Determinant Massey Products

Algebra and Geometry seminar
Stockholm University

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Talk plan

1. **Motivation**
2. **Massey products for dgas**
3. **Massey products for cdgas**
4. **The commutative and associative quasi-isomorphism problem revisited**

Motivation

Weak equivalence and formality

Definition

Let $\mathscr{P}$ be an operad in dg-R-modules. We say two $\mathscr{P}$-algebras A, B are **weakly equivalent** or **quasi-isomorphic** as $\mathscr{P}$-algebras if there is a zig-zag

$$A \xrightarrow{\sim} C_1 \xleftarrow{\sim} C_2 \xrightarrow{\sim} \cdots \xleftarrow{\sim} C_n \xrightarrow{\sim} B$$

where C_i are $\mathscr{P}$-algebras and the maps are $\mathscr{P}$-algebra maps that induce an isomorphism on cohomology groups.

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Definition

Let $\mathscr{P}$ be an operad with a map $\mathscr{P} \to H^*(\mathscr{P})$ (for example, the differentials on $\mathcal{P}$ are all 0). We say a $\mathscr{P}$-algebra A is **formal** if it is weakly equivalent to its cohomology algebra $H^*(A)$.

The quasi-isomorphism problem

Problem

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Applications:

- (Sullivan, 1977) When $\mathscr{P} = \mathrm{Com}$ and $R = \mathbb{Q}$, equivalent to classification of topological spaces up to rational weak homotopy equivalence.
- (Mandell, 2006) When $\mathscr{P} = E_\infty$ and $R = \mathbb{Z}$, implies classification of finite type, nilpotent topological spaces up to weak homotopy equivalence.

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- Additional algebra structure allows you classify structures that live on spaces (Poisson manifolds).
- Quasi-isomorphisms of L_∞ algebras show up in deformation theory: deformation problems are classified by quasi-isomorphisms and Maurer-Cartan equivalence classes.
- Mathematical physics.
- The homotopy/infinity category of $\mathscr{P}$-algebras is an intrinsically interesting object.

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of associative algebras where the C_i are not commutative algebras but A and B are commutative.

Question (The original commutative and associative quasi-isomorphism problem)

If two commutative algebras are quasi-isomorphic as associative algebras are they are quasi-isomorphic as a commutative algebras?

The commutative and associative quasi-isomorphism problem: Resolution

The problem was resolved in two stages via homotopy transfer theorem methods.

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Over $\mathbb{Q}$, if a commutative algebra is formal as an associative algebra, then it is formal as a commutative algebra.

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Over $\mathbb{Q}$, if two commutative algebras are quasi-isomorphic as associative algebras, they are quasi-isomorphic as commutative algebras.

Slogan: In characteristic 0, commutativity is a **property** not a **structure**. Or is it?
Remark: The theorems fail in characteristic p due to secondary operations living above the Frobenius map via arguments similar to ones we shall see today.

Massey products for DGAs

The Massey triple product

Definition

Let A be a dg-algebra. Let $x, y, z \in H^\bullet(A)$ by such that $xy = 0$ and $yz = 0$. Let $\bar{x}, \bar{y}, \bar{z}$ be cocycles representing x, y, z and suppose $d\bar{u} = \bar{x}\bar{y}$ and $d\bar{v} = \bar{y}\bar{z}$. Then the set

$$\langle x, y, z\rangle = \{[\bar{u}\bar{z} - \bar{x}\bar{v}] : \forall \bar{u}, \bar{v} \in A \text{ such that } d\bar{u} = \bar{x}\bar{y}, d\bar{v} = \bar{y}\bar{z}\} \subseteq H^{|x|+|y|+|z|-1}(A)$$

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is called the **Massey product set**. It represents a well-defined equivalence class of

$$\frac{H^{|x|+|y|+|z|-1}(A)}{xH^{|y|+|z|-1}(A) + H^{|x|+|y|-1}(A)z}$$

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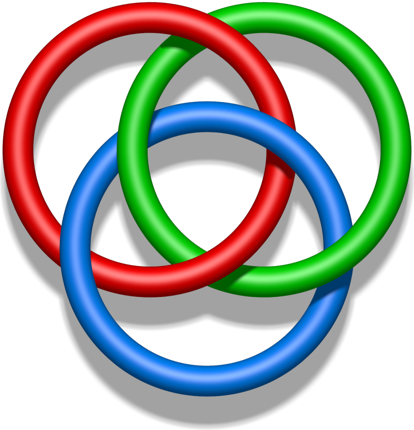
The quotient is called the **indeterminacy** of the operation.
Inutitively Massey products detect linking behaviour.

