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<u>My talk</u> Corecognition for iterated suspensions

Recollections (Reminder: Jt work w/ F. Wierstra)

<u>Ref</u> "The geometry of iterated loop spaces" (May, 1972)

"The homology of iterated loop spaces"

We want

1. Motivations: May wanted

i) a recognition principle for n-fold loop spaces (<u>suspensions</u>)

An appropriate ~~internal~~ external structure on $X \iff X$ is an n-fold loop space

ii) Usable geometric approximation to $\Omega^n \Sigma^n X$ ($\Sigma^n \Omega^n X$)

generalising James construction

(n-akin)

iii) A theory that leads to the "simple" development of homology (homotopy) operations

cut

<u>Thm</u> Every n-fold loop space is a $\mathcal{C}_n$-algebra, and ~~if a grouplike~~ every connected (grouplike) D_n-algebra has the weak homotopy type of an n-fold loop space

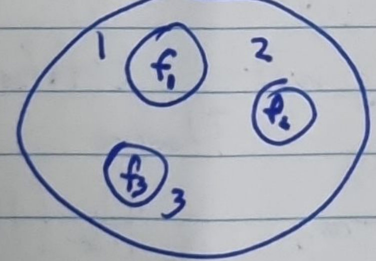
<u>Rmk</u> Beck's recognition theorem (replace D_n with $\Omega^n\Sigma^n$)

$\boxed{\Rightarrow}$ $D_n(r) \times (\Omega^n X)^{\times r} \to \Omega^n X$

$(y\,;\, f_1 \cdots f_n) \to F$

The $f_i : S^n \to X$

$X =$



<u>Def</u> Let P be an operad. For $X \in Top$, let $\simeq$ be the equivalence relation on the disjoint union $\coprod_{r\geq 0} P(r) \times X^{\times r}$ generated by

- $(\sigma_i\, c, y) = (c, s_i y)$ for $c \in C(r)$ & $y \in X^{\times(r-1)}$
- $(c\sigma, y) \simeq (c, \sigma y)$, $\sigma \in \Sigma_i$

$s_i : X^{\times(r-1)} \to X^{\times r}$

$(x_1 \cdots x_{r-1}) \mapsto (x_1 \cdots x_{i-1}, id, \cdots x_r)$

σ_i: forget i^{th} disc

$CX = \coprod_{r\geq 0} C(r) \times X^{\times r} / \simeq$

- CX has comonad structure map (μ, η) induced by the usual comonad maps on the Schur functor

Thm (Approx theorem) There is a natural map of comonads $\alpha_n : D_n X \to \Omega^n \Sigma^n X$ & α_n is a weak homotopy equivalence if X is connected

Proof A very messy induction using the p.t.2e an operadic version of $\Omega^n X \to PX \to X$ fibre sequence (12 pages)

and $D_n X \to D_n(\Omega^n \Sigma^n X) \to \Omega^n \Sigma^n X$

~~Reducing the recognition principle~~

- $X \leftarrow B(D, D, X) \to B(\Omega^n \Sigma^n, D, X) \xrightarrow{\gamma^n} \Omega^n B(\Sigma^n, D, X)$

- Beck's recognition theorem: n-~~fold~~ loop space

~~use~~ Same as recognition principle but for $\Omega^n\Sigma^n$-monad

4. Cohomology operations

- The comonad $\bigvee_{r=0}^{\infty} D_n(r) \times X^{\times r}/\sim$ has an obvious filtration

$$F^n C(X) = \bigvee_{r=0}^{n} D_n(r) \times X^{\times r}$$

The maps $F^n C(X) \hookrightarrow F^{n+1} C(X)$ turn out to be cofibrations with cofiber $D_n(r) \times (X - *)^{\times r}$. Therefore one can compute $H_*(\Omega^n \Sigma^n X)$ from $H_*(D_n(r))$

computed at each prime by Cohen

Eckmann-Hilton duality

1. - A yoga that says to every theorem there is dual theorem given by replacing words and diagrams by their opposites. For example: coalgebra $\Leftrightarrow$ algebra cohomology $\Leftrightarrow$ homotopy. $\vee \Leftrightarrow \wedge$

2. • Rationally, this is explained by the Koszul duality between Com (controls cochains / homology) & Lie (controls homotopy groups)

   • Probably there is an integral form.

