THE VECTORIAL PARAMETERIZATION OF ROTATION AND MOTION

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Abstract. In this work a complete formulation of the vectorial parameterization of motion is presented. This is a novel concept extending the vectorial parameterization of rotation, a class in which fall many techniques developed over the years for the analysis of problems characterized by an orientation, or attitude, field. Typical applications are seen in the derivation of the governing equations for rigid bodies, beams, shells, and general flexible multibody systems, in view of the application of numerical integration procedures that preserve the nonlinear configuration manifold of these systems. The material is presented by first reviewing the vectorial parameterization of rotation, considered as a generalization of the exponential map of rotation. After this, the vectorial parameterization of motion is derived by generalizing to a 6-D space what is done in a 3-D space for pure rotation. Again, the connection and formal analogy with the exponential map of motion is stressed. We present all the formulæ needed for a straightforward implementation in a computer program.

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Date: November 19, 2002.
Key words and phrases. Exponential map, Cayley's transform, Gibbs vector, Rodrigues parameters, Wiener vector, Milenkovic parameters, conformal rotation vector, reduced Euler parameters, linear parameters, rigid motion, finite rotations, kinematics of polar continua.

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1. Introduction

1.1. **Coordinates for Rotation.** Rotational motion can be regarded as the motion of a particle within a particular non-linear manifold (in particular, a Lie group), the special orthogonal group of 3-D space. Therefore, it cannot be described trivially by using standard coordinates as those commonly employed for motion evolving in a linear space. Rotations have to be parameterized using suitable *charts* (sometimes called rotational 'quasi-coordinates'), which are inherently not global and/or non-singular (see Ref. [13]). Over the years, numerous techniques have been developed to cope with the description of rotational motion, following different approaches (see Refs. [2, 7, 12]). Among these we found the Cayley, or Gibbs, or Rodrigues parameterization; the Milenkovic, or modified Rodrigues, or conformal rotation vector parameterization; the linear parameterization; the Euler-Rodrigues, or unit quaternion parameterization, the Eulerian angles parameterization (Euler angles are only one of several possible choices within this class, Cardan and Bryant angles being other choices); the Cayley-Klein parameterization; the direction cosine parameterization; and so on.

All these techniques show a certain balance between advantages and drawbacks when compared to each other. Usually both theoretical and computational issues can play a meaningful role in the choice, which is also influenced by the possible specific requirements of its application. Within this somewhat unexpectedly large set, however, it is possible to draw a separation of the various techniques into two broad classes: the *vectorial* parameterizations, and the *non-vectorial* parameterizations.

The vectorial parameterizations feature a set of three parameters that define the cartesian components of a vector. This does not apply when dealing with non-vectorial techniques (*e.g.*, Euler angles are three scalars that cannot be understood as components of a geometric vector). Note that vectorial parameterizations are *minimal*, *i.e.* are based on the smallest possible set of parameters, since the dimension of the special orthogonal group of 3-D space is three. Non-minimal parameterizations include the Euler-Rodrigues and Cayley-Klein parameterizations (4 scalar parameters related by an algebraic constraint) and the direction cosine parameterization (9 scalar parameters related by 6 algebraic constraints).

In Ref. [3] the vectorial parameterization of rotation was explored as the common framework for a number of known techniques, sharing identical general formulæ. Here, we present a different, expanded implementation of the same idea, stressing some formal properties of the vectorial parameterization mapping in general. In section 2, we review the basic concepts related to rotation description, Euler's theorem, the exponential map that naturally evolves from it, and finally the vectorial parameterization that can be seen as an extension of the exponential map under many respects. Appendix A deals with further expressions that can be useful when implementing the vectorial parameterization of rotation, while Appendix B presents the specific formulæ for some specific parameterizations that are widely known in the scientific community.

1.2. **Coordinates for Rigid Motion.** The representation of rotation naturally impacts on the description of more general rigid displacements, where rotations are coupled with translations. This means that a large number of fields in theoretical and applied mechanics, such as analytical mechanics, structural dynamics, multibody dynamics, flight mechanics, and so on, can profit from the capability of an accurate description of rotational motion.

In Ref. [4], it has been shown that an inherently coupled description of rigid displacements can be obtained by extending the formal apparatus of rotations to general rigid motions by working in the 6-D kinematic space. Also, Ref. [11] approaches the subject in an analogous way, while in Ref. [1] dual numbers and tensors are invoked to yield similar results.[1] In these works, the exponential parameterization, the parameterization based on the Cayley transform, and the Euler-Rodrigues parameterizations are devised. These all feature a set of parameters (minimal, *i.e.* six, for the exponential and Cayley techniques, non-minimal for the latter) that account for rotation and translation in a coupled way. In fact, rigid motion may be described as a motion evolving on a six-dimensional non-linear manifold (again, a Lie group), the group of special rigid transformations of the 3-D space, or *Euclidean group.* Remarkably, the formulæ for general rigid motion under this coupled framework are identical to those for pure rotation. This is also evident in Ref. [10], where higher order derivatives of rotation and rigid displacement tensors are derived.

Note that this represents a major depart from the common practice, which calls for parameterizing the translational part of the motion simply by selecting the cartesian coordinates of the linear displacement of a material point from a reference configuration with respect to an inertial frame, and then assume any of the cited parameterizations of rotation to describe the orientation, or attitude, of the system.

The 'coupled' approach is followed here to obtain the *vectorial parameterization of motion*, as a direct extension and generalization of the vectorial parameterization of rotation. In section 3, we review the basic concepts related to rigid motion description, Mozzi-Chasles' theorem, the exponential map that naturally evolves from it, and again finally the vectorial parameterization. Appendix B presents the relevant formulæ for specific parameterizations that extend to complete rigid motion some of the most commonly adopted parameterizations of rotation.

1.3. **Notation.** We indicate with $\mathbb{N}$ the set of natural numbers, with $\mathbb{R}$ the set of real numbers, with $\mathcal{E}^3$ the 3-D Euclidean point space, with $\mathbb{E}^3$ the 3-D euclidean vector space (*i.e.* the translation space of $\mathcal{E}^3$), with $\mathrm{Lin}(\mathbb{E}^3)$ the space of linear transformations (or tensors) of $\mathbb{E}^3$. The Lie group of special orthogonal transformations of $\mathbb{E}^3$ is denoted by $\mathrm{SO}(\mathbb{E}^3) \subset \mathrm{Lin}(\mathbb{E}^3)$. The Lie algebra of $\mathrm{SO}(\mathbb{E}^3)$, *i.e.* the linear space of all skew-symmetric tensors, is denoted as $\mathrm{so}(\mathbb{E}^3) \subset \mathrm{Lin}(\mathbb{E}^3)$. The skew-symmetric part of a tensor $\bullet \in \mathrm{Lin}(\mathbb{E}^3)$ is denoted by $\mathrm{skw}(\bullet)$, while the symmetric part is indicated by $\mathrm{sym}(\bullet)$. We denote as $(\bullet\times) \in \mathrm{so}(\mathbb{E}^3)$ the skew-symmetric

[1] In passing, let us state that the dual number format can be employed to get an elegant and fully equivalent formulation to that presented here. The choice on the 6-D format is adopted in continuity with Ref. [4].

tensor obtained from the cross product operator applied to vector $\bullet \in \mathbb{E}^3$, *i.e.*, $(\bullet\times)\star = \bullet \times \star, \forall \bullet, \star \in \mathbb{E}^3$, and as $\mathrm{axial}_\times(\bullet)$ its inverse mapping, *i.e.* the 'extraction' of the axial vector from the skew-symmetric part of the tensor $\bullet \in \mathrm{Lin}(\mathbb{E}^3)$. The eigenvalues of a tensor $\bullet \in \mathrm{Lin}(\mathbb{E}^3)$ will be denoted as $\{\lambda_k(\bullet)\}_{k=1,2,3}$. We denote as $\mathrm{tr}(\bullet)$ the trace of tensor $\bullet \in \mathrm{Lin}(\mathbb{E}^3)$.

The following statements refer to sets and operations defined in Ref. [4]. We indicate with $\mathbb{K}^6$ the 6-D kinematic space, $\mathbb{K}^6 = \mathbb{E}^3 \times \mathbb{E}^3$, with $\mathrm{Lin}(\mathbb{K}^6)$ the space of linear transformations (or tensors) of $\mathbb{K}^6$. The Lie group of special rigid transformations of $\mathbb{K}^6$ is denoted by $\mathrm{SR}(\mathbb{K}^6) \subset \mathrm{Lin}(\mathbb{K}^6)$. The Lie algebra of $\mathrm{SR}(\mathbb{K}^6)$, *i.e.* the linear space of all North-East cross product tensors, is denoted as $\mathrm{sr}(\mathbb{K}^6) \subset \mathrm{Lin}(\mathbb{K}^6)$. We denote as $(\bullet\times\!\!\!\!\times) \in \mathrm{sr}(\mathbb{K}^6)$ the North-East cross product tensor obtained from the cross product operator applied to vector $\bullet \in \mathbb{K}^6$, *i.e.*, $(\bullet\times\!\!\!\!\times)\star = \bullet\times\!\!\!\!\times\star, \forall \bullet, \star \in \mathbb{K}^6$, and as $\mathrm{axial}_{\times\!\!\!\!\times}(\bullet)$ its inverse mapping, *i.e.* the 'extraction' of the axial vector from the North-East cross product part of the tensor $\bullet \in \mathrm{Lin}(\mathbb{K}^6)$. Vectors in $\mathbb{K}^6$ will be represented by 3-D components as $\bullet = (\bullet_L; \bullet_A)$, where subscripts $_L$ and $_A$ stand for translational and rotational vector components and the semicolon notation denote a stacking by rows and is used to skip the more cumbersome $\bullet = (\bullet_L^T, \bullet_A^T)^T$.

The identity and null tensors, in both $\mathrm{Lin}(\mathbb{E}^3)$ and $\mathrm{Lin}(\mathbb{K}^6)$, are denoted by $\mathbf{I}$ and $\mathbf{O}$, respectively. The symbol δ^{hk} represents the components of $\mathbf{I}$ with respect to any base. Einstein's summation convention on repeated indices is employed throughout.

2. Rotation

2.1. **Representation of rotation.** A rotation, seen as a linear operation over 3-D vectors, may be represented by a *rotation tensor* $\mathbf{R} \in \mathrm{SO}(\mathbb{E}^3)$. Let us recall that this entails the fundamental properties $\mathbf{R}^{-1} = \mathbf{R}^T$ and $\det(\mathbf{R}) = +1$.

Given any vector $\mathbf{u} \in \mathbb{E}^3$, the corresponding vector after a rotation by $\mathbf{R}$ is $\mathbf{v} = \mathbf{R}\,\mathbf{u}$. Given any orthonormal base $\mathcal{B} = \{\mathbf{i}_k\}_{k=1,2,3}$ the component expressions for $\mathbf{u}$, $\mathbf{R}$, $\mathbf{v}$ are

$$\mathbf{u} = u^k \mathbf{i}_k, \tag{1}$$

$$\mathbf{R} = R^{hk} \mathbf{i}_h \otimes \mathbf{i}_k, \tag{2}$$

$$\mathbf{v} = v^k \mathbf{i}_k. \tag{3}$$

Therefore

$$v^h = R^{hk} v^k. \tag{4}$$

A rotation also can be seen as an affine map acting on points in the 3-D space $\mathcal{E}^3$ when a placement $\mathbf{c} \in \mathcal{E}^3$ is specified as the center of rotation. Given any point $\mathbf{x} \in \mathcal{E}^3$, the corresponding point after a rotation by $\mathbf{R}$ around $\mathbf{c}$ is $\mathbf{y} = \mathbf{c} + \mathbf{R}\,(\mathbf{x} - \mathbf{c})$. The relative coordinates of $\mathbf{x}$, $\mathbf{c}$, $\mathbf{y}$ with respect to a specified origin $\mathbf{o}$ along the axes of $\mathcal{B}$ are

$$x^k = (\mathbf{x} - \mathbf{o}) \cdot \mathbf{i}_k, \qquad k = 1, 2, 3 \tag{5}$$

$$c^k = (\mathbf{c} - \mathbf{o}) \cdot \mathbf{i}_k, \qquad k = 1, 2, 3 \tag{6}$$

$$y^k = (\mathbf{y} - \mathbf{o}) \cdot \mathbf{i}_k, \qquad k = 1, 2, 3 \tag{7}$$

Therefore

$$y^h = (\delta^{hk} - R^{hk})\, c^k + R^{hk} x^k. \tag{8}$$

The rotation as an affine map can therefore be identified with the pair $(\mathbf{R}, \mathbf{c})$. The following sections will deal with rotation as a linear mapping on vectors. Consequently, no mention of the center of rotation will be made.

2.2. **Euler's Theorem.** Let us recall the fundamental theorem on rotation by Euler: *"any rigid motion leaving a point fixed may be represented by a single planar rotation about a suitable axis passing through that point"*. We denote with $\mathbf{e} \in \mathbb{E}^3$ the unit vector of the axis of rotation and with $\varphi \in [0, 2\pi)$ the angle of rotation with respect to a reference configuration.[2] Whenever $\varphi = 0$, the axis $\mathbf{e}$ is not uniquely defined. Note that, since 2 scalar parameters are needed to represent a constant magnitude vector, such as $\mathbf{e}$, a generic rotation can be described at least with 3 scalar parameters, *i.e.* the dimension of the manifold $\mathrm{SO}(\mathbb{E}^3)$.