Borromean rings



Source: Jim.belk; Wikipedia

Obstructions to formality and quasi-isomorphism

Proposition (Massey, 1958)

If for some $x, y, z \in H^\bullet(A)$, the Massey product set $\langle x, y, z\rangle$ is nonempty and does not contain 0, then A is not formal.

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Proposition (Massey, 1958)

If for some $x, y, z \in H^\bullet(A)$, the Massey product set $\langle x, y, z\rangle$ is nonempty and does not contain 0, then A is not formal.

Proposition (Massey products obstruct quasi-isomorphism)

Let A and B be dg-algebras with isomorphic cohomology rings. If for some $x, y, z \in H(A)$ and all choices of ring isomorphism $f : H(A) \to H(B)$, the Massey product set $f(\langle x, y, z\rangle) \subseteq H(B)$ differs from the Massey product set $\langle f(x), f(y), f(z)\rangle \subseteq H(B)$ then A and B are not weakly equivalent.

Obtaining finer invariants of DGAs

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Obtaining finer invariants of DGAs

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1. **Higher order primitive Massey products**: The Massey product we have just seen is a cocycle because A satisfies the associative relation $(ab)c = a(bc)$. There are *higher order* operations associated to syzgies: higher relations between relations.
2. **Matric Massey products**: For the Massey triple product, we had $ab = 0$ and $bc = 0$. But what if our relations look like $ab + cd = 0$? You get new *secondary* operations this way.

Higher order Massey products

Let (A, d) be a dg-algebra and $x_1, \ldots, x_n \in H^*(A)$. A *defining system* consists of cochains $\bar{x_{ij}}$ for $1 \leq i < j \leq n$, $1 < j - i < n$, such that

$$d\bar{x_{ij}} = \sum_{k=i+1}^{j-1} (-1)^{|x_{ik}|} x_{ik} x_{kj}.$$

$$\bar{x_{1,k}} = \bar{x_k}$$

Then

$$\langle x_1, \ldots, x_n \rangle := \left\{ \left[\sum_{k=2}^{n-1} \bar{x_{1k}} \bar{x_{kn}} \right] \,\middle|\, (x_{ij}) \text{ a defining system} \right\} \subseteq H^*(A).$$

Matric Massey products

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- Choose representatives $\bar{X}, \bar{Y}, \bar{Z} \in \mathrm{Mat}(A)$ where each coordinate of each matrix is replaced with its lift. Then $\bar{X}\bar{Y} = d\bar{P}$ and $\bar{Y}\bar{Z} = d\bar{Q}$. Define the matric Massey product set to be $\langle X, Y, Z \rangle$

$$\langle X, Y, Z \rangle := \left\{ \bar{P}\bar{Z} - \bar{X}\bar{Q} : \forall \bar{P}, \bar{Q} \in \mathrm{Mat}(A) \text{ such that } \bar{X}\bar{Y} = d\bar{P} \text{ and } \bar{Y}\bar{Z} = d\bar{Q} \right\}$$

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- Higher order matric operations are constructed similarly through defining systems and same inductive relations.

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Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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Then on the cocycle level

$$\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} \bar{e} & \bar{f} \\ \bar{g} & \bar{h} \end{pmatrix} = \begin{pmatrix} \overline{p_{1,1}} & \overline{p_{1,2}} \\ \overline{p_{2,1}} & \overline{p_{2,2}} \end{pmatrix} \qquad \begin{pmatrix} \bar{e} & \bar{f} \\ \bar{g} & \bar{h} \end{pmatrix} \begin{pmatrix} \bar{i} & \bar{j} \\ \bar{k} & \bar{l} \end{pmatrix} = \begin{pmatrix} \overline{q_{1,1}} & \overline{q_{1,2}} \\ \overline{q_{2,1}} & \overline{q_{2,2}} \end{pmatrix}$$

Example continued

The matric Massey product set is given by:

$$\left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} \overline{q_{1,1}} & \overline{q_{1,2}} \\ \overline{q_{2,1}} & \overline{q_{2,2}} \end{pmatrix} + \begin{pmatrix} \overline{p_{1,1}} & \overline{p_{1,2}} \\ \overline{p_{2,1}} & \overline{p_{2,2}} \end{pmatrix} \begin{pmatrix} \bar{i} & \bar{j} \\ \bar{k} & \bar{l} \end{pmatrix} \right\}$$

over all choices of lifts for $\overline{p_{i,j}}, \overline{q_{i,j}}$.

