	0	0
1	∞	10	10	5	5	5	5	5
2	∞	∞	∞	10	6	5	5	5
3	∞	∞	∞	∞	11	7	6	6
4	∞	∞	∞	∞	∞	14	10	9
5	∞	∞	12	8	7	7	7	7
6	∞	∞	9	9	9	9	9	9
7	∞	8	8	8	8	8	8	8



- What if there was a negative cycle?
- Distance would continue to decrease

Bellman-Ford Algorithm

v	$D(v)$							
0	0	0	0	0	0	0	0	0
1	∞	10	10	5	5	5	5	5
2	∞	∞	∞	10	6	5	5	5
3	∞	∞	∞	∞	11	7	6	6
4	∞	∞	∞	∞	∞	14	10	9
5	∞	∞	12	8	7	7	7	7
6	∞	∞	9	9	9	9	9	9
7	∞	8	8	8	8	8	8	8



- What if there was a negative cycle?
- Distance would continue to decrease
- Check if update n reduces any $D(v)$

Complexity

- Initialing `infinity` takes $O(n^2)$ time

```
def bellmanford(WMat,s):
  (rows,cols,x) = WMat.shape
  infinity = np.max(WMat)*rows+1
  distance = {}
  for v in range(rows):
    distance[v] = infinity

  distance[s] = 0

  for i in range(rows):
    for u in range(rows):
      for v in range(cols):
        if WMat[u,v,0] == 1:
          distance[v] = min(distance[v],distance[u]
                                          +WMat[u,v,1])
  return(distance)
```

Complexity

- Initialing `infinity` takes $O(n^2)$ time
- The outer update loop runs $O(n)$ times

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Complexity

- Initialing `infinity` takes $O(n^2)$ time
- The outer update loop runs $O(n)$ times
- In each iteration, we have to examine every edge in the graph
  - This take $O(n^2)$ for an adjacency matrix

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- Overall, $O(n^3)$

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Complexity

- Initialing `infinity` takes $O(n^2)$ time
- The outer update loop runs $O(n)$ times
- In each iteration, we have to examine every edge in the graph
  - This take $O(n^2)$ for an adjacency matrix
- Overall, $O(n^3)$
- If we shift to adjacency lists
  - Initializing `infinity` is $O(m)$
  - Scanning all edges in each update iteration is $O(m)$

```
def bellmanfordlist(WList,s):
  infinity = 1 + len(WList.keys())*
               max([d for u in WList.keys()
                       for (v,d) in WList[u]])
  distance = {}
  for v in WList.keys():
    distance[v] = infinity

  distance[s] = 0

  for i in WList.keys():
    for u in WList.keys():
      for (v,d) in WList[u]:
        distance[v] = min(distance[v],distance[u] + d)
  return(distance)
```

Complexity

- Initialing `infinity` takes $O(n^2)$ time
- The outer update loop runs $O(n)$ times
- In each iteration, we have to examine every edge in the graph
  - This take $O(n^2)$ for an adjacency matrix
- Overall, $O(n^3)$
- If we shift to adjacency lists
  - Initializing `infinity` is $O(m)$
  - Scanning all edges in each update iteration is $O(m)$
- Now, overall $O(mn)$

```
def bellmanfordlist(WList,s):
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                max([d for u in WList.keys()
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      for (v,d) in WList[u]:
        distance[v] = min(distance[v],distance[u] + d)
  return(distance)
```

Summary

- Dijkstra's algorithm assumes non-negative edge weights
  - Final distance is frozen each time a vertex "burns"
  - Should not encounter a shorter route discovered later
- Without negative cycles, every shortest route is a path
- Every prefix of a shortest path is also a shortest path
- Iteratively find shortest paths of length $1, 2, \ldots, n-1$
- Update distance to each vertex with every iteration — Bellman-Ford algorithm
- $O(n^3)$ time with adjacency matrix, $O(mn)$ time with adjacency list
- If Bellman-Ford algorithm does not converge after $n-1$ iterations, there is a negative cycle

All-Pairs Shortest Paths

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Programming, Data Structures and Algorithms using Python

Week 5

Shortest paths in weighted graphs

Two types of shortest path problems of interest

Single source shortest paths

- Find shortest paths from a fixed vertex to every other vertex
- Transport finished product from factory (single source) to all retail outlets
- Courier company delivers items from distribution centre (single source) to addressees

All pairs shortest paths

- Find shortest paths between every pair of vertices i and j
- Optimal airline, railway, road routes between cities

Shortest paths in weighted graphs

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- Dijkstra's algorithm (non-negative weights), Bellman-Ford algorithm (allows negative weights)

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- Find shortest paths between every pair of vertices i and j
- Optimal airline, railway, road routes between cities
- Run Dijkstra or Bellman-Ford from each vertex

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- Dijkstra's algorithm (non-negative weights), Bellman-Ford algorithm (allows negative weights)

All pairs shortest paths

- Find shortest paths between every pair of vertices i and j
- Optimal airline, railway, road routes between cities
- Run Dijkstra or Bellman-Ford from each vertex
- Is there is another way?

Transitive closure

- Adjacency matrix A represents paths of length 1
- Matrix multiplication, $A^2 = A \times A$
  - $A^2[i,j] = 1$ if there is a path of length 2 from i to j
  - For some k, $A[i,k] = A[k,j] = 1$
- In general, $A^{\ell+1} = A^{\ell} \times A$,
  - $A^{\ell+1}[i,j] = 1$ if there is a path of length $\ell+1$ from i to j
  - For some k, $A^{\ell}[i,k] = 1$, $A[k,j] = 1$
- $A^{+} = A + A^2 + \cdots + A^{n-1}$

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An alternative approach

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An alternative approach

- $B^k[i,j] = 1$ if there is path from i to j via vertices $\{0, 1, \ldots, k-1\}$
  - Constraint applies only to intermediate vertices between i and j
  - $B^0[i,j] = 1$ if there is a direct edge
  - $B^0 = A$

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  - Constraint applies only to intermediate vertices between i and j
  - $B^0[i,j] = 1$ if there is a direct edge
  - $B^0 = A$
- $B^{k+1}[i,j] = 1$ if
  - $B^k[i,j] = 1$ — can already reach j from i via $\{0,1,\ldots,k-1\}$
  - $B^k[i,k] = 1$ and $B^k[k,j] = 1$ — use $\{0,1,\ldots,k-1\}$ to go from i to k and then from k to j

