Example 1. The N^{th} Dirichlet kernel is the trigonometric polynomial defined for $x \in [-\pi, \pi]$ by

$$D_N(x) = \sum_{n=-N}^{N} e^{inx}$$

A simple sum splitting yields

$$\begin{aligned} D_N(x) &= \sum_{n=-N}^{N} e^{inx} \\ &= \sum_{n=0}^{N} e^{inx} + \sum_{n=-N}^{-1} e^{inx} \end{aligned}$$

Setting $\omega = e^{ix}$, we can write both of these as finite geometric series

$$\sum_{n=0}^{N} e^{inx} = 1 + \omega + \cdots + \omega^N = \left(\frac{1-\omega^{N+1}}{1-\omega}\right)$$

$$\sum_{n=-N}^{-1} e^{inx} = \omega^{-1} + \cdots + \omega^{-N} = \left(\frac{1}{w}\right)\left(\frac{1-\omega^{-N}}{1-\omega^{-1}}\right) = \left(\frac{1-\omega^{-N}}{\omega-1}\right) = \left(\frac{\omega^{-N}-1}{1-\omega}\right)$$

Their sum is then

$$\begin{aligned} \sum_{n=-N}^{N} e^{inx} &= \left(\frac{1-\omega^{N+1}}{1-\omega}\right) + \left(\frac{\omega^{-N}-1}{1-\omega}\right) \\ &= \left(\frac{1-\omega^{N+1}+\omega^{-N}-1}{1-\omega}\right) \\ &= \left(\frac{\omega^{-N}-\omega^{N+1}}{1-\omega}\right) \\ &= \left(\frac{\omega^{-1/2}}{\omega^{-1/2}}\right)\left(\frac{\omega^{-N}-\omega^{N+1}}{1-\omega}\right) \\ &= \left(\frac{\omega^{-(N+1/2)}-\omega^{N+1/2}}{\omega^{-1/2}-\omega^{1/2}}\right) \end{aligned}$$

Using Euler's identities, we have

$$\omega^{-(N+1/2)} = e^{-(N+1/2)xi} = \cos(-(N+1/2)x) + i\sin(-(N+1/2)xi)$$

$$\omega^{N+1/2} = e^{(N+1/2)xi} = \cos((N+1/2)x) + i\sin((N+1/2)xi)$$

$$\omega^{-(N+1/2)} - \omega^{N+1/2} = -2i\sin((N+1/2)xi)$$

Similarly

$$\omega^{-1/2} = e^{-(1/2)xi} = \cos(-(1/2)x) + i\sin(-(1/2)x)$$

$$\omega^{1/2} = e^{(1/2)xi} = \cos((1/2)x) + i\sin((1/2)x)$$

$$\omega^{-1/2} - \omega^{1/2} = -2i\sin((1/2)x)$$

Hence

$$\left(\frac{\omega^{-(N+1/2)} - \omega^{N+1/2}}{\omega^{-1/2} - \omega^{1/2}}\right) = \left(\frac{-2i\sin((N+1/2)xi)}{-2i\sin((1/2)x)}\right) = \frac{\sin((N+1/2)x)}{\sin(x/2)}$$

This says that the Dirichlet kernel has the following closed form formula

$$D_N(x) = \frac{\sin((N+1/2)x)}{\sin(x/2)}$$

Note that this kernel has the property that $a_n = 1$ if $|n| \le N$ and $a_n = 0$ otherwise.

Example 2. Now we move on to discuss the Poisson kernel defined by the following

$$P_r(\theta) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in\theta}$$

where $0 \le r < 1$. It is easy to see that this converges uniformly and absolutely for all $\theta \in [-\pi, \pi]$, as

$$\sum_{n=-\infty}^{\infty} |r^{|n|} e^{in\theta}| \le \sum_{n=-\infty}^{\infty} |r^{|n|}| < +\infty$$

and uniform convergence follows from the Weierstrass M test. The Fourier coefficients are then given by

$$\begin{aligned} a_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} r^{|n|} e^{in\theta} e^{-in\theta} d\theta \\ &= \frac{2\pi}{2\pi} r^{|n|} \\ &= r^{|n|} \end{aligned}$$

And its Fourier series is given by

$$\sum_{-\infty}^{\infty} r^{|n|} e^{in\theta}$$

Similar to what we did with the Dirichlet kernel, we have

$$\sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} = \sum_{n=0}^{\infty} r^{|n|} e^{in\theta} + \sum_{-\infty}^{-1} r^{|n|} e^{in\theta}$$

