A New Cramer-Rao Lower Bound for TOA-based Localization

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Abstract— **In this paper, we derive the Cramer-Rao lower bound (CRLB) for the 2-dimensional (2D) time-of-arrival (TOA) based localization[1]. Unlike previous work on the CRLB, we consider a more practical propagation channel and relate it to inter-node range estimate through a distance-dependent variance model. We demonstrate that this will impact the derivation of the Fisher information matrix (FIM), eventually leading to a CRLB different from the existing derivations. This new theoretical framework provides us with additional insights on the immediate impact of the geometric configuration of anchor nodes on the localization accuracy.**

Keywords: Cramer-Rao lower bound (CRLB), localization, time-of-arrival (TOA), least-squares (LS).

I. Introduction

Node localization is a critical requirement for the successful deployment of wireless sensor networks (WSN) in many envisioned applications such as environment monitoring, precision agriculture, and emergency-rescue personnel tracking. In traditional infrastructure-based position location techniques, a mobile device derives its location by making measurements to several known locations, e.g., satellites in GPS [1] or base stations in E-911 [2]. The same notion holds true in WSNs, where an unlocalized node communicates with and obtains range or angle estimates to other anchor nodes whose locations are known and then estimates its own location. This is often referred to as non-collaborative localization. On the other hand, in collaborative localization, unlocalized nodes also obtain range or angle estimates among themselves, then jointly estimate their locations. Collaborative localization is not only shown to have the potential to improve localization accuracy [3], but in fact is necessary in many practical situations where only a small portion of sensor nodes have the luxury of knowing their locations due to the complexity and cost constraints. How to effectively utilize these measurements between unlocalized nodes remains a challenging issue for collaborative localization.

On the other hand, the fundamental limit on the achievable localization accuracy can serve as a benchmark for any unbiased location estimator. In [3] and [4], the CRLB for localization accuracy based on range estimates was derived, based on modeling range estimates as being corrupted by zero-mean Gaussian noise. In [5], the authors considered the CRLB in a non-line-of-sight (NLOS) environment and showed that the minimum variance unbiased estimator (MVUE) simply discards all NLOS range estimates and uses only LOS range estimates. One assumption that has been made in these works is that the variance of the range estimate is not dependent on the actual inter-node distance. However, in practical situations, signal power decays as the propagation distance increases. This means nodes that are further away from each other obtain range estimates with a higher noise level.

In this paper, we derive a new CRLB based on the distance-dependent variance model for range estimation noise. As demonstrated later, this distance-dependence does impact the derivation of the Fish Information Matrix (FIM), and finally leads to a CRLB different from existing work, e.g., those in [3] and [4]. We also show that compared to the previous CRLB, the new CRLB is lower, due mainly to the increased Fisher information, especially when the path loss exponent is large. Since this is more likely the case for indoor localization, the new CRLB offers a more accurate lower bound on the localization error. Furthermore, through the distance-dependent noise variance, the new CRLB provides more insights into the immediate impact of geometric configuration of anchor nodes on the localization accuracy.

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The rest of the paper is organized as follows. In Section II, we briefly describe the signal propagation and the variance model for range estimates. The derivation of the new CRLB is given in Section III. In Section IV, we present some numerical results. Concluding remarks are given in Section V.

II. Range estimate model

We consider a 2D network of unlocalized nodes and anchors, denoted by the sets $\mathcal{U}$ and $\mathcal{A}$, respectively. For brevity, we will focus on anchored localization, i.e., $|\mathcal{A}| \geq 3$, where $|\cdot|$ denotes cardinality. A CRLB-type bound for anchor-free localization can be found in [4] and the extension of our work to anchor-free localization is straightforward.