3. Counterexample: $\Sigma\Omega S^n \simeq \bigvee S^i$ but $\Omega\Sigma K(n,\mathbb{Z}) \not\simeq$ ~~Π~~ ~~K(~~ product of Eilenberg-Maclane spaces

Restart

4. (Corecognition)

Thm connected Every connected n-fold suspension is a C_n-coalgebra, and if a connected pointed space is a C_n-coalgebra then it is homotopy equivalent to an n-fold suspension

Rmk Also a version of Beck's recognition theorem. $(C_n \Leftrightarrow \Sigma\Omega$

~~Thm~~ No connectivity assumptions

Thm For every $n \geq 1$, there is a natural morphism of comonads

$$\alpha_n : \Sigma^n \Omega^n \to C_n$$

Furthermore, for every pointed space, X there is an explicit natural homotopy retract of pointed spaces

$$\Sigma^n \Omega^n X \rightleftarrows C_n(X)$$

- In particular α_n is a weak equivalence

Previous work

1. **Conj** (Barratt – Stasheff) A co-H-space with A_∞-comultiplication has the ~~weak~~ homotopy type of a suspension

Thm (Berstein – Hilton; Ganea; Saito; Klein – Schwänzl – Vogt, 1997)
A $(n-1)$-connected co-H-space of dimension less than or equal to $k(n-1)+3$ with an A_k-multiplication is homotopy equivalent to a suspension

2. (May – Quinn) Not possible to directly Eckmann – Hilton dualise Recognition Principle

3. **Thm** (Blomquist – Harper, 22) A $(n+1)$-connected space that is a coalgebra over the $\Sigma^n\Omega^n$-comonad is h-eq to a n-fold suspension

Pf Cobar constructions + connectivity arguments
May-style arguments

Rmk Connectivity not optimal.

with a distinguished element $\theta \in \mathcal{P}(0)$ (After next page)

Comonad associated to an operad

Def Let $\mathcal{P}$ be an operad & X be a pointed space

let $Tot(\mathcal{P}, X) = \prod_{n \geq 0} Map_{S_n}(\mathcal{P}(n), X^{\vee n})$

The endofunctor

$$C_{\mathcal{P}} : Top_* \longrightarrow Top_*$$
$$X \longrightarrow C_{\mathcal{P}}(X)$$

where

$C_{\mathcal{P}}(X) = \{ \alpha = (f_1, f_2 \cdots) \in Tot(\mathcal{P}, X) \mid \pi_i f_n = f_{n-1} d_i$ for $n \geq 2$ & $1 \leq i \leq n \}$

$$\begin{tikzcd} \mathcal{P}(n) \arrow[r] \arrow[d, "d_i"'] & X^{\vee n} \arrow[d, "\pi_i"] \\ \mathcal{P}(n-1) \arrow[r] & X^{\vee (n-1)} \end{tikzcd}$$

<u>Functoriality</u>

Let $X \xrightarrow{f} Y$, then $C_{\mathcal{P}}(f) : C_{\mathcal{P}}(X) \to C_{\mathcal{P}}(Y)$

$C_{\mathcal{P}}(f)(\alpha) = (f \circ f_1, (f \vee f) \circ f_2, f^{\vee 3} \circ f_3 \cdots)$

