Higher commutativity in algebra and algebraic topology

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Abstract

This thesis consists of seven parts.

1. The first part consists of an introduction to the themes of this thesis, followed by a summary of the following parts. It is written both in French and in English.
2. The second part is about a corecognition principle for iterated suspensions and is based on an article written with Felix Wierstra and José Moreno-Fernandez, and is currently under review.
3. The third part is about a generalisation of Massey products to Koszul operads and is based on an article written with José Moreno-Fernandez, and is currently under review.
4. The fourth part introduces cotriple products and is about an obstruction theory for commutative algebras. It is intended for publication with a preprint currently available on ArXiv.
5. The fifth part is about the construction and properties of p-adic version of the de Rham complex. It is intended for publication.
6. The sixth part is devoted to a counterexample. We exhibit two commutative algebras that are quasi-isomorphic as E_∞-algebras but not as commutative algebras. It is intended for publication.
7. The seventh part is about a generalisation of the Hochschild-Konstant-Rosenberg theorem and the construction of Poisson coalgebra structures on chain level on the Hochschild complex. It is intended for publication.

Contents

Introduction

0.1 Introduction: a rather irreverent and incomplete account of the life and times of algebraic topology

In the beginning, Gauss[1] created topological spaces and manifolds. Now, it subsequently turned out that this may have been jumping the gun a bit, because manifolds are beautiful, mysterious and *very, very* complicated and topological spaces are often just pathological. Try finding an exotic 4-sphere or wrestling with a space-filling curve. Eventually something had to give, and most sensible people decided to study compact, Hausdorff topological spaces with the homotopy type of a CW-complex.

The main goal seemed rather humble. It would be nice just to be able to tell them apart. The trouble is that topological spaces are masters of disguise and can look both very similar when they are different and, as the old joke known to every maths undergrad about topologists, bagels and coffee cups suggests, extremely different even when the same. This meant that people needed to come up with global geometric invariants. Starting with Enrico Betti in 1871, Henri Poincaré and Camille Jordan, people stepped up to the plate and invented not one but two of them - cohomology and homotopy groups. Roughly speaking, these are groups that *count the number of holes* in your object in two slightly different ways.

At first, everything was brilliant and the future seemed bright[2]. But cracks quickly started to emerge. Firstly, it was noticed that cohomology wasn't a complete invariant. For example, it can't tell the difference between $S^2 \vee S^1 \vee S^1$, two circles and a sphere glued together, and a torus, which makes the aforementioned undergrad joke slightly more embarrassing. Secondly, in 1931, Heinz Hopf discovered the Hopf fibration, a nontrivial map from $S^3 \to S^2$ built from decomposing S^3 into a monstrous union of interlocked circles indexed by S^2 , demonstrating conclusively the equation

$$\text{cohomology} \neq \text{homotopy}$$

and, incidentally, also demonstrating the futility of some of my previous analogies (it is rather disobliging to think of the Hopf fibration as a type of hole). Hopf opened Pandora's box, which

[1] and many others.

[2] for the subject that is, not so much for algebraic topologists who would have been out of a job.

turned out to be full of such maps, each more progressively horrifying than the last. Like the ghost of Anne Boleyn, these have been terrifying topologists for generations at this point. We still don't know the homotopy groups of spheres, but it's no surprise that our best tools are *spectral* sequences.

The first problem was easier to deal with. The answer was to add more stucture. Alexander, Čech, Whitney and Eilenberg observed that the diagonal map $X \to X \times X$ actually carried a lot of structure, after you flattened it to make it work algebraically. In particular, it turned the cohomology groups $\bigoplus_{i=0}^{\infty} H^i(X)$ into a commutative ring. One problem: this still wasn't a complete invariant of space. While it can at least deal with the previous example, it still can't tell $\left(\mathbb{C}P^{\infty} \times S^1\right) / \left(\{x_0\} \times S^1\right)$ from $\mathbb{C}P^{\infty} \times S^3$.

In 1958, Massey discovered that the first *higher invariant*, the Massey triple product. Computationally the idea is pretty simple. Suppose you have $a, b, c \in \bigoplus_{i=0}^{\infty} H^i(X)$ such that $ab = 0$ and $bc = 0$. Then on the cochain level, this *happens for a reason*, you have $\bar{a}\bar{b} = du$ and $\bar{b}\bar{c} = dv$. Then, up to sign, $\bar{a}v + u\bar{c}$, is a cocycle and represents something in cohomology. This *something* is not well-defined up to homotopy, but the failure is predictable, and an invariant of spaces (in fact, of differential graded algebras) can be extracted from it. Massey used it to show that the Borromean rings are linked. If you recall that the cup product can tell if two rings are pairwise interlinked, that should give you the intuition behind what it does. It actually just the first in an infinite family of invariants for dgas that can be extracted this way.

The idea of Massey products found final fruition in the pioneering work of Quillen and Sullivan in rational homotopy theory. Sullivan showed that, given a simplicial set X, one can build a commutative graded algebra $A_{PL}(X)$ over the rationals, modelled on de Rham forms, that captures the complete homotopy type of finite type, nilpotent, rational spaces. Essentially, all the information contained in these models can be summed up as being Massey products along with coherence information. This approach is incredibly elegant and efficient; one has unique *minimal models* for spaces that allows one to detect homotopy equivalences of spaces as isomorphisms of algebras.

At the same time, Stasheff, Boardman-Vogt and May were studying the same picture from a slightly different angle. They were trying to understand the structure of that of iterated loop spaces ie. for k-connected X, $k \geq n$, the mapping space $\Omega^n X = \mathsf{Map}(S^n, X)$. In particular, they wanted to compute $H^*(\Omega^n X)$ given $H^*(X)$ and some structure. To do this, May, Boardman-Vogt and others developed the notion of an operad. For an undergrad in mathematics, an operad can be thought of as a bit like a group: it abstracts away the properties of some kind of multiplicative structure like an associative or Lie algebra. Then you have algebras, which are a bit like group representations in this bad analogy, they are concrete examples of objects equipped with such a multiplication. Using this new idea, he was able to prove his celebrated recognition theorem: *Every grouplike space is an algebra over the little n-discs operad if and only if it is an n-fold loop space.* Algebras over the little n-discs operad are also called E_n-algebras. This is because they are not commutative on the nose, but are up to homotopy. And those homotopies are commutative up to homotopy and... (repeat this sentence another $n-2$ times, if you have the patience).

The notion of an operad was applied immediately more generally. In particular, it suddenly made it possible to talk about algebras that were commutative up to coherent homotopy, objects now called E_∞-algebras. Most importantly, Mandell showed that this is precisely the correct notion that describes the cochain-level cup product on spaces and Fresse and Berger gave explicit combinatorial models for it. In 2006, in a result that can only be described as stunning, Mandell went one step further and solved algebraic topology[3]. He showed that the homotopy type of the singular cochain complex as an E_∞-algebra is a complete homotopy invariant of finite type, nilpotent spaces. Topology thus evaporates, leaving only higher algebra in its wake. The golden age of algebraic topology ended and the era of homotopy theory began.

With that rather dramatic finishing thought, we move a more technical discussion of the contents of this thesis.

0.2 Synopsis of the thesis

0.2.1 Corecognition for suspensions

In this chapter, which is based on an article [33] which is joint work with Moreno-Fernández and Wierstra, we prove a recognition principle for iterated suspensions. The main result is the following, where $\mathscr{C}_n$ is the little n-discs operad, which is a model for the E_n-operad.

Theorem A. *Every n-fold suspension is a $\mathscr{C}_n$-coalgebra, and if a pointed space is a $\mathscr{C}_n$-coalgebra then it is homotopy equivalent to an n-fold suspension.*

In particular, the reader should note that our proof methods are not dual to May's and most of our statements are slightly stronger, generally giving homotopy equivalence rather than weak equivalence.

0.2.1.1 Background

The main piece of background to the result is May's recognition principle [66], which states that a grouplike space is an E_n-algebra with respect to the Cartesian product $\times$ and only if it has the weak homotopy type of an n-fold loop space. As an intermediate step in his proof, May also had an *approximation theorem*, which states that the monad associated to the little n-cubes operad is weakly equivalent to the monad $\Omega^n\Sigma^n$, where Ω is the loop space functor and Σ is the suspension functor.

0.2.1.2 Coalgebras over topological operads: approach 1

We begin by defining the notion of coalgebra over an operad. The context is that we wish to work in the category of pointed topological spaces $(\mathsf{Top}_*, \vee)$ equipped with the categorical coproduct $\vee$. In $(\mathsf{Top}_*, \vee)$, every object has an associated operad, given by

$$\mathrm{CoEnd}(X)(n) = \mathsf{Map}_{Top_*}\left(X, X^{\vee n}\right)$$

The symmetric group acts by permutation of factors in the wedge sum and the operadic composition comes from the function composition $(f; f_1, f_2, \ldots f_n) \mapsto \left(f_1 \vee \cdots \vee f_n\right) \circ f$. Let $\mathscr{P}$ be a (unpointed) operad in topological spaces. Then a *coalgebra over $\mathscr{P}$* is a morphism of

[3] only a slight exaggeration

operads $\mathscr{P} \to \mathrm{CoEnd}(X)$. This fits very well with the *common sense* definition of a coalgebra being a collection of maps

$$\triangle_n : \mathscr{P}(n) \times X \to X^{\vee n}.$$

0.2.1.3 Coalgebras over the little n-cubes operad

The main example of the previous definition is n-fold suspensions. In particular, n-spheres are equipped with such a structure via the pinch map, and this can be generalised to n-fold suspensions. The precise statement is as follows.

Theorem B. *The n-fold reduced suspension of a pointed space X is a $\mathscr{C}_n$-coalgebra. More precisely, there is a natural and explicit operad map*

$$\nabla : \mathscr{C}_n \to \mathsf{CoEnd}_{\Sigma^n X},$$

where $\mathrm{CoEnd}_{\Sigma^n X}$ *is the coendomorphism operad of $\Sigma^n X$. The map ∇ encodes the homotopy coassociativity and homotopy cocommutativity of the classical pinch map $\Sigma^n X \to \Sigma^n X \vee \Sigma^n X$. In particular, the pinch map is an operation associated to an element of $\mathscr{C}_n(2)$. Furthermore, for any based map $X \to Y$, the induced map $\Sigma^n X \to \Sigma^n Y$ extends to a morphism of $\mathscr{C}_n$-coalgebras.*

0.2.1.4 Coalgebras over topological operads: approach 2

The second approach to coalgebras is to define a comonad associated to a unitary operad $(\mathscr{P}, *)$. The comonad then has a naturally associated co-Eilenberg-Moore category, and one can define coalgebras to be precisely the elements of this category.

To make this precise, let $(\mathscr{P}, *)$ be a unitary operad, where $* \in \mathscr{P}(0)$ is the unit. We shall define the comonad to be a subspace of the following construction.

$$\mathsf{Tot}(\mathscr{P}, X) := \prod_{n \geq 0} \mathsf{Map}_{S_n}\left(\mathscr{P}(n), X^{\vee n}\right).$$

To define the subspace, we need the *restriction operators*, given by inserting the unique point $* \in \mathscr{P}(0)$ at the i-th component:

$$\begin{aligned} \mathscr{P}(n) &\xrightarrow{\ d_i\ } \mathscr{P}(n-1) \\ \theta &\longmapsto \gamma(\theta; \mathrm{id}, \ldots, *, \ldots, \mathrm{id}). \end{aligned}$$

We also need the *wedge collapse maps*, which are given by collapsing the i-th factor in the wedge as follows:

$$\begin{aligned} X^{\vee n} &\xrightarrow{\ \pi_i\ } X^{\vee (n-1)} \\ (x_1, \ldots, x_n) &\longmapsto (x_1, \ldots, \widehat{x_i}, \ldots, x_n). \end{aligned}$$

Let $\mathscr{P}$ be a unitary operad in Top. Now the endofunctor in pointed spaces

$$\begin{aligned} C_{\mathscr{P}} : \mathsf{Top}_* &\longrightarrow \mathsf{Top}_* \\ X &\longmapsto C_{\mathscr{P}}(X), \end{aligned}$$

where

$$C_{\mathscr{P}}(X) = \left\{\alpha = (f_1, f_2, \ldots) \in \operatorname{Tot}(\mathscr{P}, X) \;\middle|\; \pi_i f_n = f_{n-1} d_i \text{ for all } n \geq 2 \text{ and } 1 \leq i \leq n\right\}$$

is the subspace of $\operatorname{Tot}(\mathscr{P}, X)$ formed by those sequences $(f_1, f_2, \ldots)$ that commute with the restriction operators and wedge collapse maps.

To produce the comonadic structure on this object, it is necessary to analyse it a little. It turns out that all points $\alpha = (f_1, f_2, \ldots) \in C_{\mathscr{P}}(X)$ are determined by extending the $f_1 : \mathscr{P}(1) \to X$ component in a unique way[4]. So one can define the comonadic structure by defining a

$$\triangle(f_1) : \mathscr{P}(1) \to \mathsf{Map}(\mathscr{P}(1), X)$$

coming from the adjoint of the operadic composition $\mathscr{P}(1) \times \mathscr{P}(1) \xrightarrow{\gamma} \mathscr{P}(1)$.

0.2.1.5 Coalgebras over topological operads: conclusions

The reader should not be surprised to note that the two categories of coalgebras described above coincide. There is an immediate important consequence of the following obervation. Firstly, using the latter definition, one can show that if $\mathscr{P}(1) = *$, then the only coalgebra over $\mathscr{P}$ is the point $*$. In particular, there are no nontrivial strictly cocommutative or coassociative coalgebras in spaces and therefore no analogue of the Moore loop space for suspensions. This (among other complexities relating to extensive use of quasi-fibrations) makes May's approach to the recognition principle non-dualisable.

0.2.1.6 The coapproximation theorem

An intermediate result that is interesting in its own right is the following, which is dual to May's approximation theorem. May's version of this result was key to the computation of the cohomology of iterated loop spaces, and we hope this result may hold similar promise. It is proven by directly constructing the desired homotopy via geometric arguments.

Theorem C. For every $n \geq 1$, there is a natural morphism of comonads

$$\alpha_n : \Sigma^n \Omega^n \longrightarrow C_n.$$

Furthermore, for every pointed space X, there is an explicit natural homotopy retract of pointed spaces

$$\begin{tikzcd} \Sigma^n \Omega^n X \arrow[r, shift left] & C_n(X) \arrow[l, shift left] \arrow[loop above] \end{tikzcd}$$

In particular, $\alpha_n(X)$ is a weak equivalence.

[4] Though the reader should be warned that not all maps $f_1 : \mathscr{P}(1) \to X$ extend to such an α

0.2.1.7 The recognition theorem

The paper contains two recognition theorems. The first is a recognition principle over the $\Sigma^n\Omega^n$-comonad. This is dual to Beck's recognition principle.

Theorem D. Let X be a $\Sigma^n\Omega^n$-coalgebra. Then X is naturally isomorphic to the n-fold reduced suspension of a space $P_n(X)$ which can be computed as the equalizer of the following pair of maps:

$$\Omega^n X \underset{\eta_{\Omega^n X}}{\overset{\Omega^n\gamma}{\rightrightarrows}} \Omega^n\Sigma^n\Omega^n X.$$

Here, η is the unit of the (Σ^n, Ω^n) adjunction, and γ is the $\Sigma^n\Omega^n$-coalgebra structure map of X.

The second, which is deduced from it via categorical arguments, is dual to May's recognition principle.

Theorem E. Let X be a $\mathscr{C}_n$-coalgebra. Then there is a pointed space $\Gamma^n(X)$, naturally associated to X, together with a weak equivalence of $\mathscr{C}_n$-coalgebras

$$\Sigma^n\Gamma^n(X) \xrightarrow{\simeq} X,$$

which is a retract in the category of pointed spaces. Therefore, every $\mathscr{C}_n$-coalgebra has the homotopy type of an n-fold reduced suspension.

0.2.2 Koszul duality and Massey products

In this chapter, which is based on an article [31] with Moreno-Fernandez, we generalise Massey products to arbitary Koszul operads and compute some examples.

0.2.2.1 Background

Massey triple products were introduced by Massey in 1958 [60]. Their main purpose is in rational homotopy theory, where they detect and quantify the non-formality of differential graded algebras. They are the first in an infinite sequence of higher operations that perform the same function. Similar operations have been defined for Lie algebras by Allday and Retah [2]. Massey triple products were generalised to arbitary quadratic operads by Muro [70].

0.2.2.2 Higher $\mathscr{P}$-Massey products

Massey products for a Koszul operad $\mathscr{P}$ are in correspondence with cooperations in the Koszul dual cooperad $\mathscr{P}^{\text{i}}$. The precise correspondence is somewhat technical. First, recall that $\mathscr{P}^{\text{i}}$ is naturally weight-graded. We shall define Massey products by induction on weight using the following map to do the induction.

Definition 0.2.1. The *Massey inductive map* is the degree -1 map

$$D : \mathscr{F}^c(sE) \xrightarrow{\Delta^+} \mathscr{F}^c(sE) \circ \mathscr{F}^c(sE) \xrightarrow{\kappa\circ\mathrm{id}} E \circ \mathscr{F}^c(sE).$$

Applied to some cooperation μ, we shall write

$$D(\mu) = \sum (\zeta; \zeta_1, \ldots, \zeta_m; \sigma), \tag{1}$$

where $\zeta \in E(m)$, $\zeta_i \in \mathscr{F}^c(sE)(v_i)$, $\sigma \in \mathbb{S}_m$ and $v_1 + \cdots + v_m$ is equal to the arity of μ.

Next, we shall need the following set, which shall provide the indices that we shall use when defining the products.

Definition 0.2.2. Let $\Gamma^c \in \mathscr{P}^{\mathrm{i}}(r)$ be a weight-homogeneous cooperation. For each permutation $(k_1,...,k_r) \in \mathbb{S}_r$, we define the Γ^c*-indexing set* $I(\Gamma^c, (k_1,...,k_r))$ by induction on the weight $w(\Gamma^c)$ of Γ^c as follows.

- If $w(\Gamma^c) = 0$, then $I(\Gamma^c,) = \emptyset$.
- If $w(\Gamma^c) = 1$, then $I(\Gamma^c,) = \{(\mathrm{id}, (1)), ..., (\mathrm{id}, (r))\}$.

Assume next that $I(\Gamma^c, (k_1,...,k_r))$ has been defined for cooperations up to weight n, and suppose Γ^c is of weight $n+1$. If

$$D\left(\Gamma^c\right) = \sum (\zeta; \zeta_1, \ldots, \zeta_m; \sigma)$$

as in Equation (2.5), and the leaves on top of each ζ_i are labeled $l_1,...,l_{v_i}$, then

$$I\left(\Gamma^c, (k_1,...,k_r)\right) := \bigcup_{i=1}^{m} I\left(\zeta_i, \left(k_{l_1},...,k_{l_{v_i}}\right)\right) \cup \left\{\left(\zeta_i, \left(k_{l_1},...,k_{l_{v_i}}\right)\right)\right\}.$$

Finally, we can define the $\mathscr{P}$-Massey products on an algebra A.

Definition 0.2.3. Let A be a $\mathscr{P}$-algebra, $\Gamma^c \in \left(\mathscr{P}^{\mathrm{i}}\right)^{(n)}(r)$ with $n \geq 2$, and $x_1,...,x_r$ homogeneous elements of $H_*(A)$. Then:

1. The Γ^c-*Massey product* associated to a Γ^c-defining system $\{a_\alpha\}$ and $x_1,...,x_r$ is the homology class of the cycle

$$a_{\Gamma^c,(1,...,r)} := \sum (-1)^\gamma \zeta\left(a_{\zeta_1,(\sigma^{-1}(1),\sigma^{-1}(2),...,\sigma^{-1}(v_1))}, \ldots, a_{\zeta_m,(\sigma^{-1}(v_1+\cdots+v_{m-1}+1),...,\sigma^{-1}(r))}\right), \quad (2)$$

   where $D(\Gamma^c) = \sum (\zeta; \zeta_1, \ldots, \zeta_m; \sigma)$, and the sign is given by

$$\gamma = \alpha + \sum_{i=2}^{m} \left(|\zeta_i| - \mathrm{wgt}(\zeta_i)\right)\left(\sum_{k=1}^{v_1+\cdots+v_{i-1}} |x_{\sigma^{-1}(k)}|\right) + 1, \qquad \alpha = \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} |x_i||x_j|,$$

   where $\mathrm{wgt}(\zeta_i)$ is the weight of ζ_i.

2. The Γ^c-*Massey product set* $\langle x_1, \ldots, x_r\rangle_{\Gamma^c}$ is the (possibly empty) subset of $H_*(A)$ formed by the homology classes arising from all possible choices of Γ^c-defining systems $\{a_\alpha\}$ associated to $x_1,...,x_r$.

0.2.2.3 Examples

In the cases of associative and Lie algebras, we recover the classical examples of Massey products as defined by Massey, and Allday and Retah. The secondary operations that we obtain are precisely those defined by Muro.

A novel example in our paper is the *dual numbers operad* $\mathscr{D}$ is concentrated in arity 1, and has bicomplexes as algebras over it. The $\mathscr{D}$-Massey products are precisely differentials in the associated spectral sequence. We also compute a fourth order Massey product for the Poisson operad. In general, computing $\mathscr{P}$-Massey products is very difficult to do by hand, when the operad $\mathscr{P}$ is generated by more than one operation.

0.2.2.4 Properties of $\mathscr{P}$-Massey products

The $\mathscr{P}$-Massey products have all the properties that you would expect. Most importantly, they are invariants of homotopy-type

Theorem F. There is a bijection between the Massey product sets of weakly-equivalent $\mathscr{P}$-algebras.

This immediately implies the following corollary.

Corollary 0.2.4. *If a $\mathscr{P}$-algebra A has a nontrivial Massey product, then it is not formal. That is to say, A is not quasi-isomorphic to its homology (viewed as a $\mathscr{P}$-algebra with the obvious induced structure) as a $\mathscr{P}$-algebra.*

0.2.2.5 The Eilenberg-Moore spectral sequence

Let A be an algebra over a Koszul operad $\mathscr{P}$ and $H = H_*(A)$ be its homology. There is a canonical homology theory associated to algebras over an operad known as Quillen homology. The *$\mathscr{P}$-Eilenberg–Moore spectral sequence*, that computes the Quillen homology of A (as long as A is positively graded of finite type). It is constructed as follows. The operadic chain complex $\mathscr{P}^{\text{i}}(A)$ admits the ascending filtration

$$F_p\mathscr{P}^{\text{i}}(A) = \bigoplus_{n=1}^{p} \mathscr{P}^{\text{i}}(A)^{(n)}.$$

This filtration is bounded below and exhaustive, and so the associated spectral sequence, as a graded module, converges to the operadic homology of A. The complex $\mathscr{P}^{\text{i}}(A)$ also has the structure of a conilpotent cofree $\mathscr{P}^{\text{i}}$-coalgebra with comultiplication Δ, which respects the filtration in the sense that

$$\Delta\left(F_p\mathscr{P}^{\text{i}}(A)\right) \subseteq \bigoplus_{k=1}^{p} \bigoplus_{i_1+\cdots+i_k=p} \mathscr{P}^{\text{i}}(k) \otimes \left(F_{i_1}\mathscr{P}^{\text{i}}(A) \otimes \cdots \otimes F_{i_k}\mathscr{P}^{\text{i}}(A)\right).$$

This further implies that each page of the spectral sequence inherits a $\mathscr{P}^{\text{i}}$-coalgebra structure, and furthermore, the spectral sequence converges as a $\mathscr{P}^{\text{i}}$-coalgebra.

The relationship with our Massey products is the following

Theorem G. Let A be a $\mathscr{P}$-algebra, and $x_1, \ldots x_r$ homology classes such that the Massey product set $\langle x_1, \ldots, x_r\rangle_{\Gamma^c}$ is defined for a cooperation $\Gamma^c \in \mathscr{P}^{\text{i}}(r)^{(n)}$. Then the element

$$\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r \in \left(\mathscr{P}^{\text{i}}\right)^{(n)}(r) \otimes H_*(A)^{\otimes r}$$

survives to the E^{n-1} page in the $\mathscr{P}$-Eilenberg–Moore spectral sequence, and for $x \in \langle x_1, \ldots x_n\rangle$

$$d^{n-1}\left(\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r\right) \in (-1)^{n-2}\,[\mathrm{id} \otimes x]\,.$$

For the case of the associative operad, this recovers a known result in the literature.

0.2.2.6 The relationship with $\mathscr{P}_\infty$-structures

Classically, it is well-known that Massey products on a $\mathscr{P}$-algebra A have a very close relationship with the transferred $\mathscr{P}_\infty$-structures on the homology. We make precise the relationship with the following theorem

Theorem H. Let A be an algebra over a reduced Koszul operad $\mathscr{P}$, and let H be its homology. Let $\Gamma^c \in \left(\mathscr{P}^{¡}\right)^{(n)}(r)$, and assume that $x_1, ..., x_r$ are $r \geq 3$ homogeneous elements of H for which the Γ^c-Massey product set $\langle x_1, \dots x_r \rangle_{\Gamma^c}$ is defined. Let $x \in \langle x_1, \dots x_r \rangle_{\Gamma^c}$. Then:

(i) For any $\mathscr{P}_\infty$ structure δ on H quasi-isomorphic to A, we have

$$\delta^{(n)}\left(\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r\right) = x + \Phi,$$

where $\Phi \in \sum_{i=1}^{n-1} \mathrm{Im}\left(\delta^{(i)}\right)$.

(ii) If $\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_l}$ are linearly independent in the corresponding copy of $\mathscr{P}^{¡} \otimes_{\mathbb{S}_l} A^{\otimes l}$, where $(\mu, (i_1, ..., i_l)) \in I(\Gamma^c)$, then there is a choice of $\mathscr{P}_\infty$ structure δ on H which recovers x.

In the case of associative algebras, this recovers a theorem of Buijs, Moreno-Fernández and Murillo [16].

0.2.3 Cotriple products and strictly commutative algebras

In this chapter, which is based on the recently uploaded ArXiV prprint [29], we study how the theory of strictly commutative algebras in positive characteristic. This is merely the most practical example. We believe our methods should work more generally.

0.2.3.1 Cotriple products

In characteristic 0, Sullivan's theory of minimal models tells us that the homotopy type of an algebra over an operad essentially consists of Massey products along with some coherence data. Unfortunately, the same is not true in positive characteristic because the symmetric action creates extra data, which are usually called Steenrod operations. These are only visible at the algebra level.

This is where the idea of cotriple products comes in. Let $\mathscr{P}$ be an operad such that

$$V \cong W \implies \mathscr{P}(V) \cong \mathscr{P}(W)$$

and let A be a $\mathscr{P}$-algebra. The *cotriple resolution* $\mathsf{Res}_{\mathscr{P}}(A)$, is a free resolution of A in the category of simplicial $\mathscr{P}$-algebras. Filtering this by the skeletal filtration, we obtain a naturally associated *cotriple spectral sequence*. The differentials in the sequence are defined to be the *cotriple operations*.

0.2.3.2 Alternative description via Sullivan algebras

There is an alternative description of cotriple products that is easier to do computations with and which makes sense even when $\mathscr{P}$ does not reflect homotopy equivalences.

Definition 0.2.5. Let $\mathscr{P}$ be an operad over a field and A is a $\mathscr{P}$-algebra. An *N-step Sullivan model* for A is a semi-free algebra $f : (\mathscr{P}(\bigoplus_{i=0}^{N} V_i), d)$ such that

- the map $f|_{V_0} : V_0 \to A$ is a is a weak equivalence of dg-vector spaces. In particular $V_0 = H^*(A)$.
- the differential satisfies $d(V_k) \subseteq (\mathscr{P}(\bigoplus_{i=0}^{k-1} V_i), d)$.
- the map $V_k \oplus (\mathscr{P}(\bigoplus_{i=0}^{k-1} V_i) \to A$ is a weak equivalence for each $k \leq N$.

The connection between this and cotriple products is given by the following theorem.

Theorem I. Let $\mathscr{P}$ be an operad that reflects homotopy equivalences. Let A be an $\mathscr{P}$-algebra and fix a choice of $f : (\mathscr{P}(\bigoplus_{i=0}^{N} V_i), d) \xrightarrow{\sim} A$ a N-step Sullivan model for A. Let $\sigma \in I(\bigoplus_{i=1}^{N} V_i)$ be a cocycle. Then there exists an element

$$G(\sigma) \in \mathscr{P}^{\circ N}(H)$$

which survives to the E_N-term of the $\mathscr{P}$-cotriple spectral sequence, and

$$d_{N-1}([G(\sigma)]) \in (-1)^{N-2}\left[\mathrm{id} \otimes H^*(f(\sigma))\right].$$

0.2.3.3 Secondary cotriple operations for commutative algebras

From the previous definition, one has the following description of cotriple products for strictly commutative algebras.

Proposition 0.2.6. *All secondary primitive cotriple products on a commutative dg-algebra A over $\mathbb{F}_p$ are linear combinations of*

- *classical Massey products.*
- *Type 1 secondary Frobenius operations*
- *Type 2 secondary Frobenius operations.*

See Subsection 3.4.1 for the precise definition of the Frobenius operations. These extra two operations that are appear can be used to construct a number of counterexamples to characteristic 0 behaviour. Most notably, there examples of commutative dg-algebras that are formal over $\mathbb{Q}$ but not $\mathbb{F}_p$ and an example of an algebra that has a divided power structure on its cohomology is nonetheless not quasi-isomorphic to a divided power algebra. Finally we have the following counterexample that answers a question of Campos, Petersen, Robert-Nicoud and Wierstra [17, Section 0.3].

Theorem J. There exists A and B be two commutative dg algebras over a field of characteristic two which may be distingushed via their type 1 Frobenius operation. Nonetheless, there exists an associative algebra C such that there is a zig-zag of associative weak equivalences

$$A \xleftarrow{\sim} C \xrightarrow{\sim} B$$

0.2.3.4 Rectifiability

Finally, we set out to answer the following question. We say that an $\mathscr{E}$-algebra is *rectifiable* if it is weakly equivalent to a strictly commutative algebra. In characteristic p, not every $\mathscr{E}$-algebra is rectifiable, there are obstructions given by Steenrod operations and operations ccrresponding to syzygies between them.

Definition 0.2.7. Let A be an E_∞-algebra over $\mathbb{F}_p$. Then the higher Steenrod operations *vanish coherently* if for every (or any) Sullivan resolution $(\mathscr{E}(\bigoplus_{i=0}^{\infty} V_i), d)$ for A, there exists a splitting $V_i = X_i \bigoplus Y_i$, with $X_0 = V_0$; such that $(\mathrm{Sym}(\bigoplus_{i=0}^{\infty} X_i), d)$ is a Sullivan algebra and the kernel of

$$(\mathscr{E}(\bigoplus_{i=0}^{\infty} V_i), d) \to (\mathrm{Sym}(\bigoplus_{i=0}^{\infty} X_i), d)$$

is acyclic.

The cocycles appearing in the kernel represent Steenrod operations. For example, the kernel of the $\mathscr{E}(V_0) \to \mathrm{Sym}(V_0)$ component are precisely the Steenrod operations and the definition of a Sullivan algebra immediately implies that these extra cocycles are killed by Y_1.

Then we are able to prove the expected but satisfying following result.

Theorem K. Let A be an E_∞-algebra over $\mathbb{F}_p$. Then A is rectifiable if and only if its higher Steenrod operations vanish coherently.

0.2.4 Comparing the homotopy category of E_∞-algebras and commutative algebras

This section is the shortest of the thesis. In it, working in characteristic 2, we exhibit an explicit example of two commutative algebras A and B that are not weakly equivalent as commutative algebras but are weakly equivalent as E_∞-algebras.

To be more specific, A and B can be distingushed via a third order cotriple product. The operation in question is essentially the Frobenius operation iterated twice. However, this operation has greater indeterminacy in the category of E_∞-algebras than commutative algebras. This means that we can find an E_∞-algebra C such that there is a zig-zag of E_∞-algebras

$$A \xleftarrow{\sim} C \xrightarrow{\sim} B$$

The reader should note the similarity of this argument with the proof of Theorem 3.4.15.

This demonstrates that that the homotopy category of commutative algebras does not embed into the homotopy category of E_∞-algebras. In this sense, E_∞-algebras do not generalise commutative algebras.

0.2.5 Approximating the singular cochains complex with commutative algebras

In this section, we study a generalisation of Sullivan's algebra of piecewise polynomial functions functor A_{PL} in positive characteristic.

0.2.5.1 The p-adic forms in positive characteristic

Sullivan's A_{PL}-functor does not model the singular cochains in positive characteristic. The problem is the usual one with commutative algebras in positive characteristic, namely that the functor Sym$(-)$ does not send weak equivalences of cochain complexes to weak equivalences of commutative algebras. To fix this, we pass to divided power algebras and work with the following generalisation of the A_{PL}-functor.

$$\Omega_n^* = \left(\frac{\widehat{\mathbb{Z}_p}\langle x_0, \dots x_n \rangle \otimes \Lambda\left(dx_0, \dots, dx_n\right)}{\left(x_0 + \dots + x_n - p, dx_0 + \dots dx_n\right)} \right), \ |x_i| = 0, \ |dx_i| = 1.$$

One can then extend this construction by left Kan extension to produce a strictly commutative dg-algebra associated to any simplicial set X. For reasons that will become apparent, we shall call this construction *the p-adic forms on X* and denote it by $\Omega^*(X)$.

The reader should note that there is a key change here; instead of quotienting by $x_0 + \dots + x_n = 1$, we quotient by $x_0 + \dots + x_n = p$. This will play a significant role in the subsequent development of the theory.

0.2.5.2 The cohomology of the p-adic forms on X

It turns out that the p-adic forms on X compute the cohomology on X. Explicitly, we have the following theorem.

Theorem L. Let X be a simplicial set. The cohomology ring of $\Omega^*(X)$ is isomorphic to the singular cohomology of X. In other words, one has a ring isomorphism

$$H^*\left(\Omega^*(X)\right) \cong H^*\left(X, \widehat{\mathbb{Z}_p}\right).$$

0.2.5.3 Relationship with the singular chains complex

The non-vanishing of the zeroth Steenrod operation P^0 ensures that it is impossible to produce a strictly commutative algebra weakly equivalent to the singular cochains complex. Therefore, one can only hope to produce various kinds of approximation. This is what $\Omega^*(X)$ does. More precisely, the approximation that $\Omega^*(X)$ computes is the following.

Definition 0.2.8. Let X be a simplicial set. We define the *p-shifted singular cochain algebra* $\mathscr{D}^*\left(X, \widehat{\mathbb{Z}_p}\right)$ to be the following subalgebra of the singular cochains $C^*\left(X, \widehat{\mathbb{Z}_p}\right)$.

$$\mathscr{D}^n(X) = \left\langle p^i \sigma : \text{ for } \sigma \in C^n\left(X, \widehat{\mathbb{Z}_p}\right) \text{ and } \begin{cases} i = n & \text{if } d\sigma = 0. \\ i = n+1 & \text{otherwise.} \end{cases} \right\rangle$$

The differential and the $\mathscr{E}$ structure are that induced by those on $C^*\left(X, \widehat{\mathbb{Z}_p}\right)$.

Our theorem explicitly says the following.

Theorem M. For every simplicial set X, the $\mathscr{E}$-algebras $\Omega^*(X)$ and $\mathscr{D}^*\left(X, \widehat{\mathbb{Z}_p}\right)$ are weakly equivalent.

0.2.5.4 Applications to Massey products and formality

The p-adic de Rham forms allow us to quickly generalise a lot of theory from rational setting. In particular, they contain information about the Massey products.

Theorem N. Suppose that $\sigma \in H^*(X, \mathbb{Q})$ be the higher Massey product of $\langle x_1, x_2, \ldots, x_n\rangle \in H^*(A_{PL}(X), \mathbb{Q})$. Then there exists an $n > 0$ such that $p^n\sigma \in H^*\left(X, \widehat{\mathbb{Z}_p}\right)$ is the higher Massey product of $\langle p^n x_1, p^n x_2, \ldots, p^n x_n\rangle \in H^*\left(A_{PL}(X), \widehat{\mathbb{Z}_p}\right)$ computed in $\Omega^*(X)$.

Moreover, when X is formal rationally, the p-adic de Rham forms are *almost always* formal. Explicitly, we have the following result.

Theorem O. Let X be a finite simplicial set such that $A_{PL}(X)$ is formal over $\mathbb{Q}$. For all but finitely many primes, $\Omega^*(X)$ is formal over $\widehat{\mathbb{Z}_p}$ as a dg-commutative dg-algebra.

0.2.6 A higher Hochschild-Konstant-Rosenberg theorem and the Deligne conjecture

In this chapter, we study the higher Hochschild homology in rational homotopy theory, which was first defined by Pirashvili. We prove a HKR-theorem in this context, generalizing the usual one for the circle and Hochschild homology. which makes explicit the reliance of HKR-type statements on formality. We go on to use this statement to construct an $n+1$-Poisson algebra structure on the cotangent complex, generalising the cup product on Hochschild homology.

0.2.6.1 The classical Hochschild-Konstant-Rosenberg theorem

The classical HKR theorem states the following.

Theorem 0.2.9. *Let $\mathbb{k}$ be a field of characteristic 0 and let A be a commutative $\mathbb{k}$-algebra which is essentially of finite type and smooth over $\mathbb{k}$. Then there is an isomorphism of graded $\mathbb{k}$-algebras*

$$\Phi : HH_*((A, A) \xrightarrow{\sim} \Omega^*(A, \mathbb{k})$$

between the Hochschild homology and the module of Kähler differentials.

This theorem can be generalised in several directions. First, there are chain level statements. The smoothness assumption, which is used primarily as a cofibrancy condition in the proof, can be dropped in favour of working with the cotangent complex, which is essentially a derived version.

0.2.6.2 Generalising the Hochschild chain complex

As the name suggests, the Hochschild homology is the homology of a certain chain complex called the Hochschild chain complex. Pirashvili has shown that one such way to construct said complex is via a derived tensor product between a simplicial set and E_∞-algebra. This fundamentally arises from the fact that the category of E_∞-algebras is enriched over simplicial sets. Our first proposition is that this observation can be pushed further.

Theorem P. Let $X \in \mathsf{sSet}$ and $A \in \mathscr{E} - \mathbf{alg}$. Then there exists an is a weak equivalence of $\mathscr{E}$-algebras between $X \boxtimes A$ and $C_*(X) \otimes^{\mathbb{L}}_{\mathscr{E}} A$. In other words, the diagram

$$\begin{tikzcd} \mathsf{sSet} \times \mathscr{E} - \mathbf{alg} \arrow[rr, "\boxtimes"] \arrow[dr, "C_* \times id"'] & & \mathscr{E} - \mathbf{alg} \\ & \mathscr{E} - \mathbf{coalg} \times \mathscr{E} - \mathbf{alg} \arrow[ur, "\otimes^{\mathbb{L}}_{\mathscr{E}}"'] & \end{tikzcd}$$

commutes up to homotopy.

0.2.6.3 A higher Hochschild-Konstant-Rosenberg theorem theorem

In this context, our version of the Hochschild-Konstant-Rosenberg theorem is the following.

Theorem Q. Let X be a formal simplicial set of finite type in each degree. Let A be a CDGA. Suppose that $(\mathrm{Sym}(V), d)$ is a cofibrant, quasi-free resolution of A. Then there is a natural equivalence of chain complexes

$$A \boxtimes X \xrightarrow{\sim} \mathrm{Sym}(V \otimes H_*(X), d_X)$$

We call $\mathrm{Sym}(V \otimes H_*(X), d_X)$ the higher X-shaped tangent complex of $A \cong (\mathrm{Sym}(V), d)$. The differential d_X is defined with explicit dependence on the comultiplicative structure of $H_*(X)$. Moreover this equivalence is functorial with respect to formal maps.

When $X = S^1$, one recovers the classical HKR-theorem.

0.2.6.4 A n-Poisson coalgebra structure on the cohomology of n-fold suspensions

We first define a notion of coalgebra that takes into account the diagonal map on spaces. Then, using this notion of coalgebra, we prove the following result.

Theorem R. For $\Sigma^n X$ an n-fold suspension, the dg-algebra $C_*(\Sigma^n X)$ is equipped with a Pois_n-coalgebra structure. This structure extends to the cotangent complex and has trivial cobracket on Hochschild homology.

0.2.6.5 A $n+1$-cotangent complex structure on Hochschild homology

Our final theorem of the chapter states the following.

Theorem S. Let $X = \Sigma^n Y$ be an n-fold suspension. Then the cotangent complex

$$(\mathrm{Sym}(H_*(X) \otimes V), d_X)$$

is a coalgebra over Pois_{n+1}

This can be viewed as a higher version of the Deligne conjecture, as the E_n-operad is formal in characteristic 0, and so an n-Poisson structure is equivalent to a full E_n-structure. Secondly, the HKR-theorem states that the cotangent complex is equivalent to the Hochschild chain complex.

0.2.6.6 Appendix: A coendomorphism operad

In the appendix, we construct the potentially useful gadget of an explicit model for the coendomorphism operad of a coassociative algebra in the ∞-category of coassociative algebras.

CHAPTER 1

A recognition principle for iterated suspensions as coalgebras over the little cubes operad

Abstract

Our main result is a recognition principle for iterated suspensions as coalgebras over the little cubes operads. Given a topological operad, we construct a comonad in pointed topological spaces endowed with the wedge product. We then prove an approximation theorem that shows that the comonad associated to the little n-cubes operad is weakly equivalent to the comonad $\Sigma^n\Omega^n$ arising from the suspension-loop space adjunction. Finally, our recognition theorem states that every little n-cubes coalgebra is homotopy equivalent to an n-fold suspension. These results are the Eckmann–Hilton dual of May's foundational results on iterated loop spaces.

1.1 Introduction

Since the invention of operads, they have played an essential role in many parts of mathematics and physics. The first application and the original motivation for their invention was for the study of iterated loop spaces (see [66] and [12]). Operads provide a coherent framework for studying objects equipped with many "multiplications", i.e. operations with multiple inputs and one output, satisfying certain homotopical coherences. An important class of such objects are the n-fold loop spaces, which are algebras over the little n-cubes operad. May showed in his recognition principle a homotopical converse, namely that every little n-cubes algebra is weakly equivalent to an n-fold loop space; and further proved an approximation theorem which asserts that the monad associated to the little n-cubes operad is weakly equivalent to the monad $\Omega^n\Sigma^n$. This approximation theorem reduced the study of operations on the homology of iterated loop spaces to the combinatorics of the little cubes operads. This perspective unraveled their complete algebraic structure (see [21]).

It has long been suspected that the recognition principle and the approximation theorem should have their corresponding Eckmann–Hilton dual versions. Indeed, work on this topic predates May's recognition theorem itself. By the end of the 1950s, Barratt and Stasheff a preliminary version of these questions, trying to characterize n-fold suspensions and co-H-spaces in terms of their algebraic structure. May's proof of the recognition principle reignited interest and there were immediate attempts to prove the Eckmann–Hilton dual; some of this story can be found in the comments on the MathOverflow question [47]. We are also aware of other more recent attempts to tackle the problem, but a solution has remained evasive until now. .

The goal of this paper is to prove the Eckmann–Hilton dual results of May's work on iterated loop spaces. Our proof is the consequence of two key new insights. Firstly, in general, without the added assumption of conilpotency, cofree coalgebra functors are notoriously difficult to construct and almost impossible to concretely work with. We were able to surmount this difficulty by proving that, in our case, elements of a cofree coalgebra are determined by their arity 1 component (see Lemma 1.2.5). This is a very special feature of our setting which is surprising compared to the more algebraic setting. It is this fact that enabled us to cleanly define the cofree cooperation and perform the concrete manipulations that made the proof possible. Secondly, we were able to show that the corelations in our comonad lie in arity 2, something we were able to interpret in a very concrete way (see Proposition 1.2.18.) The reader should note that the Eckmann–Hilton dual of these facts both fail.

First of all, we construct a comonad in the category of pointed spaces associated to an operad. Next, we show that n-fold suspensions are coalgebras over the little n-cubes operad $\mathscr{C}_n$. More precisely we prove the following result.

Theorem T. *The n-fold reduced suspension of a pointed space X is a $\mathscr{C}_n$-coalgebra. More precisely, there is a natural and explicit operad map*

$$\nabla : \mathscr{C}_n \to \mathsf{CoEnd}_{\Sigma^n X},$$

where $\mathrm{CoEnd}_{\Sigma^n X}$ is the coendomorphism operad of $\Sigma^n X$. The map ∇ encodes the homotopy coassociativity and homotopy cocommutativity of the classical pinch map $\Sigma^n X \to \Sigma^n X \vee \Sigma^n X$. In particular, the pinch map is an operation associated to an element of $\mathscr{C}_n(2)$. Furthermore, for any based map $X \to Y$, the induced map $\Sigma^n X \to \Sigma^n Y$ extends to a morphism of $\mathscr{C}_n$-coalgebras.

In this new setting, the Eckmann–Hilton dual of May's celebrated recognition of iterated loop spaces reads as follows.

Theorem U. *Let X be a $\mathscr{C}_n$-coalgebra. Then there is a pointed space $\Gamma^n(X)$, naturally associated to X, together with a weak equivalence of $\mathscr{C}_n$-coalgebras*

$$\Sigma^n \Gamma^n(X) \xrightarrow{\simeq} X,$$

which is a retract in the category of pointed spaces. Therefore, every $\mathscr{C}_n$-coalgebra has the homotopy type of an n-fold reduced suspension.

Together, our theorems T and U provide the following intrinsic characterization of n-fold reduced suspensions as $\mathscr{C}_n$-coalgebras.

Corollary. *Every n-fold suspension is a $\mathscr{C}_n$-coalgebra, and if a pointed space is a $\mathscr{C}_n$-coalgebra then it is homotopy equivalent to an n-fold suspension.*

It is worth noting that this result already exists at the level of $\Sigma^n\Omega^n$ coalgebras , see Theorem 1.4.9. Let us turn our attention to the other celebrated result in [66], the approximation theorem. It constitutes an essential step for proving the recognition principle for n-fold loop spaces, and it is also the key to unlocking certain computations on the homology of iterated loop spaces. Roughly speaking, the approximation theorem for loop spaces asserts that the free $\mathscr{C}_n$-algebra on a pointed space X is weakly equivalent to $\Omega^n\Sigma^n X$. We also prove the Eckmann–Hilton dual of this result. It reads as follows.

Theorem V. *For every $n \geq 1$, there is a natural morphism of comonads*

$$\alpha_n : \Sigma^n\Omega^n \longrightarrow C_n.$$

Furthermore, for every pointed space X, there is an explicit natural homotopy retract of pointed spaces

$$\begin{tikzcd} \Sigma^n\Omega^n X \arrow[r, shift left] & C_n(X) \arrow[l, shift left] \arrow[loop above] \end{tikzcd}$$

In particular, $\alpha_n(X)$ is a weak equivalence.

The comonad C_n in the statement above is constructed in a natural way from the little n-cubes operad. It is a non-trivial Eckmann–Hilton dualization of May's monad associated to $\mathscr{C}_n$. To our knowledge, this comonad has not been studied elsewhere, and it seems to be an exciting new object that might shed light on further understanding n-fold reduced suspensions, as well as on other objects that support a coaction of the little n-cubes operad.

Let us complete a bit more of the historical context. It has been known for a long time that any $(n-1)$-connected CW complex of dimension less than or equal to $(2n-1)$ has the homotopy type of a (1-fold) suspension. In [8], [79], [37] and finally [51], this result was successively improved on. In modern language, these authors showed that an $(n-1)$-connected co-H-space equipped with an A_k comultiplication which is of dimension less than or equal to $k(n-1)+3$ is a suspension. The case of $k=\infty$ in [51] can be thought of as the E_1-version of Theorem U, although our proof strategy is very different. From a different angle, the case of iterated suspensions considered as coalgebras over (a homotopical version of) the $\Sigma^n\Omega^n$-comonad was recently treated in [11], where the authors obtained a recognition principle for $(n+1)$-connected, n-fold (simplicial) suspensions. This last result differs from our Theorem U in several key respects. Firstly; our notions of coalgebra differ as they pass to a derived functor in the homotopy category of pointed spaces, while we consider only $\Sigma^n\Omega^n$-coalgebras in the classical sense of coalgebras over comonads. Secondly; our result has the sharpest possible connectivity requirement. The most striking difference with all previous scholarships is that we make heavy use of the little n-cubes operad and the comonad C_n; whereas these objects do not seem to have appeared in previous literature on the homotopy theory of iterated suspensions (with the exception of [40] in a very different context). In particular, there is no approximation theorem in [11].

To conclude, a few remarks are in order. The first remark is that to prove our theorems *U* and *V*, we do not follow an Eckmann–Hilton dual approach to May's proof in the case of iterated loop spaces. We have found a framework and proof which depends on explicit homotopies and hence avoids the use of quasi-fibrations and the construction of auxiliary spaces. In this sense, our approach is technically simpler. The approximation of suspensions is an independent result that we believe might have potential side applications. Finally, most of the results of this paper could have been stated using little n-disks instead of little n-cubes. However, using cubes significantly simplify many of the explicit formulae that appear when proving our results, and therefore we choose to present things this way.

1.1.1 Notation and conventions

All topological spaces are compactly generated and Hausdorff. We denote by I the unit interval in $\mathbb{R}$ and by J its interior:

$$J = (0,1) \subseteq [0,1] = I.$$

The symmetric group on n letters is denoted S_n.

For $X = (X,*)$ a pointed space, it will be convenient to identify the r-fold wedge $X^{\vee r}$ as a subspace of the cartesian product $X^{\times r}$. To do so, consider

$$X^{\vee r} = \bigcup_{i=1}^{r} \{*\} \times \cdots \times \underbrace{X}_{i} \times \cdots \times \{*\} \subset X^{\times r}.$$

A point x in the i-th factor of the wedge $X^{\vee r}$ is therefore identified with the point $(*,...,*,x,*,...,*)$ having x at its i-th component and the base point at all others. We further use the convention that both $X^{\vee 0}$ and $X^{\times 0}$ are equal to the base point. Given pointed maps $\varphi_1,...,\varphi_r : X \to Y$, we denote by $\left(\varphi_1,...,\varphi_r\right)$ the induced map $X \to Y^{\times r}$ to the product. Here, we implicitly used the diagonal map $d : X \to X^{\times r}$ given by $d(x) = (x,...,x)$. To simplify the notation we will omit the diagonal from the notation when this is clear from the context. If the image of this map lands in the wedge subspace $Y^{\vee r}$, we denote the corresponding restriction by $\left\{\varphi_1,...,\varphi_r\right\}$. Thus, the curly brackets notation emphasizes that the map lands in the wedge rather than the product. We reserve the notation $\varphi_1 \vee \cdots \vee \varphi_r$ for the induced map $X^{\vee r} \to Y^{\vee r}$ given by

$$\left(\varphi_1 \vee \cdots \vee \varphi_r\right)(*,...,*,x_i,*,...,*) = \left(*,...,*,\varphi_i\left(x_i\right),*,...,*\right).$$

We frequently use the identification $\Sigma^n X = S^n \wedge X$ for the n-fold reduced suspension of a pointed space X. Thus, points in $\Sigma^n X$ will be denoted $[t,x]$, where $t \in S^n$ and $x \in X$. Since points in the suspensions are equivalence classes, we use the square brackets notation. From now on, we implicitly assume all suspensions are reduced.

We assume the reader is familiar with operad theory, especially in topological spaces, and we refer to [36]. We use the following conventions. An operad $\mathscr{P}$ in a symmetric monoidal category $\mathscr{M} = (\mathscr{M}, \otimes, \mathbb{1})$ is *unitary* if $\mathscr{P}(0) = \mathbb{1}$, and *non-unitary* if $P(0)$ is not defined (i.e., the underlying symmetric sequence of $\mathscr{P}$ starts in arity 1). We borrow this nomenclature from [36, Section 2.2]. We will make heavy use of the operad of little n-cubes $\mathscr{C}_n$, considered as a unitary operad where $\mathscr{C}_n(0) = *$ is a single point.

1.2 Coalgebras over topological operads

Given a unitary topological operad $\mathscr{P}$, we construct an explicit comonad $C_{\mathscr{P}}$ in *pointed* spaces. In Section 1.2.1, we carefully construct this comonad and study some of its basic properties. The comonad $C_{\mathscr{P}}$ gives rise to the category of coalgebras over $\mathscr{P}$, also called $\mathscr{P}$-coalgebras. There is a second way of defining $\mathscr{P}$-coalgebras by using the coendomorphism operad that does not require the explicit construction of the comonad $C_{\mathscr{P}}$. This alternative construction has the advantage that it can be defined for all operads even when they are not necessarily unitary. The disadvantage is that it is not clear how to get an explicit comonad out of this definition. We explain this alternative construction and show that in the case of unitary operads it gives an equivalent notion of $\mathscr{P}$-coalgebras in Section 1.2.2. We specialize to the case in which $\mathscr{P}$ is the operad $\mathscr{C}_n$ of little n-cubes in Section 1.2.3, producing the central comonad of this paper. Finally, we prove Theorem T in Section 1.2.4 - that the n-fold reduced suspension of a pointed space is naturally a $\mathscr{C}_n$ coalgebra. Therefore, the n-fold reduced suspensions are the paradigmatic examples of $\mathscr{C}_n$-coalgebras.

Remark 1.2.1. In our constructions of coalgebras, we are mixing pointed and unpointed spaces. All our operads live in the category of unpointed spaces while the coalgebras over the operads and associated comonads live in the category of pointed spaces.

1.2.1 Construction of topological comonads

In this section, we construct the mentioned comonad $C_{\mathscr{P}}$ in pointed spaces out of a unitary operad $\mathscr{P}$ in unpointed spaces.

Let us first establish some preliminary notation. Denote

$$\mathsf{Top} = \left(\mathsf{Top}, \times, \{*\}\right) \quad \text{and} \quad \mathsf{Top}_* = \left(\mathsf{Top}_*, \vee, \{*\}\right)$$

the symmetric monoidal categories of spaces endowed with the cartesian product $\times$, and pointed spaces endowed with the wedge product $\vee$, respectively. Let $\mathscr{P}$ be a unitary operad in Top with composition map γ and denote the unitary operation by $* \in \mathscr{P}(0)$. Define the *restriction operators*, for all $n \geq 1$ and $1 \leq i \leq n$, by inserting the unique point $* \in \mathscr{P}(0)$ at the i-th component:

$$\begin{aligned} \mathscr{P}(n) &\xrightarrow{\ d_i\ } \mathscr{P}(n-1) \\ \theta &\longmapsto \gamma\left(\theta; \mathrm{id}, \ldots, *, \ldots, \mathrm{id}\right). \end{aligned}$$

Let $X \in \mathsf{Top}_*$. The *wedge collapse maps*, defined for all $n \geq 1$ and $1 \leq i \leq n$, are given by collapsing the i-th factor in the wedge as follows:

$$\begin{aligned} X^{\vee n} &\xrightarrow{\ \pi_i\ } X^{\vee(n-1)} \\ (x_1, \ldots, x_n) &\longmapsto (x_1, \ldots, \widehat{x_i}, \ldots, x_n). \end{aligned}$$

Here, the r-fold wedge is seen inside the r-fold cartesian product, and the notation $\widehat{x_i}$ means that we are sending the i-th component to the basepoint.

Notation 1.2.2. If $\mathscr{P}$ is a unitary operad and X is a pointed space, we denote

$$\mathrm{Tot}(\mathscr{P}, X) := \prod_{n \geq 0} \mathsf{Map}_{S_n}\left(\mathscr{P}(n), X^{\vee n}\right).$$

Each space $\mathsf{Map}_{S_n}\left(\mathscr{P}(n), X^{\vee n}\right)$ consists of the equivariant maps from the arity n component of $\mathscr{P}$ equipped with its usual S_n-action to the n-fold wedge of X with itself endowed with the S_n-action that permutes the coordinates of its points by $\sigma \cdot (x_1, \ldots, x_n) =$

$\left(x_{\sigma(1)},\dots,x_{\sigma(n)}\right)$. We frequently disregard the 0-th component in the infinite product above, since the mapping space $\mathsf{Map}(\mathscr{P}(0), X^{\vee 0})$ is just a point. It can therefore be ignored in all computations that follow. Thus, the point $\left(f_0, f_1, f_2, \dots\right) \in \mathrm{Tot}\,(\mathscr{P}, X)$ will be denoted $\left(f_1, f_2, \dots\right)$. The topology on the space $\mathrm{Tot}\,(\mathscr{P}, X)$ is the usual product topology.

We are ready to define the underlying endofunctor of our comonad $C_{\mathscr{P}}$.

Definition 1.2.3. Let $\mathscr{P}$ be a unitary operad in Top. Define the endofunctor in pointed spaces

$$\begin{aligned} C_{\mathscr{P}} : \mathsf{Top}_* &\longrightarrow \mathsf{Top}_* \\ X &\longmapsto C_{\mathscr{P}}\,(X)\,, \end{aligned}$$

where

$$C_{\mathscr{P}}(X) = \left\{\alpha = \left(f_1, f_2, \dots\right) \in \mathrm{Tot}\,(\mathscr{P}, X) \mid \pi_i f_n = f_{n-1} d_i \text{ for all } n \geq 2 \text{ and } 1 \leq i \leq n\right\}$$

is the subspace of $\mathrm{Tot}\,(\mathscr{P}, X)$ formed by those sequences $\left(f_1, f_2, \dots\right)$ that commute with the restriction operators and wedge collapse maps. That is, for all $n \geq 2$ and $1 \leq i \leq n$, the following diagram commutes:

$$\begin{tikzcd} \mathscr{P}(n) \arrow[r, "f_n"] \arrow[d, "d_i"'] & X^{\vee n} \arrow[d, "\pi_i"'] \\ \mathscr{P}(n-1) \arrow[r, "f_{n-1}"] & X^{\vee(n-1)} \end{tikzcd}$$

The base point of $C_{\mathscr{P}}\,(X)$ is the sequence $\alpha = \left(f_1, f_2, \dots\right)$ where each f_r has image the base point of $X^{\vee r}$. Since the base point of the wedge $X^{\vee r}$ is fixed by the S_r-action, the base point is well-defined. If $f : X \to Y$ is a pointed map, then $C_{\mathscr{P}}\left(f\right) : C_{\mathscr{P}}\,(X) \to C_{\mathscr{P}}\,(Y)$ is defined by

$$C_{\mathscr{P}}\left(f\right)(\alpha) = \left(f \circ f_1,\ \left(f \vee f\right) \circ f_2, \dots, \left(f \vee \dots \vee f\right) \circ f_n, \dots\right).$$

The nth term in the sequence above is given by

$$\left(f \vee \dots \vee f\right) \circ f_n : \mathscr{P}(n) \xrightarrow{f_n} X^{\vee n} \xrightarrow{f \vee \dots \vee f} Y^{\vee n}.$$

Remarks 1.2.4.

1. The idea of defining $C_{\mathscr{P}}$ above as a subspace of $\mathrm{Tot}\,(\mathscr{P}, X)$ arises from an Eckmann–Hilton dualization of May's definition of the monad associated to an operad [66]. Recall that the monad M_n in pointed spaces defined in *loc. cit.* by using the little n-cubes operad is given by

$$M_n(X) = \left(\coprod_{r \geq 0} \mathscr{C}_n(r) \times X^{\times r}\right) / \sim,$$

where $\sim$ is the equivalence relation that glues level r to level $r+1$ by combining the restriction operators with the insertion of the base point, $(d_i(c), y) \sim (c, s_i(y))$, and imposing the compatibility with the group action, $(c \cdot \sigma, y) \sim (c, \sigma \cdot y)$. [1]

[1] Here, $(c, y) \in \mathscr{C}_n(r) \times X^{\times(r-1)}$, $s_i(y)$ is the point of $X^{\times r}$ where we insert the base point at the i-th component, and $\sigma \in S_r$.

2. The compatibility condition of a sequence $\alpha \in \mathrm{Tot}(\mathscr{P}, X)$ with the restriction operators and wedge collapse maps,

$$\pi_i f_n = f_{n-1} d_i, \quad \text{for all } n \geq 1 \text{ and } 1 \leq i \leq n \tag{1.1}$$

is the precise condition needed to incorporate a counit to the coalgebras in pointed spaces that result from the comonad $C_{\mathscr{P}}$. See Remark 1.2.17 for further details.

3. The comonad $C_{\mathscr{P}}$ can be constructed in more general symmetric monoidal categories. For the applications that we give in this paper, we are only interested in the category of topological spaces.

Our next goal is to endow the endofunctor $C_{\mathscr{P}}$ with a comonad structure. Before doing so, we make two elementary observations that will simplify some of our proofs later on. We will use the following notation: if $h_1, \ldots, h_r$ is a family of maps such that the composition

$$h_1 \circ \cdots \circ h_{i-1} \circ h_{i+1} \circ \cdots \circ h_r$$

makes sense, then we denote the expression above by

$$h_1 \cdots \widehat{h_i} \cdots h_r.$$

That is, the hat $\widehat{(-)}$ on top of the i-th map indicates that this component is removed from the composition. The first observation is the following.

Lemma 1.2.5. *A sequence $(f_1, f_2, \ldots) \in C_{\mathscr{P}}(X)$ is determined by its first component $f_1 : \mathscr{P}(1) \to X$. That is, we can recursively write, for all $r \geq 2$,*

$$f_r = \left\{ f_1 \widehat{d_1} d_2 \cdots d_r \ , \ f_1 d_1 \widehat{d_2} d_3 \cdots d_r \ , \ldots, \ f_1 d_1 d_2 \cdots d_{r-1} \widehat{d_r} \right\},$$

where the d_i's are the maps that insert $* \in \mathscr{P}(0)$ into the ith entry.

Recall that the term on the right hand side above follows the notation from Section 1.1.1.

Proof. Let $\alpha = (f_1, f_2, \ldots) \in C_{\mathscr{P}}(X)$. Before we give a general proof of the lemma we first work out the the $r = 2$ case since this makes the general argument clearer. Let

$$f_2 : \mathscr{P}(2) \to X \vee X$$

be the second component of α. Denote by $q_i : X \vee X \to X$ the projection onto the i-th factor of the wedge, for $i = 1, 2$. There are identifications $q_i = \pi_{3-i}$, where $\pi_1, \pi_2 : X \vee X \to X$ are the corresponding wedge collapse maps. Then,

$$f_2 = \{q_1 f_2, q_2 f_2\} = \{\pi_2 f_2, \pi_1 f_2\} = \{f_1 d_2, f_1 d_1\} = \left\{ f_1 \widehat{d_1} d_2, f_1 d_1 \widehat{d_2} \right\}.$$

In the third equality above, we used the Equation (1.1) for $n = 2$. The proof for general f_r follows a slight generalization of the case just proven, where we recursively use the identities of Equation (1.1) for all n between 2 and r. Thus, let

$$f_r : \mathscr{P}(r) \to X^{\vee r}$$

be the rth component of α. Denote by $q_i : X^{\vee r} \to X$ the projection onto the i-th factor of the wedge, for $i = 1, \ldots, r$. There are identifications

$$q_i = \pi_1 \pi_2 \cdots \widehat{\pi_i} \cdots \pi_r, \quad \text{for all } i = 1, \ldots, r.$$

Recall the hat $\widehat{\pi_i}$ indicates that we omit the i-th term. There is a slight but harmless abuse of notation above, since the π_j's that appear in the expression of q_i have different domains. Then,

$$
\begin{aligned}
f_r &= \{q_1 f_r,\ q_2 f_r, ...,\ q_r f_r\} \\
&= \{\widehat{\pi_1}\pi_2\pi_3\cdots\pi_r f_r\ ,\ \pi_1\widehat{\pi_2}\pi_3\pi_4\cdots\pi_r f_r\ ,\ ..., \pi_1\pi_2\cdots\pi_{r-1}\widehat{\pi_r} f_r\} \\
&= \{\widehat{\pi_1}\pi_2\pi_3\cdots(\pi_r f_r)\ ,\ \pi_1\widehat{\pi_2}\pi_3\pi_4\cdots(\pi_r f_r)\ ,\ ..., \pi_1\pi_2\cdots(\pi_{r-1} f_r)\} \\
&= \{\widehat{\pi_1}\pi_2\pi_3\cdots(f_{r-1} d_r)\ ,\ \pi_1\widehat{\pi_2}\pi_3\pi_4\cdots(f_{r-1} d_r)\ ,\ ..., \pi_1\pi_2\cdots(f_{r-1} d_{r-1})\} \\
&= \qquad \cdots \\
&= \left\{\widehat{\pi_1}\pi_2\left(f_2 d_3\cdots d_r\right)\ ,\ \pi_1\widehat{\pi_2}\left(f_2 d_3\cdots d_r\right)\ , \pi_1 f_2\left(d_2\widehat{d_3}d_4\cdots d_r\right)\ ,\ \pi_1 f_2\left(d_2\cdots d_{r-1}\widehat{d_r}\right)\right\} \\
&= \left\{f_1\widehat{d_1}d_2\cdots d_r\ ,\ f_1 d_1\widehat{d_2}d_3\cdots d_r\ , ...,\ f_1 d_1 d_2\cdots d_{r-1}\widehat{d_r}\right\}.
\end{aligned}
$$

This completes the proof. □

The result above tells us that any sequence $\alpha = (f_1, f_2, ...) \in C_{\mathscr{P}}(X)$ can be written as

$$\alpha = (f_1, f_2, f_3, ...) = (f_1, \{f_1 d_2, f_1 d_1\}, \{f_1 d_2 d_3, f_1 d_1 d_3, f_1 d_1 d_2\}, ...).$$

However, it does not assert that any map $\mathscr{P}(1) \to X$ can be extended to a sequence in $C_{\mathscr{P}}(X)$ whose first component is the given map. In fact, that is usually not the case. Below, we give a characterization when $\mathscr{P}$ is a unitary operad in topological spaces.

Let us point out the second observation. We need the following notation. If X is a pointed space, and $f : \mathscr{P}(1) \to X$ is any map, define for all $r \geq 2$ and $1 \leq i \leq r$ the collection of maps

$$f_r^i := f\left(d_1\cdots\widehat{d_i}\cdots d_r\right) : \mathscr{P}(r) \to X.$$

The map

$$f_r := \{f_r^1, ..., f_r^r\} : \mathscr{P}(r) \to X^{\vee r}$$

is then defined by first applying the diagonal map $\mathscr{P}(r) \to \mathscr{P}(r)^{\times r}$ and then the product of the f_r^i. The map above usually lands in the product but it restricts to the wedge if, and only if, the map belongs to the underlying space of the comonad.

Proposition 1.2.6. *Let X be a pointed space. Then the space $C_{\mathscr{P}}(X)$ is homeomorphic to the subspace of* $\mathsf{Map}(\mathscr{P}(1), X)$ *given by all those maps $f_1 : \mathscr{P}(1) \to X$ such that for any fixed $r \geq 2$, the maps f_r^i are all the base point except for at most a single index i. In particular, the image of the map*

$$f_r := (f_r^1, ..., f_r^r) : \mathscr{P}(r) \to X^{\times r}$$

is contained in the subspace $X^{\vee r} \subseteq X^{\times r}$. Furthermore, each

$$f_r : \mathscr{P}(r) \to X^{\vee r}$$

is S_r-equivariant. Under this identification, the value $C_{\mathscr{P}}(\phi)$ on a pointed map $\phi : X \to Y$ is the postcomposition with ϕ:

$$
\begin{aligned}
C_{\mathscr{P}}(X) &\xrightarrow{C_{\mathscr{P}}(\phi)} C_{\mathscr{P}}(Y) \\
f &\longmapsto C_{\mathscr{P}}(\phi)(g) = \phi \circ f.
\end{aligned}
$$

Proof. The fact that for a fixed $r \geq 2$, the map f_r^i is the base point for all indexes i except for at most one, implies that the map

$$f_r = \left(f_r^1, \dots, f_r^r\right) : \mathscr{P}(r) \to X^{\times r}$$

has its image in the wedge. Thus, it is correct to write $f_r = \{f_r^1, \dots, f_r^r\}$.