Each node has a limited communication range R_{comm} and obtains range estimates to its neighbors if the distance between them is less than R_{comm}. We use $N(i)$ to denote the set of nodes that are neighbors of the ith node. Range estimates are modeled as the true inter-node distances being corrupted by a zero-mean Gaussian noise, written as

$$r_{ij} = d_{ij} + n_{ij} \tag{1}$$

where

$$d_{ij} = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \tag{2}$$

is the true distance between the ith and jth nodes whose true 2D positions are (x_i, y_i) and (x_j, y_j), respectively. $n_{ij} \sim N(0, \sigma_{ij}^2)$ is a Gaussian random variable, whose variance is modeled as

$$\sigma_{ij}^2 = K_E\, d_{ij}^{\beta}. \tag{3}$$

In (3), K_E is a proportionality constant to capture the combined physical layer effect on the range estimate and β is the path loss exponent. Note that K_E is unitless for $\beta = 2$ and will have unit for other values of β. For simplicity, we assume both K_E and β to be constant. However, they can be connection-specific for being more realistic, while not affecting our derivations. It is worth mentioning that although (3) is simple, it is the very basic model to capture the physics-related characteristic of range estimate noise and serves as a starting point for any sophisticated model. As shown later, this model helps us better understand the true impact of node geometric configuration on the localization accuracy.

Therefore, based on the observed data and prior knowledge of the anchor positions given as

$$\mathcal{R} = \{r_{ij} \,|\, i \in \mathcal{U}, j \in \mathcal{U} \cup \mathcal{A},\, d_{ij} < R_{\text{comm}}\} \tag{4}$$

$$\mathcal{P}_a = \{a_n \,|\, a_n = (x_n, y_n)^T,\, n \in \mathcal{A}\}, \tag{5}$$

the parameters that need to be estimated are

$$\Theta = \{\theta_i \,|\, \theta_i = (x_i, y_i)^T,\, i \in \mathcal{U}\}. \tag{6}$$

We also assume that the range estimate is symmetric, i.e., $r_{ij} = r_{ji}$. In addition, we assume there is no NLOS bias, since the CRLB will simply ignore any NLOS range estimate [5].

III. CRLB for localization error

The probability density function (pdf) for r_{ij}, conditioned on θ_i and θ_j can be written as

$$f(r|\theta_i, \theta_j) = \frac{1}{\sqrt{2\pi}\sigma_{ij}} \exp\left(-\frac{(r - d_{ij})^2}{2\sigma_{ij}^2}\right) \tag{7}$$

where d_{ij} and σ_{ij}^2 are given by (2) and (3), respectively.

To derive the FIM, we have the following log-likelihood function for the pdf in (7),

$$\begin{aligned} \log f(r|\theta_i, \theta_j) = &- \log \sqrt{2\pi K_E} \\ &- \frac{\beta}{4} \log\left[(x_i - x_j)^2 + (y_i - y_j)^2\right] \\ &- \frac{1}{2K_E} \times \frac{\left(r - \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}\right)^2}{\left[(x_i - x_j)^2 + (y_i - y_j)^2\right]^{\beta/2}} \end{aligned} \tag{8}$$

Using (8), the detailed derivation of the FIM is given in Appendix A. For $i = 1, 2, \ldots, |\mathcal{U}|$, the diagonal elements of the FIM are given by

$$J_{2i-1,2i-1} = \sum_{j \in N(i)} \frac{w_{ij} \cos^2 \alpha_{ij}}{\sigma_{ij}^2},$$

$$J_{2i,2i} = \sum_{j \in N(i)} \frac{w_{ij} \sin^2 \alpha_{ij}}{\sigma_{ij}^2},$$

$$J_{2i-1,2i} = J_{2i,2i-1} = \sum_{j \in N(i)} \frac{w_{ij} \cos \alpha_{ij} \sin \alpha_{ij}}{\sigma_{ij}^2},$$

where

$$w_{ij} = 1 + \frac{\beta^2 K_E}{2} d_{ij}^{\beta - 2} \tag{9}$$

is a distance-dependent scaling factor.

On the other hand, for $i, j = 1, 2, \ldots, |\mathcal{U}|$ and $i \neq j$, the non-diagonal elements of the FIM are given by

$$J_{2i-1,2j-1} = J_{2j-1,2i-1} = -\frac{w_{ij} \cos^2 \alpha_{ij}}{\sigma_{ij}^2},$$

$$J_{2i,2j} = J_{2j,2i} = -\frac{w_{ij} \sin^2 \alpha_{ij}}{\sigma_{ij}^2},$$

$$\begin{aligned} J_{2i-1,2j} = J_{2j,2i-1} &= J_{2i,2j-1} = J_{2j-1,2i} \\ &= \sum_{j \in N(i)} \frac{w_{ij} \cos \alpha_{ij} \sin \alpha_{ij}}{\sigma_{ij}^2}, \end{aligned}$$