Matric Massey products obstruct to formality and quasi-isomorphism

Proposition (May, 1969)

If for some $X, Y, Z \in \operatorname{Mat} H^\bullet(A)$*, the Massey product* $\langle X, Y, Z\rangle$ *is nonempty and does not contain the zero matrix, then* A *is not formal.*

Proposition (Matric Massey products obstruct quasi-isomorphism)

Let A *and* B *be dg-algebras with isomorphic cohomology rings. If for some* $X, Y, Z \in \operatorname{Mat}(H^\bullet(A))$ *and all choices of ring isomorphism* $f : H(A) \to H(B)$*, the Massey product* $f(\langle X, Y, Z\rangle) \subseteq H(B)$ *differs from the Massey product* $\langle f(X), f(Y), f(Z)\rangle \subseteq H(B)$ *then* A *and* B *are not weakly equivalent.*

Eilenberg-Moore spectral sequence

The bar construction is

$$B(A) = \left(\bigoplus_{i=0}^{\infty} A^{\otimes i}, \partial_1 + \partial_2 \right)$$

It has a natural filtration $F_N B(A) = \bigoplus_{i=0}^{N} A^{\otimes i}$. Therefore there is a naturally associated **Eilenberg-Moore spectral sequence**.

Theorem (Gugenheim–May, 1974)

The differentials in the Eilenberg-Moore spectral sequence are in 1-to-1 correspondence with higher matric Massey products.

Generalizations of Massey products to other operads

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3. Primitive Massey products have generalized to quadratic (Koszul) operads $\mathcal{P}$; they are related to the combinatorics of the tree monomials appearing the Koszul dual cooperad of $\mathcal{P}$. (F.C.–Moreno).
4. Matric Massey products have not been generalised to other flavours of algebra: **even secondary operations**.

Advantages and disadvantages of Massey products

The advantages of Massey products are:

- Conceptually simple.
- Local
- Easy to calculate in practical situations.
- Almost always enough to show two spaces or algebras are different. (non-formality of Swiss-cheese operad, configurations spaces of homotopy equivalent lens spaces)
- The same philosophy works over arbitrary rings (**cotriple products**).
- They obstruct phenomena more subtle than formality.

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- The same philosophy works over arbitrary rings (**cotriple products**).
- They obstruct phenomena more subtle than formality.

The disadvantage of Massey products is:

- **Massey products are not a complete obstruction to formality or quasi-isomorphism.** They lose some global data about the algebra. Therefore they **cannot** normally be used to prove that two $\mathcal{P}$-algebras are the same. Invariants constructed in other ways: such as Halperin-Stasheff obstructions or Kaledin classes are needed for such problems.

Determinant Massey products

Massey products for CDGAs

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3. This is controllable: there is a canonical correspondence between the vector space generated by the $(n-2)^{th}$ order Com-Massey products and Lie(n).
4. What are matric Massey products in this setting?

Com-Eilenberg-Moore spectral sequence

The (operadic) bar construction is

$$B(A) = \left(\bigoplus_{i=0}^{\infty} \mathrm{Lie}(i) \otimes_{\mathbb{S}_i} A^{\otimes i}, \partial_1 + \partial_2 \right)$$

It has a natural filtration $F_N B(A) = \bigoplus_{i=0}^{N} \mathrm{Lie}(i) \otimes_{\mathbb{S}_i} A^{\otimes i}$. Therefore there is a naturally associated **Com-Eilenberg-Moore spectral sequence**.

Proposition (FC.-Moreno)

The primitive Com*-Massey products can be identified with certain differentials of this spectral sequence.*

Problem

What are the others?

