Associative and commutative dg-algebras in positive characteristic

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Abstract

In this expository piece, we construct an example in characteristic p of two commutative dg-algebras which are quasi-isomorphic as associative but not commutative dg-algebras.[1]

1 Introduction

The goal of this article is to explore the following recent problem in homotopy theory [2].

Question 1.1. If two commutative algebras A and B over $\mathbb{F}_p$ are quasi-isomorphic as associative algebras are they quasi-isomorphic as commutative algebras?

This article will strive to be as self-contained as possible and accessible to any working mathematician. We therefore start by introducing a little bit of the background in homological algebra. By the end of this, we hope the non-initiated reader will completely understand the question at hand. We shall then provide a very brief sketch as to how to solve the problem. The approach is surprisingly straightforward and avoids heavy technical machinery; instead, we make use of elementary combinatorics, linear algebra and number theory.

To commence, we begin with a short crash course in homological algebra. We will recall what we need, but as this is a vast subject, we refer the reader to [5] for a more detailed introduction.

Definition 1.2. A *cochain complex* over a field is a family of vector spaces $A^i : i \in \mathbb{Z}$ and *(linear) differential maps* $d^n : A^n \to A^{n+1}$ such that $d^{n+1} \circ d^n = 0$ for all n, where 0 is the zero map.

$$\cdots \longrightarrow A^i \xrightarrow{d} A^{i+1} \xrightarrow{d} A^{i+2} \xrightarrow{d} A^{i+3} \xrightarrow{d} A^{i+4} \xrightarrow{d} \cdots$$

A chain complex is the dual notion and has the same definition except the maps go down a degree $\partial_n : A^n \to A^{n-1}$. For notational convenience, we generally suppress the superscript n when the target and source of the map are implicitly understood. A map of cochain complexes $A \to B$ is a collection of maps f_n that commute with the differential. In other words, the following diagram commutes.

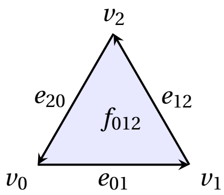
$$\begin{tikzcd}
\cdots \arrow[r] & A^i \arrow[r, "d"] \arrow[d, "f"] & A^{i+1} \arrow[r, "d"] \arrow[d, "f"] & A^{i+2} \arrow[r, "d"] \arrow[d, "f"] & A^{i+3} \arrow[r, "d"] \arrow[d, "f"] & A^{i+4} \arrow[r, "d"] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & B^i \arrow[r, "d"] & B^{i+1} \arrow[r, "d"] & B^{i+2} \arrow[r, "d"] & B^{i+3} \arrow[r, "d"] & B^{i+4} \arrow[r, "d"] & \cdots
\end{tikzcd}$$

The reader somewhat acquainted with differential geometry will already know at least one example of a cochain complex. If you have a smooth manifold M, the collection of differential forms on M form a cochain complex $\Omega^\bullet(M)$ usually called *the de Rham complex*, where $\Omega^n(M)$ is the vector space of differential n-forms and the differential is the exterior derivative.

The reader who is less comfortable with differential geometry may be more familiar with a more elementary way that (co)chain complexes arise in nature, from simplices.

Example 1.3. Consider the standard 2-simplex (in other words a triangle).

[1]We thank Chase Ford for helpful feedback on the first version of this manuscript



This consists of:

- Three 0-simplices (vertices): v_0, v_1, v_2
- Two 1-simplices (oriented edges): e_{01}, e_{12}, e_{20}
- A single 2-simplex (oriented face): f_{012}

We define the chain groups (which are vector spaces over $\mathbb{R}$):

$$C_2 = \mathbb{R}\langle f_{012}\rangle \cong \mathbb{R}, \quad C_1 = \mathbb{R}\langle e_{01}, e_{12}, e_{20}\rangle \cong \mathbb{R}^3, \quad C_0 = \mathbb{R}\langle v_0, v_1, v_2\rangle \cong \mathbb{R}^3$$

So each vector space has a basis C_i given by the simplices of dimension i. Next, we define boundary maps $\partial_i : C_i \to C_{i-1}$:

$$\begin{aligned}
\partial_2(f_{012}) &= e_{01} + e_{12} + e_{20} \\
\partial_1(e_{01}) &= v_1 - v_0 \\
\partial_1(e_{12}) &= v_2 - v_1 \\
\partial_1(e_{20}) &= v_0 - v_2 \\
\partial_0(v_i) &= 0
\end{aligned}$$

