# Mathematics for Computer Science: Category Theory Tutorial Sheet

September 2025

1. Do sets and **injective** maps form a category? What about sets and strict inclusions?

2. Consider the set of **bracketed** words on some alphabet $\{e, a_1, a_2, \cdots a_n\}$ where, for any word $w$, we have that $we = ew = w$. Define

$$\mathrm{Hom}(w_1, w_2) = \{u \mid (w_1)(u) = w_2\}$$

Define $u \circ v = (v)(u)$ Does this form a category? What about **unbracketed** words?

3. A **unital monoid** is a set $X$ equipped with a binary operation $X \times X \to X$ that is associative and which has an identity element.

   (a) Show that a vector space $V$ and the map $+ : V \times V \to V$ is a monoid. What is its identity element?

   (b) Show that a vector space $V$ and all linear maps $V \to V$ form a 1-object category.

   (c) Show that there is a one-to-one correspondence between unital monoids and categories with one element.

4. A morphism $f \colon X \to Y$ in a category $\mathcal{C}$ is called **constant** if

$$f \circ a = f \circ b$$

for all morphisms $a, b \colon A \to X$. Constant morphisms are often what we expect, but not always.

   (a) Show that constant morphisms in **Set** are precisely the constant maps, that is, maps $f \colon X \to Y$ with $f(u) = f(v)$ for all $u, v \in X$.

(b) Show that if $Y$ has at least one element, then $f\colon X \to Y$ is constant if and only if there is an element $a \in Y$ with $f(u) = a$ for all $u \in X$.

(c) Show that there is exactly one constant morphism between objects in $\mathbf{Set}_\bullet$.

5. Is the forgetful functor $\mathbf{Vect} \to \mathbf{Set}$ **faithful**? Can you obtain every map of sets this way?

6. Let $\mathcal{C}$ be a category. An **isomorphism** $f : A \to B$ is a morphism such that there exists a morphism $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. Show that functors preserve isomorphisms.

7. Let $X$ be a set. We denote by $\mathcal{P}(X)$ the **powerset** of $X$, that is, the set of all subsets of $X$. Analogously, we denote by $\mathcal{P}_f(X)$ the **finite powerset**, given by

$$\mathcal{P}_f(X) = \{\, U \subseteq X \mid U \text{ finite} \,\}.$$

Show that both $\mathcal{P}$ and $\mathcal{P}_f$ give rise to functors

$$\mathbf{Set} \to \mathbf{Set}.$$

8. What linear transformations are of the form $F(f)$ for $f : X \to Y$ for the free functor $F : \mathsf{Set} \to \mathsf{Vect}$. Hint: Express it in terms of matrices.