By Euler's theorem, the two quantities $(\varphi, \mathbf{e})$ completely define the rotational displacement represented by $\mathbf{R}$. In fact, following geometrical arguments (see, *e.g.*, Ref. [2]), it may be shown that the rotation tensor allows for an expression in terms of $(\varphi, \mathbf{e})$ known as the *Euler-Rodrigues formula* (Ref. [9]), given by

$$\mathbf{R} = \mathbf{I} + \sin\varphi\,(\mathbf{e}\times) + (1 - \cos\varphi)\,(\mathbf{e}\times)^2. \tag{9}$$

Note that the rotation corresponding to $(-\varphi, \mathbf{e})$ is equivalent to that corresponding to $(\varphi, -\mathbf{e})$, hence it is represented by tensor $\mathbf{R}^T \equiv \mathbf{R}^{-1}$.

Also, note that, since the rotation tensor leaves any vector parallel to the rotation axis unchanged, the following *unit eigenvalue property* holds:

$$\mathbf{R}\,\mathbf{e} = \mathbf{e}. \tag{10}$$

We label by $\lambda_3(\mathbf{R}) = +1$ the unit eigenvalue. The other eigenvalues are

$$\lambda_{1,2}(\mathbf{R}) = \pm e^{i\varphi} = \cos\varphi \pm i\sin\varphi. \tag{11}$$

The corresponding eigenvectors define a plane normal to the rotation axis.

Among the important results derived from eq. 9 we find

$$\mathrm{tr}(\mathbf{R}) = 1 + 2\,\cos\varphi, \tag{12}$$

$$\mathrm{axial}_\times(\mathbf{R}) = \sin\varphi\,\mathbf{e}. \tag{13}$$

These equations can be employed to construct a solution algorithm for the inverse problem, *i.e.* the recovery of $(\varphi, \mathbf{e})$ from $\mathbf{R}$ (see Ref. [4]).

Rotation composition is in important issue in applications: when two rotations $\mathbf{R}_A$, $\mathbf{R}_B$ are composed to obtain $\mathbf{R} = \mathbf{R}_B\mathbf{R}_A$, the quantities $(\varphi, \mathbf{e})$ that are found applying Euler's theorem for the composed rotation $\mathbf{R}$ are related to $(\varphi_A, \mathbf{e}_A)$, $(\varphi_B, \mathbf{e}_B)$, *i.e.* those corresponding to $\mathbf{R}_A$, $\mathbf{R}_B$, by

$$\cos\frac{\varphi}{2} = \cos\frac{\varphi_A}{2}\,\cos\frac{\varphi_B}{2} - \sin\frac{\varphi_A}{2}\,\sin\frac{\varphi_B}{2}\,\mathbf{e}_B\cdot\mathbf{e}_A, \tag{14}$$

$$\sin\frac{\varphi}{2}\,\mathbf{e} = \cos\frac{\varphi_A}{2}\,\sin\frac{\varphi_B}{2}\,\mathbf{e}_B + \cos\frac{\varphi_B}{2}\,\sin\frac{\varphi_A}{2}\,\mathbf{e}_A + \sin\frac{\varphi_A}{2}\,\sin\frac{\varphi_B}{2}\,\mathbf{e}_B\times\mathbf{e}_A. \tag{15}$$

This can be verified by elementary geometrical arguments (see, *e.g.*, Ref. [2]).

Under appropriate smoothness assumptions, the time derivative of tensor $\mathbf{R}$ may be put in the well-known form

$$\dot{\mathbf{R}} = \boldsymbol{\omega}\times\mathbf{R}, \tag{16}$$

[2] The two quantities $(\varphi, \mathbf{e})$ are sometimes labeled as the 'principal' axis of rotation and the 'principal' angle of rotation, respectively.

where vector $\boldsymbol{\omega} := \text{axial}_\times(\dot{\mathbf{R}}\,\mathbf{R}^{-1}) \in \mathbb{E}^3$ is termed the rotational velocity, or *spin*. Combining the above equation with Euler-Rodrigues formula, eq. 9 and noting that $\dot{\mathbf{e}} \cdot \mathbf{e} = 0$ since $\mathbf{e}$ is a constant magnitude vector, we get the following expression:

$$\boldsymbol{\omega} = \dot{\varphi}\,\mathbf{e} + \big(\sin\varphi\,\mathbf{I} + (1-\cos\varphi)\,(\mathbf{e}\times)\big)\,\dot{\mathbf{e}}, \tag{17}$$

where $\boldsymbol{\omega}$ is expressed in terms of $(\varphi, \mathbf{e})$ and their time derivatives. It is also interesting to consider the *co-rotated image* of the spin,[3] $\overline{\boldsymbol{\omega}} := \mathbf{R}^{-1}\boldsymbol{\omega}$. In terms of $\overline{\boldsymbol{\omega}}$, the time derivative of $\mathbf{R}$ results

$$\dot{\mathbf{R}} = \mathbf{R}\,\overline{\boldsymbol{\omega}}\times, \tag{18}$$

and it is straightforward to obtain

$$\overline{\boldsymbol{\omega}} = \dot{\varphi}\,\mathbf{e} + \big(\sin\varphi\,\mathbf{I} - (1-\cos\varphi)\,(\mathbf{e}\times)\big)\,\dot{\mathbf{e}}. \tag{19}$$

2.3. **Exponential Parameterization.** The exponential parameterization of rotation is based on the specialization of the exponential map, defined by the tensorial power series

$$\exp(\bullet) := \sum_{k=0}^{\infty} \frac{\bullet^k}{k!}, \tag{20}$$

to the case of rotation, $\exp : \text{so}(\mathbb{E}^3) \longrightarrow \text{SO}(\mathbb{E}^3)$. For any vector $\boldsymbol{\varphi} \in \mathbb{E}^3$, and hence any skew-symmetric tensor $(\boldsymbol{\varphi}\times) \in \text{so}(\mathbb{E}^3)$, we get a rotation tensor $\mathbf{R} \in \text{SO}(\mathbb{E}^3)$ by

$$\mathbf{R} = \exp(\boldsymbol{\varphi}\times). \tag{21}$$

Vector $\boldsymbol{\varphi}$ is termed the *rotation vector*[4] corresponding to tensor $\mathbf{R}$.

It is very straightforward to see that this parameterization technique is closely related to that inspired by the Euler theorem (see Ref. [4], where an explanation is given in terms of the solution of a constant-coefficient ordinary differential equation). In fact, the unit vector $\boldsymbol{\varphi}/\|\boldsymbol{\varphi}\|$ of the rotation vector coincides with the unit vector $\mathbf{e}$ that identifies the axis of rotation, while its magnitude $\|\boldsymbol{\varphi}\|$ coincides with the angle of rotation φ:[5]

$$\boldsymbol{\varphi} = \varphi\,\mathbf{e}. \tag{22}$$

Given the rotation tensor $\mathbf{R}$, the corresponding rotation vector $\boldsymbol{\varphi}$ can be formally recovered by the inverse formula

$$\boldsymbol{\varphi} := \text{axial}_\times\big(\log(\mathbf{R})\big), \tag{23}$$

where the logarithm function is defined by the tensorial power series

$$\log(\bullet) := -\sum_{k=1}^{\infty} \frac{1}{k}\,(\mathbf{I} - \bullet)^k. \tag{24}$$

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[3]In applied mechanics, the spin and its co-rotated image are often addressed as the 'spatial' and 'material' angular velocities.

[4]We prefer this simple, self-explaining denomination among the manifold names adopted in the literature, including the Euler vector, the 'principal' rotation vector, the 'finite' rotation vector, the rotational vector, the equivalent axis representation vector, and so on. Most of these names imply the direct relation that exists between this parameterization technique and Euler's theorem.

[5]For these reasons, the exponential parameterization of rotation appears as the most direct representation among all possible vectorial parameterizations discussed in the following, and is also addressed as the *natural parameterization* for $\text{SO}(\mathbb{E}^3)$.

Note that, when applied to rotations, the $\exp(\bullet)$ and $\log(\bullet)$ maps are not one-to-one, therefore a restriction over all possible (∞) determinations of $\boldsymbol{\varphi}$ for a given $\mathbf{R}$ must be imposed. This is accomplished selecting the *principal value* of $\boldsymbol{\varphi}$, *i.e.* the single vector within all possible solutions of eq. 23 that has a magnitude in $[0, 2\pi)$.

In practice, none of the eqs. 20 or 24 are used to get $\mathbf{R}$ from $\boldsymbol{\varphi}$ and vice-versa. In fact, a 'finite form' representation is available as

$$\mathbf{R} = \mathbf{I} + R_1\,(\boldsymbol{\varphi}\times) + R_2\,(\boldsymbol{\varphi}\times)^2, \tag{25}$$

where the scalar coefficients R_1, R_2, depending evenly on φ, are given by

$$R_1(\varphi) := \frac{\sin\varphi}{\varphi}, \tag{26}$$

$$R_2(\varphi) := \frac{1-\cos\varphi}{\varphi^2}. \tag{27}$$

Eq. 25 is clearly Euler-Rodrigues formula, eq. 9, rewritten in terms of the rotation vector.[6] It has the advantage, in comparison to the former, to hold also in the degenerate case when $\varphi = 0 \Longrightarrow \boldsymbol{\varphi} = \mathbf{0}$, yielding $\mathbf{R} = \mathbf{I}$. This may represent an sensitive issue in numerical applications.

The eigenvalues $\lambda_{1,2}(\mathbf{R})$ are written in terms of R_1, R_2 as

$$\lambda_{1,2}(\mathbf{R}) = (1 - \varphi^2 R_2) \pm i\,\varphi\,R_1 \tag{28}$$

and $\det(\mathbf{R}) = (1-\varphi^2 R_2)^2 + (\varphi\,R_1)^2 = 1$. The exponential map has an associated differential map (see Ref. [4], where an explanation is given in terms of the variation of a constant-coefficient ordinary differential equation) defined by the tensorial power series

$$\operatorname{dexp}(\bullet) := \sum_{k=0}^{\infty} \frac{\bullet^k}{(k+1)!}. \tag{29}$$

The exponential map and its associated differential map enjoy remarkable properties, valid beyond the particular application to rotations presented here. Among these we recall:

$$\exp(\bullet) = \operatorname{dexp}(\bullet)\,\operatorname{dexp}(-\bullet)^{-1} = \operatorname{dexp}(-\bullet)^{-1}\operatorname{dexp}(\bullet), \tag{30}$$

$$\exp(\bullet) = \mathbf{I} + \bullet\,\operatorname{dexp}(\bullet) = \mathbf{I} + \operatorname{dexp}(\bullet)\ \bullet\,. \tag{31}$$

Note that the second property above expresses the 'symbolical' definition of the associated differential map as the 'derivative' of the exponential map in the neighborhood of the identity,

$$\operatorname{dexp}(\bullet) \overset{sym}{=} \frac{\exp(\bullet) - \mathbf{I}}{\bullet}. \tag{32}$$

We denote as $\mathbf{S}$ the differential tensor associated to $\mathbf{R}$,

$$\mathbf{S} = \operatorname{dexp}(\boldsymbol{\varphi}\times). \tag{33}$$

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[6] This is obtained by taking into consideration the recursive property of the cross product:

$$(\boldsymbol{\varphi}\times)^{2m-a} = (-1)^{m-1}\varphi^{2(m-1)}(\boldsymbol{\varphi}\times)^a,$$

for any $m \in \mathbb{N}$ and $a = 1, 2$. Using the previous results we get

$$\exp(\boldsymbol{\varphi}\times) = \mathbf{I} + \sum_{m=1}^{\infty} \frac{(-1)^{m-1}\varphi^{2(m-1)}}{(2m-1)!}\,(\boldsymbol{\varphi}\times) + \sum_{m=0}^{\infty} \frac{(-1)^{m-1}\varphi^{2(m-1)}}{(2m)!}\,(\boldsymbol{\varphi}\times)^2,$$

and therefore eq. 25 after recognition of the power expansions in terms of those for $\sin\varphi$ and $\cos\varphi$.