Warshall's Algorithm

- $B^k[i,j] = 1$ if there is path from i to j via vertices $\{0, 1, \ldots, k-1\}$
- $B^0[i,j] = A[i,j]$
  - Direct edges, no intermediate vertices
- $B^{k+1}[i,j] = 1$ if
  - $B^k[i,j] = 1$, or
  - $B^k[i,k] = 1$ and $B^k[k,j] = 1$

- The algorithm on the left also computes transitive closure — Warshall's algorithm

Warshall's Algorithm

- $B^k[i,j] = 1$ if there is path from i to j via vertices $\{0, 1, \ldots, k-1\}$
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- The algorithm on the left also computes transitive closure — Warshall's algorithm
- $B^n[i,j] = 1$ if there is some path from i to j with intermediate vertices in $\{0, 1, \ldots, n-1\}$

Warshall's Algorithm

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- $B^{k+1}[i,j] = 1$ if
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- The algorithm on the left also computes transitive closure — Warshall's algorithm
- $B^n[i,j] = 1$ if there is some path from i to j with intermediate vertices in $\{0, 1, \dots, n-1\}$
- $B^n = A^+$
- We adapt Warshall's algorithm to compute all-pairs shortest paths

Floyd-Warshall Algorithm

- Let $SP^k[i,j]$ be the length of the shortest path from i to j via vertices $\{0, 1, \ldots, k-1\}$
- $SP^0[i,j] = W[i,j]$
  - No intermediate vertices, shortest path is weight of direct edge
  - Assume $W[i,j] = \infty$ if $(i,j) \notin E$

Floyd-Warshall Algorithm

- Let $SP^k[i,j]$ be the length of the shortest path from i to j via vertices $\{0, 1, \ldots, k{-}1\}$
- $SP^0[i,j] = W[i,j]$
  - No intermediate vertices, shortest path is weight of direct edge
  - Assume $W[i,j] = \infty$ if $(i,j) \notin E$
- $SP^{k+1}[i,j]$ is the minimum of
  - $SP^k[i,j]$

    Shortest path using only $\{0, 1, \ldots, k{-}1\}$
  - $SP^k[i,k] + SP^k[k,j]$

    Combine shortest path from i to k and k to j

Floyd-Warshall Algorithm

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- $SP^0[i,j] = W[i,j]$
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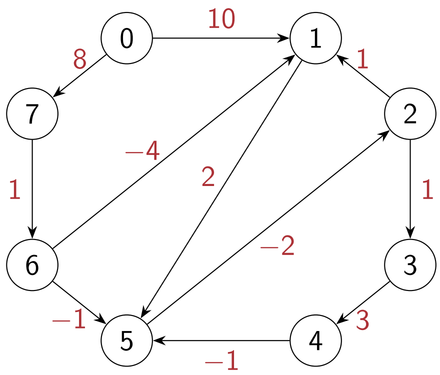
  - Assume $W[i,j] = \infty$ if $(i,j) \notin E$
- $SP^{k+1}[i,j]$ is the minimum of
  - $SP^k[i,j]$

    Shortest path using only $\{0,1,\ldots,k{-}1\}$
  - $SP^k[i,k] + SP^k[k,j]$

    Combine shortest path from i to k and k to j
- $SP^n[i,j] = 1$ is the length of the shortest path overall from i to j
  - Intermediate vertices lie in $\{0,1,\ldots,n{-}1\}$

Floyd-Warshall Algorithm

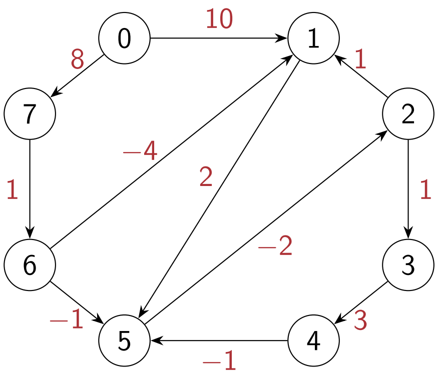
SP^0	0	1	2	3	4	5	6	7
0	∞	10	∞	∞	∞	∞	∞	8
1	∞	∞	∞	∞	∞	2	∞	∞
2	∞	1	∞	1	∞	∞	∞	∞
3	∞	∞	∞	∞	3	∞	∞	∞
4	∞	∞	∞	∞	∞	-1	∞	∞
5	∞	∞	-2	∞	∞	∞	∞	∞
6	∞	-4	∞	∞	∞	-1	∞	∞
7	∞	∞	∞	∞	∞	∞	1	∞



Floyd-Warshall Algorithm

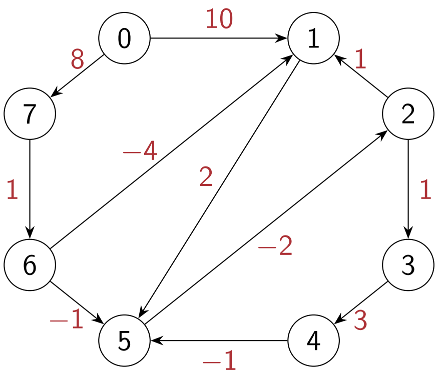
SP^0	0	1	2	3	4	5	6	7
0	∞	10	∞	∞	∞	∞	∞	8
1	∞	∞	∞	∞	∞	2	∞	∞
2	∞	1	∞	1	∞	∞	∞	∞
3	∞	∞	∞	∞	3	∞	∞	∞
4	∞	∞	∞	∞	∞	-1	∞	∞
5	∞	∞	-2	∞	∞	∞	∞	∞
6	∞	-4	∞	∞	∞	-1	∞	∞
7	∞	∞	∞	∞	∞	∞	1	∞

SP^1	0	1	2	3	4	5	6	7
0	∞	10	∞	∞	∞	∞	∞	8
1	∞	∞	∞	∞	∞	2	∞	∞
2	∞	1	∞	1	∞	∞	∞	∞
3	∞	∞	∞	∞	3	∞	∞	∞
4	∞	∞	∞	∞	∞	-1	∞	∞
5	∞	∞	-2	∞	∞	∞	∞	∞
6	∞	-4	∞	∞	∞	-1	∞	∞
7	∞	∞	∞	∞	∞	∞	1	∞