$\boxed{\Rightarrow}$ Let $(f_1, f_2, \dots) \in C_{\mathscr{P}}(X)$, then we want to show that f_r^i is the base point for all i except for at most one. It is a straightforward consequence of Lemma 1.2.5 that the component f_1 of the sequence gives rise to the family of maps $\{f_r^i\}$ of the statement, with $f_r = \{f_r^1, \dots, f_r^r\}$. So, the implication follows.

$\boxed{\Leftarrow}$ Let $f_1 : \mathscr{P}(1) \to X$ be a map giving rise to the family of maps $\{f_r^i\}$ and f_r satisfying the hypotheses of the statement. Then we want to show that this indeed belongs to $C_{\mathscr{P}}(X)$. Form the sequence

$$(f_1, f_2, \dots) \in \operatorname{Tot}(\mathscr{P}, X).$$

It suffices to check that for every $r \geq 2$ and $1 \leq i \leq r$, the identity $f_{r-1} d_i = \pi_i f_r$ holds. To do so, we will make use of the following fact and notation for maps induced onto a wedge of pointed spaces: given pointed spaces W, Y, Z and maps $\varphi_1, \dots, \varphi_n : Y \to Z$ such that $\{\varphi_1, \dots, \varphi_r\} : Y \to Z^{\vee n}$ is well-defined, then for any map $g : W \to Y$, we have

$$\{\varphi_1, \dots, \varphi_r\} \circ g = \{\varphi_1 \circ g, \dots, \varphi_r \circ g\} : W \to Z^{\vee n}.$$

Thus, fix some $r \geq 2$ and $1 \leq i \leq r$. On the one hand,

$$\pi_i f_r = \pi_i \left\{ f_1 \widehat{d_1} \cdots d_r \,,\, \dots \,,\, f_1 d_1 \cdots \widehat{d_r} \right\} = \left\{ f_1 \widehat{d_1} \cdots d_r \,,\, \dots \,,\, \cancel{f_1 d_1 \cdots \widehat{d_i} \cdots d_r} \,,\, \dots \,,\, f_1 d_1 \cdots \widehat{d_r} \right\}. \tag{1.2}$$

Above, the strike-through indicates that the i-th component is not part of the sequence. On the other hand,

$$f_{r-1} d_i = \left\{ f_1 \widehat{d_1} \cdots d_{r-1} \,,\, \dots \,,\, f_1 d_1 \cdots \widehat{d_{r-1}} \right\} \circ d_i = \left\{ f_1 \widehat{d_1} \cdots d_{r-1} \circ d_i \,,\, \dots \,,\, f_1 d_1 \cdots \widehat{d_{r-1}} \circ d_i \right\}. \tag{1.3}$$

It suffices to check that, for any j with $1 \leq j \leq r-1$, the j-th component of the sequence (1.2) is equal to the j-th component of the sequence (1.3). This is a straightforward check, taking into account whether $j \leq i$ or $j \geq i$, and using the simplicial identities satisfied by the d_k's - namely, that $d_i d_j = d_{j-1} d_i$ for $i < j$. □

Proposition 1.2.6 above is very useful, as we will see in Section 1.3. Remark that this result identifies the space $C_{\mathscr{P}}(X)$ as the subspace of $\mathsf{Map}(\mathscr{P}(1), X)$ formed by those maps satisfying an extra property. Bear in mind that, under this identification, the evaluation of $C_{\mathscr{P}}$ on a morphism $\phi : X \to Y$ corresponds to the postcomposition with ϕ.

Before going on, we introduce some notation that will be useful later.

Notation 1.2.7. We will occasionally use the following notation for the composition of the restriction operators:

$$D_i = d_1 \cdots \widehat{d_i} \cdots d_r : \mathscr{P}(r) \to \mathscr{P}(1).$$

These choices will simplify the formulae in what follows, making our results more readable. Remark also that, for any operation $\theta \in \mathscr{P}(r)$, the resulting operation $D_i(\theta) \in \mathscr{P}(1)$ is exactly

$$D_i(\theta) = \gamma(\theta; *, \dots, *, \underbrace{\mathrm{id}_{\mathscr{P}}}_{i}, *, \dots, *),$$

where γ is the composition map of $\mathscr{P}$, the element $\mathrm{id}_{\mathscr{P}} \in \mathscr{P}(1)$ is the operadic unit, and $* \in \mathscr{P}(0)$ is the unitary operation. In other words, $D_i(\theta)$ retains the unary operation determined by the i-th input of θ. For example, if $\mathscr{P} = \mathscr{C}_n$ is the little n-cubes operad and $\theta = (c_1, ..., c_r) \in \mathscr{C}_n(r)$ is a configuration of r little n-cubes, then $D_i(\theta) = c_i$ is the i-th little n-cube of the configuration, seen as an element of $\mathscr{C}_n(1)$.

Let us finally equip the endofunctor $C_{\mathscr{P}}$ with natural transformations $\varepsilon : C_{\mathscr{P}} \to \mathrm{id}_{\mathsf{Top}_*}$ and $\Delta : C_{\mathscr{P}} \to C_{\mathscr{P}} \circ C_{\mathscr{P}}$ that makes it a comonad. From now on, to lighten notation, we denote $C = C_{\mathscr{P}}$, assuming that the operad $\mathscr{P}$ is understood.

Definition 1.2.8. Let $C = C_{\mathscr{P}} : \mathsf{Top}_* \to \mathsf{Top}_*$ be the endofunctor of Definition 1.2.3. Define the natural transformations

$$\varepsilon : C \to \mathrm{id}_{\mathsf{Top}_*} \quad \text{and} \quad \Delta : C \to C \circ C$$

level-wise as follows.
• The counit structure map is defined by

$$\begin{aligned} \varepsilon_X : C(X) &\longrightarrow X \\ \alpha = \big(f_1, f_2, ...\big) &\longmapsto \varepsilon_X(\alpha) := f_1(\mathrm{id}_{\mathscr{P}}). \end{aligned}$$

Here, $\mathrm{id}_{\mathscr{P}} \in \mathscr{P}(1)$ is the operadic unit.
• We next define the coproduct structure map

$$\Delta_X : C(X) \to C(C(X)).$$

To do so, let $\alpha = \big(f_1, f_2, ...\big) \in C(X)$. Then $\Delta_x(\alpha) = \big(\bar{f}_1, \bar{f}_2, ...\big)$ is an element of the space $C(Z)$, with $Z = C(X)$. Thus, it is formed by a sequence of maps

$$\bar{f}_r : \mathscr{P}(r) \to C(X)^{\vee r}$$

satisfying the compatibility conditions

$$\pi_i \bar{f}_r = \bar{f}_{r-1} d_i, \quad \text{for } r \geq 2 \text{ and } 1 \leq i \leq r.$$

Because of Lemma 1.2.5 we only need to define the arity one component $\bar{f}_1 : \mathscr{P}(1) \to C(X)$, and extend it as a sequence by the formula

$$\bar{f}_r = \big\{\bar{f}_1 D_1, ..., \bar{f}_1 D_r\big\},$$

where $D_i = d_1 \cdots \widehat{d_i} \cdots d_r$.

For the definition above to be complete and correct, we require two steps:

Step 1. Define $\bar{f}_1 : \mathscr{P}(1) \to C(X)$.

Step 2. Check that $\bar{f}_1 D_i = *$ is the base point for all indexes i, except for at most a single one.

Where Step 2 follows from Proposition 1.2.6.
$\boxed{\text{Step 1}}$ Denote by γ the operadic composition map of $\mathscr{P}$. Define $\bar{f}_1 : \mathscr{P}(1) \to C(X)$ by

$$\bar{f}_1(\mu) = \big(g_1^{\mu}, g_2^{\mu}, ...\big) \quad \text{for all } \mu \in \mathscr{P}(1),$$

where the maps $g_r^\mu : \mathscr{P}(r) \to X^{\vee r}$ in the sequence are as follows. The first one is:

$$g_1^\mu : \mathscr{P}(1) \to X, \qquad g_1^\mu(\theta) := f_1\left(\gamma\left(\mu;\theta\right)\right),$$

for $\theta \in \mathscr{P}(1)$. That is, $g_1^\mu = f_1\left(\gamma\left(\mu;-\right)\right)$. The rest of the maps g_r^μ are recursively defined by the formula

$$g_r^\mu : \mathscr{P}(r) \to X^{\vee r}$$
$$g_r^\mu(\theta) = \left\{g_1^\mu D_1(\theta), \dots, g_1^\mu D_r(\theta)\right\} = \{g_1^\mu(\gamma(\theta; \mathrm{id}_{\mathscr{P}}, *, .., *)), \dots, g_1^\mu(\gamma(\theta; *, .., *, \mathrm{id}_{\mathscr{P}}))$$

For $\theta \in \mathscr{P}(r)$. We will check below that the image of g_r^μ is indeed contained in the wedge $X^{\vee r}$. The family of maps g_r^μ can be explicitly described. Let us first describe $g_2^\mu : \mathscr{P}(2) \to X \vee X$. Using, in the order given, the recursive definition of g_2^μ, the definitions of D_i and of g_1^μ, and the associativity of γ, we can write

$$\begin{aligned} g_2^\mu(\theta) &= \left\{g_1^\mu D_1(\theta), g_1^\mu D_2(\theta)\right\} = \left\{g_1^\mu\left(\gamma(\theta; \mathrm{id}_{\mathscr{P}}, *)\right), g_1^\mu\left(\gamma(\theta; *, \mathrm{id}_{\mathscr{P}})\right)\right\} \\ &= \left\{f_1\left(\gamma\left(\mu; \gamma(\theta; \mathrm{id}_{\mathscr{P}}, *)\right)\right), f_1\left(\gamma\left(\mu; \gamma(\theta; *, \mathrm{id}_{\mathscr{P}})\right)\right)\right\} \\ &= \left\{f_1\left(\gamma\left(\gamma\left(\mu;\theta\right); \mathrm{id}_{\mathscr{P}}, *\right)\right), f_1\left(\gamma\left(\gamma\left(\mu;\theta\right); *, \mathrm{id}_{\mathscr{P}}\right)\right)\right\}. \end{aligned}$$

Thus,

$$g_2^\mu = \left\{f_1 D_1\left(\gamma(\mu;-)\right), f_1 D_2\left(\gamma(\mu;-)\right)\right\}.$$

Next we need to show that $\overline{f_2}$ has its image in the wedge $C(X) \vee C(X)$. Since $\alpha = \left(f_1, f_2, \dots\right)$ is an element of $C(X)$, it follows that all $f_1 D_i = *$ are the base point, except for at most a single index i. Therefore, indeed, g_2^μ has its image in the wedge. Furthermore, so defined, g_2^μ is S_2-equivariant. In general, exactly the same steps as for the $r = 2$ case show that the explicit formula for g_r^μ is

$$g_r^\mu(\theta) = \left\{f_1\left(\gamma\left(\gamma(\mu;\theta); \mathrm{id}_{\mathscr{P}}, *, \dots, *\right)\right), \dots, f_1\left(\gamma\left(\gamma(\mu;\theta); *, \dots, *, \underbrace{\mathrm{id}_{\mathscr{P}}}_{j}, *, \dots, *\right)\right), \dots, f_1\left(\gamma\left(\gamma(\mu;\theta); *, \dots, *, \mathrm{id}_{\mathscr{P}}\right)\right)\right\}.$$

Above, the j-th component in the wedge has the identity $\mathrm{id}_{\mathscr{P}} \in \mathscr{P}(1)$ at the j-th component.

$\boxed{\text{Step 2}}$ Let us check that $\bar{f}_1 D_i = *$ is the base point for all indexes i except for at most a single one. Recall that for fixed i, the map

$$\bar{f}_1 D_i : \mathscr{P}(r) \to C(X)$$

evaluated at some operation $\mu \in \mathscr{P}(r)$ is the previously defined sequence

$$\bar{f}_1 D_i(\mu) = \left(g_1^{D_i(\mu)}, g_2^{D_i(\mu)}, \dots\right).$$

First, observe that for any $\theta \in \mathscr{P}(1)$ and index i, with $1 \le i \le r$, we have

$$\gamma\left(D_i(\mu);\theta\right) = D_i\left(\gamma(\mu; \mathrm{id}_{\mathscr{P}}, .., \theta, \dots, \mathrm{id}_{\mathscr{P}})\right).$$

Therefore, the first component of the sequence $\bar{f}_1 D_1(\mu)$ can be written as

$$g_1^{D_i(\mu)} = f_1\left(D_i\left(\gamma\left(\mu;-\right)\right)\right).$$

Since the sequence $\left(f_1, f_2, \dots\right)$ is an element of the space $C(X)$, it follows that $f_1 D_i$ is the base point for all i except for at most one, and therefore, the same holds for the family $\left\{g_1^{D_1(\mu)}, \dots, g_1^{D_i(\mu)}, \dots\right\}$, which implies that $\bar{f}_1 D_i = *$ is the base point for almost all i.

Remark 1.2.9. In Proposition 1.2.6, we gave an identification of $C(X)$ as a certain subspace of $\mathrm{Map}(\mathscr{P}(1), X)$. From this point of view, the comultiplication $\Delta = \Delta_X : C(X) \to CC(X)$ is given as follows. Let $f \in C(X) \subseteq \mathrm{Map}(\mathscr{P}(1), X)$. Then, $\Delta(f)$ is given by:

$$\begin{aligned} \Delta(f) : \mathscr{P}(1) &\longrightarrow C(X) \\ c &\longmapsto \Delta(f)(c) : \mathscr{P}(1) \longrightarrow X \\ &\qquad\qquad d \longmapsto f\left(\gamma(c;d)\right). \end{aligned}$$

That is, given $f \in C(X)$, and $c, d \in \mathscr{P}(1)$, the map $\Delta(f)$ is explicitly given by

$$\Delta(f)(c)(d) = f\left(\gamma(c;d)\right).$$

Proposition 1.2.10. *With the notation before, the triple* (C, ε, Δ) *is a comonad in* Top_*.

Proof. We prove the coassociativity and counit axioms object-wise. For a pointed space X, these axioms are described by the following diagrams:

$$\begin{tikzcd} C(X) \arrow[r, "\Delta_X"] \arrow[d, "\Delta_X"'] & C(C(X)) \arrow[d, "\Delta_{C(X)}"] \\ C(C(X)) \arrow[r, "C(\Delta_X)"] & C(C(C(X))) \end{tikzcd} \qquad \begin{tikzcd} C(X) \arrow[r, "\Delta_X"] \arrow[d, "\Delta_X"'] \arrow[dr, "\mathrm{id}"] & C(C(X)) \arrow[d, "\varepsilon_{C(X)}"] \\ C(C(X)) \arrow[r, "C(\varepsilon_X)"] & C(X), \end{tikzcd}$$

where the left diagram gives the coassociativity condition and the right diagram the counit condition.

Let $\alpha = (f_1, f_2, \ldots) \in C(X)$ then we will check that it satisfies the diagrams.

▷ Coassociativity. We must check that

$$(C(\Delta_X) \circ \Delta_X)(\alpha) = \left(\Delta_{C(X)} \circ \Delta_X\right)(\alpha). \tag{1.4}$$

We analyze $\Delta_X(\alpha)$ first, given that it appears on both sides of the equation above, and then look at each of the sides of the equation above. By Lemma 1.2.5, it suffices to check that the arity one term of the sequences arising from both sides of Equation (1.4) agree. This will ultimately follow from the associativity of the operadic composition γ of the operad $\mathscr{P}$.

- Description of $\Delta_X(\alpha)$.

$$\begin{aligned} \Delta_X : C(X) &\longrightarrow C(C(X)) \\ \alpha &\longmapsto \Delta_X(\alpha) = \left(\bar{f}_1, \bar{f}_2, \ldots\right) \end{aligned}$$

By Lemma 1.2.5, the sequence $\left(\bar{f}_1, \bar{f}_2, \ldots\right)$ is determined by its first component $\bar{f}_1$. It is given as follows:

$$\begin{aligned} \bar{f}_1 : \mathscr{P}(1) &\longrightarrow C(X) \\ \mu &\longmapsto \bar{f}_1(\mu) = \left(g_1^\mu, g_2^\mu, \ldots\right) \end{aligned} \qquad \begin{aligned} g_1^\mu : \mathscr{P}(1) &\longrightarrow X \\ \theta &\longmapsto g_1^\mu(\theta) = f_1\left(\gamma(\mu;\theta)\right) \end{aligned}$$

- The left hand side of Equation (1.4) reads:

$$(C(\Delta_X) \circ \Delta_X)(\alpha) = C(\Delta_X)(\Delta_X(\alpha)) = C\left(\bar{f}_1, \bar{f}_2, \ldots\right) = \left(\Delta_X \circ \bar{f}_1, \{\Delta_X, \Delta_X\} \circ \bar{f}_2, \ldots\right).$$

Here, given maps $\varphi_i : X_i \to Y$, we are denoting the induced map by $\{\varphi_1, \ldots, \varphi_n\} : X_1 \vee \ldots \vee X_n \to Y$. We have:

$$\begin{array}{rcl} \Delta_X \circ \bar{f}_1 : \mathscr{P}(1) & \longrightarrow C(X) \longrightarrow & C(C(X)) \\ \mu & \longmapsto \bar{f}_1(\mu) = \left(g_1^\mu, g_2^\mu, \ldots\right) \mapsto & \left(\bar{g}_1^\mu, \bar{g}_2^\mu, \ldots\right) \end{array}$$

The map $\bar{g}_1^\mu$ above is determined by:

$$\begin{array}{rclcrcl} \bar{g}_1^\mu : \mathscr{P}(1) & \longrightarrow & C(X) & \qquad & h_1 : \mathscr{P}(1) & \longrightarrow & X \\ \theta & \longmapsto & \bar{g}_1^\mu(\theta) := (h_1, h_2, \ldots) & & \lambda & \longmapsto & h_1(\lambda) = g_1^\mu\left(\gamma(\theta;\lambda)\right) \end{array}$$

• The right hand side of Equation (1.4) reads:

$$\left(\Delta_{C(X)} \circ \Delta_X\right)(\alpha) = \Delta_{C(X)}\left(\Delta_X(\alpha)\right) = \Delta_{C(X)}\left(\bar{f}_1, \bar{f}_2, \ldots\right) = \left(\bar{\bar{f}}_1 \bar{\bar{f}}_2, \ldots\right).$$

Here,

$$\begin{array}{rclcrcl} \bar{\bar{f}}_1 : \mathscr{P}(1) & \longrightarrow & C(C(X)) & \qquad & l_1^\mu : \mathscr{P}(1) & \longrightarrow & C(X) \\ \mu & \longmapsto & \bar{\bar{f}}_1(\mu) = \left(l_1^\mu, l_2^\mu, \ldots\right) & & \theta & \longmapsto & l_1^\mu(\theta) = \bar{f}_1\left(\gamma(\mu;\theta)\right) \end{array}$$

As mentioned, to check the coassociativity condition it suffices to check that $\bar{\bar{f}}_1 = \Delta_X \circ \bar{f}_1$. By Lemma 1.2.5 again, our problem reduces to checking that $\ell_1^\mu = \bar{g}_1^\mu$. And once more, using the same lemma, this reduces to checking that the sequence $\bar{f}_1\left(\gamma\left(\mu;\theta\right)\right)$ has first term equal to $h_1(\lambda)$ described before. The first term is explicitly given by

$$f_1\left(\gamma\left(\gamma\left(\mu;\theta\right);\lambda\right)\right). \tag{1.5}$$

On the right hand side, the first nested term of $g_1^\mu\left(\gamma(\theta;\lambda)\right)$ is explicitly given by

$$f_1\left(\gamma\left(\mu;\gamma(\theta;\lambda)\right)\right). \tag{1.6}$$

By the associativity of the operadic composition γ, the term inside f_1 in Equation (1.5) is the same as the term inside f_1 in Equation (1.6). Thus, these two maps are equal. This proves the coassociativity of the comultiplication.

▷ Counit. We must check two identities:

1. $(C(\varepsilon_X) \circ \Delta_X)(\alpha) = \alpha$.
   Indeed,

   $$(C(\varepsilon_X) \circ \Delta_X)(\alpha) = C(\varepsilon_X)(\Delta_X(\alpha)) = C(\varepsilon_X)\left(\bar{f}_1, \bar{f}_2, \ldots\right) = \left(\varepsilon_X \circ \bar{f}_1, \{\varepsilon_X, \varepsilon_X\} \circ \bar{f}_2, \ldots\right).$$

   Let us check that $\varepsilon_X \circ \bar{f}_1 = f_1$ as maps $\mathscr{P}(1) \to X$. If $\mu \in \mathscr{P}(1)$, then:

   $$\left(\varepsilon_X \circ \bar{f}_1\right)(\mu) = \varepsilon_X\left(\bar{f}_1(\mu)\right) = \varepsilon_X\left(g_1^\mu, g_2^\mu, \ldots\right) = g_1^\mu(\mathrm{id}_{\mathscr{P}}) = f_1\left(\gamma(\mu;\mathrm{id}_{\mathscr{P}})\right) = f_1(\mu).$$

2. $\left(\varepsilon_{C(X)} \circ \Delta_X\right)(\alpha) = \alpha$.
   In this case,

   $$\left(\varepsilon_{C(X)} \circ \Delta_X\right)(\alpha) = \varepsilon_{C(X)}\left(\Delta_X(\alpha)\right) = \varepsilon_{C(X)}\left(\bar{f}_1, \bar{f}_2, \ldots\right) = \bar{f}_1(\mathrm{id}_{\mathscr{P}}).$$

   We must check that $\bar{f}_1(\mathrm{id}_{\mathscr{P}}) = f_1$ as maps $\mathscr{P}(1) \to X$. Indeed, if $\theta \in \mathscr{P}(1)$, then

   $$\bar{f}_1(\mathrm{id})(\theta) = g_1^1(\theta) = f_1\left(\gamma(\mathrm{id};\theta)\right) = f_1(\theta).$$

The proposition is therefore proven. $\square$

For the sake of completeness, we recall here the well-known fact that comonads explicitly create the cofree coalgebras of the underlying category (see for instance [72, Corollary 5.4.23]).

Theorem 1.2.11. *Let X be a pointed space. Then, $C(X)$ is the* cofree C-coalgebra *on X. That is, for any C-coalgebra A in pointed spaces, there is a natural bijection*

$$\mathrm{Hom}_{\mathsf{Top}_*}(A, X) \cong \mathrm{Hom}_{\mathsf{C-Coalg}}(A, C(X)).$$

In Section 1.2.3 we will give a few explicit examples of how this comonad looks like in the case of the associative operad and the little n-cubes operad.

1.2.2 Alternative definitions of coalgebra over an operad

Let $\mathscr{P}$ be a unitary operad in Top. The comonad $C = C_{\mathscr{P}}$ constructed in Section 1.2.1 naturally gives rise to a category of coalgebras in Top_*. The objects in this category are pointed spaces X together with a coalgebra structure map $c : X \to C(X)$. We call the objects of this category $\mathscr{P}$-coalgebras. There is an equivalent way of defining a $\mathscr{P}$-coalgebra by using the coendomorphism operad that does not require the explicit construction of the comonad C. In this alternative definition, the objects are pointed spaces X together with an operad map $\mathscr{P} \to \mathrm{CoEnd}_X$, where CoEnd_X is the coendomorphism operad asssociated to the pointed space X. In this section, we present the alternative definition of $\mathscr{P}$-coalgebra in terms of coendomorphisms, and show that for unitary operads this is equivalent to the comonadic definition. The definition of $\mathscr{P}$-coalgebras in terms of the coendomorphism operad is much more intuitive and defines the coalgebra structure in terms of explicit cooperations, i.e. maps $X \to X^{\vee r}$. On the other hand, the comonad definition has the benefit that it will be much easier to compare it to the $\Sigma^n\Omega^n$-comonad, making it more suitable for proving the approximation and recognition theorems later in this paper.

We start by defining the category of $\mathscr{P}$-coalgebras using the comonad $C_{\mathscr{P}}$.

Definition 1.2.12. Let $\mathscr{P}$ be a unitary operad in Top. The category $C_{\mathscr{P}}-\mathsf{Coalg}$ of coalgebras in Top_* associated to the comonad $C_{\mathscr{P}}$ is called the *category of (comonadic) $\mathscr{P}$-coalgebras.* The objects in this category are triples (X, c, ϵ), where $c : X \to C(X)$, called the coalgebra structure map of X and $\epsilon : C_{\mathscr{P}}(X) \to X$ the counit, are maps of pointed spaces satisfying counit and coassociativity axioms:

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[dr, "\mathrm{id}"'] & C(X) \arrow[d, "\varepsilon_X"] \\ & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "c"] \arrow[d, "c"] & C(X) \arrow[d, "C(c)"] \\ C(X) \arrow[r, "\Delta_X"] & C(C(X)) \end{tikzcd}$$

The morphisms between these objects are pointed maps $X \to Y$ that make the obvious square commute.

Before giving the alternative definition of $\mathscr{P}$-coalgebras, we must give the definition of the coendomorphism operad associated with a pointed space.

Definition 1.2.13. Let X be a pointed space. The *coendomorphism operad* CoEnd_X in pointed topological spaces with the wedge sum, has arity r component

$$\mathrm{CoEnd}_X(r) := \mathsf{Map}_*\left(X, X^{\vee r}\right),$$

the based mapping space from X to the r-fold wedge sum of X with itself. For $r = 0$, set $\mathrm{CoEnd}_X(0) = *$. The operadic composition maps are defined as

$$\gamma : \mathsf{Map}_*\left(X, X^{\vee n}\right) \times \mathsf{Map}_*\left(X, X^{\vee m_1}\right) \times \cdots \times \mathsf{Map}_*\left(X, X^{\vee m_n}\right) \to \mathsf{Map}_*\left(X, X^{\vee \sum m_i}\right),$$

$$\gamma\left(f, g_1, \ldots, g_n\right) := \left(g_1 \vee \ldots \vee g_n\right) \circ f.$$

The symmetric group action on $\mathrm{CoEnd}_X(r)$ permutes the wedge factors in the output of a map $f : X \to X^{\vee r}$. The unit $\eta : I \to \mathrm{CoEnd}_X$ is determined by mapping the base point in $I(1) = \{*\}$ to the identity map in $\mathrm{CoEnd}_X(1) = \mathsf{Map}_*(X, X)$.

It is straightforward to check that CoEnd_X is an operad in pointed spaces and we leave this to the reader. The coendomorphism operad gives an alternative definition of $\mathscr{P}$-coalgebras.

Definition 1.2.14. Let $\mathscr{P}$ be a not necessarily unitary operad in Top. A $\mathscr{P}$-coalgebra is a pointed topological space X together with an operad map $\mathscr{P} \to \mathrm{CoEnd}_X$. A morphism of $\mathscr{P}$-coalgebras is a pointed map $f : X \to Y$ such that the following diagram commutes for all n:

$$\begin{array}{ccc}
\mathscr{P}(n) \times X & \xrightarrow{\Delta_n} & X \vee \ldots \vee X \\
\downarrow{\scriptstyle \mathrm{id} \times f} & & \downarrow{\scriptstyle f \vee \ldots \vee f} \\
\mathscr{P}(n) \times Y & \xrightarrow{\Delta'_n} & Y \vee \ldots \vee Y
\end{array}$$

Here, Δ_n and Δ'_n are the coalgebra structure maps of X and Y, respectively, which are written arity-wise by using the mapping space-product adjunctions

$$\mathsf{Map}\left(\mathscr{P}(n) \times Z, Z^{\vee r}\right) \cong \mathsf{Map}\left(\mathscr{P}(n), \mathsf{Map}\left(Z, Z^{\vee r}\right)\right),$$

where Z is any pointed topological space. Note that since we are mixing pointed and unpointed spaces we are viewing $\mathsf{Map}_*(X, X^{\vee r})$ as a subspace of the unpointed mapping space so that we are able to use the $\times - \mathsf{Map}$-adjunction.

Remark 1.2.15. Note that this definition of a $\mathscr{P}$-coalgebra is more general than the one using the comand from the previous section. In particular, we do not require the operad to be unitary so these coalgebras are defined for a larger class of operads.

By using the mapping space-product adjunction for S_r-spaces, we see that there are several equivalent ways of unpacking the definition of a coendomorphism $\mathscr{P}$-coalgebra. The definition of a coalgebra as a sequence of coproduct maps

$$\Delta_r : \mathscr{P}(r) \times X \to X^{\vee r}$$

is also equivalent to a sequence of maps

$$\Delta'_r : X \to \mathsf{Map}\left(\mathscr{P}(r), X^{\vee r}\right)^{\mathsf{S}_r},$$

satisfying certain conditions. Here $\mathsf{Map}(\mathscr{P}(r), X^{\vee r})^{\mathsf{S}_r}$ is the subspace of S_r-invariant maps.

Versions of the coendomorphism operad have been explicitly used before in for example [6] in the category of chain complexes. The notion of coalgebra in the category of pointed spaces with the wedge product has also appeared before in [51], however they do do not use the coendomorphism operad or construct an explicit comonad.

The following result asserts that for unitary operads both definitions of $\mathscr{P}$-coalgebras are equivalent.

Proposition 1.2.16. *Let $\mathscr{P}$ be a unitary operad in* Top. *Then the definition of a $\mathscr{P}$-coalgebra via the comonad from Section 1.2.1 is equivalent to definition of a $\mathscr{P}$-coalgebra via the coendomorphism operad from Definition 1.2.14.*

Proof. Indeed, we can identify operad maps $\rho : \mathscr{P} \to \mathrm{CoEnd}_X$ with coalgebra structure maps $c : X \to C(X)$ by the following rule: for any $\theta \in \mathscr{P}(r)$ and $x \in X$,

$$\rho_r(\theta)(x) = f_r^x(\theta).$$

Here, ρ_r is the arity r component of ρ, and f_r^x is the rth-term of the sequence $c(x) = (f_1^x, f_2^x, \ldots)$. The formula above turns a coendomorphism coalgebra into a comonad coalgebra and vice versa. It is further straightforward to check that this definition commutes with morphisms. $\square$

From now on, we always use the shorter notation $\mathscr{P}-\mathsf{Coalg}$ for the category of $\mathscr{P}$-coalgebras.

Remark 1.2.17. The $\mathscr{P}$-coalgebras defined in this section are *canonically counital*. That is, they come equipped with the unique map $\varepsilon : X \to *$, and this map behaves as a counit with respect to the rest of the structure. This explains the compatibility conditions of Equation (1.1). Indeed, if X is a $\mathscr{P}$-coalgebras, then the following diagram commutes:

$$\begin{tikzcd}
P(n) \times X \arrow[r, "\Delta_r"] \arrow[d, "d_i \times \mathrm{id}"'] & X^{\vee r} \arrow[r, "\mathrm{id} \vee \cdots \vee \varepsilon \vee \cdots \vee \mathrm{id}"] & X^{\vee (r-1)} \arrow[d, "\mathrm{id}"] \\
P(n-1) \times X \arrow[rr, "\Delta_{r-1}"] & & X^{\vee (r-1)}
\end{tikzcd}$$

In the diagram above, Δ_r is the arity r coalgebra structure map of X, and $\mathrm{id} \vee \cdots \vee \varepsilon \vee \cdots \vee \mathrm{id}$ is precisely π_i. Note that the counit of a coalgebra is *unique*, i.e. since $*$ is the terminal object there is only one possible map from X to $X^{\vee 0} = *$. This is in high contrast with the (unpointed) algebra case in which there are many possibilities for a unit, i.e. there are many maps from $X^{\times 0} = *$ to X since $*$ is not the initial object in unpointed spaces.

1.2.3 The comonad associated to the little n-cubes operad

In this section, we take a closer look at the comonad constructed in Section 1.2.1, in the particular case of $\mathscr{P} = \mathscr{C}_n$ being the little n-cubes operad. Although we assume familiarity with this operad, there are a number of small variations in the literature. We give a brief summary below in order to carefully fix our conventions and establish the notation. We will consistently denote by C_n the comonad in pointed spaces associated to the little n-cubes operad $\mathscr{C}_n$. In Proposition 1.2.18, we give a geometric characterization of $C_n(X)$ as an explicit subspace of $\mathsf{Map}(\mathscr{C}_n(1), X)$.

Denote by I^n the unit n-cube of $\mathbb{R}^n$ and by J^n its interior. A *little n-cube* is a rectilinear embedding $h : I^n \to I^n$ of the form $h = h_1 \times \cdots \times h_n$, where each component h_i is given by

$$h_i(t) = (y_i - x_i)t + x_i, \quad \text{for} \quad 0 \le x_i < y_i \le 1. \tag{1.7}$$

The image $h(J^n)$ of the interior of I^n under a rectilinear embedding h will be denoted $\mathring{h}$. So although the operad is called the little n-cubes operad it is technically the little n-rectangle operad.

For each $n \geq 1$, the *little n-cubes operad* $\mathscr{C}_n$ is an operad in Top. It was introduced independently by Boardman–Vogt and May [12, 66] for studying iterated loop spaces. A comprehensive modern reference is [36]. We consider the unitary version of this operad, i.e., $\mathscr{C}_n(0) = *$ is the one-point space. For each $r \geq 1$, the arity r component $\mathscr{C}_n(r)$ of $\mathscr{C}_n$ is the subspace of the mapping space

$$\mathscr{C}_n(r) \subseteq \mathsf{Map}\left(\coprod_r I^n, I^n\right)$$

given by those rectilinear embeddings for which the images of the interiors of different cubes are pairwise disjoint. That is,

$$\mathscr{C}_n(r) = \left\{(c_1, \ldots, c_r) \mid \text{each } c_i \text{ is a little } n\text{-cube, and } \mathring{c}_i \cap \mathring{c}_j = \emptyset \text{ for all } i \neq j\right\}.$$

The symmetric group S_r acts on a configuration $c = (c_1, \ldots, c_r)$ of little cubes by permuting its components, $(c_1, \ldots, c_r) \cdot \sigma = \left(c_{\sigma^{-1}(1)}, \ldots, c_{\sigma^{-1}(r)}\right)$. The operadic unit $1 \in \mathscr{C}_n(1)$ is the identity map $I^n \to I^n$, and the partial composition products are explicitly given by

$$(c_1, \ldots, c_r) \circ_i (d_1, \ldots, d_s) = (c_1, \ldots, c_{i-1}, c_i \circ d_1, \ldots, c_i \circ d_s, c_{i+1}, \ldots, c_r).$$

That is: we re-scale and insert the little n-cubes $d_1, \ldots, d_s$ in place of the little n-cube c_i, which is removed, and then relabel accordingly.

Recall from Proposition 1.2.6 that the underlying space of the comonad $C_{\mathscr{P}}(X)$ associated to a unitary topological operad $\mathscr{P}$ and a pointed space X is characterized as a certain subspace of $\mathsf{Map}(\mathscr{P}(1), X)$. In the particular case of the comonad C_n associated to the little n-cubes operad, there is a very geometrical characterization. We need the following preliminary notation. First, recall that

$$D_i = d_1 \cdots \widehat{d_i} \cdots d_r : \mathscr{C}_n(r) \to \mathscr{C}_n(1)$$

denotes the composition of the restriction operators omitting the i-th term, which evaluated at a configuration $\theta = (c_1, \ldots, c_n) \in \mathscr{C}_n(r)$, recovers the i-th little n-cube c_i. Now, let X be a pointed space. Given $f : \mathscr{C}_n(1) \to X$ any map, define for all $r \geq 2$ and $1 \leq i \leq r$ the collection of maps

$$f_r^i := f \circ D_i : \mathscr{P}(r) \to X \quad \text{and} \quad f_r := \left(f_r^1, \ldots, f_r^r\right) : \mathscr{P}(r) \to X^{\times r}. \tag{1.8}$$

The mentioned characterization is the following.

Proposition 1.2.18. *Let X be a pointed space, and C_n the comonad associated to the little n-cubes operad. Then a map $f : \mathscr{C}_n(1) \to X$ belongs to $C_n(X)$ if, and only if, f satisfies the following property:*

(D) *If $c_1, c_2 \in \mathscr{C}_n(1)$ are little n-cubes such that $\mathring{c}_1 \cap \mathring{c}_2 = \emptyset$, then $f(c_1) = *$ or $f(c_2) = *$.*

That is, taking $f = f_1$, each map f_r in (1.8) has its image in the r-fold wedge $X^{\vee r}$, it is S_r-equivariant, and the compatibility conditions $f_{r-1} d_i = \pi_i f_r$ are satisfied for all $r \geq 2$ and $1 \leq i \leq r$.

Proof. Assume $f = f_1 : \mathscr{C}_n(1) \to X$ satisfies property (D). Fix an arbitrary $r \geq 2$, and some $1 \leq i \leq r$. Define $f_r : \mathscr{C}_n(r) \to X^{\times r}$ by

$$f_r = \left(f_1 D_1, \ldots, f_1 D_r\right).$$

Let us check that f_r has its image in the wedge. Indeed, for any $\theta = (c_1, ..., c_r) \in \mathscr{C}_n(r)$, it follows from the definition of the space $\mathscr{C}_n(r)$ that $\mathring{c}_k \cap \mathring{c}_j = \emptyset$ for all $j \neq k$. Furthermore, for each index j between 1 and r, we can write

$$c_j = \left(d_1 \circ \cdots \widehat{d_i} \cdots \circ d_r\right)(\theta) = D_i(\theta).$$

Therefore, condition (D) applied to each pair (j, k) with $j \neq k$ implies that at most a single component $f_1(c_j)$ is not the basepoint. Said differently: f_r has its image in the wedge. The map f_r is S_r-equivariant. Indeed, for any $\sigma \in S_r$, one has

$$f_r \cdot \sigma = \left\{f_1 D_1, ..., f_1 D_r\right\} \cdot \sigma = \left\{f_1 D_1 \cdot \sigma, ..., f_1 D_r \cdot \sigma\right\} = \left\{f_1 D_{\sigma}(1), ..., f_1 D_{\sigma}(r)\right\} = \sigma \cdot \left\{f_1 D_1, ..., f_1 D_r\right\}.$$

Since σ permutes the coordinates of the wedge factors, the claim is proven.

For the converse, assume that $(f_1, f_2, ...) \in C_n(X)$, and that $c_1, c_2 \in \mathscr{C}_n(1)$ are little n-cubes such that $\mathring{c}_1 \cap \mathring{c}_2 = \emptyset$. This is precisely the condition needed to ensure that (c_1, c_2) is an element of $C_n(2)$. Consider $f_2(c_1, c_2) \in X \vee X$. From the comonadic compatibility conditions, one has

$$f_1(c_1) = \pi_1 f_2(c_1, c_2)$$

$$f_1(c_2) = \pi_2 f_2(c_1, c_2)$$

and therefore one of $f_1(c_1)$ or $f_1(c_2)$ must be the base point. Therefore f_1 satisfies property (D). □

In the next remark, we point out the obvious fact that non-trivial strictly coassociative coalgebras do not exist in pointed spaces.

Remark 1.2.19. Recall that a pointed space X is a co-H-space if it comes equipped with a map $c: X \to X \vee X$ that is a factorization up to homotopy of the identity map $X \to X$:

$$\begin{tikzcd} X \arrow[r, "c"] \arrow[dr, "\mathrm{id}"'] & X \vee X \arrow[d, "q_i"] \\ & X \end{tikzcd}$$

That is, $q_1 c \simeq \mathrm{id} \simeq q_2 c$, where $q_i : X \vee X \to X$ is the projection onto the ith factor of the wedge. If we try to strictify this diagram, considering $q_1 c = \mathrm{id} = q_2 c$, then for any $x \in X$ we would have the following situation. The coproduct $c(x)$ is either a point in the first wedge factor, $(x_1, *)$, or it is a point in the second wedge factor, $(*, x_2)$. Without loss of generality, we may assume that it is of the form $c(x) = (x_1, *)$, we would then have

$$q_2 c(x) = q_2(x_1, *) = *.$$

If X has more than one point, we will not have $q_2 c(x) = x$ for $x \neq *$. Thus, the unique strictly coassociative counital coalgebra is the one point space. This is a significant contrast with the algebra case, where for example, the James construction [49] gives a strictly associative monoid in pointed spaces modelling $\Sigma\Omega X$. The classical Moore loop space is another important example of a pointed space endowed with a strictly associative product. We conclude that there is no possible "rectification" of a counital homotopy coassociative-coalgebra into a counital strictly coassociative coalgebra. Aside from the elementary proof given here, the non-existence of strictly coassociative coalgebras in Top_* will also follow from Proposition 1.2.21, a more general statement asserting that reduced operads produce trivial comonads, leaving no place for non-trivial counital coassociative coalgebras. Remark that it is the counit that is causing all the problems in the discussion above. Since there are non-trivial non-counital strictly coassociative coalgebras, the argument above does not apply. It is therefore not known whether strictly coassociative rectifications exist in the case of non-counital coalgebras, but this is beyond the scope of this paper.

The particular instance of Theorem 1.2.11 in this case gives the following important observation.

Theorem 1.2.20. *Let X be a pointed space. Then, $C_n(X)$ is the* cofree C_n-coalgebra *on X. That is, for any C_n-coalgebra A, there is a natural bijection*

$$\mathrm{Hom}_{\mathsf{Top}_*}(A, X) \cong \mathrm{Hom}_{C_n-\mathsf{Coalg}}(A, C(X)).$$

1.2.3.1 Reduced topological operads and weak equivalences

In this section, we prove that for *reduced* unitary topological operads (i.e. $\mathscr{P}(1) = \{*\}$) , the comonad $C_{\mathscr{P}}$ is always the trivial one-point comonad. Therefore, the associated category of $\mathscr{P}$-coalgebras is trivial (Proposition 1.2.21). This is a striking difference with the construction of C_n in the case of the little n-cubes operad $\mathscr{C}_n$, whose category of coalgebras is rich and interesting. As a consequence, we readily see that the comonad construction does not respect weak equivalences in the Berger–Moerdijk model structure [7] on topological operads. That is, if $\mathscr{P} \to \mathscr{Q}$ is a morphism of unitary operads in Top_* which is a weak equivalence in each arity, it does not necessarily follow that the induced map $C_{\mathscr{P}}(X) \to C_{\mathscr{Q}}(X)$ is a weak equivalence for each pointed space X. For example, the associative operad Ass is reduced, producing a trivial category of coalgebras, but there is a a well-known weak equivalence of operads $\mathscr{C}_1 \twoheadrightarrow \mathsf{Ass}$. Said differently, a weak equivalence of unitary operads does not imply an equivalence of categories of coalgebras (even of up to homotopy algebras)

Proposition 1.2.21. *If $\mathscr{P}$ is a reduced unitary topological operad, then $C_{\mathscr{P}}$ is the trivial comonad. That is, $C_{\mathscr{P}}(X)$ is the one-point space for all pointed spaces X. In particular, the comonads C_{Ass} and C_{Com} produced respectively from the associative and commutative operads are trivial.*

Proof. Let $\mathscr{P}$ be an operad as in the statement. Fix a pointed space X, and consider an arbitrary sequence $\alpha = (f_1, f_2, \ldots) \in C_{\mathscr{P}}(X)$. Then,

$$f_1 : \mathscr{P}(1) \to X$$

specifies some point $f_1(*) = x_0 \in X$. Recall (Lemma 1.2.5) that the higher terms f_r in the sequence α are determined by the recursive formula

$$f_r = \{f_1 D_1, \ldots, f_1 D_r\}. \tag{1.9}$$

In particular, for any $\theta \in \mathscr{P}(2)$,

$$f_2(\theta) = \{f_1 d_2(\theta), f_1 d_1(\theta)\} = \{x_0, x_0\}.$$

Therefore, for f_2 to be well-defined (i.e., having its image in the wedge), the point x_0 must be the base point of X. It then follows from the recursive formula (1.9) that for all $r \geq 2$ and $\theta \in \mathscr{P}(r)$, we have

$$f_r(\theta) = \{f_1 D_1(\theta), \ldots, f_1 D_r(\theta)\} = \{x_0, \ldots, x_0\}.$$

That is, α is the trivial sequence. □

1.2.4 Iterated suspensions are coalgebras over the little cubes operad

In this section, we show that the n-fold reduced suspension $\Sigma^n X$ of a pointed space X is a coalgebra over the little n-cubes operad. These are the paradigmatic examples of $\mathscr{C}_n$-coalgebras. To show our results, we use the coendomorphism version of $\mathscr{C}_n$-coalgebras. At the end of the section, we explain how the results in this paper allows us to swiftly recover the classical $\mathscr{C}_n$-algebra structure on n-fold loop spaces as a convolution structure. The $\mathscr{C}_n$-coaction on S^n that we describe in this section has previously appeared, in the context of the factorization homology, in [40].

Theorem 1.2.22. *The n-fold reduced suspension of a pointed space X is a $\mathscr{C}_n$-coalgebra. More precisely, there is a natural and explicit operad map*

$$\nabla : \mathscr{C}_n \to \mathsf{CoEnd}_{\Sigma^n X}$$

that encodes the homotopy coassociativity and homotopy cocommutativity of the classical pinch map $\Sigma^n X \to \Sigma^n X \vee \Sigma^n X$. In particular, the pinch map is an operation associated to an element of $\mathscr{C}_n(2)$. Furthermore, for any based map $X \to Y$, the induced map $\Sigma^n X \to \Sigma^n Y$ extends to a morphism of $\mathscr{C}_n$-coalgebras.

The first step in proving the result above consists in showing that the sphere S^n, with $n \geq 1$, is a coalgebra over the little n-cubes operad. That is, we first show that the statement above is true for $X = S^n = \Sigma^n S^0$.

Proposition 1.2.23. *For every $n \geq 1$, there is a natural and explicit morphism of operads*

$$\nabla : \mathscr{C}_n \to \mathsf{CoEnd}_{S^n}$$

turning the n-sphere into a $\mathscr{C}_n$-coalgebra, so that all properties of Theorem 1.2.22 for $\Sigma^n X = S^n$ hold true.

Proof. Let us define the arity r component of ∇. This is a map

$$\nabla_r : \mathscr{C}_n(r) \to \mathsf{CoEnd}_{S^n}(r) = \mathsf{Map}_*\left(S^n, S^n \vee \ldots \vee S^n\right).$$

For $c = (c_1, \ldots, c_r) \in \mathscr{C}_n(r)$ a configuration of little n-cubes, we define the pointed map

$$\begin{aligned} \nabla_r(c) : S^n &\longrightarrow (S^n)^{\vee r} \\ t &\longmapsto \nabla_r(c)(t) \end{aligned}$$

as follows. Identify $S^n = I^n / \partial I^n$. Then $t \in S^n$ is either the base point $t = \{\partial I^n\}$ or else it is an interior point of the n-cube I^n. If t is interior, then is at most a single cube c_i such that $t \in \mathring{c}_i$. We define

$$\nabla_r(c)(t) = \begin{cases} \left[c_i^{-1}(t)\right] & \text{if } t \in \mathring{c}_i, \\ * & \text{otherwise} \end{cases}$$

Here, $\left[c_i^{-1}(t)\right]$ denotes the point in the i-th wedge factor of $S^n \vee \ldots \vee S^n$ followed by its inclusion as the i-th factor of the wedge. So defined, the maps $\nabla_r(c)$ are pointed, continuous and turn this into a morphism of operads. The fact that this is a morphism of operads is straightforward to check and left to the reader. $\square$

We prove next that the little n-cubes coalgebra structure on the sphere S^n just described induces the little n-cubes coalgebra structure on an arbitrary n-fold reduced suspension.

Proof of Theorem 1.2.22: Let $\Sigma^n X$ be the n-fold reduced suspension of the pointed space X. Write $\Sigma^n X = S^n \wedge X$, and recall that for any three pointed spaces X, Y and Z, the wedge and smash product distribute over each other [44, S. 4.F], i.e.

$$X \wedge (Y \vee Z) \cong (X \wedge Y) \vee (X \wedge Z)\,.$$

In particular, when we take X to be S^n

$$\Sigma^n (Y \vee Z) \cong \Sigma^n Y \vee \Sigma^n Z.$$

Then, for $c \in \mathscr{C}_n(r)$, define the map $\Sigma^n X \to (\Sigma^n X)^{\vee r}$ as the composition

$$\Sigma^n X \cong S^n \wedge X \xrightarrow{\nabla_r(c)\wedge \mathrm{id}_X} \left(\left(S^n\right)^{\vee r}\right) \wedge X \xrightarrow{\cong} \left(S^n \wedge X\right)^{\vee r} \cong \left(\Sigma^n X\right)^{\vee r},$$

where ∇_r is the arity r component of the map ∇ defined in Proposition 1.2.23. All these maps are continuous, commute with the symmetric group actions and the operadic composition maps, producing a functorial construction. Alternatively, one can define the operad map

$$\mathrm{CoEnd}_{S^n} \to \mathrm{CoEnd}_{\Sigma^n X}$$

given (up to isomorphism) by $f \mapsto f \wedge \mathrm{id}_X$, and precompose it with the operad map of Proposition 1.2.23. Doing this, one ends up with the map we described before. In this sense, the $\mathscr{C}_n$-coalgebra structure of an n-fold suspension always factors through the $\mathscr{C}_n$-coalgebra structure of S^n. □

Remark 1.2.24. The defined operad map $\nabla : \mathscr{C}_n \to \mathrm{CoEnd}_{\Sigma^n X}$ is determined by its arity 1 component $\nabla_1 : \mathscr{C}_n(1) \times \Sigma^n X \to \Sigma^n X$. Being more precise, as a consequence of Proposition 1.2.5, the following formula holds for all $c \in \mathscr{C}_n(r)$ and $z \in \Sigma^n X$:

$$\pi_i\left(\nabla_r(c,z)\right) = \nabla_{r-1}\left(d_i(c), z\right),$$

where π_i and d_i are the wedge collapse and restriction operators from Section 1.2.1.

In the remainder of the section, we explain how the coalgebraic framework introduced in this work let us swiftly recover the classical result by May that iterated loop spaces are algebras over the little n-cubes operad. For this, we first need to define fold algebras in the category of pointed spaces with the wedge product $\vee$.

Definition 1.2.25. Let X be a pointed space. The *fold endomorphism operad* $\mathrm{End}_X^{\vee}$ is the operad whose arity r component is given by

$$\mathrm{End}_X^{\vee}(r) = \mathsf{Map}_*\left(X^{\vee r}, X\right),$$

with the composition map given by inserting the output of a map into the input, and the symmetric group action is given by permuting the inputs. If $\mathscr{P}$ is an operad in unpointed spaces, then a fold $\mathscr{P}$-algebra is a pointed space X together with a morphism of operads $\mathscr{P} \to \mathsf{End}_X^{\vee}$.

We leave it to the reader to check that the definition above gives an operad. Every pointed space is canonically a commutative fold-algebra, where the products are given by the canonical fold maps (which explains the name).

Using the definiton of a fold $\mathscr{P}$-algebra, we can now define a convolution algebra between a $\mathscr{P}$-coalgebra and a fold $\mathscr{Q}$-algebra, for operads $\mathscr{P}$ and $\mathscr{Q}$. Denote by $\mathscr{P} \times \mathscr{Q}$ the arity-wise product of $\mathscr{P}$ and $\mathscr{Q}$. This allows us to define convolution algebras in pointed spaces as follows.

Proposition 1.2.26. *Let $\mathscr{P}$ and $\mathscr{Q}$ be operads in unpointed spaces. Let X be a $\mathscr{P}$-coalgebra and Y a fold $\mathscr{Q}$-algebra. Then the pointed mapping space $\mathsf{Map}_*(X,Y)$ is a $\mathscr{P} \times \mathscr{Q}$-algebra. The structure maps*

$$\gamma : \mathscr{P}(r) \times \mathscr{Q}(r) \times \mathsf{Map}_*(X,Y)^{\times r} \to \mathsf{Map}_*(X,Y)$$

applied to pointed maps $f_1,\ldots,f_r : X \to Y$ are explicitly given by

$$\gamma\left((\theta,\nu); f_1,\ldots,f_r\right) = \left(\nu \circ \left(f_1 \vee \cdots \vee f_r\right) \circ \Delta\right)(\theta).$$

Here, $(\theta,\nu) \in \mathscr{P}(r) \times \mathscr{Q}(r)$, i.e., the canonical map from the r-fold coproduct of X onto X, and $\Delta : \mathscr{P} \to \mathrm{CoEnd}_X$ is the $\mathscr{P}$-coalgebra structure map of X.

Proof. This is similar to the construction in Section 1 of [7] and is left to the reader. □

In particular, n-fold loop spaces fall into the framework described in the previous result. Since every pointed space is canonically a commutative fold algebra, and the arity-wise product of $\mathscr{C}_n$ with the commutative operad is isomorphic to $\mathscr{C}_n$, we recover May's classical $\mathscr{C}_n$-algebra structure on loop spaces as follows (see [66]).

Corollary 1.2.27. *Let $\Omega^n X$ be an n-fold loop space. Then, the $\mathscr{C}_n$-algebra structure on*

$$\Omega^n X = \mathsf{Map}_*\left(S^n, X\right)$$

induced by the $\mathscr{C}_n$-coalgebra strucuture of S^n and the fold Com-algebra structure on X as a convolution algebra is exactly the classical $\mathscr{C}_n$-algebra structure on loop spaces.

Proof. By definition, each map $S^n \to S^n \vee \cdots \vee S^n$ arising from the $\mathscr{C}_n$ coalgebra structure of S^n induces the following convolution product on an n-fold loop space $\Omega^n X$. Given $\alpha_1,\ldots,\alpha_r : S^n \to X$ and $\theta \in \mathscr{C}_n(r)$, define $\gamma(\alpha_1,\ldots,\alpha_r)$ as

$$S^n \xrightarrow{\nabla(\theta)} \left(S^n\right)^{\vee r} \xrightarrow{\alpha_1 \vee \ldots \vee \alpha_r} X^{\vee r} \xrightarrow{\mu_r} X,$$

where $\mu_r \in \mathsf{Com}(r)$ is the rth fold map. Here, Com is the commutative operad. One checks that these maps are exactly the maps described in [66, Section 5]. □

1.3 The Approximation Theorem

To prove the recognition principle for n-fold loop spaces, as well as to develop a unified theory of homology operations for them, May proved the *approximation theorem* [66, Theorem 6.1]. This consists of giving a morphism of monads from the monad M_n associated to the little n-cubes operad to the monad $\Omega^n\Sigma^n$, and proving that this natural transformation is a weak equivalence on connected spaces. In this section, we prove an Eckmann–Hilton dual result to approximate the comonad $\Sigma^n\Omega^n$.

Theorem 1.3.1. *For every $n \geq 1$, there is a natural morphism of comonads*

$$\alpha_n : \Sigma^n \Omega^n \longrightarrow C_n.$$

Furthermore, for every pointed space X, there is an explicit natural homotopy retract of pointed spaces

$$\begin{tikzcd} \Sigma^n\Omega^n X \arrow[r, shift left] & C_n(X) \arrow[l, shift left] \arrow[loop above] \end{tikzcd}$$

In particular, $\alpha_n(X)$ is a weak equivalence.

The proof of the result above does not consist of a dualization of the corresponding proof of May's proof in the case of loop spaces. We take a different route which has the advantage that it gives us explicit homotopies and does not require auxiliary spaces as is needed in May's original approach. It is at the moment not clear whether these methods can also be used to give an alternative proof of the loop space approximation theorem.

Let $n \geq 1$ be a fixed integer. The natural transformation $\alpha = \alpha_n : \Sigma^n\Omega^n \to C_n$ is defined object-wise as the composition

$$\alpha_X : \Sigma^n\Omega^n X \xrightarrow{\gamma} C_n\left(\Sigma^n\Omega^n X\right) \xrightarrow{C_n(\eta_X)} C_n(X),$$

where γ is the $\mathscr{C}_n$-coalgebra structure map of $\Sigma^n\Omega^n X$ (Theorem 1.2.22), and η_X is the evaluation at X of the counit $\eta : \Sigma^n\Omega^n \to \mathrm{id}_{\mathsf{Top}_*}$ of the (Σ^n, Ω^n)-adjunction. Unraveling the definitions, we readily see that $\alpha = \alpha_X$ is explicitly given on a point $[t, \ell] \in \Sigma^n\Omega^n X = S^n \wedge \mathsf{Map}_*(S^n, X)$ as the map $\alpha[t,\ell] : \mathscr{C}_n(1) \to X$ that acts on a little n-cube $c \in \mathscr{C}_n(1)$ by

$$\alpha[t,\ell](c) = \begin{cases} \ell\left(c^{-1}(t)\right) & \text{if } t \in \mathring{c} \\ * & \text{otherwise} \end{cases}$$

See Proposition 1.5.2 for more details on the definition of α.

Proof of Theorem 1.3.1: The proof consists of the following two steps.

(i) We must check that α defines a morphism of comonads. This is not complicated, but it is lengthy. Because of this, we postponed this proof to Appendix 1.5 (Proposition 1.5.2).

(ii) We must check that for a fixed pointed space X, the space $\Sigma^n\Omega^n X$ is a retract of spaces of $C_n(X)$. To do so, we give a pointed map (of spaces, not comonads) $\Psi = \Psi_n : C_n(X) \to \Sigma^n\Omega^n X$ and a homotopy $\mathscr{H} : C_n(X) \times I \to C_n(X)$ such that

$$\Psi \circ \alpha = \mathrm{id}_{\Sigma^n\Omega^n X} \quad \text{and} \quad \alpha \circ \Psi \simeq \mathrm{id}_{C_n(X)}. \tag{1.10}$$

To define Ψ and the homotopy $\mathscr{H} = \mathscr{H}_n : \alpha \circ \Psi \simeq \mathrm{id}_{C_n(X)}$, we introduce below for each $f \in C_n(X)$ a certain subset of the n-cube I^n which we name the *cubical support of f* and denote $\mathrm{CSupp}(f)$. In the case of interest, the cubical support of a map f will be non-empty and has a well-defined *center*, which is a point

$$\mathrm{Cent}(f) \in \mathrm{CSupp}(f) \subseteq I^n.$$

Theorem 1.3.1 will then follow from the two items just described. Since the first item is proved in the mentioned appendix, it remains to prove the second one. We do this in what follows.

$\boxed{\text{Definition of } \Psi}$

The pointed map Ψ is defined as follows.

$$\begin{aligned} \Psi : C_n(X) &\longrightarrow \Sigma^n \Omega^n X \\ f &\longmapsto \Psi(f) = [\mathrm{Cent}(f), \ell]. \end{aligned}$$

Here, we need to explain what the two components above are:

$$t := \mathrm{Cent}(f) \in S^n \quad \text{and} \quad \ell : S^n \to X, \quad s \mapsto \ell(s) := f\left(c_{s,\mathrm{Cent}(f)}\right).$$

Since we are identifying $S^n = I^n/\partial I^n$, we are denoting by $\mathrm{Cent}(f)$ a certain point of the n-cube I^n that we are denoting in the same way and is going to be explained below. On the other hand, the little n-cube $c_{s,\mathrm{Cent}(f)}$ that depends both on f and s, follows a certain construction to be explained below too.

Let us start with the following auxiliary definition. The *cubical support* of an arbitrary map $f : \mathscr{C}_n(1) \to X$ is the intersection of the images of all little n-cubes $c : I^n \to I^n$ such that f acts non-trivially on c:

$$\mathrm{CSupp}(f) = \bigcap_{\substack{c \in \mathscr{C}_n(1) \\ f(c) \neq *}} \mathrm{Im}(c) \subseteq I^n.$$

If the family over which we are taking the intersection above is empty, then we define $\mathrm{CSupp}(f) = \emptyset$. If f is an element of $C_n(X)$, then this happens only when f is the trivial map. In this case, we define $\Psi(f)$ to be the base point of $\Sigma^n \Omega^n X$. The cubical support of f is closely related to its classical support, namely, the set of points of the domain of f where f acts non-trivially:

$$\mathrm{Supp}(f) = \bigcap_{\substack{c \in \mathscr{C}_n(1) \\ f(c) \neq *}} c \subseteq \mathscr{C}_n(1).$$

Indeed, since each $c \in \mathscr{C}_n(1)$ defines the subset $\mathrm{Im}(c) \subseteq I^n$, the cubical support of f is the subset of I^n determined by the classical support of f. Recall also that an n-*rectangle* is a subspace of $\mathbb{R}^n$ which is rectilinearly homeomorphic to I^n or a singleton. An n-rectangle that does not reduce to a single point is determined by the set of its 2^n vertices, but also more efficiently by $2n$ numbers that describe the length of the sides and their position. In other words, an n-rectangle R is simply a cartesian product of closed intervals:

$$R = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_i \leq x_i \leq b_i \text{ for all } i = 1, \ldots, n\right\} = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

for certain $a_i, b_i \in \mathbb{R}$ satisfying $a_i \leq b_i$.

Claim 1: The cubical support $\mathrm{CSupp}(f)$ *of a map* $f \in C_n(X)$ *is empty if, and only if,* f *is the trivial map. Furthermore, if* f *is non-trivial, then its cubical support is a point or an* n*-rectangle.*

Proof of Claim 1: Let $f \in C_n(X)$ be any map. If $\mathrm{CSupp}(f) \neq \emptyset$, then obviously $f \neq *$. Let us check the converse. Assume therefore that $f \neq *$, and let us check that $\mathrm{CSupp}(f) \neq \emptyset$. Indeed: since $f \neq *$, there is some little n-cube d such that $f(d) \neq *$. Thus, the family $\{\mathrm{Im}(c) \mid f(c) \neq *\}$ over which we are taking the intersection in the definition of the cubical support is non-empty. Now, from Proposition 1.2.18, it follows that if $c_1, c_2 \in \mathscr{C}_n(1)$ are such that both $f(c_1) \neq *$ and $f(c_2) \neq *$, then necessarily $\mathring{c}_1 \cap \mathring{c}_2 \neq \emptyset$. The intersection of the interiors of any two n-rectangles that do not reduce to a point is either empty, or it is again an n-rectangle that does not reduce to a point. From this fact, it follows that $\mathrm{CSupp}(f)$ is non-empty.

To check the furthermore assertion in Claim 1, let $c \in \mathscr{C}_n(1)$ be a little n-cube, and write $c = \left(g_1, ..., g_n\right)$ in terms of its coordinate functions $g_i : I \to I$. Then, the image of the cube c is the n-rectangle

$$\operatorname{Im}(c) = \left[g_1(0), g_1(1)\right] \times \cdots \times \left[g_n(0), g_n(1)\right] \subseteq I^n.$$

There is an obvious canonical identification between little n-cubes and n-rectangles contained in I^n that do not reduce to a single point. The cubical support of a fixed map $f : \mathscr{C}_n(1) \to X$ is therefore the n-rectangle

$$\operatorname{CSupp}\left(f\right) = [a_1, b_1] \times \cdots \times [a_n, b_n],$$

where for each $i = 1, ..., n$

$$a_i := \sup\left\{g_i(0) \mid c = \left(g_1, ..., g_n\right) \in \mathscr{C}_n(1) \text{ and } f(c) \neq *\right\},$$
$$b_i := \inf\ \left\{g_i(1) \mid c = \left(g_1, ..., g_n\right) \in \mathscr{C}_n(1) \text{ and } f(c) \neq *\right\}.$$

This finishes the proof of Claim 1. □

Every non-empty n-rectangle R has a *center* $\operatorname{Cent}(R)$. If $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$, then its center is the point determined by the midpoint of each of the intervals,

$$\operatorname{Cent}(R) = \left(\frac{a_1 + b_1}{2}, ..., \frac{a_n + b_n}{2}\right).$$

Observe that, if $R = (x_1, ..., x_n)$ is a singleton, then $\operatorname{Cent}(R) = (x_1, ..., x_n)$. Assuming furthermore that $R = \operatorname{CSupp}\left(f\right)$ for some f, then we define $\operatorname{Cent}(f)$, the center of f, as

$$\operatorname{Cent}(f) := \operatorname{Cent}\left(\operatorname{CSupp}\left(f\right)\right) = \operatorname{Cent}(R).$$

Examples 1.3.2. Let us compute the cubical support $\operatorname{CSupp}\left(f\right)$ for several maps f.

1. Let $C_n(*)$ be the cofree C_n-coalgebra on a single point. Then, $C_n(X) = *$ reduces to the trivial one-point space. Thus, the unique map $f : \mathscr{C}_n(1) \to *$ collapses all little n-cubes to the base point, and therefore, $\operatorname{CSupp}\left(f\right) = \emptyset$.
2. Consider the map $f : \mathscr{C}_1(1) \to I$ given by

$$f(c) = \begin{cases} 0 & \text{if } r \leq 1/2 \\ r - 1/2 & \text{if } r \geq 1/2 \end{cases}$$

   Here, $r = c(1) - c(0)$ is the size of the little 1-cube c. By Proposition 1.2.18, f defines an element in $C_1(I)$, and one readily checks that $\operatorname{Cent}\left(f\right) = \operatorname{CSupp}\left(f\right) = \left\{\frac{1}{2}\right\}$. By varying r, it is possible to construct a map having as center any chosen point in $(0, 1)$.

3. Define $f : \mathscr{C}_1(1) \to I$ as in the example above replacing $1/2$ by any real number $a \in \left[\frac{1}{2}, 1\right)$. By Proposition 1.2.18, f defines a map in $C_1(I)$. Its cubical support is the interval $[1 - a, a]$. In the case where $\alpha = \frac{1}{2}$, we see again that $\operatorname{Cent}\left(f\right) = \frac{1}{2}$.

The examples above can be generalized to higher-dimensional cubes.

Another important example of cubical support is that of n-fold suspensions.

Proposition 1.3.3. *Let $\Sigma^n X$ be the n-fold reduced suspension of a pointed space X, and let $\gamma : \Sigma^n X \to C_n(\Sigma^n X)$ be its $\mathscr{C}_n$-coalgebra structure map. Then, for every non-base point $[t, x] \in \Sigma^n X$, we have that*

$$\mathrm{CSupp}\big(\gamma[t,x]\big) = \{t\}.$$

Proof. First, we prove the result for spheres. If $\gamma : S^n \to C_n(S^n)$ is the $\mathscr{C}_n$-coalgebra structure map, we explicitly have

$$\gamma(t)(c) = \begin{cases} c^{-1}(t) & \text{if } t \in \mathring{c} \\ * & \text{otherwise,} \end{cases}$$

where $t \in S^n$ and we identify S^n with $I^n/\partial I^n$, the ambient cube of c modulo its boundary. By definition, $\mathrm{CSupp}\big(\gamma(t)\big)$ is the intersection of the family

$$\big\{\mathrm{Im}(c) \mid c \in \mathscr{C}_n(1) \text{ and } \gamma(t)(c) \neq *\big\}.$$

The image $\mathrm{Im}(c)$ of a little n-cube is non-trivial if, and only if, $t \in \mathrm{Im}(c)$. Thus, the cubical support $\mathrm{CSupp}\big(\gamma(t)\big)$ is the intersection of all non-trivial cubes containing t, and therefore, it is the singleton $\{t\}$.

Now, for an arbitrary n-fold reduced suspension $\Sigma^n X$, factorize its coalgebra structure map as follows:

$$\Sigma^n X = S^n \wedge X \xrightarrow{\gamma_{S^n} \wedge \mathrm{id}_X} C_n(S^n) \wedge X \xrightarrow{F} C_n(S^n \wedge X).$$

The second map F above is given by

$$F\big(f, x\big) = \big[f(-), x\big], \quad \text{for } f : \mathscr{C}_n(1) \to S^n \text{ and } x \in X.$$

The final composition is therefore explicitly given by

$$\begin{aligned} \gamma[t,x] : \mathscr{C}_n(1) &\longrightarrow S^n \wedge X \\ c &\longmapsto \big[\gamma(t)(c), x\big]. \end{aligned}$$

Here, the cubical support $\mathrm{CSupp}\big(\gamma[t,x]\big)$ is the intersection of the family

$$\big\{\mathrm{Im}(c) \mid c \in \mathscr{C}_n(1) \text{ and } \big[\gamma(t)(c), x\big] \neq *\big\}.$$

Similar to the case of the spheres, we have

$$\big[\gamma(x)(c), x\big] = \begin{cases} \big[c^{-1}(t), x\big] & \text{if } t \in \mathring{c} \\ * & \text{otherwise} \end{cases}$$

We readily see from here that a little n-cube c has non-trivial image if, and only if, $\mathring{c}$ contains the component t of the sphere. Thus, the intersection of them all yields the singleton $\{t\}$. □

We also need the following auxiliary result. It explicitly describes the little n-cube that appears in the loop $\ell : S^n \to X$ of the second component of Ψ.