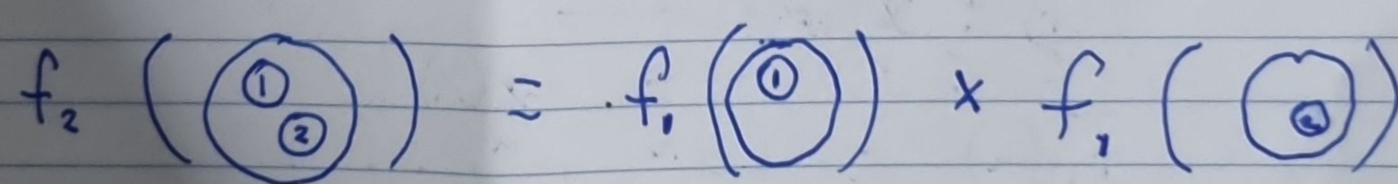
<u>Comonadic structure</u>

<u>Lemma</u> Let $\alpha := (f_1, f_2, \cdots) \in C_{\mathcal{P}}(X)$

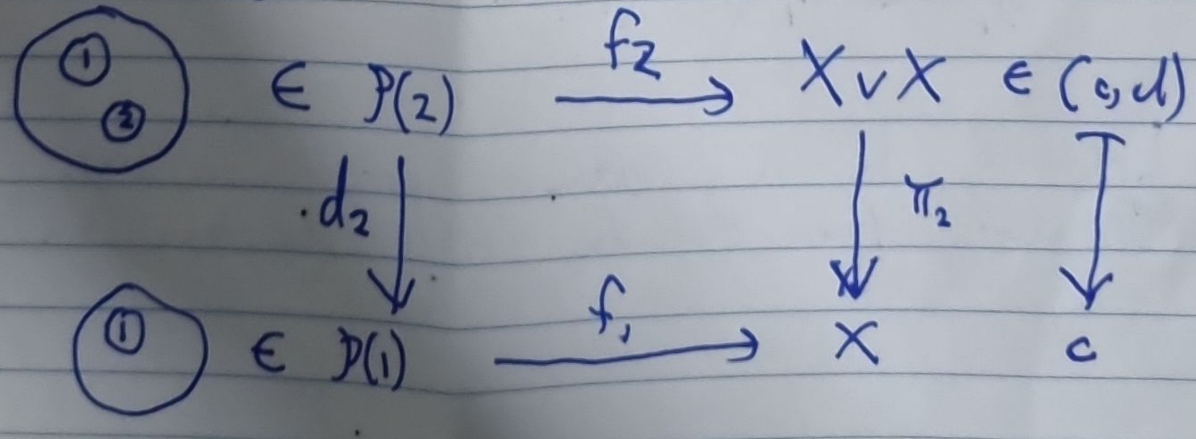
Then we can recursively write

$$f_r := f_1 \widehat{d_1} d_2 \cdots d_r \times \cdots \times f_1 d_1 d_2 \cdots \widehat{d_r}$$

Example



Proof sketch $r = 2$ case



Note $f_1 : \mathcal{P}(1) \longrightarrow X$ determines $(f_1, f_2 \cdots)$ but not every map $f_1 : \mathcal{P}(1) \to X$ can be extended to $\alpha = (f_1, f_2 \cdots)$

Note This shows there are no non-trivial strictly cocommutative coalgebras

Prop A map $f_1 : \mathcal{C}_n(1) \longrightarrow X$ belongs to $\mathcal{C}_n(X) \iff \forall c_1, c_2 \in \mathcal{C}_n(1)$ s.t. $c_1 \cap c_2 = \emptyset$ then $f_1(c_1) = *$ or $f_2(c_2) = *$

Example $f_1 : \mathcal{C}_n(1) \longrightarrow I$

$$f_1(c) = \begin{cases} 0 & \text{if radius } (c) \leq \frac{1}{2} \\ c - \frac{1}{2} & \text{if radius } (c) \geq \frac{1}{2} \end{cases}$$

Rmk We can identify $\mathcal{C}_n(X) \subseteq Map(\mathcal{P}(1), X)$

Def

$$\varepsilon_X : \mathcal{C}_{\mathcal{P}}(X) \longrightarrow X$$
$$f_1 \longmapsto f_1(id_{\mathcal{P}(1)})$$

$$\Delta_X : \mathcal{C}_{\mathcal{P}} \longrightarrow \mathcal{C}_{\mathcal{P}} \circ \mathcal{C}_{\mathcal{P}}(X)$$