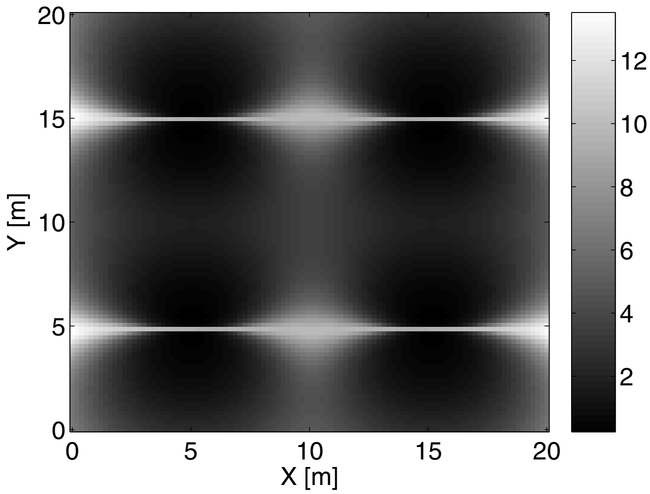

Fig. 1. Old CRLB for $K_E = 0.001$ and $\beta = 4$

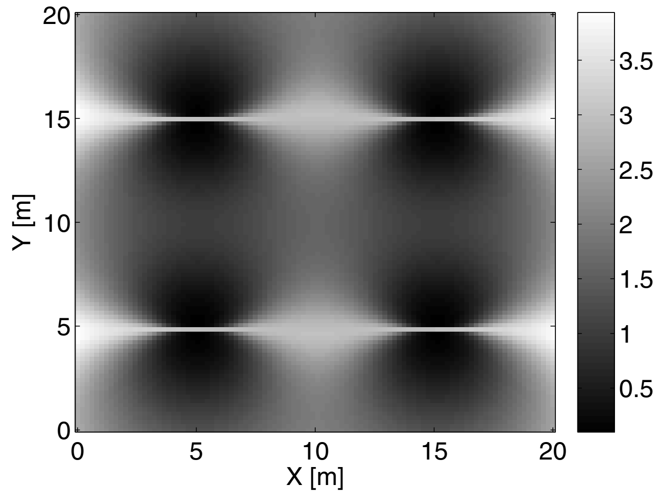

Fig. 2. New CRLB for $K_E = 0.001$ and $\beta = 4$

We observe that the derived FIM differs from the result in [4], by having the additional distance-dependent scaling factor shown in (9). The CRLB for the localization error of the ith node can be described by

$$\mathbf{E}[(\hat{x}_i - x_i)^2 + (\hat{y}_i - y_i)^2] \geq J^{-1}_{2i-1,2i-1} + J^{-1}_{2i,2i}. \quad (10)$$

The more useful metric is the root localization error Ω_i, satisfying

$$\Omega_i \geq \sqrt{J^{-1}_{2i-1,2i-1} + J^{-1}_{2i,2i}}, \quad (11)$$

where $J^{-1}_{i,j}$ denotes the $(i,j)^{\text{th}}$ element of the inverse matrix of J.

On the other hand, as we mentioned earlier, existing work on the CRLB has ignored this distance-dependence of the noise variance. Specifically, in [3], the authors simply ignored this effect. In [4], although the derived CRLB distinguish different noise variances for different range measurements, its dependence on the inter-node distance was not considered in the derivation of the FIM. This translates to the fact that when taking derivatives of (8) with respect to (θ_i, θ_j), only the mean d_{ij} is considered to be dependent on the θ_i, θ_j. However, from a practical point of view, the variance σ^2_{ij} should also be considered as dependent on the θ_i, θ_j. In the following, we will compare our results with the CRLB derived in [4], simply referred to as the old CRLB from now on. We need to emphasize that the old CRLB *does* use distance-dependence noise variance, but only when calculating the CRLB values in the final step, i.e., not in deriving the CRLB.

IV. Results and discussions

In this section, we present numerical results of the CRLB we derived. We will compare it with the old CRLB which has ignored the noise variance's dependence on inter-node distance. In doing so, we demonstrate the benefit of our new CRLB in providing more insight on the direct impact of node geometric configuration on the localization accuracy. Finally, we compare the popular least-squares (LS) solution with both the new and old CRLBs.