Claim 2: For each pair of points $s, t \in I^n - \partial I^n$, there is a unique little n-cube $c = c_{s,t} : I^n \to I^n$, depending continuously on (s, t), such that:

1. *$c(s) = t$,*

2. $\mathrm{Im}(c)$ *is the* n*-rectangle of maximum size contained in* I^n *and touching all the faces of the boundary* ∂I^n*. More precisely, we require that for each coordinate at least one side of the embedded rectangle touches a side of the ambient cube.*

If s or t lies in the boundary ∂I^n, we will not need to construct the cube $c_{s,t}$. Indeed, in this case Ψ will map the pair $[t,\ell]$ to the base point of $C_n(X)$.

Proof of Claim 2: Let us explicitly construct c. Recall from Equation (1.7) that the rectilinear embedding c is of the form

$$c(x_1,\ldots,x_n) = \big((b_1-a_1)\,x_1 + a_1,\ldots,(b_n-a_n)\,x_n + a_n\big),$$

where $0 \le a_i < b_i \le 1$ for all i. Thus, each component c_i of c is determined by the numbers a_i and b_i. Imposing that $c(s) = t$, we get the relations

$$(b_i - a_i)s_i + a_i = t_i \quad \text{for each } i.$$

A second constraint on each component i determines the numbers a_i, b_i uniquely. Since c touches each face of ∂I^n, at each component c_i we must have one of the following two options:

1. $c_i(0) = 0$, and then we deduce that
$$c_i(x_i) = \frac{t_i}{s_i}\cdot x_i,$$
or else
2. $c_i(1) = 1$, and then we deduce that
$$c_i(x_i) = 1 - \frac{(1-t_i)(1-x_i)}{1-s_i}.$$

Now, there is no choice to be made here. Rather, the option is determined by the relationship between s and t. That is, we are considering the separate cases where $s_i > t_i$ or $s_i < t_i$. More precisely, if for a fixed i, we have that $0 < t_i/s_i < 1$, then the first formula gives a well-defined affine linear map onto the interval, but the second formula does not (because its image lands outside the unit interval). If, on the other hand, the inequality $0 < t_i/s_i < 1$ does not hold, then it follows that $0 < \frac{(1-t_i)(1-x_i)}{1-s_i} < 1$, and the second formula does define an affine linear map (while the first one does not). To finish, observe that the formulae agree when $s_i = t_i$, which makes the construction of c a continuous function of s and t. Of course, in the case $s_i = t_i$, we are taking the identity map at the i-th coordinate. This finishes the proof of Claim 2. $\square$

Having explained in full detail what all the items defining Ψ are, the map Ψ is given by:

$$\begin{aligned} \Psi : C_n(X) &\longrightarrow \Sigma^n\Omega^n X \\ f &\longmapsto \Psi(f) = [\mathrm{Cent}(f), \ell], \end{aligned}$$

where ℓ is defined as

$$\ell : S^n \to X,$$
$$s \mapsto \ell(s) := f\left(c_{s,\mathrm{Cent}(f)}\right).$$

Our arguments so far show that the resulting function is a pointed continuous function of f.

$\boxed{\text{Definition of the homotopy } \mathscr{H}}$

The next step in the proof of the approximation theorem is to construct a homotopy $\mathscr{H} : C_n(X) \times I \to C_n(X)$ such that

$$\mathscr{H}_0 = \mathrm{id}_{C_n(X)}, \qquad \mathscr{H}_1 = \alpha \circ \Psi, \qquad \mathscr{H}(*, t) = * \ \forall t \in I. \tag{1.11}$$

The following auxiliary construction is a key ingredient for the homotopy $\mathscr{H}$. Intuitively speaking, the idea is to construct a homotopy from maps whose cubical support is more than a point to maps whose cubical support is exactly a point. We construct this homotopy by enlarging the cubes in $C_n(1)$ until they hit the boundary while also preserving the center. This is made precise in the following auxiliary construction.

Auxiliary construction: The rectilinear expansion of a little n-cube $c \in \mathscr{C}_n(1)$ induced by a map $f \in C_n(X)$ whose center $\mathrm{Cent}(f)$ *belongs to* $\mathring{c}$*.*

Proof and explanations for the auxiliary construction: Let us explain the construction for a little 1-interval $c \in \mathscr{C}_1(1)$; the general case is an application of this construction at each coordinate of a little n-cube. Let $c \in \mathscr{C}_1(1)$, so that

$$c(t) = (b - a)t + a$$

for some a, b with $0 \le a < b \le 1$. Let

$$x_1 = \mathrm{dist}(\mathrm{Im}(c), \partial I) = \min\{a, 1 - b\}$$

be the distance from $\mathrm{Im}(c)$ to the boundary of the interval.

Definition 1.3.4. Let $c \in \mathscr{C}_1(1)$. The *rectilinear expansion* of c induced by a map $f \in C_1(X)$ whose center $\mathrm{Cent}(f)$ belongs to $\mathring{c}$ is the unique path $\gamma = \gamma_c^f : I \to \mathscr{C}_1(1)$ satisfying:

- $\gamma(0) = c$,
- for every $s \in (0, 1]$,
  - the cube $\gamma(s)$ is a rectilinear embedding that increases the size of c by $\min\{s, \mathrm{dist}(\mathrm{Im}(c), \partial I)\}$ while keeping the ratios between the sides equal, and
  - the center $\mathrm{Cent}(f)$ is fixed by $\gamma(s)$, i.e. if $z = c^{-1}(\mathrm{Cent}(f))$, then $\gamma(s)(z) = \mathrm{Cent}(f)$.

Let us explicitly describe the path above. For each $s \in I$, we have $\gamma(s) \in \mathscr{C}_1(1)$ of the form

$$\gamma(s)(t) = (b(s) - a(s))\, t + a(s) \quad \forall\ t \in I.$$

For a fixed $s \in I$, two conditions on $a(s)$ and $b(s)$ determine $\gamma(s)$ uniquely. We impose the two mentioned conditions, namely that

$$\gamma(s)(p) = p,$$

where for simplicity we denote $p = \mathrm{Cent}(f)$, and that the radius of $\gamma(s)$ is that of c increased by $\min\{s, a, 1 - b\}$:

$$(b(s) - a(s)) - (b - a) = \min\{s, a, 1 - b\}.$$

These conditions produce the linear system of equations

$$\begin{cases} (1-p)a(s)+pb(s)=p \\ \qquad -a(s)+b(s)=\alpha(s) \end{cases}$$

where $\alpha(s)=\min\{s,a,1-b\}$. The unique solution to the system above is

$$\begin{aligned} a(s) &= p(1-\alpha(s)) \\ b(s) &= \alpha(s)-\alpha(s)p+p. \end{aligned}$$

Therefore, for a fixed $s\in I$, the little 1 interval $\gamma(s)$ is given by

$$\gamma(s)(t)=\alpha(s)t+p-p\alpha(s) \quad \forall\ t\in I.$$

This finishes the construction for a little 1-interval. In the general case, given $c\in\mathscr{C}_n(1)$ of the form

$$c(t_1,\dots,t_n)=\left((b_1-a_1)t_1+a_1,\dots,(b_n-a_n)t_n+a_n\right)$$

and $f\in C_n(X)$, define $\gamma=\gamma_c^f:I\to\mathscr{C}_n(1)$ to be the path such that

$$\gamma(s)(t_1,\dots,t_n)=\left(\alpha_1(s)t_1+p_1-p_1\alpha_1(s),\dots,\alpha_n(s)t_n+p_n-p_n\alpha_n(s)\right) \quad \forall\ (t_1,\dots,t_n)\in I^n.$$

This finishes the construction of the auxiliary path $\gamma_c^f:I\to\mathscr{C}_n(1)$, and therefore the proof and explanations for the auxiliary construction. $\square$

Now, we are ready to define the homotopy $\mathscr{H}:C_n(X)\times I\to C_n(X)$. For each $(f,t)\in C_n(X)\times I$, this is the map

$$\mathscr{H}(f,t):\mathscr{C}_n(1)\to X$$

whose image on a little n-cube $c\in\mathscr{C}_n(1)$ is

$$\mathscr{H}(f,t)(c)=f\left(\gamma_c^f(t)\right)$$

Here, γ_c^f is the rectilinear expansion of c induced by f. Note that this rectilinear expansion shrinks the cubical support of f to a point. We must check that $\mathscr{H}$ is well-defined, continuous, and satisfies the requirements for being a pointed homotopy from $\mathrm{id}_{C_n(X)}$ to $\alpha\Psi$. To check that $\mathscr{H}$ is well-defined, we must corroborate that for each (f,t), the map $\mathscr{H}(f,t)$ indeed defines an element in $C_n(X)$. Recall from Proposition 1.2.18 that given $c_1,c_2\in\mathscr{C}_n(1)$ with $\mathring{c}_1\cap\mathring{c}_2=\emptyset$, this amounts to checking that

$$\mathscr{H}(f,t)(c_1)=* \quad \text{or} \quad \mathscr{H}(f,t)(c_2)=*.$$

But this is immediate: if $\mathring{c}_1\cap\mathring{c}_2=\emptyset$, then $\mathrm{Cent}(f)$ cannot be in both c_1 and c_2 at the same time. Therefore, by definition $\mathscr{H}(f,t)$ vanishes on the little cube c_i not having $\mathrm{Cent}(f)$ in its image. We conclude that $\mathscr{H}$ is well-defined. It is straightforward to check that $\mathscr{H}$ is indeed continuous and we leave this to the reader. Similarly, it follows directly from the definitions that the identities of Equation (1.11) hold.

We have therefore explained in full detail the definition of $\mathscr{H}$, and checked it gives a pointed homotopy between $\mathrm{id}_{C_n(X)}$ and $\alpha\circ\Psi$.

$\boxed{\text{Proving the equality } \Psi\circ\alpha=\mathrm{id}_{\Sigma^n\Omega^n X}}$

Let $[t,\ell]\in\Sigma^n\Omega^n X$. By definition,

$$\Psi\alpha[t,\ell]=[\mathrm{Cent}(\alpha[t,\ell]),L], \tag{1.12}$$

where $L \colon S^n \to X$ is the loop

$$s \mapsto L(s) = \alpha[t,\ell]\left(c_{s,\mathrm{Cent}(\alpha[t,\ell])}\right).$$

Assume that X is not the one-point space and that ℓ is not the constant loop; otherwise the result is trivial. We must check the two components in the right hand side of Equation (1.12) are, respectively, t and ℓ.

1. Let us check that $\mathrm{Cent}\,(\alpha[t,\ell]) = t$. To do so, it suffices to check that $\mathrm{CSupp}\,(\alpha[t,\ell])$ reduces to the single point $\{t\}$. Indeed: if $c \in \mathscr{C}_n(1)$ is such that $\alpha[t,\ell](c) \neq *$, it follows from the definition of $\alpha[t,\ell]$ that $t \in \mathring{c}$ (recall Equation (1.15)). Thus, $t \in \mathrm{Im}(c)$ for all little n-cubes c such that $\alpha[t,\ell](c) \neq *$. Therefore, t is in the intersection of all such images, namely $\mathrm{CSupp}\,(\alpha[t,\ell])$. Now, if $t_0 \neq t$, then we can always find a little n-cube $\widetilde{c}$ such that $t_0 \notin \mathrm{Im}(\widetilde{c})$ and $t \in \mathrm{Im}(\widetilde{c})$, and furthermore $\ell\left((\widetilde{c})^{-1}(t)\right) \neq *$ (possibly after reparametrization: it might be the case that the loop ℓ passes through the basepoint of X, but we are assuming ℓ is not the constant loop).
2. Let us check that $L(s) = \ell(s)$ for all $s \in S^n$. Indeed: for $t = \mathrm{Cent}\,(\alpha[t,\ell])$, the little n-cube $c = c_{s,\alpha[t,\ell]}$ is such that $c(s) = t$. Said differently, $c^{-1}(t) = s$. Therefore, by definition:
$$L(s) = \alpha[t,\ell]\,(c) = \begin{cases} * & \text{if } t \notin \mathring{c} \\ \ell\left(c^{-1}(t)\right) & \text{otherwise} \end{cases} = \ell(s).$$

To summarise: we have explained the definition of the map Ψ and the homotopy $\mathscr{H}$, and have shown the retract requirements of Equation (1.10) hold. Thus, the proof of Theorem 1.3.1 is now complete. □

Remark 1.3.5. In this section we have chosen to prove the approximation theorem for the little n-cubes (rectangles) operad, but the ideas could easily be modified to other little convex bodies operads, like the little n-disks operad. Here some small modification would be needed to explain what exactly is meant by the center and how the expansion is defined. For simplicity, we have decided to only look at the little cubes operads.

1.4 The Recognition Principle for n-fold reduced suspensions

In this section, we prove the recognition principle for n-fold reduced suspensions. The precise statement is the following.

Theorem 1.4.1. *Let X be a $\mathscr{C}_n$-coalgebra. Then there is a pointed space $\Gamma^n(X)$, naturally associated to X, together with a weak equivalence of $\mathscr{C}_n$-coalgebras*

$$\Sigma^n\Gamma^n(X) \xrightarrow{\ \simeq\ } X,$$

which is a retract in the category of pointed spaces. Therefore, every $\mathscr{C}_n$-coalgebra has the homotopy type of an n-fold reduced suspension.

The result above is the converse of Theorem 1.2.22, where it was proven that n-fold reduced suspensions are $\mathscr{C}_n$-coalgebras. Summarizing, we are providing the following intrinsic characterization of n-fold reduced suspensions as $\mathscr{C}_n$-coalgebras.

Corollary 1.4.2. *Every n-fold suspension is a $\mathscr{C}_n$-coalgebra, and if a pointed space is a $\mathscr{C}_n$-coalgebra then it is homotopy equivalent to an n-fold suspension.*

Remark 1.4.3. Compared to other statements in the literature, see for example [11, 51], Theorem 1.4.1 does not require any additional connectivity assumptions, and it is therefore the sharpest possible result. This follows from the fact that every $\mathscr{C}_n$-coalgebra is $(n-1)$-connected. Indeed, let X be a $\mathscr{C}_n$-coalgebra with structure map $c : X \to C_n(X)$. By the approximation theorem, the space $C_n(X)$ is homotopic to $\Sigma^n\Omega^n X$, and thus $(n-1)$-connected. Since the composition $X \xrightarrow{c} C_n(X) \xrightarrow{\varepsilon_X} X$ is the identity on X by the counit axiom, it follows that X is $(n-1)$-connected.

For readability, we shall give the proof of Theorem 1.4.1 straightaway, making reference to the results and notation of the following two subsections.

Proof. By Theorem 1.3.1, there is a natural morphism of comonads $\alpha_n : \Sigma^n\Omega^n \longrightarrow C_n$, and $\Sigma^n\Omega^n X$ is a retract of $C_n(X)$. Since $\Sigma^n\Omega^n$ preserves equalizers (Proposition 1.4.10), it follows from Lemma 1.4.8 that the counit map $(\alpha_n)_*\alpha_n^!(X) \to X$ is a C_n-algebra morphism which is a retract of pointed spaces. Since $(\alpha_n)_*$ preserves the underlying topological space, it follows that the $\Sigma^n\Omega^n$-coalgebra $\alpha_n^!(X)$ is a retract of X as a pointed space. It then follows from Theorem 1.4.9 together with the approximation theorem that $\alpha_n^!(X)$ is naturally isomorphic to an n-fold suspension, and so the counit map $(\alpha_n)_*\alpha_n^!(X) \to X$ is an C_n-coalgebra map from a n-fold reduced suspension to X. In particular, Γ^n is the functor $P_n(\alpha_n)_*\alpha_n^!$. □

We give a second proof of Theorem 1.4.1 in Section 1.4.3 using explicit formulae very similar to those appearing in the approximation theorem. This alternative proof is more concrete, and has the further benefit of giving a characterization in terms of a certain C_n-subcoalgebra.

1.4.1 The change of coalgebra structures induced by a comonad morphism

In this section, we explain how a morphism of comonads $\alpha : C_1 \to C_2$ induces an adjoint pair

$$\alpha_* : C_1 - \mathsf{Coalg} \leftrightarrows C_2 - \mathsf{Coalg} : \alpha^!$$

between the corresponding categories of coalgebras (under reasonable hypotheses on the underlying ambient category). The final goal is to prove the technical Lemma 1.4.8, which is an essential ingredient for proving Theorem 1.4.1.

Suppose that C_1 and C_2 are two comonads over a category $\mathscr{M}$ which admits finite limits, and that $\alpha : C_1 \to C_2$ is a morphism of comonads. The *change of coalgebra functor*

$$\alpha_* : C_1 - \mathsf{Coalg} \longrightarrow C_2 - \mathsf{Coalg}$$

is given by mapping a C_1-coalgebra X to the same underlying object of $\mathscr{M}$ equipped with the C_2-coalgebra structure map given by the composition

$$X \xrightarrow{\gamma_X} C_1(X) \xrightarrow{\alpha_X} C_2(X).$$

On morphisms, α_* is the identity.

Since $\mathscr{M}$ has finite limits, by the dual of Dubuc's adjoint triangle theorem [25], the change of coalgebra functor α_* has a right adjoint $\alpha^!$ which we call the *enveloping coalgebra functor.* The C_1-coalgebra $\alpha^!(X)$ is explicitly given as the equalizer in $C_1 - \mathsf{Coalg}$ of the following pair of morphisms:

$$\begin{tikzcd}
C_1(X) \arrow[rr, "C_1(\delta_X)"] \arrow[dr, "\triangle_{C_1}"'] & & C_1C_2(X) \\
& C_1C_1(X) \arrow[ur, "C_1(\alpha_X)"'] &
\end{tikzcd}$$

Above, δ_X is the structure map of X as a C_2-coalgebra. The following proposition, which is the dual of [13, Prop. 4.3.2], gives conditions for this equalizer to be preserved by the forgetful functor to $\mathscr{M}$.

Proposition 1.4.4. *Let C be a comonad on $\mathscr{M}$ and let $U : C - \mathsf{Coalg} \to \mathscr{M}$ be the forgetful functor. Let $G : D \to C - \mathsf{Coalg}$ be a diagram such that UG has a limit in $\mathscr{M}$ that is preserved by C and $C \circ C$. Then G has a limit in $C - \mathsf{Coalg}$ that is preserved by U.*

Proof. The proof of this result is dual to that of [13, Prop. 4.3.2], and it is left to the reader. □

We will need the following auxiliary definition.

Definition 1.4.5. A *cosplit equalizer* in a category is a diagram

$$\begin{tikzcd}
A \arrow[r, "p"] & B \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[l, bend left, dashed, "h"] & C \arrow[l, bend left, dashed, "s"]
\end{tikzcd}$$

where

$$sg = \mathrm{id}_B, \quad hp = \mathrm{id}_A \quad \text{and} \quad sf = ph. \tag{1.13}$$

The notion of a cosplit equalizer above is dual to that of split coequalizer, and it plays in comonad theory the analog role of split coequalizers in the theory of monads (see [57, VI. 6]). The following result is elementary but important.

Proposition 1.4.6. *The cosplit equalizer of two morphisms is always an equalizer of the two morphisms; and any functor preserves cosplit equalizers.*

Proof. Assume we have a cosplit equalizer with the notation from Definition 1.4.5. To prove the first assertion, assume that φ is any map such that $f\varphi = g\varphi$. Then,

$$\varphi = hp\varphi = sf\varphi = ph\varphi$$

factors through p. Since $hp = \mathrm{id}_A$, this factorization is unique. The second assertion is a straightforward consequence of the fact that functors preserve the associativity of the composition and the identity on objects. □

Next, we relate cosplit equalizers with coalgebra structures.

Proposition 1.4.7. *Let C be a comonad in an arbitrary category, and let X be a C-coalgebra. Then, the coalgebra structure map $\gamma : X \to C(X)$ fits into a cosplit equalizer diagram*

$$X \xrightarrow{\gamma} C(X) \underset{\Delta_X}{\overset{C(\gamma)}{\rightrightarrows}} CC(X).$$

Proof. Let X be a C-coalgebra with structure map γ. As a consequence of the coassociativity axiom for γ, we have the fork in the statement. By Proposition 1.4.6, we are done as soon as we give cosplitting maps h, s satisfying the identities of Equation (1.13), taking $f = C(\gamma)$ and $g = \Delta_X$. These cosplittings h and s are respectively given by the corresponding counits

$$\varepsilon_X : C(X) \to X \qquad \text{and} \qquad \varepsilon_{C(X)} : CC(X) \to C(X).$$

Let us check that the identities in Equation (1.13) hold. The identity $hp = \mathrm{id}_A$ becomes $\varepsilon_X \circ \gamma = \mathrm{id}_X$, which holds because it is precisely the counital axiom of the C-coalgebra X. Similarly, the identity $sg = \mathrm{id}_B$ becomes $\varepsilon_{C(X)} \circ \Delta_X = \mathrm{id}_{C(X)}$, which is exactly the counit axiom at $C(X)$. It remains to check the identity $sf = ph$, that is, $\varepsilon_{C(X)} \circ C(\gamma) = \gamma \circ \varepsilon_X$. This follows from the fact ε is a natural transformation and so one has the diagram

$$\begin{tikzcd} C(X) \arrow[r, "C(\gamma)"] \arrow[d, "\varepsilon_X"] & CC(X) \arrow[d, "\varepsilon_{C(X)}"] \\ X \arrow[r, "\gamma"] & C(X). \end{tikzcd}$$

We have checked the three identities of Equation (1.13). Therefore, the mentioned diagram is a cosplit equalizer, and the proof is complete. □

Finally, the following technical lemma allows us to directly compare C_1 and C_2-coalgebras in pointed spaces under certain conditions. It constitutes an essential ingredient in the proof of Theorem 1.4.1.

Lemma 1.4.8. *Let $\alpha : C_1 \to C_2$ be a morphism of comonads in Top_* which is a retract of pointed spaces at each level. If C_1 preserves equalizers, then the counit $\alpha_* \alpha^! \to \mathrm{id}_{C_2-\mathsf{Coalg}}$ of the $(\alpha_*, \alpha^!)$ adjunction is a retract of pointed spaces at each level. In particular, for every C_2-coalgebra X, the underlying map of pointed spaces $\alpha_* \alpha^!(X) \to X$ is a retract.*

Proof. Let X be a C_2-coalgebra. Let us prove that the underlying map of pointed spaces of the C_2-coalgebra morphism $\alpha_* \alpha^!(X) \to X$ is a retract. Since α_* is the identity on the underlying pointed space, this underlying map is $\alpha^!(X) \to X$. Recall from Proposition 1.4.7 that the C_2-coalgebra structure γ on X is given by presenting X as the (cosplit) equalizer of the following diagram:

$$\begin{tikzcd} C_2(X) \arrow[r, shift left, "C_2(\gamma)"] \arrow[r, shift right, "\Delta_X"'] & C_2C_2(X). \end{tikzcd}$$

Here, Δ_X is the comultiplication of the C_2 comonad at X. This equalizer is taken in $C_2 - \mathsf{Coalg}$, but we can compute the underlying topological space via the same limit in the category of pointed topological spaces. This is because this limit is a cosplit equalizer, and therefore an equalizer which is preserved by the forgetful functor (see Proposition 1.4.6). Since C_1 is assumed to preserve equalizers, by Proposition 1.4.4, and using a similar argument, the underlying topological space of $\alpha^!(X)$ may be computed as the equalizer of the diagram

$$\begin{tikzcd} C_1(X) \arrow[r, shift left, "C_1(\gamma)"] \arrow[r, shift right, "C_1(\alpha_X)\circ\Delta_{C_1}"'] & C_1C_2(X) \end{tikzcd}$$

in the category of pointed topological spaces. The retract provided by α thus extends to a map (in the category of pointed topological spaces) between the diagram defining $\alpha^!(X)$ and one defining X, namely,

$$\begin{tikzcd}
C_1(X) \arrow[r, shift left, "C_1(\gamma)"] \arrow[r, shift right, "C_1(\alpha_X)\circ\Delta_{C_1}"'] \arrow[d, "\alpha_X"'] & C_1C_2(X) \arrow[d, "\alpha_{C_2(X)}"] \\
C_2(X) \arrow[r, shift left, "C_2(\gamma)"] \arrow[r, shift right, "\Delta_X"'] & C_2C_2(X)
\end{tikzcd}$$

The corresponding map of limits is thus precisely the desired map $\alpha^!(X) \to X$. Since retracts are preserved under limits, we conclude that this map is a retract of pointed spaces. □

1.4.2 The $\Sigma^n\Omega^n$-coalgebras are n-fold reduced suspensions

In this section, we completely characterize the coalgebras over the $\Sigma^n\Omega^n$-comonad (Theorem 1.4.9).

A warning on the notation: in other parts of this paper, we have consistently denoted by Δ and ε the comonadic structure maps of the comonad C_n constructed from the little n-cubes operad; while Δ' and ε' were used for the comonaic structure maps of the comonad $\Sigma^n\Omega^n$. Since there will be only a single comonad appearing in this section, namely $\Sigma^n\Omega^n$, we make an exception here and denote by Δ and ε the comonadic structure maps of $\Sigma^n\Omega^n$ to make the reading easier.

Theorem 1.4.9. *Let X be a $\Sigma^n\Omega^n$-coalgebra. Then X is naturally isomorphic to the n-fold reduced suspension of a space $P_n(X)$ which can be computed as the equalizer of the following pair of maps:*

$$\Omega^n X \underset{\eta_{\Omega^n X}}{\overset{\Omega^n\gamma}{\rightrightarrows}} \Omega^n\Sigma^n\Omega^n X.$$

Here, η is the unit of the (Σ^n,Ω^n) adjunction, and γ is the $\Sigma^n\Omega^n$-coalgebra structure map of X.

Theorem 1.4.9 is essentially a consequence of the fact that reduced suspensions, despite being left adjoint, preserve equalizers. Next, we give a proof of this elementary fact for completeness.

Proposition 1.4.10. *The n-fold reduced suspension functor $\Sigma^n : \mathsf{Top}_* \to \mathsf{Top}_*$ commutes with equalizers. In other words, if $\mathrm{Eq}(f,g) \hookrightarrow X$ is the equalizer of the diagram*

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y,$$

then $\Sigma^n \mathrm{Eq}(f,g) \hookrightarrow \Sigma^n X$ is the equalizer of the diagram

$$\Sigma^n X \underset{\Sigma^n g}{\overset{\Sigma^n f}{\rightrightarrows}} \Sigma^n Y.$$

Since Ω^n is right adjoint and thus preserves limits, it further follows that $\Sigma^n\Omega^n$ preserves equalizers.

Proof. Recall that, as a set, the equalizer of f and g is given by

$$\mathrm{Eq}\left(f,g\right)=\left\{x\in X\mid f(x)=g(x)\right\}.$$

Since we tacitly work in the category CGH of compactly generated Hausdorff spaces, the topology on this set is not necessarily the subspace topology, but might be finer. Explicitly, its topology is given by applying the k-ification functor $k(-)$, see for example [67, Chapter 5]. This functor is the right adjoint of the inclusion of CGH into ordinary topological spaces. This change in the underlying topology is not an issue, because taking n-fold reduced suspension commutes with the k-ification functor. Indeed, if X and Y are any compactly generated Hausdorff spaces and X is locally compact, then $X\times Y$ is a compactly generated Hausdorff space ([84, Thm. 4.3]). Since the sphere S^n is locally compact, the product $S^n\times X$ is compactly generated Hausdorff for any compactly generated Hausdorff space X. Since the smash product $S^n\wedge X$ is the pushout of the inclusion $S^n\vee X\hookrightarrow S^n\times X$ along the collapse map $S^n\vee X\to *$, it follows that $S^n\wedge X=\Sigma^n X$ is compactly generated Hausdorff. Thus,

$$\Sigma^n\,\mathrm{Eq}\left(f,g\right)=S^n\wedge\mathrm{Eq}\left(f,g\right).$$

Points in the suspension above are of the form $[t,x]$, with $t\in S^n$ and $x\in X$ such that $f(x)=g(x)$. On the other hand,

$$\mathrm{Eq}\left(\Sigma^n f,\Sigma^n g\right)=\left\{[t,x]\in\Sigma^n X\mid\left[t,f(x)\right]=\left[t,g(x)\right]\right\}.$$

Under the two identifications above, the natural map

$$\Sigma^n\,\mathrm{Eq}\left(f,g\right)\to\mathrm{Eq}\left(\Sigma^n f,\Sigma^n g\right)$$

is a homeomorphism. □

Recall from Proposition 1.4.7 that every coalgebra structure map is characterized as a cosplit equalizer. In particular, we have the following result.

Proposition 1.4.11. *Let X be a $\Sigma^n\Omega^n$-coalgebra with structure map γ. Then, as a pointed space, X is the (cosplit) equalizer of the following pairs of maps*

$$\Sigma^n\Omega^n X\ \underset{\Delta_X}{\overset{\Sigma^n\Omega^n\gamma}{\rightrightarrows}}\ \Sigma^n\Omega^n\Sigma^n\Omega^n X.$$

Here, Δ is the comonadic comultiplication of $\Sigma^n\Omega^n$.

Proof. As mentioned, this is a particular case of Proposition 1.4.7. The following diagram is a cosplit equalizer:

$$X\xrightarrow{\ \gamma\ }\Sigma^n\Omega^n X\ \underset{\Delta_X}{\overset{\Sigma^n\Omega^n\gamma}{\rightrightarrows}}\ \Sigma^n\Omega^n\Sigma^n\Omega^n X,$$

where the cosplittings h and s are respectively given by the corresponding counits

$$\varepsilon_X:\Sigma^n\Omega^n X\to X\qquad\text{and}\qquad\varepsilon_{\Sigma^n\Omega^n X}:\Sigma^n\Omega^n\Sigma^n\Omega^n X\to\Sigma^n\Omega^n X.$$

□

Let us finally prove the main result of this section.

Proof of Theorem 1.4.9. Use, in the order given, Proposition 1.4.11, that the comonadic coproduct Δ_X is explicitly given by $\Sigma^n\eta_{\Omega^n(X)}$, and Proposition 1.4.10 to obtain that

$$X=\mathrm{Eq}\left(\Sigma^n\Omega^n\gamma,\Delta_X\right)=\mathrm{Eq}\left(\Sigma^n\Omega^n\gamma,\Sigma^n\eta_{\Omega^n X}\right)=\Sigma^n\,\mathrm{Eq}\left(\Omega^n\gamma,\eta_{\Omega^n X}\right).$$

This is exactly what we wished to prove. □

1.4.3 A point-set description of the recognition principle

We give here an alternative proof of the recognition principle mentioned in the introduction to Section 1.4. This proof has the advantage of explicitly characterizing the n-fold suspension onto which a C_n-coalgebra retracts.

Theorem 1.4.12. *Let X be a C_n-coalgebra. Then, there is a pointed space Z together with a homotopy equivalence of C_n-coalgebras $X \simeq \Sigma^n Z$.*

The strategy of the proof is the following. First we show that every C_n-coalgebra X contains a C_n-subcoalgebra $S(X)$ which is also a $\Sigma^n\Omega^n$-coalgebra, and that there is a retract of X onto $S(X)$ (Theorem 1.4.13 and Theorem 1.4.14, respectively). Because of Theorem 1.4.9 this implies that $S(X)$ is an n-fold suspension, proving Theorem 1.4.12.

In Proposition 1.3.3, we saw that $\Sigma^n\Omega^n$-coalgebras considered as C_n-coalgebras have the property that the cubical support at each point is just a single point. In this section, we prove that the converse is also true. That is, every C_n-coalgebra of which the cubical support of every point (other than the base point) is just a single point is not just a C_n-coalgebra, but also a $\Sigma^n\Omega^n$-coalgebra.

It further turns out that the set of points whose cubical support is just a single point forms a C_n-subcoalgebra.

Theorem 1.4.13. *Let X be a C_n-coalgebra with coalgebra structure map $c : X \to C_n(X)$. Then, the subspace*

$$S(X) = \left\{x \in X \mid |\operatorname{CSupp}(c(x))| = 1\right\} \cup \{*\} \subseteq X$$

formed by the points of X whose cubical support is a single point, together with the base point, is such that the following assertions hold.

1. *The inclusion $S(X) \hookrightarrow X$ is a homotopy equivalence of pointed spaces.*
2. *The subspace $S(X)$ is a C_n-subcoalgebra, and the inclusion is a morphism of C_n-coalgebras.*

Therefore, the inclusion $S(X) \hookrightarrow X$ is a homotopy equivalence of C_n-coalgebras.

The result above tells us that any C_n-coalgebra X contains a homotopy equivalent C_n-subcoalgebra $S(X)$ with an extra property. Thus, to prove Theorem 1.4.12, the task has been reduced to showing that $S(X)$ is equivalent to an n-fold suspension as a C_n-coalgebra. It turns out that $S(X)$ is not only equivalent to a suspension, but we can say slightly more. This is the content of the next result.

Theorem 1.4.14. *Let X be a C_n-coalgebra. Then, the C_n-subcoalgebra $S(X)$ of Theorem 1.4.13 is a $\Sigma^n\Omega^n$-coalgebra.*

Since every $\Sigma^n\Omega^n$-coalgebra is an n-fold suspension (Proposition 1.4.10), Theorem 1.4.12 is proven. Thus, it suffices to show the two results mentioned, and we do that next.

Proof of Theorem 1.4.13. Denote by $i : S(X) \hookrightarrow C_n(X)$ the inclusion and by $c : X \to C_n(X)$ the coalgebra structure map.

$\boxed{\text{Item 1.}}$ Let us give a retraction (of spaces) $r : X \to S(X)$, that is, a continuous map r such that $ri = \mathrm{id}_{S(X)}$ and a homotopy $H : X \times I \to X$ between ir and id_X. The map r is given as the composition

$$r : X \hookrightarrow C_n(X) \xrightarrow{\Psi_X} \Sigma^n\Omega^n X \xrightarrow{\alpha_X} C_n(X) \xrightarrow{\varepsilon_X} X.$$

The maps above are, respectively, the coalgebra structure map of X, the natural transformations Ψ and α, and the counit ε from Section 1.3. Since the map Ψ_X reduces the cubical support of every point to a singleton, then the image of this map is exactly the subspace $S(X)$. It further follows that ri is the identity on the subspace $S(X)$ because the map Ψ_X does not change the cubical support of points whose cubical support was just a single point already.

The homotopy $\mathscr{H}$ from Theorem 1.3.1 can also be used to induce a homotopy in this case. In particular we get the following homotopy

$$\mathscr{H}: X \times I \hookrightarrow C_n(X) \times I \xrightarrow{\mathscr{H}_X} \Sigma^n \Omega^n X \xrightarrow{\alpha_X} C_n(X) \xrightarrow{\varepsilon_X} X.$$

It is straightforward to check that by exactly the same arguments as in Theorem 1.3.1 this is indeed a homotopy between ir and id_X. Therefore the inclusion $S(X)$ is a homotopy equivalence of pointed spaces.

$\boxed{\text{Item 2.}}$ To show that $S(X)$ is a C_n-subcoalgebra, we must show it is closed under the coproduct. That is, we must check that if $x \in S(X)$ then the image of the map $c(x): C_n(1) \to X$ is contained in the subspace $S(X) \subseteq X$.

To show that this is indeed the case we make the following observation. If $d, d' \in C_n(1)$ are two cubes such that $d \subset d'$, then $c(x)(d) \neq *$ implies that $c(x)(d') \neq *$. This is because of the coassociativity of the comonad. Since d is the composition of d' with some other little cube e $d = e \circ d'$ for some little cube e we have that $c(x)(d)$ is equal to

$$C_n(1) \xrightarrow{e} C_n(1) \xrightarrow{c} X,$$

evaluated at d'. So $c(x)(d) = c(x)(d' \circ e) = e(c(x))(d')$, where $e(c(x))$ is first the composition of e in the comonad and then acting with this on the coalgebra. It therefore follows that if $d \subset d'$ then if $c(x)(d) \neq *$ then $c(x)(d') \neq *$. From this it is straightforward to deduce that if the cubical support of $c(x)$ is just a single point then the image of $c(x)$ is contained in $S(X)$, otherwise the previous identity would be violated. Therefore, $S(X)$ is a C_n-subcoalgebra and the inclusion map is a homotopy equivalence of C_n-coalgebras. □

Proof of Theorem 1.4.14. To prove Theorem 1.4.14, we need to define a map $c': S(X) \to \Sigma^n \Omega^n S(X)$ and show that it satisfies the comonad identities.

We define $c': S(X) \to \Sigma^n \Omega^n S(X)$ by mapping $c'(x) := [t, \ell]$, where $t = \mathrm{Cent}_{c(x)}$ and $\ell: S^n \to S(X)$ is given by

$$\ell(s) = c(x)\left(c_{s,\mathrm{Cent}_{c(x)}}\right) = c(x)\left(c_{s,t}\right),$$

where $c_{s,\mathrm{Cent}_{c(x)}}$ is the cube from the proof of Theorem 1.3.1. Because c' is a C_n-coalgebra map, it follows that it also satisfies the coassociativity axiom to be a $\Sigma^n \Omega^n$-coalgebra, which completes the proof. □

1.5 Appendix: The map α is a morphism of comonads

In this appendix, we give the necessary definitions and prove in full detail that the natural transformation

$$\alpha_n: \Sigma^n \Omega^n \to C_n$$

appearing in Theorem 1.3.1 defines a morphism of comonads.

Definition 1.5.1. A *morphism of comonads* $\alpha : (C, \Delta, \varepsilon) \to \left(C', \Delta', \varepsilon'\right)$ in a category $\mathscr{M}$ is a natural transformation $\alpha : C \to C'$ such that for every object $X \in \mathscr{M}$, the following two diagrams commute:

$$\begin{tikzcd} C(X) \arrow[rr, "\alpha_X"] \arrow[dr, "\varepsilon_X"'] & & C'(X) \arrow[dl, "\varepsilon'_X"] \\ & X & \end{tikzcd} \qquad \begin{tikzcd} C(X) \arrow[r, "\Delta_X"] \arrow[d, "\alpha_X"'] & C(C(X)) \arrow[d, "\alpha^2_X"] \\ C'(X) \arrow[r, "\Delta'_X"] & C'(C'(X)) \end{tikzcd}$$

$$\varepsilon'_X \circ \alpha_X = \varepsilon_X \qquad\qquad \alpha^2_X \circ \Delta_X = \Delta'_X \circ \alpha_X$$

The morphism α^2_X is defined by the following diagram, which is commutative because α is a morphism of comonads.

$$\begin{tikzcd} C(C(X)) \arrow[r, "\alpha_{C(X)}"] \arrow[d, "C(\alpha_X)"'] \arrow[dr, dashed, "\alpha^2_X"] & C'C(X) \arrow[d, "C'(\alpha_X)"] \\ C(C'(X)) \arrow[r, "\alpha_{C'(X)}"] & C'(C'(X)) \end{tikzcd}$$

$$\alpha^2_X = C'(\alpha_X) \circ \alpha_{C(X)} = \alpha_{C'(X)} \circ C(\alpha_X) \tag{1.14}$$

Next, we settle the morphism of comonads assertion made in Theorem 1.3.1.

Proposition 1.5.2. *The natural transformation $\alpha_n : \Sigma^n\Omega^n \to C_n$ in Theorem 1.3.1 is a morphism of comonads.*

Proof. Fix an integer $n \geq 1$, and denote α_n by α to simplify the notation. Recall that object-wise, the natural transformation α is explicitly given by

$$\alpha_X : \Sigma^n\Omega^n X \xrightarrow{\gamma} C_n\left(\Sigma^n\Omega^n X\right) \xrightarrow{C_n(\eta_X)} C_n(X),$$

where γ is the $\mathscr{C}_n$-coalgebra structure map of $\Sigma^n\Omega^n X$ (Theorem 1.2.22), and η_X is the evaluation at X of the counit $\eta : \Sigma^n\Omega^n \to \mathrm{id}_{\mathsf{Top}_*}$ of the adjunction (Σ^n, Ω^n). Identify

$$\Sigma^n\Omega^n X \cong S^n \wedge \mathsf{Map}_*\left(S^n, X\right).$$

Under this identification, the counit $\eta_X : \Sigma^n\Omega^n X \to X$ becomes the evaluation map,

$$ev : S^n \wedge \mathsf{Map}_*\left(S^n, X\right) \to X \quad ev : [t, \ell] \mapsto \ell(t).$$

Next, identify $C_n(X)$ as a subspace of $\mathsf{Map}(\mathscr{C}_n(1), X)$. Recall that under this identification, the value of $C_n(g)$ on a map $g : \mathscr{C}_n(1) \to X$ is the postcomposition with g (Proposition 1.2.6). Then, the map $\alpha_X : \Sigma^n\Omega^n X \to C_n(X)$ is explicitly given on a point $[t, \ell]$ as the map

$$\alpha_X[t, \ell] : \mathscr{C}_n(1) \to X$$

whose image on a little n-cube $c \in \mathscr{C}_n(1)$ is

$$\alpha[t, \ell](c) = \begin{cases} \ell\left(c^{-1}(t)\right) & \text{if } t \in \mathring{c} \\ * & \text{otherwise} \end{cases} \tag{1.15}$$

Geometrically, α_X is just re-scaling the evaluation map $ev: S^n \wedge \mathsf{Map}_*(S^n, X)$ by shrinking the points of $S^n = I^n/\partial I^n$ according to the little n-cube c.

We can now check the commutativity of the diagrams in Definition 1.5.1.

$\boxed{\varepsilon'_X \circ \alpha_X = \varepsilon_X}$

Let $[t,\ell] \in \Sigma^n\Omega^n X$. Since ε'_X plugs the identity operation $\mathrm{id} \in \mathscr{C}_n(1)$, we have:

$$\begin{array}{rcl} \varepsilon'_X \circ \alpha_X : \Sigma^n\Omega^n X & \xrightarrow{\alpha_X} C_n(X) \xrightarrow{\quad \varepsilon'_X \quad} & X \\ [t,\ell] & \longmapsto \alpha_X[t,\ell] \longmapsto & \alpha_X[t,\ell](\mathrm{id}) = \ell(c(t)) \end{array}$$

The composition above is exactly the definition of $\varepsilon_X[t,\ell]$.

$\boxed{\alpha^2_X \circ \Delta_X = \Delta'_X \circ \alpha_X}$

The map α^2_X can be written as two different compositions, see Diagram (1.14). Here, we prove that

$$\alpha_{C'(X)} \circ C(\alpha_X) \circ \Delta_X = \Delta'_X \circ \alpha_X, \tag{1.16}$$

where $C = \Sigma^n\Omega^n \xrightarrow{\alpha_n} C' = C_n$. The left hand side of Equation (1.16) is the composition

$$\Sigma^n\Omega^n X \xrightarrow{\quad \Delta_X \quad} \Sigma^n\Omega^n(\Sigma^n\Omega^n X) \xrightarrow{\Sigma^n\Omega^n(\alpha_X)} \Sigma^n\Omega^n(C_n(X)) \xrightarrow{\quad \alpha_{C_n(X)} \quad} C_n(C_n(X)).$$

The maps in the composition above are given as follows.

- Denote by $\eta_X : X \to \Omega^n\Sigma^n X$ the unit of the (Σ^n, Ω^n) adjunction. Then $\Delta_X = \Sigma^n \circ \eta_X \circ \Omega^n$. Thus, a point $[t,\ell] \in \Sigma^n\Omega^n X = S^n \wedge \mathsf{Map}_*(S^n, X)$ maps to the point $[t,\bar{\ell}] \in S^n \wedge \mathsf{Map}_*(S^n, \Sigma^n\Omega^n X)$, where
$$\bar{\ell} : S^n \to \Sigma^n\Omega^n X \quad s \mapsto [s,\ell].$$
- The second map $\Sigma^n\Omega^n(\alpha_X)$ maps the point $[t,\bar{\ell}]$ to the point $[t, \alpha_X \circ \bar{\ell}]$.
- The last map takes a point $[t,\ell']$, where $\ell' : S^n \to C_n(X)$ is a loop, to the evaluation
$$\begin{array}{rcl} \alpha_{C_n(X)}[t,\ell'] : \mathscr{C}_n(1) & \longrightarrow & C_n(X) \\ c & \longmapsto & \ell'\left(c^{-1}(t)\right) \end{array}$$

Therefore, with the notation above, the full composition applied to a point $[t,\ell]$ yields

$$[t,\ell] \mapsto [t,\bar{\ell}] \mapsto [t, \alpha_X \circ \bar{\ell}] \mapsto \alpha_{C_n(X)}[t, \alpha \circ \bar{\ell}].$$

The resulting map

$$\alpha_{C_n(X)}[t, \alpha \circ \bar{\ell}] : \mathscr{C}_n(1) \to C_n(X)$$

acts on a little n-cube $c \in \mathscr{C}_n(1)$ by producing

$$c \mapsto \left(\alpha_X \circ \bar{l}\right)\left(c^{-1}(t)\right) = \alpha[c^{-1}(t), \ell] : \mathscr{C}_n(1) \to X,$$

where $c_2 \in \mathscr{C}_n(1)$ gets mapped to

$$\alpha[c^{-1}(t), \ell](c_2) = \ell\left(c_2^{-1}\left(c^{-1}(t)\right)\right).$$

The right hand side of Equation (1.16) is the composition

$$\Sigma^n\Omega^n X \xrightarrow{\alpha_X} C_n(X) \xrightarrow{\Delta'_X} C_n(C_n(X))$$

The first map in the composition above was given in Equation (1.15). The map Δ'_X, described in Proposition 1.2.10, applies an arbitrary map $h : \mathscr{C}_n(1) \to X$ to the map $\bar{h} : \mathscr{C}_n(1) \to C_n(X)$ given by

$$\mu \in \mathscr{C}_n(1) \mapsto \bar{h}(\mu) : \mathscr{C}_n(1) \to X, \quad \bar{h}(\mu)(\theta) := h\big(\gamma(\mu;\theta)\big).$$

In particular, Δ'_X applies the map $\alpha_X[t,\ell]$ to the map

$$\begin{array}{rcl} \Delta'_X(\alpha_X[t,\ell]) : \mathscr{C}_n(1) & \longrightarrow & C_n(X) \\ c & \longmapsto & \Delta'_X(\alpha_X[t,\ell])(c) = \overline{\alpha[t,\ell]}(c) : \mathscr{C}_n(1) \longrightarrow X \\ & & c_2 \longmapsto \ell\big(\gamma(c;c_2)^{-1}(t)\big) \end{array}$$

Since, by definition of the composition in the little cubes operad,

$$\ell\left(c_2^{-1}\left(c^{-1}(t)\right)\right) = \ell\left(\gamma(c;c_2)^{-1}(t)\right)$$

for all little cubes c, c_2, the claim is proven. $\square$

CHAPTER 2

Higher-order Massey products for algebras over algebraic operads

Abstract

We introduce higher-order Massey products for algebras over algebraic operads. This extends the work of Fernando Muro on secondary ones. We study their basic properties and behavior with respect to morphisms of algebras and operads and give some connections to formality. We prove that these higher-order operations represent the differentials in a naturally associated operadic Eilenberg–Moore spectral sequence. We also study the interplay between particular choices of higher-order Massey products and quasi-isomorphic $\mathscr{P}_\infty$-structures on the homology of a $\mathscr{P}$-algebra. We focus on Koszul operads over a characteristic zero field and explain how our results generalize to the non-Koszul case.

2.1 Introduction

In [61], reprinted as [62], W. S. Massey introduced the classical triple Massey product, a secondary operation on the (co)homology of differential graded associative algebras. He used this new operation to show that the Borromean rings are non-trivially linked. Similar secondary operations were defined independently by Allday and Retah on the homology of differential graded Lie algebras, see [2, 3, 78]. The existence of these higher-order products is due to the vanishing of certain equations that follow from the associativity and Jacobi relations at the chain level, respectively. Recently, F. Muro has shown that secondary operations analogous to Massey's in the case of associative algebras on the homology of differential graded algebraic structures are not ad-hoc at all [70]. Indeed, the theory of algebraic operads explains and organizes the existence and construction of these operations. An algebraic operad is an operad in the symmetric monoidal category of $\mathbb{Z}$-graded vector spaces over a characteristic zero field, and will be assumed to be Koszul. In *loc. cit.*, Muro defines secondary Massey products for algebras over algebraic operads. Given an algebraic operad $\mathscr{P}$, each quadratic relation in the presentation of $\mathscr{P}$ defines a secondary Massey-product-like operation on the homology of the $\mathscr{P}$-algebras. This secondary operation takes as many inputs as the arity of the relation. In this way, the associativity relation of the associative operad yields the classical triple Massey products, while the Jacobi identity relation of the Lie operad yields the Lie–Massey brackets. Under this new point of view, Muro uncovered secondary Massey-product-like operations for many distinct types of algebras for the first time, and gave applications to hyper-commutative and Gerstenhaber algebras.

In Muro's paradigm, there is no restriction as to the arity of the relation. Thus, a relation Γ of arity r in a presentation of an operad $\mathscr{P}$ produces a Massey-product-like operation with r inputs $\langle -, \dots, - \rangle_\Gamma$ on the homology of the $\mathscr{P}$-algebras. However, this still left the definition of *higher-order* Massey product operations unclear. This is where our work enters the picture. It is well-known that the triple Massey product is just the first in an infinite series of higher-order operations on the homology of differential graded associative algebras, roughly witnessing the different ways in which an n-fold product in homology vanishes as a consequence of associativity. These higher-order products have been shown many times to be essential in a wide range of topics where triple Massey products are not enough, see for example the survey [54]. In particular, they are concrete tools for computations when a fully-fledged A_∞-structure is not available.

In this work, we introduce and study *higher-order Massey products for algebras over algebraic operads*. These higher operations include Muro's secondary ones, and gather together to form the hierarchy of higher operations on the homology of algebras over algebraic operads mentioned before. Our approach generalizes the fruitful framework of higher-order Massey products for differential graded associative algebras to algebras over any algebraic operad, producing a new tool to perform computations in many kinds of differential graded algebras.

The importance of these higher-order operations seems to have been neglected due to a widespread misconception. This misconception consists of thinking that, whenever a higher-order Massey product set $\langle x_1, \dots, x_r \rangle$ on the homology of a differential graded associative algebra is defined, then any transferred A_∞-structure $\{m_r\}$ on the homology of this differential algebra via the homotopy transfer theorem satisfies

$$\pm m_r(x_1, \dots, x_r) \in \langle x_1, \dots, x_r \rangle. \tag{2.1}$$

This is true only for the triple Massey product, but fails in general [16]. Algebras over operads other than the associative one behave in the same manner (Theorem 2.4.2). This fact makes the higher-order operations defined in this paper important, filling a fundamental gap in the understanding of the homology of differential graded algebraic structures. Being slightly more precise, we show that if the homology of an algebra over a Koszul operad $\mathscr{P}$ is endowed with a $\mathscr{P}_\infty$-algebra structure quasi-isomorphic to the original structure, then the $\mathscr{P}_\infty$-algebra structure maps recover higher-order Massey products only up to lower-arity $\mathscr{P}_\infty$-algebra structure maps. We also prove, however, a positive result in this direction: for any choice of class in a higher-order Massey product set, one can make appropriate choices in the homotopy transfer theorem so that the induced $\mathscr{P}_\infty$ structure on the homology of the $\mathscr{P}$-algebra recovers this choice exactly by Formula (2.1).

Let us briefly explain how these higher-order Massey products arise. Let $\mathscr{P}$ be a Koszul operad with Koszul dual cooperad $\mathscr{P}^{\text{i}}$ (we explain in Remark 2.2.16 how to deal with the non-Koszul case). Each weight-homogeneous cooperation Γ^c of $\mathscr{P}^{\text{i}}$ gives rise to a partially defined higher operation $\langle -, \dots, - \rangle_{\Gamma^c}$ on the homology of any $\mathscr{P}$-algebra. The number of inputs of this operation is the arity r of Γ^c. If A is a $\mathscr{P}$-algebra, then out of homogeneous elements $x_1, \dots, x_r \in H_*(A)$, the operation gives a (possibly empty) set of homology classes

$$\langle x_1, \dots, x_r \rangle_{\Gamma^c} \subseteq H_*(A).$$

The non-emptiness depends on the vanishing, in a precise sense, of higher operations of the same kind that arise from Γ^c and have strictly lower weight-degree. We call $\langle x_1,\dots,x_r\rangle_{\Gamma^c}$ the Γ^c*-Massey product* of the classes $x_1,\dots,x_r$. The process to construct the Γ^c-Massey product operation $\langle -,\dots,-\rangle_{\Gamma^c}$ is done by a non-trivial analogy with the case of differential graded associative algebras. To wit, the cooperation Γ^c determines a set of indices $I(\Gamma^c)$ which is then used to form *defining systems.* Fixed a $\mathscr{P}$-algebra A and homogeneous elements $x_1,\dots,x_r \in H_*(A)$, where r is the arity of Γ^c, a defining system for the Γ^c-Massey product $\langle x_1,\dots,x_r\rangle_{\Gamma^c}$ is a coherent choice of elements $\{a_\alpha\}$ of A indexed by $I(\Gamma^c)$ that conspire together to create a cycle. Running over all possible choices of defining systems for $x_1,\dots,x_r$, we obtain all possible representatives of the homology classes in the set $\langle x_1,\dots,x_r\rangle_{\Gamma^c}$. This construction is the core of the paper, and it is performed in Section 2.2. Since the details are quite technical, we skip them for the moment and refer the reader to the mentioned section. There, we give explicit examples, including the case of the associative, commutative, Lie, and dual numbers operads. We prove that our framework generalizes Muro's in Proposition 2.2.9. In Section 2.2.1, we study the basic properties enjoyed by these new operations. For example, we prove that morphisms of $\mathscr{P}$-algebras preserve higher-order Massey products, and that quasi-isomorphisms induce a bijective correspondence between them. This makes the higher-order Massey products a useful tool in the study of homotopy types of algebras over operads; in particular, they can be used to study formality-type results. In Section 2.2.2, we explain how higher-order Massey products behave with respect to morphisms of operads. Under mild assumptions, higher-order Massey products can be pulled back and forward along morphisms of operads. This allows one to relate the formality (or more generally, the quasi-isomorphism class) of an algebra of a certain type to the formality (or quasi-isomorphism class) of a functorially associated algebra of a distinct type. The reader can have in mind the adjoint pair between taking the universal enveloping differential graded associative algeba of a differential graded Lie algebra, and forming the commutator bracket of a dg associative algebra. Under some hypotheses, one can relate formality and quasi-isomorphism classes in both directions.

We prove some further results related to higher-order Massey products. It is a well-known and celebrated result that higher-order Massey products for associative algebras provide a concrete description of the differentials in the Eilenberg–Moore spectral sequence. In Section 2.1.1.1, we explain how to construct an Eilenberg–Moore-type spectral sequence for any algebra over an algebraic operad. Under mild hypotheses, this spectral sequence computes the Quillen homology of the algebras over this operad. The spectral sequence is then exploited in Section 2.3. Our main result in this direction is Theorem 2.3.2, which proves that the higher-order Massey products defined in this paper provide concrete representatives for the differentials in this Eilenberg–Moore-type spectral sequence. To finish the paper, we give in Section 2.4 a precise relationship between the higher-order Massey products on the homology of a $\mathscr{P}$-algebra, and transferred $\mathscr{P}_\infty$-structures on it.

Notation and conventions

In this paper, all algebraic structures are taken over a base field $\Bbbk$ of characteristic zero. We work on the category of unbounded chain complexes over $\Bbbk$ with homological convention. That is, the differential $d: A_* \to A_{*-1}$ of a chain complex (A,d) is of degree -1. The degree of a homogeneous element x is denoted by $|x|$. The suspension of a chain complex (A,d_A) is the chain complex $(sA,d_{sA}) = (\Bbbk s\otimes A, 1\otimes d)$, where s is a formal variable of degree 1. For a homogeneous element $a\in A$, we denote $sa = s\otimes a \in sA$. Thus, $(sA)_* \cong A_{*-1}$, and

$d_{sA}(sa) = -sd_A(a)$ for every such $a \in A$. The symmetric group on n elements is denoted $\mathbb{S}_n$. The operads in this paper are taken in the symmetric monoidal category of $\mathbb{Z}$-graded vector spaces, and therefore have zero differential. In this monoidal category, we follow the Koszul sign rule. That is, the symmetry isomorphism $U \otimes V \xrightarrow{\cong} V \otimes U$ that identifies two graded vector spaces is given on homogeneous elements by $u \otimes v \mapsto (-1)^{|u||v|} v \otimes u$. Algebras over operads are always differential graded (dg) and homological. We will frequently omit the adjective "dg" and assume it is implicitly understood. The reason for choosing the operads to have trivial differential is that in this case, the homology of any dg $\mathscr{P}$-algebra is a graded (non-dg) $\mathscr{P}$-algebra again. If $f : A \to B$ is a morphism of differential graded algebras over an operad, then we denote by $f_* : H_*(A) \to H_*(B)$ the induced map in homology.

2.1.1 Preliminaries

In this section, we collect some of the prerequisites for understanding this paper. We start in Section 2.1.1.1 by giving a brief recollection of the results of operad theory that we will make use of, mainly to establish our notation. We borrow most of the notation from [56], which is an excellent reference for algebraic operads. A non-standard topic explained in this section is the construction of the Eilenberg–Moore-type spectral sequence mentioned in the introduction. In Section 2.1.1.2, we recall the higher-order Massey products for differential graded associative algebras. To finish, we briefly summarize in Section 2.1.1.3 the construction of the secondary Massey products for algebras over algebraic operads as defined by Muro in [70].

2.1.1.1 Operadic background

In this paper, we work with operads in the symmetric monoidal category of graded vector spaces. Our generic operad $\mathscr{P}$ is therefore arity-wise made up of $\mathbb{Z}$-graded vector spaces, but it has no differential. That is, we work with non-dg operads. The reason is that if A is a $\mathscr{P}$-algebra, we will need A and its homology $H_*(A)$ to be algebras over the same operad. Our operads will always satisfy $\mathscr{P}(0) = 0$, except for theorems 2.1.2 and 2.4.2, where they need to be reduced. Recall that an operad $\mathscr{P}$ is *reduced* if $\mathscr{P}(0) = 0$ and $\mathscr{P}(1) = \mathbb{k}$.

This paper will assume familiarity with the results and notation from [56], and we will adopt its notation for most of the objects used in this paper (infinitesimal compositions, twisting morphisms, weight gradings, and Koszul duality). We shall briefly sketch only those results that will be essential to understand this paper.

Quadratic and Koszul operads. A symmetric sequence E is *reduced* if $E(0) = E(1) = 0$. An operad $\mathscr{P}$ is *quadratic* if it is given by a presentation $\mathscr{F}(E, R)$, that is, if it is given as the quotient $\mathscr{F}(E)/(R)$ of the free operad $\mathscr{F}(E)$ on the reduced symmetric sequence E by the operadic ideal of relations generated by a sub $\mathbb{S}$-module of relations $R \subseteq \mathscr{F}(E)^{(2)}$. Here, $\mathscr{F}(E)^{(n)}$ is the sub $\mathbb{S}$-module of $\mathscr{F}(E)$ formed by elements of *weight* n, that is, formed by combining exactly n generating operations from E. The free operad $\mathscr{F}(E)$ comes equipped with a *weight grading* concentrated in non-negative degrees. Since the operadic ideal (R) is homogeneous with respect to the weight grading of $\mathscr{F}(E)$ and $\mathscr{P}$ is a quotient of $\mathscr{F}(E)$, the weight grading of $\mathscr{F}(E)$ naturally descends to $\mathscr{P}$. The degree n component of this weight grading on $\mathscr{P}$ will be denoted $\mathscr{P}^{(n)}$. Similarly, one can construct the cofree conilpotent cooperad $\mathscr{F}^c(E)$. To do so, consider the same underlying symmetric sequence $\mathscr{F}(E)$, endowed with the same weight-grading. Dually, we can consider the conilpotent sub-cooperad $\mathscr{F}^c(E,R)$ of $\mathscr{F}^c(E)$ which is final among the conilpotent sub-cooperads $\mathscr{C}$ of $\mathscr{F}^c(E)$ equipped with a morphism of $\mathbb{S}$-modules $\mathscr{C} \to E$ such that the composite

$$\mathscr{C} \hookrightarrow \mathscr{F}^c(E) \twoheadrightarrow \mathscr{F}^c(E)^{(2)}/R$$

is 0. The weight grading of $\mathscr{F}^c(E)$ restricts to the sub-cooperad $\mathscr{F}^c(E,R)$, and the degree n component of this weight grading on $\mathscr{F}^c(E,R)$ will be denoted $\mathscr{F}^c(E,R)^{(n)}$. In particular, and this will be important later, the weight 2 component of $\mathscr{F}^c(E,R)$ is precisely the submodule of co-relations R,

$$\mathscr{F}^c(E,R)^{(2)} = R.$$

We call $\mathscr{F}^c(E,R)$ the cofree conilpotent cooperad cogenerated by E with corelations R. A cooperad $\mathscr{C}$ is *quadratic* if it is given by a presentation $\mathscr{F}^c(E,R)$ as above, that is, if is is given as the subcooperad of $\mathscr{F}^c(E)$ just described. Let $\mathscr{P} = \mathscr{F}(E,R)$ be a quadratic operad. Its *Koszul dual cooperad* is defined as

$$\mathscr{P}^{\text{i}} = \mathscr{F}^c\left(sE, s^2R\right).$$

The *canonical twisting morphism* is the degree -1 morphism of $\mathbb{S}$-modules $\kappa : \mathscr{P}^{\text{i}} \to \mathscr{P}$ given by the composite

$$\kappa : \mathscr{F}^c\left(sE, s^2R\right) \twoheadrightarrow sE \xrightarrow{s^{-1}} E \to \mathscr{F}(E,R).$$

If $\mathscr{P}$ is augmented, then we can functorially associate to it a quasi-free differential graded conilpotent cooperad $B\mathscr{P}$, called the *bar construction* of $\mathscr{P}$. If $\mathscr{P}$ is quadratic, then it is naturally augmented, and the Koszul dual cooperad $\mathscr{P}^{\text{i}}$ is a subcooperad of $B\mathscr{P}$ with trivial differential. The operad $\mathscr{P}$ is *Koszul* if the inclusion $\mathscr{P}^{\text{i}} \hookrightarrow B\mathscr{P}$ is a quasi-isomorphism. The cooperad $B\mathscr{P}$, being differential graded, has a homology cooperad $H_*(B\mathscr{P})$. This homology admits an extra cohomological degree called the syzygy degree. It can be seen that $\mathscr{P}$ is Koszul if, and only if, $H^0(B\mathscr{P}) \cong \mathscr{P}^{\text{i}}$. The assignment of a Koszul dual cooperad is functorial on weighted operads as long as the morphisms of operads preserve the weight.

$\mathscr{P}_\infty$-structures and Quillen homology. In this section, we discuss several ways to present a $\mathscr{P}_\infty$-structure on a chain complex A for a given Koszul operad $\mathscr{P}$, and define the Quillen homology of a $\mathscr{P}$-algebra. A convenient choice of model for $\mathscr{P}_\infty$ is the cobar construction $\Omega\mathscr{P}^{\text{i}}$, where $\mathscr{P}^{\text{i}}$ is the Koszul dual cooperad of $\mathscr{P}$. Recall that the cobar construction is the right adjoint of the bar construction B, mapping onto the category of augmented differential graded operads. A $\mathscr{P}_\infty$ structure on A is therefore a morphism of differential graded operads $\Omega\mathscr{P}^{\text{i}} \to \mathrm{End}_A$, where End_A is the endomorphism operad of A. Under this point of view, we can think of a $\mathscr{P}_\infty$-algebra structure on A as a family of operations $\{A^{\otimes n} \to A\}$ parametrized by the operad $\Omega\mathscr{P}^{\text{i}}$.

By the *Rosetta Stone Theorem* [56, Theorem 10.1.13], an equivalent approach, and the one which we shall use in the rest of this document, is to define a *$\mathscr{P}_\infty$-algebra* to be a chain complex A along with a degree -1 square zero coderivation

$$\delta : \mathscr{P}^{\text{i}}(A) \to \mathscr{P}^{\text{i}}(A).$$

Briefly recall that if A is a $\mathscr{P}$-algebra, then $\mathscr{P}^{\text{i}}(A)$ is a quasi-free $\mathscr{P}^{\text{i}}$-coalgebra whose coderivation codifies the internal differential of A as well as its $\mathscr{P}$-algebra structure. The coderivation is meant as a $\mathscr{P}^{\text{i}}$-coalgebra, and we explain next how to understand this. Since it squares to zero, we might call it the codifferential of $\mathscr{P}^{\text{i}}(A)$. It will often be convenient to present δ in two different ways. Firstly, as a collection of linear maps $\delta_r : \mathscr{P}^{\text{!`}}(r) \otimes A^{\otimes r} \to A$, for $r \geq 1$, where each δ_r is the composition

$$\mathscr{P}^{\text{!`}}(r) \otimes A^{\otimes r} \hookrightarrow \bigoplus_{k\geq 1} \mathscr{P}^{\text{!`}}(k) \otimes A^{\otimes k} = \mathscr{P}^{\text{!`}}(A) \xrightarrow{\delta} \mathscr{P}^{\text{!`}}(A) \xrightarrow{\epsilon_A} A.$$

Here, ϵ is the counit of the $\mathscr{P}^{\text{!`}}$ comonad. The coderivation δ can be reconstructed from the family $\{\delta_r\}_{r\geq 1}$ as the map

$$\mathscr{P}^{\text{!`}}(A) \xrightarrow{\triangle_{(1)}} \left(\mathscr{P}^{\text{!`}} \circ_{(1)} \mathscr{P}^{\text{!`}}\right)(A) = \mathscr{P}^{\text{!`}} \circ \left(A; \mathscr{P}^{\text{!`}}(A)\right) \xrightarrow{\mathrm{id}\circ(\mathrm{id};m)} \mathscr{P}^{\text{!`}} \circ (A; A) \to \mathscr{P}^{\text{!`}}(A).$$

Here, $\triangle_{(1)}$ is the infinitesimal decomposition coproduct of $\mathscr{P}^{\text{!`}}$, see [56, §6.1.4], and $m$ is the map $(\delta_r)_{r\geq 1} : \bigoplus_{r\geq 1} \mathscr{P}^{\text{!`}}(r) \otimes A^{\otimes r} \to A$ induced by the universal property of the coproduct of the underlying graded vector spaces. Secondly, we can present δ as a collection of degree $n-2$ linear maps $\delta^{(n)} : \mathscr{P}^{\text{!`}}(A)^{(n)} \to A$, for $n \geq 1$, where each $\delta^{(n)}$ is the composition

$$\mathscr{P}^{\text{!`}}(A)^{(n)} \hookrightarrow \mathscr{P}^{\text{!`}}(A) \xrightarrow{\delta} \mathscr{P}^{\text{!`}}(A) \xrightarrow{\epsilon_A} A,$$

and where $\mathscr{P}^{\text{!`}}(A)^{(n)}$ consists of the weight $n$ part of $\mathscr{P}^{\text{!`}}(A)$,

$$\mathscr{P}^{\text{!`}}(A)^{(n)} = \bigoplus_{r\geq n} \left(\mathscr{P}^{(n)}(r) \otimes_{S_r} A^{\otimes r}\right).$$

To reconstruct δ from the family $\left\{\delta^{(n)}\right\}_{n\geq 1}$, one proceeds *mutatis mutandis* as in the case of $\{\delta_r\}_{r\geq 1}$.

The object $(\mathscr{P}^{\text{!`}}(A), \delta)$ is called the *operadic chain complex*. The *Quillen homology* of a $\mathscr{P}$-algebra $A$ is the homology $H_*\left(\mathscr{P}^{\text{!`}}(A), \delta\right)$ of this operadic chain complex. It forms a (non-differential) graded $\mathscr{P}^{\text{!`}}$-coalgebra.

