Square Factors in Projective Character Degrees

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Note: these slides have undergone minor revisions since they were initially presented.

Character Degree Sets

Let $\mathrm{cd}(G)$ be the multi-set of degrees $\dim \chi$ of irreducible characters χ of a finite group G. In the past few decades, people have discovered that some information about G can be extracted from $\mathrm{cd}(G)$:

Theorem ([10])

If G is a non-solvable finite group such that $\mathrm{cd}(G)$ consists only of prime powers, then there is some abelian A such that G is congruent to either $A_5 \times A$ or $PSL_2(8) \times A$.

Theorem ([4])

If G is a non-solvable finite group such that $\mathrm{cd}(G)$ consists only of square-free numbers, then there is some solvable R such that $G \cong A_7 \times R$.

Projective Representations

Our goal is to extend these sorts of theorems to structures that are not quite representations, called *projective representations*.

Intuitively, projective space is some given space quotiented out by scalars.

Definition

The *projective general linear group* $\mathrm{PGL}_n(\mathbb{C})$ is the quotient $\mathrm{GL}_n(\mathbb{C})/\mathbb{C}^* I$, where I is the identity matrix.

This group will be intimately related to our definition of projective representations.

Projective Representations

To define a projective representation, we will reference these quotiented-out scalars explicitly:

Definition

Given a group G, a map $\mathfrak{X} : G \to \mathrm{GL}_n(\mathbb{C})$ is a *projective representation* if there is some function $\alpha : G \times G \to \mathbb{C}^*$ such that for all $a, b \in G$ we have

$$\mathfrak{X}(a)\mathfrak{X}(b) = \alpha(a, b)\mathfrak{X}(ab).$$

The function α is known as the *factor set* of $\mathfrak{X}$.

We say that two factor sets are *equivalent* if we can get one from the other by multiplying the matrices in $\mathfrak{X}$ by scalars. We say that two projective representations are equivalent if we can get one from another by multiplying the matrices in $\mathfrak{X}$ by some scalars and conjugating them by some fixed non-singular matrix.

We call a projective representation *irreducible* if it is not equivalent to one of the form

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}.$$

Thus we can classify the irreducible projective representations of a group G of a given factor set α (up to equivalence). Let $\mathrm{cd}_\alpha(G)$ be the set of their degrees. If α is trivial then this is $\mathrm{cd}(G)$.

Before we continue, we provide the following definition:

Definition

Let $N \trianglelefteq G$, and let $\lambda \in \mathrm{Irr}(N)$. Then

$$\mathrm{Irr}(G \mid \lambda) = \left\{ \chi \in \mathrm{Irr}(G) \mid [\chi, \lambda^G] \neq 0 \right\},$$

and $\mathrm{cd}(G \mid \lambda)$ is the corresponding set of values $\dim \chi / \dim \lambda$.

This is going to be our link between ordinary representations and projective representations.

The Schur Cover

Theorem ([5])

Given a finite group G, there is a finite group Γ, known as the Schur cover *of G, satisfying the following properties:*

- *The quotient $\Gamma/Z(\Gamma)$ is congruent to G,*
- *For any factor set α, there is a unique $\lambda \in Irr(Z(\Gamma))$ such that*
$$cd_\alpha(G) = cd(\Gamma \mid \lambda).$$
This also gives a equal partition of $Irr(\Gamma)$ by the factor sets of G, weighting by the square sums of the dimensions.

Thus in order to prove statements about projective character degree sets $\mathrm{cd}_\alpha(G)$, we can tap into existing knowledge about ordinary character degrees.

The Conjecture in Question

The theorem from [10], categorizing groups whose character degree sets consist of prime powers, has been proven by [6] for all projective character degree sets. We wish to use the same logic to consider an extension of the theorem from [4] to the following:

Conjecture

Let G be a non-solvable finite group with some factor set α. If $cd_{\alpha}(G)$ consists of only square-free integers, Then G is expressible as $R \times A_7$ where R is solvable.

Reconstructing Our Theorem with Ordinary Characters

By considering the Schur cover, we note that our conjecture can be solved by proving the following claim about ordinary characters:

Conjecture

Let G be a finite group with a normal central subgroup L such that G/L is non-solvable, and let λ be some irreducible representation of L. If $cd(G \mid \lambda)$ consists of square-free numbers, then G/L is expressible as $R \times A_7$ where R is solvable.

Our Plan of Attack

We consider for the sake of contradiction of a counter-example minimizing $|G/L|$, and let N be its maximal solvable normal subgroup.

- We wish to show first that G/N is simple.
- We then wish to show that some $\nu \in \mathrm{Irr}(N \mid \lambda)$ is G-invariant. It is known that this will imply that G/L is simple and that $\mathrm{cd}(G \mid \lambda)$ will be the projective character degree set of some finite simple group.
- This will allow us to consider only projective representations of finite simple groups, or ordinary representations of their Schur covers.

We will give our proof of the first, and our progress towards the second and third.

Proving that G/N Is Simple

If G/N is not simple, there is some minimal $N \trianglelefteq M \trianglelefteq G$, so that M is non-solvable. By the minimality of $|G/L|$ we have that $M/L \cong N/L \times A_7$, by inducing the corresponding characters. Let A_M be the corresponding group such that $A_M/L \cong A_7$. If M is not unique, then we can similarly generate an A_K such that

$$G/L \trianglerighteq A_M A_K/L \cong A_7 \times A_7.$$

Proving that G/N Is Simple

Consider the following

Theorem ([9])

If N is a normal subgroup of G, and $\lambda \in Irr(N)$, then $cd(G \mid \lambda)$ contains no multiples of some prime p only if the Sylow p-subgroups of G/N are abelian.