Floyd-Warshall Algorithm

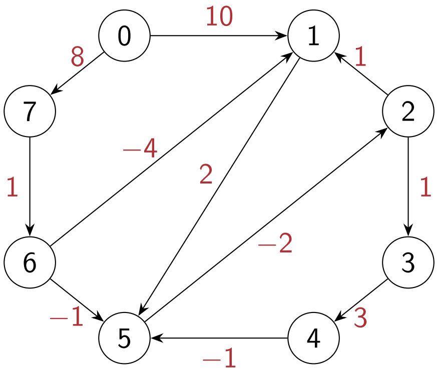
SP^1	0	1	2	3	4	5	6	7
0	∞	10	∞	∞	∞	∞	∞	8
1	∞	∞	∞	∞	∞	2	∞	∞
2	∞	1	∞	1	∞	∞	∞	∞
3	∞	∞	∞	∞	3	∞	∞	∞
4	∞	∞	∞	∞	∞	-1	∞	∞
5	∞	∞	-2	∞	∞	∞	∞	∞
6	∞	-4	∞	∞	∞	-1	∞	∞
7	∞	∞	∞	∞	∞	∞	1	∞



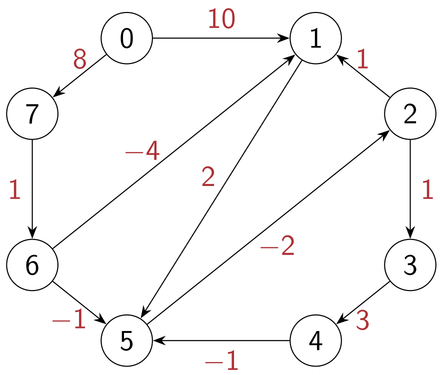
Floyd-Warshall Algorithm

SP^1	0	1	2	3	4	5	6	7
0	∞	10	∞	∞	∞	∞	∞	8
1	∞	∞	∞	∞	∞	2	∞	∞
2	∞	1	∞	1	∞	∞	∞	∞
3	∞	∞	∞	∞	3	∞	∞	∞
4	∞	∞	∞	∞	∞	-1	∞	∞
5	∞	∞	-2	∞	∞	∞	∞	∞
6	∞	-4	∞	∞	∞	-1	∞	∞
7	∞	∞	∞	∞	∞	∞	1	∞

SP^2	0	1	2	3	4	5	6	7
0	∞	10	∞	∞	∞	12	∞	8
1	∞	∞	∞	∞	∞	2	∞	∞
2	∞	1	∞	1	∞	3	∞	∞
3	∞	∞	∞	∞	3	∞	∞	∞
4	∞	∞	∞	∞	∞	-1	∞	∞
5	∞	∞	-2	∞	∞	∞	∞	∞
6	∞	-4	∞	∞	∞	-2	∞	∞
7	∞	∞	∞	∞	∞	∞	1	∞

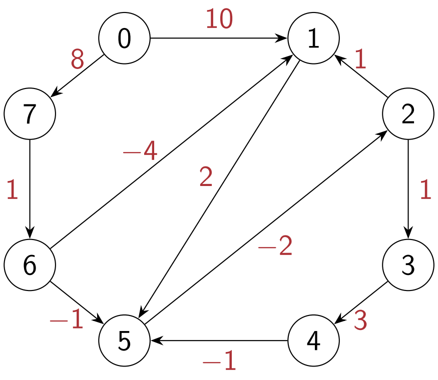


SP^2	0	1	2	3	4	5	6	7
0	∞	10	∞	∞	∞	12	∞	8
1	∞	∞	∞	∞	∞	2	∞	∞
2	∞	1	∞	1	∞	3	∞	∞
3	∞	∞	∞	∞	3	∞	∞	∞
4	∞	∞	∞	∞	∞	-1	∞	∞
5	∞	∞	-2	∞	∞	∞	∞	∞
6	∞	-4	∞	∞	∞	-2	∞	∞
7	∞	∞	∞	∞	∞	∞	1	∞



SP^2	0	1	2	3	4	5	6	7
0	∞	10	∞	∞	∞	12	∞	8
1	∞	∞	∞	∞	∞	2	∞	∞
2	∞	1	∞	1	∞	3	∞	∞
3	∞	∞	∞	∞	3	∞	∞	∞
4	∞	∞	∞	∞	∞	-1	∞	∞
5	∞	∞	-2	∞	∞	∞	∞	∞
6	∞	-4	∞	∞	∞	-2	∞	∞
7	∞	∞	∞	∞	∞	∞	1	∞

SP^3	0	1	2	3	4	5	6	7
0	∞	10	∞	∞	∞	12	∞	8
1	∞	∞	∞	∞	∞	2	∞	∞
2	∞	1	∞	1	∞	3	∞	∞
3	∞	∞	∞	∞	3	∞	∞	∞
4	∞	∞	∞	∞	∞	-1	∞	∞
5	∞	-1	-2	-1	∞	1	∞	∞
6	∞	-4	∞	∞	∞	-2	∞	∞
7	∞	∞	∞	∞	∞	∞	1	∞



Floyd-Warshall Algorithm

SP^7	0	1	2	3	4	5	6	7
0	∞	10	10	11	14	12	∞	8
1	∞	1	0	1	4	2	∞	∞
2	∞	1	1	1	4	3	∞	∞
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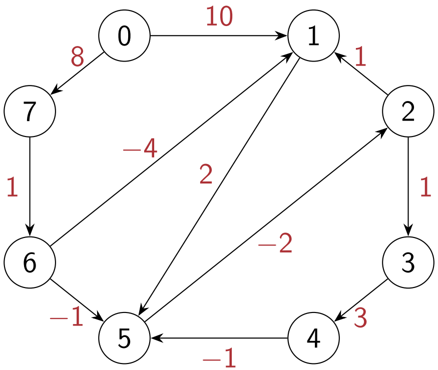

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7	∞	-3	-3	-2	1	-1	1	∞

SP^8	0	1	2	3	4	5	6	7
0	∞	5	5	6	9	7	9	8
1	∞	1	0	1	4	2	∞	∞
2	∞	1	1	1	4	3	∞	∞
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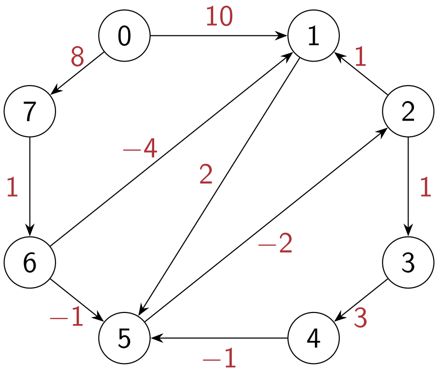

