

Determinant Massey Products

Algebra and Geometry seminar
Stockholm University

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- ① Motivation
- ② Massey products for $d\text{gas}$
- ③ Massey products for $cd\text{gas}$
- ④ The commutative and associative quasi-isomorphism problem revisited

Motivation

Weak equivalence and formality

Definition

Let \mathcal{P} be an operad in dg- R -modules. We say two \mathcal{P} -algebras A, B are **weakly equivalent** or **quasi-isomorphic** as \mathcal{P} -algebras if there is a zig-zag

$$A \xrightarrow{\sim} C_1 \xleftarrow{\sim} C_2 \xrightarrow{\sim} \cdots \xleftarrow{\sim} C_n \xrightarrow{\sim} B$$

where C_i are \mathcal{P} -algebras and the maps are \mathcal{P} -algebra maps that induce an isomorphism on cohomology groups.

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Definition

Let \mathcal{P} be an operad with a map $\mathcal{P} \rightarrow H^*(\mathcal{P})$ (for example, the differentials on \mathcal{P} are all 0). We say a \mathcal{P} -algebra A is **formal** if it is weakly equivalent to its cohomology algebra $H^*(A)$.

The quasi-isomorphism problem

Problem

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Applications:

- (Sullivan, 1977) When $\mathcal{P} = \text{Com}$ and $R = \mathbb{Q}$, equivalent to classification of topological spaces up to rational weak homotopy equivalence.
- (Mandell, 2006) When $\mathcal{P} = E_\infty$ and $R = \mathbb{Z}$, implies classification of finite type, nilpotent topological spaces up to rational weak homotopy equivalence.

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- Additional algebra structure allows you classify structures that live on spaces (Poisson manifolds),
- Quasi-isomorphism types of \mathcal{L}_∞ -algebras control deformation problems.
- Mathematical physics.
- The homotopy/infinity category of \mathcal{P} -algebras is an intrinsically interesting object.

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of associative algebras where the C_i are not commutative algebras but A and B are commutative.

Question (The original commutative and associative quasi-isomorphism problem)

If two commutative algebras are quasi-isomorphic as associative algebras are they are quasi-isomorphic as a commutative algebras?

The commutative and associative quasi-isomorphism problem: Resolution

The problem was resolved in two stages via homotopy transfer theorem methods.

Theorem (Saleh, 2017)

Over \mathbb{Q} , if a commutative algebra is formal as an associative algebra, then it is formal as a commutative algebra.

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Over \mathbb{Q} , if two commutative algebras are quasi-isomorphic as associative algebras, they are quasi-isomorphic as commutative algebras.

Slogan: In characteristic 0, commutativity is a **property** not a **structure**. Or is it?

Remark: The theorems fail in characteristic p due to secondary operations living above the Frobenius map via arguments similar to ones we shall see today.

Massey products for DGAs

The Massey triple product

Definition

Let A be a dg-algebra. Let $x, y, z \in H^\bullet(A)$ by such that $xy = 0$ and $yz = 0$. Let $\bar{x}, \bar{y}, \bar{z}$ be cocycles representing x, y, z and suppose $d\bar{u} = \bar{x}\bar{y}$ and $d\bar{v} = \bar{y}\bar{z}$. Then the set

$$\langle x, y, z \rangle = \{[\bar{u}\bar{z} - \bar{x}\bar{v}] : \forall \bar{u}, \bar{v} \in A \text{ such that } d\bar{u} = \bar{x}\bar{y}, d\bar{v} = \bar{y}\bar{z}\} \subseteq H^{|x|+|y|+|z|-1}(A)$$

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is called the **Massey product set**. It represents a well-defined equivalence class of

$$\frac{H^{|x|+|y|+|z|-1}(A)}{xH^{|y|+|z|-1}(A) + H^{|x|+|y|-1}(A)z}$$

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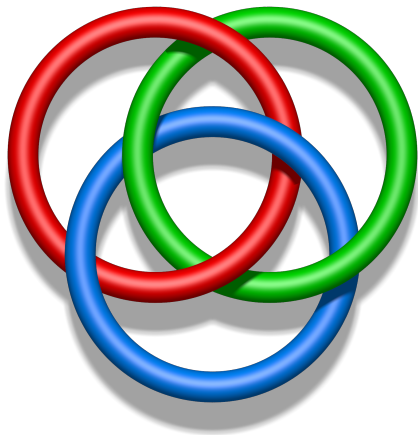
$$\frac{H^{|x|+|y|+|z|-1}(A)}{xH^{|y|+|z|-1}(A) + H^{|x|+|y|-1}(A)z}$$

that we call the Massey product.