A $\mathscr{P}_\infty$-algebra A is a *strict* $\mathscr{P}$-algebra if the map m factors through the canonical twisting morphism $\kappa : \mathscr{P}^{\text{!`}} \to \mathscr{P}$. Conversely, any $\mathscr{P}$-algebra $A$ can be seen as a $\mathscr{P}_\infty$-algebra by pulling back its algebra structure along the morphism of operads $\Omega\mathscr{P}^{\text{!`}} \to \mathscr{P}$.

A $\mathscr{P}_\infty$-*morphism* is a map of (dg) $\mathscr{P}^{\text{!`}}$-coalgebras $F : \left(\mathscr{P}^{\text{!`}}(A), \delta\right) \to \left(\mathscr{P}^{\text{!`}}(B), \delta'\right)$. As in the case of a codifferential on a $\mathscr{P}^{\text{!`}}$-coalgebra, it will often be convenient to present F as a collection of linear maps $F_n : \mathscr{P}^{\text{!`}}(n) \otimes A^{\otimes n} \to B$, for $n \geq 1$, where each F_n is the composition

$$\mathscr{P}^{\text{!`}}(n) \otimes A^{\otimes n} \hookrightarrow \bigoplus_{k\geq 1} \mathscr{P}^{\text{!`}}(k) \otimes A^{\otimes k} = \mathscr{P}^{\text{!`}}(A) \xrightarrow{F} \mathscr{P}^{\text{!`}}(B) \xrightarrow{\epsilon_B} B.$$

The map F can be reconstructed from the family $\{F_n\}_{n\geq 1}$ as the map

$$\mathscr{P}^{\text{!`}}(A) \xrightarrow{\triangle} \mathscr{P}^{\text{!`}} \circ \mathscr{P}^{\text{!`}}(A) \xrightarrow{\mathscr{P}^{\text{!`}}(f)} \mathscr{P}^{\text{!`}}(B),$$

where f is the map $(F_i)_{i\geq 1} : \bigoplus_{i\geq 1} \mathscr{P}^{\text{!`}}(n) \otimes A^{\otimes n} \to B$ induced by the universal property of the coproduct. Similarly, we can decompose by weight instead of arity to produce a collection of degree $n-1$ linear maps $F^{(n)} : \mathscr{P}^{\text{!`}}(A)^{(n)} \to B$.

The $\mathscr{P}$-Eilenberg–Moore spectral sequence. Let A be an algebra over a Koszul operad $\mathscr{P}$ and $H = H_*(A)$ be its homology. There is a spectral sequence, which we call the *$\mathscr{P}$-Eilenberg–Moore spectral sequence*, that computes the Quillen homology of A as long as A is positively graded of finite type (which is implicitly assumed whenever we speak of convergence). It is constructed as follows. The operadic chain complex $\mathscr{P}^{\text{i}}(A)$ admits the ascending filtration

$$F_p\mathscr{P}^{\text{i}}(A) = \bigoplus_{n=1}^{p} \mathscr{P}^{\text{i}}(A)^{(n)}.$$

This filtration is bounded below and exhaustive. Therefore, the associated spectral sequence converges to the operadic homology of A as a graded module. The complex $\mathscr{P}^{\text{i}}(A)$ also has the structure of a conilpotent cofree $\mathscr{P}^{\text{i}}$-coalgebra with comultiplication Δ, which respects the filtration in the sense that

$$\Delta\left(F_p\mathscr{P}^{\text{i}}(A)\right) \subseteq \bigoplus_{k=1}^{p} \bigoplus_{i_1+\cdots+i_k=p} \mathscr{P}^{\text{i}}(k) \otimes \left(F_{i_1}\mathscr{P}^{\text{i}}(A) \otimes \cdots \otimes F_{i_k}\mathscr{P}^{\text{i}}(A)\right).$$

This further implies that each page of the spectral sequence inherits a $\mathscr{P}^{\text{i}}$-coalgebra structure, and furthermore, the spectral sequence converges as a $\mathscr{P}^{\text{i}}$-coalgebra. A morphism of $\mathscr{P}^{\text{i}}$-coalgebras naturally induces a morphism of the corresponding spectral sequences. The E^0-page of this spectral sequence is explicitly given by

$$E^0_{p,q} = \left(\mathscr{P}^{\text{i}}(A)^{(p)}\right)_{p+q} \cong \left(\bigoplus_{r\geq 1} \left(\mathscr{P}^{\text{i}}\right)^{(p)}(r) \otimes_{\mathbb{S}_r} A^{\otimes r}\right)_{p+q}$$

where the $p+q$ grading is induced from the internal grading of A. Under the isomorphism above, the differential d^0 is determined by the differential d of A, and there is an isomorphism of differential bigraded modules

$$\left(E^0, d^0\right) \cong \left(\mathscr{P}^{\text{i}}(A), \delta^{(1)}\right),$$

where abusing the notation, $\delta^{(1)}$ stands for the coderivation of $\mathscr{P}^{\text{i}}(A)$ induced by the weight 1 component of the codifferential δ. Taking homology of $\left(E^0, d^0\right)$, it follows that the E^1-page of the spectral sequence is

$$E^1_{p,q} = \left(\mathscr{P}^{\text{i}}\left(H_*(A)\right)^{(p)}\right)_{p+q}$$

and the differential on this page is therefore entirely determined by the weight 2 component of the codifferential. In other words, we have that $d^1 = H_*\left(\delta^{(2)}\right)$. Taking homology again, we finally have

$$E^2_{p,q} = H_{p+q}\left(\mathscr{P}^{\text{i}}(H)^{(p)}\right) \xRightarrow{\ p\ } H_*\left(\mathscr{P}^{\text{i}}(A), \delta\right).$$

While this definition seems to be original to this paper for general operads, it has some very well-known special cases. When $\mathscr{P}$ is binary, that is, generated by operations of arity 2, the weight grading coincides with the arity grading up to a shift. So, for example, when $\mathscr{P} = \mathsf{Ass}$ is the associative operad, the $\mathscr{P}$-Eilenberg–Moore spectral sequence is exactly the classical Eilenberg–Moore spectral sequence [26]. When $\mathscr{P} = \mathsf{Lie}$ is the Lie operad, the $\mathscr{P}$-Eilenberg–Moore spectral sequence is exactly a classical Quillen spectral sequence that appears in [76, (6.9) p. 262].

Remarks 2.1.1.

1. If A is an algebra over a Koszul operad $\mathscr{P}$, there are several spectral sequences closely related to the one defined above. First, we can filter $\mathscr{P}^{\text{i}}$ by weight. This gives the spectral sequence we studied above. Second, we can filter $\mathscr{P}^{\text{i}}$ by arity. This produces a spectral sequence that coincides with the previous one up to a shift when the operad is binary generated, or more generally, when the generators of the operad are

concentrated in a single arity. However, in general, these two spectral sequences differ. Third, one can replace $\mathscr{P}^{¡}$ with the bar construction $B\mathscr{P}$ and filter similarly. Since not every operad is Koszul, this spectral sequence will be useful in those situations.

2. If A is a $\mathscr{P}_\infty$-algebra, then the construction of the spectral sequence above goes through with straightforward adjustments.

A version of the homotopy transfer theorem. In [73, Theorem 2], D. Petersen gave what probably is the most general form of T. Kadeishvili's version of the classical homotopy transfer theorem [50] for algebras over binary algebraic operads. Adapted to our needs, it reads as follows. In the statement, $\mathscr{P}$ is a reduced Koszul operad.

Theorem 2.1.2. *Let (A,d) be a $\mathscr{P}$-algebra, H its homology, and $f: H \to A$ a cycle-choosing (and therefore necessarily degree 0) linear map. Let δ_A be the degree -1 square-zero coderivation of $\mathscr{P}^{¡}(A)$ representing the $\mathscr{P}$-algebra structure on A whose arity 1 term equals the given differential d. Then there exists noncanonically a square-zero degree -1 coderivation δ of $\mathscr{P}^{¡}(H)$ whose arity 1 term vanishes, and a morphism of $\mathscr{P}^{¡}$-coalgebras $F: \mathscr{P}^{¡}(H) \to \mathscr{P}^{¡}(A)$ whose linear term F_1 is f and which is a chain map with respect to the differentials defined by δ_A and δ.*

Sketch of the proof. The homology H is equipped with the structure of a $\mathscr{P}$-algebra descending from the $\mathscr{P}$-algebra structure on A. This induces a degree -1 coderivation $\delta^1 : \mathscr{P}^{¡}(H) \to \mathscr{P}^{¡}(H)$ whose arity 1 component δ^1_1 is identically 0. Now, by induction, assume that for some $n \geq 2$, we have a degree -1 coderivation $\delta^{n-1} : \mathscr{P}^{¡}(H) \to \mathscr{P}^{¡}(H)$ and a $\mathscr{P}^{¡}$-coalgebra morphism $F^{n-1} : \mathscr{P}^{¡}(H) \to \mathscr{P}^{¡}(A)$ with $F_1 = f$, such that the restrictions of δ^{n-1} and F^{n-1} to $F_{n-1}\mathscr{P}^{¡}(H)$ satisfy

$$\begin{cases} \delta^{n-1} \circ \delta^{n-1} = 0 \\ F^{n-1} \circ \delta^{n-1} - \delta_A \circ F^{n-1} = 0. \end{cases}$$

Above, $\circ$ denotes the usual composition of maps, not the operadic circle product. Write F^1 for the coalgebra map determined by f in arity 1 and vanishing in higher arities. Then δ^1 and F^1 satisfy the identities above, providing the base case in the induction. The idea now is to modify only the arity n terms of δ^{n-1} and F^{n-1} to produce new δ^n and F^n such that the equations above are satisfied on $F_n\mathscr{P}^{¡}(A)$. One can show that there are e and e' such that

$$\left(F^{n-1} \circ \delta^{n-1} - \delta_A \circ F^{n-1}\right)_n = f \circ e + de'$$

where $e \in \mathrm{Hom}(\mathscr{P}^{¡}(n) \otimes H^{\otimes n}, H)$ and $e' \in \mathrm{Hom}(\mathscr{P}^{¡}(n) \otimes H^{\otimes n}, A)$. Therefore, we can define

$$\delta^n_i = \begin{cases} \delta^{n-1}_i & \text{for } i \neq n. \\ \delta^{n-1}_n - e & \text{for } i = n. \end{cases}$$

In fact, e may be computed as the projection of $\left(F^{n-1} \circ \delta^{n-1} - \delta_A \circ F^{n-1}\right)_n$ onto H. Similarly, we can define F^n_n to be

$$F^n_i = \begin{cases} F^{n-1}_i & \text{if } i \neq n. \\ F^n_n & \text{for any } F^n_n \text{ such that } dF^n_n = F^{n-1}_n - e' \text{ when } i = n. \end{cases}$$

So defined, the coderivation δ^n and the coalgebra map F^n satisfy the required conditions, and the proof is complete. $\square$

2.1.1.2 Higher-order Massey products for associative algebras

The triple Massey product for differential graded associative algebras was introduced in the fifties, see [88] and [61] (reprinted as [62]). Massey himself soon realized that the triple product could be extended to n-fold Massey products [60], see also [64]. Our generalization of the higher-order Massey products to algebras over algebraic operads has its roots in this definition. Therefore, we find it convenient to devote this section to recall the higher-order Massey products for differential graded associative algebras. Excellent references for this topic include [52, 64, 77].

Let (A,d) be a differential graded associative algebra, and $x_1, x_2 \in H_*(A)$ homogeneous elements. The Massey product $\langle x_1, x_2\rangle$ is defined as the singleton $\{x_1x_2\}$ formed by the product of the two classes in $H_*(A)$. It is also possible to identify the set $\{x_1x_2\}$ with the product x_1x_2 itself and define the Massey product of two homogeneous elements in homology as their ordinary product. Let us define next the triple and higher-order Massey products. First, we introduce the auxiliary notion of a defining system. A defining system in the case of the Massey product of two homology classes $\langle x_1, x_2\rangle$ is just a choice $\{b_1, b_2\}$ of cycle representatives of x_1 and x_2.

Definition 2.1.3. Let (A,d) be a differential graded associative algebra, and $x_1,\ldots,x_n$ be $n \geq 3$ homogeneous elements in $H_*(A)$. A *defining system for the n^{th}-order Massey product of the classes* $x_1,\ldots,x_n$ is a set of homogeneous elements

$$\{b_{ij}\} \subseteq A, \quad \text{for} \quad 0 \leq i < j \leq n \quad \text{and} \quad 1 \leq j-i \leq n-1,$$

defined as follows.

- (Initial step) For $i = 1,\ldots,n$ the element $b_{i-1,i}$ is a cycle representative of x_i.
- (Inductive relation) For each $0 \leq i < j \leq n$ and $1 \leq j-i \leq n-1$, the element $b_{ij} \in A$ satisfies
$$d\left(b_{ij}\right) = \sum_{0\leq i<k<j\leq n} (-1)^{|b_{ik}|+1} b_{ik}b_{kj}. \tag{2.2}$$

The *n^{th}-order Massey product of the classes* $x_1,\ldots,x_n$ is the set

$$\langle x_1,\ldots,x_n\rangle = \left\{ \left[\sum_{0\leq i<k<j\leq n} (-1)^{|b_{ik}|+1} b_{ik}b_{kn} \right] \;\middle|\; \{b_{ij}\} \text{ is a defining system} \right\} \subseteq H_{s+2+n}(A),$$

where $s = \sum_{i=1}^n |x_i|$, and the bracket $[-]$ denotes taking homology class.

The elements b_{ij} of Equation (2.2) might not exist at all, in which case the Massey product set is empty. The necessary and sufficient condition for $\langle x_1,\ldots,x_n\rangle$ to be non-empty is that for all $1 \leq i < j \leq n$ and $1 \leq j-i \leq n-2$, the Massey product sets $\langle x_i,\ldots,x_j\rangle$ are non-empty and furthermore contain the zero class.

The fact that for a fixed defining system the sum

$$\sum_{0\leq i<k<j\leq n} (-1)^{|b_{ik}|+1} b_{ik}b_{kn}$$

defines a cycle is a straightforward check by applying d and using the inductive relations. If there are no defining systems for the classes $x_1,\ldots,x_n$, their Massey product $\langle x_1,\ldots,x_n\rangle$ is defined as the empty set, or it is said to be undefined.

A similar definition for higher Lie–Massey brackets on the homology of a differential graded Lie algebra exists, see [2, 3, 15, 78, 87]. The main purpose of this paper is to provide a suitable generalization of Definition 2.1.3 to algebras over Koszul operads, see Section 2.2.

2.1.1.3 Secondary Massey products for algebras over algebraic operads

In this section, we briefly outline Muro's definition of secondary Massey products for algebras over algebraic operads. Our eventual definition of Massey products for algebras over operads, Def. 2.2.7, is shown to extend the one below in Proposition 2.2.9.

Definition 2.1.4. ([70, Def. 2.1]) Let $\mathscr{P} = \mathscr{F}(E,R)$ be a Koszul operad generated by the reduced symmetric sequence E with quadratic relations $R \subseteq \mathscr{F}(E)^{(2)}$. Fix

$$\Gamma = \sum \left(\mu^{(1)} \circ_k \mu^{(2)}\right) \cdot \sigma$$

a relation of arity r of R. Here, $\mu^{(i)} \in E(r_i)$, with $r_1 + r_2 = r+1$, the symbol $\circ_k$ denotes the k-th partial composition product, $1 \le k \le r_1$, and $\sigma \in \mathbb{S}_r$. Let A be a $\mathscr{P}$-algebra and let $x_1, ..., x_r \in H_*(A)$ be homogeneous elements such that

$$\mu^{(2)}\left(x_{\sigma^{-1}(k)}, \dots, x_{\sigma^{-1}(k+r_2-1)}\right) = 0 \tag{2.3}$$

in $H_*(A)$ for each term in the relation. For each $1 \le i \le r$, fix $y_i \in A$ a cycle representative of x_i and, for each summand in the relation, let $\rho^{(2)} \in A$ be an element such that

$$d\rho^{(2)} = \mu^{(2)}\left(y_{\sigma^{-1}(k)}, \dots, y_{\sigma^{-1}(k+r_2-1)}\right) \tag{2.4}$$

in A. Such an element exists by Equation (2.3). The Γ-*Massey product* set $\langle x_1, \dots, x_r\rangle_\Gamma$ is the set of homology classes represented by cycles of the form

$$\sum (-1)^\gamma \mu^{(1)}\left(y_{\sigma^{-1}(1)}, \dots, y_{\sigma^{-1}(k-1)}, \rho^{(2)}, y_{\sigma^{-1}(k+r_2)}, \dots, y_{\sigma^{-1}(r)}\right),$$

where

$$\gamma = \alpha + |\mu^{(1)}| + \left(|\mu^{(2)}| - 1\right) \sum_{m=1}^{k-1} |x_{\sigma^{-1}(m)}|, \qquad \alpha = \sum_{\substack{i<j \\ \sigma(i)>\sigma(j)}} |x_i||x_j|.$$

for all possible coherent choices of elements $\rho^{(2)}$.

Muro shows that the definition above recovers the usual triple Massey products for differential graded associative algebras when Γ is the associativity relation of the associative operad, and the triple Lie–Massey brackets for differential graded Lie algebras when Γ is the Jacobi relation of the Lie operad.

The perspective we take to construct higher-order Massey products for algebras over algebraic operads differs significantly from the construction of Muro just explained. Muro uses the form of relations defined using partial composition. The definition does not depend exclusively on the relation Γ, but also on a specific choice of expansion of Γ in terms of the partial compositions. This choice is not unique. Our approach is also affected by a choice in the explicit form of the higher relations. To generalize, we prefer to see such relations as the weight 2 cooperations in the Koszul dual cooperad of $\mathscr{P}$, and work with defining systems in a similar way as in Definition 2.1.3. This makes our formulas easier to write in the usual language of algebraic operads and Koszul duality theory. To take into account the dependency of the higher relations on a presentation, we will assume all through that a $\mathbb{k}$-linear basis of the symmetric sequence E has been fixed, and then there is an induced basis on $\mathscr{F}^c(sE)$ given by symmetric tree monomials. This will be recalled in the corresponding section. We show in Proposition 2.2.9 that the secondary case of our definition coincides with Muro's definition.

2.2 Higher-order operadic Massey products

In this section, we define higher-order Massey products for algebras over algebraic operads. We focus on the case of Koszul operads and explain in Remark 2.2.16 how to deal with the non-Koszul case. We recommend familiarity with the classical higher-order Massey products for differential graded associative algebras recalled in Section 2.1.1.2.

Let $\mathscr{P} = \mathscr{F}(E,R)$ be a Koszul operad with Koszul dual cooperad $\mathscr{P}^{\text{i}} = \mathscr{F}^c(sE, s^2R)$. We will assume all through the paper that a $\Bbbk$-linear basis of E has been fixed. Then, there are induced bases on $\mathscr{F}(E)$ and on $\mathscr{F}^c(sE)$ given by appropriate symmetric tree monomials, see [24, Section 2.4]. These bases will also be fixed once and for all. Since $\mathscr{P}^{\text{i}} \subseteq \mathscr{F}^c(sE)$, we will use this basis to linearly expand the elements of $\mathscr{P}^{\text{i}}$ in our results.

As mentioned in the introduction, each weight-homogeneous cooperation Γ^c of $\mathscr{P}^{\text{i}}$ creates a partially defined higher-order operation $\langle -,\dots,-\rangle_{\Gamma^c}$ on the homology of any $\mathscr{P}$-algebra, with as many inputs as the arity r of Γ^c. Out of homogeneous elements $x_1,\dots,x_r \in H_*(A)$ on the homology of a $\mathscr{P}$-algebra A, this operation creates a (possibly empty) set of homology classes

$$\langle x_1,\dots,x_r\rangle_{\Gamma^c} \subseteq H_*(A).$$

The non-emptiness depends on the vanishing, in a precise sense, of strictly lower-order operations of the same kind that depend on Γ^c. The set $\langle x_1,\dots,x_r\rangle_{\Gamma^c}$ is called the Γ^c*-Massey product* of the classes $x_1,\dots,x_r$.

To construct the Γ^c-Massey product operation $\langle -,\dots,-\rangle_{\Gamma^c}$, we proceed as follows. First, the cooperation Γ^c determines a set of indices $I(\Gamma^c)$ which is then used to form *defining systems*. A defining system for the concrete Γ^c-Massey product set $\langle x_1,\dots,x_r\rangle_{\Gamma^c}$ is a coherent choice of elements $\{a_\alpha\}$ of A indexed by $I(\Gamma^c)$ that are combined to create a cycle. The homology classes contained in $\langle x_1,\dots,x_r\rangle_{\Gamma^c}$ are obtained by running over all possible choices of defining systems for $x_1,\dots,x_r$ and taking the homology class of the associated cycle.

The section is organized as follows. First, we introduce the *Massey inductive map*. This map depends on the coproduct of $\mathscr{P}^{\text{i}}$ and a fixed twisting morphism $\kappa : \mathscr{P}^{\text{i}} \to \mathscr{P}$. It is an essential ingredient when dealing with the inductive definitions that follow. Then, we define the indexing set $I(\Gamma^c)$ associated to an arbitrary cooperation Γ^c and compute some examples. Once the concept of indexing sets is established, we proceed to explain what a defining system is and give examples of them. Then, we define the higher-order Γ^c-Massey products, and compute examples including the associative, commutative, Lie, Poisson, and dual numbers operads. Later on, we show that our higher-order Massey products framework includes Muro's [70] (Prop. 2.2.9). We study the elementary properties of these higher-order products in Section 2.2.1. These include the behavior along morphisms of $\mathscr{P}$-algebras, quasi-isomorphisms, and some connections to formality. Some further properties are explored in Section 2.2.2. There, we focus on the behavior of the higher-order Massey products along morphisms of operads and give some applications to formality.

Recall that the decomposition map $\Delta : \mathscr{C} \to \mathscr{C} \circ \mathscr{C}$ of any counital cooperad $\mathscr{C}$ can be uniquely written as

$$\Delta(c) = \Delta^+(c) + (\mathrm{id}; c)$$

for every arity-homogeneous $c \in \mathscr{C}$. Here, $\mathrm{id} \in \mathscr{C}(1)$ is the element that corresponds to the identity element 1 of the ground field $\Bbbk$ under the linear isomorphism $\mathscr{C}(1) \to \Bbbk$ induced by the counit. We call Δ^+ the half-reduced decomposition map of $\mathscr{C}$.

Definition 2.2.1. The *Massey inductive map* is the degree -1 map

$$D : \mathscr{F}^c(sE) \xrightarrow{\Delta^+} \mathscr{F}^c(sE) \circ \mathscr{F}^c(sE) \xrightarrow{\kappa \circ \mathrm{id}} E \circ \mathscr{F}^c(sE).$$

Applied to some cooperation μ, we shall write

$$D(\mu) = \sum (\zeta; \zeta_1, \dots, \zeta_m; \sigma), \tag{2.5}$$

where $\zeta \in E(m)$, $\zeta_i \in \mathscr{F}^c(sE)(v_i)$, $\sigma \in \mathbb{S}_m$ and $v_1 + \cdots + v_m$ is equal to the arity of μ.

The sum in Equation (2.5) is indexed over all ζ along the chosen basis of E, and each term may have a $\Bbbk$-coefficient (possibly 0). The map D is inductive in the sense that, for any cooperation μ, the cooperations $\zeta_1, \dots, \zeta_m$ appearing on the terms of $D(\mu)$ will each always have weight strictly less than that of μ. This will allow us to establish the inductive relations of our defining systems later on. If $\mathscr{P}$ is a Koszul operad, then the fact that $\mathscr{P}^{\mathrm{i}}$ is a subcooperad of $\mathscr{F}^c(sE)$ allows us to restrict the Massey inductive map to a map

$$D : \mathscr{P}^{\mathrm{i}} \xrightarrow{\Delta^+} \mathscr{P}^{\mathrm{i}} \circ \mathscr{P}^{\mathrm{i}} \xrightarrow{\kappa \circ \mathrm{id}} E \circ \mathscr{P}^{\mathrm{i}}.$$

Abusing the notation, we call this restriction the Massey inductive map too, and use the same symbols to denote the maps that constitute it.

As mentioned before, the cofree conilpotent cooperad $\mathscr{F}^c(sE)$ has a fixed combinatorial description in terms of rooted tree monomials whose internal vertices are labeled by elements of sE. Each such tree monomial has a *first vertex*, which is the unique child of the root and corresponds to the first generating cooperation to be applied. The action of D is determined by sending any tree monomial T to $(s^{-1}x; T_1, \dots T_m)$, where $x \in (sE)(m)$ is the label of the first vertex of T, and $T_1, \dots, T_m$ are the tree monomials attached to this first vertex of T. Intuitively, the Massey inductive map is trimming level 1 edges. See figures 2.1 and 2.2.

Next, we introduce the set associated with a cooperation of $\mathscr{P}^{\mathrm{i}}$ that will provide the indices for our defining systems. It is defined by induction on the weight of arity-homogeneous cooperations of $\mathscr{P}^{\mathrm{i}}$, with the Massey inductive map providing the necessary inductive step.

Definition 2.2.2. Let $\Gamma^c \in \mathscr{P}^{\mathrm{i}}(r)$ be a weight-homogeneous cooperation. For each permutation $(k_1, \dots, k_r) \in \mathbb{S}_r$, we define the Γ^c*-indexing set* $I(\Gamma^c, (k_1, \dots, k_r))$ by induction on the weight $w(\Gamma^c)$ of Γ^c as follows.

- If $w(\Gamma^c) = 0$, then $I(\Gamma^c,) = \emptyset$.
- If $w(\Gamma^c) = 1$, then $I(\Gamma^c,) = \{(\mathrm{id}, (1)), \dots, (\mathrm{id}, (r))\}$.

Assume next that $I(\Gamma^c, (k_1, \dots, k_r))$ has been defined for cooperations up to weight n, and suppose Γ^c is of weight $n+1$. If

$$D(\Gamma^c) = \sum (\zeta; \zeta_1, \dots, \zeta_m; \sigma)$$

as in Equation (2.5), and the leaves on top of each ζ_i are labeled $l_1, \dots, l_{v_i}$, then

$$I\left(\Gamma^c, (k_1, \dots, k_r)\right) := \bigcup_{i=1}^{m} I\left(\zeta_i, \left(k_{l_1}, \dots, k_{l_{v_i}}\right)\right) \cup \left\{\left(\zeta_i, \left(k_{l_1}, \dots, k_{l_{v_i}}\right)\right)\right\}.$$

The super index c in Γ^c indicates that we are seeing the corresponding element in the Koszul dual *cooperad* of $\mathscr{P}$. At a later place, we will see this same element as a relation Γ in the free operad $\mathscr{F}(E)$. Since we will need to distinguish between these two elements, we keep the super index in the notation.

The following elementary observation will be the base case of the inductive definition of defining systems below. We record this fact before giving some explicit examples.

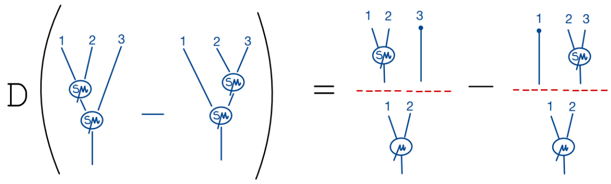

Figure 2.1: The Massey inductive map for Ass

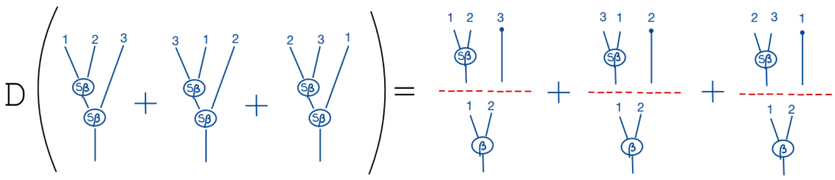

Figure 2.2: The Massey inductive map for Lie

Remark 2.2.3. If $\mathscr{P}$ is any Koszul operad and $\Gamma^c \in \left(\mathscr{P}^{\text{i}}\right)^{(1)}(r) = (sE)(r)$ is any cogenerator of arity r, then the Γ^c-indexing set is always given by

$$I\left(\Gamma^c\right) = \{(\mathrm{id},(1)), \ldots, (\mathrm{id},(r))\}.$$

Let us illustrate the definition of indexing sets with some examples.

Example 2.2.4. Let $\mathscr{P} = \mathsf{Ass}$. Then the weight n component of $\mathscr{P}^{\text{i}}$ is freely generated as an $\mathbb{S}_{n+1}$-module by a single generator $\mu_{n+1}^c \in \mathsf{Ass}^{\text{i}}(n+1)$. Recall that

$$\Delta\left(\mu_n^c\right) = \sum_{i_1+\cdots+i_k=n} (-1)^{\sum(i_j+1)(k-j)} \left(\mu_k^c; \mu_{i_1}^c, \ldots \mu_{i_k}^c; \mathrm{id}\right).$$

Here, we denote $\mu_1^c = \mathrm{id} \in \mathsf{Ass}^{\text{i}}(1)$. Since $\kappa\left(\mu_2^c\right) = \mu_2$ and $\kappa\left(\mu_k^c\right) = 0$ for $k \geq 3$, this implies that

$$D\left(\mu_n^c\right) = \sum_{i_1+i_2=n} (-1)^{i_1+1} \left(\mu_2; \mu_{i_1}^c, \mu_{i_2}^c; \mathrm{id}\right).$$

This means that the defining system $I\left(\mu_n^c\right)$ contains the elements

$$(\mu_{i_1}^c, (1, 2, \ldots, i_1)) \quad \text{and} \quad (\mu_{i_2}^c, (n - i_2, n - i_2 + 1, \ldots, n)),$$

where $i_1 + i_2 = n$. By iterating this process, we see that

$$I\left(\mu_n^c\right) = \left\{\left(\mu_k^c, (i, i+1, \ldots, i+k-1)\right) \mid k < n \text{ and } i \in \{1, 2, \ldots, n-k-1\}\right\}.$$

$\square$

Example 2.2.5. Let $\mathscr{P} = \mathsf{Lie}$. Then the weight n part of $\mathscr{P}^{\text{i}}$ is one-dimensional and generated by $\tau_{n+1}^c \in \mathsf{Lie}^{\text{i}}(n+1)$. Recall that

$$\Delta\left(\tau_n^c\right) = \sum_{\substack{i_1+\cdots+i_k=n \\ \sigma \in \overline{Sh}^{-1}(i_1,\ldots,i_k)}} (-1)^{\sum(i_j+1)(k-j)} \operatorname{sgn}(\sigma) \left(\tau_k^c; \tau_{i_1}^c, \ldots \tau_{i_k}^c; \sigma\right),$$

where $\overline{Sh}^{-1}(i_1, \ldots, i_k)$ is the set of reduced unshuffles. Here, an unshuffle is the inverse of a shuffle, and *reduced* signifies that we are considering only those shuffles that fix the position of the first element, i.e. $\sigma(1)=1$. Since $\kappa(\tau_2^c)=\tau_2$ and $\kappa(\tau_k^c)=0$ for $k\geq 3$, this implies that

$$D\left(\tau_n^c\right)=\sum_{\substack{i+j=n\\ \sigma\in\overline{Sh}^{-1}(i,j)}}(-1)^{i+1}\operatorname{sgn}(\sigma)\left(\tau_2;\tau_i^c,\tau_j^c;\sigma\right).$$

This means that the defining system $I\left(\tau_n^c\right)$ contains the elements

$$\left(\tau_i^c,(\sigma(1),\sigma(2),\ldots,\sigma(i))\right)\quad\text{and}\quad\left(\tau_j^c,\left(\sigma(n-j),\sigma(n-j+1),\ldots,\sigma(n)\right)\right)$$

for each reduced shuffle $\sigma\in\overline{Sh}\left(i,j\right)$ with $i+j=n$. In this step, we changed from using unshuffles to shuffles, because there is an inversion involved. By iterating this process, we find that

$$I\left(\tau_n^c\right)=\left\{\left(\tau_k^c,(i_1,\ldots i_k)\right)\mid k<n\text{ and }1\leq i_1\leq\cdots\leq i_l\leq n\right\}.$$

□

As mentioned before, each cooperation Γ^c of weight n in the Koszul dual cooperad $\mathscr{P}^{\text{i}}$ of $\mathscr{P}$ produces a partially defined n-th order operation $\langle -,\ldots,-\rangle_{\Gamma^c}$ on the homology $H_*(A)$ of a $\mathscr{P}$-algebra A. This higher operation has r inputs, where r is the arity of Γ^c, and the output is the set of homology classes created from all possible choices of *defining systems*, generalizing the case of associative algebras of Section 2.1.1.2. Our next task is to explain what the defining systems are. Each defining system will depend on a weight-homogeneous cooperation Γ^c of arity r and r homogeneous homology classes $x_1,\ldots,x_r\in H_*(A)$. Their definition is given by induction on the weight of the cooperation.

Definition 2.2.6. Let $\Gamma^c\in\left(\mathscr{P}^{\text{i}}\right)^{(n)}(r)$ for some $n\geq 1$, A a $\mathscr{P}$-algebra, and $x_1,\ldots,x_r\in H_*(A)$ homogeneous elements. A Γ^c*-defining system (associated to* $x_1,\ldots,x_r$*)* is a collection $\{a_\alpha\}_{\alpha\in I(\Gamma^c)}$ of elements of A indexed by $I(\Gamma^c)$ such that:

1. Each $a_{(\mathrm{id},(i))}\in A$ is a cycle representative for $x_i\in H_*(A)$.
2. For each index $\left(\mu,(k_1,\cdots,k_i)\right)\in I(\Gamma^c)$ with $\mu\neq\mathrm{id}$, the corresponding element $a_{\left(\mu,(k_1,\cdots,k_i)\right)}$ is such that

$$d\left(a_{\mu,(k_1,\cdots,k_i)}\right)=\sum\zeta\left(a_{\left(\zeta_1,\left(k_{\sigma^{-1}(1)},\ldots,k_{\sigma^{-1}(v_1)}\right)\right)},\ldots,a_{\left(\zeta_m,\left(k_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,k_{\sigma^{-1}(i)}\right)\right)}\right),$$

where $D\left(\mu\right)=\sum(\zeta;\zeta_1,\ldots,\zeta_m;\sigma)$.

Next, we use the defining systems explained above to define the Γ^c-Massey products. If the cooperation Γ^c is of weight 1 and arity r, that is, a cogenerator, then $\Gamma=\kappa(\Gamma^c)$ is a generator of $\mathscr{P}$. For any homogeneous elements $x_1,\ldots,x_r\in H_*(A)$, we define their Γ^c-*Massey product* as the set

$$\langle x_1,\ldots,x_r\rangle_{\Gamma^c}:=\{\Gamma(x_1,\ldots,x_r)\}.$$

We may also identify this set with its unique element $\Gamma(x_1,\ldots,x_r)\in H_*(A)$. Let us define the Γ^c-Massey products for elements of weight ≥ 2.

Definition 2.2.7. Let A be a $\mathscr{P}$-algebra, $\Gamma^c\in\left(\mathscr{P}^{\text{i}}\right)^{(n)}(r)$ with $n\geq 2$, and $x_1,\ldots,x_r$ homogeneous elements of $H_*(A)$. Then:

1. The Γ^c*-Massey product* associated to a Γ^c-defining system $\{a_\alpha\}$ and $x_1,\ldots,x_r$ is the homology class of the cycle

$$a_{\Gamma^c,(1,\ldots,r)}:=\sum(-1)^\gamma\zeta\left(a_{\zeta_1,(\sigma^{-1}(1),\sigma^{-1}(2),\ldots,\sigma^{-1}(v_1))},\ldots,a_{\zeta_m,(\sigma^{-1}(v_1+\cdots+v_{m-1}+1),\ldots,\sigma^{-1}(r))}\right),\tag{2.6}$$

where $D(\Gamma^c) = \sum(\zeta;\zeta_1,\dots,\zeta_m;\sigma)$, and the sign is given by

$$\gamma = \alpha + \sum_{i=2}^{m} \left(|\zeta_i| - \mathrm{wgt}(\zeta_i)\right)\left(\sum_{k=1}^{v_1+\cdots+v_{i-1}} |x_{\sigma^{-1}(k)}|\right) + 1, \qquad \alpha = \sum_{\substack{i<j\\ \sigma(i)>\sigma(j)}} |x_i||x_j|,$$

where $\mathrm{wgt}(\zeta_i)$ is the weight of ζ_i.

2. The Γ^c-*Massey product set* $\langle x_1,\dots,x_r\rangle_{\Gamma^c}$ is the (possibly empty) subset of $H_*(A)$ formed by the homology classes arising from all possible choices of Γ^c-defining systems $\{a_\alpha\}$ associated to $x_1,\dots,x_r$.

The next result shows that the proposed definition is correct. As a consequence of it, we readily see from the definition of defining systems that the Γ^c-Massey product set $\langle x_1,\dots,x_r\rangle_{\Gamma^c}$ is non-empty if, and only if, for all $(\mu;k_1,\dots k_i) \in I(\Gamma^c)$, the Massey product set $\langle x_{k_1},\dots,x_{k_i}\rangle_\mu$ is defined and contains the zero class.

Proposition 2.2.8. *Let A be a $\mathscr{P}$-algebra, $\Gamma^c \in \left(\mathscr{P}^{\text{i}}\right)^{(n)}(r)$ for some $n \geq 2$, and $x_1,\dots,x_r$ homogeneous elements of $H_*(A)$. Then the Γ^c-Massey product x associated to any Γ^c-defining system for $x_1,\dots,x_r$ is a cycle.*

Proof. Let $\{a_\alpha\}$ be a defining system, and denote by x the associated cycle given by formula (2.6),

$$x = \sum(-1)^\gamma \zeta\left(a_{\zeta_1,(\sigma^{-1}(1),\sigma^{-1}(2),\dots,\sigma^{-1}(v_1))},\dots,a_{\zeta_m,(\sigma^{-1}(v_1+\cdots+v_{m-1}+1),\dots,\sigma^{-1}(r))}\right).$$

Let us compute dx in terms of the Massey inductive map D and terms of the form $a_{\mu,(k_1,\dots k_i)}$. Recall that the differential of A fits into the commutative diagram

$$\begin{tikzcd} \mathscr{P}\circ A \arrow[r, "\gamma_A"] \arrow[d, "\mathrm{id}\circ' d"'] & A \arrow[d, "d"] \\ \mathscr{P}\circ A \arrow[r, "\gamma_A"] & A \end{tikzcd}$$

where $\circ'$ is the infinitesimal composite. From here, it follows that

$$dx = d\left(\sum\zeta\left(a_{\zeta_1,(\sigma^{-1}(1),\sigma^{-1}(2),\dots,\sigma^{-1}(v_1))},\dots,a_{\zeta_m,(\sigma^{-1}(v_1+\cdots+v_{m-1}+1),\dots,\sigma^{-1}(r))}\right)\right)$$

is equal to

$$\sum\sum_{i=1}^{m}(-1)^{\epsilon_i}\zeta\left(a_{\zeta_1,(\sigma^{-1}(1),\sigma^{-1}(2),\dots,\sigma^{-1}(v_1))},\dots,d\left(a_{\zeta_i,(\sigma^{-1}(v_1+\cdots+v_{i-1}+1),\dots,\sigma^{-1}(v_1+\cdots+v_i))}\right),\dots a_{\zeta_m,(\sigma^{-1}(v_1+\cdots+v_{m-1}+1),\dots,\sigma^{-1}(r))}\right)$$

where

$$\epsilon_i = |\zeta| + |a_{\zeta_1,(\sigma^{-1}(1),\sigma^{-1}(2),\dots,\sigma^{-1}(v_1))}| + \cdots + |a_{\zeta_{i-1},(\sigma^{-1}(v_1+\cdots+v_{i-2}+1),\dots,\sigma^{-1}(v_1+\cdots+v_{i-1}))}|.$$

Each term $d\left(a_{\zeta_i,(\sigma^{-1}(v_1+\cdots+v_{i-1}+1)),\dots,\sigma^{-1}(v_1+\cdots+v_i))}\right)$ appearing in the sum above can be rewritten in terms of $a_{\mu,(k_1,\dots k_i)}$ of lower order, by using the inductive relation of the defining system (Def 2.2.6, item 2). In particular, if we consider the composite

$$G : \mathscr{P}^{\text{i}} \xrightarrow{\Delta^+} \mathscr{P}^{\text{i}}\circ\mathscr{P}^{\text{i}} \xrightarrow{\kappa\circ\mathrm{id}} \mathscr{P}\circ\mathscr{P}^{\text{i}} \xrightarrow{f} \mathscr{P}\circ\left(\mathscr{P}^{\text{i}};\mathscr{P}^{\text{i}}\right) \xrightarrow{\mathrm{id}\circ(\mathrm{id};\Delta^+)} \mathscr{P}\circ(\mathscr{P}^{\text{i}};\mathscr{P}^{\text{i}}\circ\mathscr{P}^{\text{i}}) \xrightarrow{\mathrm{id}\circ(\mathrm{id};\kappa\circ\mathrm{id})}$$
$$\mathscr{P}\circ\left(\mathscr{P}^{\text{i}};\mathscr{P}\circ\mathscr{P}^{\text{i}}\right) \xrightarrow{p} \mathscr{P}\circ\left(\mathscr{P}^{\text{i}};\mathscr{P}^{\text{i}}\right) \xrightarrow{q} \mathscr{P}\circ\mathscr{P}^{\text{i}},$$

where f is the natural inclusion, p is induced by the partial composition in $\mathscr{P}$, and q is the forgetful map, then the element dx is given by

$$\sum \xi \left(a_{\xi_1,(\sigma^{-1}(1),\ldots,\sigma^{-1}(v_1))}, \ldots, a_{\xi_m,(\sigma^{-1}(v_1+\cdots+v_{m-1}+1),\ldots,\sigma^{-1}(r))} \right),$$

where $G(\Gamma^c) = \sum (\xi; \xi_1, \ldots, \xi_m; \sigma)$. So to prove the result, it suffices to show that G is identically 0. We shall do this by showing that $\operatorname{Im} G \subseteq R \circ \mathscr{P}^{\text{i}}$, where R is the sub-module of relations in the fixed presentation $\mathscr{P} = \mathscr{F}(E, R)$. Recall that $\mathscr{P}^{\text{i}}$ can be thought of as a subset of the tree module and all the maps defining G descend from maps on or between the free operad on E and the cofree conilpotent cooperad on sE. It follows that we may describe G combinatorially by giving its action on individual basis tree monomials T of $\mathscr{F}^c(sE)$. This action is as follows.

1. First, apply the Massey inductive map D. This is sending the tree monomial T to a sum of tree monomials of the form $\left(s^{-1}e; T_1, \ldots T_m\right)$, where $e \in (sE)(m)$ is the label of the first vertex and $T_1, \ldots, T_m$ are its children.
2. Now, repeat this procedure on each T_i individually, thereby obtaining sums of tree monomials of the form $\left(s^{-1}e_i; T_{i,1}, \ldots T_{i,m_i}\right)$, and take for each individual tree monomial T_i the sum over the results to obtain

   $$(-1)^{\epsilon_i} \sum_{i=1}^{m} \left(s^{-1}e; T_1, \ldots, \left(s^{-1}e_i; T_{i,1}, \ldots T_{i,m_i}\right), \ldots T_m\right).$$

   Here, each e_i is the first vertex of the corresponding T_i.
3. The final step is to apply the partial composition in the free operad $\mathscr{F}(E)$ in order to obtain

   $$\sum_{i=1}^{m} \left(s^{-1}e \circ_i s^{-1}e_i; T_1, \ldots, T_{i-1}, T_{i,1}, \ldots T_{i,m_i}, T_{i+1}, \ldots T_m\right).$$

   This time, without the suspension.

From this description, it follows that there is another equivalent way to describe G:

- First, directly apply the cooperadic reduced decomposition map Δ^+ to T to obtain

  $$\Delta^+(T) = \sum (S; S_1, \ldots S_k).$$

- Then, project the first component of $\mathscr{F}^c(sE) \circ \mathscr{F}^c(sE)$ into weight 2. That is, map S to itself if it is in weight 2, and map it to 0 otherwise. This produces

  $$\sum_{i=1}^{m} \left(e \circ_i e_i; T_1, \ldots, T_{i-1}, T_{i,1}, \ldots T_{i,m_i}, T_{i+1}, \ldots T_m\right).$$

- Desuspend the tree monomial $e \circ_i e_i$ twice.

From this description, it follows that

$$\sum_{i=1}^{m} \left(e \circ_i e_i; T_1, \ldots, T_{i-1}, T_{i,1}, \ldots T_{i,m_i}, T_{i+1}, \ldots T_m\right) \in \mathscr{P}^{\text{i}(2)} \circ \mathscr{P}^{\text{i}},$$

and thus that

$$\sum_{i=1}^{m} \left(s^{-1}e \circ_i s^{-1}e_i; T_1, \ldots, T_{i-1}, T_{i,1}, \ldots T_{i,m_i}, T_{i+1}, \ldots T_m\right) \in R \circ \mathscr{P}^{\text{i}}.$$

This is exactly what we wanted to prove. $\square$

In the next result, we show that our higher-order Massey products recover the secondary Massey products defined by Muro in [70] when restricting to cooperations of weight 2, up to a sign. The construction of Muro is recalled in Section 2.1.1.3, and we stick to the notation used there.

Proposition 2.2.9. *Let $\mathscr{P}$ be a Koszul operad with fixed presentation $\mathscr{F}(E,R)$. Let*

$$\Gamma = \sum \left(\mu^{(1)} \circ_l \mu^{(2)}\right) \cdot \sigma \in R(r)$$

be a quadratic relation of arity r, and denote the corresponding weight 2 element of the Koszul dual cooperad $\mathscr{P}^{¡}$ by Γ^c, so that

$$\Gamma^c := s^2(\Gamma) = \sum (-1)^{|\mu^{(1)}|} \left(s\mu^{(1)} \circ_l s\mu^{(2)}\right) \cdot \sigma.$$

Let A be a $\mathscr{P}$-algebra, and let $x_1, ..., x_r \in H_*(A)$ be homogeneous elements. Then the Γ-Massey product set $\langle x_1, \ldots, x_r \rangle_\Gamma$ of Def. 2.1.4 and the Γ^c-Massey product set $\langle x_1, \ldots, x_r \rangle_{\Gamma^c}$ of Def. 2.2.7 are the same up to a sign,

$$\langle x_1, \ldots, x_r \rangle_\Gamma = \pm \langle x_1, \ldots, x_r \rangle_{\Gamma^c}.$$

In particular, the Massey product set $\langle x_1, \ldots, x_r \rangle_\Gamma$ is non-empty if, and only if, the Γ^c-Massey product set $\langle x_1, \ldots, x_r \rangle_{\Gamma^c}$ is non-empty.

Proof. One can directly verify that

$$\Delta^+\left(\Gamma^c\right) = \sum (-1)^{|\mu^{(1)}|} (s\mu^{(1)}; \mathrm{id}, \ldots, \mathrm{id}, s\mu^{(2)}, \mathrm{id}, \ldots, \mathrm{id}; \sigma).$$

Since $\mu^{(1)}$ has weight 1, it follows that $\kappa(s\mu^{(1)}) = \mu^{(1)}$. Therefore, a cycle representing the Γ^c-Massey product associated to the elements $x_1, \ldots x_r$ is of the form

$$\sum (-1)^\gamma \mu^{(1)} \left(a_{\mathrm{id},\sigma^{-1}(1)}, \ldots, a_{\mathrm{id},\sigma^{-1}(l-1)}, a_{s\mu^{(2)},\sigma^{-1}(l)}, a_{\mathrm{id},\sigma^{-1}(l+r_1)}, \ldots, a_{\mathrm{id},\sigma^{-1}(r)}\right),$$

as in Equation (2.6). Now, the $a_{\mathrm{id},(i)}$ are just cycle representatives of the x_i. To finish, we will check that the element $a_{s\mu^{(2)},(l)}$ satisfies exactly Condition (2.4) in Muro's construction (Def. 2.1.4), so it corresponds to the element $\rho^{(2)}$ there. Indeed, since $s\mu^{(2)}$ has weight 1, it follows that $\Delta^+\left(s\mu^{(2)}\right) = \left(s\mu^{(2)}; \mathrm{id}, \mathrm{id} \ldots, ; \mathrm{id}\right)$, and so $D\left(s\mu^{(2)}\right) = \left(\mu^{(2)}; \mathrm{id}, \mathrm{id} \ldots, ; \mathrm{id}\right)$. Therefore,

$$d\, a_{s\mu^{(2)},(l)} = \mu^{(2)}\left(a_{\mathrm{id},l}, \ldots a_{\mathrm{id},l+r_1-1}\right).$$

The sign $(-1)^\gamma$ that appears in Equation (2.6) gives exactly Muro's sign plus one because for binary operads, the weight equals the arity degree minus one. This completes the proof. □

In the following examples, we explain how our operadic framework for defining systems recovers the classical framework in the associative and Lie cases, and then explain how it creates completely new higher-order operations.

Example 2.2.10. When $\mathscr{P} = \mathrm{Ass}$ is the associative operad, our framework recovers the classical definition of higher-order Massey products as in Def. 2.1.3. To see this, recall from Example 2.2.4 that the weight n component of $\mathrm{Ass}^{¡}$ is freely generated as an $\mathbb{S}_{n+1}$-module by a single generator μ^c_{n+1}, and that the μ^c_n-indexing set attached to a cooperation is given by

$$\left\{\left(\mu^c_k, (i, i+1, \ldots, i+i-1)\right) \mid 1 \le k < n \text{ and } i \in \{1, 2, \ldots, n-k+1\}\right\}.$$

We show next that fixing a particular differential graded associative algebra (A, d) and homogeneous homology classes $x_1, ..., x_n \in H_*(A)$, there is a bijective correspondence between the classical defining systems $\{b_{ij}\}$ for the higher-order Massey product $\langle x_1,\dots,x_n\rangle$, and the defining systems $\{a_\alpha\}$ for the μ_n^c-Massey product $\langle x_1,\dots,x_n\rangle_{\mu_n^c}$ as defined in this paper. Indeed, the correspondence is given by

$$b_{i,j} = a_{\mu^c_{j-i},(i+1,i+2,\dots,i+(j-i)=j)} \quad \text{for all} \quad 0\le i<j\le n \quad \text{and} \quad 1\le j-i\le n-1.$$

To finish, it suffices to compare the boundaries of the elements in these sets. Recall that

$$D\left(\mu_r^c\right) = \sum_{l_1+l_2=r} (-1)^{l_1+1}\left(\mu_2;\mu^c_{l_1},\mu^c_{l_2};\mathrm{id}\right).$$

Therefore, by directly applying Definition 2.2.6, we see that

$$\begin{aligned} db_{ij} &= \sum_{k=i+1}^{j-1}\bar b_{ik}b_{kj} = \sum_{k=i+1}^{j-1}(-1)^{|b_{ik}|+1}b_{ik}b_{kj} = \sum_{k=i+1}^{j-1}(-1)^{|b_{ik}|+1}a_{\mu^c_{k-i},(i+1,i+2,\dots,k)}\cdot a_{\mu^c_{j-k},(k+1,k+2,\dots,j)}\\ &= \sum_{k=i+1}^{j-1}(-1)^{|a_{\mu^c_{k-i},(i+1,i+2,\dots,k)}|+1}a_{\mu^c_{k-i},(i+1,i+2,\dots,k)}\cdot a_{\mu^c_{j-k},(k+1,k+2,\dots,j)}\\ &= \sum_{l_1+l_2=j-i}(-1)^{l_1+1}(-1)^{\gamma}a_{\mu^c_{l_1},(k-l_1+1,k-l_2+2,\dots,k)}\cdot a_{\mu^c_{l_2},(k+1,k+2,\dots,k+l_2)} = da_{\mu^c_{j-i},(i+1,i+2,\dots,i+(j-i)=j)}. \end{aligned}$$

where $\gamma = |x_{i+1}|+|x_{i+2}|+\cdots+|x_{i+l_1}|$ and the change of sign from the second to the third line follows from the equality

$$|a_{\mu^c_{k-i},(i+1,i+2,\dots,k)}| = |x_{i+1}|+|x_{i+2}|+\cdots+|x_k|+|\mu^c_{k-i}|+1.$$

□

The observant reader will likely have spotted that the above is just one of the several linearly independent Massey products that Ass possesses. In fact, there are different, linearly independent Massey products for each permutation $\sigma\in\mathbb{S}_n$, since $\tau_n^c\cdot\sigma$ is linearly independent of τ_n^c. Up to a sign, these are related by $\langle x_1,\dots x_n\rangle_{\tau_n^c\cdot\sigma} = \left\langle x_{\sigma^{-1}(1)},\dots x_{\sigma^{-1}(n)}\right\rangle_{\tau_n^c}$, see Prop. 2.2.21. Similarly, different presentations of an operad (in the associative case, one could take for example the Livernet–Loday presentation [56, Prop. 9.1.1]) give rise to seemingly distinct Massey products, which are just the same expressed with respect to a different basis.

Example 2.2.11. When $\mathscr{P} = \mathsf{Lie}$ is the Lie operad, our framework recovers the classical definition of higher Lie–Massey brackets as in [3, 78] (see also [4, 87]). To see this, recall that the weight n part of $\mathsf{Lie}^{\text{¡}}$ is one-dimensional and generated by $\tau^c_{n+1}\in\mathsf{Lie}^{\text{¡}}(n+1)$. Recall also from Example 2.2.5 that in this case, the τ_n^c-indexing set is

$$I\left(\tau_n^c\right) := \left\{(\tau_k^c,(i_1,\dots i_k))\mid\ k\le n \text{ and } 1\le i_1\le\cdots\le i_l\le n\right\}.$$

We show next that fixed a particular differential graded Lie algebra (L,d) and homogeneous elements $x_1,\dots,x_n\in H_*(L)$, there is a bijective correspondence between the classical defining systems $\{x_{j_1,\dots,j_l}\}$ of [3] for the higher-order Whitehead product $[x_1,\dots,x_n]$, and the defining systems $\{a_\alpha\}$ for the τ_n^c-Massey product as defined in this paper. Indeed, the correspondence is given by

$$x_{j_1,\dots,j_l} = a_{\tau^c_l,(j_1,\dots,j_l)} \quad \text{for all } 1\le j_1\le\cdots\le j_l\le n.$$

Recall from Example 2.2.5 that

$$D\left(\tau_n^c\right) = \sum_{\substack{r_1+r_2=n \\ \sigma\in\overline{Sh}^{-1}(r_1,r_2)}} (-1)^{r_1+1}\operatorname{sgn}(\sigma)\left(\tau_2;\tau_{r_1}^c,\tau_{r_2}^c;\sigma\right).$$

Therefore, by directly applying Definition 2.2.6, we see that

$$\begin{aligned} dx_{j_1,\dots,j_l} &= \sum_{p=1}^{l}\sum_{\sigma\in\overline{Sh}(p,l-p)} \epsilon(\sigma)\left[x_{j_{\sigma(1),\dots,\sigma(p)}}, x_{j_{\sigma(p+1),\dots,\sigma(l)}}\right] \\ &= \sum_{\substack{r_1+r_2=l \\ \sigma\in\overline{Sh}^{-1}(r_1,r_2)}} (-1)^{r_1+1}\operatorname{sgn}(\sigma)\tau_2\left(a_{\tau_{r_1}^c,\left(j_{\sigma^{-1}(1)},j_{\sigma^{-1}(2)},\dots,j_{\sigma^{-1}(r_1)}\right)}, a_{\tau_{r_2}^c,\left(j_{\sigma^{-1}(r_1+1)},\dots,j_{\sigma^{-1}(l)}\right)}\right) \\ &= da_{\tau_l^c,(j_1,\dots,j_l)}. \end{aligned}$$

□

As likely expected, the higher-order Massey products for commutative differential graded associative algebras coincide with those formed by forgetting that the structure is commutative. This can be seen as a consequence of the theory developed in the next section, see Example 2.2.25.

In [70], Muro contributed a new kind of triple Massey-product operation for Gerstenhaber and/or Poisson algebras. Our framework recovers this triple operation as a consequence of Proposition 2.2.9. It follows from our results that all the higher-order analogs of this new operation also exist. Although we will not give closed formulas, we hope these higher products will be successfully applied in the future in cases where the triple-product operation defined by Muro does not suffice.

Example 2.2.12. Recall that the Poisson operad Pois is self-Koszul dual, generated by a commutative associative product $\wedge$ and a Lie bracket $[-,-]$, both of degree zero, which are compatible via the Poisson relation,

$$[x\wedge y, z] = x\wedge[y,z] + (-1)^{|y||z|}[x,z]\wedge y.$$

The inner combinatorics of this operad are complex, and multiplying base elements frequently involves complicated rewriting procedures. Therefore, we cannot hope to write down formulas that are quite as clean as in examples 2.2.10 and 2.2.11. Nonetheless, it is possible to compute Poisson Massey products inductively in low weight.

For example, if one considers $[-,-]\wedge - \in \mathsf{Pois}^{\mathsf{i}}(3)$, where we are taking the Koszul suspensions to be implicit, one has

$$D([-,-]\wedge -) = (\wedge;[-,-],\mathrm{id}) - ([-,-];\mathrm{id},\wedge,) - ([-,-];\mathrm{id},\wedge,)\cdot(2,3).$$

Recall that $\kappa[-,-] = \wedge$ and $\kappa(\wedge) = [-,-]$. A defining system for the Massey product $\langle x_1,x_2,x_3\rangle$ associated to the cooperation above in a Poisson algebra is therefore a set of elements $\{z_1,z_2,z_3,y_1,y_2,y_3\}$ where each z_i is a cycle representative of x_i for $i=1,2,3$, and

$$dy_1 = z_1\wedge z_2, \qquad dy_2 = [z_2,z_3] \qquad dy_3 = [z_1,z_3].$$

The cycle representative associated to this defining system is

$$[y_1,z_3] - (-1)^{|z_1|}z_1\wedge y_2 - (-1)^{|z_2|+|z_1||z_2|}z_2\wedge y_3.$$

Similarly, if we consider $[-,- \wedge -] \in \mathsf{Pois}^{\mathsf{i}}(3)$,

$$D([-,-\wedge -]) = ([-,-];\mathrm{id},\wedge) + (\wedge;[-,-],\mathrm{id}) + (\wedge;\mathrm{id},[-,-])\cdot(1,2)$$

A defining system for the Massey product $\langle x_1, x_2, x_3\rangle$ associated to this cooperation in a Poisson algebra is therefore a set of elements $\{z_1, z_2, z_3, y_1, y_2, y_3\}$ where each z_i is a cycle representative of x_i for $i = 1,2,3$, and

$$dy_1 = [z_2, z_3], \qquad dy_2 = z_1 \wedge z_2 \qquad dy_3 = z_1 \wedge z_3.$$

The cycle representative associated to this defining system is

$$z_1 \wedge y_1 - (-1)^{|z_1|}[y_2, z_3] - (-1)^{|z_2|+|z_1||z_2|}[z_2, y_3].$$

□

Example 2.2.13. We continue the previous example by computing the higher Massey product corresponding to $\wedge \circ ([-,-],[-,-]) \in \mathsf{Pois}^{\mathsf{i}}(4)$. If one takes the Koszul suspensions and Koszul signs to be implicit, one has:

$$\begin{aligned}
&\triangle^{+}(\wedge\circ([-,-],[-,-]))\\
&\quad= (\wedge;[-,-],[-,-]) + (\wedge\circ(-,[-,-]);[-,-],\mathrm{id},\mathrm{id}) + (\wedge\circ([-,-],-);\mathrm{id},\mathrm{id},[-,-],)\\
&\qquad + (\wedge;\mathrm{id},[-,-\wedge -]) + (-\wedge[-,-];\mathrm{id},\mathrm{id},\wedge) + (\wedge;[-,-\wedge -],\mathrm{id})\cdot(2,4,3)\\
&\qquad + ([-,-]\wedge -;\mathrm{id},-\wedge -,\mathrm{id})\cdot(2,4,3) + 2(\wedge;\mathrm{id},-\wedge[-,-],)\cdot(2,4,3) + 2(-\wedge-\wedge -;\mathrm{id},\mathrm{id},[-,-])\cdot(2,4,3)\\
&\qquad + 2(\wedge;-\wedge -,[-,-])\cdot(2,4,3) + 2(\wedge;\mathrm{id},[-,-]\wedge -) + 2(-\wedge-\wedge -;\mathrm{id},[-,-],\mathrm{id})\\
&\qquad + 2(\wedge;-\wedge[-,-],\mathrm{id}) + 2(\wedge;\mathrm{id},[-,-]\wedge -)\cdot(1,2,4,3) + 2(-\wedge-\wedge -;\mathrm{id},[-,-],\mathrm{id})\cdot(1,2,4,3)\\
&\qquad + 2(\wedge;-\wedge[-,-],\mathrm{id})\cdot(1,2,4,3) + (\wedge;\mathrm{id},[-\wedge -,-],\mathrm{id})\cdot(1,2,3) + (-\wedge[-,-];\mathrm{id},-\wedge -,\mathrm{id})\cdot(1,2,3)\\
&\qquad + (\wedge;[-\wedge -,-],\mathrm{id}) + ([-,-]\wedge -;\wedge,\mathrm{id})
\end{aligned}$$

This means that

$$\begin{aligned}
&D(\wedge\circ([-,-],[-,-]))\\
&\quad= (\wedge;[-,-],[-,-]) + (\wedge;\mathrm{id},[-,-\wedge -]) + (\wedge;[-,-\wedge -],\mathrm{id})\cdot(2,4,3) + 2(\wedge;\mathrm{id},-\wedge[-,-],)\cdot(2,4,3)\\
&\qquad + 2(\wedge;-\wedge -,[-,-])\cdot(2,4,3) + 2(\wedge;\mathrm{id},[-,-]\wedge -) + 2(\wedge;-\wedge[-,-],\mathrm{id}) + 2(\wedge;\mathrm{id},[-,-]\wedge -)\cdot(1,2,4,3)\\
&\qquad + 2(\wedge;-\wedge[-,-],\mathrm{id})\cdot(1,2,4,3) + (\wedge;\mathrm{id},[-\wedge -,-],\mathrm{id})\cdot(1,2,3) + (\wedge;[-\wedge -,-],\mathrm{id})
\end{aligned}$$

Now we compute the product corresponding to the equation above. This means that the Massey product may be computed as being, up to Koszul sign

$$\begin{aligned}
&= [y_{\wedge,(1,2)}, y_{\wedge,(3,4)}] + [z_1, x_{b,(2,3,4)}] + [x_{b,(1,3,4)}, z_2] + 2[z_1, x_{a,(3,4,2)}]\\
&\quad + 2[y_{[-],(1,3)}, y_{\wedge,(4,2)}] + 2[z_1, x_{a,(2,3,4)}] + 2[x_{a,(1,2,3)}, z_4] + 2[z_3, x_{a,(1,4,2)}]\\
&\quad + 2[x_{a,(1,4,3)}, z_2] + [x_{b,(2,3,1)}, z_4] + [x_{b,(3,1,2)}, z_4]
\end{aligned} \tag{2.7}$$

where

$$dy_{\wedge,(i,j)} = z_i \wedge z_j, \qquad y_{[-],(i,j)} = [z_i, z_j]$$

$$dx_{a,(i,j,k)} = [y_{\wedge,(i,j)}, z_k] - (-1)^{|z_i|} z_j \wedge y_{[-],(j,k)} - (-1)^{|z_j|+|z_i||z_j|} z_j \wedge y_{[-],(i,k)}$$

$$dx_{b,(i,j,k)} = z_i \wedge y_{[-],(j,k)} - (-1)^{|z_i|}[y_{\wedge,(i,j)}, z_k] - (-1)^{|z_j|+|z_i||z_k|}[z_j, y_{\wedge,(i,k)}]$$

□

The signs missing in each term of (2.7) can be computed as follows. These signs arise in three ways:

- Firstly, the products in Δ^+ come with the usual Koszul signs.
- Secondly, one has those signs corresponding to γ in Equation (2.6).
- Thirdly, to simplify the expression, for the three terms on the final line, we use the (anti)commutativity of the generating cooperations. This introduces signs coming from the (signed) identities

$$[x, y] = -(-1)^{|x||y|}[y, x] \qquad \text{and} \qquad x \wedge y = (-1)^{|x||y|} y \wedge x.$$

Our final example will illustrate the close connection of Massey products with spectral sequences. We point the reader to [56, 10.3.7] for some of the basic background on this example.

Definition 2.2.14. The *dual numbers operad* is the quadratic operad $\mathscr{D}$ presented as

$$\mathscr{D} := \mathscr{F}(\Bbbk\triangle, \triangle \circ \triangle),$$

where $\triangle$ is an arity 1 element of homological degree 1.

Algebras over this operad are precisely the bicomplexes, i.e., chain complexes (A, d) equipped with an operation $\triangle : A \to A$ such that $\triangle^2 = 0$ and $d\triangle + \triangle d = 0$. The dual numbers operad is Koszul, and its Koszul dual cooperad is cofree conilpotent on a single generator,

$$\mathscr{D}^{\text{i}} = \mathscr{F}^c(s\triangle).$$

In particular, this cooperad has no corelations and is concentrated in degree 1.

Example 2.2.15. We shall compute the Massey products of the dual numbers operad. The arity 1 component of $\mathscr{D}^{\text{i}}$ is

$$\mathscr{D}^{\text{i}}(1) = \bigoplus \Bbbk\delta_n,$$

where δ_n has weight n and degree $2n$. Since

$$\Delta^+(\delta_n) = \sum_{k+l=n} (\delta_k; \delta_l),$$

where $l \geq 1$ and $k \geq 0$, it follows that

$$D(\delta_n) = (\triangle; \delta_{n-1}) \quad \text{for all } n \geq 2.$$

Therefore, the δ_n-indexing system is given by $\{a_{\delta_i} : 0 < i < n\}$, with the relation $da_{\delta_i} = \triangle(a_{\delta_{i-1}})$. This is almost the definition of the d_{n-1}-differential in the spectral sequence associated to the bicomplex $(A, d, \triangle)$. More precisely, one can check that if $x \in \langle y \rangle_{\delta_n}$ is defined in $H_*(A)$, then y survives to the E_{n-1}-page of the associated spectral sequence and $d_{n-1}(y) = [x]$. □

Remark 2.2.16. In our higher-order Massey products framework for Koszul operads, there is nothing special about the Koszul dual cooperad $\mathscr{P}^{\text{i}}$ aside from it being a very useful resolution. In principle, starting with any conilpotent cooperad $\mathscr{C}$ together with a choice of twisting morphism $\tau : \mathscr{C} \to \mathscr{P}$, it is possible to define a *relative Massey inductive map*

$$D : \mathscr{C} \xrightarrow{\Delta^+} \mathscr{C} \circ \mathscr{C} \xrightarrow{\tau \circ \text{id}} \mathscr{P} \circ \mathscr{C}.$$

From this, one defines *relative Massey products* following, *mutatis mutandis*, the same recipe we gave in the Koszul case. Taking $\mathscr{C} = B\mathscr{P}$ to be the bar construction of $\mathscr{P}$ and $\tau : B\mathscr{P} \to \mathscr{P}$ the canonical twisting morphism, this allows for defining Massey products for non-Koszul operads.

2.2.1 Elementary properties of the operadic Massey products

In this section, we collect some elementary properties of the operadic Massey products. First, we show that Massey product sets do not depend on the initial choice of cycles in the defining system (Prop. 2.2.17). Then, that morphisms of $\mathscr{P}$-algebras preserve Massey products (Prop. 2.2.18). In particular, quasi-isomorphisms induce bijections of the corresponding Massey product sets. This provides an obstruction for two $\mathscr{P}$-algebras to be weakly equivalent. In particular, a nontrivial Massey product provides an obstruction to formality. At the end of the section, we collect a few elementary properties of the Massey products that might be useful elsewhere (Prop. 2.2.21).

For the next few results, we fix a cooperation $\Gamma^c \in \left(\mathscr{P}^{\text{i}}\right)^{(n)}(k)$. We say that a Massey product set $\langle x_1, \ldots, x_k \rangle_{\Gamma^c}$ is *defined* if it is non-empty, that is, if there is some defining system for the Massey product; *trivial* if it contains the zero homology class; and *non-trivial* if it is defined and does not contain the zero homology class.