A 'finite form' formula similar to eq. 25 holds for the associated differential tensor:[7]

$$\mathbf{S} = \mathbf{I} + S_1\,(\boldsymbol{\varphi}\times) + S_2\,(\boldsymbol{\varphi}\times)^2, \tag{34}$$

where the scalar coefficients S_1, S_2, again depending evenly on φ, are given by

$$S_1(\varphi) := \frac{1-\cos\varphi}{\varphi^2}, \tag{35}$$

$$S_2(\varphi) := \frac{\varphi-\sin\varphi}{\varphi^3}. \tag{36}$$

Note that $S_1 \equiv R_2$, $S_2 = (1-R_1)/\varphi^2$. For future convenience, note that S_2 can be written in terms of the half rotation angle as

$$S_1(\varphi) \equiv R_2(\varphi) = \frac{1}{2}\,\frac{\sin(\varphi/2)}{(\varphi/2)^2}. \tag{37}$$

As for tensor $\mathbf{R}$, vector $\mathbf{e}$ is an eigenvector of tensor $\mathbf{S}$, and the corresponding eigenvalue is $+1$,

$$\mathbf{S}\,\mathbf{e} = \mathbf{e}, \tag{38}$$

We label by $\lambda_3(\mathbf{S}) = +1$ the unit eigenvalue. The other eigenvalues are

$$\lambda_{1,2}(\mathbf{S}) = (1-\varphi^2 S_2) \pm i\,\varphi\,S_1 = R_1 \pm i\,\varphi\,R_2, \tag{39}$$

or, in terms of the half rotation angle,

$$\lambda_{1,2}(\mathbf{S}) = \frac{\sin(\varphi/2)}{\varphi/2}\,\big(\cos(\varphi/2) \pm i\sin(\varphi/2)\big). \tag{40}$$

Note also that $\det(\mathbf{S}) = 2\,R_2 = \sin^2(\varphi/2)/(\varphi/2)^2$, so that $\mathbf{S}$ is singular at $\varphi = \pi$, while for $\varphi = 0$ we get $\mathbf{S} = \mathbf{R} = \mathbf{I}$.

The associated differential map relates the derivative $\dot{\boldsymbol{\varphi}}$ of the rotation vector with the spin $\boldsymbol{\omega}$. In fact, we get

$$\boldsymbol{\omega} = \operatorname{dexp}(\boldsymbol{\varphi}\times)\,\dot{\boldsymbol{\varphi}}, \tag{41}$$

as can be easily verified from eq. 17. Also, it relates the derivative $\dot{\boldsymbol{\varphi}}$ of the rotation vector with the co-rotated image of the spin $\overline{\boldsymbol{\varpi}}$ as

$$\overline{\boldsymbol{\varpi}} = \operatorname{dexp}(-\boldsymbol{\varphi}\times)\,\dot{\boldsymbol{\varphi}}. \tag{42}$$

Given the definition 29, we may rewrite eqs. 41, 42 as $\boldsymbol{\omega} = \mathbf{S}\,\dot{\boldsymbol{\varphi}}$, $\overline{\boldsymbol{\varpi}} = \mathbf{S}^T\,\dot{\boldsymbol{\varphi}}$.

The inverse of the associated differential map can be expressed as the tensorial power series

$$\operatorname{dexp}(\bullet)^{-1} = \mathbf{I} - \frac{1}{2}\,\bullet - \sum_{k=1}^{\infty} \frac{B_k}{(2\,k)!}\,\bullet^k, \tag{43}$$

where the scalar coefficients B_k are the Bernoulli numbers. Again, a finite form expression is found as

$$\mathbf{S}^{-1} = \mathbf{I} - \frac{1}{2}\,(\boldsymbol{\varphi}\times) + \frac{1}{\varphi^2}\left(1 - \frac{1}{2}\,\frac{R_1}{R_2}\right)(\boldsymbol{\varphi}\times)^2. \tag{44}$$

[7] The same reasoning seen above for $\mathbf{R}$ applies to $\mathbf{S}$.

Note that the coefficient of $(\boldsymbol{\varphi}\times)^2$ can be expressed in terms of the half rotation angle, since $R_1/(2\,R_2) = (\varphi/2)/\tan(\varphi/2)$, to yield the following equivalent expression for $\mathbf{S}^{-1}$:

$$\mathbf{S}^{-1} = \mathbf{I} - \frac{1}{2}\,(\boldsymbol{\varphi}\times) + \frac{1}{\varphi^2}\left(1 - \frac{\varphi/2}{\tan(\varphi/2)}\right)(\boldsymbol{\varphi}\times)^2. \tag{45}$$

2.4. **Vectorial Parameterization.** The vectorial parameterization of rotation is a general class of techniques based on a minimal set of parameters. This set consists of the pair $(p, \mathbf{e})$, where $p = p(\varphi)$ is the *generating function* of the parameterization. The generating function must be an odd function of the rotation angle and must present the limit behavior

$$\lim_{\varphi \to 0} \frac{p(\varphi)}{\varphi} = \kappa, \tag{46}$$

where $\kappa \in \mathbb{R}$ is a constant called the *normalization factor* of the parameterization. The parameters are used to construct the *rotation parameter vector* $\mathbf{p} \in \mathbb{E}^3$ as

$$\mathbf{p} = p\,\mathbf{e}. \tag{47}$$

We denote the *vectorial parameterization map of rotation* as $\mathrm{rot} : \mathrm{so}(\mathbb{E}^3) \longrightarrow \mathrm{SO}(\mathbb{E}^3)$. Thus, given a rotation parameter vector $\mathbf{p}$, we get a rotation tensor $\mathbf{R}$ by

$$\mathbf{R} = \mathrm{rot}(\mathbf{p}\times). \tag{48}$$

The explicit expression of the vectorial parameterization map is easily obtained from Euler-Rodrigues formula, eq. 9, as

$$\mathbf{R} = \mathbf{I} + P_1\,(\mathbf{p}\times) + P_2\,(\mathbf{p}\times)^2, \tag{49}$$

where the scalar coefficients P_1, P_2, depending evenly on φ, read

$$P_1(\varphi) := \frac{\sin\varphi}{p(\varphi)}, \tag{50}$$

$$P_2(\varphi) := \frac{1 - \cos\varphi}{p(\varphi)^2}. \tag{51}$$

As with the exponential map, we remark that formula 49 holds also for the case $\mathbf{p} = \mathbf{0}$, yielding $\mathbf{R} = \mathbf{I}$.

The eigenvalues $\lambda_{1,2}(\mathbf{R})$ are written in terms of P_1, P_2 as

$$\lambda_{1,2}(\mathbf{R}) = (1 - p^2 P_2) \pm i\,p\,P_1 \tag{52}$$

and $\det(\mathbf{R}) = (1 - p^2 P_2)^2 + (p\,P_1)^2 = 1$.

In view of the description of the composition of rotations, as well as other formulæ, we introduce an alternative expression of the vectorial parameterization map as

$$\mathbf{R} = \mathbf{I} + \left(\gamma\,\mathbf{I} + \frac{1}{2}\,(\nu\mathbf{p}\times)\right)(\nu\mathbf{p}\times), \tag{53}$$

where the scalar coefficients γ, ν, depending evenly on φ, are defined as

$$\gamma(\varphi) := \cos(\varphi/2), \tag{54}$$

$$\nu(\varphi) := 2\,\frac{\sin(\varphi/2)}{p(\varphi)}. \tag{55}$$

Clearly, $\gamma = P_1/\sqrt{2P_2}$, $\nu = \sqrt{2P_2}$. Therefore,

$$\lambda_{1,2}(\mathbf{R}) = \left(1 - \frac{1}{2}\,(\nu p)^2\right) \pm i\,\gamma\,(\nu p) \tag{56}$$

and $\det(\mathbf{R}) = \big(1 - (\nu p)^2/2\big) + \gamma^2(\nu p)^2 = 1$.

Note that eqs. 53 and 56 purposely highlight the role of vector $(\nu\mathbf{p})$ and its norm (νp). This will be also evident in the subsequent formulæ.

Indeed, in the case of a composed rotation $\mathbf{R} = \mathbf{R}_B\mathbf{R}_A$, the rotation parameter vector $\mathbf{p}$ corresponding to $\mathbf{R}$ is found in terms of $\mathbf{p}_A$, $\mathbf{p}_B$ corresponding to $\mathbf{R}_A$, $\mathbf{R}_B$, respectively, by

$$\gamma = \gamma_A\,\gamma_B - \frac{1}{4}\,(\nu_B\mathbf{p}_B)\cdot(\nu_A\mathbf{p}_A), \tag{57}$$

$$\nu\,\mathbf{p} = \gamma_A\,(\nu_B\mathbf{p}_B) + \gamma_B\,(\nu_A\mathbf{p}_A) + \frac{1}{2}\,(\nu_B\mathbf{p}_B)\times(\nu_A\mathbf{p}_A), \tag{58}$$

having set $\gamma_A := \gamma(\varphi_A)$, $\gamma_B := \gamma(\varphi_B)$, $\nu_A := \nu(\varphi_A)$, $\nu_B := \nu(\varphi_B)$, and with the slight abuses $\gamma = \gamma(\varphi)$, $\nu = \nu(\varphi)$.

As seen with the exponential parameterization, it is possible to associate a differential map $\mathrm{drot} : \mathrm{so}(\mathbb{E}^3) \longrightarrow \mathrm{Lin}(\mathbb{E}^3)$ to the vectorial parameterization map such that, symbolically,

$$\mathrm{drot}(\bullet) \overset{sym}{=} \frac{\mathrm{rot}(\bullet) - \mathbf{I}}{\bullet}, \tag{59}$$

and the following properties are satisfied:

$$\mathrm{rot}(\bullet) = \mathrm{drot}(\bullet)\,\mathrm{drot}(-\bullet)^{-1} = \mathrm{drot}(-\bullet)^{-1}\mathrm{drot}(\bullet), \tag{60}$$

$$\mathrm{rot}(\bullet) = \mathbf{I} + \bullet\;\mathrm{drot}(\bullet) = \mathbf{I} + \mathrm{drot}(\bullet)\,\bullet, \tag{61}$$

in complete analogy with the exponential map.

We denote as $\mathbf{H}$ the differential tensor associated to $\mathbf{R}$,

$$\mathbf{H} := \mathrm{drot}(\mathbf{p}\times). \tag{62}$$

The explicit expression for the associated differential map is

$$\mathbf{H} = \mu\,\mathbf{I} + H_1\,(\mathbf{p}\times) + H_2\,(\mathbf{p}\times)^2. \tag{63}$$

The coefficients μ, H_1, H_2, depending evenly on φ, are defined as

$$\mu(\varphi) := \frac{1}{p'(\varphi)}, \tag{64}$$

$$H_1(\varphi) := \frac{1 - \cos\varphi}{p(\varphi)^2}, \tag{65}$$

$$H_2(\varphi) := \frac{\mu(\varphi)\,p(\varphi) - \sin(\varphi)}{p(\varphi)^3}, \tag{66}$$

where $p' := \mathrm{d}p/\mathrm{d}\varphi$. Note that $H_1 \equiv P_2$ and that $H_2 = (\mu - P_1)/p^2$.

An equivalent expression for $\mathbf{H}$ is found as

$$\mathbf{H} = \mu\,\mathbf{I} + \left(\frac{\nu}{2}\,\mathbf{I} + \frac{\mu - \gamma\nu}{(\nu p)^2}\,(\nu\mathbf{p}\times)\right)(\nu\mathbf{p}\times). \tag{67}$$

Coefficient μ is the eigenvalue of tensor $\mathbf{H}$ corresponding to the eigenvector $\mathbf{e}$,

$$\mathbf{H}\,\mathbf{e} = \mu\,\mathbf{e}, \tag{68}$$

and we get $\mu(0) = 1/\kappa$. We label by $\lambda_3(\mathbf{H}) = \mu$ this eigenvalue. The other eigenvalues are

$$\lambda_{1,2}(\mathbf{H}) = (\mu - p^2 H_2) \pm i\, p\, H_1 = P_1 \pm i\, p\, P_2, \tag{69}$$

or, in terms of γ, ν,

$$\lambda_{1,2}(\mathbf{H}) = \nu\, \left(\gamma \pm i\, \frac{1}{2}\, (\nu p)\right). \tag{70}$$

Note also that $\det(\mathbf{H}) = 2\, \mu\, P_2 = \mu\, \nu^2$, so that $\mathbf{H}$ is singular at the values of φ that annihilate μ or ν, while for $\varphi = 0$ we get $\mathbf{H} = (1/\kappa)\, \mathbf{I}$.

The associated differential map relates the derivative $\dot{\mathbf{p}}$ of the rotation parameter vector with the spin $\boldsymbol{\omega}$ and its co-rotated image $\overline{\boldsymbol{\omega}}$:

$$\boldsymbol{\omega} = \mathrm{drot}(\mathbf{p}\times)\, \dot{\mathbf{p}}, \tag{71}$$

$$\overline{\boldsymbol{\omega}} = \mathrm{drot}(-\mathbf{p}\times)\, \dot{\mathbf{p}}. \tag{72}$$

Given eq. 63, we may rewrite eqs. 71, 72 as $\boldsymbol{\omega} = \mathbf{H}\, \dot{\mathbf{p}}$, $\overline{\boldsymbol{\omega}} = \mathbf{H}^T\, \dot{\mathbf{p}}$.

The inverse of the associated differential tensor may be expressed as

$$\mathbf{H}^{-1} = \frac{1}{\mu}\, \mathbf{I} - \frac{1}{2}\, (\mathbf{p}\times) + \frac{1}{p^2}\, \left(\frac{1}{\mu} - \frac{1}{2}\, \frac{P_1}{P_2}\right)\, (\mathbf{p}\times)^2. \tag{73}$$

An equivalent expression is given by

$$\mathbf{H}^{-1} = \frac{1}{\mu}\, \mathbf{I} - \frac{1}{2}\, (\mathbf{p}\times) + \frac{1}{p^2}\, \left(\frac{1}{\mu} - \frac{\gamma}{\nu}\right)\, (\mathbf{p}\times)^2. \tag{74}$$

We end this section remarking the strikingly close analogy in the mathematical formalization of all the results developed up to here for the vectorial parameterization of rotation compared to the exponential parameterization of rotation.