Letting $\omega = e^{in\theta}$, we have

$$\sum_{n=0}^{\infty} r^{|n|} e^{in\theta} = 1 + r\omega + r^2\omega^2 + \dots$$

Using the formula for the sum of an infinite geometric series, we have

$$\sum_{n=0}^{\infty} r^{|n|} e^{in\theta} = \frac{1}{1 - r\omega}$$

Similarly

$$\sum_{-\infty}^{-1} r^{|n|} e^{in\theta} = r\omega^{-1} + r^2\omega^{-2} + \dots$$

$$\sum_{-\infty}^{-1} r^{|n|}e^{in\theta} = \frac{r\omega^{-1}}{1-r\omega^{-1}}$$

Combining these yields

$$\sum_{-\infty}^{\infty} r^{|n|}e^{in\theta} = \frac{1}{1-r\omega} + \frac{r\omega^{-1}}{1-r\omega^{-1}} = \frac{1-r\omega^{-1}+r\omega^{-1}-r^2}{1-r\omega^{-1}-r\omega+r^2}$$

Again, by Euler's identities, we see that

$$r\omega^{-1} + r\omega = r\cos(\theta) - ir\sin(\theta) + r\cos(\theta) + ir\sin(\theta) = 2r\cos(\theta)$$

So

$$\sum_{-\infty}^{\infty} r^{|n|}e^{in\theta} = \frac{1-r^2}{1-r\omega^{-1}-r\omega+r^2} = \frac{1-r^2}{1-2r\cos\theta+r^2}$$

Now we shall return to the question of convergence, letting $S_N(f)$ denote the partial sums of the Fourier series of f, where f is defined on $[-\pi,\pi]$ and Riemann integrable, we want to know if

$$\lim_{N\to\infty} S_N(f)(\theta) = f(\theta)$$

for every θ. In other words, we want to know if S_N converges to f pointwise. Note that this may be thought of as a "weaker" type of convergence when compared with uniform convergence. It is easy to see that this is not true in general if we recall the following result:

Proposition 2.1. Let $f, \tilde{f}$ be real valued functions defined on $[a,b]$ such that f is Riemann integrable. If the set of points at which $\tilde{f}$ differs from f is at most countable, then $\tilde{f}$ is also Riemann integrable with

$$\int_a^b f = \int_a^b \tilde{f}$$

Remark 2.1. Note that the above result comes from the fact that countable and finite sets have measure zero, i.e. the same argument applies for certain uncountable sets (e.g. the Cantor set).

This implies that f and $\tilde{f}$ can differ at countably many points but still have the same Fourier coefficients. Motivated by this, we might ask the same question assuming that f is continuous and periodic (however the answer would still be "no"). Though we will later see that the Fourier series of f **does** converges to f uniformly given that f is continuously differentiable.

A natural question to ask at this point is whether a function is uniquely defined by its Fourier coefficients. Of course this wouldn't be true unless we put some restrictions on f. However the following result is true:

Theorem 3. Suppose that f is an integrable function on the circle with $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$. Then $f(\theta_0) = 0$ whenever f is continuous at θ_0.

Proof. We shall now outline a proof, due to Lebesgue, of the theorem stated above. Suppose f is real valued, assume, for the sake of contradiction, that $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}$, but there exists θ_0 such that f is continuous at θ_0 but $f(\theta_0) \neq 0$. Without loss of generality, take $\theta_0 = 0$ and $f(0) > 0$. By continuity, there exists $\delta > 0$ such that

$$f(\theta) > f(0)/2$$

whenever $\theta \in (-\delta, \delta)$. Then, choose $\epsilon > 0$ such that

$$|\epsilon + \cos\theta| < 1 - \epsilon/2$$

whenever

$$\delta \leq |\theta| \leq \pi$$

set

$$p(\theta) = \epsilon + \cos\theta$$

We know that there exists $\eta > 0$ such that

$$p(\theta) = \epsilon + \cos\theta > \epsilon/2 + 1$$

for

$$|\theta| < \eta$$

since $p(\theta)$ is continuous and has value $\epsilon + 1$ at $\theta = 0$. Now we set

$$p_k(\theta) = (\epsilon + \cos\theta)^k$$

By construction, we see that each p_k is a trig polynomial (in terms of cosine). Since $\hat{f}(n) = 0$ for all n by assumption, we have that

$$\int_{-\pi}^{\pi} f(\theta) p_k(\theta) d\theta = 0$$

for all k. This is because every $p_k(\theta)$ can be written as a linear combination of the harmonics, and the rest follows from the fact that integration is a linear operator. However, we can bound the following