First, we examine the old and the new CRLB for a simple single-node localization scenario. We assume the network is a $20 \times 20\,\mathrm{m}^2$ square area and four anchors are placed at (5, 5), (15, 15), (5, 15) and (15, 5), respectively. A single unlocalized node is placed within the network area and is assumed to be able to communicate with all four anchors, i.e., the effect of limited R_{comm} is not considered. Figs. 1 and 2 present the old and the new CRLB for each unknown location with the network, when $K_E = 0.001$ and $\beta = 4$. Note that *darker* regions correspond to those having *smaller* CRLB values. As we can observe, both CRLBs have larger values for the locations at certain regions, which correspond to bad anchor geometry for the position of interest. Furthermore, the new CRLB is always smaller than the old one. In fact, our approach of introducing distance-dependence into the noise variance no longer allows the extraction of geometric dilution of precision (GDOP), since GDOP is a pure geometric quantity derived by assuming all range estimates have the *same* noise variance [3]. This is a reasonable assumption for GPS based localization, since the distances from a GPS receiver to its visible satellites are not much different from each other. However, this assumption does not hold for indoor localization, where the difference between the distances from an unlocalized node to anchors may cause non-negligible difference in noise variance, especially when the path loss exponent is large, e.g., 3 or 4, which is exactly the case for many indoor environments. Therefore, we argue that it may not be proper to attempt to de-couple geometric information from the noise variance for indoor localization and GDOP may no longer be a good metric to gain insight into the localization system performance. In this regard, the existing derivation of the CRLB is not accurate, due to ignoring the distance-dependence in deriving the FIM. Our approach, on the other hand, is more accurate and offers direct insight into the impact of geometric configuration on the localization system performance.

Next, we examine the CRLB for a network of unlocalized nodes. Both network size and anchors are the same as before. The communication range is now limited to be $R_{\text{comm}} = 8\,\text{m}$. We randomly generated 50 unlocalized nodes within the network. In Figs. 3 and 4, we present the new and the old CRLBs with respect to different path loss exponents, for $K_E = 0.005$ and 0.01, respectively. A larger value of K_E indicates the presence of more range estimate noise. We observe that the CRLB increases as the path loss exponent becomes larger, which is a straightforward result because of the increased noise variance. In addition, it is seen that the new and the old CRLBs are almost identical when $\beta = 2$, while the difference becomes larger as β changes from 2 to 4. Furthermore, the new CRLB is always smaller than the old one and the difference becomes larger as K_E increases from 0.005 to 0.01. This can be seen from the fact that w_{ij} in (9) is always larger than 1 when $\beta \geq 2$, which effectively increases the amount of information of the observable about the parameters to be estimated. This ultimately leads to a smaller value in the CRLB.

Finally, we present a comparison between the CRLB with the popular LS estimator [6] in Fig. 5, for localizing a single unlocalized node using different numbers of anchors. The root localization error for the LS estimator is obtained by averaging over 1000 noise realizations. We assume the unlocalized node is able to communicate with all anchors. As we can see, the CRLB serves as a benchmark to evaluate how well a practical estimator performs. The more anchors being used, the better the localization accuracy becomes.