The only choice needed in order to obtain a specific parameterization technique is the actual form of $p(\varphi)$. In the following, we list some possible choices, including the most employed techniques that fall into the class of the vectorial parameterization of rotation:

- *Exponential map*

  $$p(\varphi) = \varphi.$$

  This represents the simplest possible choice, being the exponential parameterization seen above: rot $\equiv$ exp.
- *Cayley/Gibbs/Rodrigues parameterization*

  $$p(\varphi) = 2\, \kappa\, \tan(\varphi/2).$$

  This is variously referred to as *Cayley parameterization*, *Gibbs parameterization*, or *Rodrigues parameterization*. In the following we shall refer to this technique as the 'Cayley/Gibbs/Rodrigues parameterization'. Note that when $\kappa = 1/2$, rot $\equiv$ cay.
- *Wiener/Milenkovic parameterization*

  $$p(\varphi) = 4\, \kappa\, \tan(\varphi/4).$$

  This is variously referred to as *Milenkovic parameterization*, *modified Rodrigues parameterization*, or *conformal rotation vector (CRV) parameterization*, although it was first proposed by Wiener (see Ref. [15]). In the following we shall refer to this technique as the 'Wiener/Milenkovic parameterization'.

- *Linear parameterization*

$$p(\varphi) = \sin\varphi.$$

This is known as the *linear parameterization.*

- *Reduced Euler-Rodrigues parameterization*

$$p(\varphi) = 2\,\kappa\,\sin(\varphi/2).$$

This is variously referred to as *reduced Euler parameterization* or *reduced Euler-Rodrigues parameterization*, being closely related to the Euler-Rodrigues parameterization (a well known non-vectorial technique based on unit quaternions, see, *e.g.*, Ref. [3]). In the following we shall refer to this technique as the 'reduced Euler-Rodrigues parameterization' when the normalization factor is chosen as $\kappa = 1$ and as the 'normalized reduced Euler-Rodrigues parameterization' when $\kappa = 2$.

The reader is addressed to the Appendix B for a detailed presentation of each parameterization technique cited above.

3. Rigid motion

3.1. **Representation of rigid motion.** Consider a rigid displacement composed by a rotation around some point followed by a (uniform) translation. This is clearly an affine map acting on points in the 3-D space $\mathcal{E}^3$. Given any point $\mathbf{x} \in \mathcal{E}^3$, the corresponding point after a rotation by $\mathbf{R}$ around $\mathbf{c}$ followed by a translation by $\mathbf{l} \in \mathbb{E}^3$ is $\mathbf{y} = \mathbf{c} + \mathbf{R}\,(\mathbf{x} - \mathbf{c}) + \mathbf{l}$. If the component expression for $\mathbf{l}$ is written as

$$\mathbf{l} = l^k \mathbf{i}_k, \tag{75}$$

we obtain

$$y^h = (\delta^{hk} - R^{hk})\,c^k + R^{hk} x^k + l^h. \tag{76}$$

Note that the reverse procedure, *i.e.* an uniform translation by $\mathbf{l}$ followed by a rotation by $\mathbf{R}$ around $\mathbf{c}$ yields $\mathbf{y} = \mathbf{c} + \mathbf{R}\,(\mathbf{x} - \mathbf{c} + \mathbf{l})$, so that

$$y^h = (\delta^{hk} - R^{hk})\,c^k + R^{hk}(x^k + l^k). \tag{77}$$

Both procedures represent possible descriptions of a general rigid displacement. In the following, we shall concentrate on the first one, *i.e.* the composition of a rotation and a subsequent translation. This displacement will then be identified by the set $(\mathbf{l}, \mathbf{R}, \mathbf{c})$.

As shown in detail in Ref. [4], a rigid displacement can be seen as an linear operation over 6-D vectors in both the kinematic and co-kinematic spaces, $\mathbb{K}^6$ and $\mathbb{K}^{6^*}$. In this case, it represents the change of both axes and pole for the reduction of velocities and moments. In particular, the displacement $(\mathbf{t}, \mathbf{R}, \mathbf{o})$, where $\mathbf{o} \in \mathcal{E}^3$ is the base pole, fixed for all times, is represented by the *displacement tensor* $\mathbf{D} \in \mathrm{SR}(\mathbb{K}^6)$,

$$\mathbf{D} := \begin{bmatrix} \mathbf{R} & \mathbf{t} \times \mathbf{R} \\ \mathbf{O} & \mathbf{R} \end{bmatrix}, \tag{78}$$

when acting on kinematic vectors, and by $\mathbf{D}^{-T}$ when acting on co-kinematic vectors. The base pole *translation vector* $\mathbf{t}$ is then given by $\mathbf{t} = \mathbf{l} + (\mathbf{I} - \mathbf{R})\,(\mathbf{c} - \mathbf{o})$.

A special case is that of $\mathbf{R}\,\mathbf{l} = \mathbf{l}$, *i.e.* when the direction of translation coincides with the axis of rotation. Such a case is termed *screw motion* (see Ref. [5]). In this case the two alternative descriptions coincide.

3.2. **Mozzi-Chasles' Theorem.** Let us recall the fundamental theorem on rigid motion by Mozzi and Chasles: *"any rigid motion may be represented by a planar rotation about a suitable axis, followed by an uniform translation along that same axis"*. We start from the notation laid down for Euler's theorem denoting as $\mathbf{e} \in \mathbb{E}^3$ the unit vector of the screw axis (*i.e.* the axis of rotation and translation, see Ref. [8]), with $\varphi \in [0, 2\,\pi)$ the angle of rotation with respect to a reference configuration, adding new quantities: the translation $\tau \in \mathbb{R}$ along $\mathbf{e}$, and a point $\mathbf{a} \in \mathcal{E}^3$ along the axis. Contrary to our previous treatment of rotation, here the position of the screw axis in $\mathcal{E}^3$ is crucial. We shall therefore refer to the screw axis as represented by the pair $(\mathbf{e}, \mathbf{a})$. Now, even when $\varphi = 0$, the axis $\mathbf{e}$ remains uniquely defined by the direction of translation. It is clearly undefined whenever both φ and τ vanish. Note that, since 2 scalar parameters are needed to represent the position of a point along a given line, a generic screw motion can be described with no less than 6 scalar parameters (3 for the rotation, 2 for the position of $\mathbf{a}$ along $\mathbf{e}$, and 1 for τ), *i.e.* the dimension of the manifold $\mathrm{SR}(\mathbb{K}^6)$.

By Mozzi-Chasles' theorem, the four quantities $(\varphi, \mathbf{e}, \tau, \mathbf{a})$ completely define the rigid displacement represented by $\mathbf{D}$. In the following we shall delve on the exploitation of this relationship.

We define, for future convenience, the following quantities: the *screw magnitude*[8] tensor $\mathbf{\Phi}$,

$$\mathbf{\Phi} := \begin{bmatrix} \varphi\,\mathbf{I} & \tau\,\mathbf{I} \\ \mathbf{O} & \varphi\,\mathbf{I} \end{bmatrix}; \tag{79}$$

the *axis moment* $\mathbf{m} \in \mathbb{E}^3$ with respect to the base pole $\mathbf{o}$,

$$\mathbf{m} := \big(\mathbf{a} - \mathbf{o}\big) \times \mathbf{e}; \tag{80}$$

and the *screw axis vector* $\mathbf{h} \in \mathbb{K}^6$,

$$\mathbf{h} := (\mathbf{m}; \mathbf{e}). \tag{81}$$

Note that $\mathbf{m}$ does not depend on the specific point $\mathbf{a}$ chosen along the screw axis, and hence the same applies for $\mathbf{h}$. The column vector representing $\mathbf{h}$ with respect to a given base coincides with the so-called *Plücker coordinates* of the screw axis $(\mathbf{e}, \mathbf{a})$ (see Ref. [6]). The pair $(\mathbf{\Phi}, \mathbf{h})$ fully synthesizes the information offered by Mozzi-Chasles' theorem through the quantities $(\varphi, \mathbf{e}, \tau, \mathbf{a})$, appearing as a generalization of the pair $(\varphi, \mathbf{e})$ employed in the analysis of rotation.

It can be shown that a straightforward generalization of the Euler-Rodrigues formula, eq. 9, holds for a general rigid displacement:

$$\mathbf{D} = \mathbf{I} + \sin\mathbf{\Phi}\,(\mathbf{h}\times) + (\mathbf{I} - \cos\mathbf{\Phi})\,(\mathbf{h}\times)^2. \tag{82}$$

We refer to eq. 82 as the *generalized Euler-Rodrigues formula*, and to the representation of the rigid displacement based on $\mathbf{\Phi}$ and $\mathbf{h}$ as the *helicoidal representation* of frame motion.

[8]The term 'magnitude' is used generalizing the notion of magnitude for a vector. It must be noted that tensor $\mathbf{\Phi}$ does not represent a norm in the usual sense.

Note that tensor coefficients $\cos\mathbf{\Phi}$ and $\sin\mathbf{\Phi}$, are obtained by the extension to tensorial arguments of the standard definitions:

$$\cos\bullet := \frac{1}{2}\big(\exp(i\,\bullet) + \exp(-i\,\bullet)\big), \tag{83}$$

$$\sin\bullet := \frac{1}{2i}\big(\exp(i\,\bullet) - \exp(-i\,\bullet)\big), \tag{84}$$

and therefore are given by

$$\cos\mathbf{\Phi} = \begin{bmatrix} \cos\varphi\,\mathbf{I} & -\tau\,\sin\varphi\,\mathbf{I} \\ \mathbf{O} & \cos\varphi\,\mathbf{I} \end{bmatrix}, \tag{85}$$

$$\sin\mathbf{\Phi} = \begin{bmatrix} \sin\varphi\,\mathbf{I} & \tau\,\cos\varphi\,\mathbf{I} \\ \mathbf{O} & \sin\varphi\,\mathbf{I} \end{bmatrix}. \tag{86}$$

Also, note that, by eq. 82, the following *unit eigenvalue property* holds:

$$\mathbf{D}\,\mathbf{h} = \mathbf{h}. \tag{87}$$

The unit eigenvalue, as well as the other eigenvalues, clearly has multiplicity 2.

To prove eq. 82, let us compute the expression of the translation vector $\mathbf{t}$ in terms of the quantities $(\varphi, \mathbf{e}, \tau, \mathbf{m})$ related to Mozzi-Chasles' theorem. To this end, note that equation

$$\mathbf{o} + \mathbf{R}\,(\mathbf{x} - \mathbf{o}) + \mathbf{t} = \mathbf{c} + \mathbf{R}\,(\mathbf{x} - \mathbf{c}) + \tau\,\mathbf{e}, \tag{88}$$

states the equivalence of the rigid displacement obtained by rotating around $\mathbf{o}$ and translating by $\mathbf{t}$ with rotating around $\mathbf{c}$ and translating by $\mathbf{l}$. Recalling that the screw representation of the rigid displacement offered by Mozzi-Chasles' theorem implies $\mathbf{l} = \tau\,\mathbf{e}$ and taking into account that $\mathbf{m}\cdot\mathbf{e} = 0$, we are led to the desired expression for the translation vector $\mathbf{t}$:

$$\mathbf{t} = \tau\,\mathbf{e} + \big(\sin\varphi\,\mathbf{I} + (1 - \cos\varphi)\,(\mathbf{e}\times)\big)\,\mathbf{m}. \tag{89}$$

The preceding result implies the definition for τ as

$$\tau := \mathbf{t}\cdot\mathbf{e}. \tag{90}$$

For future convenience, we note that

$$\mathbf{t}\times\mathbf{e} = \big((1 - \cos\varphi)\,\mathbf{I} - \sin\varphi\,(\mathbf{e}\times)\big)\,\mathbf{m}. \tag{91}$$

At this point, verification that $\mathbf{D}$ has the expression given in eq. 82 in terms of $(\mathbf{\Phi}, \mathbf{h})$, once eq. 9 and eq. 89 are taken into account, is a trivial application of linear algebra.

Under appropriate smoothness assumptions, the time derivative of tensor $\mathbf{D}$ may be put in the form

$$\dot{\mathbf{D}} = \mathbf{w}\times\mathbf{D}, \tag{92}$$

where vector $\mathbf{w} := (\mathbf{v}; \boldsymbol{\omega}) \in \mathbb{K}^6$ is termed the *generalized velocity*. It is a kinematic vector composed by the *base pole velocity* $\mathbf{v} := (\dot{\mathbf{t}} + \mathbf{t}\times\boldsymbol{\omega}) \in \mathbb{E}^3$ and by the spin $\boldsymbol{\omega}$.