The quotient is called the **indeterminacy** of the operation.

Intuitively Massey products detect linking behaviour.

Borromean rings



Source: Jim.belk; Wikipedia

Obstructions to formality and quasi-isomorphism

Proposition (Massey, 1958)

If for some $x, y, z \in H^\bullet(A)$, the Massey product set $\langle x, y, z \rangle$ does not contain 0, then A is not formal.

Obstructions to formality and quasi-isomorphism

Proposition (Massey, 1958)

If for some $x, y, z \in H^\bullet(A)$, the Massey product set $\langle x, y, z \rangle$ does not contain 0, then A is not formal.

Proposition (Massey products obstruct quasi-isomorphism)

Let A and B be dg-algebras with isomorphic cohomology rings. If for some $x, y, z \in H(A)$ and all choices of ring isomorphism $f : H(A) \rightarrow H(B)$, the Massey product set $f(\langle x, y, z \rangle) \subseteq H(B)$ differs from the Massey product set $\langle f(x), f(y), f(z) \rangle \subseteq H(B)$ then A and B are not weakly equivalent.

Obtaining finer invariants of DGAs

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- ① **Higher order primitive Massey products:** The Massey product we have just seen is a cocycle because A satisfies the associative relation $(ab)c = a(bc)$. There are *higher order* operations associated to syzgies: higher relations between relations.

Obtaining finer invariants of DGAs

There are two conceptually different (but related) ways to generalise Massey triple products:

- 1 **Higher order primitive Massey products:** The Massey product we have just seen is a cocycle because A satisfies the associative relation $(ab)c = a(bc)$. There are *higher order* operations associated to syzgies: higher relations between relations.
- 2 **Matric Massey products:** For the Massey triple product, we had $ab = 0$ and $bc = 0$. But what if our relations look like $ab + cd = 0$? You get new *secondary* operations this way.

Higher order Massey products

Let (A, d) be a dg-algebra and $[x_1], \dots, [x_n] \in H^*(A)$. The **n -order Massey product** $\langle [x_1], \dots, [x_n] \rangle$ is

Higher order Massey products

Let (A, d) be a dg-algebra and $[x_1], \dots, [x_n] \in H^*(A)$. The **n -order Massey product** $\langle [x_1], \dots, [x_n] \rangle$ is defined if all lower-order products $\langle [x_i], \dots, [x_j] \rangle$ with $j - i < n - 1$ contain 0. A *defining system* consists of cochains x_{ij} for $1 \leq i < j \leq n$, $j - i < n$, such that

$$dx_{ij} = \sum_{k=i+1}^{j-1} (-1)^{|x_{ik}|} x_{ik} x_{kj}.$$

Then

$$\langle [x_1], \dots, [x_n] \rangle := \left\{ \left[\sum_{k=2}^{n-1} (-1)^{|x_{1k}|} x_{1k} x_{kn} \right] \mid (x_{ij}) \text{ a defining system} \right\} \subseteq H^*(A).$$

Matric Massey products

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- Suppose you have three matrices $X, Y, Z \in \text{Mat}(H(A))$ such that $XY = 0$ and $YZ = 0$.
- Choose representatives $\bar{X}, \bar{Y}, \bar{Z} \in \text{Mat}(A)$ where each coordinate of each matrix is replaced with its lift. Then $\bar{X}\bar{Y} = d\bar{P}$ and $\bar{Y}\bar{Z} = d\bar{Q}$. Define the matric Massey product set to be $\langle X, Y, Z \rangle$

$$\langle X, Y, Z \rangle := \{ \bar{P}\bar{Z} - \bar{X}\bar{Q} : \forall \bar{P}, \bar{Q} \in \text{Mat}(A) \text{ such that } \bar{X}\bar{Y} = d\bar{P} \text{ and } \bar{Y}\bar{Z} = d\bar{Q} \}$$

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- Higher order matric operations are constructed similarly through defining systems and same inductive relations.