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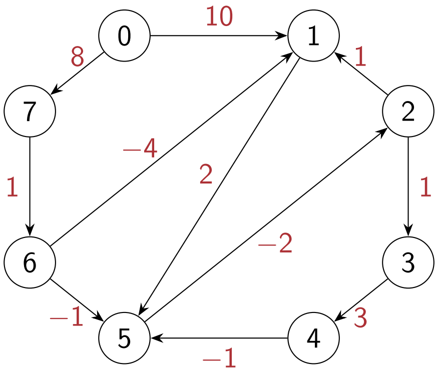

Implementation

- Shortest path matrix SP is $n \times n \times (n+1)$

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- Time complexity is $O(n^3)$
- We only need $SP[i,j,k-1]$ to compute $SP[i,j,k]$
- Maintain two "slices" $SP[i,j]$, $SP'[i,j]$, compute SP' from SP, copy SP' to SP, save space

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```

Summary

- Warshall's algorithm is an alternative way to compute transitive closure
  - $B^k[i,j] = 1$ if we can reach j from i using vertices in $\{0, 1, \ldots, k-1\}$
- Adapt Warshall's algorithm to compute all pairs shortest paths
  - $SP^k[i,j]$ is the length of the shortest path from i to j using vertices in $\{0, 1, \ldots, k-1\}$
  - $SP^n[i,j]$ is the length of the overall shortest path
  - Floyd-Warshall algorithm
- Works with negative edge weights, assuming no negative cycles
- Simple nested loop implementation, time $O(n^3)$
- Space can be limited to $O(n^2)$ by reusing two "slices" SP and SP'

Minimum Cost Spanning Trees

Madhavan Mukund

https://www.cmi.ac.in/~madhavan

Programming, Data Structures and Algorithms using Python

Week 5

Examples

Roads

- District hit by cyclone, roads are damaged
- Government sets to work to restore roads
- Priority is to ensure that all parts of the district can be reached
- What set of roads should be restored first?

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Roads

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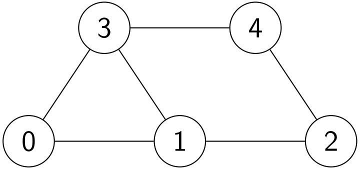
- What set of roads should be restored first?

Fibre optic cables

- Internet service provider has a network of fibre optic cables
- Wants to ensure redundancy against cable faults
- Lay secondary cables in parallel to first
- What is the minimum number of cables to be doubled up so that entire network is connected via redundant links?

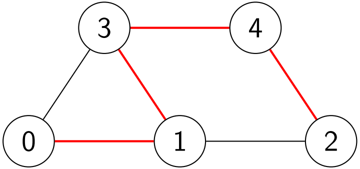
Spanning trees

- Retain a minimal set of edges so that graph remains connected



Spanning trees

- Retain a minimal set of edges so that graph remains connected
- Recall that a minimally connected graph is a tree
    - Adding an edge to a tree creates a loop
    - Removing an edge disconnects the graph



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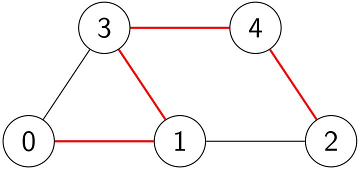

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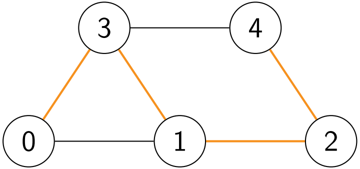
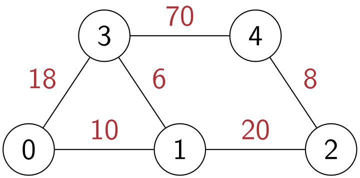
- Want a tree that connects all the vertices — spanning tree
- More than one spanning tree, in general



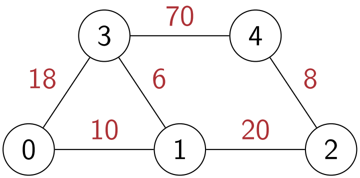
Spanning trees with costs

- Restoring a road or laying a fibre optic cable has a cost



Spanning trees with costs

- Restoring a road or laying a fibre optic cable has a cost
- Minimum cost spanning tree
  - Add the cost of all the edges in the tree
  - Among the different spanning trees, choose one with minimum cost



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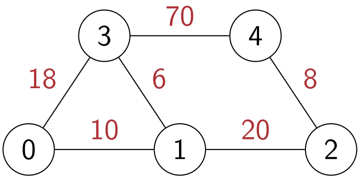

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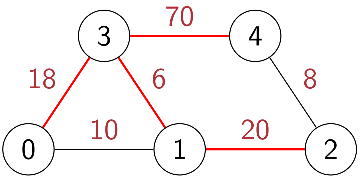
  - Spanning tree, Cost is 114 — not minimum cost spanning tree



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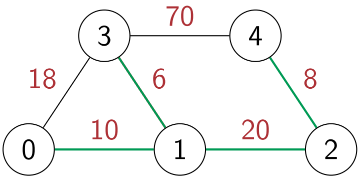

Some facts about trees

Definition A tree is a connected acylic graph.

Fact 1

A tree on n vertices has exactly $n-1$ edges

- Initially, one single component
- Deleting edge (i,j) must split component
  - Otherwise, there is still a path from i to j, combine with (i,j) to form cycle
- Each edge deletion creates one more component
- Deleting $n-1$ edges creates n components, each an isolated vertex

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Fact 2

Adding an edge to a tree must create a cycle.

- Suppose we add an edge (i,j)
- Tree is connected, so there is already a path from i to j
- The new edge (i,j) combined with this path from i to j forms a cycle

Some facts about trees

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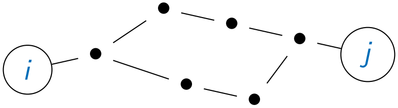
Fact 3

In a tree, every pair of vertices is connected by a unique path.