$\cancel{f_1 \longmapsto f_1(\gamma(-; 1))}$

$\alpha = (f_1, f_2 \cdots) \Rightarrow \Delta_X(\alpha) = (\bar{f}_1, \bar{f}_2 \cdots)$

Now

$\cancel{\bar{f}_1 : \mathcal{C}_n(1) \longrightarrow}$

$$\bar{f}_1 : \mathcal{P}(1) \longrightarrow \mathcal{C}_{\mathcal{P}}(X)$$
$$\bar{f}_1(\mu) = (g_1^{\mu}, g_2^{\mu} \cdots)$$
$$g_1^{\mu} = f_1(\gamma(\mu; -))$$

Prop Let $\mathcal{P}$ be a unitary operad in Top. Then our two definitions of coalgebras are equivalent

4 Coalgebras & suspensions

~~Def Let $X \in Top_*$ & $\mathcal{P}$ be an operad. Then~~

Def The CoEndomorphism operad $CoEnd(X)$ is defined as

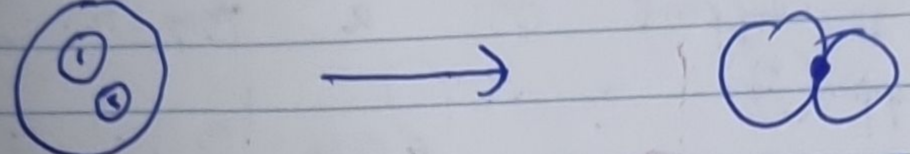
$$CoEnd(X)(r) := Map(X, X^{\vee r})^{\Sigma_r}$$ mention problems in Σ_r

$$CoEnd(r) \times CoEnd(n_1) \times \cdots \times CoEnd(n_r) \to CoEnd(n_1 + \cdots + n_r)$$

$$(f; f_1, \dots f_r) \longrightarrow (f_1 \vee \cdots \vee f_r) \circ f$$

* Mention what a coalgebra is here *

There is a coaction (Ginot, Tradler, Zeinalian)



$$\Delta_2 : D_n(2) \times S^n \to S^n \vee S^n \Rightarrow D_n \to CoEnd(S^n)$$

$$\downarrow$$

$$D_n(2) \times \Sigma^n X = D_n(2) \times (S^n \wedge X) \to (S^n \vee S^n) \wedge X = \Sigma^n X \vee \Sigma^n X$$

$$D_n \to CoEnd(\Sigma^n X)$$

The approximation theorem

$$\Sigma^n \Omega^n X \underset{\Psi}{\overset{\alpha_n}{\rightleftarrows}} C_n X \circlearrowleft H$$

~~Def The cubical support of a morphism~~ $\cancel{f : C_n(1) \to X}$

$$\Psi : C_n X \longrightarrow \Sigma^n \Omega^n X$$
$$f \longmapsto [\mathrm{Cent}(f), f]$$

Def The cubical support of a map $f : C_n(1) \to X$ is
$$\mathrm{CSupp}(f) = \bigcap_{\substack{c \in C_n(1) \\ f(c) \neq *}} \mathrm{Im}(c) \subseteq I^n$$

Fact $\mathrm{CSupp}(f) = \phi \iff f$ is the trivial map

f non-trivial $\Rightarrow$ $\mathrm{CSupp}(f)$ is a point or n-rectangle

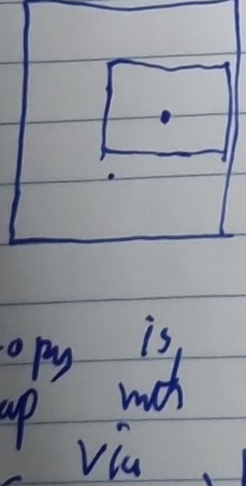
Def $\mathrm{Cent}(f) = \text{centre}(\mathrm{CSupp})$

Claim: For each pair of points $s, t \in I^n - \partial I^n$, there is a unique n-cube $c = c_{s,t} : I^n \to I^n$ s.t.

- $c(s) = t$
- $\mathrm{Im}(c)$ is the n-rectangle of maximum size contained in I^n and ~~touching~~ touching all faces of the boundary ∂I^n.

