Conventional Beamforming (for uniform linear arrays)

Consider an antenna array comprised of N identical antenna elements that are in a linear configuration (see Figure 1 below) with the elements indexed from 0 to $N-1$. The elements have equal spacing d.

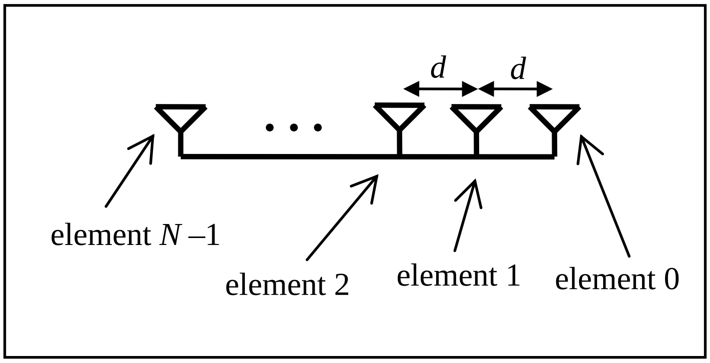

Figure 1. Uniform Linear Array Antenna

As shown in Figure 2, we will establish the geometry such that the *spatial angle* ϕ for a received plane wave is defined as 0° for the direction normal to the array. This is also known as the array boresight. Spatial receive directions for $\phi = -90°$ and $\phi = +90°$ are also shown.

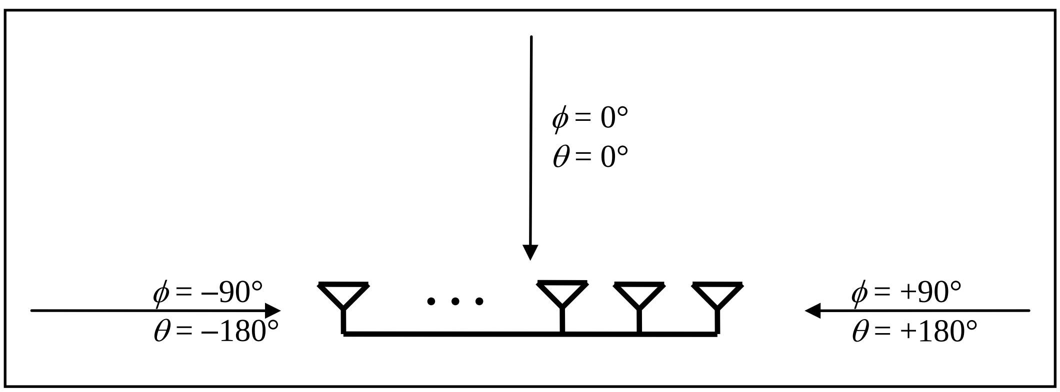

Figure 2. Receive Signal Geometry for Uniform Linear Array
(half-wavelength element spacing)

We shall assume that, for an incident signal, the difference of arrival time between the elements at the ends of the array (the two farthest elements) is much less than the reciprocal of the bandwidth. This is the well-known *narrowband array assumption*. By making this assumption, we can treat the received signal at any antenna element as simply a phase shift of the received signal on any other element. As such we may also define an *electrical angle* θ that characterizes the phase shift between antenna elements. For a uniform linear array (which implies equal element spacing), the relationship between spatial angle ϕ and electrical angle θ is therefore

$$\theta = \frac{2\pi d}{\lambda}\sin\phi \qquad (1)$$

where λ is the wavelength of the incident signal. Figure 2 depicts the relationship between spatial angle and electrical angle when $d = \lambda / 2$ (known as half-wavelength spacing). Note that a spatial equivalent to Nyquist sampling exists such that $d \leq \lambda / 2$ yields no aliasing (known as grating lobes for array processing).

Figure 3 illustrates what occurs when a plane wave is incident from a particular spatial angle ϕ_o, which corresponds to electrical angle θ_o. Relative to element 0, each subsequent antenna element encounters a phase delay depending on how far away it is from the reference element due to the time delay involved with traveling the additional distance.