First, we shall show that Massey product sets do not depend on the initial choice of cycles in the defining system.

Proposition 2.2.17. *Let A be a $\mathscr{P}$-algebra. Suppose that $x_1, \ldots, x_k \in H_*(A)$ are homogeneous elements such that the Massey product set $\langle x_1, \ldots, x_k \rangle_{\Gamma^c}$ is defined. For each $x \in \langle x_1, \ldots, x_k \rangle_{\Gamma^c}$ and each choice of cycle representative $\overline{x_i}$ for x_i, one has a defining system $\{a_\beta\}$ for x such that $a_{\text{id},(i)} = \overline{x_i}$*

Proof. Let $\{b_\beta\}$ be a defining system for a Massey product $x \in \langle x_1, \ldots, x_k \rangle_{\Gamma^c}$. We shall construct, by induction on the weight of the elements of the defining system, a defining system $\{a_\beta\}$ for a Massey product $x \in \langle x_1, \ldots, x_k \rangle_{\Gamma^c}$ such that $a_{\text{id},(i)} = \overline{x_i}$ and that $a_{\Gamma^c,(1,\ldots,k)}$ is homologous to $b_{\Gamma^c,(1,\ldots,k)}$.

For the first step, simply fix $a_{\text{id},(i)} = \overline{x_i}$. Since $a_{\text{id},(i)}$ and $b_{\text{id},(i)}$ are both choices of representative for x_i, it follows that $a_{\text{id},(i)} - b_{\text{id},(i)}$ is nullhomologous, which means that there is a $c_{\text{id},(i)} \in A$ such that

$$d c_{\text{id},(i)} = a_{\text{id},(i)} - b_{\text{id},(i)}.$$

The family $\{a_{\text{id},(i)}\}$ gives the first inductive step. Now, suppose that for all indexes $\left(\mu, (i_1, \ldots, i_k)\right) \in I(\Gamma^c)$ with the weight of μ strictly less than N, with $1 < N < n$, where n is the weight of Γ^c, we have constructed $a_{\mu,(i_1,\ldots,i_k)}, c_{\mu,(i_1,\ldots,i_k)} \in A$ such that

$$d\left(a_{\mu,(i_1,\cdots,i_k)}\right) = \sum \zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)}, \ldots, a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right),$$

where $D(\mu) = \sum (\zeta; \zeta_1, \ldots, \zeta_m; \sigma)$, and

$$d c_{\mu,(i_1,\ldots,i_k)} = a_{\mu,(i_1,\ldots,i_k)} - b_{\mu,(i_1,\ldots,i_k)} + Q_{\mu,(i_1,\ldots,i_k)},$$

where $Q_{\mu,(i_1,\ldots,i_k)}$ is the sum:

$$\sum\sum \zeta\left(x_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)}, \ldots, x_{\zeta_j,\left(i_{\sigma^{-1}(v_1+\cdots+v_{j-1}+1)},\ldots,i_{\sigma^{-1}(v_1+\cdots+v_{j-1}+v_j)}\right)} \ldots, x_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right).$$

Here, the outer summation is indexed by

$$D(\mu) = \sum (\zeta; \zeta_1, \ldots, \zeta_m; \sigma),$$

and for each term $(\zeta;\zeta_1,\ldots,\zeta_m;\sigma)$, the inner sum is taken over every possible choice of tuple

$$\left(x_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots,i_{\sigma^{-1}(v_1)}\right)},\ldots,x_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}\ldots,x_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right),$$

where one of the

$$x_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}$$

is precisely

$$c_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)},$$

anything to the left of it in the tuple is

$$a_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)},$$

and anything to its right is

$$b_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}.$$

Now, let $\left(\mu,(i_1,\ldots,i_k)\right)\in I(\Gamma^c)$ have μ of weight N. Then, $Q_{\mu,(i_1,\ldots,i_k)}$ is well defined, because the cooperations appearing in its defining tuple have weight strictly less than μ. Its boundary is as follows:

$$dQ_{\mu,(i_1,\ldots,i_k)}=db_{\mu,(i_1,\ldots,i_k)}+\sum\zeta\left(a_{\zeta_1,(i_{\sigma^{-1}(1)}),\ldots i_{\sigma^{-1}(v_1)}},\ldots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right). \quad (2.8)$$

Indeed, the sum $dQ_{\mu,(i_1,\ldots,i_k)}$ can be separated into two parts: a telescoping part that converges to the right-hand side of the equation above, and a second part that can be divided into subsums each vanishing by arguments similar to the proof of Theorem 2.2.8. Now, from Equation (2.8), we deduce that the element

$$\sum\zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)},\ldots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right)$$

where the sum ranges over $D(\mu)=\sum(\zeta;\zeta_1,\ldots,\zeta_m;\sigma)$, is a cycle. Therefore, there is an element $a'_{\mu,(i_1,\ldots,i_k)}\in A$ such that

$$da'_{\mu,(i_1,\ldots,i_k)}=\sum\zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)},\ldots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(k)}\right)}\right).$$

Define a cycle

$$e_{\mu,(i_1,\ldots,i_k)}=a'_{\mu,(i_1,\ldots,i_k)}-b_{\mu,(i_1,\ldots,i_k)}+Q_{\mu,(i_1,\ldots,i_k)}.$$

Then there is an element $e'_{\mu,(i_1,\ldots,i_k)}\in A$ such that $e'_{\mu,(i_1,\ldots,i_k)}$ is homologous to $e_{\mu,(i_1,\ldots,i_k)}$, that is, such that

$$dc_{\mu,(i_1,\ldots,i_k)}=e'_{\mu,(i_1,\ldots,i_k)}-e_{\mu,(i_1,\ldots,i_k)},$$

and

$$a_{\mu,(i_1,\ldots,i_k)}=a'_{\mu,(i_1,\ldots,i_k)}-e'_{\mu,(i_1,\ldots,i_k)}.$$

It follows that

$$d\left(a_{\mu,(i_1,\cdots,i_k)}\right) = \sum \zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\dots i_{\sigma^{-1}(v_1)}\right)},\dots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\dots+v_{m-1}+1)},\dots,i_{\sigma^{-1}(k)}\right)}\right),$$

where the sum ranges over $D\left(\mu\right) = \sum\left(\zeta;\zeta_1,\dots,\zeta_m;\sigma\right)$, and furthermore, that

$$dc_{\mu,(i_1,\dots,i_k)} = a_{\mu,(i_1,\dots,i_k)} - b_{\mu,(i_1,\dots,i_k)} + Q_{\mu,(i_1,\dots,i_k)}.$$

This concludes the induction step. To finish, consider the element $Q_{\Gamma^c,(i_1,\dots,i_k)}$. This is defined by the same logic as above, and its boundary satisfies

$$dQ_{\Gamma^c,(i_1,\dots,i_k)} = a_{\Gamma^c,(i_1,\dots,i_k)} - b_{\Gamma^c,(i_1,\dots,i_k)}.$$

Therefore, the elements $a_{\Gamma^c,(i_1,\dots,i_k)}$ and $b_{\Gamma^c,(i_1,\dots,i_k)}$ are homologous, as we wanted to prove. □

Proposition 2.2.18. *A morphism of $\mathscr{P}$-algebras $f : A \to B$ preserves Massey products. That is, if $x_1,\dots,x_k \in H_*(A)$ are homogeneous elements such that the Massey product set $\langle x_1,\dots,x_k\rangle_{\Gamma^c}$ is defined, then $\left\langle f_*(x_1),\dots,f_*(x_k)\right\rangle_{\Gamma^c}$ is also defined, and moreover*

$$f_*\langle x_1,\dots,x_k\rangle_{\Gamma^c} \subseteq \left\langle f_*(x_1),\dots,f_*(x_k)\right\rangle_{\Gamma^c}.$$

If furthermore f is a quasi-isomorphism, then f_* induces a bijection between the corresponding Massey product sets.

Proof. Assume that the Massey product set $\langle x_1,\dots,x_k\rangle_{\Gamma^c}$ is defined. Then, any defining system $\{a_\alpha\}$ for $\langle x_1,\dots,x_k\rangle_{\Gamma^c}$ produces a defining system $\left\{f(a_\alpha)\right\}$ for $\left\langle f_*(x_1),\dots,f_*(x_k)\right\rangle_{\Gamma^c}$, because f commutes with the operadic structure maps and the differentials. Therefore, if $\langle x_1,\dots,x_k\rangle_{\Gamma^c}$ is defined, then $\left\langle f_*(x_1),\dots,f_*(x_k)\right\rangle_{\Gamma^c}$ is also defined, and the containment $f_*\langle x_1,\dots,x_k\rangle_{\Gamma^c} \subseteq \left\langle f_*(x_1),\dots,f_*(x_k)\right\rangle_{\Gamma^c}$ follows.

Next, assume that f is a quasi-isomorphism and let us prove that the corresponding Massey product sets are in bijective correspondence. Let $\left\{b_\beta\right\}$ be a defining system for a Massey product $y \in \left\langle f_*(x_1),\dots,f_*(x_k)\right\rangle_{\Gamma^c}$. We shall construct, by induction on the weight of the elements of the defining system, a defining system $\left\{a_\beta\right\}$ for a Massey product $x \in \langle x_1,\dots,x_k\rangle_{\Gamma^c}$ such that $f\left(a_{\Gamma^c,(1,\dots,k)}\right)$ is homologous to $b_{\Gamma^c,(1,\dots,k)}$, and therefore $f_*(x) = y$.

Let $a_{\mathrm{id},(i)}$ be any representative for x_i. This means that $f(a_{\mathrm{id},(i)}) - b_{\mathrm{id},(i)}$ is nullhomologous, which means that there is a $c_{\mathrm{id},(i)} \in B$ such that

$$dc_{\mathrm{id},(i)} = f\left(a_{\mathrm{id},(i)}\right) - b_{\mathrm{id},(i)}.$$

The family $\left\{a_{\mathrm{id},(i)}\right\}$ gives the first inductive step. Now, suppose that for all indexes $\left(\mu,(i_1,\dots,i_k)\right) \in I\left(\Gamma^c\right)$ with the weight of μ strictly less than N, with $1 < N < n$, where n is the weight of Γ^c, we have constructed $a_{\mu,(i_1,\dots,i_k)} \in A$ and $c_{\mu,(i_1,\dots,i_k)} \in B$ such that

$$d\left(a_{\mu,(i_1,\cdots,i_k)}\right) = \sum \zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\dots i_{\sigma^{-1}(v_1)}\right)},\dots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\dots+v_{m-1}+1)},\dots,i_{\sigma^{-1}(k)}\right)}\right),$$

where $D\left(\mu\right) = \sum\left(\zeta;\zeta_1,\dots,\zeta_m;\sigma\right)$, and

$$dc_{\mu,(i_1,\dots,i_k)} = f\left(a_{\mu,(i_1,\dots,i_k)}\right) - b_{\mu,(i_1,\dots,i_k)} + Q_{\mu,(i_1,\dots,i_k)},$$

where $Q_{\mu,(i_1,\ldots,i_k)}$ is the sum:

$$\sum\sum\zeta\left(x_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)},\ldots,x_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}\ldots,x_{\zeta_m,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{m-1}+1\right)},\ldots,i_{\sigma^{-1}(k)}\right)}\right).$$

Here, the outer summation is indexed by

$$D(\mu)=\sum(\zeta;\zeta_1,\ldots,\zeta_m;\sigma),$$

and for each term $(\zeta;\zeta_1,\ldots,\zeta_m;\sigma)$, the inner sum is taken over every possible choice of tuple

$$\left(x_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots,i_{\sigma^{-1}(v_1)}\right)},\ldots,x_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}\ldots,x_{\zeta_m,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{m-1}+1\right)},\ldots,i_{\sigma^{-1}(k)}\right)}\right),$$

where one of the

$$x_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}$$

is precisely

$$c_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)},$$

anything to the left of it in the tuple is

$$f\left(a_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}\right),$$

and anything to its right is

$$b_{\zeta_j,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+1\right)},\ldots,i_{\sigma^{-1}\left(v_1+\cdots+v_{j-1}+v_j\right)}\right)}.$$

Now, let $\left(\mu,(i_1,\ldots,i_k)\right)\in I(\Gamma^c)$ have μ of weight N. Then, $Q_{\mu,(i_1,\ldots,i_k)}$ is well defined, because the cooperations appearing in its defining tuple have weight strictly less than μ. Its boundary is as follows:

$$dQ_{\mu,(i_1,\ldots,i_k)}=db_{\mu,(i_1,\ldots,i_k)}+\sum\zeta\left(f(a_{\zeta_1,(i_{\sigma^{-1}(1)}),\ldots i_{\sigma^{-1}(v_1)})}),\ldots,f(a_{\zeta_m,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{m-1}+1\right)},\ldots,i_{\sigma^{-1}(k)}\right)})\right). \tag{2.9}$$

Indeed, the sum $dQ_{\mu,(i_1,\ldots,i_k)}$ can be separated into two parts: a telescoping part that converges to the right-hand side of the equation above, and a second part that can be divided into subsums each vanishing by arguments similar to the proof of Theorem 2.2.8. Now, from Equation (2.9), we deduce that the element

$$\sum\zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)},\ldots,a_{\zeta_m,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{m-1}+1\right)},\ldots,i_{\sigma^{-1}(k)}\right)}\right)$$

where the sum ranges over $D(\mu)=\sum(\zeta;\zeta_1,\ldots,\zeta_m;\sigma)$, is a cycle. Therefore, there is an element $a'_{\mu,(i_1,\ldots,i_k)}\in A$ such that

$$da'_{\mu,(i_1,\ldots,i_k)}=\sum\zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots i_{\sigma^{-1}(v_1)}\right)},\ldots,a_{\zeta_m,\left(i_{\sigma^{-1}\left(v_1+\cdots+v_{m-1}+1\right)},\ldots,i_{\sigma^{-1}(k)}\right)}\right).$$

Define a cycle
$$e_{\mu,(i_1,\dots,i_k)} = f\left(a'_{\mu,(i_1,\dots,i_k)}\right) - b_{\mu,(i_1,\dots,i_k)} + Q_{\mu,(i_1,\dots,i_k)}.$$
Then there is an element $e'_{\mu,(i_1,\dots,i_k)} \in A$ such that $f\left(e'_{\mu,(i_1,\dots,i_k)}\right)$ is homologous to $e_{\mu,(i_1,\dots,i_k)}$, that is, such that
$$dc_{\mu,(i_1,\dots,i_k)} = f\left(e'_{\mu,(i_1,\dots,i_k)}\right) - e_{\mu,(i_1,\dots,i_k)},$$
and
$$a_{\mu,(i_1,\dots,i_k)} = a'_{\mu,(i_1,\dots,i_k)} - e'_{\mu,(i_1,\dots,i_k)}.$$
It follows that
$$d\left(a_{\mu,(i_1,\cdots,i_k)}\right) = \sum \zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\dots i_{\sigma^{-1}(v_1)}\right)},\dots,a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\dots,i_{\sigma^{-1}(k)}\right)}\right),$$
where the sum ranges over $D(\mu) = \sum(\zeta;\zeta_1,\dots,\zeta_m;\sigma)$, and furthermore, that
$$dc_{\mu,(i_1,\dots,i_k)} = f\left(a_{\mu,(i_1,\dots,i_k)}\right) - b_{\mu,(i_1,\dots,i_k)} + Q_{\mu,(i_1,\dots,i_k)}.$$
This concludes the induction step. To finish, consider the element $Q_{\Gamma^c,(i_1,\dots,i_k)}$. This is defined by the same logic as above, and its boundary satisfies
$$dQ_{\Gamma^c,(i_1,\dots,i_k)} = f\left(a_{\Gamma^c,(i_1,\dots,i_k)}\right) - b_{\Gamma^c,(i_1,\dots,i_k)}.$$
Therefore, the elements $f\left(a_{\Gamma^c,(i_1,\dots,i_k)}\right)$ and $b_{\Gamma^c,(i_1,\dots,i_k)}$ are homologous, as we wanted to prove. □

Recall that two $\mathscr{P}$-algebras are *weakly-equivalent*, or *quasi-isomorphic*, if there is a zig-zag of $\mathscr{P}$-algebra quasi-isomorphisms between them. From the two previous results, we can deduce the following.

Corollary 2.2.19. *There is a bijection between the Massey product sets of weakly-equivalent $\mathscr{P}$-algebras.*

Proof. Suppose one has a zig-zag of quasi-isomorphisms of $\mathscr{P}$-algebras
$$A \xrightarrow{f} B \xleftarrow{g} C.$$
By Proposition 2.2.18, there is a bijection between a Massey product set $\langle x_1,\dots,x_k\rangle_{\Gamma^c}$ in A and the corresponding Massey product set $\left\langle f_*(x_1),\dots,f_*(x_k)\right\rangle_{\Gamma^c}$ at B. Since g is a quasi-isomorphism, there exists y_i such that $g_*(y_i) = f_*(x_i)$. Therefore, a second application of Proposition 2.2.18 yields that the Massey product set $\left\langle f_*(y_1),\dots,f_*(y_k)\right\rangle_{\Gamma^c}$ is in bijection with $\langle x_1,\dots,x_k\rangle_{\Gamma^c}$. □

Recall that, for an operad $\mathscr{P}$ without differential, the homology of a $\mathscr{P}$-algebra is also a $\mathscr{P}$-algebra. We say that a $\mathscr{P}$-algebra is *formal* if it is weakly equivalent to its homology endowed with the induced $\mathscr{P}$-algebra structure (and trivial differential). In general, the Massey products of $\mathscr{P}$-algebras with trivial differential are always trivial because they have no relations that exist at the chain level but not at the homological level. In particular, the Massey products of the homology of a $\mathscr{P}$-algebra are all trivial whenever they are defined. From this, we immediately deduce the following result.

Corollary 2.2.20. *If a $\mathscr{P}$-algebra has a nontrivial Massey product, then it is not formal.*

Proof. Assume that a $\mathscr{P}$-algebra A has a nontrivial Massey product. Since the homology of A has a zero differential, it must be that all of its Massey products are trivial. Therefore, by Corollary 2.2.19 it cannot be quasi-isomorphic to A. $\square$

Next, we collect some elementary properties satisfied by the operadic Massey products. These are similar to some of those explained in [53] and, more recently, in [77, p. 325]. The proofs follow from the definitions, and are left to the reader.

Proposition 2.2.21. *Let A be a $\mathscr{P}$-algebra, $\Gamma^c \in \left(\mathscr{P}^{i}\right)^{(n)}(r)$, and $x_1,\ldots,x_r \in H_*(A)$ be homogeneous elements such that $\langle x_1,...,x_r\rangle_{\Gamma^c}$ is defined. Then the following assertions hold.*

1. *(Homological linearity) If $k \in \mathbb{k}$ is a scalar, then for all $1 \le i \le r$,*

$$k\langle x_1,\ldots,x_r\rangle_{\Gamma^c} \subseteq \langle x_1,\ldots,kx_i,\ldots,x_r\rangle_{\Gamma^c}.$$

2. *(Equivariance) For every permutation $\sigma \in \mathbb{S}_r$, there is a bijection*

$$\langle x_1,\ldots,x_r\rangle_{\Gamma^c_n\cdot\sigma} = (-1)^{\varepsilon(\sigma^{-1})}\langle x_{\sigma^{-1}(1)},\ldots x_{\sigma^{-1}(r)}\rangle_{\Gamma^c_n},$$

where $(-1)^{\varepsilon(\sigma^{-1})}$ is the Koszul sign appearing by permuting the variables according to σ^{-1}.

Proof. 1. Let $\{a_\alpha\}$ be a defining system for a Massey product $x \in \langle x_1,\ldots,x_r\rangle_{\Gamma^c}$. Consider a new defining system

$$b_{(\zeta,j_1,\ldots j_s)} = \begin{cases} ka_{(\zeta,j_1,\ldots j_s)} & \text{if } j_l = i \text{for some } l. \\ a_{(\zeta,j_1,\ldots j_s)} & \text{otherwise.}\end{cases}$$

In particular, one has

$$b_{(\mathrm{id},j)} = \begin{cases} ka_{(\mathrm{id},i)} & \text{for } i=j. \\ a_{(\mathrm{id},j)} & \text{otherwise.}\end{cases}$$

so $\{b_\alpha\}$ is a defining system for a Massey product in $\langle x_1,\ldots,kx_i,\ldots,x_r\rangle_{\Gamma^c}$. Furthermore, $b_{(\Gamma^c,1,2,\ldots,r)} = ka_{(\Gamma^c,1,2,\ldots,r)}$ so the corresponding Massey product is kx.

2. Let $\{a_\alpha\}$ be a defining system for a Massey product $x \in \langle x_1,\ldots,x_r\rangle_{\Gamma^c}$. Consider a new defining system

$$b_{(\zeta,j_1,\ldots j_s)} := a_{(\zeta,\sigma^{-1}(j_1),\ldots\sigma^{-1}(j_s))}.$$

This is then a defining system for $(-1)^{\varepsilon(\sigma^{-1})}\langle x_{\sigma^{-1}(1)},\ldots x_{\sigma^{-1}(r)}\rangle_{\Gamma^c_n}$ and the result follows. $\square$

2.2.2 Massey products along morphisms of operads and formality

In this section, we shall discuss pullbacks and pushforwards of Massey products along morphisms of operads and give some applications to formality.

Before we begin, it will be helpful to remark some observations. Let $f:\mathscr{P}\to\mathscr{Q}$ be a morphism of weighted operads. In this case, taking Koszul dual cooperad is functorial, and therefore there is an induced map $f^{\mathrm{i}}:\mathscr{F}(E,R)=\mathscr{P}^{\mathrm{i}}\to\mathscr{Q}^{\mathrm{i}}=\mathscr{F}(F,S)$. Moreover, there is a commutative diagram

$$\begin{tikzcd} \mathscr{P}^{\mathrm{i}} \arrow[r, "\Delta^+"] \arrow[d, "f^{\mathrm{i}}"] & \mathscr{P}^{\mathrm{i}}\circ\mathscr{P}^{\mathrm{i}} \arrow[r, "\kappa\circ\mathrm{id}"] \arrow[d, "f^{\mathrm{i}}\circ f^{\mathrm{i}}"] & \mathscr{P}\circ\mathscr{P}^{\mathrm{i}} \arrow[d, "f\circ f^{\mathrm{i}}"] \\ \mathscr{Q}^{\mathrm{i}} \arrow[r, "\Delta^+"] & \mathscr{Q}^{\mathrm{i}}\circ\mathscr{Q}^{\mathrm{i}} \arrow[r, "\kappa'\circ\mathrm{id}"] & \mathscr{Q}\circ\mathscr{Q}^{\mathrm{i}} \end{tikzcd}$$

From this, we conclude that the Massey inductive map D commutes with f^{i}. Secondly, because the category of graded vector spaces admits finite colimits, on the level of algebras, f descends to an adjoint pair

$$f_! : \mathscr{P} - \mathsf{Alg} \leftrightarrows \mathscr{Q} - \mathsf{Alg} : f^*.$$

The functor f^* preserves the underlying chain complex of the $\mathscr{Q}$-algebras, and therefore there is a chain map $f^*(A) \to A$ which is just the identity morphism. We define next another chain map $h : A \to f_!(A)$. Given a $\mathscr{P}$-algebra A, the unit of the adjunction above is a morphism of $\mathscr{P}$-algebras

$$A \to f^* f_!(A).$$

Forgetting the $\mathscr{P}$-algebra structure and recalling that f^* preserves the underlying chain complex, there is a chain map

$$h : A \to f_!(A).$$

Pullbacks of Massey products. For any $\mathscr{Q}$-algebra B, the $\mathscr{P}$-Massey products on $f^*(B)$ induce $\mathscr{Q}$-Massey products on B. Since the underlying chain complex of both algebras is the same, we can prove the following result.

Proposition 2.2.22. *Let $f : \mathscr{P} \to \mathscr{Q}$ be a morphism of weighted operads, B a $\mathscr{Q}$-algebra, and $\Gamma^c \in \left(\mathscr{P}^{i}\right)^{(n)}(r)$. Suppose that $x_1, \ldots, x_r \in H_*\left(f^*(B)\right)$ are homogeneous elements such that the Massey product set $\langle x_1, \ldots, x_r \rangle_{\Gamma^c}$ is defined. If $\mathscr{P}$ is finite type arity-wise and $f^{i}(\Gamma^c) \neq 0$, then under the identification of $f^*(B)$ and B as chain complexes, we have*

$$\langle x_1, \ldots, x_r \rangle_{\Gamma^c} \subseteq \langle x_1, \ldots, x_r \rangle_{f^{i}(\Gamma^c)}.$$

If f^{i} is injective, this is an equality.

Proof. Let $x \in \langle x_1, \ldots, x_r \rangle_{\Gamma^c}$ have a defining system $\left\{b_{\mu,(i_1,\ldots,i_r)}\right\}$. We shall prove the statement by constructing a defining system for x as a $f^{\text{i}}(\Gamma^c)$-Massey product. If f^{i} is injective, then the statement is trivial. Indeed, since D commutes with f^{i}, we may obtain the desired defining set $\left\{b_{f^{\text{i}}(\mu),(i_1,\ldots,i_r))}\right\}$ by setting $b_{f^{\text{i}}(\mu),(i_1,\ldots,i_r))} := b_{\mu,(i_1,\ldots,i_r))}$. The converse is also true; each defining set $\left\{b_{f^{\text{i}}(\mu),(i_1,\ldots,i_r))}\right\}$ is a defining set for a Γ^c-Massey product.

If f^{i} is not injective, then the set before may fail to be a defining system. Two problems may arise. Firstly, $f^{\text{i}}(\mu)$ may be zero. In this case, however, any term coming from D in which $f^{\text{i}}(\mu)$ plays a role will also vanish, so we may safely remove any term of the form $b_{f^{\text{i}}(\mu),(i_1,\ldots,i_r))}$ from the defining system altogether. Secondly, f^{i} may fail to preserve linear independence. We circumvent this problem as follows. To fix notation, write $\mathscr{P} = \mathscr{F}(E,R)$ and $\mathscr{Q} = \mathscr{F}(F,S)$. The map f^{i} is a map of weighted quadratic cooperads and, in particular, it sends cogenerators to cogenerators, $\left(f^{\text{i}}\right)^{(1)} : E \to F$. We shall assume that $\Bbbk$-linear bases of E and F are chosen such that the image of the basis elements of E are precisely the first m basis vectors $\{u_i\}$ of F, and further that the other basis elements of F are not in the image of $\left(f^{\text{i}}\right)^{(1)}$, and that the rest of the elements of the basis of E are in the kernel of $\left(f^{\text{i}}\right)^{(1)}$. These bases now, as explained in the second paragraph of Section 2.2, extend to bases of the operads $\mathscr{P}$ and $\mathscr{Q}$ using appropriate symmetric tree monomials.

The image of f^{i} now entirely lies in the span of tree monomials labeled by the first m basis elements of F. This means that there is now a canonical (with respect to this choice of basis) linear section s of f^{i} defined only on this codomain that preserves the cocomposition. This section is given by sending sums of tree monomials labeled by the first m basis elements of F to sums of tree monomials of the same shape labeled by the corresponding first m basis elements of E.

The section s induces a bijection between the indexing sets $I\left(f^{\mathrm{i}}(\Gamma^c),(1,\ldots,r)\right)$ and $I(\Gamma^c,(1,\ldots,r))$. Define $b_{\mu,(i_1,\ldots i_n)}$ to be $b_{s(\mu),(i_1,\ldots i_n)}$. This provides a defining system for x. $\square$

Remark 2.2.23. This means that if f^{i} is injective and $f^*(B)$ has nontrivial Massey products, then so does B.

Example 2.2.24. Consider the natural weighted operad morphism $f:\mathsf{Lie}\to\mathsf{Ass}$. For any differential graded associative algebra A, the differential graded Lie algebra $f^*(A)$ is the chain complex A equipped with the bracket $[a,b]=ab-(-1)^{|a||b|}ba$ for all homogeneous $a,b\in A$. Now, recall from Example 2.2.11 that $\mathsf{Lie}^{\mathrm{i}}(n)^{(n-1)}$ is generated by an element denoted τ_n^c. Since on the level of Koszul dual cooperads, the map $f^{\mathrm{i}}:\mathsf{Lie}^{\mathrm{i}}\to\mathsf{Ass}^{\mathrm{i}}$ is the linear dual of the canonical operad map $\mathsf{Ass}\to\mathsf{Com}$, one can verify that $f^{\mathrm{i}}(\tau_n^c)=\sum_{\sigma\in\mathbb{S}_n}\mu_n^c\cdot\sigma$, where μ_n^c is the generator of $\left(\mathsf{Ass}^{\mathrm{i}}\right)^{(n-1)}(n)$ as an $\mathbb{S}_n$-module. This map is injective and therefore, it follows from Proposition 2.2.22 that

$$\langle x_1,\ldots,x_n\rangle_{\tau_n^c}=\langle x_1,\ldots,x_n\rangle_{\sum_{\sigma\in\mathbb{S}_n}\mu_n^c\cdot\sigma}.$$

This can be used in two ways. Firstly, we can deduce that if $f^*(A)$ admits a nontrivial Lie–Massey bracket, then A admits a (nonclassical) associative bracket and so is not formal. In general, on the other hand, most of the time if A has a nontrivial (classical) Massey product, we cannot deduce the existence of a Massey product on $f^*(A)$ or its formality. However, if A admits a Massey product of the form $\langle x,x,\ldots,x\rangle_{\mu_n^c}$, referred to in the literature as *Kraine's* $\langle x\rangle^n$ *product*, or *iterated* Massey product, then it follows that it admits a product of the form $\langle x,\ldots,x\rangle_{\sum_{\sigma\in\mathbb{S}_n}\mu_n^c\cdot\sigma}$, and so we can deduce that $f^*(A)$ is not formal. $\square$

Example 2.2.25. We can use Prop 2.2.22 to compute the Massey products for the commutative operad Com. Consider the canonical weighted operad map $f:\mathsf{Ass}\to\mathsf{Com}$. As mentioned in the example before, the map $f^{\mathrm{i}}:\mathsf{Ass}^{\mathrm{i}}\to\mathsf{Com}^{\mathrm{i}}$ is the linear dual of the natural operad morphism $g:\mathsf{Lie}\to\mathsf{Ass}$. This last operad map is an embedding, so it follows that f^{i} is surjective. Thus, for any $\tau\in\left(\mathsf{Com}^{\mathrm{i}}\right)^{(n)}(r)$ there exists $\mu\in\left(\mathsf{Ass}^{\mathrm{i}}\right)^{(n)}(r)$ such that $f^{\mathrm{i}}(\mu)=\tau$, and it follows from Proposition 2.2.22 that

$$\langle x_1,\ldots,x_k\rangle_\tau\subseteq\langle x_1,\ldots,x_k\rangle_\mu,$$

whenever the products above make sense. $\square$

Pushforwards of Massey products. For any $\mathscr{P}$-algebra A, the $\mathscr{P}$-Massey products on A induce $\mathscr{Q}$-Massey products on $f_!(A)$.

Proposition 2.2.26. *Let $f:\mathscr{P}\to\mathscr{Q}$ be a morphism of weighted operads, A a $\mathscr{P}$-algebra, and $\Gamma^c\in\left(\mathscr{P}^i\right)^{(n)}(r)$. Suppose that $x_1,\ldots,x_r\in H_*(A)$ are homogeneous elements such that the Massey product set $\langle x_1,\ldots,x_r\rangle_{\Gamma^c}$ is defined. Then, the $\mathscr{Q}$-Massey product set $\langle h_*(x_1),\ldots,h_*(x_r)\rangle_{f^{\mathrm{i}}(\Gamma^c)}$ is also defined, and*

$$h_*\langle x_1,\ldots,x_r\rangle_{\Gamma^c}\subseteq\langle h_*(x_1),\ldots,h_*(x_r)\rangle_{f^{\mathrm{i}}(\Gamma^c)}.$$

Proof. One constructs a defining system for a $f^{\mathrm{i}}(\Gamma^c)$-Massey product in essentially the same manner as in the proof of Proposition 2.2.22.

Let $x\in\langle x_1,\ldots,x_r\rangle_{\Gamma^c}$ have a defining system $\left\{b_{\mu,(i_1,\ldots,i_r)}\right\}$. We shall prove the statement by constructing a defining system for x as a $f^{\mathrm{i}}(\Gamma^c)$-Massey product. If f^{i} is injective, then the statement is trivial. Indeed, since D commutes with f^{i}, we may obtain the desired defining set $\left\{b_{f^{\text{i}}(\mu),(i_1,\dots,i_r))}\right\}$ by setting $b_{f^{\text{i}}(\mu),(i_1,\dots,i_r))} := h\left(b_{\mu,(i_1,\dots,i_r))}\right)$. If f^{i} is not injective, then the set before may fail to be a defining system. Two problems may arise. Firstly, $f^{\text{i}}(\mu)$ may be zero. In this case, however, any term coming from D in which $f^{\text{i}}(\mu)$ plays a role will also vanish, so we may safely remove any term of the form $b_{f^{\text{i}}(\mu),(i_1,\dots,i_r))}$ from the defining system altogether. Secondly, f^{i} may fail to preserve linear independence. We fix this problem as follows.

To fix notation, write $\mathscr{P} = \mathscr{F}(E,R)$ and $\mathscr{Q} = \mathscr{F}(F,S)$. The map f^{i} is a map of weighted quadratic cooperads and, in particular, it sends cogenerators to cogenerators, $\left(f^{\text{i}}\right)^{(1)} : E \to F$. We shall assume that $\mathbb{k}$-linear bases of E and F are chosen such that the image of the basis elements of E are precisely the first m basis vectors $\{u_i\}$ of F, and further that the other basis elements of F are not in the image of $\left(f^{\text{i}}\right)^{(1)}$, and that the rest of the elements of the basis of E are in the kernel of $\left(f^{\text{i}}\right)^{(1)}$. These bases now, as explained in the second paragraph of Section 2.2, extend to bases of the operads $\mathscr{P}$ and $\mathscr{Q}$ using appropriate symmetric tree monomials.

The image of f^{i} now entirely lies in the span of tree monomials labeled by the first m basis elements of F. This means that there is now a canonical (with respect to this choice of basis) linear section s of f^{i} defined only on this codomain that preserves the cocomposition. This section is given by sending sums of tree monomials labeled by the first m basis elements of F to sums of tree monomials of the same shape labeled by the corresponding first m basis elements of E.

The section s induces a bijection between the indexing sets $I\left(f^{\text{i}}(\Gamma^c),(1,\dots,r)\right)$ and $I(\Gamma^c,(1,\dots,r))$. Define $b_{\mu,(i_1,\dots i_n)}$ to be $\left(b_{s(\mu),(i_1,\dots i_n)}\right)$. This provides a defining system for x. $\square$

Example 2.2.27. Consider the natural operad map $f : \mathsf{Lie} \to \mathsf{Ass}$. For any differential graded Lie algebra $\mathfrak{g}$, the differential graded associative algebra $f_!(\mathfrak{g})$ is the universal enveloping algebra of A. Recall that there is an embedding of graded vector spaces (in fact, graded Lie algebras) $h : \mathfrak{g} \to f_!(\mathfrak{g})$. Since on the level of Koszul dual cooperads, the map $f^{\text{i}} : \mathsf{Lie}^{\text{i}} \to \mathsf{Ass}^{\text{i}}$ is the linear dual of the forgetful functor $\mathsf{Ass} \to \mathsf{Com}$, one can verify that $f^{\text{i}}(\tau_n^c) = \sum_{\sigma\in\mathbb{S}_n} \mu_n^c \cdot \sigma$, where μ_n^c is the generator of $\left(\mathsf{Ass}^{\text{i}}\right)^{(n-1)}(n)$ as an $\mathbb{S}$-module. Therefore,

$$h_*\langle x_1,\dots,x_k\rangle_{\tau_n^c} \subseteq \langle h_*(x_1),\dots,h_*(x_k)\rangle_{\sum_{\sigma\in\mathbb{S}_n}\mu_n^c\cdot\sigma}.$$

$\square$

A criterion for formality. In this section, we characterize the formality of a $\mathscr{P}$-algebra in terms of its Sullivan model, whenever it makes sense (Prop. 2.2.29 below). Although the result is presumably well-known to experts, we could not find a precise statement in the literature. The connection of the characterization with this paper is that it gives us a method to construct non-formal algebras with vanishing higher operadic Massey products of all orders. We leave the task of finding explicit examples to the interested reader.

The Sullivan model of a $\mathscr{P}$-algebra exists after imposing some connectivity assumptions on the operad and the algebra itself. To our knowledge, the first work in this direction is [55], where $\mathscr{P}$ is assumed to be Koszul and concentrated in degree 0, while the most general results are achieved in [20], were $\mathscr{P}$ is not required to be Koszul, but satisfy a mild connectivity requirement called being *tame*. We stick to the setting of [20], but will also require $\mathscr{P}$ to be Koszul to make use of infinity structures. An operad $\mathscr{P}$ is *r-tame* for a fixed integer $r \geq 0$ if for every $n \geq 2$,

$$\mathscr{P}(n)_q = 0 \quad \text{for all } q \geq (n-1)(1+r).$$

The operads Ass, Com and Lie are examples of 0-tame operads, as well as their minimal models. The Gerstenhaber operad Gerst is 1-tame. The main results of [20] combine to read as follows.

Theorem 2.2.28. [20] *Every r-connected algebra over an r-tame operad has a Sullivan minimal model, unique to isomorphism.*

Now, suppose that $\mathscr{P}$ is an r-tame Koszul operad and A is an r-connected finite type algebra for some $r \geq 0$. Furthermore, suppose that A is $\mathscr{P}_\infty$-quasi-isomorphic to a minimal $\mathscr{P}_\infty$-algebra H with differential δ whose components $\delta^{(n)}$ vanish for all $n \geq 2$. Then there is a quasi-isomorphism of $\mathscr{P}^{\text{i}}$-coalgebras

$$(\mathscr{P}^{\text{i}}(A), \delta) \xrightarrow{\simeq} (\mathscr{P}^{\text{i}}(H), \delta').$$

Taking the linear dual, one obtains a quasi-isomorphism of $\mathscr{P}^!$-algebras

$$(\mathscr{P}^!(H), d') \xrightarrow{\simeq} (\mathscr{P}^!(A), d).$$

The differential d' is decomposable and concentrated in weight 2. Therefore, $(\mathscr{P}^!(H), d)$ is a minimal Sullivan model for $(\mathscr{P}^!(A), d)$, which is the dual of the bar construction on A. This model is unique up to isomorphism, as mentioned before. We sum this discussion up in the following characterization.

Proposition 2.2.29. *Let $\mathscr{P}$ be an r-tame Koszul operad for some $r \geq 0$, and A an r-connected finite type $\mathscr{P}$-algebra. Then A is formal if, and only if, the Sullivan minimal model of the dual of the bar construction on A admits a differential concentrated in weight 2.*

2.3 Differentials in the $\mathscr{P}$-Eilenberg–Moore spectral sequence

Aside from providing obstructions to formality, one of the major uses of higher Massey products is in providing a concrete description of the differentials in the classical Eilenberg–Moore spectral sequence. The following is a classical result of May [63], compare also [83], but adapted to the notation of this paper.

Theorem 2.3.1. *Let A be a differential graded associative algebra, and let $x_1, \dots x_n$ be homology classes such that the Massey product set $\langle x_1, \dots, x_n \rangle$ is non-empty. Then, the element $[sx_1 | \cdots | sx_n]$ survives to the E^{n-1}-page of the Eilenberg–Moore spectral sequence of A, and furthermore, the suspension sx of any representative of $x \in \langle x_1, \dots x_n \rangle$ is a representative for $d^{n-1}[sx_1 | \cdots | sx_n]$.*

An analogous statement for differential graded Lie algebras appears in [3]. Our following result generalizes these statements to all algebras over a Koszul (in fact, quadratic) operad. Recall from Section 2.1.1.1 the construction of the spectral sequence. We will sometimes confuse homology classes with representatives to lighten the notation.

Theorem 2.3.2. *Let A be a $\mathscr{P}$-algebra, and $x_1, \dots x_r$ homology classes such that the Massey product set $\langle x_1, \dots, x_r \rangle_{\Gamma^c}$ is defined for a cooperation $\Gamma^c \in \mathscr{P}^i(r)^{(n)}$. Then the element*

$$\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r \in \left(\mathscr{P}^i\right)^{(n)}(r) \otimes H_*(A)^{\otimes r}$$

survives to the E^{n-1} page in the $\mathscr{P}$-Eilenberg–Moore spectral sequence, and for $x \in \langle x_1, \dots x_n \rangle$

$$d^{n-1}\left(\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r\right) \in (-1)^{n-2}\,[\text{id} \otimes x]\,.$$

Our proof of this is an adaption of the classical one, so we shall therefore make use of the Staircase Lemma [53, Lemma 2.1], which we briefly recall next.

Lemma 2.3.3. *Let $A = (A_{*,*}, d', d'')$ be a bicomplex, denote by d the differential on its total complex, and fix $c_1, \ldots, c_n$ homogeneous elements in A. Suppose that $d' c_s = d'' c_{s+1}$ for $1 \leq s \leq n-1$, and define $c := c_1 - c_2 + \cdots + (-1)^{n-1} c_n$. Then, $dc = d'c + d''c = d'' c_1 + (-1)^{n-1} d' c_n$, and furthermore, in the spectral sequence $\{(E^r, d^r)\}$ associated to the bicomplex, if $d'' c_1 = 0$ then c_1 survives to E^n, and $d^n[c_1] = (-1)^{n-1}[d' c_n]$.*

Our approach to proving Theorem 2.3.2 is therefore to construct a sequence $c_1, \ldots c_{r-1}$ satisfying the conditions of the Staircase Lemma.

Proof of Theorem 2.3.2. First, fix a defining system $\{a_{\mu,(k_1,\ldots,k_i)}\}$ for the element $x \in \langle x_1, \ldots, x_r \rangle_{\Gamma^c}$. For each s between 1 and $n-1$, we will define c_s in terms of this defining system and the auxiliary maps

$$\triangle_s : \mathscr{P}^{\mathrm{i}} \xrightarrow{\triangle} \mathscr{P}^{\mathrm{i}} \circ \mathscr{P}^{\mathrm{i}} \xrightarrow{p_s \circ \mathrm{id}} \left(\mathscr{P}^{\mathrm{i}}\right)^{(s)} \circ \mathscr{P}^{\mathrm{i}},$$

where p_s is the projection onto the weight s component. More precisely, the element c_s is defined as

$$c_s := \sum \left[\zeta^{(n-s)} \otimes a_{\mu_1,(\sigma^{-1}(1),\ldots\sigma^{-1}(i_1))} \otimes \cdots \otimes a_{\mu_m,(\sigma^{-1}(i_1+\cdots+i_{m-1}+1),\ldots\sigma^{-1}(r))}\right],$$

where $\triangle_{n-s}(\Gamma^c) = \sum \left(\zeta^{(s)}; \mu_1, \ldots, \mu_m; \sigma\right)$. In particular, $c_1 = [\Gamma^c \otimes a_{\mathrm{id},(1)} \otimes \cdots \otimes a_{\mathrm{id},(r)}]$, and

$$c_{n-1} = \sum \left(s\zeta^{(1)}\right) \otimes a_{\mu_1,(\sigma^{-1}(1),\ldots,\sigma^{-1}(v_1))} \otimes \cdots \otimes a_{\mu_m,(\sigma^{-1}(v_1+\cdots+v_{m-1}+1),\ldots,\sigma^{-1}(r))},$$

where $D(\Gamma^c) = \sum \left(\zeta^{(1)}; \mu_1, \ldots, \mu_m; \sigma\right)$ with $\zeta^{(1)} \in E$ and thus $s\zeta^{(1)} \in sE \subset \left(\mathscr{P}^{\mathrm{i}}\right)^{(1)}$. To finish, we must verify that the conditions of the Staircase Lemma 2.3.3 are met. Denote by ∂ the external differential on $\mathscr{P}^{\mathrm{i}}(A)$, and by $d^\bullet$ its internal differential. Then, since $d(a_{\mathrm{id},(i)}) = 0$ for each i, it follows that $d^\bullet c_1 = 0$. A routine calculation shows that $d^\bullet c_{s+1} = \partial c_s$ for each s. It follows from the Staircase Lemma that

$$d_{n-1}[c_1] = (-1)^n[\partial c_{n-1}] = (-1)^n[x].$$

In the expression of c_{n-1}, the element ζ is in the image of the twisting morphism κ. In particular, this implies that it is of weight 1, and so $\partial c_{n-1} \in \mathrm{id} \otimes \langle x_1, \ldots, x_r \rangle_{\Gamma^c}$. This finishes the proof. $\square$

The formality of a dg algebra of some type is well-known to be related to the collapse of the associated Eilenberg–Moore-type spectral sequence, see for instance [43] for the commutative case, and [28] for the Lie case. The following statement generalizes these results.

Theorem 2.3.4. *The Eilenberg–Moore spectral sequence of a formal $\mathscr{P}$-algebra over a Koszul operad collapses at the E^2-page. The same is true for formal $\mathscr{P}_\infty$-algebras.*

Proof. Since every $\mathscr{P}$-algebra is a $\mathscr{P}_\infty$-algebra, we prove the result for $\mathscr{P}_\infty$-algebras. Let A be a formal $\mathscr{P}_\infty$-algebra, and denote by $H = H_*(A)$ its homology as a $\mathscr{P}$-algebra. Then there are $\mathscr{P}_\infty$-quasi-isomorphisms $A \leftrightarrows H$, or equivalently, $\mathscr{P}^{\mathrm{i}}$-coalgebra quasi-isomorphisms

$$\mathscr{P}^{\mathrm{i}}(A) \leftrightarrows \mathscr{P}^{\mathrm{i}}(H).$$

Recall that the codifferential $\delta_H(\mu, -)$ of $\mathscr{P}^{\mathrm{i}}(H)$ vanishes unless $\mu \in \mathscr{P}^{\mathrm{i}}$ has weight 1. By comparison, both Eilenberg–Moore spectral sequences are isomorphic from the first page.

Therefore, it suffices to consider the case where A has no internal differential and $\delta_A^{(i)}$ vanishes when $i \neq 1$. We now check that the differential d^i in the Eilenberg–Moore spectral sequence vanishes for $i \geq 2$. To do so, we will use the standard relative cycles and boundaries spaces,

$$Z_p^r = F_p\mathscr{P}^{\mathrm{i}}(A) \cap \delta^{-1}\left(F_{p-r}\mathscr{P}^{\mathrm{i}}(A)\right) \quad \text{and} \quad D_p^r = F_p\mathscr{P}^{\mathrm{i}}(A) \cap \delta\left(F_{p+r}\mathscr{P}^{\mathrm{i}}(A)\right).$$

The differential d^r in the successive pages of the spectral sequence is induced by the restrictions of the differential δ of $\mathscr{P}^{\mathrm{i}}(A)$ to Z_p^r, as shown below:

$$\begin{tikzcd}
Z_p^r \arrow[r, "\delta"] \arrow[d, two heads] & Z_{p-r}^r \arrow[d, two heads] \\
E_p^r = Z_p^r / Z_{p-1}^{r-1} + D_p^{r-1} \arrow[r, dashed, "d^r"] & E_{p-r}^k = Z_{p-r}^r / Z_{p-r-1}^{r-1} + D_{p-r}^{r-1}
\end{tikzcd}$$

Fix some $r \geq n$. To check that $d^r = 0$, we will fix an element $x \in Z_p^r$ and find a representative y of the class $[x] \in E_p^r$ such that

$$\delta(y) \in Z_{p-r-1}^{r-1} + D_{p-r}^{r-1} = F_{p-r-1}\mathscr{P}^{\mathrm{i}}(A) \cap \delta^{-1}\left(F_{p-r-1}\mathscr{P}^{\mathrm{i}}(A)\right) + F_{p-r}\mathscr{P}^{\mathrm{i}}(A) \cap \delta\left(F_{p-1}\mathscr{P}^{\mathrm{i}}(A)\right).$$

Indeed, write $x = x_1 + \cdots + x_p$ where each $x_i \in \mathscr{P}^{\mathrm{i}}(i) \otimes A^{\otimes i}$. Now, since $\delta(x) \in F_{p-r}\mathscr{P}^{\mathrm{i}}(A)$, it follows that $\delta\left(x_{p-r+1} + \cdots + x_p\right) = 0$. Thus, we take $y = x - \left(x_{p-r+1} + \cdots + x_p\right)$ as a representative of the form we needed, finishing the proof. □

The converse to Theorem 2.3.4 in general is not true, it fails even in the associative case and some examples are computed in some of the references given before the statement of the theorem.

Remark 2.3.5. Massey products sometimes completely determine formality. For the case of the dual numbers operad, the Massey products are precisely the differentials in the spectral sequence associated to the bicomplex. So if the differentials all vanish, the spectral sequence must collapse on the E^2-page.

2.4 Higher-order operadic Massey products and $\mathscr{P}_\infty$-structures

In this section, we fix a Koszul operad $\mathscr{P}$. In this case, there is a natural relationship between the higher-order operadic Massey products and $\mathscr{P}_\infty$-structures on the homology of the $\mathscr{P}$-algebras.

Let A be a $\mathscr{P}$-algebra, and denote by H its homology. Since $\mathscr{P}$ has no operadic differential, H is a $\mathscr{P}$-algebra in a natural way. It is well-known that the homotopy transfer theorem (in its various forms) extend this $\mathscr{P}$-algebra structure on H to a $\mathscr{P}_\infty$-structure that retains the quasi-isomorphism class of A as a $\mathscr{P}_\infty$-algebra. In this paper, we mainly focus on D. Petersen's extension [73] of T. Kadeishvili's classical transfer theorem [50], which is recalled in Theorem 2.1.2. See also [56, Section 10.3]. It is a common misconception to expect that higher-order Massey products sets of the sort $\langle x_1, \ldots, x_r \rangle$ are related to $\mathscr{P}_\infty$-structure maps θ_r induced on the homology H via the homotopy transfer theorem by the clean formula

$$\pm\theta_r(x_1, \ldots, x_r) \in \langle x_1, \ldots, x_r \rangle.$$

At this level of generality, the assertion is incorrect. However, it is true for secondary Massey products, as shown in [70, Theorem 3.9].

Let us make the connection between infinity structures and higher-order Massey products more precise. First, recall that codifferentials on the cofree conilpotent $\mathscr{P}^{¡}$-coalgebra $\mathscr{P}^{¡}(A)$ are in bijective correspondence with $\mathscr{P}_\infty$-structures on the chain complex A [56, Theorem 10.1.13].

Definition 2.4.1. Let A be a $\mathscr{P}$-algebra, $\Gamma^c \in \left(\mathscr{P}^{¡}\right)^{(n)}(r)$, and $x_1,...,x_r$ homogeneous elements of $H = H_*(A)$ for which the Γ^c-Massey product set $\langle x_1,\ldots,x_r\rangle_{\Gamma^c}$ is defined. A given $\mathscr{P}_\infty$ structure δ on H for which A and H are quasi-isomorphic is said to *recover* the Massey product element $x \in \langle x_1,\ldots,x_r\rangle_{\Gamma^c}$ if, up to sign,

$$\delta_r\left(\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r\right) = x.$$

We begin by showing below that given a higher-order Massey product $x \in \langle x_1,\ldots,x_r\rangle_{\Gamma^c}$, there is always a choice of $\mathscr{P}_\infty$ structure on H quasi-isomorphic to A which recovers x. In general, however, an arbitrary $\mathscr{P}_\infty$ structure on H quasi-isomorphic to A only recovers a given higher-order Massey product element up to multiplications of lower arity. Our proof strategy is very similar to the proof in [16], where the authors demonstrated this result in the associative case. In the result below, we require the operad to be reduced for Theorem 2.1.2 to apply.

Theorem 2.4.2. *Let A be an algebra over a reduced Koszul operad $\mathscr{P}$, and let H be its homology. Let $\Gamma^c \in \left(\mathscr{P}^{i}\right)^{(n)}(r)$, and assume that $x_1,...,x_r$ are $r \geq 3$ homogeneous elements of H for which the Γ^c-Massey product set $\langle x_1,\ldots x_r\rangle_{\Gamma^c}$ is defined. Let $x \in \langle x_1,\ldots x_r\rangle_{\Gamma^c}$. Then:*

(i) For any $\mathscr{P}_\infty$ structure δ on H quasi-isomorphic to A, we have

$$\delta^{(n)}\left(\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r\right) = x + \Phi,$$

where $\Phi \in \sum_{i=1}^{n-1} \operatorname{Im}\left(\delta^{(i)}\right)$.

(ii) If $\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_l}$ are linearly independent in the corresponding copy of $\mathscr{P}^{i} \otimes_{\mathbb{S}_l} A^{\otimes l}$, where $(\mu,(i_1,...,i_l)) \in I(\Gamma^c)$, then there is a choice of $\mathscr{P}_\infty$ structure δ on H which recovers x.

Proof. (i) We will construct a $\mathscr{P}_\infty$ structure on H recovering x via the procedure established in the proof of Theorem 2.1.2. We shall continue to use the notation of that proof. First, we choose a defining system $\{a_\alpha\}$ for the Massey product element $x \in \langle x_1,\ldots,x_r\rangle_{\Gamma^c}$. We proceed by induction on arity, starting in arity 1 with δ^1 initially defined as the coderivation corresponding to the strict $\mathscr{P}$-algebra structure on H induced from it being the homology of A, and defining f as any chain quasi-isomorphism $H \to A$ extending the choice $f(x_1) = a_{\mathrm{id},(1)}, f(x_2) = a_{\mathrm{id},(2)},\ldots, f(x_r) = a_{\mathrm{id},(r)}$. This defines a map $F_1 : \mathscr{P}^{¡}(1) \otimes H \to A$, since $\mathscr{P}^{¡}(1) = \Bbbk$. We give next the arity 2 step. This step is not needed for the inductive procedure, but we include it because we think it sheds light on the general case. Recall that the algorithm of Theorem 2.1.2 automatically determines the multiplication on H, but there are choices for F_2. First, we make the following observation. If $\left(s\mu,(i,j)\right)$ appears in the Γ^c-indexing system, then $\overline{\gamma_A}(\mu; x_i, x_j) = 0$. This is because $D\left(s\mu\right) = \left(\mu, \mathrm{id}, \mathrm{id}\right)$, and therefore

$$d\,a_{s\mu,(i,j)} = \kappa\left(s\mu\right)\left(a_{\mathrm{id},(i)}, a_{\mathrm{id},(j)}\right),$$

which implies that $\overline{\gamma_A}(\mu; x_i, x_j) \in H$ admits a lift to A which is a coboundary, which implies that it is 0 on homology. It therefore follows that $\left(F^1 \circ \delta^1\right)_2$ is 0 when applied to $[s\mu \otimes x_i \otimes x_j]$. On the other hand,

$$\left(\delta^1 \circ F^1\right)_2 [s\mu \otimes x_i \otimes x_j] = \kappa\left(s\mu\right)\left(F_1\left(x_i\right), F_1\left(x_j\right)\right) = d a_{s\mu,(i,j)}.$$

So we choose $F_2 : \mathscr{P}^{\text{i}}(2) \otimes H^{\otimes 2} \to A$ to extend the choice $F_2\left(\mu, x_i, x_j\right) = a_{\mu,(i,j)}$. The general case is similar. Our inductive hypothesis has the following two parts:

$$F_l\left[\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_l}\right] = a_{\mu,(i_1,\ldots,i_l)}, \text{where} \left(\mu, (i_1, \ldots i_l)\right) \in I\left(\Gamma^c\right) \text{and } l < n, \tag{2.10}$$

and

$$\delta_l^{k-1}\left[\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_l}\right] = 0, \text{ where } \left(\mu, (i_1, \ldots i_l)\right) \in I\left(\Gamma^c\right) \text{ for } l \leq n. \tag{2.11}$$

We verified these two items in the arity 2 case in the previous paragraph. Next, we shall compute $(\delta_A \circ F)_n\left[\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}\right]$. The map $(\delta_A \circ F)_n$ is precisely the composite

$$\mathscr{P}^{\text{i}}(H) \xrightarrow{\Delta(H)} \mathscr{P}^{\text{i}} \circ \mathscr{P}^{\text{i}}(H) \xrightarrow{\mathscr{P}^{\text{i}}(f)} \mathscr{P}^{\text{i}}(A) \xrightarrow{\kappa(A)} \mathscr{P}(A) \xrightarrow{\gamma_A} A.$$

The arity n component of f is 0, and in particular $\mathscr{P}^{\text{i}}(f)\left(\text{id}; \mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}\right) = 0$. It follows that $(\delta_A \circ F)_n$ sends the class of $\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}$ to the same element as the following map does:

$$\mathscr{P}^{\text{i}}(H) \xrightarrow{\Delta^+} \mathscr{P}^{\text{i}} \circ \mathscr{P}^{\text{i}}(H) \xrightarrow{\mathscr{P}^{\text{i}}(f)} \mathscr{P}^{\text{i}}(A) \xrightarrow{\kappa(A)} \mathscr{P}(A) \xrightarrow{\gamma_A} A.$$

The map above is tightly related to the Massey inductive map D. Indeed, the image of $\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}$ is given by

$$\sum \zeta\left(f\left(\zeta_1 \otimes x_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes x_{i_{\sigma^{-1}(v_1)}}\right), \ldots, f\left(\zeta_m \otimes x_{i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)}} \otimes \cdots \otimes x_{i_{\sigma^{-1}(n)}}\right)\right),$$

where

$$D\left(\mu\right) = \sum \left(\zeta; \zeta_1, \ldots, \zeta_m; \sigma\right).$$

By the first assumption of our inductive hypothesis (2.10), this is equal to

$$\sum \zeta\left(a_{\zeta_1,\left(i_{\sigma^{-1}(1)},\ldots,i_{\sigma^{-1}(v_1)}\right)}, \ldots, a_{\zeta_m,\left(i_{\sigma^{-1}(v_1+\cdots+v_{m-1}+1)},\ldots,i_{\sigma^{-1}(n)}\right)}\right).$$

It follows from the definition of a defining system that this is equal to

$$d a_{\mu,(x_{i_1},\ldots,x_{i_n})}.$$

The second assumption of our inductive hypothesis (2.11) implies that $\left(F^{n-1} \circ \delta^{n-1}\right)_l\left[\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_l}\right] = 0$, so we have that

$$\left(F \circ \delta^{n-1} - \delta_A \circ F\right)_n\left[\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}\right] = -d a_{\mu,(x_{i_1},\ldots,x_{i_n})}.$$

Therefore, there is no obstruction to obtaining a lift F_n such that $F_n(\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}) = a_{\mu,(x_{i_1},\ldots,x_{i_n})}$. Notice that the algorithm also tells us that $\delta_n^n\left[\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_n}\right] = 0$ (the projection of a boundary in homology).

Next, we shall verify that $\delta_{n+1}^n\left(\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_{n+1}}\right) = 0$ when $(\mu, (i_1, \ldots, i_{n+1})) \in I(\Gamma^c)$. Because the arity $(n+1)$-component of δ^n comes from the $\mathscr{P}$-algebra structure induced on H from A, we have that

$$\delta^n\left(\mu \otimes x_{i_1} \otimes \cdots \otimes x_{i_{n+1}}\right) = \overline{\gamma_A}\left(\kappa\left(\mu\right); x_{i_1}, \ldots, x_{i_{n+1}}\right).$$

But if $\kappa(\mu)$ is non-zero, then μ must be of weight 1. It then follows that $D(\mu) = (\mu; \mathrm{id}, \dots, \mathrm{id})$. So, by the same argument as in the arity 2 case, we conclude that $\overline{\gamma_A}(\kappa(\mu); x_{i_1}, \dots, x_{i_{n+1}}) = 0$.

(ii) Consider any $\mathscr{P}_\infty$ quasi-isomorphism $H \xrightarrow{\simeq} A$ and the corresponding quasi-isomorphism of $\mathscr{P}^{\text{i}}$-coalgebras

$$\mathscr{P}^{\text{i}}(H) \xrightarrow{\simeq} \mathscr{P}^{\text{i}}(A).$$

The induced morphism of $\mathscr{P}$-Eilenberg–Moore spectral sequences is, at the E_1 level, the identity on $\mathscr{P}^{\text{i}}(H)$. By comparison, all the terms in both spectral sequences are also isomorphic. Now, it follows from Theorem 2.3.2 that if $\langle x_1, \dots x_r\rangle_{\Gamma^c}$ is nonempty, then the element $[\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r]$ survives to the $(n-1)$-page (E^{n-1}, d^{n-1}). Moreover, given any $x \in \langle x_1, \dots x_r\rangle_{\Gamma^c}$, one has

$$d^{n-1}\overline{\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r} = (-1)^r \overline{x}.$$

Here, $\overline{\cdot}$ denotes the class in E^{n-1}. In other words, there exists $\Phi \in F_{n-1}\mathscr{P}^{\text{i}}(H)$ such that

$$\delta_H\left(\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r + \Phi\right) = x.$$

Applying the counit $\epsilon_H : \mathscr{P}^{\text{i}}(H) \to H$ to both sides, we obtain

$$m_H\left(\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r + \Phi\right) = x.$$

Write $m_H = \sum_{i\geq 2} \partial_H^{(i)}$, and decompose $\Phi = \sum_{i=2}^{r-1} \phi_i$ with $\phi_i \in \mathscr{P}^{\text{i}}(H)^{(i)}$. By a word length argument,

$$\delta_H^{(n)}\left(\Gamma^c \otimes x_1 \otimes \cdots \otimes x_r\right) + \sum_{i=2}^{r-1} \delta_H^{(i)}(\phi_i) = x.$$

This completes the proof. $\square$

CHAPTER 3

An obstruction theory for strictly commutative algebras in positive characteristic

Abstract

This is the first in a sequence of articles exploring the relationship between commutative algebras and E_∞-algebras in characteristic p and mixed characteristic. In this paper we lay the groundwork by defining a new class of cohomology operations over $\mathbb{F}_p$ called cotriple products, generalising Massey products. We compute the secondary cohomology operations for a strictly commutative dg-algebra and the obstruction theories these induce, constructing several counterexamples to characteristic 0 behaviour, one of which answers a question of Campos, Petersen, Robert-Nicoud and Wierstra. We construct some families of higher cotriple products and comment on their behaviour. Finally, we distingush a subclass of cotriple products that we call higher Steenrod operations and conclude with our main theorem, which says that E_∞-algebras can be rectified if and only if the higher Steenrod operations vanish coherently.

3.1 Introduction

Since its introduction by Quillen [76] and Sullivan [85]; rational homotopy theory has probably become the single most successful subfield of algebraic topology. The latter approach to the theory reduces the study of rational topological spaces to that of commutative dg-algebras.

One of the central observations of [85] was that it was possible to replace the E_∞-algebra $C^\bullet(X,\mathbb{Q})$ with a strictly commutative model $A_{PL}(X)$. In positive characteristic, this is not possible because the Steenrod operations act as obstructions, and, in particular, for spaces we have that the zeroth Steenrod power operation P^0 never vanishes (see Proposition 3.5.1). The main goal of this article is to investigate the precise relationship between strictly commutative and E_∞-algebras in positive characteristic.

It is here that we introduce the key idea of this article; that is, that rectifiablity in characteristic p should be studied in a similar manner to formality in characteristic 0. In characteristic

0, Massey products provide higher obstructions to formality. Massey triple products correspond to relations in the cohomology algebra and higher Massey products correspond to syzygies between those relations. For an E_∞-algebra to be rectifiable, its Steenrod operations must vanish, but we also will need to impose conditions ensuring that syzygies between them should vanish as well. This will be complicated by the fact that strictly commutative dg-algebras do have one Steenrod operation - the Frobenius map - and some higher cohomology operations coming from that.

It is worth noting that, despite not modelling spaces directly, commutative algebras retain some significant advantages over E_∞-algebras. The most significant of these is computational. Commutative algebras, due to the fact that they are generated by a single operation with the simplest possible behaviour, of all the algebraic objects appearing in geometry and topology, are the most amenable to computer algebra techniques. In contrast, E_∞-algebras are generated by infinitely many operations, are generally large, unwieldy, and very few operations, with some notable exceptions [69], on them have been, or are likely to be, automated.

We develop an obstruction theory, determined by *cotriple products*, a non-linear generalisation of both Massey products and Steenrod operations, designed to handle the extra invariants given by the Frobenius map. More precisely, these operations are defined to be differentials in a spectral sequence associated to the operadic cotriple resolution. We show that they also admit a description that is highly reminscent of defining systems for Massey products, complete with well-defined indeterminacies.

We then study cotriple resolutions in the context of the commutative operad in positive characteristic. The main result is that the secondary cotriple products consist of traditional Massey products along with two additional types of operation coming from the Frobenius map. We compute the indeterminacy of these additional operations and use them to construct examples of commutative dg-algebras that are formal over $\mathbb{Q}$ (Example 3.4.10) but not over $\mathbb{F}_p$, and algebras with a divided power structure on cohomology that are not weakly equivalent to a divided powers algebra (Example 3.4.11).

In [17, Section 0.3], the authors pose the following question: *If two commutative dg algebras are quasi-isomorphic as associative dg algebras, must they be quasi-isomorphic also as commutative dg algebras?* They then settle the question in characteristic 0.

Theorem 3.1.1. *[17, Theorem A] Let A and B be two commutative dg algebras over a field of characteristic zero. Then, A and B are quasi-isomorphic as associative dg algebras if and only if they are also quasi-isomorphic as commutative dg algebras.*

It is perhaps unsurprising that cotriple products offer a useful perspective in characteristic p. In particular, we are able to provide an explicit counterexample (Theorem 3.4.15) in characteristic 2 which extends to a general method for constructing counterexamples in characteristic p for odd primes.

We then turn to the study of higher order operations, this turns out to be much more subtle than with Massey produts, as the existence of a higher *homotopy invariant* operation does not necessarily follow automatically from the vanishing of a lower order one (Example 3.4.26). We define an infinite family of higher operations coming from the iterations of the Frobenius map and compute their indeterminacy.

Finally, we conclude with a necessary and sufficient condition for an E_∞-algebra to be quasi-isomorphic to a commutative algebra. We first define higher Steenrod operations as subset of the cotriple operations. Then we have the following rectification result.

Theorem W. Let A be an E_∞-algebra over $\mathbb{F}_p$. Then A is rectifiable if and only if its higher Steenrod operations vanish coherently.

The coherent vanishing condition, which is inspired by the following theorem of Deligne, Griffiths, Morgan and Sullivan.

Theorem 3.1.2. *[23] Let A be a commutative dg-algebra over $\mathbb{Q}$. Let $\mathfrak{m} = (\mathrm{Sym}(\bigoplus_{i=0}^\infty V_i), d)$ be the minimal model for A. Then A is formal if and only if, there is in each V_i a complement B_i to the cocycles Z_i, $V_i = Z_i \oplus B_i$, such that any closed form, a, in the ideal, $I((\bigoplus_{i=0}^\infty B_i)$, is exact.*

One can also ask whether the homotopy type of a commutative dg-algebra coincides with its homotopy type as an E_∞-algebra. While this is true in characteristic 0, we believe that in positive characteristic the answer is no. This can be shown with the aid of third order cotriple products, and we intend to return to it in separate work.

Structure of the article

First we recall some preliminaries on commutative dg-algebras, E_∞-algebras and Steenrod operations. Then in Section 3, we define cotriple products, our non-linear generalisation of Massey products, and show that they are homotopy invariant for well-behaved classes of algebras. In Section 4, we study the case of strictly commutative dg-algebras and construct various counterexamples. Finally in Section 5, we prove our rectification result.

Notation and conventions

In this paper, we work on the category of unbounded chain complexes over some base field or ring with cohomological convention. That is, the differential $d : A^* \to A^{*+1}$ of a chain complex (A, d) is of degree 1. The degree of a homogeneous element x is denoted by $|x|$. The symmetric group on n elements is denoted $\mathbb{S}_n$. We follow the Koszul sign rule. That is, the symmetry isomorphism $U \otimes V \xrightarrow{\cong} V \otimes U$ that identifies two graded vector spaces is given on homogeneous elements by $u \otimes v \mapsto (-1)^{|u||v|} v \otimes u$. Algebras over operads are always differential graded (dg) and cohomological. We will frequently omit the adjective "dg" and assume it is implicitly understood. Finally, when A is a ring of characteristic p for a prime p, A^p will be the subring $\{a^p : a \in A\}$.

