On the divisibility of sums of Fibonacci numbers

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Abstract

We show that for infinitely many odd integers n, the sum of the first n Fibonacci numbers is divisible by n. This resolves a conjecture of Fatehizadeh and Yaqubi.

1 Introduction

In a 2022 paper from the *Journal of Integer Sequences* [1], the authors propose the following conjecture on the divisibility of sums of Fibonacci numbers by their index.

Conjecture 1.1. There are infinitely many odd integers n that divide the sum of the first n Fibonacci numbers.

The main purpose of the current note is to provide a brief construction (Theorem 3.3) resolving this conjecture. Our main theorem states

Theorem 1.2. *Suppose that $n = 2p$ or $n = 4p$ where p is a prime such that*

$$p \equiv 2 \mod 3 \qquad p \equiv \pm 2 \mod 5.$$

Then

$$F_n \equiv 1 \mod 2 \qquad \sum_{i=1}^{F_n} F_i \equiv 0 \mod F_n.$$

We also mildly strengthen and reprove several results of [1].

Our proof is based on studying those indices n for which F_n divides $\sum_{i=1}^{n} F_i$; we refer to the resulting sequence as the *self-summable Fibonacci numbers.* . To our knowledge, this sequence has not previously appeared in the literature or in the Online Encyclopedia of Integer Sequence.

We comment briefly on some related work. There is the aforementioned [1, 9], which resolved the problem in the even case, by proving that

$$3 \cdot 2^{n+3} \mid \sum_{i=1}^{3\cdot 2^{n+3}} F_i.$$

A similar sequence, consisting of those integers n such $n \mid F_n$ has been considered by several authors [6, 7, 3] and seems to be called the *self-Fibonacci numbers* and appears as sequence A023172 in the OEIS.

Notation and conventions

The n^{th} Fibonacci number is denoted by F_n where Fibonacci sequence is taken to start at 1, ie. $F_1 = 1, F_2 = 1$. The Pisano period of n is denoted $\pi(n)$. The sum of the first n Fibonacci numbers is denoted S_n, ie.

$$S_n = \sum_{i=1}^{n} F_i$$

This is a short note and we do not intend to overburden it with recollections, we therefore refer to the reference [2] for basic information about the Fibonacci sequence.

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2 Sums of Fibonacci numbers

In this section, we recall some basic facts about sums of Fibonacci numbers and perform some elementary computations. In what follows, we have the following immediate proposition which strengthens [1, Theorem 7].

Proposition 2.1. *Let $n \in \mathbb{N}$. Then*

$$S_n = F_{n+2} - 1.$$

Proof. The proof is a straightforward induction. For the base case,

$$S_1 = 1 = 2 - 1 = F_3 - 1.$$

Then by induction, suppose

$$S_n = F_{n+2} - 1.$$

We have that

$$S_{n+1} = S_n + F_{n+1} = F_{n+2} + F_{n+1} - 1 = F_{n+3} - 1$$

as desired, where we have used $F_{n+2} + F_{n+1} = F_{n+3}$. □

We can now prove a stronger version [1, Theorem 12] with a shorter proof using the Binet formula.

Theorem 2.2. *Let $p > 5$ be an odd prime number:*

$$S_p = \begin{cases} 3 \mod p & \text{if } p = 1,4 \mod 5 \\ 1 \mod p & \text{if } p = 2,3 \mod 5 \end{cases}$$

In particular, S_p is not divisible by p.