- If there are two paths from i to j, there must be a cycle



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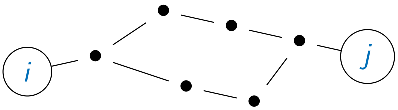

Observation

Any two of the following facts about a graph G implies the third

- G is connected
- G is acyclic
- G has $n-1$ edges

Building minimum cost spanning trees

- We will use these facts about trees to build minimum cost spanning trees

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Building minimum cost spanning trees

- We will use these facts about trees to build minimum cost spanning trees
- Two natural strategies
- Start with the smallest edge and "grow" a tree
  - Prim's algorithm
- Scan the edges in ascending order of weight to connect components without forming cycles
  - Kruskal's algorithm

Minimum Cost Spanning Trees: Prim's Algorithm

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Programming, Data Structures and Algorithms using Python

Week 5

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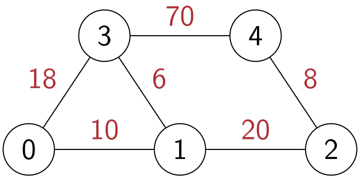

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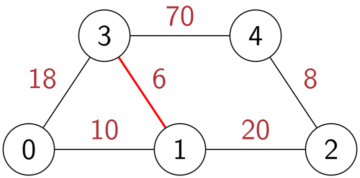

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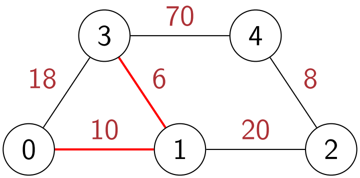

- Start with smallest edge, $(1, 3)$
- Extend the tree with $(1, 0)$

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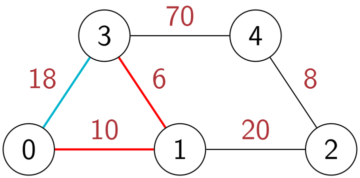

- Start with smallest edge, $(1, 3)$
- Extend the tree with $(1, 0)$
- Can't add $(0, 3)$, forms a cycle

Minimum cost spanning tree (MCST)

- Weighted undirected graph, $G = (V, E), W : E \to \mathbb{R}$
  - G assumed to be connected
- Find a minimum cost spanning tree
  - Tree connecting all vertices in V
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  - Incrementally grow the minimum cost spanning tree
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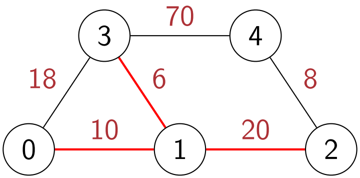

- Start with smallest edge, $(1, 3)$
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- Instead, extend the tree with $(1, 2)$

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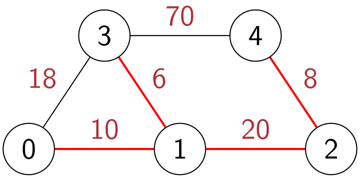

- Start with smallest edge, $(1, 3)$
- Extend the tree with $(1, 0)$
- Can't add $(0, 3)$, forms a cycle
- Instead, extend the tree with $(1, 2)$
- Extend the tree with $(2, 4)$

Prim's algorithm

- $G = (V, E)$, $W : E \to \mathbb{R}$

Prim's algorithm

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  - $TV \subseteq V$: tree vertices, already added to MCST
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  - Choose minimum weight edge $f = (u, v)$ such that $u \in TV$, $v \notin TV$
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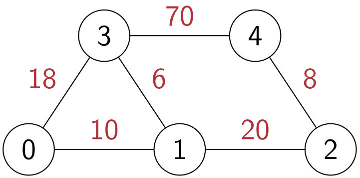

$TV = \emptyset$

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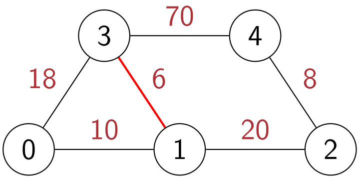

$TV = \{1, 3\}$

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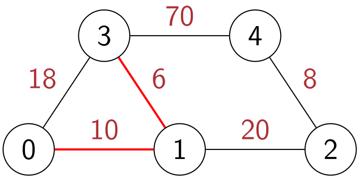

$TV = \{1, 3, 0\}$

$TE = \{(1, 3), (1, 0)\}$

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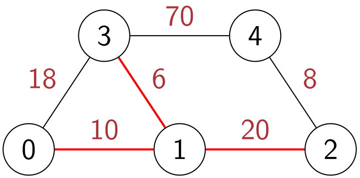
Example



$TV = \{1, 3, 0, 2\}$

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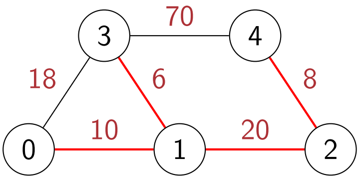

$TV = \{1, 3, 0, 2, 4\}$

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Correctness of Prim's algorithm

Minimum Separator Lemma

- Let V be partitioned into two non-empty sets U and $W = V \setminus U$
- Let $e = (u, w)$ be the minimum cost edge with $u \in U$, $w \in W$
- Every MCST must include e

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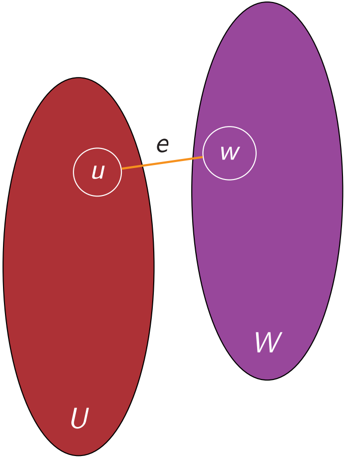

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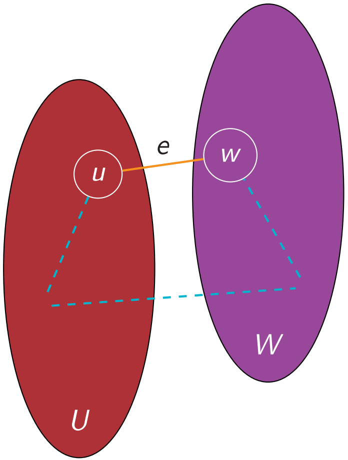

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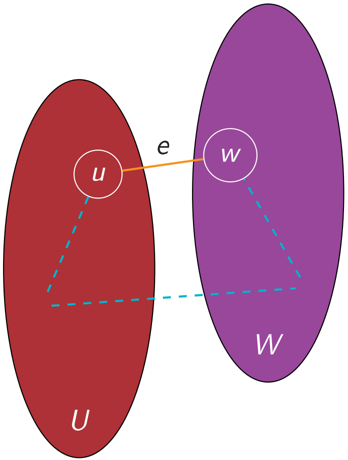

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  - p starts in U, ends in W