An example of matrix Massey product

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Then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

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Then on the cocycle level

$$\begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} \bar{e} & \bar{f} \\ \bar{g} & \bar{h} \end{pmatrix} = \begin{pmatrix} \overline{p_{1,1}} & \overline{p_{1,2}} \\ \overline{p_{2,1}} & \overline{p_{2,2}} \end{pmatrix} \quad \begin{pmatrix} \bar{e} & \bar{f} \\ \bar{g} & \bar{h} \end{pmatrix} \begin{pmatrix} \bar{i} & \bar{j} \\ \bar{k} & \bar{l} \end{pmatrix} = \begin{pmatrix} \overline{q_{1,1}} & \overline{q_{1,2}} \\ \overline{q_{2,1}} & \overline{q_{2,2}} \end{pmatrix}$$

Example continued

The matric Massey product set is given by:

$$\left\{ \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} \overline{q_{1,1}} & \overline{q_{1,2}} \\ \overline{q_{2,1}} & \overline{q_{2,2}} \end{pmatrix} + \begin{pmatrix} \overline{p_{1,1}} & \overline{p_{1,2}} \\ \overline{p_{2,1}} & \overline{p_{2,2}} \end{pmatrix} \begin{pmatrix} \bar{i} & \bar{j} \\ \bar{k} & \bar{l} \end{pmatrix} \right\}$$

over all choices of lifts for $\overline{p_{i,j}}, \overline{q_{i,j}}$.

Matric Massey products obstruct to formality and quasi-isomorphism

Proposition (May, 1958)

If for some $X, Y, Z \in \text{Mat } H^\bullet(A)$, the Massey product $\langle X, Y, Z \rangle$ is nonzero, then A is not formal.

Proposition (Matric Massey products obstruct quasi-isomorphism)

Let A and B be dg-algebras with isomorphic cohomology rings. If for some $X, Y, Z \in \text{Mat}(H^\bullet(A))$ and all choices of ring isomorphism $f : H(A) \rightarrow H(B)$, the Massey product $f(\langle X, Y, Z \rangle) \subseteq H(B)$ differs from the Massey product $\langle f(X), f(Y), f(Z) \rangle \subseteq H(B)$ then A and B are not weakly equivalent.

Eilenberg-Moore spectral sequence

The bar construction is

$$B(A) = \left(\bigoplus_{i=0}^{\infty} A^{\otimes i}, \partial_1 + \partial_2 \right)$$

It has a natural filtration $F_N B(A) = \bigoplus_{i=0}^N A^{\otimes i}$. Therefore there is a naturally associated **Eilenberg-Moore spectral sequence**.

Theorem (Gugenheim–May, 1974)

The differentials in the Eilenberg-Moore spectral sequence are in 1-to-1 correspondence with higher matrix Massey products.

Generalizations of Massey products to other operads

- 1 Primitive Massey products have been generalized to dg-Lie algebras (Allday and Retah, 1977).
- 2 Massey triple products have been generalised to quadratic operads (Muro, 2023).
- 3 Primitive Massey products have generalized to quadratic (Koszul) operads \mathcal{P} ; they are related to the combinatorics of the tree monomials appearing the Koszul dual cooperad of \mathcal{P} . (F.C., Moreno-Fernandez).
- 4 Matric Massey products have not been generalised to other flavours of algebra: **even secondary operations**.

Advantages and disadvantages of Massey products

The advantages of Massey products are:

- Conceptually simple.
- Easy to calculate in practical situations.
- Almost always enough to show two spaces or algebras are different. (non-formality of Swiss-cheese operad, configurations spaces of homotopy equivalent lens spaces)
- The same philosophy works over arbitrary rings (**cotriple products**).
- They obstruct phenomena more subtle than formality.