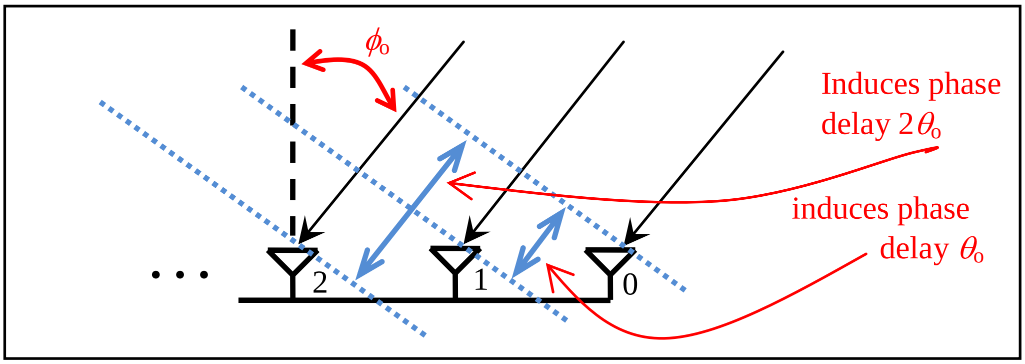

Figure 3. Phase delays introduced by path length differences

In general, for a received signal at some electrical angle θ, the received (and time sampled) signal at sample index n can be expressed as the $N \times 1$ vector

$$\mathbf{x}(n) = \left[x(n) \;\; x(n)e^{-j\theta} \;\; x(n)e^{-j2\theta} \;\; \cdots \;\; x(n)e^{-j(N-1)\theta} \right]^T \tag{2}$$

where $(\bullet)^T$ is the vector transpose operation. Noting that the term $x(n)$ appears in every term of (2) (due to our previous narrowband array assumption), it can be re-expressed as

$$\mathbf{x}(n) = x(n)\,\mathbf{s}_\theta = x(n)\left[1 \;\; e^{-j\theta} \;\; e^{-j2\theta} \;\; \cdots \;\; e^{-j(N-1)\theta}\right]^T \tag{3}$$

where $\mathbf{s}_\theta = \left[1 \;\; e^{-j\theta} \;\; e^{-j2\theta} \;\; \cdots \;\; e^{-j(N-1)\theta}\right]^T$ is denoted as the *spatial steering vector*.

Just like the filtering of time-domain signal, we can also filter spatial signals. Defining a length N spatial filter as the $N \times 1$ vector

$$\mathbf{w} = \left[w_0 \;\; w_1 \;\; w_2 \;\; \cdots \;\; w_{N-1}\right]^T, \tag{4}$$

the output of the spatial filter at discrete-time index n is

$$y(n) = \mathbf{w}^H \mathbf{x}(n) = \sum_{k=0}^{N-1} w_k^* \left[x(n)\, e^{-jk\theta} \right] = x(n)\, \mathbf{w}^H \mathbf{s}_\theta \tag{5}$$

where $(\bullet)^H$ is the complex-conjugate transpose (or Hermitian) operation. There are numerous approaches to designing the spatial filter $\mathbf{w}$ that we shall discuss. However, a simple approach can be borrowed from FIR time-domain filters.

It is often the desire to "steer" the receive beam such that it maximally collects a desired incident signal. If the direction-of-arrival (DOA) of a signal is known, then its electrical angle θ can be easily determined using (1). Hence, the filter that maximizes the receive signal-to-noise ratio (also known as the *matched filter*) is

$$\mathbf{w} = \mathbf{s}_\theta . \tag{6}$$

However, this filter produces *spatial sidelobes* that follow a sinc shape (*i.e.* high sidelobes). High spatial sidelobes mean that interference from other spatial angles may more easily corrupt the desired receive signal.

Alternatively, if we express one of the window functions (*e.g.* Hamming, Hanning, Blackman-Tukey, etc) as a vector denoted as **t**, then the filter can be modified as

$$\mathbf{w} = \mathbf{t} \odot \mathbf{s}_\theta \tag{7}$$

where $\odot$ is the Hadamard product (term-by-term multiplication of the elements in the respective vectors). Thus the spatial filter in (7) will yield lower spatial sidelobes, though this comes at the cost of a wider *mainbeam* (and thus degraded *spatial resolution*). In the array processing nomenclature, the window **t** is called a *taper*.

For a given spatial filter $\mathbf{w}$, the *beampattern* can be determined by plotting $\left| \mathbf{w}^H \mathbf{s}_\theta \right|$ for $-\pi \le \theta \le \pi$. Figures 4 and 5 provide a depiction of the beampattern (in absolute and dB, respectively) for a spatial filtering employing a rectangular taper for the mainbeam pointed at spatial angle $\phi = 0°$ ($\theta = 0°$) and $N = 10$.