The intuition behind these maps is that given an object, they return a formal sum of the objects forming its boundary. Then:

$$\partial_1 \circ \partial_2(f_{012}) = \partial_1(e_{01} + e_{12} + e_{20}) = (v_1 - v_0) + (v_2 - v_1) + (v_0 - v_2) = 0$$

Thus, we have a chain complex:

$$\cdots \longrightarrow 0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \longrightarrow 0 \longrightarrow \cdots$$

To obtain a cochain complex, we dualize the chain complex by applying the functor $\mathrm{Hom}(-, \mathbb{R})$ to each term, reversing the arrows:

$$\cdots \longrightarrow 0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \longrightarrow 0 \longrightarrow \cdots$$

The de Rham forms can be used to compute some interesting global homotopy invariants of manifolds.

Definition 1.4. The cohomology of a cochain complex A is the following family of abelian groups

$$H^i(A) = \ker d^i / \operatorname{Im} d^{i-1}.$$

where $\ker d^i$ is the kernel of d^i and $\operatorname{Im} d^{i-1}$ is the image of d^{i-1}. The group operation is induced by the vector space addition on A. We refer to elements in $\ker d^i$ as *cocycles* and elements in $\operatorname{Im} d^{i-1}$ as *coboundaries*. A map of cochain complexes $f : A \to B$ induces group homomorphisms

$$H(f^i) : H^i(A) \to H^i(B).$$

The remarkable thing about the cohomology groups of $\Omega^\bullet(M)$ is that they are a *homotopy invariant* of the manifold M. In other words, they are preserved by squashing and squeezing (but not cutting) it. A very similar invariant - *singular cohomology* - of locally compact, Hausdorff topological spaces can be produced by probing a topological space with simplices of varying dimensions to produce the *singular cochain complex*.

The attentive reader probably notices an issue here: while all homotopy equivalent manifolds or topological spaces have the same cohomology groups, the converse is not true. For example, $S^1 \times S^1$, a torus, has the same cohomology groups as $S^2 \vee S^1 \vee S^1$, a sphere and two circles glued together at a point, but these spaces are not homotopy equivalent. It would be desirable to have finer invariants, capable of distinguishing more spaces.

As it turns out, the de Rham forms actually have a little more structure: a commutative operation called the *wedge product* that behaves well with respect to the differential. The singular cochains on a topological space X have a similar operation induced by the diagonal map of a

$$X \to X \times X$$

$$x \mapsto (x, x).$$

Definition 1.5. An *associative dg-algebra* is a cochain complex A equipped with a binary associative multiplication $- \cup - : A^p \otimes A^q \to A^{p+q}$ and d satisfies the Leibniz rule from differentiation ie.

$$d(x \cup y) = d(x) \cup y + (-1)^{|x|} x \cup d(y).$$

where $|x|$ is the degree of x ie. $|x| = n$ if $x \in A^n$. If the product is also graded commutative[2], ie.

$$x \cup y = (-1)^{|x||y|} y \cup x$$

we call A a *commutative dg-algebra*.

Remark 1.6. An associative algebra structure on a cochain complex A induces a graded multiplication on the cohomology of A,

$$H^i(A) \otimes H^j(A) \to H^{i+j}(A)$$

on the cohomology groups, turning the direct sum $\bigoplus H^i(A)$ into a ring. If the algebra structure is commutative, then the ring will be commutative too.

Again, a delicate question is how one extracts a homotopy invariant from this algebra structure. To do this, we shall need to introduce the notion of *quasi-isomorphism*.

A quasi-isomorphism between cochain complexes A to B is a map that

a) induces an isomorphism on cohomology.

b) preserves any underlying algebraic structure.

A small problem is that quasi-isomorphisms do not necessarily admit inverses.