Determinant Massey products

Suppose that a, b, c, e, l, v are even degree elements in the cohomology of an associative algebra A such that

$$ab + ce \quad = \quad bl - cv \quad = \quad el + av \quad = \quad 0.$$

Choose cocycles $\bar{a}, \bar{b}, \bar{c}, \bar{e}, \bar{l}, \bar{v}$ representing the corresponding classes, and assume $\bar{x}, \bar{y}, \bar{z}$ are elements such that

$$d\bar{x} = \bar{a}\bar{b} + \bar{c}\bar{e}, \qquad d\bar{y} = \bar{b}\bar{l} - \bar{c}\bar{v}, \qquad d\bar{z} = \bar{e}\bar{l} + \bar{a}\bar{v}.$$

Then

$$d\left(\bar{x}\bar{l} - \bar{a}\bar{y} - \bar{c}\bar{z}\right) = \bar{a}\bar{c}\bar{v} - \bar{c}\bar{a}\bar{v} = \left(\bar{a}\bar{c} - \bar{c}\bar{a}\right)\bar{v}.$$

So this vanishes if $\bar{a}\bar{c} - \bar{c}\bar{a} = 0$. So it **always vanishes** in the commutative case but **not necessarily** if A is associative. In the commutative case, it gives rise to an operation with indeterminacy

$$l \cdot H^{|\bar{x}|}(A) + a \cdot H^{|\bar{y}|}(A) + c \cdot H^{|\bar{z}|}(A).$$

but **not in the associative case.**

Associated matric Massey product

There is a closely related matric Massey product. Since that cohomology is associative, we have $ac - ca = 0$. Using this identity we see that the Massey triple product

$$\left\langle \begin{pmatrix} a & c \end{pmatrix}, \begin{pmatrix} b & c \\ e & -a \end{pmatrix}, \begin{pmatrix} l \\ -v \end{pmatrix} \right\rangle$$

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is defined. However, it has a **different indeterminacy**. The relation $ac - ca = 0$ is a relation on the algebra **not** the operad. So it is only defined up to homotopy. So the indeterminacy is

$$l \cdot H^{|\bar{x}|}(A) + a \cdot H^{|\bar{y}|}(A) + c \cdot H^{|\bar{z}|}(A) + v \cdot H^{|a|+|c|-1|}(A) \,.$$

So these are genuinely different operations.
These new Massey products are sensitive to commutative structure.

Determinant formalism

There is a formalism for these new operations using determinants. Define

$$U_{\mathcal{D}} = (-1)^{|a|} a\zeta_a - (-1)^{|b|+|b||a|} b\zeta_b + (-1)^{|c|+|c||a|+|c||b|} c\zeta_c.$$

$$d\zeta_a = yc - bz, \quad d\zeta_b = xc - az, \quad d\zeta_c = xb - ay$$

Then

$$\begin{aligned} dU_{\mathcal{D}} &= a\,(d\zeta_a) - (-1)^{|a||b|} b(d\zeta_b) + (-1)^{|a||c|+|b||c|} c(d\zeta_c) \\ &= a\,(yc - bz) - (-1)^{|b||a|}\,(xc - az) + (-1)^{|c||a|+|c||b|} c\,(xb - ay) \\ &= a\underbrace{\begin{vmatrix} y & b \\ z & c \end{vmatrix}}_{d\zeta_a} - b\underbrace{\begin{vmatrix} x & a \\ z & c \end{vmatrix}}_{d\zeta_b} + c\underbrace{\begin{vmatrix} x & a \\ y & b \end{vmatrix}}_{d\zeta_c} = \begin{vmatrix} a & x & a \\ b & y & b \\ c & z & c \end{vmatrix}. \end{aligned}$$

This vanishing of minors argument can be extended to define higher order products.

The commutative and associative quasi-isomorphism problem revisited

A_∞-algebras

Definition (Stasheff, 1963)

An A_∞*-algebra* is a graded vector space V along with a degree 1, square-zero coderivation

$$D : T^c V[1] \to T^c V[1]$$

where $T^c sV$ is the cofree conilpotent coalgebra on the suspension of V ie. $\bigoplus_{n\geq 1}(V[1])^{\otimes n}$

By cofreeness, D is determined by its value on cogenerators ie. by a succession of maps m_n of degree $2-n$:

$$\begin{aligned} m_1 &: V \to V \\ m_2 &: V^{\otimes 2} \to V \\ m_3 &: V^{\otimes 2} \to V \\ &\cdots \end{aligned}$$

The square zero condition ensures these operations must satisfy the *Stasheff relations*.