Given eq.89, the base pole velocity reads

$$\begin{aligned} \mathbf{v} \;&=\; \dot\varphi\,\mathbf{m} + \dot\tau\,\mathbf{e} + \big(\sin\varphi\,\mathbf{I} + (1 - \cos\varphi)\,(\mathbf{e}\times)\big)\,\dot{\mathbf{m}} \\ &+\; \Big((1 - \cos\varphi)\,(\mathbf{m}\times) - \tau\,\big(\cos\varphi\,\mathbf{I} - \sin\varphi\,(\mathbf{e}\times)\big)\Big)\,\dot{\mathbf{e}}, \end{aligned} \tag{93}$$

so that, grouping this result together with eq. 17, we obtain

$$\mathbf{w} = \dot{\mathbf{\Phi}}\,\mathbf{h} + \big(\sin\mathbf{\Phi} + (\mathbf{I} - \cos\mathbf{\Phi})\,(\mathbf{h}\times)\big)\,\dot{\mathbf{h}}. \tag{94}$$

It is also interesting to consider the *co-moved image*[9] of the generalized velocity, $\overline{\mathbf{w}} := \mathbf{D}^{-1}\mathbf{w}$. In terms of $\overline{\mathbf{w}}$, the time derivative of $\mathbf{D}$ results

$$\dot{\mathbf{D}} = \mathbf{D}\,\overline{\mathbf{w}}\times. \tag{95}$$

It is immediate to obtain $\overline{\mathbf{w}}$ as

$$\overline{\mathbf{w}} = \dot{\mathbf{\Phi}}\,\mathbf{h} + \big(\sin\mathbf{\Phi} - (\mathbf{I} - \cos\mathbf{\Phi})\,(\mathbf{h}\times)\big)\,\dot{\mathbf{h}}, \tag{96}$$

with $\overline{\mathbf{w}} := (\overline{\mathbf{v}}; \overline{\boldsymbol{\varpi}})$, where $\overline{\mathbf{v}} := \mathbf{R}^{-1}\dot{\mathbf{t}}$ is the co-moved image of the base pole velocity, *i.e.* the co-rotated image of the moving pole velocity:

$$\begin{aligned} \overline{\mathbf{v}} &= \dot{\varphi}\,\mathbf{m} + \dot{\tau}\,\mathbf{e} + \big(\sin\varphi\,\mathbf{I} - (1-\cos\varphi)\,(\mathbf{e}\times)\big)\,\dot{\mathbf{m}} \\ &+ \Big((1-\cos\varphi)\,(\mathbf{m}\times) - \tau\,\big(\cos\varphi\,\mathbf{I} - \sin\varphi\,(\mathbf{e}\times)\big)\Big)\,\dot{\mathbf{e}}. \end{aligned} \tag{97}$$

The previous equations are expressively written to underline the complete analogy in formulæ with the pure rotation case. In fact, eqs. 82, 92, 94, 95, 96 are identical to eqs. 9, 16, 17, 18, 19 when $\{\varphi, \mathbf{e}, \mathbf{R}, \boldsymbol{\omega}, \overline{\boldsymbol{\varpi}}\}$ are substituted with $\{\mathbf{\Phi}, \mathbf{h}, \mathbf{D}, \mathbf{w}, \overline{\mathbf{w}}\}$, respectively.

3.3. **Exponential Parameterization.** The exponential parameterization of motion is based on the specialization of the exponential map to the case of rigid motion, $\exp : \mathrm{sr}(\mathbb{K}^6) \longrightarrow \mathrm{SR}(\mathbb{K}^6)$. For any vector $\boldsymbol{\nu} \in \mathbb{K}^6$, and hence any North-East cross product tensor $(\boldsymbol{\nu}\times) \in \mathrm{sr}(\mathbb{K}^6)$, we get a displacement tensor $\mathbf{D} \in \mathrm{SR}(\mathbb{K}^6)$ by

$$\mathbf{D} = \exp(\boldsymbol{\nu}\times). \tag{98}$$

Vector $\boldsymbol{\nu}$ is termed the *generalized screw vector* corresponding to tensor $\mathbf{D}$ (see Ref. [4]). It can be shown that $\boldsymbol{\nu} = (\boldsymbol{\rho}; \boldsymbol{\varphi})$ where $\boldsymbol{\varphi}$ is the rotation vector corresponding to $\mathbf{R}$ while $\boldsymbol{\rho}$ is given by

$$\boldsymbol{\rho} = \mathrm{dexp}(\boldsymbol{\varphi}\times)^{-1}\,\mathbf{t}. \tag{99}$$

It is straightforward to see that the exponential parameterization is closely related to that inspired by the Mozzi-Chasles' theorem. Indeed,

$$\boldsymbol{\nu} = \mathbf{\Phi}\,\mathbf{h}, \tag{100}$$

a result analogous to eq. 22. In fact, from eq. 99 it is immediate to get

$$\boldsymbol{\rho}\cdot\mathbf{e} = \tau. \tag{101}$$

On the other hand, from the same equation, we get

$$\boldsymbol{\rho}\times\mathbf{e} = \varphi\,\mathbf{m}\times\mathbf{e}, \tag{102}$$

so that[10]

$$\boldsymbol{\rho} = \varphi\,\mathbf{m} + \tau\,\mathbf{e} \tag{103}$$

and hence eq. 100.

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[9] This represents the analog of the co-rotated image of a 3-D vector, accounting for both rotation and translation.

[10] The following result is obtained applying the general decomposition of any vector in two vector components: one aligned with a given direction, and the other perpendicular to it. Given vector $\bullet \in \mathbb{E}^3$, its decomposition reads

$$\bullet = (\mathbf{e}\cdot\bullet)\,\mathbf{e} + (\mathbf{e}\times\bullet)\times\mathbf{e}$$

holding for any unit vector $\mathbf{e}$.

Given the displacement tensor $\mathbf{D}$, the corresponding generalized screw vector $\boldsymbol{\nu}$ can be formally recovered by the inverse formula

$$\boldsymbol{\nu} := \mathrm{axial}_{\times}\big(\log(\mathbf{D})\big). \tag{104}$$

However, as with pure rotation, it is not necessary to use eqs. 98 and 104 in practice, since a finite form expression is available for the exponential map of motion as

$$\mathbf{D} = \mathbf{I} + \hat{\mathbf{R}}_1\,(\boldsymbol{\nu}\times) + \hat{\mathbf{R}}_2\,(\boldsymbol{\nu}\times)^2, \tag{105}$$

where the tensorial coefficients $\hat{\mathbf{R}}_1$, $\hat{\mathbf{R}}_2$ are defined as

$$\hat{\mathbf{R}}_1(\boldsymbol{\Phi}) := \boldsymbol{\Phi}^{-1}\,\sin\boldsymbol{\Phi}, \tag{106}$$

$$\hat{\mathbf{R}}_2(\boldsymbol{\Phi}) := \boldsymbol{\Phi}^{-2}\,(\mathbf{I} - \cos\boldsymbol{\Phi}). \tag{107}$$

These definitions entail the matricial form

$$\hat{\mathbf{R}}_k(\boldsymbol{\Phi}) = \begin{bmatrix} R_k(\varphi)\,\mathbf{I} & \tau\,U_k(\varphi)\,\mathbf{I} \\ \mathbf{O} & R_k(\varphi)\,\mathbf{I} \end{bmatrix}, \qquad k = 1, 2, \tag{108}$$

where the scalar coefficients U_1, U_2 are defined as[11]

$$U_1(\varphi) := \cos\varphi - R_1(\varphi), \tag{109}$$

$$U_2(\varphi) := R_1(\varphi) - 2\,R_2(\varphi). \tag{110}$$

Note that $U_k = \varphi\,R_k'$ for $k = 1, 2$. Eq. 105 is clearly the generalized Euler-Rodrigues formula, eq. 82, rewritten in terms of the generalized screw vector.

As in the case of rotation, we introduce the associated differential map of motion, denoting as $\mathbf{E}$ the differential tensor associated to $\mathbf{D}$,

$$\mathbf{E} = \mathrm{dexp}(\boldsymbol{\nu}\times). \tag{111}$$

Its 'finite form' formula reads

$$\mathbf{E} = \mathbf{I} + \hat{\mathbf{S}}_1\,(\boldsymbol{\nu}\times) + \hat{\mathbf{S}}_2\,(\boldsymbol{\nu}\times)^2, \tag{112}$$

where the tensorial coefficients $\hat{\mathbf{S}}_1$, $\hat{\mathbf{S}}_2$ are defined as

$$\hat{\mathbf{S}}_1(\boldsymbol{\Phi}) := \boldsymbol{\Phi}^{-2}\,(\mathbf{I} - \cos\boldsymbol{\Phi}), \tag{113}$$

$$\hat{\mathbf{S}}_2(\boldsymbol{\Phi}) := \boldsymbol{\Phi}^{-3}\,(\boldsymbol{\Phi} - \sin\boldsymbol{\Phi}). \tag{114}$$

Note that $\hat{\mathbf{S}}_1 \equiv \hat{\mathbf{R}}_2$, $\hat{\mathbf{S}}_2 = \boldsymbol{\Phi}^{-2}\,(\mathbf{I} - \hat{\mathbf{R}}_1)$. These definitions entail the matricial form

$$\hat{\mathbf{S}}_k(\boldsymbol{\Phi}) = \begin{bmatrix} S_k(\varphi)\,\mathbf{I} & \tau\,V_k(\varphi)\,\mathbf{I} \\ \mathbf{O} & S_k(\varphi)\,\mathbf{I} \end{bmatrix}. \qquad k = 1, 2, \tag{115}$$

where the scalar coefficients V_1, V_2 are defined as

$$V_1(\varphi) := R_1(\varphi) - 2\,S_1(\varphi), \tag{116}$$

$$V_2(\varphi) := S_1(\varphi) - 3\,S_2(\varphi). \tag{117}$$

Note that $V_k = \varphi\,S_k'$ for $k = 1, 2$ and $V_1 \equiv U_2$.

As for tensor $\mathbf{D}$, vector $\boldsymbol{\nu}$ is an eigenvector of tensor $\mathbf{D}$, and the corresponding eigenvalue is $+1$,

$$\mathbf{E}\,\boldsymbol{\nu} = \boldsymbol{\nu}, \tag{118}$$

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[11] Note that the second definition allows to write

$$\mathbf{S}^{-1} = \mathbf{I} - \frac{1}{2}\left(\mathbf{I} + \frac{U_2}{R_2}\,(\boldsymbol{\varphi}\times)\right)(\boldsymbol{\varphi}\times).$$

as further possible expression for eq. 44.

Note also that $\det(\mathbf{E}) = \det(\mathbf{S})^2$, so that $\mathbf{E}$ is singular at $\varphi = \pi$.

The associated differential map relates the derivative $\dot{\boldsymbol{\nu}}$ of the generalized screw vector with the generalized velocity $\mathbf{w}$. In fact, we get

$$\mathbf{w} = \operatorname{dexp}(\boldsymbol{\nu}\times)\,\dot{\boldsymbol{\nu}}, \tag{119}$$

as can be easily verified from eq. 94. Also, it relates the derivative $\dot{\boldsymbol{\nu}}$ of the generalized screw vector with the co-moved image of the generalized velocity $\overline{\mathbf{w}}$ as

$$\overline{\mathbf{w}} = \operatorname{dexp}(-\boldsymbol{\nu}\times)\,\dot{\boldsymbol{\nu}}. \tag{120}$$

The inverse of the associated differential map can be expressed in finite form as

$$\mathbf{E}^{-1} = \mathbf{I} - \frac{1}{2}\,(\boldsymbol{\nu}\times) + \boldsymbol{\Phi}^{-2}\left(\mathbf{I} - \frac{1}{2}\,\hat{\mathbf{R}}_2^{-1}\hat{\mathbf{R}}_1\right)(\boldsymbol{\nu}\times)^2. \tag{121}$$

Note that the coefficient of $(\boldsymbol{\varphi}\times)^2$ can be expressed in terms of the half screw magnitude, since $\hat{\mathbf{R}}_2^{-1}\hat{\mathbf{R}}_1/2 = \tan(\boldsymbol{\Phi}/2)^{-1}(\boldsymbol{\Phi}/2)$. Therefore an equivalent expression for $\mathbf{E}^{-1}$ reads

$$\mathbf{E}^{-1} = \mathbf{I} - \frac{1}{2}\,(\boldsymbol{\nu}\times) + \boldsymbol{\Phi}^{-2}\left(\mathbf{I} - \tan(\boldsymbol{\Phi}/2)^{-1}\boldsymbol{\Phi}/2\right)(\boldsymbol{\nu}\times)^2, \tag{122}$$

where $\tan(\bullet) := (1/i)\,(\exp(i\,\bullet) - \exp(-i\,\bullet))\,(\exp(i\,\bullet) + \exp(-i\,\bullet))^{-1}$, *i.e.* the standard definition extended to tensorial arguments.