Example 1.7. For example, we can consider the following differential graded algebras[3]

$$A : \mathbb{R} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{R}x \xrightarrow{0} \mathbb{R}y \xrightarrow{[y \mapsto x^2]} \mathbb{R}x^2 \xrightarrow{0} \mathbb{R}xy \xrightarrow{[xy \mapsto x^3]} \mathbb{R}x^3 \longrightarrow \cdots$$

$$B : \mathbb{R} \xrightarrow{0} 0 \xrightarrow{0} \mathbb{R}z \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} 0 \xrightarrow{0} \cdots$$

To recap, A is the polynomial algebra $\mathbb{R}[x, y]/(y^2)$, where the degree of $|x| = 2$ and $|y| = 3$. This cochain complex has a differential generated by the rules

$$dx = 0 \qquad dy = x^2$$

and extended via the Leibniz rule. For example,

$$d(xy) = d(x)y + (-1)^2 x d(y) = x^3.$$

[2] Note that a commutative algebra is also required to be associative.

[3] The reader who is well versed in rational homotopy theory may recognize A as the minimal model for S^2 and B as its cohomology algebra.

The dg-algebra B is $\mathbb{R}[z]/(z^2)$, which is two dimensional. The cohomology of both of these complexes is

$$H^i(A) = H^i(B) = \begin{cases} \mathbb{R} & \text{if } i = 0,2. \\ 0 & \text{otherwise.} \end{cases}$$

Elements of the cohomology group $H^2(A)$ are, by definition, equivalences classes of elements in $\ker d^2 \subseteq \mathbb{R}x$. In particular, $H^2(A)$, as a one-dimensional vector space, has *cochain-level* basis of the form $\{x\}$, and $H^2(B)$ has a cochain-level basis given by $\{z\}$. When we say *cochain-level basis of* $H^2(A)$, we mean a set of elements of A^2 which become a basis of $H^2(A)$ upon applying the equivalence relation.

There is a quasi-isomorphism of commutative algebras from A to B given as follows.

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "0"] \arrow[d, "\mathrm{id}"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{R}x \arrow[r, "0"] \arrow[d, "{[x\mapsto z]}"] & \mathbb{R}y \arrow[r, "{[y\mapsto x^2]}"] \arrow[d, "0"] & \mathbb{R}x^2 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{R}xy \arrow[r, "{[xy\mapsto x^3]}"] \arrow[d, "0"] & \mathbb{R}x^3 \arrow[r] \arrow[d, "0"] & \cdots \\
\mathbb{R} \arrow[r, "0"] & 0 \arrow[r, "0"] & \mathbb{R}z \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & \cdots
\end{tikzcd}$$

However, there is no quasi-isomorphism from $g : B \to A$. This is because the induced map on cohomology is a quasi-isomorphism, and, in particular, one must have an isomorphism $H^2(B) \cong \mathbb{R} \to H^2(A) \cong \mathbb{R}$. For this to be true, on the cochain level, the basis element z has to be sent to ax for some scalar $a \in \mathbb{R}$ and $a \neq 0$. But then $g(0) = g(z^2) = g(z)^2 = a^2x^2$ as g must preserve the algebraic structure. But then g fails to be a linear map as it sends zero to something non-zero!

Quasi-isomorphism is transitive and reflexive, but as the previous example shows, it is unfortunately not symmetric. Therefore it is not an equivalence relation. But we can "pretend it is" by saying that cochain complexes A and B are quasi-isomorphic as associative dg-algebras if there is a *zig-zag*

$$A = C_1 \xleftarrow{\ f_1\ } C_2 \xrightarrow{\ f_2\ } C_3 \xleftarrow{\ f_3\ } \cdots \xleftarrow{\ f_{n-2}\ } C_{n-1} \xrightarrow{\ f_{n-1}\ } C_n = B.$$

The key point here is that each of the C_i is an associative dg-algebra and each quasi-isomorphism f_i preserves the algebra structure ie. $f_i(ab) = f_i(a)f_i(b)$. One way to understand this is to view the quasi-isomorphism relation as being formally made symmetric by adding formal inverses to each quasi-isomorphism.

So now, a natural question is to try to classify when are two associative dg-algebras A and B are quasi-isomorphic **as associative algebras**?