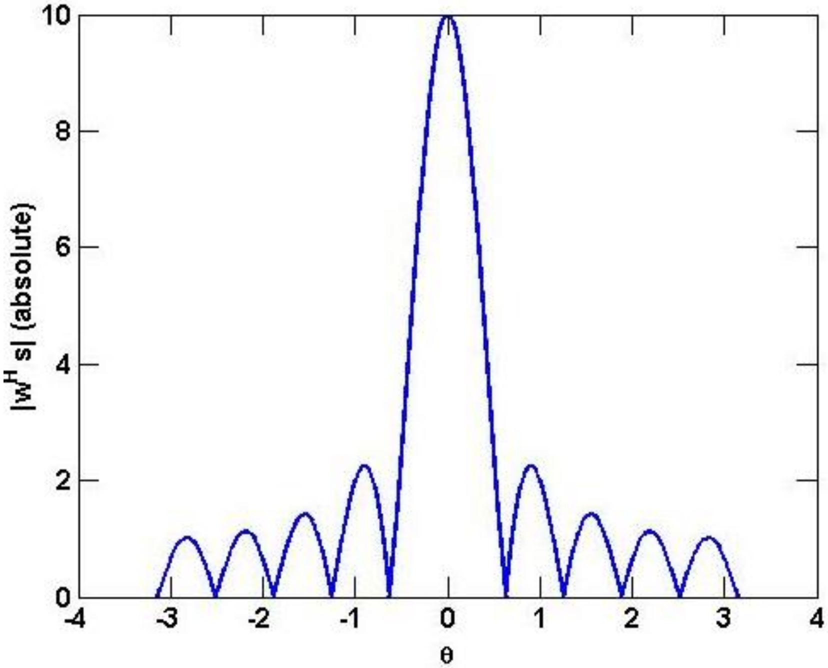

Figure 4. Absolute beampattern for rectangular taper steered to $\phi = 0°$ ($\theta = 0°$) with $N = 10$ elements (angle plotted in radians)

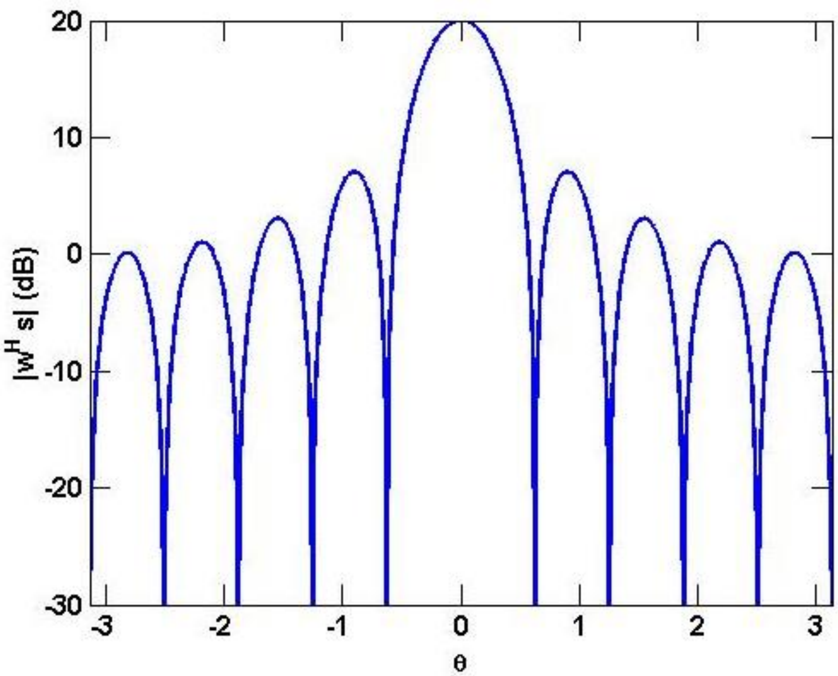

Figure 5. Beampattern (in dB, which is $20\log_{10}\left|\mathbf{w}^H \mathbf{s}_\theta\right|$) for rectangular taper steered to $\phi = 0°$ ($\theta = 0°$) with $N = 10$ elements (angle plotted in radians)