Associative algebras

Example

Every associative algebra is an A_∞-algebra with $m_1 : A \to A$ being the differential and m_2 being the multiplication on $A \otimes A \to A$ and $m_n = 0$ for $n \geq 2$.

Theorem

The choice of model for the A_∞-operad is $\mathrm{B}\Omega\,\mathrm{Ass}$

Proposition (Rectification of A_∞-algebras)

Over $\mathbb{Q}$, every A_∞-algebra is weakly equivalent to a strictly associative algebra and every C_∞-algebra is weakly equivalent to a strictly commutative algebra .

The ∞-commutative and associative quasi-isomorphism problem

Theorem (Saleh, 2017)

Over $\mathbb{Q}$, if a C_∞-algebra is formal as an A_∞-algebra, then it is formal as a C_∞-algebra.

Theorem (Campos–Robert-Nicoud–Petersen–Wierstra, 2024)

Over $\mathbb{Q}$, if two C_∞-algebras are quasi-isomorphic as A_∞-algebras, they are quasi-isomorphic as a C_∞-algebras.

Homotopy commutativity and associativity

The A_∞-operad comes equipped with a natural filtration

$$\mathcal{A}_2 \hookrightarrow \mathcal{A}_3 \hookrightarrow \mathcal{A}_4 \hookrightarrow \cdots \hookrightarrow \mathcal{A}_\infty.$$

An $\mathcal{A}_n$-**algebra** is an dg-algebra equipped with operations $m_2, \cdots, m_n$ where $m_i : A^{\otimes i} \to s^{i-2}A$. These operations must satisfy those Stasheff relations involving only $m_2, \cdots, m_n$.

Slogan: An $\mathcal{A}_n$-algebra is associative up to $(n-2)^{th}$ coherent homotopy.

The C_∞-operad has a similar filtration.

$$\mathcal{C}_2 \hookrightarrow \mathcal{C}_3 \hookrightarrow \mathcal{C}_4 \hookrightarrow \cdots \hookrightarrow \mathcal{C}_\infty.$$

Concretely, a $\mathcal{C}_n$-**algebra** is $\mathcal{A}_n$-algebra where m_n vanishes on shuffles. In particular, the binary product m_2 is strictly graded-commutative.

Example

Definition

An $\mathcal{A}_2$-algebra consists of a cochain complex (A, d) equipped with a bilinear, (not-necessarily-associative) binary, degree 0, product satisfying the relation

$$d(ab) = (da)b + (-1)^{|a|}a(db).$$

These are better known as **dg-magmas.** A $\mathcal{C}_2$-algebra is an $\mathcal{A}_2$-algebra whose product is strictly graded-commutative.

The $\mathcal{A}_n$ and $\mathcal{C}_n$ quasi-isomorphism problem

Question

If two $\mathcal{C}_n$-algebras are quasi-isomorphic as $\mathcal{A}_n$-algebras, are they are quasi-isomorphic as $\mathcal{C}_n$-algebras?

The $\mathcal{A}_n$ and $\mathcal{C}_n$ quasi-isomorphism problem

Question

If two $\mathcal{C}_n$-algebras are quasi-isomorphic as $\mathcal{A}_n$-algebras, are they are quasi-isomorphic as $\mathcal{C}_n$-algebras?

Answer

$$\begin{cases} \textit{Yes} & \textit{for } n = 2, \infty. \\ \textit{No} & \textit{otherwise.} \end{cases}$$

The $n = 2$ case

Theorem (FC–Moreno–Muro)

Let A be either a $\mathcal{C}_2$ or $\mathcal{A}_2$-algebra in characteristic zero. Then A is formal.

Proof sketch: The cooperad $\Omega\,\mathcal{A}_2$ is tiny. Use this to construct an explicit $(\mathcal{A}_2)_\infty$-map between A and its cohomology.

Corollary

Two $\mathcal{C}_2$-algebras are quasi-isomorphic as $\mathcal{C}_2$-algebras if and only if they are quasi-isomorphic as $\mathcal{A}_2$-algebras.