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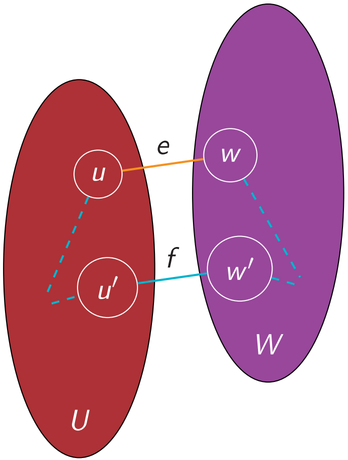

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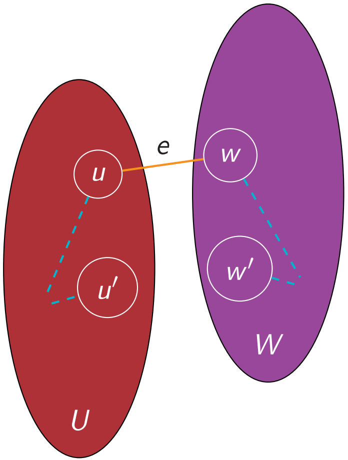
  - Let $f = (u', w')$ be the first edge on p crossing from U to W
  - Drop f, add e to get a cheaper spanning tree



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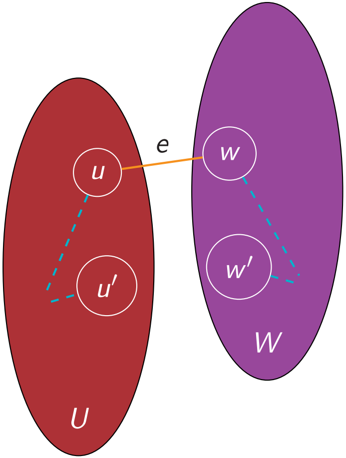

- Assume for now, all edge weights distinct
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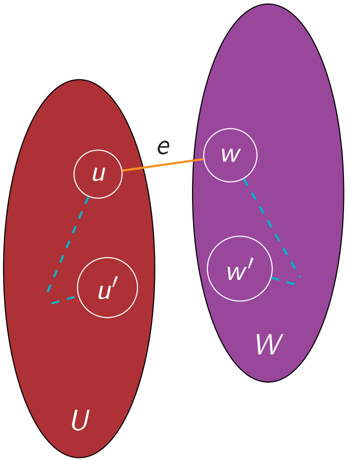

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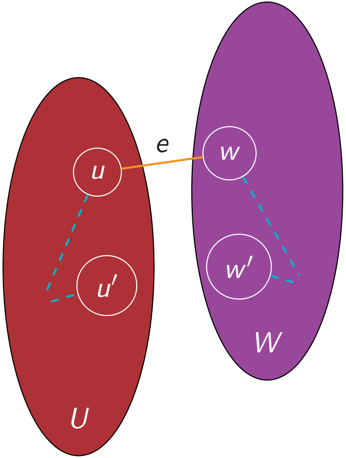

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- Assign each edge a unique index from 0 to $m - 1$
- Define $(e, i) < (f, j)$ if $W(e) < W(j)$ or $W(e) = W(j)$ and $i < j$

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- We started with overall minimum cost edge
- Instead, can start at any vertex v, with $TV = \{v\}$ and $TE = \emptyset$
- First iteration will pick minimum cost edge from v

Implementation

- Keep track of
  - visited[v] – is v in the spanning tree?
  - distance[v] – shortest distance from v to the tree
  - TreeEdges – edges in the current spanning tree

```
def primlist(WList):
  infinity = 1 + max([d for u in WList.keys()
                          for (v,d) in WList[u]])
  (visited,distance,TreeEdges) = ({},{},[])
  for v in WList.keys():
    (visited[v],distance[v]) = (False,infinity)
  visited[0] = True
  for (v,d) in WList[0]:
    distance[v] = d
  for i in WList.keys():
    (mindist,nextv) = (infinity,None)
    for u in WList.keys():
      for (v,d) in WList[u]:
        if visited[u] and (not visited[v]) and d < mindist:
          (mindist,nextv,nexte) = (d,v,(u,v))
    if nextv is None:
      break
    visited[nextv] = True
    TreeEdges.append(nexte)
    for (v,d) in WList[nextv]:
      if not visited[v]:
        distance[v] = min(distance[v],d)
  return(TreeEdges)
```

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- Keep track of
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  - `TreeEdges` – edges in the current spanning tree
- Initialize `visited[v]` to `False`, `distance[v]` to `infinity`
- First add vertex `0` to tree
- Find edge `(u,v)` leaving the tree where `distance[v]` is minimum, add it to the tree, update `distance[w]` of neighbours

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Complexity

- Initialization takes $(O(n)$

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Complexity

- Initialization takes ($O(n)$)
- Loop to add nodes to the tree runs $O(n)$ times

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      for (v,d) in WList[u]:
        if visited[u] and (not visited[v]) and d < mindist:
          (mindist,nextv,nexte) = (d,v,(u,v))
    if nextv is None:
      break
    visited[nextv] = True
    TreeEdges.append(nexte)
    for (v,d) in WList[nextv]:
      if not visited[v]:
        distance[v] = min(distance[v],d)
  return(TreeEdges)
```

Complexity

- Initialization takes ($O(n)$)
- Loop to add nodes to the tree runs $O(n)$ times
- Each iteration takes $O(m)$ time to find a node to add

```
def primlist(WList):
  infinity = 1 + max([d for u in WList.keys()
                           for (v,d) in WList[u]])
  (visited,distance,TreeEdges) = ({},{},[])
  for v in WList.keys():
    (visited[v],distance[v]) = (False,infinity)
  visited[0] = True
  for (v,d) in WList[0]:
    distance[v] = d
  for i in WList.keys():
    (mindist,nextv) = (infinity,None)
    for u in WList.keys():
      for (v,d) in WList[u]:
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Complexity

- Initialization takes $(O(n)$
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- Overall time is $O(mn)$, which could be $O(n^3)$!

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- Loop to add nodes to the tree runs $O(n)$ times
- Each iteration takes $O(m)$ time to find a node to add
- Overall time is $O(mn)$, which could be $O(n^3)$!
- Can we do better?