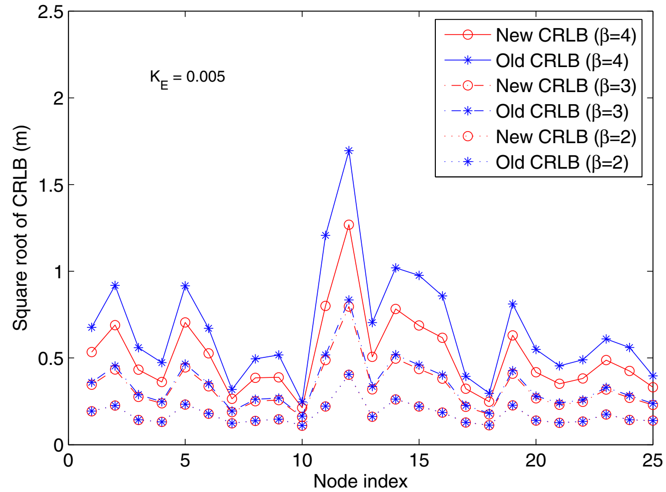

Fig. 3. CRLB comparison $R_{\text{comm}} = 8\,\text{m}$ and $K_E = 0.005$

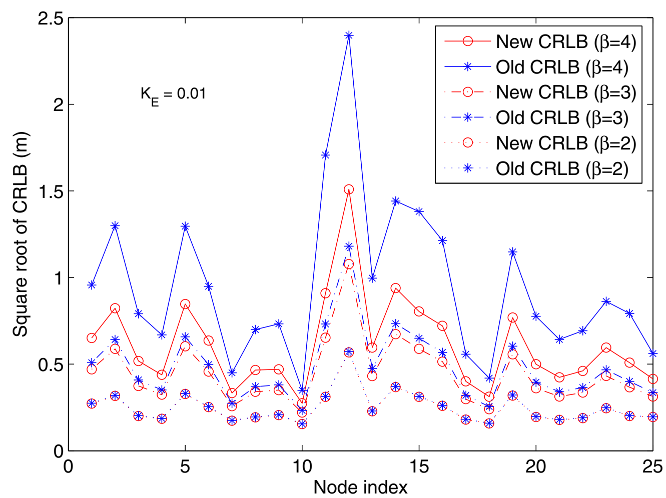

Fig. 4. CRLB comparison $R_{\text{comm}} = 8\,\text{m}$ and $K_E = 0.01$

Furthermore, there is a relatively large performance gap between the popular LS estimator and that indicated by the CRLB. In fact, it is easy to analytically show that the LS estimator is not even an unbiased estimator. Our new CRLB indicates the possibility that the localization accuracy can be further improved. However, the search for a good estimator remains a challenging issue.

V. Concluding Remarks

In this paper, we derive a new CRLB based on a distance-dependent noise variance modeling. This physics based model impacts the FIM and ultimately leads to a different CRLB. We argue that the traditional way of computing GDOP, by assuming equal variance on all range estimates, may not be an appropriate

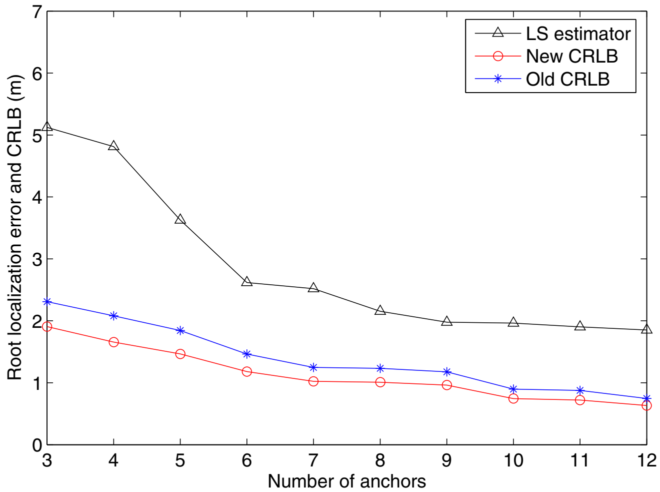

Fig. 5. Comparing the LS estimator's performance with the CRLBs, for $K_E = 0.001$ and $\beta = 4$.

way to characterize the impact of node geometric configuration for indoor localization, where the path loss exponent can be high. Finally, the fact that the new CRLB is always smaller than the old one indicates the possibility of further improving the localization accuracy.

Appendix A: Derivation of the FIM

Using (8), we define the following two terms

$$A = -\frac{\beta}{4}\log\left[(x_i - x_j)^2 + (y_i - y_j)^2\right] \tag{12}$$

$$B = -\frac{\left[r - \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}\right]^2}{2K_E\left[(x_i - x_j)^2 + (y_i - y_j)^2\right]^{\beta/2}}. \tag{13}$$

Therefore, let $L(\theta_i, \theta_j) \triangleq \log f(r|\theta_i, \theta_j)$, we have

$$\frac{\partial L(\theta_i, \theta_j)}{\partial x_i} = \frac{\partial A}{\partial x_i} + \frac{\partial B}{\partial x_i} \tag{14}$$

It is easy to derive that

$$\frac{\partial A}{\partial x_i} = -\frac{\beta}{2} \times \frac{x_i - x_j}{(x_i - x_j)^2 + (y_i - y_j)^2} \tag{15}$$

$$\begin{aligned} \frac{\partial B}{\partial x_i} &= \frac{1}{K_E} \times \frac{x_i - x_j}{[(x_i - x_j)^2 + (y_i - y_j)^2]^{1+\beta/2}} \\ &\times \left[r - \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}\right] \\ &\times \left[\frac{\beta}{2}r + (1 - \frac{\beta}{2})\sqrt{(x_i - x_j)^2 + (y_i - y_j)^2}\right] \end{aligned} \tag{16}$$