$$\left| \int_{\delta \leq |\theta|} f(\theta) p_k(\theta) d\theta \right| \leq \int_{\delta \leq |\theta|} |f(\theta) p_k(\theta)| \, d\theta \leq B \int_{\delta \leq |\theta|} |p_k(\theta)| \, d\theta \leq 2\pi B (1 - \epsilon/2)^k$$

where B is chosen such that $|f(\theta)| < B$ for all $\theta \in [-\pi, \pi]$, and 2π is an upper bound on the length of the interval $[-\pi, \pi]$ with $(-\delta, \delta)$ removed. Moreover, observe that

$$\int_{\eta \leq |\theta| < \delta} f(\theta) p_k(\theta) d\theta \geq 0$$

And finally

$$\int_{|\theta| \leq \eta} f(\theta) p_k(\theta) d\theta \geq \int_{|\theta| \leq \eta} \frac{f(0)}{2} (1 + \epsilon/2)^k d\theta = 2\eta \frac{f(0)}{2} (1 + \epsilon/2)^k$$

It is easy to see that these integrals combine to give

$$\begin{aligned}\int_{-\pi}^{\pi} f(\theta)p_k(\theta)d\theta &= \int_{|\theta|\le\eta} f(\theta)p_k(\theta)d\theta + \int_{\eta\le|\theta|<\delta} f(\theta)p_k(\theta)d\theta + \int_{\delta\le|\theta|} f(\theta)p_k(\theta)d\theta \\ &= \int_{|\theta|\le\eta} f(\theta)p_k(\theta)d\theta + \int_{\delta\le|\theta|} f(\theta)p_k(\theta)d\theta \\ &\ge 2\eta\frac{f(0)}{2}(1+\epsilon/2)^k - 2\pi B(1-\epsilon/2)^k\end{aligned}$$

Observe that

$$\lim_{k\to+\infty} 2\eta\frac{f(0)}{2}(1+\epsilon/2)^k - 2\pi B(1-\epsilon/2)^k = +\infty$$

So

$$\int p_k(\theta)f(\theta)d\theta \to \infty$$

contradicting the fact that $\int p_k(\theta)f(\theta)d\theta = 0$ for all k. If f is real valued, the proof is complete. Suppose f is complex valued, we can always write

$$f(\theta) = u(\theta) + iv(\theta)$$

where $u(\theta)$ and $v(\theta)$ are real valued, and the result follows. ☺
We now state an important result as a corollary to this theorem:

Corollary 4. Suppose that f is a continuous function on the circle and that the Fourier coefficients of f is absolutely convergent, that is

$$\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty$$

Then the Fourier series converges uniformly to f

$$\lim_{N\to\infty} \|f - S_N(f)\|_\infty = 0$$

It should be noted that this says the Fourier series not only converges uniformly to some limit, but it converges to the function f itself. Before proceeding with the proof we shall recall two important lemmas:

Lemma 5. Let (f_n) be a sequence of functions, and suppose there exists f such that

$$\lim_{n\to\infty} \|f - f_n\|_\infty = 0$$

then (f_n) is said to converge uniformly to f. Moreover, if f_n is continuous for each n, then f must also be continuous.

Lemma 6. (Weierstrass M-Test) Let (f_n) be a sequence of functions, and suppose there exists a sequence (M_n) such that

$$|f_n| < M_n$$

for all n, and $\sum M_n$ converges, then f_n converges uniformly.

Proof. By our assumption

$$\sum_{-\infty}^{\infty} |\hat{f}(n)| < \infty$$

so the Fourier series of f converges uniformly to some limit $g(\theta)$, where

$$g(\theta) = \sum_{-\infty}^{\infty} \hat{f}(n) e^{in\theta}$$

is continuous on the circle. Using our previous theorem, we see that $g - f$ has Fourier coefficients that vanish everywhere. Therefore, by continuity, we may conclude that $g = f$ on the circle, which is what we wanted to show. ☺ Before introducing yet another important corollary, recall that

$$\hat{f}(n) = O(1/n^2)$$

as $|n| \to \infty$ means that there exists $C > 0$ with

$$|\hat{f}(n)| \leq C/|n|^2$$

for n large enough.

Corollary 7. Suppose that f is a twice continuously differentiable function on the circle. Then

$$\hat{f}(n) = O(1/|n|^2)$$

as $|n| \to \infty$. So that the Fourier Series converges absolutely and uniformly to f.

To end this section, we introduce the notion of the Hölder condition. f is said to satisfy a Hölder condition of order α if

$$\| f(\theta + t) - f(\theta) \|_{\infty} \leq A|t|^{\alpha}$$

for all t, where A is a constant. It turns out that if f satisfies a Hölder condition of order $\alpha > 1/2$, then the Fourier series of f converges absolutely and uniformly to f.