The disadvantage of Massey products is:

- **Massey products are not a complete obstruction to formality or quasi-isomorphism.** They lose global data about the algebra. Therefore they **cannot** normally be used to prove that two \mathcal{P} -algebras are the same. Invariants constructed in other ways: such as Halperin-Stasheff obstructions or Kaledin classes are needed for such problems.

Determinant Massey products

Massey products for CDGAs

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- ③ This is controllable: there is a canonical correspondence between the vector space generated by the $(n - 2)^{th}$ order Com-Massey products and $\text{Lie}(n)$.
- ④ What are matrix Massey products in this setting?

Com-Eilenberg-Moore spectral sequence

The (operadic) bar construction is

$$B(A) = \left(\bigoplus_{i=0}^{\infty} \mathrm{Lie}(i) \otimes_{\mathbb{S}_i} A^{\otimes i}, \partial_1 + \partial_2 \right)$$

It has a natural filtration $F_N B(A) = \bigoplus_{i=0}^N \mathrm{Lie}(i) \otimes_{\mathbb{S}_i} A^{\otimes i}$. Therefore there is a naturally associated **Com-Eilenberg-Moore spectral sequence**.

Proposition (FC.-Moreno-Fernandez)

The primitive Com-Massey products can be identified with certain differentials of this spectral sequence.

Problem

What are the others?

Determinant Massey products

Suppose that a, b, c, e, l, v are even degree elements in the cohomology of an associative algebra A such that

$$ab + ce = bl - cv = el + av = 0.$$

Choose cocycles $\bar{a}, \bar{b}, \bar{c}, \bar{e}, \bar{l}, \bar{v}$ representing the corresponding classes, and assume $\bar{x}, \bar{y}, \bar{z}$ are elements such that

$$d\bar{x} = \bar{a}\bar{b} + \bar{c}\bar{e}, \quad d\bar{y} = \bar{b}\bar{l} - \bar{c}\bar{v}, \quad d\bar{z} = \bar{e}\bar{l} + \bar{a}\bar{v}.$$

Then

$$d(\bar{x}\bar{l} - \bar{a}\bar{y} - \bar{c}\bar{z}) = \bar{a}\bar{c}\bar{v} - \bar{c}\bar{a}\bar{v} = (\bar{a}\bar{c} - \bar{c}\bar{a})\bar{v}.$$

So this vanishes if $\bar{a}\bar{c} - \bar{c}\bar{a} = 0$. So it **always vanishes** in the commutative case but **not necessarily** if A is associative. In the commutative case, it gives rise to an operation with indeterminacy

$$l \cdot H^{|\bar{x}|}(A) + a \cdot H^{|\bar{y}|}(A) + c \cdot H^{|\bar{z}|}(A).$$

but **not in the associative case.**

Associated matrix Massey product

There is closely related matrix Massey product. Since that cohomology is associative, we have $ac - ca = 0$. Using this identity we see that the Massey triple product

$$\left\langle \begin{pmatrix} a & c \end{pmatrix}, \begin{pmatrix} b & c \\ e & -a \end{pmatrix}, \begin{pmatrix} I \\ -v \end{pmatrix} \right\rangle$$

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is defined. However, it has a **different indeterminacy**. The relation $ac - ca = 0$ is a relation on the algebra **not** the operad. So it is only defined up to homotopy. So the indeterminacy is

$$I \cdot H^{|\bar{x}|}(A) + a \cdot H^{|\bar{y}|}(A) + c \cdot H^{|\bar{z}|}(A) + v \cdot H^{|a|+|c|-1}(A).$$

So these are genuinely different operations.

These new Massey products are sensitive to commutative structure.