3.4. **Vectorial Parameterization.** The vectorial parameterization of motion introduced here is an extension of the vectorial parameterization of rotation, consisting in a general class of techniques based on a minimal set of parameters. This set is represented by $(\mathbf{P}, \mathbf{h})$, where $\mathbf{P}$ is the *screw parameter magnitude* tensor defined as

$$\mathbf{P}(\boldsymbol{\Phi}) := \begin{bmatrix} p(\varphi)\,\mathbf{I} & \tau\,p'(\varphi)\,\mathbf{I} \\ \mathbf{O} & p(\varphi)\,\mathbf{I} \end{bmatrix} = \begin{bmatrix} p(\varphi)\,\mathbf{I} & \sigma(\tau,\varphi)\,\mathbf{I} \\ \mathbf{O} & p(\varphi)\,\mathbf{I} \end{bmatrix}, \tag{123}$$

where p is the generating function of the underlying vectorial parameterization of rotation, and σ is the *parameter translation* defined as

$$\sigma(\tau,\varphi) := \frac{\tau}{\mu(\varphi)}, \tag{124}$$

where μ is defined by eq. 64. Note that $\sigma(\tau, 0) = \kappa\,\tau$. Given the behavior of the generating function, eq. 46, we get

$$\lim_{\varphi\to 0}\left(\boldsymbol{\Phi}^{-1}\mathbf{P}(\boldsymbol{\Phi})\right) = \kappa\,\mathbf{I}, \tag{125}$$

where κ is the normalization factor of the underlying vectorial parameterization of rotation. We define the *generalized screw parameter vector* $\mathbf{q} := (\mathbf{r};\mathbf{p}) \in \mathbb{K}^6$ combining $\mathbf{P}$ and $\mathbf{h}$ as

$$\mathbf{q} = \mathbf{P}\,\mathbf{h}. \tag{126}$$

Therefore,

$$\mathbf{r} = p\,\mathbf{m} + \sigma\,\mathbf{e}. \tag{127}$$

We denote the *vectorial parameterization map of motion* as $\operatorname{mot} : \operatorname{sr}(\mathbb{K}^6) \longrightarrow \operatorname{SR}(\mathbb{K}^6)$. Thus, given a generalized screw parameter vector $\mathbf{q}$, we get a displacement tensor $\mathbf{D}$ by

$$\mathbf{D} = \operatorname{mot}(\mathbf{q}\times). \tag{128}$$

The explicit expression of the vectorial parameterization map is easily obtained from the generalized Euler-Rodrigues formula, eq. 82, as

$$\mathbf{D} = \mathbf{I} + \hat{\mathbf{P}}_1\,(\mathbf{q}\times) + \hat{\mathbf{P}}_2\,(\mathbf{q}\times)^2, \tag{129}$$

where the tensorial coefficients $\hat{\mathbf{P}}_1$, $\hat{\mathbf{P}}_2$ are defined as

$$\hat{\mathbf{P}}_1(\mathbf{\Phi}) := \mathbf{P}(\mathbf{\Phi})^{-1}\,\sin\mathbf{\Phi}, \tag{130}$$

$$\hat{\mathbf{P}}_2(\mathbf{\Phi}) := \mathbf{P}(\mathbf{\Phi})^{-2}\,(\mathbf{I} - \cos\mathbf{\Phi}). \tag{131}$$

From eq. 129 it is easy to obtain

$$\mathbf{r} = \mathrm{drot}(\mathbf{p}\times)^{-1}\mathbf{t}. \tag{132}$$

In view of the description of the composition of rigid displacements, as well as other formulæ, we introduce an alternative expression of the vectorial parameterization map as

$$\mathbf{D} = \mathbf{I} + \left(\mathbf{\Gamma} + \frac{1}{2}\,(\mathbf{N}\mathbf{q}\times)\right)\,(\mathbf{N}\mathbf{q}\times), \tag{133}$$

where the tensorial coefficients $\mathbf{\Gamma}$, $\mathbf{N}$, are defined as

$$\mathbf{\Gamma}(\mathbf{\Phi}) := \cos\,(\mathbf{\Phi}/2), \tag{134}$$

$$\mathbf{N}(\mathbf{\Phi}) := 2\,\mathbf{P}(\mathbf{\Phi})^{-1}\sin\,(\mathbf{\Phi}/2). \tag{135}$$

Note that eq. 133 purposely highlights the role of vector $(\mathbf{N}\mathbf{q})$. This will be also evident in the formulæ that deal with the composition of rigid displacements. Indeed, if $\mathbf{D} = \mathbf{D}_B\mathbf{D}_A$, the generalized screw parameter vector $\mathbf{q}$ corresponding to $\mathbf{D}$ is found in terms of $\mathbf{q}_A$, $\mathbf{q}_B$ corresponding to $\mathbf{D}_A$, $\mathbf{D}_B$, respectively, by

$$\mathbf{\Gamma} = \mathbf{\Gamma}_A\,\mathbf{\Gamma}_B - \frac{1}{4}\,(\mathbf{N}_B\mathbf{q}_B)\cdot(\mathbf{N}_A\mathbf{q}_A), \tag{136}$$

$$\mathbf{N}\,\mathbf{q} = \mathbf{\Gamma}_A\,(\mathbf{N}_B\mathbf{q}_B) + \mathbf{\Gamma}_B\,(\mathbf{N}_A\mathbf{q}_A) + \frac{1}{2}\,(\mathbf{N}_B\mathbf{q}_B)\times(\mathbf{N}_A\mathbf{q}_A). \tag{137}$$

An associated differential map dmot : $\mathrm{sr}(\mathbb{K}^6) \longrightarrow \mathrm{Lin}(\mathbb{K}^6)$ can be defined for the vectorial parameterization map such that, symbolically,

$$\mathrm{dmot}(\bullet) \overset{sym}{=} \frac{\mathrm{mot}(\bullet) - \mathbf{I}}{\bullet}, \tag{138}$$

and the following properties are satisfied:

$$\mathrm{mot}(\bullet) = \mathrm{dmot}(\bullet)\,\mathrm{dmot}(-\bullet)^{-1} = \mathrm{dmot}(-\bullet)^{-1}\mathrm{dmot}(\bullet), \tag{139}$$

$$\mathrm{mot}(\bullet) = \mathbf{I} + \bullet\;\mathrm{dmot}(\bullet) = \mathbf{I} + \mathrm{dmot}(\bullet)\,\bullet, \tag{140}$$

in complete analogy with the exponential map .

We denote as $\mathbf{\Theta}$ the differential tensor associated to $\mathbf{D}$,

$$\mathbf{\Theta} := \mathrm{dmot}(\mathbf{q}\times). \tag{141}$$

An explicit expression for the associated differential map is found as

$$\mathbf{\Theta} = \mathbf{M} + \hat{\mathbf{H}}_1\,(\mathbf{q}\times) + \hat{\mathbf{H}}_2\,(\mathbf{q}\times)^2. \tag{142}$$

The tensorial coefficients $\mathbf{M}$, $\hat{\mathbf{H}}_1$, $\hat{\mathbf{H}}_2$ are defined as

$$\mathbf{M}(\boldsymbol{\Phi}) := \begin{bmatrix} \mu(\varphi)\,\mathbf{I} & \tau\,\mu'(\varphi)\,\mathbf{I} \\ \mathbf{O} & \mu(\varphi)\,\mathbf{I} \end{bmatrix}, \tag{143}$$

$$\hat{\mathbf{H}}_1(\boldsymbol{\Phi}) := \mathbf{P}(\boldsymbol{\Phi})^{-2}(\mathbf{I} - \cos\boldsymbol{\Phi}), \tag{144}$$

$$\hat{\mathbf{H}}_2(\boldsymbol{\Phi}) := \mathbf{P}(\boldsymbol{\Phi})^{-3}\big(\mathbf{M}(\boldsymbol{\Phi})\,\mathbf{P}(\boldsymbol{\Phi}) - \sin(\boldsymbol{\Phi})\big), \tag{145}$$

where $\mu' := \mathrm{d}\mu/\mathrm{d}\varphi$. Note that $\hat{\mathbf{H}}_1 \equiv \hat{\mathbf{P}}_2$, $\hat{\mathbf{H}}_2 = \mathbf{P}(\boldsymbol{\Phi})^{-2}(\mathbf{M} - \hat{\mathbf{P}}_1)$.

An equivalent expression for $\boldsymbol{\Theta}$ is found as

$$\boldsymbol{\Theta} = \mathbf{M} + \left(\frac{1}{2}\,\mathbf{N} + (\mathbf{N}\mathbf{P})^{-2}(\mathbf{M} - \boldsymbol{\Gamma}\mathbf{N})\,(\mathbf{N}\mathbf{q}\times)\right)(\mathbf{N}\mathbf{q}\times). \tag{146}$$

Tensor $\mathbf{M}$ represents a sort of generalized tensorial eigenvalue of tensor $\boldsymbol{\Theta}$ corresponding to the generalized eigenvector $\mathbf{h}$,

$$\boldsymbol{\Theta}\,\mathbf{h} = \mathbf{M}\,\mathbf{h}, \tag{147}$$

and we get $\lim_{\varphi\to 0}\mathbf{M} = (1/\kappa)\,\mathbf{I}$. Note also that $\det(\boldsymbol{\Theta}) = \det(\mathbf{H})^2$, so that $\boldsymbol{\Theta}$ is singular at the values of φ that annihilate μ or ν, while for $\varphi = 0$ we get $\boldsymbol{\Theta} = (1/\kappa)\,\mathbf{I}$.

The associated differential map relates the derivative $\dot{\mathbf{q}}$ of the generalized screw parameter vector with the generalized velocity $\mathbf{w}$ and its co-moved image $\overline{\mathbf{w}}$:

$$\mathbf{w} = \mathrm{dmot}(\mathbf{q}\times)\,\dot{\mathbf{q}}, \tag{148}$$

$$\overline{\mathbf{w}} = \mathrm{dmot}(-\mathbf{q}\times)\,\dot{\mathbf{q}}. \tag{149}$$

The inverse of the associated differential tensor may be expressed as

$$\boldsymbol{\Theta}^{-1} = \mathbf{M}^{-1} - \frac{1}{2}\,(\mathbf{q}\times) + \mathbf{P}^{-2}\left(\mathbf{M}^{-1} - \frac{1}{2}\,\hat{\mathbf{P}}_2^{-1}\hat{\mathbf{P}}_1\right)(\mathbf{q}\times)^2. \tag{150}$$

An equivalent expression is given by

$$\boldsymbol{\Theta}^{-1} = \mathbf{M}^{-1} - \frac{1}{2}\,(\mathbf{q}\times) + \mathbf{P}^{-2}\left(\mathbf{M}^{-1} - \mathbf{N}^{-1}\boldsymbol{\Gamma}\right)(\mathbf{q}\times)^2. \tag{151}$$

We end this section remarking that all the formulæ derived for the vectorial parameterization of motion extend, on the one hand, the exponential parameterization of motion to the more general vectorial setting, and, on the other hand, the vectorial parameterization of rotation to complete rigid motion, under every respect.

Again, the only choice needed in order to obtain a specific parameterization technique is the actual form of $p(\varphi)$. In the following, we list as possible choices the generalizations of the techniques already cited for the rotation case:

- *Exponential map*

  $$\mathbf{P}(\boldsymbol{\Phi}) = \boldsymbol{\Phi}.$$

  This represents the simplest possible choice, being the exponential parameterization seen above: mot $\equiv$ exp.
- *Cayley/Gibbs/Rodrigues parameterization*

  $$\mathbf{P}(\boldsymbol{\Phi}) = 2\,\kappa\,\tan(\boldsymbol{\Phi}/2).$$

  Note that when $\kappa = 1/2$, mot $\equiv$ cay. This case was treated at length in Ref. [4].
- *Wiener/Milenkovic parameterization*

  $$\mathbf{P}(\boldsymbol{\Phi}) = 4\,\kappa\,\tan(\boldsymbol{\Phi}/4).$$

- *Linear parameterization*

$$\mathbf{P}(\mathbf{\Phi}) = \sin \mathbf{\Phi}.$$

- *Reduced Euler-Rodrigues parameterization*

$$\mathbf{P}(\mathbf{\Phi}) = 2\,\kappa\,\sin(\mathbf{\Phi}/2).$$

Further specific parameterizations of motion can be freely devised following a similar reasoning to that given in Ref. [3].