Example 1.8. A brief aside, if we work purely with cochain complexes over a field, and assume no additional algebraic structure, there is always a quasi-isomorphism of cochain complexes

$$A \xrightarrow{\sim} H(A).$$

So, if A has the same cohomology groups as B, we can always find a quasi-isomorphism

$$B \xrightarrow{\sim} H(B) = H(A) \xrightarrow{\sim} A.$$

Thus, two cochain complexes have the same quasi-isomorphism type as cochain complexes, if and only if, they have the same cohomology groups.

In light of Example 1.8, one might hope two associative dg-algebras A and B are quasi-isomorphic as associative (or commutative) algebras if and only they have the same cohomology ring. This turns out not to be the case, one can construct some concrete counterexamples.

When A and B are **commutative** and the zig-zag consists of **commutative** algebras, this turns out to be a very subtle question, which was elegantly answered by Sullivan [4] with his theory of minimal models, which completely answers the classification problem *rationally*. This is an amazing result, because it shows that the rational homotopy type of a topological space is completely determined by a (relatively!) small commutative dg-algebra. It formed part of the work that Sullivan was awarded an Abel prize for in 2022.

This all led Ricardo Campos, Dan Petersen, Daniel Robert-Nicoud, and Felix Wierstra to ask the following question [1]:

Question 1.9. If two commutative algebras A and B are quasi-isomorphic as **associative** algebras are they quasi-isomorphic as **commutative** algebras?

At first glance, the answer appears to be obviously no. Every zig-zag of commutative algebras

$$A = C_1 \xleftarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xleftarrow{f_3} \cdots \xleftarrow{f_{n-2}} C_{n-1} \xrightarrow{f_{n-1}} C_n = B.$$

is also a zig-zag of associative algebras, as commutative algebras are a special kind of associative algebra.

However, it is easy to find zig-zags

$$A = C_1 \xleftarrow{f_1} C_2 \xrightarrow{f_2} C_3 \xleftarrow{f_3} \cdots \xleftarrow{f_{n-2}} C_{n-1} \xrightarrow{f_{n-1}} C_n = B.$$

where A and B are commutative, but the algebras C_i are associative but not commutative.

Example 1.10. There are some very simple examples of non-commutative, associative algebras that are weakly equivalent **as associative algebras** to commutative algebras. For a concrete example, consider the following algebra A, with a linear basis which has just five elements $\{x, y, xy, yx, z\}$, with the degrees given by $|x| = |y| = 2$ and $|z| = 3$ (in particular, there are some obvious relations here like $x^2 = y^2 = 0$). We set $dz = xy - yx$.

Now, consider B to be the commutative dg-algebra with a linear basis which has just three elements $\{a, b, ab\}$, with the degrees given by $|a| = |b| = 2$. There is a quasi-isomorphism from A to B as follows.

$$\begin{tikzcd}
0 \arrow[r, "0"] \arrow[d, "\mathrm{id}"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \mathbb{R}x \oplus \mathbb{R}y \arrow[r, "0"] \arrow[d, "{[x\mapsto a] \atop [y\mapsto b]}"] & \mathbb{R}z \arrow[r, "{[z\mapsto xy-yx]}"] \arrow[d, "0"] & \mathbb{R}xy \oplus \mathbb{R}yx \arrow[r, "0"] \arrow[d, "{[xy\mapsto ab] \atop [yx\mapsto -ab]}"] & 0 \arrow[r, "0"] \arrow[d, "0"] & 0 \arrow[r] \arrow[d, "0"] & 0 \arrow[r, "0"] \arrow[d, "0"] & 0 \arrow[r, "0"] \arrow[d, "0"] & \cdots \\
0 \arrow[r, "0"] & 0 \arrow[r, "0"] & \mathbb{R}a \oplus \mathbb{R}b \arrow[r, "0"] & 0 \arrow[r, "0"] & \mathbb{R}ab \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & 0 \arrow[r, "0"] & \cdots
\end{tikzcd}$$

There are thus many more potential zig-zags in the larger category of associative dg algebras. However, if you work with commutative and associative algebras over a field of characteristic 0, they proved that the answer is actually yes [1]. In other words, if there is a zig-zag between two commutative algebras A and B where some of the C_i are associative but not necessarily commutative, there is an alternative zig-zag between A and B consisting entirely of commutative algebras. The proof, as one might expect, is very technical and makes a lot of use of a collection of methods informally referred to as the *operadic calculus*.