Proof. Recall the Binet formula which states that

$$F_n = \frac{\phi^n - \overline{\phi}^n}{\sqrt{5}}$$

where

$$\phi = \frac{1+\sqrt{5}}{2}, \qquad \overline{\phi} = \frac{1-\sqrt{5}}{2}.$$

Now, rearranging

$$F_{p+2} = \frac{\phi^{p+2} - \overline{\phi}^{p+2}}{\sqrt{5}}$$

one can directly compute F_{n+2} as

$$2^{p+1} F_{p+2} = \sum_{k=0}^{\frac{p+1}{2}} \binom{p+2}{2k+1} 5^k. \tag{1}$$

It follows from Fermat's little theorem that

$$2^{p+1} = 4 \mod p.$$

The binomial coefficients are divisible by p except for $k = 0, \frac{p-1}{2}, \frac{p-1}{2}$. Therefore one has

$$4F_{p+2} = \binom{p+2}{1} 5 + \binom{p+2}{p} 5^{\frac{p-1}{2}} + \binom{p+2}{p+2} 5^{\frac{p+1}{2}} \mod p$$

Finally, this simplifies to

$$4F_{p+2} = 10 + \left(\frac{5}{p}\right) + 5\left(\frac{5}{p}\right)$$

using the identity

$$5^{\frac{p-1}{2}} = \left(\frac{5}{p}\right)$$

where $\left(\frac{5}{p}\right)$ is the quadratic residue of 5 modulo p. Finally we know that

$$\left(\frac{5}{p}\right) = \begin{cases} 1 \mod p & \text{if } p = 1,4 \mod 5 \\ -1 \mod p & \text{if } p = 2,3 \mod 5 \end{cases}$$

The conclusion follows applying Proposition 2.1 and computing $F_{n+2} - 1 = S_n$. □

The remaining cases can be computed by hand

$$S_3 = 1 \mod 3, \qquad S_5 = 2 \mod 5.$$

Remark 2.3. An obvious approach to Conjecture 1.1 is to attempt to extend the strategy used for the proof of Theorem 2.2 to semiprime numbers $n = pq$, via the theorem of Lucas on prime divisors of binomial coefficents [4]. However, this strategy rapidly becomes combinatorially intractable. One may also attempt to approach this problem via Pisano periods of p and q and the work of [8]. This succeeds in producing sufficient (but not necessary) conditions on such primes p, q in modular arithmetic but it is not clear, with the modern tools of number theory, how to prove the infinitude of such solutions. Solving such linear Diophantine equations with prime solutions may become more tractable in the future thanks to recent advances in the field [10].

3 Self-summation Fibonacci numbers

In this section we prove Conjecture 1.1. We shall prove this by producing an explicit sequence of such odd numbers.

Definition 3.1. We say that k is an *self-summable Fibonacci number* if $F_k \mid \sum_{i=1}^{F_n} F_i$.

Remark 3.2. Clearly an alternative definition is the k such that $F_k | F_{F_k+2} - 1$. We note that the self-summable Fibonacci numbers is clearly precisely the indexes of the subsequence of A124456 of the OEIS [5] consisting of Fibonacci numbers. The first terms are

$$1, 2, 3, 12, 24, 34, 36, 46, 48, 60, 68, 72, 92, 94, 96, 106.$$

For some of the terms k of this sequence the corresponding Fibonacci number F_k is odd. The first such terms are

$$1, 2, 34, 46, 68, 92, 94, 106, 166, 188, 212, 214, 226, 274$$

We now show that one can explicitly describe a subsequence of the self-summable Fibonacci numbers where F_n is odd.

Theorem 3.3. *Suppose that $n = 2p$ or $n = 4p$, where p is a prime such that*

$$p \equiv 2 \mod 3 \quad \textit{and} \quad p \equiv \pm 2 \mod 5.$$

Then

$$F_n \equiv 1 \mod 2 \quad \textit{and} \quad F_{F_n+2} - 1 \equiv 0 \mod F_n.$$

The strategy behind this proof is essentially repeated use of the periodicity of Fibonacci numbers mod n, along with some explicit computation of relevant Pisano periods.

To prove this result, we first establish the following helpful proposition about the Pisano period of Fibonacci numbers, which may be of independent interest. Although this result seems likely to exist in the literature, we were unable to locate a reference, and the associated sequence does not appear in the OEIS.