$\mathcal{A}_3$-algebra

Definition

An $\mathcal{A}_3$-algebra is a cochain complex (A, d) equipped with two multilinear operations:

1. A (not-necessarily-associative) binary, degree 0, product satisfying the relation
$$d(ab) = (da)b + (-1)^{|a|}a(db).$$
2. A ternary degree -1 product $\alpha : A^{\otimes 3} \to A$ such that
$$d\alpha(a,b,c) = a(bc) - (ab)c - \left(\alpha(da,b,c) + (-1)^{|a|}\alpha(a,db,c) + (-1)^{|a|+|b|}\alpha(a,b,dc)\right).$$

A $\mathcal{C}_3$-algebra is an $\mathcal{A}_3$-algebra whose product is strictly graded-commutative and where m_3 vanishes on shuffles.

The $\mathcal{A}_3$ and $\mathcal{C}_3$ quasi-isomorphism problem

Theorem (FC–Moreno–Muro)

There exists a $\mathcal{C}_3$-algebra A that is formal as an $\mathcal{A}_3$-algebra but not as a $\mathcal{C}_3$-algebra.

The $\mathcal{A}_3$ and $\mathcal{C}_3$ quasi-isomorphism problem

Theorem (FC–Moreno–Muro)

There exists a $\mathcal{C}_3$-algebra A that is formal as an $\mathcal{A}_3$-algebra but not as a $\mathcal{C}_3$-algebra.

Corollary (FC–Moreno–Muro)

There are two $\mathcal{C}_3$-algebras that are quasi-isomorphic as $\mathcal{A}_3$-algebras but not as a $\mathcal{C}_3$-algebras.

Determinant Massey products for $\mathcal{C}_3$-algebras

Suppose that a, b, c, e, l, v are even degree elements in the cohomology of a $\mathcal{C}_3$-algebra A such that

$$ab + ce \quad = \quad bl - cv \quad = \quad el + av \quad = \quad 0.$$

Choose cocycles $\bar{a}, \bar{b}, \bar{c}, \bar{e}, \bar{l}, \bar{v}$ representing the corresponding classes, and assume $\bar{x}, \bar{y}, \bar{z}$ are elements such that

$$d\bar{x} = \bar{a}\bar{b} + \bar{c}\bar{e}, \qquad d\bar{y} = \bar{b}\bar{l} - \bar{c}\bar{v}, \qquad d\bar{z} = \bar{e}\bar{l} + \bar{a}\bar{v}.$$

Then

$$d\left(\bar{x}\bar{l} - \bar{a}\bar{y} - \bar{c}\bar{z} + \alpha\left(\bar{a}, \bar{b}, \bar{l}\right) + \alpha\left(\bar{c}, \bar{e}, \bar{l}\right) + \alpha\left(\bar{a}, \bar{c}, \bar{v}\right)\right) = 0$$

This gives rise to an operation with indeterminacy

$$l \cdot H^{|\bar{x}|}(A) + a \cdot H^{|\bar{y}|}(A) + c \cdot H^{|\bar{z}|}(A).$$

The $\mathcal{A}_3$ and $\mathcal{C}_3$ quasi-isomorphism problem: proof strategy

The proof is by building a zig-zag of $\mathcal{A}_3$-algebras, where A is a commutative algebra and B is an $\mathcal{A}_3$-algebra.

$$H(A) \xleftarrow{\sim} B \xrightarrow{\sim} A$$

The algebras $H(A)$ and A are distinguished by a determinant Massey product p. The algebra B only has the associated matric Massey product which has a larger indeterminacy and in particular contains 0 (you deliberately set the product $p = \{uv\}$ in A for some u so $p = \{uv, 0\}$ in B).

Further questions

On the result

1. Do $\mathcal{C}_n$-algebras appear in nature?
2. What do matric Massey products look like for other operads (in particular, for Lie algebras).
3. A completeness result: describe all of the differentials in the operadic Eilenberg-Moore spectral sequence.

On things mentioned in passing

1. Geometrically interpret the category of E_∞-algebras, ie. find a Quillen adjunction from a (possibly valued in some monoidal category $(\mathcal{C}, \otimes)$ not **Set**) model presheaf category

$$\mathsf{PSh}\, X \rightleftarrows E_\infty - \mathrm{alg}$$

that induces an equivalence on the homotopy categories at the level of objects.