The derivation of the second-order partial derivatives are more involved. After a few steps, we can show that

$$\frac{\partial^2 A}{\partial x_i^2} = \frac{\beta}{2} \times \frac{(x_i - x_j)^2 - (y_i - y_j)^2}{d_{ij}^4} \tag{17}$$

$$\begin{aligned} \frac{\partial^2 B}{\partial x_i^2} &= \frac{1}{K_E\, d_{ij}^{4+\beta}} \times \Big[(1-\beta)r(x_i - x_j)^2 d_{ij} \\ &- (2-\beta)(x_i - x_j)^2 d_{ij}^2 + \frac{\beta r^2}{2} d_{ij}^2 - (1 - \frac{\beta}{2})d_{ij}^4 \\ &- \frac{\beta r}{2}(2+\beta)(x_i - x_j)^2 (r - d_{ij}) \\ &- 2(1 - \frac{\beta^2}{4})(x_i - x_j)^2 (r - d_{ij}) d_{ij}\Big] \end{aligned} \tag{18}$$

Now, using (17) to (18), and the fact that

$$\mathbf{E}[r] = d_{ij}, \tag{19}$$

$$\mathbf{E}[r^2] = d_{ij}^2 + K_E d_{ij}^{\beta}, \tag{20}$$

we can obtain

$$\begin{aligned} \mathbf{E}\left[\frac{\partial^2 L(\theta_i, \theta_j)}{\partial x_i^2}\right] &= \mathbf{E}\left[\frac{\partial^2 A}{\partial x_i^2}\right] + \mathbf{E}\left[\frac{\partial^2 B}{\partial x_i^2}\right] \\ &= -\frac{\cos^2\alpha_{ij}}{\sigma_{ij}^2}\left[1 + \frac{\beta^2 K_E}{2} d_{ij}^{\beta-2}\right]. \end{aligned}$$

Similarly, we can derive

$$\mathbf{E}\left[\frac{\partial^2 L(\theta_i, \theta_j)}{\partial y_i^2}\right] = -\frac{\sin^2\alpha_{ij}}{\sigma_{ij}^2}\left[1 + \frac{\beta^2 K_E}{2} d_{ij}^{\beta-2}\right],$$

$$\mathbf{E}\left[\frac{\partial^2 L(\theta_i, \theta_j)}{\partial x_i y_i}\right] = -\frac{\cos\alpha_{ij}\sin\alpha_{ij}}{\sigma_{ij}^2}\left[1 + \frac{\beta^2 K_E}{2} d_{ij}^{\beta-2}\right].$$

All other elements of the FIM can be derived in a similar manner. Finally, the FIM can easily obtained by using the above results.

References

[1] B. W. Parkinson, J. J. Spilker, Jr., P. Axelrad, and P. Enge., *The global positioning system : theory and applications*. American Institute of Aeronautics and Astronautics, 1995.

[2] J. Reed, K. Krizman, B. Woerner, and T. Rappaport, "An overview of the challenges and progress in meeting the E-911 requirement for location service," in *IEEE Personal Communications Magazine*, vol. 5, pp. 30 – 37, April 1998.

[3] N. Patwari, A. O. Hero III, M. Perkins, N. S. Correal, and R. J. O'Dea, "Relative location estimation in wireless sensor networks," *IEEE Transactions on signal processing*, vol. 51, no. 8, pp. 2137–2148, 2003.

[4] C. Chang and A. Sahai, "Estimation bounds for localization," *Sensor and Ad Hoc Communications and Networks, 2004. IEEE SECON 2004.*, pp. 415–424, Oct. 2004.

[5] Y. Qi and H. Kobayashi, "Cramer-rao lower bound for geolocation in non-line-of-sight environment," in *ICASSP'02*, vol. 3, pp. 2473–2476, 2002.

[6] J. J. Caffery, "A new approach to the geometry of TOA location," in *Proceedings of IEEE VTC*, vol. 4, pp. 1943–1949, 2000.