Proposition 3.4. *Suppose that $n > 1$ and $2 \mid n$. Then $\pi(F_n) \mid 2n$.*

Proof. The fact that the Fibonacci numbers form a strong divisibility sequence and that $n \mid 2n$ implies that $F_n \mid F_{2n}$. It therefore suffices to establish that

$$F_n \mid F_{2n+1} - 1.$$

By the addition rule for Fibonacci numbers, we can establish

$$F_{n+(n+1)} = F_n^2 + F_{n+1}^2.$$

Therefore, it suffices to show that

$$F_n \mid F_{n+1}^2 - 1.$$

By the Catalan identity

$$F_{n+1}^2 - F_n F_{n-1} = (-1)^n,$$

we get

$$F_{n+1}^2 \equiv (-1)^n \mod F_n.$$

When n is even, $(-1)^n = 1$, so:

$$F_{n+1}^2 \equiv 1 \mod F_n \quad \Rightarrow \quad F_n \mid (F_{n+1}^2 - 1).$$

This completes the proof. □

Remark 3.5.

The following immediate corollary of Proposition 3.4 is likely well-known to experts but we could not locate it in the literature.

Corollary 3.6. *One has*

$$\lim_{N \to \infty} \min\left\{\frac{\pi(n)}{n} : n \in \mathbb{N}, n < N\right\} = 0$$

Proof. Clearly

$$\min\left\{\frac{\pi(n)}{n} : n \in \mathbb{N}, n < F_M + 1\right\} < \frac{\pi(F_M)}{F_M} = \frac{2M}{F_M}.$$

and the right hand side of the inequality goes to 0 as M goes to infinity. □

We are now ready to prove our main result.

Proof of Theorem 3.3. The conclusion that $F_n = 1$ is odd follows inductively from the fact that $\pi(2) = 3$ with the base case being that $F_2 = 1$ is odd.

To prove that $F_{F_n+2} - 1 = 0 \mod F_n$, we shall use the periodicity of the Fibonacci numbers mod F_n. We consider first the case that $n = 2p$. This implies that

$$F_{F_n+2} \equiv F_k \mod F_n$$

for any k such that

$$F_n + 2 \equiv k \mod \pi(F_n).$$

In particular, as $\pi(F_n) \mid 2n$ by Proposition 3.4, one may choose any k such that

$$F_n + 2 \equiv k \mod 2n.$$

We are interested in precisely two cases: when $k = 1, 2$, as then one has

$$F_{F_n+2} - 1 \equiv F_k - 1 \equiv 1 - 1 \equiv 0 \mod F_n.$$

as $F_1 = F_2 = 1$. It therefore suffices to establish that

$$F_n \equiv -1, 0 \mod 2n. \tag{2}$$

when n is of the form given in the statement. Note that, in this case $\pi(2n) = \pi(4p)$ and

$$\pi(4p) \mid \mathrm{lcm}(6, \pi(p))$$

by the multiplicativity of $\pi(-)$ and the fact $\pi(4) = 6$, where lcm is the least common multiple of its arguments. Next, we apply [8, Theorem 7], which states that when $p \equiv \pm 2 \mod 5$, one has

$$\pi(p) \mid 2(p+1).$$

The integer $p+1$ is divisible by 2 since it is a prime and by 3 since $p \equiv 2 \mod 3$. We can conclude that

$$\pi(2n) \mid 2(p+1).$$

It therefore follows from the periodicity of the Fibonacci sequence mod $2n$ that

$$F_n \equiv F_{k'} \mod 2n$$

for any $k' \equiv n \mod 6(p+1)$. Now

$$n \equiv 2p \equiv -2 \mod 2(p+1).$$

Finally, as $F_{-1} \equiv 1 \mod u$ for all $u \in \mathbb{N}$, we have that

$$F_{-2} \equiv -1 \mod 2n.$$

The conclusion follows from Equation (2). A similar argument goes through for $n = 4p$. □

Corollary 3.7. *There are infinitely many odd integers n that divide the sum of the first n Fibonacci numbers.*

Proof. By Dirichlet's theorem on primes in arithmetic progressions, there are infinitely many prime numbers satisfying the conditions of Theorem 2.2. The conclusion follows from Proposition 2.1. □

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