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```

Improved implementation

- For each v, keep track of its nearest neighbour in the tree
  - visited[v] – is v in the spanning tree?
  - distance[v] – shortest distance from v to the tree
  - nbr[v] – nearest neighbour of v in tree

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def primlist2(WList):
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Improved implementation

- For each `v`, keep track of its nearest neighbour in the tree
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- Scan all non-tree vertices to find `nextv` with minimum distance

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- Then (nbr[nextv],nextv) is the tree edge to add

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- Scan all non-tree vertices to find `nextv` with minimum distance
- Then `(nbr[nextv],nextv)` is the tree edge to add
- Update `distance[v]` and `nbr[v]` for all neighbours of `nextv`

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```

Improved implementation — complexity

- Now the scan to find the next vertex to add is $O(n)$

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- Now the scan to find the next vertex to add is $O(n)$
- Very similar to Dijkstra's algorithm, except for the update rule for distance

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Improved implementation — complexity

- Now the scan to find the next vertex to add is $O(n)$
- Very similar to Dijkstra's algorithm, except for the update rule for distance
- Like Dijkstra's algorithm, this is still $O(n^2)$ even for adjacency lists

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Improved implementation — complexity

- Now the scan to find the next vertex to add is $O(n)$
- Very similar to Dijkstra's algorithm, except for the update rule for distance
- Like Dijkstra's algorithm, this is still $O(n^2)$ even for adjacency lists
- With a more clever data structure to extract the minimum, we can do better

```
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  return(nbr)
```

Summary

- Prim's algorithm grows an MCST starting with any vertex
- At each step, connect one more vertex to the tree using minimum cost edge from inside the tree to outside the tree
- Correctness follows from Minimum Separator Lemma
- Implementation similar to Dijkstra's algorithms
  - Update rule for distance is different
- Complexity is $O(n^2)$
  - Even with adjacency lists
  - Bottleneck is identifying unvisited vertex with minimum distance
  - Need a better data structure to identify and remove minimum (or maximum) from a collection

Minimum Cost Spanning Trees: Kruskal's Algorithm

Madhavan Mukund

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Programming, Data Structures and Algorithms using Python

Week 5

Minimum cost spanning tree (MCST)

- Weighted undirected graph, $G = (V, E), W : E \to \mathbb{R}$
  - G assumed to be connected
- Find a minimum cost spanning tree
  - Tree connecting all vertices in V

Minimum cost spanning tree (MCST)

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- Strategy 2
  - Start with n components, each a single vertex
  - Process edges in ascending order of cost
  - Include edge if it does not create a cycle

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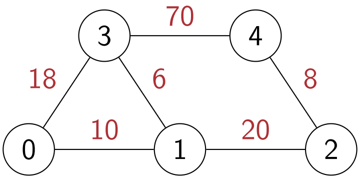
Example



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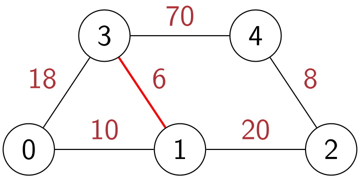

- Start with smallest edge, $(1, 3)$

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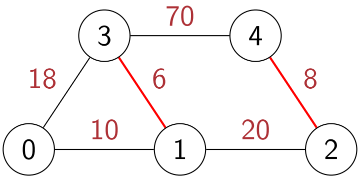

- Start with smallest edge, $(1, 3)$
- Add next smallest edge, $(2, 4)$

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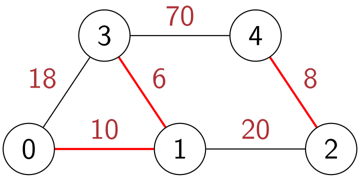

- Start with smallest edge, $(1, 3)$
- Add next smallest edge, $(2, 4)$
- Add next smallest edge, $(0, 1)$

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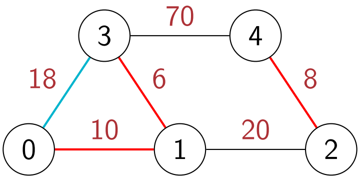

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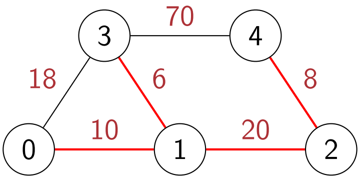

- Start with smallest edge, $(1, 3)$
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- Add next smallest edge, $(0, 1)$
- Can't add $(0, 3)$, forms a cycle
- Add next smallest edge, $(1, 2)$

Kruskal's algorithm

- $G = (V, E)$, $W : E \to \mathbb{R}$

Kruskal's algorithm

- $G = (V, E)$, $W : E \to \mathbb{R}$
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- Scan E from e_0 to e_{m-1}
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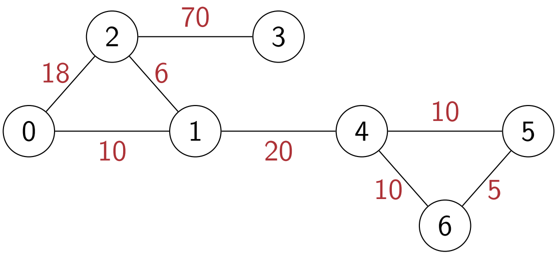

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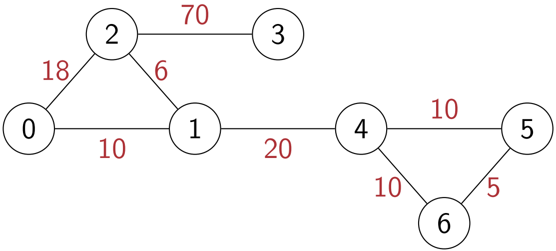

Sort E as
$\{(5,6),(1,2),(0,1),(4,5),(4,6),(0,2),(1,4),(2,3)\}$

Set $TE = \emptyset$

- $G = (V, E)$, $W : E \to \mathbb{R}$
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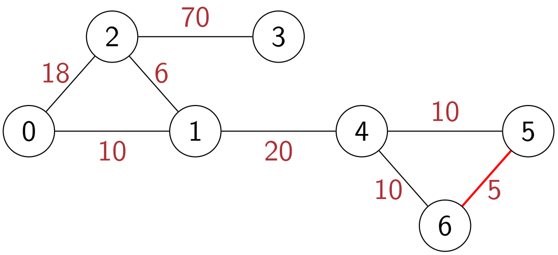