This is a short article and we do not intend to load it excessively with recollections; so therefore we refer to [56] for the definition of an operad and other basic notions.

3.2 Preliminaries

3.2.1 Three flavours of algebra over an operad

Divided power algebras were first introduced for the commutative operad by Cartan [18], generalised to the general operadic setting by Fresse [34] and studied further by Ikonicoff [48]. In this section, we carefully define them and, in particular, we explain the free graded commutative divided powers algebra on a free module.

Recall that an algebra A over a operad $\mathscr{P}$ is defined to be an algebra over the following monad

$$\mathscr{P}(V) = \bigoplus_{n\geq 0} \left(\mathscr{P}(n) \otimes V^{\otimes n}\right)_{\mathbb{S}_n},$$

where the coinvariants are taken with respect to the action of the symmetric group on $V^{\otimes n}$ given by permutation of factors in the tensor product. In characteristic zero, assuming $\mathscr{P}(n)$ is finitely generated as a representation of $\mathbb{S}_n$, this is the same monad taking invariants

$$\mathscr{P}(V) = \bigoplus_{n\geq 0} \left(\mathscr{P}(n) \otimes V^{\otimes n}\right)^{\mathbb{S}_n}.$$

However, when we are not working over a field of characteristic zero, these notions do not coincide. This motivates the following definition.

Definition 3.2.1. [34] Let A be dg-module over a commutative unital ring. We say that A is a *$\mathscr{P}$-algebra* if it is an algebra over the monad

$$\mathscr{P}(V) = \bigoplus_{n\geq 0} \left(\mathscr{P}(n) \otimes V^{\otimes n}\right)_{\mathbb{S}_n}.$$

An algebra over the monad

$$\Gamma\mathscr{P}(V) = \bigoplus_{n\geq 0} \left(\mathscr{P}(n) \otimes V^{\otimes n}\right)^{\mathbb{S}_n},$$

is referred to as a *divided powers-$\mathscr{P}$ algebra.*

In general, there is a universal map from coinvariants to invariants, the norm map

$$\mathscr{P}(V) \to \Gamma\mathscr{P}(V)$$

It follows that every divided powers $\mathscr{P}$-algebra is a $\mathscr{P}$-algebra. The image of the norm map is usually denoted $\Lambda\mathscr{P}(V)$ and is the third flavour of algebras.

We shall mainly be interested in the case $\mathscr{P} = \mathsf{Com}$, so it will be useful to have explicit descriptions in this case.

Example 3.2.2. When $\mathscr{P} = \mathsf{Com}$ and $V = \mathbb{F}_p$, with the single basis element x,

$$\Gamma\mathsf{Com}(V) = \begin{cases} \mathbb{F}_p[x_1, x_2, \dots]/(x_1^p, x_2^p, \dots) & \text{with } |x_k| = k|x|, \text{ when } |x| \text{ is even.} \\ \mathbb{F}_p[x]/(x^2) & \text{otherwise.} \end{cases}$$

3.2.2 E_∞-algebras and Steenrod operations

As a representation of $\mathbb{S}_n$, $\mathsf{Com}(n) = \Bbbk$ is not free. Unfortunately this means that the associated Schur functor behaves poorly in positive characteristic (see Example 3.4.1). The traditional approach to remedying this is to find a weak equivalence of operads $\mathscr{E} \xrightarrow{\sim} \mathsf{Com}$ such that, for each n, the action of $\mathbb{S}_n$ on $\mathscr{E}(n)$ is free. Any such operad is called an E_∞-operad. The model we shall use in this article is the Barratt-Eccles operad; where the action of the symmetric group is free, which we briefly recall. For more details, see [6].

Definition 3.2.3. The simplicial sets defining the Barratt-Eccles operad in each arity are of the form

$$\mathscr{E}(r)_n = \{(w_0, \dots, w_n) \in \mathbb{S}_r \times \cdots \times \mathbb{S}_r\}$$

equipped with face and degeneracy maps

$$d_i(w_0, \dots, w_n) = (w_0, \dots, w_{i-1}, \hat{w}_i, w_{i+1}, \dots, w_n)$$
$$s_i(w_0, \dots, w_n) = (w_0, \dots, w_{i-1}, w_i, w_i, w_{i+1}, \dots, w_n).$$

The group $\mathbb{S}_r$ acts on $\mathscr{E}(n)$ diagonally, that is to say if $\sigma \in \mathbb{S}_n$ and $(w_0, \dots, w_n) \in \mathscr{E}(n)$ then

$$(w_0, \dots, w_n) * \sigma = (w_0 * \sigma, \dots, w_n * \sigma)$$

Finally the compositions are also defined componentwise via the explicit composition law of

$$\gamma : \mathbb{S}(r) \otimes \mathbb{S}(n_1) \otimes \cdots \otimes \mathbb{S}(n_r) \to \mathbb{S}(n_1 + \cdots + n_r)$$
$$(\sigma, \sigma_1, \dots, \sigma_r) \mapsto \sigma_{n_1 \cdots n_r} \circ (\sigma_1 \times \cdots \times \sigma_r)$$

where $\sigma_{n_1 \cdots n_r}$ is the permutation that acts on $n_1 + \cdots + n_r$ elements, by dividing them into r blocks, the first of length n_1, the second of length n_2 and so on. It then rearranges the blocks according to σ, maintaining the order within each block.

Remark 3.2.4. The Barratt-Eccles operad as defined above is an operad in simplicial sets. However it becomes an operad in non-negatively graded chain complexes after applying the singular chains functor. When we work in cohomological grading and cochain complexes, it is therefore concentrated in non-positive degrees and is unbounded below. In this article, the notation $\mathscr{E}$ shall always refer to the operad in chain complexes.

One can check that the free algebra functor $\mathscr{E}(-)$ respects homotopy equivalences. In other words, if $V \xrightarrow{\sim} W$ is a homotopy equivalence of dg-modules; then $\mathscr{E}(V) \xrightarrow{\sim} \mathscr{E}(W)$ is a homotopy equivalence. Furthermore, the cohomology of $\mathscr{E}(V)$ has an additional grading induced by the operadic degree

$$\mathscr{E}(V)_i = \bigoplus_{r=1}^{\infty} \mathscr{E}(r)_i \otimes_{\mathbb{S}_r} V^{\otimes r}.$$

There is then an isomorphism $\mathrm{Sym}(H^*(V)) \xrightarrow{\sim} H^* \mathscr{E}(V)_0$, where $\mathrm{Sym}(-)$ is the symmetric algebra functor.

Definition 3.2.5. [65] Let V be a dg-module over $\mathbb{F}_p$. The Steenrod algebra on V is the cohomology group $\mathscr{A}(V) = H^*(\mathscr{E}(V))$.

Remark 3.2.6. Let V be a non-negatively graded dg-module. Then the Steenrod algebra $\mathscr{A}(V)$ will not be bounded below. However, if V is a non-positively graded dg-module, $\mathscr{A}(V)$ will also be concentrated in non-positive degrees.

Given an E_∞-algebra A, one has a map

$$\mathscr{A}(H^*(A)) \xrightarrow{H^*(\gamma)} H^*(A).$$

where γ is the E_∞-algebra map $\mathscr{E}(A) \xrightarrow{\gamma} A$. In other words, the cohomology of an E_∞-algebra always carries an action of the Steenrod algebra.

3.2.2.1 Rectification

There is a weak equivalence of operads $\phi : \mathscr{E} \xrightarrow{\sim} \mathsf{Com}$, so it is natural to ask whether or not the pair $(\phi^*, \phi_!)$ forms a Quillen equivalence between E_∞-algebras and Com-algebras. If there is, then *rectification* is said to occur. With coefficients in $\mathbb{Q}$, this is indeed the case; see for example [89]. In particular, this implies that that every E_∞-algebra A has a commutative model given by $\phi_!(A)$.

3.2.2.2 The homotopy theory of E_∞-algebras and commutative dg-algebras

In this subsection we shall discuss the existence of model structures on categories of $\mathscr{P}$-algebras and specialise to the cases of E_∞-algebras. One has the following general fact.

Theorem 3.2.7. *[45] Let $\mathscr{P}$ be an $\mathbb{S}$-split (or cofibrant) operad over a commutative ring R. Then the category of $\mathscr{P}$-algebras over R is a closed model category with quasi-isomorphisms as the weak equivalences and surjective maps as fibrations.*

The Barratt-Eccles operad is $\mathbb{S}$-split. This immediately gives the model structure on E_∞-algebras over $\mathbb{F}_p$.

Definition 3.2.8. The model category $E_\infty-\mathsf{alg}$ of E_∞-algebras is the category of algebras over the operad given by cochains on the Barratt-Eccles operad, in chain complexes over $\mathbb{F}_p$. It has quasi-isomorphisms of cochain complexes as weak equivalences and surjective maps as fibrations.

We have already mentioned that in characteristic 0, the homotopy theory of commutative dg-algebras coincides with that of E_∞-algebras. In positive characteristic the relationship is much more complex.

Quasi-isomorphisms of commutative dg-algebras are those algebra maps that are quasi-isomorphisms of chain complexes. The category of commutative algebras thus has a well-defined homotopy category given by localising at these quasi-isomorphisms. Promoting this to a model category is possible [82] if one considers the category of Com-algebras over the Eilenberg-MacLane spectrum $H\mathbb{F}_p$. This is equivalent to the category of dg-$\mathscr{P}$-algebras in R-modules. A weak equivalence in this setting is a weak equivalence of chain complexes, but neither fibrations or cofibrations are easy to describe.

3.3 Cotriple products

3.3.1 Sullivan algebras

In this subsection, we explain how to construct a semi-free, and therefore, in the presence of a model category, cofibrant, resolution of an algebra over an arbitrary operad $\mathscr{P}$. The following results are likely well known to experts but we could not locate a proof in the literature.

Definition 3.3.1. Let $\mathscr{P}$ be an operad over a field and A is a $\mathscr{P}$-algebra. A *Sullivan model* for A is a quasi-free algebra $f : (\mathscr{P}(\bigoplus_{i=0}^{\infty} V_i), d) \xrightarrow{\sim} A$ such that

- the map $f|_{V_0} : V_0 \to A$ is a is a weak equivalence of dg-vector spaces.
- the differential satisfies $d(V_k) \subseteq \mathscr{P}(\bigoplus_{i=0}^{k-1} V_i)$ *(the nilpotence condition)*. In particular $V_0 = H^*(A)$.
- the map $V_k \oplus (\mathscr{P}(\bigoplus_{i=0}^{k-1} V_i) \to A$ is a weak equivalence for each k.

Every $\mathscr{P}$-algebra A over a field admits a Sullivan model. The argument is essentially the same as [27, Proposition 12.1].

Proposition 3.3.2. *Suppose either that $\mathscr{P}$ be an operad over a field and A is an arbitrary $\mathscr{P}$-algebra or that $\mathscr{P}$ be an operad over a ring and A is $\mathscr{P}$-algebra equipped with a cochain map $H^*(A) \to A$. Then A has a Sullivan resolution $m : (\mathscr{P}(V), d) \xrightarrow{\sim} A$.*

Proof. Let $V_0 = H^*(A)$ and choose a map

$$m_0 : (\mathscr{P}(V_0), 0) \to A$$

such that $V_0 \to A$ is an isomorphism on cohomology.

Suppose that m_0 has been extended to $m_k : \left(\mathscr{P}\left(\bigoplus_{i=0}^k V_i\right), d\right) \to A$. Let z_α be cocycles in $\mathscr{P}\left(\bigoplus_{i=0}^\infty V_i\right)$ such that $[z_\alpha]$ is a basis for $\ker H(m_k)$. Let V_{k+1} be a graded space with basis $\{v_\alpha\}$ in $1-1$ correspondence with the z_α, and with $|v_\alpha| = |z_\alpha| - 1$. Extend d to a derivation in $\mathscr{P}\left(\bigoplus_{i=0}^k V_i\right)$ by setting $dv_\alpha = z_\alpha$. Since d has odd degree, d^2 is a derivation. Since $d^2 v_\alpha = dz_\alpha = 0$, we have that $d^2 = 0$.

Since $H(m_k)[z_\alpha] = 0$, $m_k z_\alpha = da_\alpha, a_\alpha \in A$. Extend m_k to a graded $\mathscr{P}$-algebra morphism $m_{k+1} : \mathscr{P}\left(\bigoplus_{i=0}^{k+1} V_i\right) \to A$ by setting $m_{k+1} v_\alpha = a_\alpha$. Then $m_{k+1} dv_\alpha = dm_{k+1} v_\alpha$, and so we havc $m_{k+1} d = dm_{k+1}$.

We conclude our construction by setting $V = \bigoplus_{i=0}^\infty V_i$. We have a map $m : (\mathscr{P}(V), d) \xrightarrow{\sim} A$ such that $m|_{V_k} = m_k$. Since $m|_{\mathscr{P}(V_0)} = m_0$ and $H(m_0)$ is surjective, $H(m)$ is surjective too. If $H(m)[z] = 0$ then, since z must be in some $\mathscr{P}\left(\bigoplus_{i=0}^k V_i\right)$, one has $H(m_k)[z] = 0$. But then, by construction z is a coboundary in $\mathscr{P}\left(\bigoplus_{i=0}^{k+1} V_i\right)$, and so is a coboundary in $(\mathscr{P}(V), d)$. Therefore $H(m)$ is an isomorphism.

The nilpotence condition on d is built into the construction. □

Definition 3.3.3. We refer to the semifree algebra appearing in the previous proof

$$m_k : \left(\mathscr{P}\left(\bigoplus_{i=0}^k V_i\right), d\right) \to A$$

as a *step k Sullivan resolution* of A.

Sullivan algebras are essentially cofibrant objects.

Theorem 3.3.4. *Let $m : (\mathscr{P}(V), d)$ be a Sullivan algebra. Then the map $0 \to (\mathscr{P}(V), d)$ has the left lifting property against all surjective weak equivalences $p : C \to D$.*

Proof. We have the following diagram.

$$\begin{tikzcd} 0 \arrow[r, "f"] \arrow[d, "i"] & C \arrow[d, "p"] \\ (\mathscr{P}(V), d) \arrow[r, "g"] \arrow[ur, dotted, "h"] & D \end{tikzcd}$$

The map h, is determined by the image of the generators $v \in V$. We do this inductively. For $v \in V_0$, we define $h_0 : \mathscr{P}(V_0) \to C$ by $f(v) = c$ where $p(c) = g(v)$, which is defined since p is surjective. Now assume that we have defined $\left(\mathscr{P}\left(\bigoplus_{i=0}^k V_i\right), d\right) \to C$. For $v_\alpha \in V_{k+1}$, we claim it possible to find $c_\alpha \in C$ such that $p(c_\alpha) = g(v_\alpha)$ and $dc_\alpha = h(z_\alpha)$. Let $dv_\alpha = z_\alpha$, then $h(z_\alpha)$ is a cocycle. Choose any lift c'_α such that $dc'_\alpha = h(z_\alpha)$. Moreover, we have $p(h(z_\alpha)) = g(z_\alpha) = g(dv_\alpha) = dg(v_\alpha)$. Therefore $g(v_\alpha) - p(c'_\alpha)$ is nullhomologous in D. There is therefore $K_\alpha \in D$ such that $dK_\alpha = g(v_\alpha) - p(c'_\alpha)$. The cochain K_α has a preimage in $L_\alpha \in C$ and we define $c_\alpha = L_\alpha + c'_\alpha$. Finally, we define $h(v_\alpha) = c_\alpha$. □

Remark 3.3.5. In particular, the surjective weak equivalences are the acyclic fibrations in the setting of Theorem 3.2.7. Therefore, Sullivan models are cofibrant objects in this model category.

Remark 3.3.6. An observation that will be important later is that map h constructed in the proof satisfies the conditions of Definition 3.3.1.

In what follows, we say that an operad $\mathscr{P}$ *reflects homotopy equivalences* if the free algebra functor $\mathscr{P}(-)$ sends quasi-isomorphisms of cochain complexes to quasi-isomorphisms of $\mathscr{P}$-algebras.

Proposition 3.3.7. *Let $\mathscr{P}$ be an operad over a field that reflects homotopy equivalences. Let A be a $\mathscr{P}$-algebra and let $m : (\mathscr{P}(V), d) \xrightarrow{\sim} A$ be a Sullivan resolution. Then, for any $\mathscr{P}$-algebra B weakly equivalent to A there exists a m' such that $m' : (\mathscr{P}(V), d) \xrightarrow{\sim} A$ is a Sullivan resolution.*

Proof. The $\mathscr{P}$-algebra B must be connected to A via zig-zags of quasi-isomorphisms. So it suffices to show that Sullivan resolutions can be transferred across quasi-isomorphisms in both directions. So if there is a quasi-isomorphism $f : A \xrightarrow{\sim} B$, then $m' = f \circ m$. Now, suppose there is a quasi-isomorphism $f : B \xrightarrow{\sim} A$. One can associate an acyclic fibration to f by choosing an acyclic complement W in A to the image of f ie. $A = \mathrm{Im}\, f \oplus B$. Then the map $f' : B \oplus \mathscr{P}(W) \to A$ is an acyclic fibration and so is the projection $\pi : B \oplus \mathscr{P}(W) \to B$. The desired map is then $m' = \pi \circ h$, where $h : (\mathscr{P}(V), d) \xrightarrow{\sim} B \oplus \mathscr{P}(W)$ is the Sullivan resolution coming from applying from Remark 3.3.6 to f'. □

3.3.2 Cotriple products in positive characteristic

In this subsection we introduce a theory of higher Massey-like products for algebras over operads over a field of arbitrary characteristic paralleling that of [31, 60, 70]. The operads here are permitted to have a differential. The underlying idea is similar to that in [31]; we simply wish to define a higher operation for every syzygy. In the setting of Koszul operads over $\mathbb{Q}$, the existence of minimal models for operads made this straightforward. In this context, Massey products correspond to differentials in the Eilenberg-Moore spectral sequence.

In the positive characteristic setting, things are much more complicated. The existence of the Frobenius map produces syzygies that mix the algebra and the operad in ways that were not possible in zero characteristic. We therefore define a non-linear generalisation of a Massey product called a *cotriple product* that captures this phenomenon.

The cotriple resolution. Let $\mathscr{P}$ be an operad and let A be a $\mathscr{P}$-algebra. The *cotriple resolution* $\mathsf{Res}_{\mathscr{P}}(A)$, which is a model for A in the category of free $\mathscr{P}$-algebras, is a simplicial $\mathscr{P}$-algebra defined as follows. In simplicial degree n, one has

$$\mathsf{Res}_{\mathscr{P}}(A)_n = \mathscr{P}^{\circ(n+1)}(A)$$

where $\mathscr{P}^{\circ i}$ indicates that the free algebra functor is applied i times. The face maps are given by

$$d_n^i : \mathscr{P}^{\circ(n+1)}(A) \to \mathscr{P}^{\circ n}(A)$$

$$d_n^i = \begin{cases} \mathscr{P}(A)^{\circ i} \circ \gamma_{\mathscr{P}} \circ \mathscr{P}(A)^{\circ(n-i-1)}(\mathbb{k}) & \text{for } n = 0, \dots, n-1. \\ \mathscr{P}^{\circ(n-1)} \circ \gamma_A & \text{for } i = n. \end{cases}$$

where $\gamma_{\mathscr{P}}$ is the operadic composition map and $\gamma_A : \mathscr{P}(A) \to A$ is the $\mathscr{P}$-algebra map. The degeneracy maps are defined by

$$s_n^i : \mathscr{P}^{\circ(n+1)}(A) \to \mathscr{P}^{\circ(n+2)}(A)$$

$$s_n^i = \mathscr{P}^{\circ i} \circ u \circ \mathscr{P}^{\circ(n-i)}(A)$$

where $u : \mathscr{P} \to \mathscr{P} \circ \mathscr{P}$ is the unit map. This object can be realized as a chain complex $(|\mathsf{Res}_{\mathscr{P}}(A)|, d + \partial)$ where the d is the internal differential on A and ∂ is the differential coming from the simplicial structure we have just defined. We remark that the cotriple resolution has also been studied by Fresse [35].

The cotriple spectral sequence. The cotriple resolution $(|\mathsf{Res}_{\mathscr{P}}(A)|, d + \partial)$ is defined as the realization of a simplicial object and therefore admits a skeletal filtration. The *cotriple* *spectral sequence* is the associated spectral sequence. A morphism of augmented $\mathscr{P}$-algebras naturally induces a morphism of the corresponding spectral sequences. The E^0-page of this spectral sequence is explicitly given by

$$E_0^{p,q} = \mathscr{P}^{\circ p}(A)^{p+q}$$

where the $p+q$ grading is the total grading. The differential d^0 is therefore the usual differential on $\mathscr{P}^{\circ p}(A)$.

Suppose now that the free algebra functor $\mathscr{P}$ reflects homotopy equivalences and therefore $H^*(\mathscr{P}(A)) = \mathscr{B}(H^*(A))$ for a functor $\mathscr{B}$; it follows that the E_1-page of the spectral sequence is

$$E_1^{p,q} = \left(\mathscr{B}^{(p)}(H^*(A))\right)^{p+q}$$

and the differential on this page is therefore entirely determined by the depth 2 component of the codifferential. For $\mathscr{P} = \mathscr{E}$, $E^2_{p,q}$ is the free Steenrod algebra applied p times to the cohomology of A.

Cotriple products. We shall refer to the higher differentials in this spectral sequence as *cotriple products*. When $\mathscr{P}$ reflects homotopy equivalences, homotopy invariance is immediate; as any weak equivalence of $\mathscr{P}$-algebras induces an isomorphism of E_1-pages. Cotriple products are therefore well-defined as elements of $E_n^{p,q}$ in the spectral sequence.

Cotriple products in terms of Sullivan algebras. We shall give a second description which is more in line with the classical definition of Massey products in terms of defining systems. Theorem 3.3.12 gives the proof of the correspondence.

Definition 3.3.8. Let A be a $\mathscr{P}$-algebra and fix a choice of $f : (\mathscr{P}(\bigoplus_{i=0}^N V_i), d) \xrightarrow{\sim} A$ a N-step Sullivan model for A. Consider the ideal $I(\bigoplus_{i=1}^N V_i)$ in $(\mathscr{P}(\bigoplus_{i=0}^N V_i), d)$ generated by $\bigoplus_{i=1}^N V_i$. Let $\sigma \in I(\bigoplus_{i=1}^N V_i)$ be a cocycle. A *defining system* for σ is a $\mathscr{P}$-algebra map

$$g_N : (\mathscr{P}(\bigoplus_{i=0}^N V_i), d) \to A$$

such that $H^*(g_N)|_{V_0} = H^*(f_N)|_{V_0}$. The *$\sigma$-cotriple product set* is given by the collection of $H^*(g_N)(\sigma)$ where g_N ranges across all choices of defining systems.

Remark 3.3.9. Both definitions have advantages. The spectral sequence definition immediately shows that cotriple products are homotopy invariant. The Sullivan algebra definition shows that Massey products are examples of cotriple products.

Rather like usual Massey products, one only needs to compute the σ-cotriple product set rather than the indeteminacy in the spectral sequence to use cotriple products.

Proposition 3.3.10. *Let $\mathscr{P}$ be an operad that reflects homotopy equivalences. A morphism of $\mathscr{P}$-algebras $f : A \to B$ preserves cotriple product sets. If furthermore f is a quasi-isomorphism, then f_* induces a bijection between the corresponding cotriple product sets.*

Proof. The first statement is straightforward as one can verify that given a σ-defining system

$$g_N : (\mathscr{P}(\bigoplus_{i=0}^N V_i), d) \to A,$$

postcomposing by f gives a defining system h_N on B such that $f^* H^*(g_N)|_{V_0} = H^*(h_N)|_{V_0}$. Therefore the σ-cotriple product set on B is a subset of that on A.

To prove the second statement, first observe that if the quasi-isomorphism $f : A \to B$ is surjective, one may lift any defining system on $g_N : (\mathscr{P}(\bigoplus_{i=0}^N V_i), d) \to B$ via the algorithm of

Theorem 3.3.4. The general case follows from a similar argument to Proposition 3.3.7, as in the proof of that result, one may replace the map f with a zig-zag of quasi-isomorphisms $A \xleftarrow{g} C \xrightarrow{h} B$ where B is surjective. One can then lift the defining system to C and then push it forward to A. □

Remark 3.3.11. This definition allows us to extend the notion of cotriple products to algebras over other operads, like the commutative operad, where $\mathscr{P}$ is not a functor that reflects homotopy equivalences. In this case, we cannot necessarily deduce the homotopy invariance of such products automatically (and indeed, frequently they will not be, see Example 3.4.26). Therefore, each time we define such a product, we shall need to manually check homotopy invariance.

The next theorem essentially states that this formulation of cotriple products is equivalent to the spectral sequence one.

Theorem 3.3.12. *Let $\mathscr{P}$ be an operad that reflects homotopy equivalences. Let A be a $\mathscr{P}$-algebra and fix a choice of $f : (\mathscr{P}(\bigoplus_{i=0}^{N} V_i), d) \xrightarrow{\sim} A$ a N-step Sullivan model for A. Let $\sigma \in I(\bigoplus_{i=1}^{N} V_i)$ be a cocycle. Then there exists an element*

$$G(\sigma) \in \mathscr{P}^{\circ N}(H^*(A))$$

which survives to the E_N-term of the $\mathscr{P}$-cotriple spectral sequence, and

$$d_{N-1}([G(\sigma)]) \in (-1)^{N-2}\left[\mathrm{id} \otimes H^*(f(\sigma))\right].$$

To prove this we shall make use of the Staircase Lemma [53, Lemma 2.1], which we briefly recall next.

Lemma 3.3.13. *Let $A = \left(A_{*,*}, d', d''\right)$ be a bicomplex, denote by d the differential on its total complex, and fix $c_1, \dots, c_n$ homogeneous elements in A. Suppose that $d'c_s = d''c_{s+1}$ for $1 \le s \le n-1$, and define $c := c_1 - c_2 + \cdots + (-1)^{n-1}c_n$. Then, $dc = d'c + d''c = d''c_1 + (-1)^{n-1}d'c_n$, and furthermore, in the spectral sequence $\{(E^r, d^r)\}$ associated to the bicomplex, if $d''c_1 = 0$ then c_1 survives to E^n, and $d^n[c_1] = (-1)^{n-1}[d'c_n]$.*

Our approach to proving Theorem 3.3.12 is therefore to construct a sequence $c_1, \dots c_{r-1}$ satisfying the conditions of the Staircase Lemma.

Proof of Theorem 3.3.12. Our first step is to recursively define a sequence $x_0, \dots, x_N$ where

$$x_i \in \mathscr{P}^{\circ i}\left(\bigoplus_{j=0}^{r(i)} V_j\right)$$

for some $r(i) \le k$. Firstly, let $x_0 = \sigma \in \mathscr{P}(\bigoplus_{i=0}^{k} V_i) = \mathscr{P}^{\circ 1}(\bigoplus_{i=0}^{k} V_i)$. Then one obtains an element of $x_1 \in \mathscr{P}^{\circ 2}(\bigoplus_{i=0}^{k-1} V_i)$ by replacing every occurrence of an element in $v_n \in V_k$ by the formula for $dv_n \in \mathscr{P}(\bigoplus_{i=0}^{k-1} V_i)$ and identifying $\mathscr{P} \circ \mathscr{P}(\bigoplus_{i=0}^{k-1} V_i) = \mathscr{P}^{\circ 2}(\bigoplus_{i=0}^{k-1} V_i)$. Continuing this procedure, we obtain x_i for $i = 0, \cdots k$. There are maps

$$g_k : \mathscr{P}^{\circ l}(\bigoplus_{i=0}^{k-l} V_i) \to \mathscr{P}^{\circ l}(A)$$

defined on generators $v \in V_i$ by

$$v \mapsto f(v).$$

We define $c_i = g_{n-i-1}(x_{n-i-1})$. The element $G(\sigma)$ in the proof statement is $d(c_0)$.

To finish, we must verify that the conditions of the Staircase Lemma 3.3.13 are met. Denote by ∂ the external differential on $\mathscr{P}(A)$, and by d^* its internal differential. Then, since $dv = 0$ for each $v \in V_0$, it follows that $d^* c_1 = 0$. A routine calculation shows that $d^* c_{s+1} = \partial c_s$ for each s. It follows from the Staircase Lemma that

$$d_{n-1}[c_1] = (-1)^n[\partial c_{n-1}] = (-1)^n[f(\sigma)].$$

This concludes the proof.

□

3.4 Cotriple products for strictly commutative dg-algebras

In this section, we shall apply the theory of cotriple products developed in the last section to the case of strictly commutative algebras.

3.4.1 Secondary cotriple products

Fundamentally, the main difference between the rational and p-adic commutative dg-algebras is that the Sym functor does not behave well homotopically in positive characteristic.

Example 3.4.1. The functor $\mathrm{Sym} : \mathsf{dg}-R-\mathsf{mod} \to \mathsf{dg}-R-\mathsf{Com}-\mathsf{alg}$ does not reflect homotopy equivalence. For example, when $R = \mathbb{F}_p$, the dg-modules $V = 0$ and $W = [\mathbb{F}_p x \to \mathbb{F}_p dx]$ are homotopy equivalent. However $H^*(\mathrm{Sym}(V)) = 0$, while $H^*(W)$ is non-zero as x^p represents a nontrivial cohomology class.

The reader should be warned that the theory of commutative dg-algebras in positive characteristic is less gentle than the rational case. For example, commutative dg-algebras over $\mathbb{F}_p$ possess *higher commutative operations*, that is cotriple products that do not arise in the same way as classical higher Massey products of [60] and [31]. We shall use these to produce some examples; notably examples of dg-algebras over $\mathbb{Z}$ with torsion-free cohomology that are formal over $\mathbb{Q}$ but not over $\mathbb{F}_p$. The first examples of these is the following.

Definition 3.4.2. Let A be a commutative dg-algebra over $\mathbb{F}_p$. Let $x, y \in H^*(A)$ be homogeneous elements such that $xy = 0$. Choose cocycles $a, b \in A$ representing x, y respectively. Then there exists $c \in A$ such that $dc = xy$. Then c^p is a cocycle which we call the *type 1 secondary Frobenius product* of x and y. Like classical Massey products, this operation has indeterminancy. Indeed, if we choose another lift $c' = ab$ it must differ from the c by a cocycle $\sigma \in Z^{|x|+|y|-1}$. So $c'^p = c^p + \sigma^p$. So, for $p = 2$, c^p represents a well defined element of

$$\frac{H^{p(|x|+|y|-1)}(A)}{H^{(|x|+|y|-1)}(A)^p + x^p H^{p(|y|-1)}(A) + y^p H^{p(|x|-1)}(A)}$$

where the term $x^p H^{p(|y|-1)}(A) + y^p H^{p(|x|-1)}(A)$ in the denominator accounts for the choice of representatives x and y. If the prime p odd, assume without loss of generality, that $|x|$ is even and $|y|$ is odd. Then c^p is a well-defined class of

$$\frac{H^{p(|x|+|y|-1)}(A)}{H^{(|x|+|y|-1)}(A)^p + y^p H^{p(|x|-1)}(A)}.$$

Remark 3.4.3. The striking difference between Type 1 Frobenius operations and ordinary Massey products in characteristic 0 is the dependence of the indeterminacy on the initial choice of cocycles. For a concrete example of this in practice, see Example 3.4.26. This is an added complication with the development of cotriple products in positive characteristic. The underlying reason for this problem is very simple, as an E_∞-algebra over $\mathbb{F}_2$, $c \otimes c$ is not always a cocycle, but if we work with the cup-1 algebras defined in Subsection 3.4.2

$$c \otimes c + c \cup_1 (a \otimes b) + a^{\otimes 2} \otimes K + L \otimes b^{\otimes 2}$$

is, where a, b, c are as above, $dL = a \cup_1 a$ and $dK = b \cup_1 b$. The extra term comes from the fact that, in a cup-1-algebra, one can add cocycles to K and L.

For odd primes, there is a second type of secondary cohomology operation on commutative dg-algebras.

Definition 3.4.4. Let p be an odd prime and A be a commutative dg-algebra over $\mathbb{F}_p$. Then there is a *type 2 secondary Frobenius product* defined for homogeneous elements $x, y \in H^*(A)$ such that $xy = 0$. Choose cocycles $a, b \in A$ representing x, y respectively. Then there exists $c \in A$ such that $dc = xy$. Then, it follows from the antisymmetry of multiplication that $c^{p-1}ab$ is a cocycle which we call the *type 2 secondary Frobenius product* of x and y. In this case, the operation represents a well-defined element of

$$\frac{H^{p(|x|+|y|-1)+|x|+|y|}(A)}{H^{(|x|+|y|-1)}(A)^{p-1}}$$

Such operations do not exist over $\mathbb{F}_2$ as there is no reason for $c^{p-1}ab$ to be a cocycle. This is because the relation $x^2 = 0$ for $|x|$ odd does not hold for commutative dg-algebras in characteristic 2.

Remark 3.4.5. Observe that $d(\frac{1}{p}c^p) = c^{p-1}ab$. Therefore type 2 secondary Frobenius products vanish on divided power algebras. Therefore this kind of operation provides an obstruction for a commutative dg-algebra A to be weakly equivalent to a divided power algebra. It is obvious that non-zero type 1 Frobenius operations are also obstructions as $(-)^p$ vanishes on divided power algebras.

The next lemma is a short verification that our operation is well defined.

Lemma 3.4.6. *Up to indeterminacy, the secondary Type 1 and Type 2 Frobenius products of $x, y \in H^*(A)$ does not depend on the choice of cocycles representing x or y.*

Proof. Let a' and b' respectively be an alternative choice of cocycles representing x, y. Then $a' - a = dK$ and $b' - b = dL$ are coboundaries. Let $c' = c + aL + b'K$. Then we have $dc' = a'b'$. Moreover, we have $(c')^p = c^p + a^pL^p + (b')^pK^p$. Observe that K^p and L^p are cocycles and therefore represent elements of $H^{p(|x|-1)}(A)$ and $H^{p(|y|-1)}(A)$ respectively. We therefore have that c^p and $(c')^p$ represent the same element of

$$\frac{H^{p(|x|+|y|-1)}(A)}{H^{(|x|+|y|-1)}(A)^p + x^pH^{p(|y|-1)}(A) + y^pH^{p(|x|-1)}(A)}.$$

If p is odd, it follows that $|K|$ is odd, and we have that c^p and $(c')^p$ represent the same element of

$$\frac{H^{p(|x|+|y|-1)}(A)}{H^{(|x|+|y|-1)}(A)^p + y^pH^{p(|x|-1)}(A)}.$$

Now we consider the case of Type 2 operations. Let a', b', c', L, K all be defined as before. Then we have

$$(c')^{p-1}a'b' = (c + aM + MdL + bL)^{p-1}(dL + a)(dM + b)$$

This can be written as

$$(c)^{p-1}ab + d\left(\sum_{\substack{i+j=p\\ i,j\neq 0}} \frac{1}{i}\binom{p-1}{i} c^i\,(aM + MdL + bL)^j\right)$$

We have therefore that $(c')^{p-1}a'b'$ and $(c)^{p-1}ab$ represent the same element of

$$\frac{H^{p^n(|x|+|y|-1)+|x|+|y|}(A)}{H^{(|x|+|y|-1)}(A)^{p^n-1}}.$$

This proves the lemma. $\square$

The following lemma is straightforward consequence of the previous one.

Lemma 3.4.7. *Secondary Frobenius products are homotopy invariant. That is, if $x, y \in H^*(A_1)$ are homogeneous elements such that their secondary Frobenius product z is defined then for any zig-zag of quasi-isomorphisms*

$$A_1 \xrightarrow{f_1} A_2 \xleftarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_n$$

the Massey product of $f_1^*(f_2^*)^{-1}\cdots f_{n-1}^*(x)$ and $f_1^*(f_2^*)^{-1}\cdots f_{n-1}^*(y)$ is, up to indeterminacy, equal to $f_1^*(f_2^*)^{-1}\cdots f_{n-1}^*(z)$ in $H^*(A_n)$.

The most immediate application of these cohomology operations is that they provide extra obstructions when comparing commutative dg-algebras. Our first example is an algebra with no torsion in its cohomology that is formal over $\mathbb{Q}$ but not $\mathbb{F}_2$.

Example 3.4.8. Consider the case $p = 2$ and consider the following dg algebras over $\mathbb{Z}$.

$$A = \mathrm{Sym}(x, y, z)/(xy, xz, yz) \qquad B = \mathrm{Sym}(x, y, z, t)/(t^p - z, tz, xz, yz, t^{p+1}, t^{p-1}x, t^{p-1}y)$$

where x, y, z are cocycles, we have $dt = xy$ and $|x| = |t| = 2$, $|y| = 1$ and $|z| = 2p$.

A basis for A is given by $\{x, x^2, x^3, \cdots, y, y^2, y^3, \cdots, z, z^2, z^3, \cdots\}$ and its cohomology ring is A itself. For B, one has a basis given by

$$\{z^i \text{for } i \geq 1\} \cup \{t^i x^j y^k, \text{for } p-2 \geq i \geq 1 \text{ and } j, k \geq 0\}$$

Therefore, by direct computation, one has that the cohomology of B is equal to A. However, by Lemma 3.4.7, they are not quasi-isomorphic as commutative dg-algebras as all the secondary Frobenius products in A vanish, while in B the secondary Frobenius product of x and y is $\{z\}$. However, these algebras are quasi-isomorphic over $\mathbb{Q}$ via the zig-zag

$$A \xleftarrow{f} \mathrm{Sym}(x, y, z, t)/(xz, yz) \xrightarrow{g} B$$

where $dt = xy$ and f, g are the obvious projection maps. So B is formal over $\mathbb{Q}$ but not $\mathbb{F}_2$.

Remark 3.4.9. Formal algebras have vanishing Massey products. The previous example therefore demonstrates that Frobenius products are a different set of invariants to classical Massey products.

The problem with extending Example 3.4.8 to $\mathbb{F}_p$ directly is that the element $ty \in B$ becomes the Massey product $< x, y, y >$ as $y^2 = 0$. We therefore alter it slightly to produce an example of two algebras with the same cohomology that are homotopic over $\mathbb{Q}$ but not $\mathbb{F}_p$

Example 3.4.10. Consider the following dg algebra over $\mathbb{Z}$.

$$A = \mathrm{Sym}(x, y, z, t)/(x, y, z^2, xz, yz, tz, t^p, t^{p-1}x, t^{p-1}y)$$

where $dt = xy$, $|x| = |t| = 2$, $|y| = 1$ and $|z| = 2p$.. A basis for A is given by

$$\{x, x^2, x^3, \cdots, y, z\} \cup \{t^i x^j y^k \text{ for } p > i \geq 0, j > 0, k \in \{0, 1\}\}$$

With coefficients in either $\mathbb{Q}$ or $\mathbb{F}_p$, the cohomology of A is therefore

$$\mathrm{Sym}(x, y, z)/(xy, yz, xz) \cup (s_1, s_2, \dots s_{p-1})$$

where the added elements are Massey products $s_i = t^i y =< x, y, s_{i-1} >$ in A.

Our second algebra is

$$B = \mathrm{Sym}(x, y, z, t)/(t^p - z, tz, xz, yz, t^{p+1}, t^{p-1}x, t^{p-1}y,)$$

where x, y, z are cocycles, we have $dt = xy$ and $ds = ty$ and $|s| = |x| = |t| = 2$, $|y| = 1$ and $|z| = 2p$.

For B, one has a basis given by

$$\{z^i \text{for } i \geq 1\} \cup \{t^i x^j y^k, \text{for } p-2 \geq i \geq 1 \text{ and } j, k \geq 0\}$$

Therefore, by direct computation, one has that the cohomology of B is equal to A. However, by Lemma 3.4.7, they are not quasi-isomorphic as commutative dg-algebras as all the secondary Frobenius products in A vanish, while in B the secondary Frobenius product of x and y is $\{z\}$. However, these algebras are quasi-isomorphic over $\mathbb{Q}$ via the zig-zag

$$A \xleftarrow{f} \mathrm{Sym}(x, y, z, t)/(xz, yz) \xrightarrow{g} B$$

where $dt = xy$ and f, g are the obvious projection maps.

Lastly, we give a counterexample constructed using a Type 2 Frobenius product.

Example 3.4.11. Here we give an example of an algebra that has a divided power structure on its cohomology is nonetheless not quasi-isomorphic to a divided power algebra. For this we use type 2 Frobenius products. Consider the following dg-algebra

$$A = \mathrm{Sym}(x, y, z, t)/(xz, yz, zs_i, s_{p-1}x, s_{p-1}y, s_i s_j - xs_{i+j}, s_p x, s_p y)$$

$$\mathrm{Sym}(\mathbb{F}_p\langle x, y, z\rangle, t)/(t^p, t^{p-1}y - z, tx^2)$$

where $dt = xy$ and the degrees $|x|, |t|$ are even and $|y|, |z|$ are odd. The cohomology of this is given by $(\mathbb{F}_p\langle x, y, z, c_1, c_2, \dots c_{p-2}\rangle/(xy, c_i c_j, xc_i, yc_i, zc_i))$, where the $c_i = t^i y$, which is a divided powers algebra. Nonetheless, the type 2 Frobenius product of x, y is xz so by Remark 3.4.5 it cannot be quasi-isomorphic to a commutative algebra that can be equipped with the structure of a divided powers algebra.

We conclude this section with a brief completeness result.

Definition 3.4.12. We call a cotriple product *primitive* if it arises from monomial relations in cohomology.

Proposition 3.4.13. *All secondary primitive cotriple products on a commutative dg-algebra A over $\mathbb{F}_p$ are linear combinations of*

- *classical Massey products.*

- *Type 1 secondary Frobenius operations*
- *Type 2 secondary Frobenius operations.*

Proof. Let $\mathrm{Sym}(V_0 \oplus V_1) \to A$ be a step 2 Sullivan resolution. We recall that secondary cohomology operations are precisely given by terms representing cohomology in $I(\mathrm{Sym}(V_1))$. These can be directly verified to be linear combinations of elements of the following form

$$x^{p^n}, \qquad x^{p^n-1}dx, \qquad uc - av$$

where $x, u, v \in V_1$ and $du = ab$ and $dv = bc$ for $a, b, c \in \mathrm{Sym}(V_0)$. These corresponds to type 1 and type 2 Frobenius products and classical Massey products respectively. □

3.4.2 The relationship between associative and commutative algebras

In [17], the authors raise the question of whether two commutative dg-algebras are quasi-isomorphic as associative dg algebras if and only if they are also quasi-isomorphic as commutative dg algebras. They prove the following theorem.

Theorem 3.4.14. *[17, Theorem A] Let A and B be two commutative dg algebras over a field of characteristic zero. Then, A and B are quasi-isomorphic as associative dg algebras if and only if they are also quasi-isomorphic as commutative dg algebras.*

We give a counterexample this in positive characteristic using Frobenius product obstruction theory. Similar examples should hold at all primes. More precisely, our statement is the following.

Theorem 3.4.15. *There exists A and B be two commutative dg algebras over a field of characteristic two which may be distingushed via their type 1 Frobenius operation. Nonetheless, there exists an associative algebra C such that there is a zig-zag of associative weak equivalences*

$$A \xleftarrow{\sim} C \xrightarrow{\sim} B$$

Since it must have commutative cohomology, such an algebra C must be commutative up to homotopy (in the most naive sense possible). We shall capture this idea using the notion of a lax cup-1-algebra which we introduce in subsubsection 3.4.2.1. The key idea is that type 1 Frobenius products may be defined on such algebras but they have a different indeterminacy there than on commutative algebras; we study this phenomenon in Proposition 3.4.20. In subsubsection 3.4.2.3, we define the commutative algebras A and B. In subsection C, we define the lax cup-1-algebra C and compute its cohomology. Finally in subsubsection 3.4.2.5, we give the maps appearing in the zig-zag in Theorem 3.4.15.

3.4.2.1 Cup-1-algebras

We recall the following flavour of algebra. They are essentially commutative algebras up to homotopy (but not coherently) and are similar to those appearing in [75].

Definition 3.4.16. A *lax cup-1-algebra* is a chain complex A equipped with two binary operations $\cup$ and $\cup_1$. The operation $- \cup -$ is degree 0 and associative. The second $- \cup_1 -$ is degree -1 and associative. These are intertwinned by two identities. First we have the Hirsch identity, namely that

$$(u \cup v) \cup_1 w = u \cup (v \cup_1 w) + (u \cup_1 w) \cup v \tag{3.1}$$

Secondly, we have the *Steenrod relation*

$$d(u \cup_1 v) = (du \cup_1 v) + (u \cup_1 dv) + (u \cup v) + (v \cup u) \tag{3.2}$$

A cup-1-algebra is described as *strict* if $\cup_1$ is graded commutative. A morphism of cup-1-algebras is a morphisms that preserves both operations.

Lemma 3.4.17. *A strict cup-1-algebra structure on A extends to an $\mathscr{E}$-algebra structure.*

Proof. In the surjection operad, which is a quotient of $\mathscr{E}$ defined in [6], $\cup$ corresponds to the operation $(1,2) \in \mathscr{X}(2)_0$ and $\cup_1$ corresponds to $(1,2,1) \in \mathscr{X}(2)_1$. Consider the quotient operad of $\mathscr{X}$ given by quotienting by all operations not generated by these operations. Since $(1,2,1,2) \in \mathscr{X}(2)_2$ vanishes, we obtain the commutativity of $\cup_1$. The Steenrod and Hirsch relations can now be obtained by routine computations. □

Remark 3.4.18. Strictly commutative algebras are examples of cup-1-algebras such that the $\cup_1$ operation is identically 0. The Steenrod relation then ensures strict commutativity.

The following definition will be useful for our later computations.

Definition 3.4.19. Let $U = \mathrm{Cup}(X)/(R)$ be (lax) cup-1-algebra presented in terms of generators and relations. Let m be a monomial in A, constructed from the generators using both $\cup_1$ and $\cup$. Then m is *reduced* if it is written as

$$m = m_1 \cup m_2 \cup \cdots \cup m_n$$

where each m_i is a monomial constructed only using the $\cup_1$ operation.

Clearly, the free cup-1-algebra has a basis consisting of reduced monomials.

3.4.2.2 Frobenius products for cup-1-algebras

Next, we describe the Frobenius products that exist on cup-1-algebras.

Proposition 3.4.20. *Let A be a cup-1-algebra that is quasi-isomorphic to a strictly commutative dg-algebra. Then the following is a cocycle*

$$c \cup c + c \cup_1 (a \cup b) + K$$

where $dc = a \cup b$, $dK = (a \cup b) \cup_1 (a \cup b)$. If, furthermore, A is a strict cup-1-algebra, then this operation is equivalent to

$$c \cup c + c \cup_1 (a \cup b) + a^2 \cup K' + L' \cup b^2.$$

where $dK' = b \cup_1 b$ and $dL' = a \cup_1 a$.

Remark 3.4.21. The equivalence of A to a strictly commutative dg-algebra guarantees the existence of K and L as $b \cup_1 b$ and $a \cup_1 a$ represent Steenrod operations and these must vanish for a strictly commutative algebra.

Proof. The proof is a completely straightforward computation which is nonetheless pedagogic.

$$dc \cup c = (a \cup b) \cup c + c \cup (a \cup b)$$

This is nonzero as $\cup$ is not commutative. Next, one has, by the Steenrod identity

$$d(c \cup_1 (a \cup b)) = (a \cup b) \cup c + c \cup (a \cup b) + (a \cup b) \cup_1 (a \cup b)$$

Finally one has

$$d\left(a^2 \cup K + L \cup b^2\right) = a \cup (b \cup_1 (a \cup b)) + (a \cup_1 (a \cup b)) \cup b = (a \cup b) \cup_1 (a \cup b)$$

or, if A is strict, this can be written as

$$d\left(a^2 \cup K' + L' \cup b^2\right) = a^2 \cup (b \cup_1 b) + (a \cup_1 a) \cup b^2 = (a \cup b) \cup_1 (a \cup b)$$

where the last equality follows from the Hirsch identity applied twice. Summing all three expressions one gets zero; which proves that the expression is a cocycle. □

The expression above allows one to compute the indeterminacy of this higher cup-1-product operation which can be computed on strict algebras to be

$$\frac{H^{2(|x|+|y|-1)}(A)}{H^{(|x|+|y|-1)}(A)^2 + x^2 H^{2(|y|-1)}(A) + y^2 H^{2(|x|-1)}(A)}.$$

In other words, it is the same as in the commutative case. The main difference here is that, in this case, they fill out the complete indeterminacy.

The point of using lax cup-1-algebras however, is that K does not split apart in the same way as in the strict case. This means that the operation has larger indeterminacy.

3.4.2.3 The commutative algebras

In this subsubsection, we shall construct the commutative algebras A and B from Theorem 3.4.15. Consider the following strictly commutative dg-algebras over $\mathbb{F}_2$,

$$A = \mathrm{Sym}(x, y, z)/(x^3, y^3, xy, z^2, x^2y, yx^2, x^2y, yx^2, yz, z^2)$$

where $|x| = |y| = 2$, $|z| = 4$. The algebra A has the following linear basis $\{x, x^2, y, y^2, z, xz, a\}$ and coincides with its cohomology. Then we have

$$B = \mathrm{Sym}(x, y, z, t)/(x^3, y^3, az, tx^i y^j, t^3, t^2 - xyz^2, x^2y, yx^2, x^2y, yx^2, az, yz, z^2, ax, ay, ata^2)$$

such that $|t| = 3$ and $dt = xy$ and where i, j range over the positive integers such that $i + j = 2$. It is easy to explicitly write down a basis for B as

$$\{x, x^2, y, y^2, z, xz, xy, x^2y, xy^2, x^2y^2, t, tx, ty, txy\}$$

and one then easily verifies that its cohomology is equal to A. Finally we have

$$B = \mathrm{Sym}(x, y, z, t)/(x^3, y^3, az, tx^i y^j, t^3, t^2 + xyz^2, x^2y, yx^2, x^2y, yx^2, az, yz, z^2, ax, ay, ata^2)$$

Next one computes the Frobenius operation of x and x^2. These operation is clearly strictly defined. Moreover, for A it is $\{0\}$, but for B it is $\{xy\}$. These are both in different indeterminacy classes, so it follows that the algebras cannot be quasi-isomorphic either as commutative algebras.

3.4.2.4 The cup-1-algebra C

In this subsubsection, we shall construct the associative algebra C from Theorem 3.4.15. Consider the lax cup-1-algebra C generated by the elements x, y, t subject to the following

- x, y, z are cocycles, $dt = x \cup y$
- One has $|x| = |y| = 2$, $|z| = 4$, $|t| = 3$.
- We set $S = x \cup_1 t + t \cup_1 x$ and $T = y \cup_1 t + t \cup_1 y$. Clearly $dT = (y \cup_1 xy), dS_2 = (x \cup_1 xy)$. This seem unnecessarily confusing now, but will greatly simplify the notation for describing the maps.
- We quotient out by every monomial containing z, except $z, x \cup z$ and $z \cup x$; and we also quotient by $x \cup z + y \cup z$.
- x and y commute and we quotient out by $x \cup_1 x, x \cup_1 y, y \cup_1 x, y \cup_1 y$.
- Ignoring z, we introduce a new degree called x*-word length*, denoted $|-|_x$ where $|y|_x = 0$, $|x|_x = |T|_x = 1, |t|_x = 1, |S| = 2$. Similarly, we have y*-word length*, denoted $|-|_y$ where $|x|_y = 0$, $|y|_y = |S|_y = 1, |t|_y = 1$ and $|T|_y = 2$. We consider word-length to be additive under both $\cup$ and $\cup_1$. The *total word length* is the sum of the word lengths. The differential can easily be checked to preserve word length. We kill all monomials of x- or y-word length 3 or greater.
- Finally, we impose the relation that $t \cup t + t \cup xy + x \cup T + y \cup S = 0$

In the proof of the following proposition, we also present a linear basis for this algebra with 32 elements. This is probably significantly easier to parse.

Proposition 3.4.22. *The cohomology of C is equal to*

$$\mathrm{Sym}(x, y, z, a)/(x^3, y^3, xy, z^2, x^2y, yx^2, x^2y, yx^2, az, yz, z^2, ax, ay, a^2)$$

where $|x| = |y| = 2$, $|z| = 4$ and $|a| = 6$ and, in our previous notation, where $a = t \cup t + t \cup x^2 + L_1 + L_2$.

Proof. Observe that the differential preserves x- or y-word-length and that our relations are homogeneous in word length since they are monomial. It follows that every cocycles can be written as the sum of cocycles that are homogeneous in word length. Therefore, if $C = \bigoplus_{i=1}^{6} C_i$ then $H^*(C) = \bigoplus_{i=1}^{6} H^*(C_i)$. We proceed by computing a basis of reduced monomials.

Before proceeding further in the calculation, we make the following observation: one always has a linear generating set consisting of reduced monomials. We can also ignore the monomials containing z as there is only three of them. Therefore we can do a direct computation.

First of all, one has that $C_1 = \mathbb{F}_2 x$. It is completely straightforward to directly check the following by hand on reduced monomials.

Total word length 1

Degree	Basis
2	x, y

Total word length 2

Degree	Basis
3	t
4	x^2, xy, y^2

Total word length 3

Degree	Basis
4	$x \cup_1 t, y \cup_1 t, T, S$
5	$x \cup t, t \cup x, y \cup t, t \cup y, x \cup_1 (xy), y \cup_1 (xy)$
6	$x^2 y, xy^2,$

Total word length 4

Degree	Basis
6	$x \cup T, T \cup x, S \cup y, (x \cup_1 t) \cup y, (y \cup_1 t) \cup x, x \cup (y \cup_1 t), t \cup_1 t$
7	$xy \cup t, t \cup xy, x \cup t \cup y, y \cup t \cup x, x \cup (y \cup_1 xy), (y \cup_1 xy) \cup x, (x \cup_1 xy) \cup y, t \cup_1 xy$
8	$x^2 y^2$

In the last table, $t \cup t = t \cup xy + x \cup S + T \cup y$. We have also used that

$$x \cup T + y \cup S + T \cup x = y \cup S$$

$$(x \cup_1 t) \cup y + (y \cup_1 t) \cup x + x \cup (y \cup_1 t) = y \cup (x \cup_1 t)$$

which come from the Hirsch identities. □

3.4.2.5 The zig-zag

In this subsubsection, we shall construct the weak equivalences in the zig-zag from the statement of Theorem 3.4.15. Clearly, the algebra C is associative with respect to cup product $\cup$ by the definition of lax cup-1 algebras. The first map

$$f : C \to A$$

is given by sending x, y, z to themselves, and t, S, T to 0. We send all elements in the basis we computed that contain a $\cup_1$ to 0. This map is a quasi-isomorphism as it sends the generators of the cohomology to themselves. The other map is

$$g : C \to B$$

which is given by sending x, y, z to themselves, $T \mapsto y$ and $t, S \mapsto 0$. We send all elements in the basis we computed that contain a $\cup_1$ to 0. This map is also a quasi-isomorphism. The reader should note that these are maps of associative algebras, they are not maps of lax cup-1 algebras as T and S are not independent of x and y in the cup-1-algebras.

Since these maps are quasi-isomorphisms of associative algebras, we conclude that A and B are weakly equivalent as associative algebras but not as commutative algebras.

3.4.3 Higher order cotriple products

In this section, we shall define and study some families of primitive higher order cotriple products. As we saw in Subsection 3.4.1, secondary cotriple products generally behave quite well for commutative dg-algebras. The only property that they lack is filling out the whole indeterminacy. Unfortunately, this failure then directly implies that the higher order operations will not be definable on every choice of model. This then breaks the homotopy invariance of such operations.

Example 3.4.23. There are cotriple products of all orders. One way to see this is to choose a vanishing n^{th} order Massey product $m_n(x_1, x_2, \cdots, x_n) = dc$. Then c^p is a Type 1 higher operation of order $n+1$.

The next example gives an explicit family arising directly from Type 1 Frobenius operations. The reader should be warned that we shall see shortly that this family is not always homotopy invariant.

Definition 3.4.24. Let A be a commutative dg-algebra over $\mathbb{F}_p$. Let $x, y \in H^*(A)$ be homogeneous elements such that $xy = 0$. A defining system for a n^{th} *order type 1 Frobenius product* is a collection $\{a, b, c_1, \ldots c_{n-1}\}$ such that a, b are choices of cocycle representatives for x, y, $dc_1 = ab$ and $c_i^p = dc_{i+1}$. The n^{th} *order type 1 Frobenius product* is then c_n^p. In particular, second order type 1 Frobenius products coincide with those defined in Definition 3.4.2.

In order to be a useful class of operations it is important to compute the indeterminacy of the class. However, this is more complicated than it appears, because, in general, secondary cotriple products for the commutative operad do not completely fill out their indeterminacy.

Example 3.4.25. For example, if it exists, the third order type 1 Frobenius product on a given algebra A will be a well-defined element of

$$\frac{H^{4(|x|+|y|)-6}(A)}{H^{2(|x|+|y|)-3}(A)^2 + x^4 H^{4|y|-6}(A) + y^4 H^{4|x|-6}(A)}.$$

if there does not exist $u, v \in H^*(A)$, both nonzero, such that $x^2u + y^2v = 0$. Otherwise each relation $x^2u + y^2v = 0$ in cohomology will give rise to extra secondary non-primary Frobenius operations in the obvious way. These will have some indeterminacy X and the denominator of the above quotient will be $H^{2(|x|+|y|)-3}(A)^2 + x^4 H^{4|y|-6}(A) + y^4 H^{4|x|-6}(A) + X$.

This can be seen as follows: Firstly, we can add any choice of cocycle to c_2. This accounts for the $H^{2(|x|+|y|)-3}(A)^2$ term. Then let a' and b' respectively be an alternative choice of cocycles representing x, y. Then $a' - a = dK$ and $b' - b = dL$ are coboundaries. Let $c_1' = c_1 + aL + b'K$. Then we have $dc_1' = a'b'$. Moreover, we have $(c_1')^2 = c^2 + a^2L^2 + (b')^2K^2$. Suppose now that

$$dc_2 = c_1$$

then there exists $dc_2' = c_1'$ if and only if there exists an R such that

$$dR = a^2L^2 + (b')^2K^2.$$

If there does not exist $u, v \in H^*(A)$, both nonzero, such that $x^2u + y^2v = 0$, we must have that the cocycles L^2 and K^2 are both zero in cohomology and hence, there exists S, T such that

$$dS = L^2 \qquad dT = K^2$$

and therefore

$$R = a^2S + (b')^2T$$

and therefore $(c_2+R)^2 = c_2^2 + a^4S^2 + (b')^4T^2$, from whence comes the $x^4H^{4|y|-6}(A) + y^4H^{4|x|-6}(A)$ term in the indeterminacy.

However, a n^{th} order Type 1 Frobenius product is not guaranteed to exist on commutative dg-algebra A even if it does on other algebras weakly equivalent to A. This prevents Frobenius products from being homotopy invariant. This is a counterexample to [31, Proposition 2.18] in positive characteristic.

Example 3.4.26. Consider the following dg-algebras over $\mathbb{F}_2$.

$$A = \mathrm{Sym}(w, x, y, z)/(x^2 w - z, xy, xz, yz)$$

$$B = \mathrm{Sym}(w, x, y, z, t)/(t^2 - z, xz, yz, t^3, y^2 w - z,)$$

where w, x, x', y, z are cocycles, we have $dt = xy$ and $ds = x - x'$ The cohomology ring of both algebras is A. In this case, one can check that type 1 Frobenius product of x and y vanishes. Define

$$C = \mathrm{Sym}(w, x, x', y, z, s, t)/(t^2 - z, xz, yz, t^3, s^3, y^2 w - z, s^2 - w).$$

There is a zig of quasi-isomorphisms

$$A \xrightarrow{\sim} C \xleftarrow{\sim} B.$$

The third order type 1 Frobenius operation associated to the relation $xy = 0$ in cohomology is defined in A and C, as the second order product set is $\{0\}$ in A and $\{0, z\}$ in C. However, it is not defined in B as the second order type 1 Frobenius product set is $\{z\}$, which contains no coboundaries.

The examples above imply that we need an extra condition to ensure that higher order operations are homotopy invariant. One such condition is that the higher operation is *strictly defined*. This condition has the added benefit of allowing easy computation of the indeterminacy.

Proposition 3.4.27. *Let A be a commutative dg-algebra and suppose $x, y \in H^*(A)$ are such that their type 1 n^{th} Frobenius product is* strictly defined, *that is, that $xy = 0$ and*

$$H^{p(|y|-1)}(A) = H^{p(|x|-1)}(A) = \{0\}$$

$$\cdots$$

$$H^{p^{n-1}(|y|)-\Sigma_{i=1}^{n-1} p^i}(A) = H^{p^{n-1}(|x|)-\Sigma_{i=1}^{n-1} p^i}(A) = \{0\}$$

and the $(n-1)^{th}$ Frobenius product is equal to 0. Then n^{th} order type 1 Frobenius product is defined and is a well-defined element of

$$\frac{H^{p^{n-1}(|x|+|y|)-\Sigma_{i=1}^{n-1} p^i}(A)}{H^{p^{n-1}(|x|+|y|)-2^{n-1}+\Sigma_{i=0}^{n-1} p^i}(A)^p + x^{p^{n-1}} H^{p^{n-1}(|y|)-\Sigma_{i=1}^{n-1} p^i}(A) + y^{p^n} H^{p^{n-1}(|x|)-\Sigma_{i=1}^{n-1} p^i}(A)}$$

and therefore is invariant under quasi-isomorphism.

Proof. The proof is by induction on the order of the operation. First observe that if it is strictly defined, the secondary Frobenius operation represents a well-defined class of

$$\frac{H^{p(|x|+|y|-1)}(A)}{H^{(|x|+|y|-1)}(A)^p + x^p H^{p(|y|-1)}(A) + y^p H^{p(|x|-1)}(A)} = \frac{H^{p(|x|+|y|-1)}(A)}{H^{(|x|+|y|-1)}(A)^p}.$$

Therefore it fills out its indeterminacy and the third order Frobenius operation is defined at every commutative algebra. Finally it has a well defined indeterminacy, which can be computed as follows . Recall from the proof of Lemma 3.4.6 that the Frobenius product set is given by all elements of the form

$$c_1^p + a^p L^p + (b')^p K^p$$

(using the notation from the proof of that proof). By assumption $dc_2 = c_1^p$. But, one can also choose $c_2' = c_2 + \sigma$ such that

$$d(c_2+\sigma) = c_1^p + a^p L^p + (b')^p K^p$$

This implies $d\sigma = a^p L^p + (b')^p K^p$. By the strictly defined hypothesis, all such terms are coboundaries. Moreover, both $a^p L^p$ and $(b')^p K^p$ are individually coboundaries. It follows that σ can be factored as $\sigma' + \tau_1 + \tau_2$, where σ_1 is a cocycle and

$$d\tau_1 = a^p L^p \qquad d\tau_2 = (b')^p K^p.$$

A priori, τ_1 and τ_2 are only defined up to cocycle, but any choice of cocycle can be added to σ', so we may assume they are unique for any given choice of a, a', b, b', L, K. Then, since $H^{p(|y|-1)}(A) = H^{p(|x|-1)}(A) = \{0\}$, one has that $dR = L^p$ and $dS = K^p$. Therefore, we have that

$$c_2' = c_2 + \sigma' + a^p R + b^p S.$$

So

$$(c_2)^p = c_2^p + (\sigma')^p + a^{p^2} R^p + b^{p^2} S^p.$$

As R^p and S^p are cocycles, one has the desired indeterminacy. Therefore we have the desired invariance under quasi-isomorphism.

Then, by induction, assume that the order k type 1 Frobenius product is defined and has the desired indeterminacy. Moreover assume that order k type 1 Frobenius product set takes the form of a subset of:

$$\begin{aligned}\{c_k^p + \sigma^p + a^{p^k} P^p + b^{p^k} Q^p : \sigma \in Z^{p^{k-1}(|x|+|y|)-\sum_{i=1}^{k-1} p^i}(A),\\ P \in C^{p^{k-1}(|y|)-\sum_{i=1}^{k-1} p^i}(A), Q \in C^{p^{k-1}(|y|)-\sum_{i=1}^{k-1} p^i}(A)\}\end{aligned}$$

for a fixed $c_k \in Z^{p^{k-1}(|y|)-\sum_{i=1}^{k-1} p^i}(A)$. Again, by the fact that the operation is strictly defined, the order k type 1 Frobenius product is a well-defined element of

$$\frac{H^{p^n(|x|+|y|)-\sum_{i=1}^{n} p^i}(A)}{H^{p^{n-1}(|x|+|y|)-2^{n-1}+\sum_{i=0}^{n} p^i}(A)^p}.$$

It follows that, if the class the order k type 1 Frobenius product in the above quotient is 0, it is always possible to find $c_k^p = dc_{k+1}$. By the same argument as before, observe that $c_k^p = d(c_{k+1} + \sigma + \tau_1 + \tau_2)$ for all cocycles σ and $d\tau_1 = a^{p^{n-1}} P^p$ and $d\tau_2 = b^{p^{n-1}} Q^p$. The cocycle σ accounts for the $H^{p^{n-1}(|x|+|y|)-2^{n-1}+\sum_{i=0}^{n} p^i}(A)^p$ in the indeterminacy calculation. The τ_1 and τ_2 accounts $x^{p^n} H^{p^n(|y|)-\sum_{i=1}^{n} p^i}(A) + y^{p^n} H^{p^n(|x|)-\sum_{i=1}^{n} p^i}(A)$ in the indeterminacy. This shows that the order $(i+1)^{th}$ operation has the correct indeterminacy. Moreover the order $k+1$ Frobenius product set has the correct form by the same reasoning as in the order 2 case. □

3.5 Higher Steenrod operations as obstructions to rectifiablity

The purpose of this subsection is to set up an obstruction theory for commutativity, paralleling the obstruction theory for formality given by Massey products. Our obstructions will be given by *higher Steenrod products*. The first application of this theory is the following well-known folklore result that the author learned from some online lecture slides of Mandell [59].

Proposition 3.5.1. *Let X be a topological space. The E_∞-algebra $C^*(X,\mathbb{F}_p)$ admits a strictly commutative model only if X is weakly equivalent to the disjoint union of contractible spaces.*

Proof. Suppose towards contradiction that E_∞-algebra $C^*(X,\mathbb{F}_p)$ admitted a commutative model in the category of $\mathbb{F}_p$-commutative dg-algebras. Recall that $C^*(X,\mathbb{F}_p)$ admits Steenrod operations on its cohomology. Such operations are preserved by quasi-isomorphisms of E_∞-algebras. All of these operations vanish on strictly commutative dg-algebras except for $\mathrm{P}^n x$ when $|x| = n$. In particular, the zeroth Steenrod operation $\mathrm{P}^0 x$ is always x on cohomology of $C^*(X,\mathbb{F}_p)$, while $\mathrm{P}^0 x$ vanishes on the cohomology of commutative dg-algebra, except when $|x| = 0$. It follows that $C^*(X,\mathbb{F}_p)$ admits a commutative model only if its cohomology is concentrated in degree 0. □

We now proceed to define *higher Steenrod operations*. These will be our obstructions to commutativity. These are essentially the subset of cotriple products given by the primary Steenrod operations, and higher obstructions formed by syzgyies of Steenrod operations.