However, this still leaves open the case of what happens when one works over a field of characteristic p. This brings us back to the question from the introduction.

Question 1.11. If two commutative algebras A and B over $\mathbb{F}_p$ are quasi-isomorphic as associative algebras are they quasi-isomorphic as commutative algebras?

Sadly and not completely unexpectedly, I should warn you that the answer here is no, but how to construct a counterexample? The construction actually ends up being mainly number theoretic and combinatorial in nature rather than homotopic.

The strategy here would be familiar to anyone who has ever sat a maths contest - one wants to look for homotopy invariants of commutative algebras that are not homotopy invariants of associative algebras. There is something very weird going on with the Frobenius map in characteristic p. Observe that, the Leibniz rule for differentiating tells us that in a commutative algebra A, we have:

$$d(x^p) = px^{p-1}dx = 0.$$

where the second equality holds because we are working over a field of characteristic p. In other words, x^p is always a cocycle. But this relation does not hold in a general associative algebra A. Instead, one has

$$d(x^p) = (dx)x^{p-1} + x(dx)x^{p-2} + \cdots + x^{p-1}dx.$$

One cannot rearrange the order of the multiplication in the terms, since A is not assumed to be commutative. So this has no reason to a cocycle and generally will not be.

This suggests the following general strategy. You know that, if two commutative algebras A and B are associative quasi-isomorphic, they have the same cohomology ring and that this ring is even commutative.

Like any commutative ring, we have relations: $ab = 0$ for $a, b \in H^\bullet(A)$. Recall the definition of the cohomology groups $H^i(A) = \ker d^i / \operatorname{Im} d^{i-1}$.

This means that, on the level of A, a represents an equivalence class, and one may choose

$$\bar{a} \in \ker d^i \subset A^i.$$

representing it. Such choice may not unique, but given any two choices of representative $\bar{a}$ and $\bar{a}'$, they must be related by the rule

$$\bar{a} - \bar{a}' \in \operatorname{Im} d^{i-1}.$$

So, we may take representatives $\bar{a}, \bar{b} \in A$ ie. for $a, b \in H(A)$ in cohomology. The product $\bar{a}\bar{b}$ in A does not need to be 0 on the nose, but it does need to represent that equivalence class. So $\bar{a}\bar{b} \in \operatorname{Im} d$. It follows that one can find $\bar{c} \in A$ such that

$$\bar{a}\bar{b} = d\bar{c}.$$

Again, $\bar{c}$ is not necessarily unique, but the choice is sufficiently constrained as to still be useful. For example, you can add any cocycle σ to c and one still has

$$d(\bar{c} + \sigma) = \bar{a}\bar{b}$$

but things break if you add an element such that $d\sigma \neq 0$. Cocycles determine elements of the cohomology, so everything is well-defined in a quotient ring of the cohomology.

Now, you can just take the p^{th} power $\bar{c}^p$ and you get a cocycle and therefore an element in the cohomology. This is is not perfectly well defined, but, by keeping track of all our previous choices, you can show, without much difficulty, that it is well-defined as an element in the quotient group

$$\frac{H(A)}{H(A)^p + a^p H(A) + b^p H(A)}.$$

Here, the $H(A)^p$ accounts for the ambiguity in the choice of $\bar{c}$, and $a^p H(A) + b^p H(A)$ accounts for the ambiguity in the choice of $\bar{a}$ and $\bar{b}$.

So now our strategy becomes clear. We just need to find a pair of commutative algebras A and B with the same cohomology, but where this invariant differs. Then we need to find an associative algebra C such that there is a zig-zag of associative weak equivalences

$$A \xleftarrow{\sim} C \xrightarrow{\sim} B.$$

Building C, and ensuring it has the same cohomology ring as A and B, involves a little combinatorial trickery. But it's fundamentally just linear algebra - a concrete example can be found in the paper [2, Section 4.2.4]. There, as vector spaces, the commutative algebras A and B are 7 and 14 dimensional respectively, and the associative algebra C ends up having 32 basis elements.

This method of finding *higher invariants* that live just above the cohomology to solve very concrete problems has a rich history, going back to Massey [3], who used his eponymous products, constructed using similar vanishing arguments, to show that the Borromean rings were pairwise unlinked but cannot be separated.

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