Determinant formalism

There is a formalism for these new operations using determinants. Define

$$U_D = (-1)^{|a|} a \zeta_a - (-1)^{|b|+|b||a|} b \zeta_b + (-1)^{|c|+|c||a|+|c||b|} c \zeta_c.$$

$$d\zeta_a = yc - bz, \quad d\zeta_b = xc - az, \quad d\zeta_c = xb - ay$$

Then

$$\begin{aligned} dU_D &= a(d\zeta_a) - (-1)^{|a||b|} b(d\zeta_b) + (-1)^{|a||c|+|b||c|} c(d\zeta_c) \\ &= a(yc - bz) - (-1)^{|b||a|} (xc - az) + (-1)^{|c||a|+|c||b|} c(xb - ay) \\ &= \underbrace{a \begin{vmatrix} y & b \\ z & c \end{vmatrix}}_{d\zeta_a} - \underbrace{b \begin{vmatrix} x & a \\ z & c \end{vmatrix}}_{d\zeta_b} + \underbrace{c \begin{vmatrix} x & a \\ y & b \end{vmatrix}}_{d\zeta_c} = \begin{vmatrix} a & x & a \\ b & y & b \\ c & z & c \end{vmatrix}. \end{aligned}$$

This vanishing of minors argument can be extended to define higher order products.

The commutative and associative quasi-isomorphism problem revisited

Definition (Stasheff, 1963)

An A_∞ -algebra is a graded vector space V along with a degree 1, square-zero coderivation

$$D : T^c V[1] \rightarrow T^c V[1]$$

where $T^c sV$ is the cofree conilpotent coalgebra on the suspension of V ie. $\bigoplus_{n \geq 1} (V[1])^{\otimes n}$

By cofreeness, D is determined by its value on cogenerators ie. by a succession of maps m_n of degree $2 - n$:

$$m_1 : V \rightarrow V$$

$$m_2 : V^{\otimes 2} \rightarrow V$$

$$m_3 : V^{\otimes 2} \rightarrow V$$

...

The square zero condition ensures these operations must satisfy the *Stasheff relations*.

Associative algebras

Example

Every associative algebra is an A_∞ -algebra with $m_1 : A \rightarrow A$ being the differential and m_2 being the multiplication on $A \otimes A \rightarrow A$ and $m_n = 0$ for $n \geq 2$.

Theorem

The minimal model for the A_∞ -operad is $B\Omega \text{ Ass}$

Proposition (Rectification of A_∞ -algebras)

Over \mathbb{Q} , every A_∞ -algebra is weakly equivalent to a strictly associative algebra and every C_∞ -algebra is weakly equivalent to a strictly commutative algebra .

The ∞ -commutative and associative quasi-isomorphism problem

Theorem (Saleh, 2017)

Over \mathbb{Q} , if a C_∞ -algebra is formal as an A_∞ -algebra, then it is formal as a C_∞ -algebra.

Theorem (Campos–Robert–Nicoud–Petersen–Wierstra, 2024)

Over \mathbb{Q} , if two C_∞ -algebras are quasi-isomorphic as A_∞ -algebras, they are quasi-isomorphic as C_∞ -algebras.

Homotopy commutativity and associativity

The A_∞ -operad comes equipped with a natural filtration

$$\mathcal{A}_2 \hookrightarrow \mathcal{A}_3 \hookrightarrow \mathcal{A}_4 \hookrightarrow \cdots \hookrightarrow \mathcal{A}_\infty.$$

An \mathcal{A}_n -**algebra** is an dg-algebra equipped with operations m_2, \dots, m_n where $m_i : A^{\otimes i} \rightarrow s^{i-2}A$. These operations must satisfy those Stasheff relations involving only m_2, \dots, m_n .

Slogan: An \mathcal{A}_n -algebra is associative up to $(n-2)^{th}$ coherent homotopy.

The C_∞ -operad has a similar filtration.

$$\mathcal{C}_2 \hookrightarrow \mathcal{C}_3 \hookrightarrow \mathcal{C}_4 \hookrightarrow \cdots \hookrightarrow \mathcal{C}_\infty.$$

Concretely, a \mathcal{C}_n -**algebra** is \mathcal{A}_n -algebra where the binary product m_2 is strictly graded-commutative.