Appendix A. Complements on rotation

A.1. **Alternative expressions.** Euler-Rodrigues formula can also be written as

$$\mathbf{R} = \cos\varphi\,\mathbf{I} + \sin\varphi\,(\mathbf{e}\times) + (1-\cos\varphi)\,(\mathbf{e}\otimes\mathbf{e}), \tag{152}$$

since $(\bullet\times)^2 = \bullet\otimes\bullet - \|\bullet\|^2\mathbf{I}, \forall\bullet\in\mathbb{E}^3$. Therefore, in the vectorial parameterization, alternative expressions for $\mathbf{R}$, $\mathbf{H}$, and $\mathbf{H}^{-1}$ in terms of the dyadic tensors $\mathbf{p}\otimes\mathbf{p}$ and $(\nu\mathbf{p})\otimes(\nu\mathbf{p})$ instead of $(\mathbf{p}\times)^2$ read

$$\left\{\begin{array}{lcccc} \mathrm{sym}(\mathbf{R}) & = & \cos\varphi\,\mathbf{I} + P_2\,(\mathbf{p}\otimes\mathbf{p}) & = & \left(1 - \dfrac{1}{2}\,(\nu p)^2\right)\mathbf{I} + \dfrac{1}{2}\,(\nu\mathbf{p})\otimes(\nu\mathbf{p}) \\ \mathrm{skw}(\mathbf{R}) & = & P_1\,\mathbf{p}\times & = & \gamma\,(\nu\mathbf{p})\times \end{array}\right., \tag{153}$$

$$\left\{\begin{array}{lcccc} \mathrm{sym}(\mathbf{H}) & = & P_1\,\mathbf{I} + H_2\,(\mathbf{p}\otimes\mathbf{p}) & = & \gamma\nu\,\mathbf{I} + (\mu-\gamma\nu)\,\dfrac{(\nu\mathbf{p})\otimes(\nu\mathbf{p})}{(\nu p)^2} \\ \mathrm{skw}(\mathbf{H}) & = & H_1\,\mathbf{p}\times & = & \dfrac{\nu}{2}\,(\nu\mathbf{p})\times \end{array}\right., \tag{154}$$

$$\left\{\begin{array}{lcc} \mathrm{sym}(\mathbf{H}^{-1}) & = & \dfrac{\gamma}{\nu}\,\mathbf{I} + \left(\dfrac{1}{\mu} - \dfrac{\gamma}{\nu}\right)\dfrac{(\nu\mathbf{p})\otimes(\nu\mathbf{p})}{(\nu p)^2} \\ \mathrm{skw}(\mathbf{H}^{-1}) & = & -\dfrac{1}{2}\,\mathbf{p}\times. \end{array}\right.. \tag{155}$$

Other interesting formulæ refer to a multiplicative decomposition of $\mathbf{R}$, $\mathbf{H}$ through a common factor $\big(\mathbf{I} + (P_2/P_1)\,\mathbf{p}\times\big) = \big(\mathbf{I} + (1/2\gamma)\,(\nu\mathbf{p})\times\big)$. In fact, we may write

$$\begin{aligned} \mathbf{R} &= \left(\mathbf{I} + \frac{P_2}{P_1}\,\mathbf{p}\times\right)\left(\mathbf{I} + \frac{1}{2}\,(P_1\,\mathbf{I} + P_2\,\mathbf{p}\times)\,(\mathbf{p}\times)\right), \\ &= \left(\mathbf{I} + \frac{1}{2\gamma}\,(\nu\mathbf{p})\times\right)\left(\mathbf{I} + \frac{1}{2}\left(\gamma\,\mathbf{I} + \frac{1}{2}\,(\nu\mathbf{p})\times\right)(\nu\mathbf{p})\times\right), \end{aligned} \tag{156}$$

$$\begin{aligned} \mathbf{H} &= \left(\mathbf{I} + \frac{P_2}{P_1}\,\mathbf{p}\times\right)\big(\mathbf{I} + H_2\,(\mathbf{p}\times)^2\big) \\ &= \left(\mathbf{I} + \frac{1}{2\gamma}\,(\nu\mathbf{p})\times\right)\left(\mathbf{I} + \frac{\mu-\gamma\nu}{(\nu p)^2}\,\big((\nu\mathbf{p})\times\big)^2\right). \end{aligned} \tag{157}$$

Note that the decomposition for $\mathbf{R}$ may be interpreted as the composition of an 'average' rotation and an incremental (linearized) rotation, as

$$\mathbf{I} + \frac{1}{2}\left(\gamma\,\mathbf{I} + \frac{1}{2}\,(\nu\mathbf{p})\times\right)(\nu\mathbf{p})\times = \frac{1}{2}\,(\mathbf{R}+\mathbf{I}). \tag{158}$$

Also, it is easily shown that

$$\mathbf{I} + \frac{1}{2\gamma}\,(\nu\mathbf{p})\times = 2\,\left(\mathbf{R}^T + \mathbf{I}\right)^{-1}. \tag{159}$$

These formulæ may be interpreted in terms of the Cayley transform, $\mathbf{R} = \mathrm{cay}(\boldsymbol{\zeta}\times)$, so that

$$\frac{1}{2}(\mathbf{R} + \mathbf{I}) = (\mathbf{I} - \boldsymbol{\zeta}\times)^{-1}, \tag{160}$$

$$2\left(\mathbf{R}^T + \mathbf{I}\right) = (\mathbf{I} + \boldsymbol{\zeta}\times)^{-1}. \tag{161}$$

where $\boldsymbol{\zeta} := (\nu\mathbf{p})/(2\,\gamma) = \tan(\varphi/2)\,\mathbf{e}$ is the Cayley rotation vector (see Ref. [4]).

A.2. **Canonical representations.** It is always possible to define an orthonormal base $\mathcal{B} := \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ such that vector $\mathbf{e}$ coincides, say, with its third element, $\mathbf{e} \equiv \mathbf{b}_3$. The matrix $[\mathbf{R}]_\mathcal{B}$ formed by the scalar components of the tensor $\mathbf{R}$ with respect to the base $\mathcal{B}$ thus assumes the *canonical form*

$$[\mathbf{R}]_\mathcal{B} = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & +1 \end{bmatrix}. \tag{162}$$

In terms of the coefficients appearing in the vectorial parameterization formulæ we obtain

$$[\mathbf{R}]_\mathcal{B} = \begin{bmatrix} 1 - p^2 P_2 & -p\,P_1 & 0 \\ p\,P_1 & 1 - p^2 P_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - (\nu p)^2/2 & -\gamma\,(\nu p) & 0 \\ \gamma\,(\nu p) & 1 - (\nu p)^2/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{163}$$

Tensor $\mathbf{H}$ can be represented in canonical form by the matrix of its scalar components with respect to the base $\mathcal{B}$ as

$$\begin{aligned} [\mathbf{H}]_\mathcal{B} &= \begin{bmatrix} \sin\varphi/p & -(1-\cos\varphi)/p & 0 \\ (1-\cos\varphi)/p & \sin\varphi/p & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} H_2 & -p\,H_1 & 0 \\ p\,H_1 & H_2 & 0 \\ 0 & 0 & \mu \end{bmatrix} = \nu \begin{bmatrix} \gamma & -(\nu p)/2 & 0 \\ (\nu p)/2 & \gamma & 0 \\ 0 & 0 & \mu/\nu \end{bmatrix}. \end{aligned} \tag{164}$$

The canonical form of the matrix of the scalar components of tensor $\mathbf{H}^{-1}$ finds elegant expressions as

$$[\mathbf{H}^{-1}]_\mathcal{B} = \begin{bmatrix} \dfrac{p/2}{\tan(\varphi/2)} & p/2 & 0 \\ -p/2 & \dfrac{p/2}{\tan(\varphi/2)} & 0 \\ 0 & 0 & p' \end{bmatrix} = \begin{bmatrix} \gamma/\nu & -p/2 & 0 \\ p/2 & \gamma/\nu & 0 \\ 0 & 0 & 1/\mu \end{bmatrix}. \tag{165}$$

If the canonical base $\mathcal{B}$ is chosen as to have, say, its second element aligned with $\dot{\mathbf{e}}$, *i.e.* $\dot{\mathbf{e}}/\|\dot{\mathbf{e}}\| = \mathbf{b}_2$, we get the matrix $[\boldsymbol{\omega}]_\mathcal{B}$ of the scalar components of $\boldsymbol{\omega}$ with respect to the base $\mathcal{B}$ as

$$[\boldsymbol{\omega}]_\mathcal{B} = \begin{bmatrix} -\Omega\,(1-\cos\varphi) \\ \Omega\,\sin\varphi \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} -\Omega\,p^2 P_2 \\ \Omega\,p P_1 \\ \dot{p}/\mu \end{bmatrix} = \begin{bmatrix} -\Omega\,(\nu p)^2/2 \\ \Omega\,\gamma\,(\nu p) \\ \dot{p}/\mu \end{bmatrix}, \tag{166}$$

where $\Omega := \|\dot{\mathbf{e}}\|$.

Appendix B. Specific parameterizations

B.1. Some widely adopted techniques.

B.1.1. *Cayley/Gibbs/Rodrigues parameterization.* This parameterization is based on the choice

$$p(\varphi) = 2\,\kappa\,\tan(\varphi/2). \tag{167}$$

The expressions of the coefficients for this parameterization are

$$\mu = \frac{1}{\kappa}\cos^2(\varphi/2) = \frac{4\kappa}{4\,\kappa^2 + p^2}, \qquad \gamma = \sqrt{\kappa\,\mu}, \qquad \nu = \sqrt{\frac{\mu}{\kappa}}. \tag{168}$$

Therefore, the relevant formulæ for $\mathbf{R}$, $\mathbf{H}$, $\mathbf{H}^{-1}$ read

$$\mathbf{R} = \mathbf{I} + \mu\left(\mathbf{I} + \frac{1}{2\,\kappa}(\mathbf{p}\times)\right)(\mathbf{p}\times), \tag{169}$$

$$\mathbf{H} = \mu\left(\mathbf{I} + \frac{1}{2\,\kappa}(\mathbf{p}\times)\right), \tag{170}$$

$$\mathbf{H}^{-1} = \frac{1}{\mu}\mathbf{I} - \frac{1}{2}\left(\mathbf{I} - \frac{1}{2\,\kappa}(\mathbf{p}\times)\right)(\mathbf{p}\times). \tag{171}$$

Note that the expression for $\mathbf{H}$ does not feature a term in $(\mathbf{p}\times)^2$ since $H_2 = 0$. Also, $P_2 = P_1/(2\,\kappa)$, $\gamma = \kappa\,\nu$, $\det(\mathbf{H}) = \mu^2/\kappa$, $\mathbf{H}^{-1} = (\mathbf{I} + \mathbf{R}^T)/(2\mu)$.

Composition of rotation, $\mathbf{R} = \mathbf{R}_B\mathbf{R}_A$, results in an analytical formula, known as *Rodrigues' formula* (Ref. [9]), a remarkable property peculiar to this parameterization only:

$$\mathbf{p} = \frac{1}{1 - \dfrac{1}{4\,\kappa^2}\,\mathbf{p}_B\cdot\mathbf{p}_A}\left(\mathbf{p}_A + \mathbf{p}_B + \frac{1}{2\,\kappa}\,\mathbf{p}_B\times\mathbf{p}_A\right). \tag{172}$$

Switching to complete rigid motion, we get

$$\sigma = \tau\,\frac{\kappa}{\cos^2(\varphi/2)} = \tau\,\frac{4\,\kappa^2 + p^2}{4\,\kappa}, \tag{173}$$

and the relevant formulæ for $\mathbf{D}$, $\mathbf{\Theta}$, $\mathbf{\Theta}^{-1}$ read

$$\mathbf{D} = \mathbf{I} + \mathbf{M}\left(\mathbf{I} + \frac{1}{2\,\kappa}(\mathbf{q}\times)\right)(\mathbf{q}\times), \tag{174}$$

$$\mathbf{\Theta} = \mathbf{M}\left(\mathbf{I} + \frac{1}{2\,\kappa}(\mathbf{q}\times)\right), \tag{175}$$

$$\mathbf{\Theta}^{-1} = \mathbf{M}^{-1} - \frac{1}{2}\left(\mathbf{I} - \frac{1}{2\,\kappa}(\mathbf{q}\times)\right)(\mathbf{q}\times). \tag{176}$$

Note that $\hat{\mathbf{H}}_2 = \mathbf{O}$, $\hat{\mathbf{P}}_2 = \hat{\mathbf{P}}_1/(2\,\kappa)$, $\mathbf{\Gamma} = \kappa\,\mathbf{N}$.

The composition of rigid displacements, $\mathbf{D} = \mathbf{D}_B\mathbf{D}_A$, results in the *generalized Rodrigues formula*:

$$\mathbf{q} = \frac{1}{1 - \dfrac{1}{4\,\kappa^2}\,\mathbf{q}_B\cdot\mathbf{q}_A}\left(\mathbf{q}_A + \mathbf{q}_B + \frac{1}{2\,\kappa}\,\mathbf{q}_B\times\mathbf{q}_A\right). \tag{177}$$

B.1.2. *Wiener/Milenkovic parameterization.* This parameterization is based on the choice

$$p(\varphi) = 4\,\kappa\,\tan(\varphi/4). \tag{178}$$

It is also know as the *conformal rotation vector parameterization* (*e.g.*, Ref. [7]) and as the *modified Rodrigues parameterization* (*e.g.*, Ref. [14]). The expressions of the coefficients for this parameterization are

$$\mu = \frac{1}{\kappa}\cos^2(\varphi/4) = \frac{16\kappa}{16\,\kappa^2 + p^2}, \qquad \gamma = 2\,\kappa\,\mu - 1, \qquad \nu = \mu. \tag{179}$$

Therefore, the relevant formulæ for $\mathbf{R}$, $\mathbf{H}$, $\mathbf{H}^{-1}$ read

$$\mathbf{R} = \mathbf{I} + \mu\left(\frac{16\,\kappa^2 - p^2}{16\,\kappa^2 + p^2}\,\mathbf{I} + \frac{\mu}{2}\,(\mathbf{p}\times)\right)(\mathbf{p}\times), \tag{180}$$

$$\mathbf{H} = \mu\,\mathbf{I} + \frac{\mu^2}{2}\left(\mathbf{I} + \frac{1}{4\,\kappa}\,(\mathbf{p}\times)\right)(\mathbf{p}\times), \tag{181}$$

$$\mathbf{H}^{-1} = \frac{1}{\mu}\,\mathbf{I} - \frac{1}{2}\left(\mathbf{I} - \frac{1}{4\,\kappa}\,(\mathbf{p}\times)\right)(\mathbf{p}\times). \tag{182}$$

Note that $\det(\mathbf{H}) = \mu^3$, $\mathbf{H}^{-1} = \mathbf{H}^T/\mu^2$. Also, since $\mathbf{R} = \mathbf{H}\,\mathbf{H}^{-T}$, we get $\mathbf{R} = (\mathbf{H}/\mu)^2$, or $\sqrt{\mathbf{R}} = \mathbf{H}/\mu$.