Definition 3.5.2. Consider the map of operads $\mathscr{E} \to \mathrm{Com}$. Let $(\mathscr{E}(\bigoplus_{i=0}^N V_i), d)$ be an N-step Sullivan model. Then the *Sullivan projection map* is the map of $\mathscr{E}$-algebras

$$\pi_N : (\mathscr{E}(\bigoplus_{i=0}^{N} V_i), d) \to (\mathrm{Sym}(\bigoplus_{i=0}^{N} V_i), d).$$

and is defined by induction on i as follows. When $i = 0$; $\pi_0 : \mathscr{E}(V_0) \to \mathrm{Sym}(V_0)$ is the $\mathscr{E}$-algebra map directly induced by the map $\mathscr{E} \to \mathrm{Com}$. Therefore we assume that there is a map $\pi_k : (\mathscr{E}(\bigoplus_{i=0}^k V_i), d) \to (\mathrm{Sym}(\bigoplus_{i=0}^k V_i), d)$. We define $(\mathrm{Sym}(\bigoplus_{i=0}^k V_i), d)$ on generators via the attachment map

$$dV_{k+1} \to \mathscr{E}(\bigoplus_{i=0}^{k} V_i), d) \to (\mathrm{Sym}(\bigoplus_{i=0}^{k} V_i), d)$$

and extend this as a derivation. The map π_k therefore also extends to

$$\pi_{k+1} : (\mathscr{E}(\bigoplus_{i=0}^{k+1} V_i), d) \to (\mathrm{Sym}(\bigoplus_{i=0}^{k+1} V_i), d)$$

by sending V_{k+1} to itself.

Definition 3.5.3. Let $\mathscr{E}$ be a model for the E_∞-operad, A be a $\mathscr{E}$-algebra and let $\sigma \in I(\mathscr{E}(\bigoplus_{i=1}^N V_i), d))$ be a cocycle. There is a unique quasi-isomorphism $\mathscr{E} \to \mathrm{Com}$. Then σ defines a *higher Steenrod operation of order N* if it appears in the kernel of the Sullivan projection map

$$\mathscr{E}(\bigoplus_{i=1}^{N} V_i), d) \to \mathrm{Sym}(\bigoplus_{i=1}^{N} V_i), d).$$

Corollary 3.5.4. *Let A be an $\mathscr{E}$-algebra. Suppose A admits a higher Steenrod operation that does not vanish as a differential in the cotriple spectral sequence. Then A is not rectifiable.*

Proof. Any non-commutative higher Steenrod operation is always identically zero on a strictly commutative dg-algebra B. This is because it can be written in terms of a defining system in which every operation vanishes on B. Moreover higher Massey operations are preserved by quasi-isomorphisms of E_∞-algebras as differentials in the cotriple spectral sequence. Therefore A cannot be quasi-isomorphic to a commutative dg-algebra. □

3.5.1 Necessary and sufficient condition for rectifiability

The purpose of this section is to show that our obstruction theory for commutativity is essentially complete. In other words, we shall give a necessary and sufficient condition for an arbitrary E_∞-algebra A over $\mathbb{F}_p$ to have a commutative model. Note that in this case, by Proposition 3.5.1 A will never have the homotopy type of a space. Our result is inspired by the following classical result which we state first.

Theorem 3.5.5. *[23] Let A be a commutative dg-algebra in $\mathbb{Q}$-vector spaces. Let $\mathfrak{m} = (\mathrm{Sym}(\bigoplus_{i=0}^{\infty} V_i), d)$ be the minimal model for A. Then A is formal if and only if, there is in each V_i a complement B_i to the cocycles Z_i, $V_i = Z_i \oplus B_i$, such that any closed form, a, in the ideal, $I((\bigoplus_{i=0}^{\infty} B_i)$, is exact.*

Remark 3.5.6. The condition stated in this theorem is often referred as to the *coherent vanishing* of Massey products. The reason for this is that any closed form a in the ideal $I((\bigoplus_{i=0}^{\infty} B_i)$ is a Massey product in the sense of Definition 3.3.8, since the minimal resolution can be upgraded to a Sullivan resolution. The condition that a is exact is precisely the requirement that it vanish in cohomology.

When we are not working in the rational setting, there is no longer a preferred choice of cofibrant resolution like the minimal model. Therefore our statement will be stated in the language of Sullivan resolutions.

Definition 3.5.7. Let A be an E_∞-algebra over $\mathbb{F}_p$. Then the higher Steenrod operations *vanish coherently* if for every Sullivan resolution $(\mathscr{E}(\bigoplus_{i=0}^{\infty} V_i), d)$ for A, there exists a splitting $V_i = X_i \bigoplus Y_i$, with $X_0 = V_0$; such that $(\mathrm{Sym}(\bigoplus_{i=0}^{\infty} X_i), d)$ is a Sullivan algebra and the kernel of

$$(\mathscr{E}(\bigoplus_{i=0}^{\infty} V_i), d) \to (\mathrm{Sym}(\bigoplus_{i=0}^{\infty} X_i), d)$$

is acyclic.

Remark 3.5.8. The cocycles appearing in the kernel represent Steenrod operations. For example, the kernel of the $\mathscr{E}(V_0) \to \mathrm{Sym}(V_0)$ component are precisely the Steenrod operations and the definition of a Sullivan algebra immediately implies that these extra cocycles are killed by Y_1.

Theorem 3.5.9. *Let A be an E_∞-algebra over $\mathbb{F}_p$. Then A is rectifiable if and only if its higher Steenrod operations vanish coherently.*

Proof. First we prove the *only if* direction. That is to say that we first suppose that A is rectifiable and we then we shall show that every Sullivan model for A admits a splitting such that the conditions of Definition 3.5.7 are satisfied. If A is rectifiable it has a strictly commutative model $\bar{A}$. Since Sullivan models are cofibrant, it follows from Proposition 3.3.7 that one has a map

$$f : (\mathscr{E}(\bigoplus_{i=0}^{\infty} V_i), d) \to \bar{A}$$

satisfying the axioms of a Sullivan algebra. We build the desired splitting by induction on i. Firstly let $X_0 = V_0$ and $Y_0 = 0$. Since $\bar{A}$ is strictly commutative, there is a factorisation

$$\begin{tikzcd} \mathscr{E}(V_0) \arrow[r, "g_0"] \arrow[dr, "f|_{\mathscr{E}(V_0)}"'] & \mathrm{Sym}(X_0) \arrow[d, "h_0"] \\ & \bar{A} \end{tikzcd}$$

Consider the map

$$V_1 \xrightarrow{d} \mathscr{E}(V_0) \xrightarrow{g_0} \mathrm{Sym}(X_0)$$

There is a splitting $V_1 = X_1 \oplus Y_1$ such that Y_1 is the kernel of this map and X_1 is some complement to it.

Now inductively, we assume the splitting $V_i = X_i \oplus Y_i$ exists for $i \leq k$ and moreover that there is a factorisation

$$\begin{tikzcd} \mathscr{E}((\bigoplus_{i=0}^k V_i), d) \arrow[r, "g_k"] \arrow[dr, "f_k"'] & \mathrm{Sym}((\bigoplus_{i=0}^k X_i), d) \arrow[d, "h_k"] \\ & \bar{A} \end{tikzcd}$$

Then there is a splitting $V_{k+1} = X_{k+1} \oplus Y_{k+1}$ such that Y_{k+1} is the kernel of the map

$$V_{k+1} \xrightarrow{d} \mathscr{E}((\bigoplus_{i=0}^{k+1} V_i), d) \xrightarrow{g_k} \mathrm{Sym}((\bigoplus_{i=0}^{k} X_i)$$

and X_{k+1} is some complement to it. The existence of the factorisation once again follows from the fact $\bar{A}$ is commutative.

Lastly, we verify that this splitting satisfies the condition that the kernel of the projection

$$(\mathscr{E}(\bigoplus_{i=0}^{\infty} V_i), d) \to (\mathrm{Sym}(\bigoplus_{i=0}^{\infty} X_i), d)$$

is acyclic. Suppose that $\sigma \in (\mathscr{E}(\bigoplus_{i=)}^{\infty} V_i), d)$ is a cocycle in the kernel. Then we have that $\sigma \in (\mathscr{E}(\bigoplus_{i=0}^{N} V_i), d)$ for some N. Since the map f_N, by construction, factors through $\mathrm{Sym}(\bigoplus_{i=1}^{N} X_i), d)$ it follows that $f(\sigma) = 0$. It then follows from the second condition of Definition 3.3.1, that there is $\tau \in V_{k+1}$ such that $d\tau = \sigma$. It is clear from our definition of the splitting that $\tau \in Y_{N+1}$.

Conversely, suppose that A is an E_∞-algebra over $\mathbb{F}_p$ such that its higher Steenrod operations vanish coherently. Then, by definition, there is a quasi-isomorphism

$$(\mathscr{E}(\bigoplus_{i=0}^{\infty} X_i \oplus Y_i), d) \xrightarrow{\sim} A.$$

where $(\mathscr{E}(\bigoplus_{i=0}^{\infty} X_i \oplus Y_i), d)$ satisfies the hypotheses of Definition 3.5.7. We claim that the projection map

$$f : (\mathscr{E}(\bigoplus_{i=0}^{\infty} X_i \oplus Y_i), d) \to \mathrm{Sym}(\bigoplus_{i=0}^{\infty} X_i)$$

is a quasi-isomorphism. The map f is surjective so, by the long exact sequence in cohomology, it suffice to prove that the kernel of f is acyclic. This is precisely the coherent vanishing condition. □

The previous result has the following corollary; which is proven similarly.

Definition 3.5.10. Let A be an $\mathscr{E}$-algebra. We say that it is *formal* if it is quasi-isomorphic to $H^*(A)$, regarding the cohomology as a commutative algebra.

Definition 3.5.11. Let A be an E_∞-algebra over $\mathbb{F}_p$. Then the cotriple products *vanish coherently* if for every Sullivan resolution $(\mathscr{E}(\bigoplus_{i=0}^{\infty} V_i), d)$ for A, the ideal $I(dV_1 \oplus \mathscr{E}(\bigoplus_{i=1}^{\infty} V_i), d)$ is acyclic. Similarly,

Corollary 3.5.12. *Let A be an E_∞-algebra over $\mathbb{F}_p$. Then A is formal as an $\mathscr{E}$-algebra if and only if its cotriple operations all vanish coherently.*

Proof. First we prove the *only if* direction. That is to say that we first suppose that A is formal and we then we shall show that every Sullivan model for A, the cotriple operations vanish coherently. It follows from Proposition 3.3.7 that one has a map

$$f : (\mathscr{E}(\bigoplus_{i=0}^{\infty} V_i), d) \to H^*(A)$$

that makes $\big((\mathscr{E}(\bigoplus_{i=0}^{\infty} V_i), d), f\big)$ a Sullivan model for $H^*(A)$. This map is clearly surjective and therefore the kernel of f must be acyclic. If we show that the kernel is isomorphic to $I(dV_1 \oplus \mathscr{E}(\bigoplus_{i=1}^{\infty} V_i), d)$, we can conclude the result by the long exact sequence in cohomology. First note that the map $f|_{V_0} = \mathrm{id}_H$. Let

$$W_i = \{v - \big(f|_{V_0}\big)^{-1}\big(f(v)\big) : v \in V_i\}$$

Then, one can easily verify that the kernel of f is equal to $I(dV_1 \oplus \mathscr{E}(\bigoplus_{i=1}^{\infty} W_i), d)$ and this is isomorphic to the ideal $I(dV_1 \oplus \mathscr{E}(\bigoplus_{i=1}^{\infty} V_i), d)$. In particular, both have the same cohomology .

Conversely, suppose that A is an E_∞-algebra over $\mathbb{F}_p$ such that its cotriple operations vanish coherently. Then, by definition, there is a quasi-isomorphism

$$(\mathscr{E}(\bigoplus_{i=0}^{\infty} V_i), d) \xrightarrow{\sim} A.$$

where $I(dV_1 \oplus \mathscr{E}(\bigoplus_{i=1}^{\infty} V_i), d)$ is acyclic. But this is the kernel of the algebra map

$$(\mathscr{E}(\bigoplus_{i=0}^{\infty} V_i), d) \to V_0 = H^*(A)$$

so, by the long exact sequence in cohomology, we conclude that this map is an isomorphism, and so A is formal. $\square$

CHAPTER 4

A p-adic de Rham complex

Abstract

This is the second in the sequence of three chapters exploring the relationship between commutative algebras and E_∞-algebras in characteristic p and mixed characteristic. Given a topological space X, we construct, in a manner analogous to Sullivan's A_{PL}-functor, a strictly commutative algebra over $\widehat{\mathbb{Z}_p}$ which we call the de Rham forms on X. We show this complex computes the singular cohomology ring of X. We prove that it is quasi-isomorphic as an E_∞-algebra to the Berthelot-Ogus-Deligne *décalage* of the singular cochains complex with respect to the p-adic filtration. We show that it can be modified slightly to provide a "best strictly commutative approximation" to the singular cochains complex. We show that one can extract concrete invariants from our model, including Massey products which live in the torsion part of the cohomology. We show that if X is formal then, except at possibly finitely many primes, the p-adic de Rham forms on X are also formal.

4.1 Introduction

Since its introduction by Quillen [76] and Sullivan [85], rational homotopy theory has probably become the single most successful subfield of algebraic topology. One of the main observations of [85], which was completely fleshed out by [17], was that it was possible to completely capture the rational homotopy theory of spaces via a strictly commutative model $A_{PL}(X)$, which behaves roughly like the de Rham cochains. This reduces the study of rational topological spaces to that of commutative dg-algebras. This has led to some spectacular practical advances; for example, the rational homotopy groups of spheres and many other spaces are now completely understood.

In a tour de force, Mandell [58] showed that it was possible to go one step further, and that the study of all nilpotent, finite type spaces *integrally* can be reduced to studying E_∞-algebras. In terms of computation, less mileage seems to have been got from this than rationally; largely because E_∞-algebras are usually very complicated objects, generated by infinitely many n-ary operations, and which are not naturally amenable to being studied computationally. We are unaware of any implementations of even simple procedures such as Groebner bases for general E_∞-algebras. In contrast, the strictly commutative algebras appearing in rational homotopy theory are, almost uniquely, suited to being studied via computer algebraic approaches such as using GAP or Sage due to the fact they are generated by a single binary operation displaying the simplest possible behaviour. Most of these techniques are not available even one step up, when working with cup-1-algebras - algebras that are commutative up to strictly commutative homotopy [29, Definition 4.18].

The goal of this chapter is therefore to provide strictly commutative models for spaces over the p-adic numbers $\widehat{\mathbb{Z}_p}$. The central problem is that it is not possible to capture all of the information about the homotopy type of the spaces this way. This because the Steenrod operations act as obstructions to strict commutivity. In particular, we have that the zeroth Steenrod power operation P^0 never vanishes on E_∞-algebras with the homotopy type of spaces. Therefore, we can only hope to study approximations that carry *some* of this information. There are multiple possible approaches. Mandell [59] has suggested for n-connected spaces X at most primes, it may be possible to truncate the E_∞-structure on $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$ to an E_n-structure and find a strictly commutative model for this truncation. While we think this is a interesting point of view and worthy of further study, in this paper we have opted for a more universal approach; we explain how to construct an explicit strictly commutative algebra representing the homotopy right adjoint to the inclusion of commutative algebras into E_∞-algebras. We further explain which well-known invariants may be extracted from it.

In this paper, we study a generalisation of Sullivan's approach to homotopy theory. Recall that this involves defining a *cochain algebra*, that is a functor

$$A_{PL}\colon \Delta \to \mathsf{CDGA}$$

which extends to

$$A_{PL}\colon \mathsf{sSet} \to \mathsf{CDGA}$$

by the universal property of simplicial sets. We shall recall this in more detail later, but for now it suffices to recall that

$$A_{\mathrm{PL}}\left(\Delta^n\right) = \frac{\mathbb{Q}\left(t_0,\dots,t_n,dt_0,\dots,dt_n\right)}{\left(\sum t_i - 1, \sum dt_i\right)}$$

The problem with doing this in positive characteristic is that Sym is not a homotopy invariant functor. In 1979, Cartan [19] generalised the work of Sullivan [85] to a slightly more general framework. In particular, Example 4 from that paper uses divided power algebras

$$\mathrm{Gr}\left(\Delta^n\right) = \frac{\mathbb{Z}\langle s\rangle\langle t_0,\dots,t_n,dt_0,\dots,dt_n\rangle}{\left(\sum t_i - s, \sum dt_i\right\rangle}$$

where $\langle -\rangle$ denotes the free divided power algebra. Cartan computes the cohomology of the extension to sSet and proves that a subring of the cohomology is isomorphic to the singular cohomology ring of X.

We, initially independently, had the same idea of modifying Sullivan's construction using divided power algebras. However, instead of working with $\mathbb{Z}\langle s\rangle$, we found it more convenient to localise at a fixed prime p and work over $\widehat{\mathbb{Z}_p}$, with p itself playing the role of s. This way, we are able to extract the singular cohomology ring of $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$ itself from the construction, which we call *the $\widehat{\mathbb{Z}_p}$-de Rham forms on X*.

Theorem 4.1.1. *Let X be a simplicial set. The cohomology ring of the p-adic de Rham complex $\Omega^*(X)$ is isomorphic to the singular cohomology of X. In other words, one has a ring isomorphism*

$$H^*\left(\Omega^*(X)\right) \cong H^*\left(X,\widehat{\mathbb{Z}_p}\right).$$

After computing the cohomology ring, from a modern perspective, the natural next step is interpret the higher information contained the $\widehat{\mathbb{Z}_p}$-de Rham forms. To that end, we show (Theorem 4.3.18) that our construction, as an E_∞-algebra, is equivalent to the following subalgebra of the singular cochains. In this sense, our work is the logical continuation of that by Cartan and sheds new light on many of the constructions of [19].

Definition 4.1.2. Let X be a simplicial set. We define the *p-shifted singular cochain algebra* $\mathscr{D}^*\left(X,\widehat{\mathbb{Z}_p}\right)$ to be the following subalgebra of the singular cochains $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$.

$$\mathscr{D}^n(X) = \left\langle p^i\sigma : \text{ for } \sigma \in C^n\left(X,\widehat{\mathbb{Z}_p}\right) \text{ and } \begin{cases} i = n & \text{if } d\sigma = 0. \\ i = n+1 & \text{otherwise.} \end{cases} \right\rangle$$

The differential and the E_∞-structure are that induced by those on $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$.

This also reveals an unexpected connection with the theory of crystalline cohomology for schemes. The same object as above can be viewed as $\eta_p\left(C^*\left(X,\widehat{\mathbb{Z}_p}\right)\right)$, where η is the Berthelot-Ogus-Deligne [9, 22] *décalage* functor, which is defined as the connective cover with respect to the Beilinson t-structure on filtered complexes. In our case we we are working in complexes over $\widehat{\mathbb{Z}_p}$ with the p-adic filtration. In Cartan's case, he was working in complexes over $\mathbb{Z}\langle s\rangle$ with the filtration generated by the ideal (s). In particular, this ties in with the work of Bhatt-Lurie-Mathew [10, Thereom 7.4.7, Example 7.6.7], which states that, in the ∞-categorical context, the fixed points of the left derived functor $L\eta_p$ of η_p acting on the derived category of p-complete dg-$\widehat{\mathbb{Z}_p}$-modules is equivalent to a 1-category. The de Rham forms appearing in our and Cartan's work can therefore be seen supplying a convenient strictly commutative model for this rectification when working with spaces.

Theorem 4.3.18 also has some immediate applications. It means that the $\widehat{\mathbb{Z}_p}$-de Rham forms can be used to compute Massey products up to a factor, including in the torsion part of the cohomology, which has proven useful, in, for example, [41] for specific classes of spaces. We conclude with a result on formality which was inspired by a conjecture of Mandell's [59].

Theorem 4.1.3. *Let X be a finite simplicial set such that $A_{PL}(X)$ is formal over $\mathbb{Q}$. For all but finitely many primes, $\Omega^*(X)$ is formal over $\widehat{\mathbb{Z}_p}$ as a dg-commutative dg-algebra.*

Finally, it turns out the $\widehat{\mathbb{Z}_p}$-de Rham forms, have the nice property of having the same cohomology as X, but otherwise, from the topological point of view, do not seem to satisfy any other universal properties. However, at the cost of changing the cohomology ring, one can produce a right adjoint to the inclusion of commutative dg-algebras into E_∞-algebras over $\widehat{\mathbb{Z}_p}$, which is defined on those E_∞-algebras with the homotopy type of cochains on a space. More precisely, we have the following result.

Theorem 4.1.4. *Let $A \in \mathsf{Com-alg}$, $X \in \mathsf{sSet}$ and $i : \mathsf{Com-alg} \to E_\infty - \mathsf{alg}$ be the inclusion functor. Then there exists a cochain algebra $\mathscr{R}^*$ taking values in strictly commutative algebras such that there is an equivalence of derived mapping spaces*

$$\mathsf{Map}_{\mathsf{Com-alg}}\left(A, \mathscr{R}^*(X)\right) \cong \mathsf{Map}_{E_\infty-\mathsf{alg}}\left(i(A), C^*\left(X,\widehat{\mathbb{Z}_p}\right)\right).$$

Structure of the chapter

This paper has the following structure. First we recall some preliminaries on rational homotopy theory, divided power algebras and E_∞-algebras. Then in part 3, we define the de Rham forms, compute ther cohomology and relate them to a subalgebra of the singular cochains complex. In part 4, we construct a related complex that acts as the best strictly commutative approximation to the cochain complex. Finally, in the last part, we examine the homotopy invariants that can be extracted from the p-adic de Rham forms and prove a formality theorem.

Notation and conventions

In this paper, we work on the category of unbounded cochain complexes over some base field or ring with cohomological convention. That is, the differential $d : A^* \to A^{*+1}$ of a cochain complex $(A, d)$is of degree 1. The degree of a homogeneous element x is denoted by $|x|$. The symmetric group on n elements is denoted $\mathbb{S}_n$. We follow the Koszul sign rule. That is, the symmetry isomorphism $U \otimes V \xrightarrow{\cong} V \otimes U$ that identifies two graded vector spaces is given on homogeneous elements by $u \otimes v \mapsto (-1)^{|u||v|} v \otimes u$. Algebras over operads are always differential graded (dg) and cohomological. We will frequently omit the adjective "dg" and assume it is implicitly understood. The ring of p-adic numbers is denoted $\widehat{\mathbb{Z}_p}$. The functor of p-adic de Rham forms $\Omega(-)$ generally depends on a prime p, but to avoid needing to specify this each time, we shall assume that p is fixed.

This is a short chapter and we do not intend to load it excessively with recollections; so therefore we refer to [56] for the definition of an operad and other basic notions.

4.2 Preliminaries

In this part, we shall discuss the basic preliminaries. First, we shall discuss E_∞-algebras and why they model spaces. Next, we shall review the basic ideas from rational homotopy theory that we shall need. Then, we shall discuss the different notions of algebra in mixed characteristic and define divided power algebras. Finally, we shall define the homotopy categories of commutative and E_∞-algebras. This last section contains some non-standard material, and is likely the only section the expert reader needs to read.

4.2.1 E_∞-algebras and spaces

Recall the notion of an E_∞-algebra from Section 3.2.2. In practice, the main reason why the Barratt-Eccles operad is useful is that the cochain complex of a space X is an algebra over it (with integral coefficients).

Theorem 4.2.1. *[6] For any simplicial set X, we have evaluation products $\mathcal{E}(r) \otimes C^*(X)^{\otimes r} \to C^*(X)$ which are functorial in X which give the cochain complex $C^*(X)$ the structure of an algebra over the Barratt-Eccles operad $\mathcal{E}$. In particular, the classical cup-product of cochains is an operation $\mu_0 : C^*(X)^{\otimes 2} \to C^*(X)$ associated to an element $\mu_0 \in \mathcal{E}(2)^0$.*

While not explicitly stated in that paper, it is a straightforward to observe that the $\mathcal{E}$-algebra structure on $C^*(X)$ constructed in [6] is *stable*. This means the following. Recall that Barratt-Eccles operad admits the Smith filtration (for the precise definition see [5, 81]).

$$\mathrm{Ass} = \mathcal{E}^{(1)} \hookrightarrow \mathcal{E}^{(2)} \hookrightarrow \cdots \hookrightarrow \mathcal{E}$$

An $\mathscr{E}$-algebra A is *stable* if for all $\gamma \in \mathscr{E}()$ and $x_1 \otimes \cdots \otimes x_r \in A^{\otimes r}$ the following condition is satisfied. Suppose that $\min\{|x_1|, |x_2|, \ldots, |x_r|\} = k$. Then if $p(\gamma) = 0$ where p is the projection $p : \mathscr{E} \to \mathscr{E}^{(k)}$ then $\gamma(x_1 \otimes \cdots \otimes x_r). = 0$. In other words, the only part of the $\mathscr{E}$-algebra structure that does not vanish on elements of degree n is the E_n-part.

4.2.2 Rational homotopy theory

In this section, we review the rational case and explain the connection between E_∞ algebras, rational topological spaces and strictly commutative. We begin by explaining the constructions Sullivan's A_{PL} functor, which will be our basis for later constructing the p-adic de Rham form functor Ω. In particular, in Proposition 4.2.2 we shall explain why Sullivan's approach does not work in positive characteristic. Next, we explain the equivalence in approach with that of singular cochains. Next, we shall discuss the rectification of E_∞-algebras with rational coefficients. Finally, we conclude by explaining Cartan's approach to cochain algebras.

4.2.2.1 Sullivan's approach to rational homotopy theory

In this section, we briefly revise Sullivan's approach to homotopy theory [85]. In general, if R is a commutative ring, we call any functor $\mathsf{sSet} \to \mathsf{CDGA}_R$ a *cochain algebra.* Recall that the Sullivan's *PL-forms functor* $A_{\mathrm{PL}} : \mathsf{sSet} \to \mathsf{CDGA}_{\mathbb{Q}}$, also called *the rational de Rham forms functor*, is explicitly defined by taking simplicial set maps against the cochain algebra $A^*_\bullet$,

$$A_{\mathrm{PL}}(X) = \mathsf{sSet}(X, A_\bullet),$$

where

$$A_n = A_{\mathrm{PL}}(\Delta^n) = \frac{\mathrm{Sym}(t_0, \ldots, t_n, dt_0, \ldots, dt_n)}{\left(\sum t_i - 1, \sum dt_i\right)} \cong \mathrm{Sym}(t_1, \ldots, t_n, dt_1, \ldots, dt_n).$$

Here, each t_i is of degree 0, and dt_i is a degree 1 generator identified with $d(t_i)$ by abuse of notation. See [14, 85]. The object $\mathsf{sSet}(X, A^*_\bullet)$ is a commutative dg-algebra where $\mathsf{sSet}(X, A^*_\bullet)_k = \mathrm{Hom}_{\mathsf{sSet}}(X, \Omega^k_\bullet)$ and the differential is induced by the differential $\Omega^k_\bullet \to \Omega^{k+1}_\bullet$. The algebras Ω_n are, in a very precise sense, the polynomial differential forms with rational coefficients on the n-simplex, and gather into a simplicial object $\Omega_\bullet$ in the category $\mathsf{CDGA}_{\mathbb{Q}}$.

In the case of A_{PL}, cochain algebras satisfy two additional key properties. First is the *Poincaré Lemma*, which asserts that

$$\widetilde{H}^*(A_n; \mathbb{Q}) = 0.$$

Second is *extendablity*; which asserts that the restriction map $A_{PL}(X) \to A_{PL}(Y)$ is surjective for every inclusion of simplicial sets $Y \subseteq X$. Although the polynomial forms exist over any base ring R, it is essential that $\mathbb{Q} \subseteq R$ for the Poincaré lemma to hold. To prove this, one can observe that

$$A_{\mathrm{PL}}(\Delta^n) \cong (\mathbb{Q}[t] \otimes \mathrm{Sym}(dt))^{\otimes n},$$

then give an explicit contraction $K : \mathbb{Q}[t] \otimes \mathrm{Sym}(dt) \xrightarrow{\simeq} \mathbb{Q}$, given by geometric integration, and extend it (non-canonically) as a contraction from the n-fold tensor product to $\mathbb{Q}$. Although there are choices for this extension, there is a choice given by geometric integration which is quite natural. For example, the explicit formulas for Δ^2 can be taken to be

$$K\left(t_j^n dt_j\right) = \frac{1}{n+1} t_j^{n+1}, \quad j = 1, 2,$$

$$K\left(t_1^n t_2^m dt_1 dt_2\right) = \frac{1}{2}\left(\frac{1}{n+1} t_1^{n+1} t_2^m dt_2 + \frac{1}{m+1} t_1^n t_2^{m+1} dt_1\right).$$

Here, we see the fundamental role played by division by n. In positive characteristic, this is impossible to achieve. That is, for every prime p, the cohomology algebra $\widetilde{H}^*\left(A_n;\mathbb{F}_p\right)$ is non-trivial, see Proposition 4.2.2 for the precise computation which we learned from José Moreno-Fernández.

Proposition 4.2.2. *The cohomology of $A_{\mathrm{PL}}(\Delta^n)$ with $\mathbb{F}_2$-coefficients is in bijection with the tuples*

$$\left(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n\right)\in\mathbb{Z}^n_{\geq 0}\times\{0,1\}^n$$

satisfying

$$\alpha_i \text{ even} \implies \beta_i=0, \qquad \text{and} \qquad \alpha_i \text{ odd} \implies \beta_i=1.$$

For a fixed tuple as above, its cocyle representative is explictly given by

$$t_1^{\alpha_1}\cdots t_n^{\alpha_n}(dt_1)^{\beta_1}\cdots(dt_n)^{\beta_n}.$$

Proof. First, we compute the cohomology with $\mathbb{F}_2$-coefficients of $A_{\mathrm{PL}}\left(\Delta^1\right)$. Identify $A_{\mathrm{PL}}\left(\Delta^1\right)=S(t,dt)$. Applying Leibniz's rule inductively, we find that

$$d\left(t^k\right)=kt^{k-1}dt \quad \text{for all} k.$$

Therefore, the non-trivial cocyles of $A_{\mathrm{PL}}\left(\Delta^1\right)$ are all the even powers t^{2k} in degree 0 and all the elements of the form $t^{2k+1}dt$ for $k\geq 0$ in degree 1. By inspection, these cohomology classes are all distinct. Thus,

$$H^n\left(A_{\mathrm{PL}}\left(\Delta^1\right);\mathbb{F}_2\right)=\begin{cases}\left[t^{2k}\right] & \forall\, k\geq 0 \quad \text{in degree } 0,\\ \left[t^{2k+1}dt\right] & \forall\, k\geq 0 \quad \text{in degree } 1.\end{cases}$$

It is well-known that $A_{\mathrm{PL}}(\Delta^n)\cong A_{\mathrm{PL}}\left(\Delta^1\right)^{\otimes n}$, with the following identifications for all $i=1,\dots,n$:

$$t_i=1\otimes\cdots\otimes\underbrace{t}_{i}\otimes\cdots\otimes 1, \qquad \text{and} \qquad dt_i=1\otimes\cdots\otimes\underbrace{dt}_{i}\otimes\cdots\otimes 1.$$

Since we are working over a field, the Künneth map is an isomorphism, so that

$$H^*\left(A_{\mathrm{PL}}(\Delta^n)\right)\cong H^*\left(A_{\mathrm{PL}}\left(\Delta^1\right)^{\otimes n}\right)\cong H^*\left(A_{\mathrm{PL}}\left(\Delta^1\right)\right)^{\otimes n}.$$

A straightforward computation gives the cohomology classes mentioned in the statement. □

4.2.2.2 Comparison between de Rham forms and singular cochains

We next explain the comparison between the A_{PL} functor and the singular cochains $C^*(-,\mathbb{Q})$ functor. The material in this section is essentially due to Sullivan [85], Bousfield-Gugenheim [14] and Mandell [58]. Recall that $C^*(\Delta^*,\mathbb{Q})$ is a simplicial $\mathscr{E}$-algebra, with the $\mathscr{E}$-algebra structure given by Theorem 4.2.1.

Definition 4.2.3. Let A^* and B^* be simplicial $\mathscr{E}$-algebras. The *tensor product* $(A\otimes B)^*$ is given by

$$(A\otimes B)^k\left(\Delta^n\right)=\bigoplus_{i+j=k}A^i\left(\Delta^n\right)\otimes B^j\left(\Delta^n\right)$$

This object is equipped with the obvious face and degeneracy maps. The $\mathscr{E}$-algebra structure on $(A\otimes B)^*(\Delta^n)$ is induced from the diagonal on $\mathscr{E}$ in the obvious way.

Proposition 4.2.4. *[85] Suppose that A^* and B^* are extendable cochain algebras that both satisfy the Poincaré lemma. Then $(A \otimes B)^*$ also satisfies the Poincaré lemma and is extendable. In particular,*

$$H^*(A \otimes B)(X) = H^*(X)$$

Now one has the following zig-zag of simplicial $\mathscr{E}$-algebras.

$$A^*_{PL}(\triangle^*) \xrightarrow{\mathrm{id}\otimes 1} (A_{PL} \otimes C^*)(\triangle^*) \xleftarrow{1\otimes \mathrm{id}} C^*(\triangle^*) \tag{4.1}$$

For all $X \in \mathsf{sSet}$, this extends to a zig-zag of $\mathscr{E}$-algebras by the universal property of simplicial sets

$$A^*_{PL}(X) \xrightarrow{\sim} (A_{PL} \otimes C)^*(X) \xleftarrow{\sim} C^*(X)$$

and by Proposition 4.2.4, these maps are quasi-isomorphisms.

4.2.2.3 Rectification

There is a weak equivalence of operads $\phi : \mathscr{E} \xrightarrow{\sim} \mathsf{Com}$, so it is natural to ask whether or not the pair $(\phi^*, \phi_!)$ forms a Quillen equivalence between $\mathscr{E}$-algebras and Com-algebras. If there is, then *rectification* is said to occur. With coefficients in $\mathbb{Q}$, this is indeed the case; see for example [89]. In particular, this implies that, in zero characteristic, that every $\mathscr{E}$-algebra A has a strictly commutative model given by $\phi_!(A)$.

4.2.2.4 Cartan's approach to cochain algebras

Outside of characteristic zero, it appears to be very difficult to find commutative cochain algebras that both satisfy the Poincaré Lemma and which are extendable. In [19], Cartan extended Sullivan's approach to more general cochain algebras. In particular, he proved the the following generalisation of Theorem 4.2.4.

Theorem 4.2.5. *[19] Let R be a commutative ring, X be a simplicial set and $A^*_\bullet$ be a simplicial cochain R-algebra. Let the simplicial cochain R-algebra $Z^k A$ be given by the kernel of the differential $d : A^k \to A^{k+1}$. Suppose further that $\pi_i(A^k)$ and $\pi_i(Z^k A)$ are zero when $i \neq k$. Then one has a natural isomorphism $H^k(A(X)) \cong H^k(X, \pi_k(Z^k A))$. Moreover this isomorphism is multiplicative when the $Z^k A$ are flat R-modules.*

4.2.3 A closer look at divided power algebras

Recall the notion of divided power algebras from Section 3.2.1. We shall mainly be interested in the case $\mathscr{P} = \mathsf{Com}$, so it will be useful to be more explicit in this case. Let R be a commutative unital ring. The cofree conilpotent coalgebra on a graded projective R-module V, also called *tensor coalgebra* on V, is the graded R-module

$$TV = \bigoplus_{k \geq 0} T^k V, \tag{4.2}$$

where $T^k V = V^{\otimes k}$ for all k, endowed with the deconcatenation coproduct,

$$\Delta[v_1|\cdots|v_n] = \sum_{i=0}^{n} [v_1|\cdots|v_i] \otimes [v_{i+1}|\cdots|v_n].$$

A basis tensor of TV is therefore denoted $[v_1|\cdots|v_n]$ rather than $v_1 \otimes \cdots \otimes v_n$. The direct sum decomposition in (4.2) is called the *word-length decomposition* of TV, and elements in $T^k V$

are said to be of word-length k. The tensor coalgebra can be endowed with the associative and commutative shuffle product $\circledast$,

explicitly given by

$$[v_1|\cdots|v_p] \circledast [v_{p+1}|\cdots|v_n] = \sum_{\sigma \in S(p,q)} \varepsilon(\sigma)\, v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

Here, $S(p,q)$ is the set of (p,q)-shuffles, given by those permutations of $p+q$ elements such that

$$\sigma(1) < \cdots < \sigma(p) \quad \text{and} \quad \sigma(p+1) < \cdots < \sigma(p+q),$$

while $\varepsilon(\sigma)$ stands for the Koszul sign associated to the permutation σ. Endowed with the deconcatenation coproduct and the shuffle product, TV is a commutative bialgebra.

There is a natural action of the symmetric group S_n on the word-length n components of TV, given by

$$\sigma \cdot [v_1|\cdots|v_n] = \varepsilon(\sigma) \cdot [v_{\sigma^{-1}(1)}|\cdots|v_{\sigma^{-1}(n)}].$$

For each n, one can form the submodule of S_n-invariants under this action, that is, the submodule generated by those word-length n homogeneous elements x with $\sigma \cdot x = x$ for all $\sigma \in S_n$. Denote by $\Gamma^n V$ this submodule of $T^n V$. Summing over all n, we form a graded submodule of TV,

$$\Gamma(V) = \bigoplus_{n \geq 0} \Gamma^n V.$$

The submodule ΓV happens to be a subalgebra of TV, and it is called the *free commutative divided powers algebra on V*. It comes equipped with set-theoretical maps $\gamma^k : \Gamma V \to \Gamma V$ determined by

$\gamma^0(v) = 1$ for all $v \in V_{2n}$,

$\gamma^n(v) = [v|\cdots|v]$($n$ times if v is of even degree and $n \geq 1$, and

$\gamma^n(v) = 0$ if v is of odd degree and $n \geq 2$.

In particular, the following two identities are satisfied on homogeneous elements (the second one only when u is of even degree):

$$\gamma^n(u+v) = \sum_{i=0}^{n} \gamma^i(u)\gamma^{n-i}(v),$$

$$\gamma^i(u)\gamma^j(u) = \binom{i+j}{i} \gamma^{i+j}(u).$$

Intuitively, the element $\gamma^n(u)$ is a replacement of the element $\frac{u^n}{n!}$ whenever it does not make sense to divide by $n!$.

Assume V is freely generated by the homogeneous elements $\{v_i\}$. Then, an R-linear basis of ΓV is explicitly given by elements of the form

$$\gamma^{k_1}(v_1)\gamma^{k_2}(v_2)\cdots\gamma^{k_r}(v_r)$$

for all $r \geq 0$, $k_i \geq 0$, with $k_i \in \{0,1\}$ if $|v_i| = 1$.

Example 4.2.6. Let $t \geq 1$. A very useful example occurs when V is a free R-module $R^{\otimes t}$ with basis $x_1, \ldots x_t$. In this case ΓV is usually called *divided power polynomial algebra* and denoted $R\langle x_1, \ldots, x_t\rangle$. Explicitly, we have that

$$R\langle x_1, \ldots, x_t\rangle := \bigoplus_{n_1, \ldots, n_t \geq 0} R x_1^{[n_1]}, \ldots, x_t^{[n_t]}$$

with multiplication is given by

$$x_i^{[n]} x_i^{[m]} = \frac{(n+m)!}{n!m!} x_i^{[n+m]}$$

We also set $x_i = x_i^{[1]}$. Note that $1 = x_1^{[0]} \cdots x_t^{[0]}$. There is an canonical R-algebra map $R\langle x_1, \ldots, x_t\rangle \to R$ sending $x_i^{[n]}$ to zero for $n > 0$. The kernel of this map is denoted $R\langle x_1, \ldots, x_t\rangle_+$

Example 4.2.7. When $R = \mathbb{F}_p$, as a commutative dg-algebras

$$\mathbb{F}_p\langle x\rangle = \begin{cases} \mathbb{F}_p[x_1, x_2, \ldots]/\left(x_1^p, x_2^p, \ldots\right) & \text{with } |x_k| = k|x|, \text{ when } |x| \text{ is even.} \\ \mathbb{F}_p[x]/\left(x^2\right) & \text{otherwise.} \end{cases}$$

Example 4.2.8. When $R = \widehat{\mathbb{Z}_p}$, the divided powers algebra $\widehat{\mathbb{Z}_p}\langle x_1, \ldots x_t\rangle$ is a subalgebra of usual polynomial algebra $\mathbb{Q}_p[x_1, \ldots x_t]$ via the injective map

$$\widehat{\mathbb{Z}_p}\langle x_1, \ldots x_t\rangle \hookrightarrow \mathbb{Q}_p[x_1, \ldots x_t]$$

$$x_i^{[n]} \mapsto \frac{1}{n!} x_i^n$$

4.2.4 The homotopy theory of $\mathscr{E}$-algebras and commutative dg-algebras

In this subsection we shall discuss the existence of model structures on categories of $\mathscr{P}$-algebras and specialise to the cases of $\mathscr{E}$-algebras. The key takeaway of this subsection is that, in this paper, we shall work with the external homotopy category of commutative algebras instead of the naive (internal) one.

4.2.4.1 The case of E_∞ algebras

One has the following general fact.

Theorem 4.2.9. *[45] Let $\mathscr{P}$ be a $\mathbb{S}$-split (or cofibrant) operad over a commutative ring R. Then the category of $\mathscr{P}$-algebras over R is a closed model category with quasi-isomorphisms as the weak equivalences and surjective maps as fibrations.*

The Barratt-Eccles operad is $\mathbb{S}$-split. This immediately gives the model structure on E_∞-algebras over $\widehat{\mathbb{Z}_p}$.

Definition 4.2.10. The model category $\mathscr{E}-\mathsf{alg}$ of E_∞-algebras is the category of algebras over the Barratt-Eccles operad, in dg-modules over $\widehat{\mathbb{Z}_p}$, equipped with the model structure of Theorem 4.2.9. It has quasi-isomorphisms of chain complexes as weak equivalences and surjective maps as fibrations.

4.2.4.2 The case of commutative dg-algebras

We have already mentioned that in characteristic 0, the homotopy theory of commutative dg-algebras coincides with that of $\mathscr{E}$-algebras. In positive characteristic the relationship is much more complex. Commutative dg-algebras come with an obvious notion of weak equivalence, that is, algebra maps that are quasi-isomorphisms of cochain complexes. Localising with respect to these maps gives a well-defined homotopy category, which we call the *internal homotopy category.* The main result of [30] shall show that this is the wrong homotopy category to consider when working with spaces. Instead, we shall consider the *external homotopy category.*

Definition 4.2.11. The *external homotopy category* of commutative algebras is defined by taking the full subcategory of $\mathscr{E}-\mathsf{alg}$ given by $\mathscr{E}$-algebras that are quasi-isomorphic to strictly commutative dg-algebras and localising it at quasi-isomorphisms of $\mathscr{E}$-algebras.

The external homotopy category of commutative algebras is clearly a subcategory of the homotopy category of $\mathscr{E}$-algebras. It works well for forming constructions such as derived mapping spaces.

4.3 The de Rham forms over $\widehat{\mathbb{Z}_p}$

We saw in Proposition 4.2.2, that Sullivan's A_{PL} functor fails to generalise to positive characteristic. This problem can partially be solved by trading the free polynomial algebra appearing in the definition for a free divided powers algebra. The resulting object, $\Omega^*(X)$, has the correct cohomology but is not quasi-isomorphic to the singular cochains on X as an $\mathscr{E}$-algebra. It is however very closely related. We shall see in the next part that it enables us to define and calculate Massey products in situations where this machinery was previously inconvenient, for example, one has Massey products arising in the torsion part of the cohomology.

This section of the paper is broken into four subsections. The first is devoted to defining $\Omega^*(X)$. In the second, we compute the cohomology of this object. In the third, we explain how it is related to the singular cochain algebra. Finally, in the fourth, we explain the universal property defining it.

4.3.1 The algebra of p-adic de Rham forms

In this subsection, we introduce the key object of this paper - a generalisation of Sullivan PL-forms to the p-adic setting. We show that this generalisation satisfies the Poincaré lemma, but not the extendable condition (as defined in Section 4.2.2.1). As mentioned in the introduction, a similar object to $\Omega^*(X)$ appears in [19, Section 4].

Definition 4.3.1. The *p-adic de Rham cohain algebra* $\Omega^*_\bullet$ is a simplicial cochain algebra that has for n-simplices

$$\Omega^*_n = \left(\frac{\widehat{\mathbb{Z}_p}\langle x_0, \ldots x_n\rangle \otimes \Lambda(dx_0, \ldots, dx_n)}{(x_0 + \cdots + x_n - p, dx_0 + \cdots dx_n)}\right)^c, \ |x_i| = 0,\ |dx_i| = 1.$$

Here, the $(-)^c$ indicates that we are taking the closure of this set under an formal interchange of variables

$$x_r \mapsto p - \sum_{i=0}^{j} x_{k_i}$$

$$dx_r \mapsto -\sum_{i=0}^{j} dx_{k_i}$$

for all r and such that the x_{k_i} are all distinct from each other and from x_r.

The differential $d : \Omega_n^* \to \Omega_n^{*+1}$ is determined by the formula

$$d(f) = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} dx_i$$

for $f \in \Gamma_p(x_0, \dots, x_n) / (x_0 + \cdots + x_n - p)$ and then extended by the Leibniz rule. The simplicial structure is defined as follows

$$d_i^n : \Omega_n^* \to \Omega_{n+1}^* : x_k \mapsto \begin{cases} x_k & \text{for } k < i. \\ 0 & \text{for } k = i. \\ x_{k-1} & \text{for } k > i. \end{cases}$$

and

$$s_i^n : \Omega_n^* \to \Omega_{n-1}^* : x_k \mapsto \begin{cases} x_k & \text{for } k < i. \\ x_k + x_{k+1} & \text{for } k = i. \\ x_{k+1} & \text{for } k > i. \end{cases}$$

Example 4.3.2. The 0-simplices $\Omega_\bullet^0$ are given by

$$\frac{\widehat{\mathbb{Z}_p}\langle x_0\rangle \otimes \Lambda(dx_0)}{(x_0 - p, dx_0)} = \widehat{\mathbb{Z}_p}[p, \frac{p^2}{2}, \dots \frac{p^k}{k!}, \dots] = \widehat{\mathbb{Z}_p}$$

On the other hand, one has that

$$\frac{\widehat{\mathbb{Z}_p}\langle x_0\rangle \otimes \Lambda(dx_0)}{(x_0 - 1, dx_0)} = \widehat{\mathbb{Z}_p}[\frac{1}{p}, \frac{1}{2\cdot p^2}, \dots \frac{1}{k!\cdot p^k}, \dots] = \widehat{\mathbb{Q}_p}.$$

This is why we must impose the condition that $x_0 + \cdots + x_n = p$ and cannot imitate the $x_0 + \cdots + x_n = 1$ condition from the definition of the algebra of piecewise polynomial forms.

The p-adic de Rham forms cochain complex have one of the two desirable properties of a cochain algebra: they satisfy the Poincaré lemma.

Proposition 4.3.3. *The simplicial cochain algebra Ω^* satisfies the Poincaré lemma. In other words:*

$$H^i(\Omega_n^*) = \begin{cases} \widehat{\mathbb{Z}_p} & \text{if } i = 0. \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Observe that one has the following isomorphism of cochain algebras

$$\widehat{\mathbb{Z}_p}\langle x_0, \dots x_n\rangle \otimes \Lambda(dx_0, \dots, dx_n) \cong \left(\widehat{\mathbb{Z}_p}\langle x\rangle \otimes \Lambda(dx)\right)^{\otimes n+1}.$$

Since $\widehat{\mathbb{Z}_p}\langle x\rangle \otimes \Lambda(dx)$ is free as a $\widehat{\mathbb{Z}_p}$-module, we can apply the Künneth theorem to deduce

$$H^*\left(\widehat{\mathbb{Z}_p}\langle x_0, \dots x_n\rangle \otimes \Lambda(dx_0, \dots, dx_n)\right) = H^*\left(\widehat{\mathbb{Z}_p}\langle x\rangle \otimes \Lambda(dx)\right)^{\otimes n+1}.$$

So the problem reduces to computing $H^*\left(\widehat{\mathbb{Z}_p}\langle x\rangle \otimes \Lambda(dx)\right)$. The elements $x^{[i-1]}dx$ form a linear base for the degree 1 part of this algebra. Further, one has $d\left(x^{[i]}\right) = x^{[i-1]}dx$. The conclusion follows. □

As in Section 4.2.2.1, we now Kan extend our cochain algebra along the inclusion $\triangle^* \to$ sSet.

Definition 4.3.4. Let X be a simplicial set and let p be a fixed prime number.. The p-adic de Rham forms on X is the commutative dg-algebra

$$\Omega^*(X) = \mathsf{sSet}\left(X, \Omega^*_\bullet\right).$$

where $\mathsf{sSet}(X, \Omega_\bullet)_k = \mathrm{Hom}_{\mathsf{sSet}}\left(X, \Omega^k_\bullet\right)$ and the differential is induced by the differential $\Omega^k_\bullet \to \Omega^{k+1}_\bullet$.

The main difficulty with this approach is that $\Omega^*_\bullet$ is not an extendable cochain algebra. In subsection 4.3.2, we shall use Theorem 4.2.5 to resolve this problem.

Proposition 4.3.5. *The cochain algebra $\Omega^*_\bullet$ is not extendable.*

Proof. Recall that $\Omega^*\left(\triangle^0\right)$ is $\widehat{\mathbb{Z}_p}$. It follows that $\Omega^*\left(\partial\triangle^1\right) = \widehat{\mathbb{Z}_p} \oplus \widehat{\mathbb{Z}_p}$. Consider the element $(1, p) \in \Omega^*\left(\partial\triangle^1\right)$. It suffices to prove that there does not exist a polynomial $f(x_0, x_1) \in \Omega^*\left(\triangle^1\right)$ such that $f(0, p) = 1$ and $f(p, 0) = p$.

Indeed, assume towards contradiction that such an f exists. Then, as p is not invertible in $\widehat{\mathbb{Z}_p}$, $f(0, p) = 1$ implies that f has a constant term which is not divisible by p. On the other hand, $f(p, 0) = p$ implies that the constant term of f is divisible by p. We have obtained the desired contradiction. We can conclude that the map $\Omega^*\left(\triangle^1\right) \to \Omega^*\left(\partial\triangle^1\right)$ is not surjective and therefore that $\Omega^*_\bullet$ is not extendable. □

4.3.1.1 Some examples

To illustrate the definition of $\Omega^*(X)$, we compute some examples for specific topological spaces X. First, we have the most trivial case.

Example 4.3.6. When X is a standard n-simplex, one has the de Rham forms $\Omega^*(\triangle^n) = \Omega^*_n$, where Ω^*_n is the algebra defined in Definition 4.3.1.

Next, we compute the next simplest group of examples, the spheres of various dimension.

Example 4.3.7. For the usual simplicial model of $S^1 = \triangle^1/\partial\triangle^1$, one has the following: the $\widehat{\mathbb{Z}_p}$-module $\Omega^0\left(S^1\right) = (x_0x_1) \oplus \widehat{\mathbb{Z}_p}$, where (x_0x_1) is the ideal generated by the monomial in

$$\frac{\widehat{\mathbb{Z}_p}\langle x_0, x_1\rangle}{\left(x_0 + x_1 - p\right)}$$

This can also be written, purely in terms of one variable as the ideal generated by $x_0^2 - px_0$. In the classical computation by Sullivan, this ideal would have been generated by $x_0^2 - x_0$. The $\widehat{\mathbb{Z}_p}$-module $\Omega^1\left(S^1\right)$ is

$$\frac{\widehat{\mathbb{Z}_p}\langle x_0, x_1\rangle dx_0 \oplus \widehat{\mathbb{Z}_p}\langle x_0, x_1\rangle dx_1}{\left(x_0 + x_1 - p, dx_0 + dx_1\right)} = \widehat{\mathbb{Z}_p}\langle x_0\rangle dx_0.$$

One can easily compute the cohomology of $\Omega^*\left(S^1\right)$. One has $H^0\left(\Omega^*\left(S^1\right)\right) = \widehat{\mathbb{Z}_p}$, which is generated by 1. One therefore also has $H^1\left(\Omega^*\left(S^1\right)\right) = \widehat{\mathbb{Z}_p}$ which is generated by dx_0.

In general it follows that, for $S^n = \triangle^n/\partial\triangle^n$, one has that

$$\Omega^i\left(S^n\right) = \begin{cases} \widehat{\mathbb{Z}_p}\langle x_0, x_1, \cdots x_{n-1}\rangle dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{n-1} & \text{for } i = n. \\ \left(\{x_{\sigma(0)}x_{\sigma(1)}\ldots x_{\sigma(i)}dx_{\sigma(i+1)} \wedge \cdots \wedge dx_{\sigma(n)} : \sigma \in \mathbb{S}_{n+1}\}\right) & \text{for } i < n. \end{cases}$$

where $\mathbb{S}_{n+1}$ acts on the set of indices $\{0,1,\cdots,n\}$ by permutation and we replace x_n with $p - x_0 + \cdots + x_{n-1}$ and dx_n with $-dx_0 + \cdots - dx_{n-1}$. One therefore recovers that $H^n\left(\Omega^*\left(S^n\right)\right) = \widehat{\mathbb{Z}_p}$ which is generated by $dx_1 \wedge \cdots \wedge dx_{n-1}$.

We conclude this section by computing an example with non-trivial torsion in its cohomology and therefore which would not have been possible to model in Sullivan's framework.

Example 4.3.8. The space $\mathbb{R}P^2$ has a simplicial model X with nondegenerate simplices given by $X_2 = \{U, V\}$, $X_1 = \{a, b, c\}$ and $X_0 = \{v, w\}$, with face maps as follows

$$\delta_0 U = b, \qquad \delta_1 U = a, \qquad \delta_2 U = c, \qquad \delta_0 V = a, \qquad \delta_1 V = b, \qquad \delta_2 V = c,$$

$$\delta_0 a = w, \qquad \delta_1 a = v, \qquad \delta_0 b = w, \qquad \delta_1 b = v, \qquad \delta_0 c = v, \qquad \delta_1 c = v$$

We therefore compute $\Omega^*(\mathbb{R}P^2)$. One can easily verify that

$$\Omega^2\left(\mathbb{R}P^2\right) = \widehat{\mathbb{Z}_p}\langle x_0, x_1\rangle dx_0 \wedge dx_1 \oplus \widehat{\mathbb{Z}_p}\langle y_0, y_1\rangle dy_0 \wedge dy_1$$

Next, one wants to compute $\Omega^1\left(\mathbb{R}P^2\right)$. Elements contained in this are clearly of the form

$$\begin{aligned} f = U_0(x_0, x_1, x_2)dx_0 + U_1(x_0, x_1, x_2)dx_0 + U_2(x_0, x_1, x_2)dx_2 + \\ V_0(y_0, y_1, y_2)dy_0 + V_1(y_0, y_1, y_2)dy_0 + V_2(y_0, y_1, y_2)dy_2 \end{aligned}$$

where $U_i, V_i \in \widehat{\mathbb{Z}_p}\langle t_0, t_1, t_2\rangle$ and must satisfy relations coming from the simplicial structure of X. Firstly $\delta_0 U = \delta_1 V$,$\delta_1 U = \delta_0 V$ and $\delta_2 U = \delta_2 V$. This implies that

$$U_1(0,t,s) = V_0(t,0,s), \qquad U_2(0,t,s) = V_2(t,0,s) \qquad V_1(0,t,s) = U_0(t,0,s), \qquad V_2(0,t,s) = U_2(t,0,s)$$

$$U_0(t,s,0) = V_0(t,s,0), \qquad U_1(t,s,0) = V_1(t,s,0)$$

Lastly, we have the bottom row of relations, which imply that

$$U_0(s,0,0) = U_1(0,s,0)$$

Similarly the elements of $\Omega^0\left(\mathbb{R}P^2\right)$ are of the form

$$f(x_0, x_1, x_2) + g(x_0, x_1, x_2)$$

with $f, g \in \widehat{\mathbb{Z}_p}\langle t_0, t_1, t_2\rangle$ and where

$$f(0,s,t) = g(s,0,t), \qquad f(s,0,t) = g(0,s,t), \qquad f(s,t,0) = g(s,t,0)$$

and

$$f(s,0,0) = f(0,s,0).$$

4.3.2 The cohomology of $\Omega^*(X)$

In this section, we compute the cohomology ring of $\Omega^*(X)$ and show that it coincides with the usual cohomology ring of X. The main result is the following theorem.

Theorem 4.3.9. *Let X be a simplicial set. The cohomology ring of $\Omega^*(X)$ is isomorphic to the singular cohomology of X. In other words, one has a ring isomorphism*

$$H^*(\Omega^*(X)) \cong H^*\left(X, \widehat{\mathbb{Z}_p}\right).$$

The arguments in this section are very similar to that in [19, Section 4]. The strategy is that to apply Theorem 4.2.5. In order to do so, it is necessary to compute the homotopy groups $\pi_i\left(\Omega^k\right)$ and $\pi_i\left(\Omega^k\right)$.

Proposition 4.3.10. *The homotopy groups of Ω^k are as follows:*

$$\pi_i\left(\Omega^k\right) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{when } i = k \\ 0 & \text{otherwise.} \end{cases}$$

with the generator of $\pi_i\left(\Omega^k\right)$ being $dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{k-1}$.

First, make the auxiliary definition.

$$\overline{\Omega}^*_n = \left(\frac{\widehat{\mathbb{Z}_p}\langle x_0, \dots x_n\rangle \otimes \Lambda(dx_0, \dots, dx_n)}{(x_0 + \cdots + x_n - p)}\right)^c, \ |x_i| = 0, \ |dx_i| = 1.$$

Here, the $(-)^c$ indicates that we are taking the closure of this set under an formal interchange of variables

$$x_r \mapsto p - \sum_{i=0}^{j} x_{k_i}$$

for all r and such that the x_{k_i} are all distinct from each other and from x_r.

Let $N_*(-)$ be the normalised chains functor. The homotopy groups $\pi_i\left(N_*\left(\overline{\Omega}^*\right)\right)$ are as follows.

Lemma 4.3.11. *The homotopy groups of $N_*\left(\overline{\Omega}^*\right)$ are as follows:*

$$\pi_k\left(N_*\left(\overline{\Omega}^*\right)\right) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & \text{when } k = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Suppose $k > 0$, then consider a k-cycle $\omega(x_0, \dots x_k) \in \Omega^k$ such that $\partial_i\omega = 0$. We may use the closure condition to rewrite $\omega(x_0, \dots x_k)$ such that $\partial_i\omega = 0$ in

$$\widehat{\mathbb{Z}_p}\langle x_0, \dots x_n\rangle \otimes \Lambda(dx_0, \dots, dx_n)$$

It then follows that the $(k+1)$-chain $\omega(x_1, \dots x_{k+1})$ is such that $\partial_0\omega(x_1, \dots x_{k+1}) = \omega(x_0, \dots x_k)$ and $\partial_i\omega(x_1, \dots x_{k+1}) = 0$ for $i > 0$.

When $k = 0$, the chains are

$$\frac{\widehat{\mathbb{Z}_p}[x_0]}{(x_0 - p)}.$$

But the image of the differential is the ideal generated by (x_0). So therefore

$$\pi_0\left(N_*\left(\overline{\Omega}^*\right)\right) = \frac{\widehat{\mathbb{Z}_p}[x_0]}{(x_0, x_0 - p)} = \mathbb{Z}/p\mathbb{Z}.$$

□

Proof of Proposition 4.3.10. Consider the ideal I_n^* of $\overline{\Omega}_n^*$ generated by $dx_0 + \cdots + dx_n$. One has the relation

$$\Omega_n^* = \overline{\Omega}_n^* / I_n^*.$$

Multiplication by $dx_0 + \cdots + dx_n$ sends $\overline{\Omega}_n^i$ to $\overline{\Omega}_n^{i+1}$. Observe that the kernel of this map is I_n^i. One therefore has an exact sequence of simplical $\widehat{\mathbb{Z}_p}$-modules

$$0 \to I^i \to \overline{\Omega}^i \to I^{i+1} \to 0$$

We therefore have that Ω^i is isomorphic to I^{i+1}. One has $I^0 = 0$, and therefore by induction, one finds that $\pi_i\left(\Omega^k\right) = 0$ when $i \neq k$ and $\pi_k\left(\Omega^k\right) = \mathbb{Z}/p\mathbb{Z}$ with the generator being $dx_0 \wedge dx_1 \wedge \cdots \wedge dx_{k-1}$. □

Now, since $d \circ d = 0$, one has a short exact sequence

$$0 \to Z^k\Omega \to \Omega^k \to Z^{k+1}\Omega \to 0.$$

where the first map is the inclusion and the last map is surjective because Ω satisfies the Poincaré Lemma.

Again one can consider the long exact sequence in homotopy. First, one observes that $\pi_i\left(Z^k\Omega\right) = 0$ when $i \neq k, k-1$ and therefore one has an exact sequence

$$0 \to \pi_k\left(Z^k\Omega\right) \to \pi_{k-1}\left(Z^{k-1}D\right) \to \pi_{k-1}\left(\Omega^{k-1}\right) \to \pi_{k-1}\left(Z^k\Omega\right) \to 0.$$

This identifies $\pi_k\left(Z^k\Omega\right)$ as a subgroup of $\pi_{k-1}\left(Z^{k-1}\Omega\right)$. A routine computation shows that $\pi_0\left(Z^0 D\right) = \widehat{\mathbb{Z}_p}$; and then one can show by induction that $\pi_{k-1}\left(Z^{k-1}D\right) \to \pi_{k-1}\left(\Omega^{k-1}\right)$ is surjective, so it follows that $\pi_{k-1}\left(Z^k\Omega\right) = 0$. The induction therefore gives that

$$\pi_k\left(Z^k\Omega\right) = p^k\widehat{\mathbb{Z}_p}.$$

Finally, we observe that there is an isomorphism $H^k\left(X, p^k\widehat{\mathbb{Z}_p}\right) = H^k\left(X, \widehat{\mathbb{Z}_p}\right)$. We phrase the all of the above as a proposition.

Proposition 4.3.12. *The cohomology ring of $\Omega^*(X)$ is isomorphic to the singular cohomology of X. In other words, one has a ring isomorphism*

$$H^*\left(\Omega^*(X)\right) = H^*\left(X, \widehat{\mathbb{Z}_p}\right).$$

Proof. The computation above gives that

$$\pi_k\left(Z^k\Omega\right) = p^k\widehat{\mathbb{Z}_p}.$$

It therefore follows from Theorem 4.2.5 that $H^*\left(\Omega^*(X)\right) = H^*\left(X, \widehat{\mathbb{Z}_p}\right)$. The $Z^k\Omega$ are submodules of the torsion-free $\widehat{\mathbb{Z}_p}$-modules Ω^k. Therefore they are torsion-free modules over a PID and so are flat. It therefore follows from Theorem 4.2.5 that the cohomology ring is as in the statement. □

Remark 4.3.13. As in the rational case, one can check that there is a zig-zag of $\mathscr{E}$-algebras.

$$\Omega^*(X) \xrightarrow{i} (C \otimes \Omega)^*(X) \xleftarrow{j} C^*(X).$$

which is induced by left Kan extending the zig-zig

$$\Omega^*\left(\triangle^*\right) \xrightarrow{1\otimes\mathrm{id}} (C \otimes \Omega)^*\left(\triangle^*\right) \xleftarrow{\mathrm{id}\otimes 1} C^*\left(\triangle^*\right).$$

along $\triangle^* \to \mathrm{sSet}$. However, these maps do not descend to isomorphisms on cohomology. In fact, one can verify that for $X = S^1$, the map $H^1(1 \otimes \mathrm{id})$ is multiplication by p. In the next subsection, we shall discuss how one can remedy this.

4.3.3 The relationship between the p-adic de Rham forms and the algebra of singular cochains

In this subsection, we upgrade the result of the previous section by explaining how to interpret $\Omega^*(X)$ as an $\mathscr{E}$-algebra. Given the nonvanishing of the Steenrod operation P^0, it has no chance of generally being weakly equivalent to the singular cochains on X. However we shall show in this section that it is quasi-isomorphic to the following subalgebra of $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$. First, it is necessary to establish some notation.

4.3.3.1 The p-shifted singular cochains

In this subsubsection, we shall define the p-shifted singular cochains algebras.

Definition 4.3.14. Let X be a simplicial set. We define the *p-shifted singular cochain algebra* $\mathscr{D}^*\left(X,\widehat{\mathbb{Z}_p}\right)$ to be the following subalgebra of the singular cochains $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$.

$$\mathscr{D}^n(X) = \left\langle p^i\sigma : \text{ for } \sigma \in C^n\left(X,\widehat{\mathbb{Z}_p}\right) \text{ and } \begin{cases} i=n & \text{if } d\sigma = 0. \\ i = n+1 & \text{otherwise.} \end{cases} \right\rangle$$

The differential and the $\mathscr{E}$ structure are that induced by those on $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$.

Remark 4.3.15. Since the singular cochains $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$ are a free graded $\widehat{\mathbb{Z}_p}$-module, recall that it splits (non-canonically) as a sum of graded submodules $C^*\left(X,\widehat{\mathbb{Z}_p}\right) = B^*\left(X,\widehat{\mathbb{Z}_p}\right) \bigoplus Z^*\left(X,\widehat{\mathbb{Z}_p}\right)$, where $Z^*\left(X,\widehat{\mathbb{Z}_p}\right)$ are the cocycles and $B^*\left(X,\widehat{\mathbb{Z}_p}\right)$ is a choice of complement. Then $\mathscr{C}^*(X)$ can be written

$$pB^0\left(X,\widehat{\mathbb{Z}_p}\right) \oplus Z^0\left(X,\widehat{\mathbb{Z}_p}\right) \to p^2B^1\left(X,\widehat{\mathbb{Z}_p}\right) \oplus pZ^1\left(X,\widehat{\mathbb{Z}_p}\right) \to \cdots$$

This equips $\mathscr{C}^*(X)$ with a splitting that is also non-canonical.

We quickly verify the basic properties of $\mathscr{C}^*(X)$; namely that $\mathscr{C}^*(X)$ is indeed a sub-$\mathscr{E}$-algebra and we compute its cohomology.

Proposition 4.3.16. *Let X be a simplicial set, then p-shifted singular cochain algebra $\mathscr{C}^*(X)$ is a sub-$\mathscr{E}$-algebra of $C^*\left(X,\widehat{\mathbb{Z}_p}\right)$ and has cohomology given by $H^*\left(X,\widehat{\mathbb{Z}_p}\right)$.*

Proof. The first claim follows from the fact that for every operation $\mu \in \mathscr{E}(r)^k$, the operation μ is linear in each variable. In particular, if $x_i \in \mathscr{D}(X)^{r_1}$ then $x_i = p^{r_1}x_i'$. Therefore $\mu(x_1,x_2,\cdots x_n) = \mu\left(p^{r_1}x_1', p^{r_2}x_2', \cdots p^{r_n}x_n'\right) = p^{r_1+\cdots+r_n}\mu(x_1,x_2,\cdots x_n) \in \mathscr{D}(X)^{r_1+\cdots r_n-k}$. The cohomology of $\mathscr{C}^*(X)$ can be directly computed as

$$\frac{p^nZ^n\left(X,\widehat{\mathbb{Z}_p}\right)}{d\left(p^nB^{n-1}\left(X,\widehat{\mathbb{Z}_p}\right)\right)} = H^*\left(X,p^n\widehat{\mathbb{Z}_p}\right) = H^*\left(X,\widehat{\mathbb{Z}_p}\right).$$

The lemma follows. □

Remark 4.3.17. The underlying cochain complex of the p-shifted singular cochains complex functor can be viewed as $\eta_p\left(C^*\left(X,\widehat{\mathbb{Z}_p}\right)\right)$, where η is the the Berthelot-Ogus-Deligne [9, 22] *décalage* functor. This is the connective cover with respect to the Beilinson t-structure on filtered complexes. In this case, we are considering the filtration given by powers of the ideal (p). In this context, Theorem 4.3.18 of this chapter can be compared with Theorem 7.4.7 and Example 7.6.7 of [10], which suggest that these objects should have strictly commutative models.

4.3.3.2 The equivalence

Now, we are ready to compute the homotopy type of $\Omega^*(X)$.

Theorem 4.3.18. *For every simplicial set X, there exists a cochain algebra $\mathcal{V}^*$ such that there is a zig-zag of quasi-isomorphisms of $\mathcal{E}$-algebras*

$$\Omega^*(X) \xrightarrow{f} \mathcal{V} \otimes \Omega^*(X) \xleftarrow{g} \mathcal{D}^*\left(X, \widehat{\mathbb{Z}_p}\right)$$

Remark 4.3.19. The same arguments go through for the complex $\mathrm{Gr}(X)$ of [19] if one adjusts the definitions of $\mathcal{D}^*$ and of $\mathcal{V}^*$ appropriately with respect to the (s)-adic filtration.