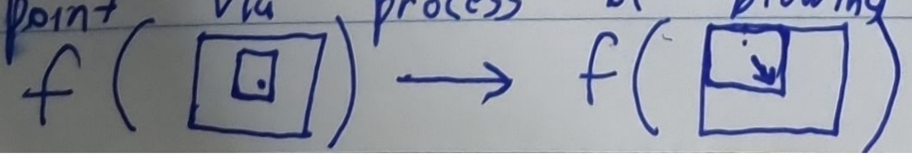
$$\tilde{f} : S^n \to X$$
$$s \longmapsto \tilde{f}(s) = f(c_{s, \mathrm{cent}(f)})$$

Intuition



Rectilinear expansion

Idea Homotopy is given deforming f into a map with r[illegible] support given by a point via process of "blowing" up cubes



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co Recognition Theorem

Thm Let X be a C_n-coalgebra. Then there is a pointed space $\Gamma^n(X)$, naturally associated to X along with a weak equivalence of C_n-coalgebras

$$\Sigma^n \Gamma^n(X) \xrightarrow{\sim} X$$

which is a retract in the category of pointed spaces. Therefore every C_n-coalgebra has the homotopy type of an n-fold suspension

1. **Thm** (Beck corecognition theorem)
Let X be a $\Sigma^n\Omega^n$-coalgebra. Then X is naturally isomorphic to the n-fold reduced suspension of a space ~~$P_n(X)$~~

$$P_n(X) := Eq\left(\Omega^n X \underset{\eta_{\Omega^n X}}{\overset{\Omega^n \gamma}{\rightrightarrows}} \Omega^n \Sigma^n \Omega^n X\right)$$

Pf

- Σ^n preserves equalisers
- ~~For a comonad C~~ For a coalgebra X over a comonad C, one has

$$X = Eq\left(CX \underset{\nabla X}{\overset{C\gamma}{\rightrightarrows}} CCX\right)$$

- Apply this to the $\Omega^n\Sigma^n$-comonad and pull the Σ^n out. □

Abstract nonsense

Let M be a category with finite limits & C_1, C_2 be two comonads & $\alpha: C_1 \to C_2$

$$\alpha_* : C_1\text{-coalg} \rightleftarrows C_2\text{-coalg} : \alpha^!$$

$\alpha^!$ is the universal coenveloping algebra

$$\alpha^! = Eq\left(\begin{tikzcd} C_1(X) \arrow[r, "C_1(\gamma_X)"] \arrow[dr, "\Delta_{C_1}"'] & C_1C_2(X) \\ & C_1C_1X \arrow[u, "C_1(\alpha_X)"'] \end{tikzcd}\right.$$

Lemma Let $C_1 \to C_2$ be a morphism of comonads which is a retract of pointed spaces at each level. If C_1 preserves equalisers, then the counit $\alpha_* \alpha^! \longrightarrow \mathrm{id}_{C_2\text{-coalg}}$ is a retract of pointed spaces at each level. In particular, for every C_2-coalg X the underlying map of pointed spaces $\alpha_* \alpha^!(X) \to X$ is a retract

Pf Play the same game as before as $\alpha^!$ is given by ~~...~~ Eq

Pf (Recognition Principle): ~~...~~ Let X be an D_n-alg. Then $(\alpha_n)_* \alpha^!(X) \to X$ retract. But $\alpha^!(X)$ is a $\Omega^n\Sigma^n$-coalg
$\Rightarrow$ (Beck) $\alpha^!(X) = \Sigma^n P_n X$

$$\Rightarrow X = \Sigma^n P_n \alpha^!$$

since α_* preserves the underlying algebra

Outlook and applications

1. One can filter comonad $F^n C(X) = \prod_{i=1}^{n} \mathrm{Map}(D_n(i), X^i)$ but the maps $F^{n+1} \to F^n$ are not fibrations

2. The E_n-structure on spheres can be studied in other contexts (i.e. higher Hochschild homology) and induces (higher) version of E_n-structure coming from Deligne conjecture