Switching to complete rigid motion, we get

$$\sigma = \tau\,\frac{\kappa}{\cos^2(\varphi/4)} = \tau\,\frac{16\,\kappa^2 + p^2}{16\,\kappa}, \tag{183}$$

and the relevant formulæ for $\mathbf{D}$, $\boldsymbol{\Theta}$, $\boldsymbol{\Theta}^{-1}$ read

$$\mathbf{D} = \mathbf{I} + \mathbf{M}\left(\left(16\,\kappa^2\mathbf{I} - \mathbf{P}^2\right)\left(16\,\kappa^2\mathbf{I} + \mathbf{P}^2\right)^{-1} + \frac{1}{2}\,\mathbf{M}\,(\mathbf{q}\times)\right)(\mathbf{q}\times), \tag{184}$$

$$\boldsymbol{\Theta} = \mathbf{M} + \frac{1}{2}\,\mathbf{M}^2\left(\mathbf{I} + \frac{1}{4\,\kappa}\,(\mathbf{q}\times)\right)(\mathbf{q}\times), \tag{185}$$

$$\boldsymbol{\Theta}^{-1} = \mathbf{M}^{-1} - \frac{1}{2}\left(\mathbf{I} - \frac{1}{4\,\kappa}\,(\mathbf{q}\times)\right)(\mathbf{q}\times). \tag{186}$$

B.1.3. *Linear parameterization.* This parameterization is based on the choice

$$p(\varphi) = \sin\varphi. \tag{187}$$

The expressions of the coefficients for this parameterization are

$$\mu = \frac{1}{\cos\varphi} = \frac{1}{\sqrt{1 - p^2}}, \qquad \gamma = \sqrt{\frac{1+\mu}{2\mu}}, \qquad \nu = \sqrt{\frac{2\mu}{1+\mu}}. \tag{188}$$

Therefore, the relevant formulæ for $\mathbf{R}$, $\mathbf{H}$, $\mathbf{H}^{-1}$ read

$$\mathbf{R} = \mathbf{I} + \left(\mathbf{I} + \frac{\mu}{1+\mu}\,(\mathbf{p}\times)\right)(\mathbf{p}\times), \tag{189}$$

$$\mathbf{H} = \mu\,\mathbf{I} + \frac{\mu}{1+\mu}\left(\mathbf{I} + \mu\,(\mathbf{p}\times)\right)(\mathbf{p}\times), \tag{190}$$

$$\mathbf{H}^{-1} = \frac{1}{\mu}\,\mathbf{I} - \frac{1}{2}\left(\mathbf{I} + \frac{\mu}{1+\mu}\,(\mathbf{p}\times)\right)(\mathbf{p}\times). \tag{191}$$

Note that $P_1 = 1$, $\nu = 1/\gamma$, $\det(\mathbf{H}) = 2\mu^2/(1+\mu)$.

Switching to complete rigid motion, we get

$$\sigma = \tau \cos\varphi = \tau \sqrt{1-p^2}, \tag{192}$$

and the relevant formulæ for $\mathbf{D}$, $\mathbf{\Theta}$, $\mathbf{\Theta}^{-1}$ read

$$\mathbf{D} = \mathbf{I} + \big(\mathbf{I} + \mathbf{M}\,(\mathbf{I}+\mathbf{M})^{-1}(\mathbf{q}\times)\big)\ (\mathbf{q}\times), \tag{193}$$

$$\mathbf{\Theta} = \mathbf{M} + \mathbf{M}\,(\mathbf{I}+\mathbf{M})^{-1}\big(\mathbf{I} + \mathbf{M}\,(\mathbf{q}\times)\big)\,(\mathbf{q}\times), \tag{194}$$

$$\mathbf{\Theta}^{-1} = \mathbf{M}^{-1} - \frac{1}{2}\ \big(\mathbf{I} + \mathbf{M}\,(\mathbf{I}+\mathbf{M})^{-1}(\mathbf{q}\times)\big)\ (\mathbf{q}\times). \tag{195}$$

Note that $\hat{\mathbf{P}}_1 = \mathbf{I}$, $\mathbf{N} = \mathbf{\Gamma}^{-1}$.

B.1.4. *Reduced Euler-Rodrigues parameterization.* This parameterization is based on the choice

$$p(\varphi) = 2\,\kappa\,\sin(\varphi/2). \tag{196}$$

The expressions of the coefficients for this parameterization are

$$\mu = \frac{1/\kappa}{\cos(\varphi/2)} = \frac{2}{\sqrt{4\,\kappa^2 - p^2}}, \qquad \gamma = \frac{1}{\kappa\,\mu}, \qquad \nu = \frac{1}{\kappa}. \tag{197}$$

Therefore, the relevant formulæ for $\mathbf{R}$, $\mathbf{H}$, $\mathbf{H}^{-1}$ read

$$\mathbf{R} = \mathbf{I} + \frac{1}{\kappa^2}\left(\frac{1}{\mu}\mathbf{I} + \frac{1}{2}\,(\mathbf{p}\times)\right)(\mathbf{p}\times), \tag{198}$$

$$\mathbf{H} = \mu\,\mathbf{I} + \frac{\mu}{2}\,\frac{1}{\kappa^2}\left(\frac{1}{\mu}\mathbf{I} + \frac{1}{2}\,(\mathbf{p}\times)\right)(\mathbf{p}\times), \tag{199}$$

$$\mathbf{H}^{-1} = \frac{1}{\mu}\mathbf{I} - \frac{1}{2}\,(\mathbf{p}\times). \tag{200}$$

Note that $\det(\mathbf{H}) = \mu/\kappa^2$, $\mathbf{H} = (\mu/2)\,(\mathbf{I}+\mathbf{R})$.

Switching to complete rigid motion, we get

$$\sigma = \tau\,\kappa\,\cos(\varphi/2) = \tau\,\frac{\sqrt{4\,\kappa^2 - p^2}}{2}, \tag{201}$$

and the relevant formulæ for $\mathbf{D}$, $\mathbf{\Theta}$, $\mathbf{\Theta}^{-1}$ read

$$\mathbf{D} = \mathbf{I} + \frac{1}{\kappa^2}\left(\mathbf{M}^{-1} + \frac{1}{2}\,(\mathbf{q}\times)\right)(\mathbf{q}\times), \tag{202}$$

$$\mathbf{\Theta} = \mathbf{M} + \frac{1}{2\,\kappa^2}\,\mathbf{M}\left(\mathbf{M}^{-1} + \frac{1}{2}\,(\mathbf{q}\times)\right)(\mathbf{q}\times), \tag{203}$$

$$\mathbf{\Theta}^{-1} = \mathbf{M}^{-1} - \frac{1}{2}\,(\mathbf{q}\times). \tag{204}$$

Note that $\mathbf{\Theta} = \mathbf{M}\,(\mathbf{I}+\mathbf{D})/2$.

B.2. **The tangent family.** Known techniques such as the Cayley/Gibbs/Rodrigues parameterization and the Wiener/Milenkovic parameterization fall into a special subclass of the vectorial parameterization introduced in Ref. [3], termed the *tangent family.* This subclass features generating functions of the form

$$p(\varphi) = m\,\kappa\,\tan(\varphi/m), \tag{205}$$

where $m \in \mathbb{N}$.[12] The corresponding values for μ are given by

$$\mu(\varphi) = \frac{1}{\kappa}\cos^2(\varphi/m). \tag{206}$$

In terms of p we get

$$\mu(p) = \frac{m^2\kappa}{m^2\kappa^2 + p^2}. \tag{207}$$

In Ref. [14], the $m = 6$ and $m = 8$ members of this subclass were defined. It is interesting to investigate the relation between the Cayley transform (see *e.g.* Ref. [4]) and its higher order generalizations (see Refs. [3, 14]. In fact, the even integer divisor, non-normalized subclass of the tangent family, *i.e.* when

$$p(\varphi) = \tan\big(\varphi/(2m)\big), \tag{208}$$

can be expressed by the formula

$$\mathrm{rot}(\mathbf{p}\times) = \mathrm{cay}_{(m)}(\mathbf{p}\times), \tag{209}$$

where $\mathrm{cay}_{(m)}(\bullet) := (\mathbf{I} + \bullet)^m(\mathbf{I} - \bullet)^{-m}$.

B.2.1. *A new parameterization.* To the author's knowledge, the $m = 1$ member of this class is unknown in the literature. In the case $\kappa = 1$, the coefficients of the parameterization read

$$\mu = \cos^2\varphi = \frac{1}{1+p^2}, \qquad \gamma = \sqrt{\frac{1+\sqrt{\mu}}{2}}, \qquad \nu = \sqrt{\frac{2\,\mu}{1+\sqrt{\mu}}}. \tag{210}$$

Therefore, the relevant formulæ for $\mathbf{R}$, $\mathbf{H}$, $\mathbf{H}^{-1}$ read

$$\mathbf{R} = \mathbf{I} + \sqrt{\mu}\left(\mathbf{I} + \frac{\sqrt{\mu}}{1+\sqrt{\mu}}(\mathbf{p}\times)\right)(\mathbf{p}\times), \tag{211}$$

$$\mathbf{H} = \mu\,\mathbf{I} + \frac{\mu}{1+\sqrt{\mu}}\big(\mathbf{I} - \sqrt{\mu}\,(\mathbf{p}\times)\big)(\mathbf{p}\times), \tag{212}$$

$$\mathbf{H}^{-1} = \frac{1}{\mu}\mathbf{I} - \frac{1}{2}\left(\mathbf{I} - \frac{2-\sqrt{\mu}}{1-\sqrt{\mu}}(\mathbf{p}\times)\right)(\mathbf{p}\times). \tag{213}$$

We leave the generalization to complete rigid motion to the reader.

B.3. **The sine family.** Known techniques such as the linear parameterization and the reduced Euler-Rodrigues parameterization fall into a special subclass of the vectorial parameterization introduced in Ref. [3], termed the *sine family.* This subclass features generating functions of the form

$$p(\varphi) = m\,\kappa\,\sin(\varphi/m), \tag{214}$$

where $m \in \mathbb{N}$.[13] The corresponding values for μ are given by

$$\mu(\varphi) = \frac{1}{\kappa}\frac{1}{\cos(\varphi/m)}. \tag{215}$$

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[12]Clearly, the Cayley/Gibbs/Rodrigues parameterization and the Wiener/Milenkovic parameterization correspond to the $m = 2$ and $m = 4$ elements of this subclass, respectively.

[13]Clearly, the linear parameterization and the reduced Euler-Rodrigues parameterization correspond to the $m = 1$ and $m = 2$ elements of this subclass, respectively.

In terms of p we get

$$\mu(p) = \frac{m}{\sqrt{m^2\kappa^2 - p^2}}. \tag{216}$$

B.3.1. *A new parameterization.* The $m = 4$ member of this class was presented in Ref. [3] as an example of application of the vectorial parameterization theory (the case of the rescaling procedure was used as a motivation to its definition). In the case $\kappa = 1$, the coefficients of the parameterization read

$$\mu = \frac{1/\kappa}{\cos(\varphi/4)} = \frac{4}{\sqrt{16\,\kappa^2 - p^2}}, \qquad \gamma = \frac{2 - \kappa^2\mu^2}{\kappa^2\mu^2}, \qquad \nu = \frac{1}{\kappa^2\mu}. \tag{217}$$

Therefore, the relevant formulæ for $\mathbf{R}$, $\mathbf{H}$, $\mathbf{H}^{-1}$ read

$$\mathbf{R} = \mathbf{I} + \frac{1}{(\kappa^2\mu)^2}\left(\frac{2 - \kappa^2\mu^2}{\mu}\,\mathbf{I} + \frac{1}{2}\,(\mathbf{p}\times)\right)(\mathbf{p}\times), \tag{218}$$

$$\mathbf{H} = \mu\,\mathbf{I} + \frac{1}{2}\,\frac{1}{\kappa^2\mu}\left(\frac{1}{\kappa^2\mu}\,\mathbf{I} + \frac{2 + \kappa^2\mu^2}{8\,\kappa^2}\,(\mathbf{p}\times)\right)(\mathbf{p}\times), \tag{219}$$

$$\mathbf{H}^{-1} = \frac{1}{\mu}\,\mathbf{I} - \frac{1}{2}\left(\mathbf{I} - \frac{\mu}{8}\,(\mathbf{p}\times)\right)(\mathbf{p}\times). \tag{220}$$

In this case also, we leave the generalization to complete rigid motion to the reader.

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