Sort E as
$\{(5,6), (1,2), (0,1), (4,5), (4,6), (0,2), (1,4), (2,3)\}$

Add $(5,6)$

Set $TE = \{(5,6)\}$

Kruskal's algorithm

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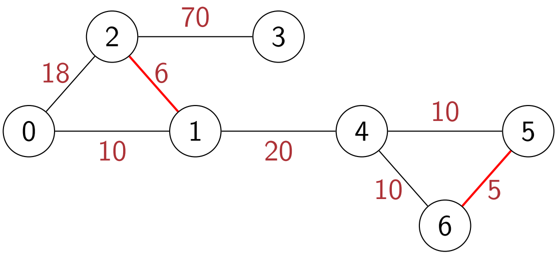

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Add $(1,2)$

Set $TE = \{(5,6), (1,2)\}$

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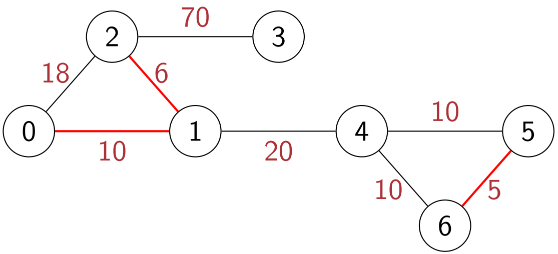
Example



Sort E as

$\{(5,6),(1,2),(0,1),(4,5),(4,6),(0,2),(1,4),(2,3)\}$

Add $(0,1)$

Set $TE = \{(5,6),(1,2),(0,1)\}$

Kruskal's algorithm

- $G = (V, E)$, $W : E \to \mathbb{R}$
- Let $E = \{e_0, e_1, \ldots, e_{m-1}\}$ be edges sorted in ascending order by weight
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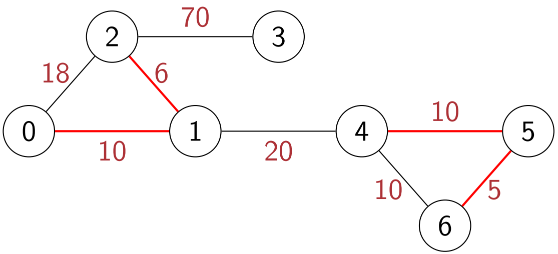

Sort E as

$\{(5,6),(1,2),(0,1),(4,5),(4,6),(0,2),(1,4),(2,3)\}$

Add $(4,5)$

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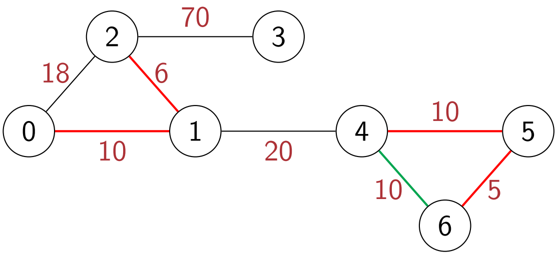

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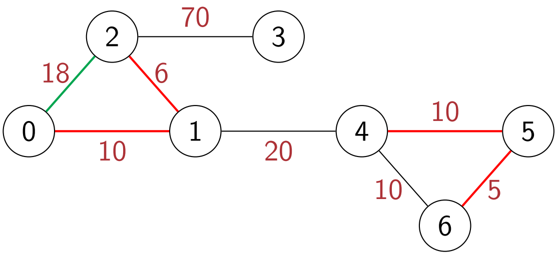

Sort E as
$\{(5,6), (1,2), (0,1), (4,5), (4,6), (0,2), (1,4), (2,3)\}$

Skip $(0,2)$

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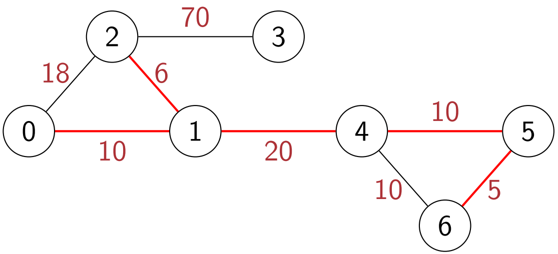

Sort E as
$\{(5,6),(1,2),(0,1),(4,5),(4,6),(0,2),(1,4),(2,3)\}$

Add $(1,4)$

Set $TE = \{(5,6),(1,2),(0,1),(4,5),(1,4)\}$

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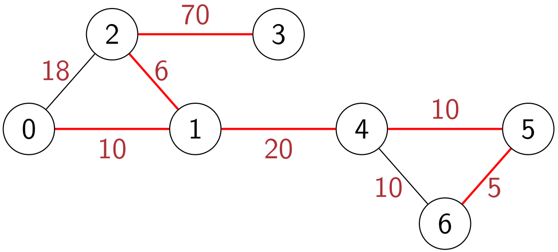

Sort E as

$\{(5,6),(1,2),(0,1),(4,5),(4,6),(0,2),(1,4),(2,3)\}$

Add $(2,3)$

Set $TE = \{(5,6),(1,2),(0,1),(4,5),(1,4),(2,3)\}$

Correctness of Krusksal's algorithm

Minimum Separator Lemma

- Let V be partitioned into two non-empty sets U and $W = V \setminus U$
- Let $e = (u, w)$ be the minimum cost edge with $u \in U$, $w \in W$
- Every MCST must include e

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- Let U be component of u, W be $V \setminus U$
  - U, W form a partition of V with $u \in U$ and $w \in W$
  - Since we are scanning edges in ascending order of cost, e is minimum cost edge connecting U and W, so it must be part of any MCST

Implementing Kruskal's algorithm

- Collect edges in a list as `(d,u,v)`
  - Weight as first component for easy sorting

```
def kruskal(WList):
    (edges,component,TE) = ([],{},[])
    for u in WList.keys():
        # Weight as first component to sort easily
        edges.extend([(d,u,v) for (v,d) in WList[u]])
        component[u] = u
    edges.sort()

    for (d,u,v) in edges:
        if component[u] != component[v]:
            TE.append((u,v))
            c = component[u]
            for w in WList.keys():
                if component[w] == c:
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    return(TE)
```

Implementing Kruskal's algorithm

- Collect edges in a list as `(d,u,v)`
  - Weight as first component for easy sorting
- Main challenge is to keep track of connected components
  - Dictionary to record component of each vertex
  - Initially each vertex is an isolated component
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- Overall, $O(n^2)$

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- Efficient union-find brings complexity down to $O(m \log n)$

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  - Consider a triangle on 3 vertices with all edges equal
- Different choices lead to different spanning trees
- In general, there may be a very large number of minimum cost spanning trees