Applying this twice, we have a multiple of 4 in $\mathrm{cd}(A_M A_K \mid \lambda)$, so M is unique. Thus the centralizer of M/N in G/N is trivial, and by the Normalizer-Centralizer theorem and our contradiction assumption we have $G/N \cong \mathrm{Aut}(A_7) \cong S_7$.

Proving that G/N Is Simple

We invoke one more

Theorem ([5])

Let $L \trianglelefteq G$ and let $\lambda \in Irr(L)$, and let $I_G(\lambda)$ be the inertia subgroup of λ, the largest group which fixes λ under conjugation. Then $cd(G \mid \lambda) = |G : I_G(\lambda)| cd_\alpha(I_G(\lambda)/L)$ for some α.

Thus there is some $\mu \in \mathrm{Irr}(A_M \mid \lambda)$ with $\mu(1)/\lambda(1) = 10$, so $2 \nmid |G : I_G(\mu)|$ implying $I_G(\mu)/N \cong S_7$. Note that any automorphism of G yields an automorphism of A_M which fixes L. Choosing an automorphism corresponding to an element x of $S_7 \backslash A_7$ of order 2 allows us to construct a group $A_M \rtimes Z_2$ not necessarily in G where μ is G-invariant and $(A_M \rtimes Z_2)/L \cong S_7$. This implies a projective representation of S_7 of degree 10, contradiction.

Progress Towards Showing that G/L Is Simple

Our strategy for showing this is as follows:

- It suffices to show that some ν of $\mathrm{Irr}(N|\lambda)$ is G-invariant.
- We consider the possible square-free $|G : I_G(\nu)|$. If this is even, then $\mathrm{cd}(I_G(\nu) \mid \nu)$ consists of odd integers, restricting its possibilities.
- The subgroups of simple groups of odd index are also relatively small, though we should be able to reduce our consideration further.

We will spend the remainder of the talk discussing strategies for simple groups G/L, which in addition to our more general problem is interesting in its own right - Huppert's conjecture, one of the most famous character degree-related conjectures, concerns itself only with character degree sets of simple groups.

Finite Simple Groups Not of Lie Type

For A_n with $n \geq 8$ there is only one non-trivial factor set up to equivalence, and there is a known representation of it which has degree a multiple of 4. Since the trivial factor sets fall under the original theorem, this suffices.

For the sporadic groups, the character tables of the Schur covers have been explicitly calculated and partitioned by the sub-extensions of the initial group [3], which is sufficient to determine the factor sets in question, and by brute force search it can be seen that a counter-example cannot be one of these groups.

Projective Representations of E_6 and E_7

We start with the following theorem, a projective generalization of a well-known fact:

Theorem ([2])

For all finite groups G and factor sets α of G, the square sum of $cd_\alpha(G)$ is $|G|$.

Say that there are n inequivalent factor sets of G, and let Γ be the Schur cover of G. If the square sum of the character degrees of Γ with square factors is more than $\frac{n-1}{n}|\Gamma|$, then G cannot be an exception by the pigeonhole principle.

The website [7] lists the character degree sets of the Schur covers of $E_6(q)$ and $E_7(q)$ as functions of q. Running a quick Java program, we see that the above pigeonhole argument applies, so that these groups cannot be exceptions.

Jordan Decomposition

Luckily, some information is known about the irreducible characters of the Schur covers of finite simple groups of Lie type is known. For every such cover Γ, there is a closely related group Γ^* known as the dual group, and a Γ^*-invariant subset of these are known as *semisimple*. Additionally, some subgroups of Γ^* have some of their irreducible characters deemed *unipotent*.

Theorem ([1])

There is a bijection between the irreducible characters χ of Γ and pairs (s, ψ), where s is a semisimple element of Γ^* and ψ is a unipotent character of the centralizer of s, and $\chi(1) = \chi_s(1)\psi(1)$, where χ_s is a function of s.

We wish to show that *most* characters χ_s have square factors, and each corresponds to *roughly* the same square sum.

Rough Justification of the Weighted Pigeonhole Argument

First, it is known that $\chi_s(1)$ is the part of the index of the centralizer of s co-prime to the characteristic of the group. In many cases, the classification of these centralizers is known. Furthermore, there is an explicit formula for the corresponding square sum for a given s in [8]. All of this information together gives a rough introduction to *Deligne-Lusztig theory*, in its form relevant to us.

This leaves us only to work out the mechanics of this formula to determine the close-ness of these square sums, to allow us to formulate a bound on the proportion of s with square-factor indices, reducing our case considerations drastically. Progress to these steps appear to validate the viability of this plan of attack.

However, the formula for these partial square sums for some groups is dense and hard to evaluate. Luckily, the groups which pose the largest problems have small numbers n of inequivalent factor sets - namely, at most 6. Thus for such a group it would suffice to find $n-1$ projective representations of different non-trivial factor sets whose degrees had square factors, or $n-1$ corresponding ordinary characters of the Schur cover.

The Weyl Representation

The Weyl representation, one proposed ordinary representation, exists on some of these Schur covers. For example, in the groups of type B with dimension n and corresponding finite field $\mathbb{F}_q$, there are 4 such representations of degrees $(q^n-1)/2$ and $(q^n+1)/2$, each with multiplicity 2. When q is odd, there are two factor sets of these groups up to equivalence - if we could show that each contained one of these representations for each degree, then we could ignore whenever n was even or q was equivalent to 1 or 7 mod 8.

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