Example

Definition

An \mathcal{A}_2 -algebra consists of a cochain complex (A, d) equipped with a bilinear, (not-necessarily-associative) binary, degree 0, product satisfying the relation

$$d(ab) = (da)b + (-1)^{|a|}a(db).$$

These are better known as **dg-magmas**. A \mathcal{C}_2 -algebra is an \mathcal{A}_2 -algebra whose product is strictly graded-commutative.

The \mathcal{A}_n and \mathcal{C}_n quasi-isomorphism problem

Question

If two \mathcal{C}_n -algebras are quasi-isomorphic as \mathcal{A}_n -algebras, are they quasi-isomorphic as \mathcal{C}_n -algebras?

The \mathcal{A}_n and \mathcal{C}_n quasi-isomorphism problem

Question

If two \mathcal{C}_n -algebras are quasi-isomorphic as \mathcal{A}_n -algebras, are they quasi-isomorphic as \mathcal{C}_n -algebras?

Answer

$$\begin{cases} \text{Yes} & \text{for } n = 2, \infty. \\ \text{No} & \text{otherwise.} \end{cases}$$

The $n = 2$ case

Theorem (FC–Moreno-Fernandez–Muro)

Let A be either a \mathcal{C}_2 or \mathcal{A}_2 -algebra in characteristic zero. Then A is formal.

Proof sketch: The cooperad $\Omega \mathcal{A}_2$ is tiny. Use this to construct an explicit $(\mathcal{A}_2)_\infty$ -map between A and its cohomology.

Corollary

Two \mathcal{C}_2 -algebras are quasi-isomorphic as \mathcal{C}_2 -algebras if and only if they are quasi-isomorphic as \mathcal{A}_2 -algebras.

Definition

An \mathcal{A}_3 -algebra is a cochain complex (A, d) equipped with two multilinear operations:

- 1 A (not-necessarily-associative) binary, degree 0, product satisfying the relation

$$d(ab) = (da)b + (-1)^{|a|}a(db).$$

- 2 A ternary degree -1 product $\alpha : A^{\otimes 3} \rightarrow A$ such that

$$d\alpha(a, b, c) = a(bc) - (ab)c - \left(\alpha(da, b, c) + (-1)^{|a|}\alpha(a, db, c) + (-1)^{|a|+|b|}\alpha(a, b, dc) \right).$$

A \mathcal{C}_3 -algebra is an \mathcal{A}_3 -algebra whose product is strictly graded-commutative.

The \mathcal{A}_3 and \mathcal{C}_3 quasi-isomorphism problem

Theorem (FC–Moreno-Fernandez–Muro)

There exists a \mathcal{C}_3 -algebra A that is formal as an \mathcal{A}_3 -algebra but not as a \mathcal{C}_3 -algebra.

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The \mathcal{A}_3 and \mathcal{C}_3 quasi-isomorphism problem

Theorem (FC–Moreno-Fernandez–Muro)

There exists a \mathcal{C}_3 -algebra A that is formal as an \mathcal{A}_3 -algebra but not as a \mathcal{C}_3 -algebra.

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The proof is by building a zig-zag of \mathcal{A}_3 -algebras, where A is a commutative algebra.

$$H(A) \xleftarrow{\sim} B \xrightarrow{\sim} A$$

The algebras $H(A)$ and A are distinguished by a (modified) determinant Massey product p . The algebra B only has the associated matrix Massey product which has a larger indeterminacy (you deliberately set the product $p = \{uv\}$ in A for some u).

On the result

- 1 Do \mathcal{C}_n -algebras appear in nature?
- 2 What do matrix Massey products look like for other operads (in particular, for Lie algebras).
- 3 A completeness result: describe all of the differentials in the operadic Eilenberg-Moore spectral sequence.

On things mentioned in passing

- 1 Geometrically interpret the category of E_∞ -algebras, ie. find a Quillen adjunction from a (possibly valued in some monoidal category (\mathcal{C}, \otimes) not **Set**) model presheaf category

$$\mathbf{PSh} X \rightleftarrows E_\infty - \mathbf{alg}$$

that induces an isomorphism on the homotopy categories at the level of objects.