The tensor product appearing in the statement is that of Definition 4.2.3. The proof strategy is to construct a zig-zag similar to that of (4.1). First, we define $\mathcal{V}^*$.

Definition 4.3.20. Let X be a simplicial set. We define the $\mathcal{V}^*(X)$ to be the following subalgebra of the singular cochains $C^*\left(X, \widehat{\mathbb{Z}_p}\right)$.

$$\mathcal{V}^n(X) = \left\langle p^i\sigma : \text{ for } \sigma \in C^n\left(X, \widehat{\mathbb{Z}_p}\right) \text{ and } \begin{cases} i = 1 & \text{if } n > 0 \text{ or } d\sigma \neq 0. \\ i = 0 & \text{if } n = 0 \text{ and } d\sigma = 0 \end{cases} \right\rangle$$

The differential and the $\mathcal{E}$ structure are that induced by those on $C^*\left(X, \widehat{\mathbb{Z}_p}\right)$.

Remark 4.3.21. Using the notation from Remark 4.3.15, $\mathcal{V}^*(X)$ can also be written as

$$Z^0\left(X, \widehat{\mathbb{Z}_p}\right) \oplus pB^1\left(X, \widehat{\mathbb{Z}_p}\right) \to pC^1\left(X, \widehat{\mathbb{Z}_p}\right) \to pC^2\left(X, \widehat{\mathbb{Z}_p}\right) \to \cdots pC^i\left(X, \widehat{\mathbb{Z}_p}\right) \to \cdots$$

This cochain algebra will reappear later, when we discuss the best commutative approximation to the singular cochains. Next, we compute the cohomology of $\mathcal{V} \otimes \Omega(X)$.

Proposition 4.3.22. *The cohomology of $\mathcal{V} \otimes \Omega(X)$ is $H^*\left(X, \widehat{\mathbb{Z}_p}\right)$.*

Proof. The strategy is to compute both $\pi_i\left((\mathcal{V} \otimes \Omega)^k\right)$ and $\pi_i\left(Z^k(\mathcal{V} \otimes \Omega^*)\right)$, and then the result will follow by an immediate application of Theorem 4.2.5. The first step is observe that one has

$$\pi_r\left(N_*\left(\mathcal{V}^k\right)\right) = \begin{cases} \mathbb{F}_p & \text{when } i = 0. \\ 0 & \text{otherwise.} \end{cases}$$

where $\pi_0\left(\mathcal{V}^0\right)$ is generated by 1. The cohomology of $N_*\left((\mathcal{V} \otimes \Omega)^k_\bullet\right)$ can then be directly computed using the Kunneth theorem. In particular one has the following short exact sequence

$$\bigoplus_{i+j=k} \bigoplus_{p+q=r} \pi_p\left(\mathcal{V}^i\right) \otimes \pi_q\left(\Omega^j\right) \to \pi_r\left((\mathcal{V} \otimes \Omega)^k_\bullet\right) \to \bigoplus_{i+j=k} \bigoplus_{p+q=r-1} \mathrm{Tor}_1\left(\pi_p\left(\mathcal{V}^i\right), \pi_q\left(\Omega^j\right)\right)$$

First observe that $\pi_p\left(\mathcal{V}^i\right) = 0$ except when $p = 0$ and $\pi_q\left(\Omega^j\right) = 0$ except when $q = j$. We can therefore deduce that

$$\bigoplus_{i+j=k} \bigoplus_{p+q=r-1} \mathrm{Tor}_1\left(\pi_p\left(\mathcal{V}^i\right), \pi_q\left(\Omega^j\right)\right) = 0$$

We conclude that

$$\pi_i\left((\mathcal{V} \otimes \Omega)^k\right) = \begin{cases} \mathbb{F}_p & \text{when } i = k \\ 0 & \text{otherwise.} \end{cases}$$

Now one has a short exact sequence

$$0 \to Z^k(\mathcal{V} \otimes \Omega) \to (\mathcal{V} \otimes \Omega)^k \to Z^{k+1}(\mathcal{V} \otimes \Omega) \to 0.$$

Again one can consider the long exact sequence in homotopy. First, one observes that $\pi_i\left(Z^k(\mathcal{V} \otimes \Omega)\right) = 0$ when $i \neq k, k-1$ and therefore one has an exact sequence

$$0 \to \pi_k\left(Z^k(\mathcal{V} \otimes \Omega)\right) \to \pi_{k-1}\left(Z^{k-1}(\mathcal{V} \otimes \Omega)\right) \to \pi_{k-1}\left((\mathcal{V} \otimes \Omega)^{k-1}\right) \to \pi_{k-1}\left(Z^k(\mathcal{V} \otimes \Omega)\right) \to 0.$$

This identifies $\pi_k\left(Z^k(\mathcal{V} \otimes \Omega)\right)$ as a subgroup of $\pi_{k-1}\left(Z^{k-1}(\mathcal{V} \otimes \Omega)\right)$. Since $\pi_0\left(Z^0(\mathcal{V} \otimes \Omega)\right) = \widehat{\mathbb{Z}_p}$ and one can show by induction that $\pi_{k-1}\left(Z^{k-1}(\mathcal{V} \otimes \Omega)\right) \to \pi_{k-1}\left((\mathcal{V} \otimes \Omega)^{k-1}\right)$ is surjective, it follows that $\pi_{k-1}\left(Z^k(\mathcal{V} \otimes \Omega)\right) = 0$. The induction therefore gives that

$$\pi_k\left(Z^k(\mathcal{V} \otimes \Omega)\right) = p^k\widehat{\mathbb{Z}_p}.$$

Therefore, since $Z^k(\mathcal{V} \otimes \Omega)$ is free, by Theorem 4.2.5, we have that $H^i(\mathcal{V} \otimes \Omega(X)) = H^i\left(X, p^i\widehat{\mathbb{Z}_p}\right) = H^i\left(X, \widehat{\mathbb{Z}_p}\right)$ as desired. □

We can now prove our main theorem.

Proof of Theorem 4.3.18. Observe that there is an obvious inclusion $i : \mathscr{C}^*(\triangle^n) \to \mathcal{V}^*(\triangle^n)$ induces a map of $\mathscr{E}$-algebras

$$f_n : \mathscr{C}^*\left(\triangle^n\right) \to (\mathcal{V} \otimes \Omega)^*\left(\triangle^n\right)$$
$$x \mapsto i(x) \otimes 1$$

and, we also have a homotopy equivalence

$$g_n : \Omega^*\left(\triangle^n\right) \to (\mathcal{V} \otimes \Omega)^*\left(\triangle^n\right)$$
$$x \mapsto 1 \otimes x$$

These maps are both compatible with the simplicial structure on the cochain algebras. For all $X \in \mathsf{sSet}$, this extends to a zig-zag of $\mathscr{E}$-algebras by the universal property of simplicial sets

$$\mathscr{C}^*(X) \xrightarrow{f} (\mathcal{V} \otimes \Omega)^*(X) \xleftarrow{g} \Omega^*(X).$$

and by Proposition 4.3.22, these maps are quasi-isomorphisms. □

4.4 The best approximation to commutative algebras

While $\Omega^*(X)$ seems to be an interesting object from the point of view of crystalline cohomology, it does not, on its own, appear to answer the motivating question of this note. That is, *what is the best strictly commutative approximation to the singular cochains?* However, the purpose of this subsection is to show that it *almost does.* To be precise, we shall prove the following theorem.

Theorem 4.4.1. *Let $A \in \mathsf{Com-alg}$, $X \in \mathsf{sSet}$ and $i : \mathsf{Com-alg} \to \mathscr{E}\mathsf{-alg}$ be the inclusion functor. Then there exists a cochain algebra $\mathscr{R}^*$ taking values in strictly commutative algebras such that the following derived mapping spaces are weakly equivalent.*

$$\mathsf{Map}_{\mathsf{Com-alg}}\left(A, \mathscr{R}^*(X)\right) \cong \mathsf{Map}_{\mathscr{E}\mathsf{-alg}}\left(A, C^*\left(X, \widehat{\mathbb{Z}_p}\right)\right).$$

Remark 4.4.2. One way to think about the previous theorem is $\mathscr{R}^*(X)$ is a "partially defined" right adjoint up to homotopy for i. It sends $\mathscr{E}$-algebras with the homotopy type of singular cochains on spaces to commutative dg-algebras. It is not defined on those $\mathscr{E}$-algebras that do not have the homotopy type of cochains of spaces.

The basic observation, roughly put, is that, because of the identity $P^0 = \mathrm{id}$, the cochain algebra $\mathscr{V}^*$ from the last section is the largest subalgebra of the singular cochains that stands any chance of being rectifiable. So, using the p-adic de Rham complex, we construct a strictly commutative algebra model $\mathscr{R}$ for it. First, we shall define $\mathscr{R}$. Then we shall explain the equivalence with $\mathscr{V}^*$ and finally we prove that $\mathscr{V}^*$ has the desired universal property.

4.4.0.1 The extended de Rham complex

In this section, we shall define the commutative algebra that will be quasi-isomorphic to $\mathscr{V}^*$. First, we define a splitting on cochains in $\Omega^*(\triangle^n)$ similar to that on the singular cochains in Remark 4.3.15. In this case, however, it is functorial.

Definition 4.4.3. As a free $\widehat{\mathbb{Z}_p}$-module $\Omega^*(\triangle^n)$ functorially admits a linear basis consisting of two kinds of monomials. Define

$$\Omega^*_B\left(\triangle^n\right) = \langle x_{i_1}\cdots x_{i_n} dx_{j_1}\wedge\cdots\wedge dx_{j_m} \in \Omega^*\left(\triangle^n\right) : n \geq 1\rangle$$

$$\Omega^*_Z\left(\triangle^n\right) = \langle dx_{j_1}\wedge\cdots\wedge dx_{j_m} \in \Omega^*\left(\triangle^n\right) : m \geq 1\rangle$$

Therefore, we have, as a sum of graded modules

$$\Omega^*\left(\triangle^n\right) = \Omega^*_B\left(\triangle^n\right) \oplus \Omega^*_Z\left(\triangle^n\right).$$

The reader should note that this splitting is clearly compatible with the simplicial structure.

Remark 4.4.4. This is not precisely a splitting into cocycles and non-cocycles. The module $\Omega^n_B(\triangle^n)$ contains some cocycles. This turns out not to matter as such cocycles turn out to be identified with cocycles in $C^n\left(X,\widehat{\mathbb{Z}_p}\right)$ which are divisible by p^{n+1}.

Definition 4.4.5. We define a commutative dg-algebra

$$\mathscr{R}^k\left(\triangle^n\right) = \frac{1}{p^k}\Omega^k_B\left(\triangle^n\right) \oplus \frac{1}{p^{k-1}}\Omega^k_Z\left(\triangle^n\right)$$

where Ω^*_B and Ω^*_Z are as in Definition 4.4.3. For a simplicial set X, the commutative dg-algebra $\mathscr{R}^*(X)$ is defined, as before, by left Kan extension. The commutative dg-algebra structure on $\Omega^*(X)$ then extends to $\mathscr{R}^*(X)$.

It is perhaps worth noting that the cohomology ring of $\mathscr{R}^*(X)$ is no longer equal to $H^*(\Omega^*(X))$ or the singular cohomology ring of X. We phrase this as a proposition, whose proof is a straightforward calculation.

Proposition 4.4.6. *The cohomology of both $\mathscr{V}^*(X)$ and $\mathscr{R}(X)$ is equal to $H^*\left(X,\widehat{\mathbb{Z}_p}\right)$ as a graded $\widehat{\mathbb{Z}_p}$-module but with multiplication m given by*

$$m\left(x,y\right) = \begin{cases} xy & \text{for both } x,y \in H^0(X) \\ pxy & \text{for one of } x,y \in H^0(X) \\ p^2xy & \text{for both of } x,y \in H^{>0}(X) \end{cases}$$

where the multiplication on the right is the usual multiplication on singular cohomology.

4.4.0.2 The universal property

In this section, we prove Theorem 4.4.1.

Proposition 4.4.7. *For every simplicial set X, there is a zig-zag of quasi-isomorphisms of $\mathcal{E}$-algebras*

$$\mathcal{R}^*(X) \xrightarrow{f} \mathcal{V} \otimes \mathcal{R}^*(X) \xleftarrow{g} \mathcal{V}^*(X)$$

Proof. *Mutatis mundadis*, the proof is exactly the same as Proposition 4.3.18. □

Now as the first step towards Theorem 4.4.1, we have the following technical proposition.

Proposition 4.4.8. *Let $X \in \mathsf{sSet}$ and let F be a cofibrant replacement functor, taking values in semi-free $\mathcal{E}$-algebras, in the category of $\mathcal{E}$-algebras. Then there exists an $\mathcal{E}$-subalgebra $\bar{E} \hookrightarrow C^*\left(X, \widehat{\mathbb{Z}_p}\right)$ weakly equivalent to $\mathcal{V}^*(X)$, such that: for every $A \in \mathsf{Com}$-alg there is an isomorphism of mapping spaces*

$$\mathrm{Map}_{\mathcal{E}-alg}\left(F \circ i(A), \bar{E}\right) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{E}-alg}\left(F \circ i(A), C^*\left(X, \widehat{\mathbb{Z}_p}\right)\right)$$

Proof. Our first remark is, since $C^*\left(X, \widehat{\mathbb{Z}_p}\right)$ is stable, to compute the derived mapping space it suffices to cofibrantly replace A in the category of *stable $\mathcal{E}$-algebras*.

The subalgebra $\bar{E}$ is defined to be the subalgebra of $C^*\left(X, \widehat{\mathbb{Z}_p}\right)$ such that the 0-cocycles of $C^*\left(X, \widehat{\mathbb{Z}_p}\right)$ are $Z^0\left(X, \widehat{\mathbb{Z}_p}\right)$, and the n-cocycles, for $n > 0$, are in the kernel of the the projection map $Z^n(X) \to H^n\left(X, \mathbb{Z}/\mathrm{p}\mathbb{Z}\right)$. The non-cocycles in $\bar{E}$ are the non-cocycles of $C^*\left(X, \widehat{\mathbb{Z}_p}\right)$. We remark that $\bar{E}$ is an ideal in $C^*\left(X, \widehat{\mathbb{Z}_p}\right)$ with respect to operations in $\mathcal{E}$.

It follows from the definition of $\bar{E}$ that the inclusion map $\mathcal{V}^*(X) \to C^*\left(X, \widehat{\mathbb{Z}_p}\right)$ must factor through $\bar{E}$. Moreover, one can easily verify that this map is an isomorphism on cohomology and so the inclusion $\mathcal{V}^*(X) \to \bar{E}$ is a homotopy equivalence.

Since A is weakly equivalent to a strictly commutative algebra, it follows that the cohomology operation Sq^0 vanishes on $H^n\left(F \circ i(A), \mathbb{Z}/\mathrm{p}\mathbb{Z}\right)$ when $n > 0$. On the other hand, this cohomology operation is the identity on $H^n\left(X, \mathbb{F}_p\right)$. But since $f : F \circ i(A) \to C^*\left(X, \widehat{\mathbb{Z}_p}\right)$ is a map of $\mathcal{E}$-algebras, it follows that the following diagram on cohomology commutes.

$$\begin{array}{ccc}
H^n\left(F \circ i(A), \mathbb{Z}/\mathrm{p}\mathbb{Z}\right) & \xrightarrow{H^*(f)} & H^n\left(X, \mathbb{F}_p\right) \\
\downarrow{\scriptstyle \mathrm{Sq}^0} & & \downarrow{\scriptstyle \mathrm{Sq}^0} \\
H^n\left(F \circ i(A), \mathbb{Z}/\mathrm{p}\mathbb{Z}\right) & \xrightarrow{H^*(f)} & H^n\left(X, \mathbb{F}_p\right)
\end{array}$$

It follows that $H^n\left(f, \mathbb{F}_p\right) = 0$ for $n > 0$ and so the map $f^n : F \circ i(A)^n \to C^n(X)$ sends cocycles to cocycles in $\bar{E}$.

The above argument does not apply to non-cocycles $b \in F \circ i(A)$. But a similar argument applies for them using higher cohomology operations. For simplicity, we consider $p = 2$. The following element of $F \circ i(A)$ is a cocycle mod 2

$$b \cup_{|b|} b + b \cup_{|b+1|} db + K.$$

where $dK = db \cup_{|b+1|} db$. Now, suppose $db = \sum_i x_i \cup y_i$ where x_i and y_i are cocycles. Since in this case, one prove, by stability, that $a \cup_n b = b \cup_n a$ when $|a| = |b| = n$ therefore cross-terms cancel and $db \cup_{|b+1|} db = \sum_i \left(x_i \cup y_i\right) \cup_{|x_i|+|y_i|} \left(x_i \cup y_i\right)$. So it suffices to consider the case $i = 1$.

Then, by the Cartan formula, $db \cup_{|b+1|} db$ is cohomologous to $\left(x \cup_{|x|} x\right) \cup \left(y \cup_{|y|} y\right)$. The relevant coboundary $R(x,y)$ has been computed in [68, Theorem 10], where it is given an explicit formula in terms of x and y. One therefore has that

$$z = b \cup_{|b|} b + b \cup_{|b+1|} db + R(x,y) + S \cup (y \cup_{|y|} y) + \sigma$$

is a cocycle in $F \circ i(A)$ where $dS = x \cup_{|x|} x$ and σ is any choice of cocycle. Since $F \circ i(A)$ is quasi-isomorphic to a commutative algebra, this must vanish in cohomology mod 2 for some choices of S, σ. Now, we consider the image of z under the map f. Observe that the following diagram commute.

$$\begin{tikzcd}
A^{\otimes 2} \arrow[r, "(\cup_n)\otimes(-)"] \arrow[d, "f^{\otimes 2}"] & \mathscr{E}(2) \otimes A^{\otimes 2} \arrow[r, "H\otimes((23)\circ d\otimes d)"] \arrow[d, "\mathrm{id}\otimes f^{\otimes 2}"] & \mathscr{E}(4) \otimes A^{\otimes 4} \arrow[r, "\gamma"] \arrow[d, "\mathrm{id}\otimes f^{\otimes 4}"] & A \arrow[d, "f"] \\
B^{\otimes 2} \arrow[r, "(\cup_n)\otimes(-)"] & \mathscr{E}(2) \otimes B^{\otimes 2} \arrow[r, "H\otimes((23)\circ d\otimes d)"] & \mathscr{E}(4) \otimes B^{\otimes 4} \arrow[r, "\gamma"] & B
\end{tikzcd}$$

where $H : \mathscr{E}(2) \to \mathscr{E}(4)$ is any choice of homotopy, $d : A \to A \otimes A$ is the map $a \mapsto a \otimes a$ and γ is the algebra map. Since it is through one such homotopy H that R is defined, it follows that $f\left(R\left(x,y\right)\right) = R(f(x), f(y))$. One also has that $f\left(b \cup_{|b+1|} db\right) = 0$, since cup-i-products vanish on singular cochains of degree less than i, one has $f(\sigma) \in \bar{E}$ by the first part and $f(S \cup (y \cup_{|y|} y))$ must also be in $\bar{E}$. Finally one can check that $f(b \cup_{|b|} b) = f(b)$ mod 2 by the explicit description of $\mathscr{E}$-algebra structure in [6]. It follows that if $f(b)$ is a cocycle, it is cohomologous to an element of $\bar{E}$ and is therefore in $\bar{E}$ itself.

Finally, $R(f(x), f(y)) = H(f(x), f(y), f(x), f(y))$. Both $f(x)$ and $f(y)$ are cohomologous to cocycles divisible by p. Therefore, modulo p, $H(f(x), f(y), f(x), f(y)) = H(dr, ds, dr, ds)$ for some choice of r and s. Then $H(dr, ds, dr, ds) = 0$ in $C^*(X, \mathbb{F}_2)$ since in an E_∞ algebra $H(dr, ds, dr, ds) + (dH)(r, ds, dr, ds) = d(H(r, ds, dr, ds))$ and then $(dH)(r, ds, dr, ds) = d(H(r, ds, dr, ds)) = 0$ because the E_n-algebra structure on $C^*(X, \mathbb{F}_2)$ vanishes on elements of degree less than n. We conclude that $f(b) \in \bar{E}$.

If $db \neq \sum_i a_i \cup b_i$, then at least one operation γ of degree greater than one must appear. One can show that $db \cup_{|db|} db = 0$ in the cofibrant replacement of A because it is stable, and therefore the above argument significantly simplifies.

We deduce that the map f uniquely factors through $\bar{E}$. The conclusion follows. $\square$

Now we can prove our main theorem.

Proof of Theorem 4.4.1. By Proposition 4.4.8, for every $A \in \mathsf{Com}\text{-alg}$, there is a weak equivalence of mapping sets

$$F_X : \mathrm{Map}_{\mathscr{E}-alg}\left(F \circ i(A), \bar{E}\right) \xrightarrow{\sim} \mathrm{Map}_{\mathscr{E}-alg}\left(F \circ i(A), C^*\left(X, \widehat{\mathbb{Z}_p}\right)\right)$$

We previously saw that there was a weak equivalence $\mathscr{V}^*(X) \to \bar{E}$. All $\mathscr{E}$-algebras are fibrant so therefore the map

$$\mathrm{Map}_{\mathscr{E}-alg}\left(F \circ i(A), \mathscr{V}^*(X)\right) \to \mathrm{Map}_{\mathscr{E}-alg}\left(F \circ i(A), \bar{E}\right)$$

is a quasi-isomorphism of simplicial sets. Finally, by Proposition 4.4.7, one has a zig-zag of weak equivalences

$$\mathscr{R}^*(X) \xrightarrow{f} \mathscr{V} \otimes \mathscr{R}^*(X) \xleftarrow{g} \mathscr{V}^*(X)$$

Therefore, when F is the cofibrant replacement functor, one has a zig-zag of weak equivalences.

$$\mathrm{Map}_{\mathscr{E}-alg}\left(F\circ i(A),\mathscr{R}^*(X)\right)\to \mathrm{Map}_{\mathscr{E}-alg}\left(F\circ i(A),\mathscr{V}\otimes\mathscr{R}^*(X)\right)\leftarrow \mathrm{Map}_{\mathscr{E}-alg}\left(F\circ i(A),\mathscr{V}^*(X)\right)$$

Since Com-alg is defined to be a full subcategory of $\mathscr{E}$-alg, it follows that $\mathrm{Map}_{\mathscr{E}-alg}(F(A),\mathscr{R}^*(X)) = \mathrm{Map}_{\mathsf{Com}\text{-alg}}(F(A),\mathscr{R}^*(X))$. Therefore, as derived mapping sets $\mathrm{Map}_{\mathsf{Com}\text{-alg}}(A,\mathscr{R}^*(X))$ is equivalent to $\mathrm{Map}_{\mathscr{E}-\mathrm{alg}}\left(A,C^*\left(X,\widehat{\mathbb{Z}_p}\right)\right)$. ◻

4.5 E_n-models for $C^*\left(X,\mathbb{F}_p\right)$

Mandell [59] has suggested for n-connected spaces X at most primes, it may be possible to truncate the E_∞-structure on the singular cochains to an E_n-structure and find a strictly commutative model for this truncation. In this section, we explain how to compute such a truncation and give an explicit commutative approximations to the E_1-structure on all spaces.

4.5.1 The E_n-truncation of the singular cochains

In this section, we explain, given a space X, how one can compute an E_n-algebra $\mathscr{B}_n(X)$ that is quasi-isomorphic to the singular chains as an E_n-algebra and which has better cofibrancy properties than the singular chains.

Definition 4.5.1. We define a simplicial cochain algebra $\left(\mathscr{B}^{(r)}\right)^*$ that has for n-simplices

$$(\mathscr{B}^{(r)})^*_n = \frac{\mathscr{E}^{(r)}(x_0,\dots x_n,dx_0,\dots,dx_n)}{(x_0+\dots+x_n-1,dx_0+\dots+dx_n)},\ |x_i|=0,\ |dx_i|=1.$$

As an E_r-algebra, this contains enough information to approximate $C^*(X)$.

Theorem 4.5.2. *Let X be a simplicial set. Then there is a zig-zag of equivalences of $\mathscr{E}^{(r)}$-algebras*

$$C^*(X)\xrightarrow{\sim}\left(C\otimes\mathscr{B}^{(r)}\right)^*(X)\xleftarrow{\sim}\left(\mathscr{B}^{(r)}\right)^*(X)$$

The result will follow from the following lemma.

Lemma 4.5.3. *The cochain algebra $\left(\mathscr{B}^{(r)}\right)^*(X)$ satisfies the Poincaré Lemma and is extendable.*

Proof. Both claims follow from adaptions of classical arguments. To verify the Poincaré Lemma, it suffices to observe that, as a cochain complex, one has

$$\frac{\left(\mathscr{B}^{(r)}\right)(x_0,\dots x_n,dx_0,\dots,dx_n)}{(x_0+\dots+x_n-1,dx_0+\dots+dx_n)}=\left(\mathscr{B}^{(r)}\right)(x_1,\dots x_n,dx_1,\dots,dx_n)$$

Finally, one has that $H^*\left(\left(\mathscr{B}^{(r)}\right)(x_1,\dots x_n,dx_1,\dots,dx_n)\right)=0$ because the cohomology of $\mathbb{F}_p x_1\oplus\dots\mathbb{F}_p x_n\oplus\mathbb{F}_p dx_1\oplus\dots\oplus\mathbb{F}_p dx_n$ is 0 and the free functor $\left(\mathscr{E}^{(r)}\right)$ reflects weak equivalences of cochain complexes.

To verify extendability, it suffices to check that the normalized chain complex $N_*\left(\left(\mathscr{B}^{(r)}\right)^p_\bullet\right)$ has zero homology for every p. Let $\omega\in N_n\left(\left(\mathscr{B}^{(r)}\right)^p_\bullet\right)$ be an n-cycle. This means that $\omega\in\left(\mathscr{B}^{(r)}\right)^p_n$ is a p-cochain such that $\delta_i(\omega)=0$ for all $0\le i\le n$. We need to show that ω is a boundary. In other words, that there exists $\nu\in\left(\mathscr{B}^{(r)}\right)^p_{n+1}$ such that $\delta_i(\nu)=0$ for $0<i\le n$ and $\partial_0(\nu)=\omega$. We claim that

$$\nu:=\sum_{j=1}^{n+1}t_j\omega(t_1,\dots,t_{j-1},t_j+t_0,t_{j+1},\dots,t_{n+1})$$

works for this. The verification is left to the reader. ◻

Proof of Theorem 4.5.2. The zig-zag

$$\left(\mathscr{B}^{(r)}\right)^*(\triangle^*) \xrightarrow{\mathrm{id}\otimes 1} \left(\mathscr{B}^{(r)}\otimes C^*\right)(\triangle^*) \xleftarrow{1\otimes\mathrm{id}} C^*(\triangle^*) \tag{4.3}$$

extends to a zig-zag of $\mathscr{E}^{(r)}$-algebras by the universal property of simplicial sets

$$\left(\mathscr{B}^{(r)}\right)^*(X) \xrightarrow{\sim} \left(\mathscr{B}^{(r)}\otimes C\right)^*(X) \xleftarrow{\sim} C^*(X)$$

and, since $\left(\mathscr{B}^{(r)}\right)^*$ is extendable and satisfies the Poincaré Lemma, it follows that by Proposition 4.2.4 that these maps are quasi-isomorphisms. □

Conjecture 4.5.4 (Refinement of a conjecture of Mandell [59])**.** If X is n-connected, it ought to be possible to find a strictly commutative algebra $\mathscr{M}_X$ such that there is a surjective map $\left(\mathscr{B}^{(r)}\right)^*(X) \to \mathscr{M}_X$. The object $\mathscr{M}_X$ should just be symmetrisation of the associative structure along with appropriate identifications of the cotriple maps from [29] determined by E_n-structure. In favourable situations, where all obstructions vanish, this should be a quasi-isomorphism.

4.6 Homotopy invariants

In this section we shall discuss some applications of the p-adic de Rham forms. First, we shall show that they recover the Massey products. Recall that Massey products, first defined in [60], are secondary operations defined on the homology of differential graded associative algebras. They are a finer invariant than the cohomology ring. For example, they can be used to show that the Borromean rings are non-trivially linked, which cannot be detected using only the cohomological cup product. We shall also discuss the relationship between $\mathbb{Q}$-formality and $\widehat{\mathbb{Z}_p}$-formality.

4.6.1 Massey products in $\Omega^*(X)$

This section discuss the homotopical applications of $\Omega^*(X)$. We shall show that that it allow us to use the machinery of Massey products in situations where they were previously unavailable, for example, in the torsion part of the cohomology of spaces. We finish this section by giving an example of a space X that is formal over $\mathbb{Q}$ but not over $\widehat{\mathbb{Z}_p}$.

We begin by showing that all traditional Massey products in $A_{PL}(X)$ (that is to say, Massey products in the sense of [60] that are defined over $\mathbb{Q}$) may also be computed using $\Omega^*(X)$.

Proposition 4.6.1. *Suppose that $\sigma \in H^*(X,\mathbb{Q})$ be the higher Massey product of $\langle x_1, x_2, \ldots, x_n\rangle \in H^*(A_{PL}(X),\mathbb{Q})$. Then there exists an $n > 0$ such that $p^n\sigma \in H^*\left(X,\widehat{\mathbb{Z}_p}\right)$ is the higher Massey product of $\langle p^n x_1, p^n x_2, \ldots, p^n x_n\rangle \in H^*\left(A_{PL}(X),\widehat{\mathbb{Z}_p}\right)$ computed in $\Omega^*(X)$.*

Proof. Let $\{a_{i,j}\}$ be a defining system for a Massey product in $A_{PL}(X)$. The inclusion

$$\widehat{\mathbb{Z}_p}\langle x\rangle \to \mathbb{Q}_p[x]$$

induces an inclusion of $\widehat{\mathbb{Z}_p}$-modules

$$f : \Omega^*(X) \hookrightarrow A_{PL}(X)\otimes\mathbb{Q}_p.$$

given by

$$x_{i_1}\cdots x_{i_n}dx_{j_1}\wedge\cdots\wedge dx_{j_m} \mapsto \frac{1}{p^{n+m}}y_{i_1}\cdots y_{i_n}dy_{j_1}\wedge\cdots\wedge dy_{j_m}.$$

Now for a sufficiently large n, the defining system $\{p^n a_{i,j}\}$ must lie in the image of f. Since f is injective, it then can be pulled back to a defining system for $p^n\sigma$ on $\Omega^*(X)$. □

One can generalise the notion of Massey products with the same definition but choosing cochains representing the torsion part of the cohomology of a space. This has already been done in some special cases. For an example with moment-angle complexes we refer the reader to [41, Example 3.21]. We expect that our construction generalises this, up to factor, and provides a convenient model for doing computations.

4.6.2 Formality of $\Omega^*(X)$

Recall that a space X is called $\mathbb{Q}$-*formal* if $A_{PL}(X)$ is quasi-isomorphic to the cohomology of X. We shall say that X is $\widehat{\mathbb{Z}_p}$-*formal*, if $\Omega^*(X)$ is quasi-isomorphic to $H^*(X)$ via a zig-zag of commutative dg-algebras. Formality is an extremely useful property in rational homotopy theory, and we hope that $\widehat{\mathbb{Z}_p}$-formality may have similar applications in future.

The main theorem of this section is the following, which is inspired by a conjecture of Mandell [59].

Theorem 4.6.2. *Let X be a finite simplicial set such that $A_{PL}(X)$ is formal over $\mathbb{Q}$. For all but finitely many primes, $\Omega^*(X)$ is formal over $\widehat{\mathbb{Z}_p}$ as a dg-commutative dg-algebra.*

Before proving this theorem, it will be convenient to introduce some notation and prove a useful lemma.

Definition 4.6.3. Let V and W be free dg-modules in $\widehat{\mathbb{Z}_p}$.We define the *mixed symmetric algebra* $\mathrm{MSym}(V_0, V_1)$ to be the smallest free commutative dg-algebra containing both $\mathrm{Sym}(V \oplus W)$ and $\Gamma\mathrm{Sym}(W)$.

Lemma 4.6.4. *Let X be a simplicial set. Suppose that a cochain $\sigma \in \Omega^*(X)$ is not a cocycle. Then there exists a cocycle c such that $(\sigma + c)^{p^n}$ is divisible by p^n.*

Proof. The noncocyles in $\Omega^*(\triangle^n)$ are easily verified to be of the form $\sigma + c$ for $c = 1, 2 \dots, p-1 \in \Omega^0(\triangle^n)$. For the general case, observe that

$$\Omega^*(X) = \mathrm{sSet}\left(X_\bullet, \Omega^*\left(\triangle^n\right)\right).$$

The result holds for each $x \in X_\bullet$ so the result must hold in the general case. □

Proof of Theorem 4.6.2. Before beginning the proof we briefly summarise the idea behind the proof. One constructs a quasi-free, and therefore cofibrant, replacement of $A_{PL}(X)$ in the category of $CDGA_{\mathbb{Q}}$ via the step-by-step procedure of [27, Proposition 12.1]. At each step one constructs a quasi-free resolution of $\Omega(X)$ with a map to the cofibrant resolution. Finally; if $A_{PL}(X)$ is formal there is a weak–equivalence from the cofibrant resolution of $A_{PL}(X)$ to its cohomology and one shows that this extends to a map on the quasi-free resolution of $\Omega(X)$ to its cohomology.

For all but finitely many primes the cohomology $H^*\left(X, \widehat{\mathbb{Z}_p}\right)$ is torsion-free and therefore projective. Assume we are working at such a prime. In this case,

$$H^*\left(X, \widehat{\mathbb{Z}_p}\right) \otimes_{\widehat{\mathbb{Z}_p}} \mathbb{Q}_p = H^*\left(X, \mathbb{Q}_p\right)$$

Then, we recall that if $A_{PL}(X)$ is formal, then $A_{PL}(X) \otimes \mathbb{Q}_p$ is formal. Now, the inclusion

$$\widehat{\mathbb{Z}_p}\langle x \rangle \to \mathbb{Q}_p[x]$$

induces an isomorphism

$$\widehat{\mathbb{Z}_p}\langle x \rangle \otimes_{\widehat{\mathbb{Z}_p}} \mathbb{Q}_p \xrightarrow{\sim} \mathbb{Q}_p[x].$$

and therefore one has a isomorphism

$$\Omega^*(X) \otimes_{\widehat{\mathbb{Z}_p}} \mathbb{Q}_p = A_{PL}(X) \otimes \mathbb{Q}_p.$$

given by

$$x_{i_1} \cdots x_{i_n} dx_{j_1} \wedge \cdots \wedge dx_{j_m} \mapsto \frac{1}{p^{n+m}} y_{i_1} \cdots y_{i_n} dy_{j_1} \wedge \cdots \wedge dy_{j_m}.$$

This isomorphism restricts to an inclusion of dg-$\widehat{\mathbb{Z}_p}$-modules

$$\Omega^*(X) \to A_{PL}(X) \otimes \mathbb{Q}_p,$$

which is clearly an quasi-isomorphism after tensoring by $\mathbb{Q}_p$. We shall commence by showing that one can build compatible Sullivan-type models for $A_{PL}(X) \otimes \mathbb{Q}_p$ and $\Omega^*(X)$ as commutative dg-algebras. Since $H^*\left(X, \widehat{\mathbb{Z}_p}\right)$ is free and therefore projective, one has a quasi-isomorphism of dg-$\widehat{\mathbb{Z}_p}$-modules

$$H^*\left(X, \widehat{\mathbb{Z}_p}\right) \to \Omega^*(X).$$

One can then choose a map $H^*\left(X, \mathbb{Q}_p\right) \to A_{PL}(X) \otimes \mathbb{Q}_p$ such that the following diagram commutes.

$$\begin{tikzcd}
H^*\left(X, \widehat{\mathbb{Z}_p}\right) \arrow[r] \arrow[d, "H^*(X,i)"'] & \Omega^*(X) \arrow[d] \\
H^*\left(X, \mathbb{Q}_p\right) \arrow[r] & A_{PL}(X) \otimes \mathbb{Q}_p.
\end{tikzcd}$$

Here, the map $i : \widehat{\mathbb{Z}_p} \to \mathbb{Q}_p$ is the usual inclusion of a ring into its field of fractions. Next, we follow the next step of the classical procedure for building a Sullivan model by extending this to a map of free commutative dg-$\widehat{\mathbb{Z}_p}$-algebras.

$$\begin{tikzcd}
\mathrm{Sym}\left(H^*\left(X, \widehat{\mathbb{Z}_p}\right)\right) \arrow[r, "f_0"] \arrow[d] & \Omega^*(X) \arrow[d] \\
\mathrm{Sym}\left(H^*\left(X, \mathbb{Q}_p\right)\right) \arrow[r, "g_0"] & A_{PL}(X) \otimes \mathbb{Q}_p.
\end{tikzcd}$$

The reader should observe that $\ker H^*\left(f_0\right) \otimes \mathbb{Q}_p = \ker H^*\left(g_0\right)$ since $H^*\left(X, \widehat{\mathbb{Z}_p}\right)$ has zero differential. Moreover, these kernels are free since we are working over a PID. Therefore, any basis of cocycles W_1 for $\ker H^*\left(f_0\right)$ is such that $W_1 \otimes \mathbb{Q}_p$ is a basis for $\ker H^*\left(g_0\right)$. Therefore one can extend the differential to

$$d : V_1 = sW_1 \to W_1 \subset \mathrm{Sym}\left(H^*\left(X, \widehat{\mathbb{Z}_p}\right)\right)$$

$$d : V_1 \otimes \mathbb{Q}_p = sV_1 \otimes \mathbb{Q}_p \to W_1 \otimes \mathbb{Q}_p \subset \mathrm{Sym}\left(H^*\left(X, \mathbb{Q}_p\right)\right)$$

that kill all surplus cocycles. Now, observe that the map

$$f_1 : V_1 \to \Omega^*(X)$$

is defined to be any choice of map such that the following diagram commutes

$$\begin{tikzcd}
V_1 \arrow[d, "d"] \arrow[dr, dotted, "f_1"] & \\
W_1 \arrow[r, "f_0"] & \Omega^*(X).
\end{tikzcd}$$

In particular, it follows from Lemma 4.6.4 that f_1 can be chosen such that for all $v \in V_1$, we have $p^n | f_1(v)^{p^n}$. Define $g_1 = f_1 \otimes \mathbb{Q}_p$. By freeness, we can produce a commutative diagram

$$\begin{tikzcd} \left(\mathrm{Sym}\left(H^*\left(X, \widehat{\mathbb{Z}_p}\right) \oplus V_1\right), \Omega\right) \arrow[r, "f_1"] \arrow[d] & \Omega^*(X) \arrow[d] \\ \left(\mathrm{Sym}\left(H^*\left(X, \mathbb{Q}_p\right) \oplus V_1 \otimes \mathbb{Q}_p\right), \Omega\right) \arrow[r, "g_1"] & A_{PL}(X) \otimes \mathbb{Q}_p. \end{tikzcd}$$

This, so far, is precisely as in [27, Proposition 12.1]. Now, we claim that the map

$$f_1 : \mathrm{Sym}\left(H^*\left(X, \widehat{\mathbb{Z}_p}\right) \oplus V_1\right) \to \Omega^*(X)$$

extends uniquely to

$$\left(\mathrm{MSym}\left(H^*\left(X, \widehat{\mathbb{Z}_p}\right), V_1\right)\right) \to \Omega^*(X).$$

The existence of such an extension is equivalent to showing that for all $v \in V_1$, the element $\left(f_1(v)\right)^{p^n}$ is divisible by p^n. This is true since $f_1(v)$ was chosen to satisfy the hypotheses of Lemma 4.6.4.

$$\begin{tikzcd} \left(\mathrm{MSym}\left(H^*\left(X, \widehat{\mathbb{Z}_p}\right), V_1\right), d\right) \arrow[r, "f_1"] \arrow[d] & \Omega^*(X) \arrow[d] \\ \left(\mathrm{Sym}\left(H^*\left(X, \mathbb{Q}_p\right) \oplus \left(V_1 \otimes \mathbb{Q}_p\right)\right), d\right) \arrow[r, "g_1"] & A_{PL}(X) \otimes \mathbb{Q}_p. \end{tikzcd}$$

It is clear we can iterate this procedure provided that two conditions. Namely, we must show that, if

- the cohomology of $\mathrm{MSym}\left(V_0, \bigoplus_{i=1}^{k} V_i\right)$ is torsion-free.
- the map $\mathrm{MSym}\left(V_0, \bigoplus_{i=1}^{k} V_i\right) \xrightarrow{f_k} \mathrm{Sym}\left(\bigoplus_{i=0}^{k} V_i \otimes \mathbb{Q}_p\right)$ is a $\mathbb{Q}_p$-quasi-isomorphism

for $k = N-1$, then the same pair of conditions hold for $k = N$. The first condition is clearly true for our construction since, by assumption, the cohomology of $\Omega^*(X)$ is torsion-free. For the second condition to hold, it suffices to observe that f_N sends cocycles to cocycles because the divided powers of the V_i for $i \geq 1$ kill all surplus cocycles.

It therefore follows that the map

$$\left(\mathrm{MSym}\left(V_0, \bigoplus_{i=1}^{\infty} V_i\right), d\right) \to \left(\mathrm{Sym}\left(\bigoplus_{i=0}^{\infty} V_i \otimes \mathbb{Q}_p\right), d\right)$$

is a $\mathbb{Q}_p$-quasi-isomorphism. Since $\left(\mathrm{Sym}\left(\bigoplus_{i=0}^{\infty} V_i \otimes \mathbb{Q}_p\right), d\right)$ is cofibrant, there is a quasi-isomorphism $\left(\mathrm{Sym}\left(\bigoplus_{i=0}^{\infty} V_i \otimes \mathbb{Q}_p\right), d\right) \to H^*\left(X, \mathbb{Q}_p\right)$. This restricts to a quasi-isomorphism of commutative dg-algebras

$$\left(\mathrm{MSym}\left(V_0, \bigoplus_{i=1}^{\infty} V_i\right), d\right) \to H^*\left(X, \widehat{\mathbb{Z}_p}\right)$$

which implies $\Omega^*(X)$ is formal as desired. $\square$

CHAPTER 5

Homotopically, E_∞- algebras do not generalise commutative algebras

Abstract

This is the third chapter exploring the relationship between commutative algebras and E_∞-algebras in characteristic p and mixed characteristic. In this paper, we show that, in characteristic 2, the homotopy category of strictly commutative dg-algebras does not form a subcategory of the homotopy category of E_∞-algebras. This is done by constructing a explicit example of two strictly commutative algebras that are quasi-isomorphic in the category of E_∞-algebras but not the category of strictly commutative algebras. The construction is based on the theory of cotriple operations.

5.1 Introduction

Clearly, the commutative algebra structure on a CDGA always extends to a full E_∞-algebra structure. In this sense, E_∞-algebras generalise commutative algebras. One can also ask if the same property holds on *homotopical level*. The categories of CDGAs and E_∞-algebras both have a natural notion of quasi-isomorphism and therefore homotopy categories. Therefore commutative algebras have two notions of homotopy type, firstly, one can say A and B have the same homotopy type if they are connected by zig-zags of quasi-isomorphisms of commutative algebras; and secondly, one can consider if there are zigs-zags of quasi-isomorphisms of E_∞-algebras where the algebras appearing in the zig-zags can be both commutative algebras and E_∞-algebras.

At first glance, these two notions seem obviously equivalent and, indeed, in characteristic 0, this is true. The purpose of this chapter is to show that this does not occur in positive characteristic by exhibiting an explicit counterexample.

Cotriple products and versions of the Frobenius map can be defined for most algebraic operads. We therefore expect the same basic method to produce similar counterexamples for most other operads in positive characteristic.

We comment briefly on related work. In [42], the authors develop operadic calculus in positive characteristic using quasi-planar operads as the backbone. This allows them to develop a general theory that allows them to study the homotopy category of algebras of all $\mathbb{S}$-projective operads. While their work does not claim such; our work shows that this is not a complete picture - there is information, in this case about indeterminacies of higher operations, in the homotopy category of algebras over an operad that cannot be recovered from the category of algebras over cofibrant replacements of said operads.

5.2 Preliminaries

5.2.0.1 Linear bases for cup-1-algebras

Recall the notion of a strict cup-1-algebra from Section 3.4.2.1 and the following definition.

Definition 5.2.1. Let $A = \mathrm{Cup}(X)/(R)$ be cup-1-algebra presented in terms of generators and relations. Let m be a monomial in A, constructed from the generators using both $\cup_1$ and $\cup$. Then m is *reduced* if it is written as

$$m = m_1 \cup m_2 \cup \cdots \cup m_n$$

where each m_i is a monomial constructed only using the $\cup_1$ operation.

Clearly, using the Hirch relation, every monomial can be written as the sum of reduced monomials. However, the chief computational problem is that such a sum may not be unique. Indeed, one has

$$a \cup c \cup (b \cup_1 d) + (a \cup_1 c) \cup b \cup d = a \cup (b \cup_1 d) = c \cup a \cup (b \cup_1 d) + (a \cup_1 c) \cup d \cup b \tag{5.1}$$

as both are different ways of expanding $(a \cup b) \cup_1 (c \cup d) = (c \cup d) \cup_1 (a \cup b)$. In what follows, we shall refer to any relation of this form as a *Hirsch relation*.

5.2.0.2 Recalling cotriple operations

For the full theory of cotriple products, we refer the reader to Section 3. For convenience, here we briefly recall some essential preliminaries about higher Frobenius operations in characteristic 2.

Definition 5.2.2. Let A be a commutative dg-algebra over $\mathbb{F}_2$. Let $x, y \in H^*(A)$ be homogeneous elements such that $xy = 0$. A defining system for a n^{th} *order Frobenius product* is a collection $\{a, b, c_1, \dots c_{n-1}\}$ such that a, b are choices of cocycle representatives for x, y, $dc_1 = ab$ and $c_i^2 = dc_{i+1}$. The n^{th} *order Frobenius product* is then c_n^2.

There is a small subtlety when computing the indeterminacy for type n Frobenius opperations when $n > 2$ (the higher operation must be *strictly defined*); but the result is given by the following theorem.

Proposition 5.2.3. *[29] Let A be a commutative dg-algebra and suppose $x, y \in H^*(A)$ are such that their type 1 n^{th} Frobenius product is* strictly defined, *that is, that $xy = 0$ and*

$$H^{p(|y|-1)}(A) = H^{p(|x|-1)}(A) = \{0\}$$

$$\dots$$

$$H^{p^{n-1}(|y|)-\sum_{i=1}^{n-1} p^i}(A) = H^{p^{n-1}(|x|)-\sum_{i=1}^{n-1} p^i}(A) = \{0\}$$

and the $(n-1)^{th}$ *Frobenius product is equal to 0. Then* n^{th} *order type 1 Frobenius product is defined and is a well-defined element of*

$$\frac{H^{p^{n-1}(|x|+|y|)-\sum_{i=1}^{n-1}p^i}(A)}{H^{p^{n-1}(|x|+|y|)-2^{n-1}+\sum_{i=0}^{n-1}p^i}(A)^p + x^{p^{n-1}}H^{p^{n-1}(|y|)-\sum_{i=1}^{n-1}p^i}(A) + y^{p^n}H^{p^{n-1}(|x|)-\sum_{i=1}^{n-1}p^i}(A)}$$

and therefore is invariant under quasi-isomorphism.

In this paper we shall be concerned with the third order Frobenius operation. This corresponds to a E_∞-cotriple operation given as follows.

Proposition 5.2.4. *Let* A *be a cup-1-algebra that is quasi-isomorphic to a strictly commutative dg-algebra. Then the following is a cocycle*

$$c \cup c + c \cup_1 (a \cup b) + a^2 \cup K + L \cup b^2$$

where $dK = b \cup_1 b$ *and* $dL = a \cup_1 a$*.*

Remark 5.2.5. The equivalence of A to a strictly commutative dg-algebra guarantees the existence of K and L as $b \cup_1 b$ and $a \cup_1 a$ represent Steenrod operations and these must vanish for a strictly commutative algebra.

We saw in Section 3.4.2.2 that the indeterminacy of this cup-1-product operation can be computed to be

$$\frac{H^{2(|x|+|y|-1)}(A)}{H^{(|x|+|y|-1)}(A)^2 + x^2 H^{2(|y|-1)}(A) + y^2 H^{2(|x|-1)}(A)}$$

which is the same as the commutative case. The more interesting example is a third order Frobenius operation, which we introduce next (with some assumptions on the cup-1-algebra designed to simplify expressions).

Definition 5.2.6. Let A be a cup-1-algebra over $\mathbb{F}_2$. Let $x, y \in H^*(A)$ be homogeneous elements such that $xy = 0$ and suppose that

$$H^{2(|y|-1)}(A) = H^{2(|x|-1)}(A) = \{0\}$$

A defining system for a *third order Frobenius product* is a collection $\{a, b, c_1, c_2\}$ such that a, b are choices of cocycle representatives for x, y and $dc_1 = ab$. For simplicity, we assume that $a \cup_1 b = a \cup_1 a = b \cup_1 b = 0$. In particular, a and b commute. Then

$$dc_2 = c_1 \cup c_1 + c_1 \cup_1 (a \cup b)$$

The third order Frobenius product is then

$$c_2 \cup c_2 + c_2 \cup_1 (c_1 \cup c_1 + c_1 \cup_1 (a \cup b)) + K$$

where

$$dK = (c_1 \cup c_1 + c_1 \cup_1 (a \cup b)) \cup_1 (c_1 \cup c_1 + c_1 \cup_1 (a \cup b)) = c_1^2 \cup c_1^2 = c_1^2 \cup (c_1 \cup_1 c_1) + (c_1 \cup_1 c_1) \cup c_1^2$$

Remark 5.2.7. The equality

$$(c_1 \cup c_1 + c_1 \cup_1 (a \cup b)) \cup_1 (c_1 \cup c_1 + c_1 \cup_1 (a \cup b)) = c_1^2 \cup c_1^2$$

follows from the fact $a \cup_1 a = b \cup_1 b = 0$. Otherwise, we would obtain a much more complicated expression. Another observation that will be important later is that $c_1 \cup_1 c_1$ is a cocycle.

The counterexample follows from the fact that one can add any cocycle to the K in Definition 5.2.6. Therefore the third order Frobenius product has a greater indeterminacy in the category of cup-1-algebras (even those with extra conditions on x and y that we have imposed) than in the category of strictly commutative algebras.

5.3 The counterexample

In this section we produce the promised explicit counterexample. The section consists of three subsections. In the first, we define two strictly commutative algebras with the same cohomology and explain why they are not quasi-isomorphic as commutative dg-algebras. In the second, we define and compute a linear basis for C.

5.3.1 The commutative algebras A and B

We shall consider the following strictly commutative dg-algebras.

$$A = \mathrm{Sym}(x, z)/(x^2, z^2, xz)$$

Because we are in characteristic 2, degrees do not make a difference as long as they are chosen so that the operations are strictly defined. The algebra A has the following linear basis $\{x, z\}$ and coincides with its cohomology. and

$$B = \mathrm{Sym}(x, t, s)/(x^3, xt, t^3, s^3, xs, ts)$$

such that $dt = x^2$, and $ds = t^2$. It is easy to explicitly write down a basis for B as

$$\{x, x^2, t, t^2, s, s^2\}$$

and one then easily verifies the cohomology of both A and B is equal to A, under the identification $s^2 = z$.

Next one computes the 3^{rd} order Frobenius operation of x and x. These operation is clearly strictly defined. Moreover, for A it is $\{0\}$, but for B it is $\{z\}$. These are both in different indeterminacy classes, so it follows from Proposition 5.2.3 that the algebras cannot be quasi-isomorphic.

5.3.2 The cup-1-algebra C

Consider the cup-1-algebra C generated by the elements $x, s, t, M, K_1, K_2, K_3, R_1, R_2 L_1, L_2$ subject to the requirements

- x is a cocycle, $dt = x^2$, $ds = t \cup t + t \cup_1 x^2$, $dM = t \cup_1 t$, $dK_1 = (x \cup_1 t) \cup (x \cup_1 t)$, $dK_2 = (t \cup x) \cup (t \cup x) + x^3 \cup_1 (t \cup x)$, $dK_3 = (x \cup t) \cup (x \cup t) + x^3 \cup_1 (x \cup t)$, $dR_1 = (t \cup_1 t) \cup (x \cup_1 t)$, $dR_2 = (x \cup_1 t) \cup (t \cup_1 t)$ $dL_1 = t^2 \cup (t \cup_1 t)$, $dL_2 = (t \cup_1 t) \cup t^2$
- One has $|x| = 5$, $|t| = 9$, $|s| = 17$, $|M| = 16$, $|K_1| = 25$, $|K_2| = |K_3| = 28$, $|R_1| = |R_2| = 29$ and $|L_1| = |L_2| = 34$.
- We introduce a new degree called *word length*, denoted $|-|_w$ where $|x|_w = 1, |t|_w = 2, |s|_w = 4, |M|_w = 4, |K_1|_w = |K_2|_w = |K_3|_w = 6$ $|R_1|_w = |R_2|_w = 7$ and $|L_1|_w = |L_2| = 8$. We consider word-length to be additive under both $\cup$ and $\cup_1$. The differential can easily be checked to preserve word length. We kill all terms of word length 9 or greater.
- We quotient by $x \cup_1 x$.
- Additionally, we kill all nonexact cocycles of word length 7. It will become clear later why we impose this condition and also why we do not need to be more specific than this.
- In word length 8, we also kill all nonexact cocycles of word length 8 except $s \cup s + s \cup t^2 + L_1 + L_2$. In particular we quotient by $s \cup_1 s$, $M \cup_1 M$, $M \cup M$, $t \cup_1 t \cup_1 t \cup_1 t$, $M \cup_1 (t \cup_1 t)$ (this is slightly stronger than killing all the cocycles as $M \cup M + M \cup_1 (t \cup_1 t) + l$ is a cocycle where $dl = t \cup_1 t \cup_1 t \cup_1 t$).

For computational reasons it will be helpful to introduce the following subcomplexes.

Definition 5.3.1. Let $C_{i,j} \subset C$ denote the subcomplex consisting of the linear span of monomials of x-word length i and y-word length j. Let $C_k \subset C$ denote subcomplex consisting of the linear span monomials of monomials of total length k.

Proposition 5.3.2. *The cohomology of C is equal to*

$$\mathrm{Sym}(x,z)/(x^2, xz, z^2)$$

where $z = s \cup s + s \cup t^2 + L_1 + L_2$.

Proof. Observe that the differential preserves word-length. It follows that every cocycles can be written as the sum of cocycles that are homogeneous in word length. Therefore, if $C = \bigoplus_{i=1}^{8} C_i$ then $H^*(C) = \bigoplus_{i=1}^{8} H^*(C_i)$.

Before proceeding further in the calculation, we make the following observation: one always has a linear generating set consisting reduced monomials. If one knows all the Hirsch relations between them, one has a basis and can do the computation directly. There are very few relations in low word length so it is possible directly compute the cohomology in word length less than or equal to 6: For the other cases, the fact we kill non-exact cocycles, which will add extra relation in relevant degrees for reasons of degree, means we can restrict to certain degrees where we can show there are few or no relations and therefore avoid the bulk of the calculations

First of all, one has that $C_1 = \mathbb{F}_2 x$. It is completely straightforward to directly check by hand that

$$H^*(C_2) = H^*(C_3) = H^*(C_4) = H^*(C_5) = 0$$

as one can use a basis of reduced monomials. The main subtlety the above calculation is that in C_4, $t \cup_1 t$ and $t \cup t + t \cup_1 x^2$ are cocycles and we add generators

$$dM = t \cup_1 t, \qquad ds = t \cup t + t \cup_1 x^2.$$

There remains three more cases, where an extra relation (Equation 5.1) makes an appearance. This identity first becomes relevant in word length 6. We also note that the only relations that appear must involve only x and t for reasons of word length. We compute the final three cases by directly computing a basis. For notational convenience, we shall use $< a_1, a_2, \ldots, a_n >$ to denote the set

$$\{a_{\sigma(1)} \cup a_{\sigma(2)} \cup \cdots \cup a_{\sigma(n)} : \text{ for all } \sigma \in \mathbb{S}_n\}$$

In what follows, we present a spanning set along with relations between all possible generators (this can be presented more easily than a basis). In each case, the generating set will be arranged by degree. In what follows, we present a spanning set along with relations between all possible generators (this can be presented more easily than a basis). In each case, the generating set will be arranged by degree.

Word length 6

Degree	Generating set
24	$t \cup_1 M$
25	$K_1, t \cup M, M \cup t, t \cup_1 t \cup_1 t, < M \cup_1 x, x >$
26	$< x \cup_1 t \cup_1 t, x >, < x \cup_1 t, x \cup_1 t >, < t \cup_1 t, t >, < M, x, x >$
27	$K_2, K_3, < x, x, t \cup_1 t >, < x, t, x \cup_1 t >, t \cup t \cup t$
28	$< x, x, x, x \cup_1 t >, < x, x, t, t >$
29	$< x, x, x, x, t >$
30	x^6

The only linear relation that can appear here is

$$x \cup t \cup (x \cup_1 t) + (x \cup_1 t) \cup x \cup t + t \cup x \cup (x \cup_1 t) + (x \cup_1 t) \cup t \cup x = 0$$

This relation implies that $dK_1 = (x \cup_1 t) \cup (x \cup_1 t)$ is a cocycle. Everything else is as expected.

Word length 7

Degree	Generating set
28	$M \cup_1 t \cup_1 x$
29	$K_1 \cup_1 x, s \cup_1 t \cup_1 x, < M \cup_1 t, x >, < M, t \cup_1 x >$
30	$K_1 \cup x, x \cup K_1 < t \cup_1 t \cup_1 t, x >, < t \cup_1 x, t \cup_1 t >, < M, t, x >, < s \cup_1 t, x >, < s, x \cup_1 t >, < s \cup_1 x, t >$
31	$K_2 \cup_1 x, K_3 \cup_1 x, < t \cup_1 t, x, t >, < x \cup_1 t, t, t >< s, t, x >, < x \cup_1 t, x \cup_1 t >, < t \cup_1 t, t >, < M, x, x, x >, < s \cup_1 x, x, x >$
32	$< K_2, x >, < K_3, x >< s, x, x, x >, < x, t, t, t >, < x, x, x, t \cup_1 t >, < x, x, t, x \cup_1 t >$
33	$< x, x, x, t, t >, < x, x, x, x, x \cup_1 t >$
34	$< x, x, x, x, x, t >$
35	x^7

There are several linear relations here.

$$x^2 \cup t \cup (x \cup_1 t) + x \cup (x \cup_1 t) \cup x \cup t + x \cup t \cup x \cup (x \cup_1 t) + x \cup (x \cup_1 t) \cup t \cup x = 0$$

$$x \cup t \cup (x \cup_1 t) \cup x + (x \cup_1 t) \cup x \cup t \cup x + t \cup x \cup (x \cup_1 t) \cup x + (x \cup_1 t) \cup t \cup x \cup x = 0$$

$$x \cup_1 (x \cup t \cup (x \cup_1 t) + (x \cup_1 t) \cup x \cup t + t \cup x \cup (x \cup_1 t) + (x \cup_1 t) \cup t \cup x) = 0$$

$$x \cup t \cup (t \cup_1 t) + t \cup x \cup (x \cup_1 t) = 0$$

$$(t \cup_1 t) \cup x \cup t + (x \cup_1 t) \cup t \cup x = 0$$

The first two linear relations create cocycles that are killed by $x \cup K_1$ $K_1 \cup x$. The third means that

$$x^2 \cup (x \cup_1 t) \cup (x \cup_1 t) + x \cup (x \cup_1 t) \cup x \cup (x \cup_1 t) + x \cup (x \cup_1 t) \cup x \cup (x \cup_1 t) + x \cup (x \cup_1 t) \cup (x \cup_1 t) \cup x = 0.$$

The final two linear equations imply that $(x \cup_1 t) \cup (x \cup_1 t) = dR_1$ and $(x \cup_1 t) \cup (t \cup_1 t) = dR_2$ are cocycles. Finally, there are linear relations arising from different expansions of $(a_1 \cup a_2 \cup a_3) \cup (a_4 \cup a_5)$ where $\{a_1, a_2, a_3, a_4, a_5\}$ is the set $\{t, t, x, x, x\}$. For degree reasons, it will not be necessary to analyse the relations coming from these in any great detail. Any cocycles produced this way are automatically killed.

Word length 8

Degree	Generating set
32	$M \cup_1 t \cup_1 t$
33	$s \cup_1 t \cup_1 t, < x, M \cup_1 x \cup_1 t >, < M \cup_1 t, t >, t \cup_1 t \cup_1 t \cup_1 t$
34	$L_1, L_2, < x, s \cup_1 x \cup_1 t >, < s \cup_1 t, t >, < M, t, t >, < M \cup_1 x, t, x >, < M \cup_1 t, x, x >$
34 (ctd)	$< M, x \cup_1 t, x >, < x, x, t \cup_1 t \cup_1 t >, < t \cup_1 t \cup_1 t, t >, < x \cup_1 t, x \cup_1 t \cup_1 t >, < R_1, x >, < R_2, x >$
34 (ctd)	$< x, x \cup_1 t \cup_1 t \cup_1 t >, < t, t \cup_1 t \cup_1 t >$
35	$< M, x, x, t >, < s, t, t >, < s \cup_1 x, t, x >, < s \cup_1 t, x, x >, < s, x \cup_1 t, x >, < x, x \cup_1 t, t \cup_1 t >, < t \cup_1 t, t, t >,$
35 (ctd)	$< M \cup_1 x, x, x, x >, < x \cup_1 t \cup_1 t, t \cup_1 t >, < x \cup_1 t \cup_1 t, x, t >, < x \cup_1 t \cup_1 t, x, x, x >, < x, x, t \cup_1 t \cup_1 t >$
36	$< s \cup_1 x, x, x, x >, < M, x, x, x, x >, < s, t, x, x >, < x, x, x \cup_1 t, x \cup_1 t >$
36 (ctd)	$< x, x, t, t \cup_1 t >, < x, x \cup_1 t, t, t > t \cup t \cup t \cup t$
37	$< x, x, x, x, t \cup_1 t >, < x, x, x, t, x \cup_1 t >, < x, x, t, t, t >, < s, x, x, x, x >$
38	$< x, x, x, x, x, x \cup_1 t >, < x, x, x, x, t, t >$
39	$< x, x, x, x, x, x, t >$
40	x^8

The terms $t^2 \cup (t \cup_1 t)$ and $(t \cup_1 t) \cup t^2$ are both cocycles because of the Hirsh identities

$$x^2 \cup t \cup (t \cup_1 t) + (x^2 \cup_1 t) \cup t \cup t = (x^2 \cup t) \cup_1 (t \cup_1 t) = t \cup x^2 \cup (t \cup_1 t) + (x^2 \cup_1 t) \cup t \cup t$$

$$(t \cup_1 t) \cup x^2 \cup t + t \cup t(x^2 \cup_1 t) = (t \cup x^2) \cup_1 (t \cup_1 t) = x^2 \cup t \cup (t \cup_1 t) + \cup t \cup t \cup (x^2 \cup_1 t)$$

Next, we discuss the relevant relations to $t^2 \cup (t \cup_1 t)$ and $(t \cup_1 t) \cup t^2$. Both terms are a cup product of $(t \cup_1 t)$ which is both a cocycle and of word length 4 with other terms of total word length 4. Our result will follow from the following statement:

Lemma 5.3.3. *The cocycles $t^2 \cup (t \cup_1 t), (t \cup_1 t) \cup t^2, t^2 \cup (t \cup_1 t) + (t \cup_1 t) \cup t^2$ are not exact in $C - \{L_1, L_2\}$, and $t^2 \cup (t \cup_1 t) + (t \cup_1 t) \cup t^2$ is not exact in $C - \{L_1\}$.*

Proof. First, observe that these cocycles are of degree 35. Observe that any linear dependence must contain either four copies of t or it must contain three copies of t and two copies of x and two uses of $\cup_1$. There is only one identity fitting either of those descriptions, that being:

$$x \cup t \cup (x \cup_1 t \cup_1 t) + (x \cup_1 t \cup_1 t) \cup x \cup t + t \cup x \cup (x \cup_1 t \cup_1 t) + (x \cup_1 t \cup_1 t) \cup t \cup x +$$
$$x \cup (t \cup_1 t) \cup (x \cup_1 t) + (x \cup_1 t) \cup x \cup (t \cup_1 t) + (t \cup_1 t) \cup x \cup (x \cup_1 t) + (x \cup_1 t) \cup (t \cup_1 t) \cup x = 0.$$

Combined with the fact that we know we have a generating set of the form

Degree	Generating set
35	$< M, x, x, t >, < s, t, t >, < s \cup_1 x, t, x >, < s \cup_1 t, x, x >,$
35 (ctd)	$< s, x \cup_1 t, x >, < x, x \cup_1 t, t \cup_t >, < t \cup_1 t, t, t >, < x, x, t \cup_1 t \cup_1 t >$
35 (ctd)	$< M \cup_1 x, x, x, x, x >, < x \cup_1 t \cup_1 t, t \cup_1 t >, < x \cup_1 t \cup_1 t, x, t >, < x \cup_1 t \cup_1 t, x, x, x, x >$

we can compute a 42 dimensional linear basis of C_8^{35}. Furthermore, we have a generating set for $C_8^{34} - \{L_1, L_2\}$ as

Degree	Generating set
34	$< x, s \cup_1 x \cup_1 t >, < s \cup_1 t, t >, < M, t, t >, < M \cup_1 x, t, x >, < M \cup_1 t, x, x >, < R_1, x >,$
34 (ctd)	$< M, x \cup_1 t, x >, < x, x, t \cup_1 t \cup_1 t >, < t \cup_1 t \cup_1 t, t >, < x \cup_1 t, x \cup_1 t \cup_1 t >, < R_2, x >$
34 (ctd)	$< x, x \cup_1 t \cup_1 t \cup_1 t >, < t, t \cup_1 t \cup_1 t >$

It suffices to check now that the three terms given are non-zero in cohomology. Our result can therefore be proven via a straightforward cohomology computation. □

Observe that in the last table $t \cup t \cup (t \cup_1 t) + (t \cup_1 t) \cup t \cup t$ is a cocycle by Equation 5.1. It follows that $H^*(C_8) = \mathbb{F}_2[s \cup s + s \cup (t^2 + x^2 \cup t + L_1 + L_2]$ □

5.3.3 The zig-zag

However, there is a map

$$f : C \to A$$

given by sending x to itself, t, s, M to zero and L to z. This map is a quasi-isomorphism. Then there is a map

$$g : C \to B$$

given by sending x, s, t to themselves and K, M to 0. This map is also a quasi-isomorphism.

CHAPTER 6

A higher Hochschild–Kostant–Rosenberg theorem and associated operations

Abstract

We prove a generalisation of the Hochschild–Kostant–Rosenberg theorem that holds for all commutative algebras and formal spaces. Using a A_∞-version of the coendomorphism operad, we define the notion of a coalgebra on the Hochschild homology. Finally, we specialise to the case of iterated suspensions and construct the E_{n+1}-coalgebra coming from the Deligne conjecture on a model for the Hochschild chain complex.

6.1 Introduction

Hochschild (co)homology [46] is a (co)homology theory for associative algebras over rings. It turns out to have a multitude of uses in geometry, topology and even physics. For example, the second cohomology group controls the deformation theory of associative algebra. The classical HKR theorem acts as a bridge between homological algebra and geometry, and is very closely related to Kontsevich formality.

The classical Hochschild–Kostant–Rosenberg theorem is normally stated as follows.

Theorem 6.1.1. *Let $\Bbbk$ be a field of characteristic 0 and let A be a commutative $\Bbbk$-algebra which is essentially of finite type and smooth over $\Bbbk$. Then there is an isomorphism of graded $\Bbbk$-algebras*

$$\Phi : HH_*((A,A) \xrightarrow{\sim} \Omega^*(A,\Bbbk)$$

between the Hochschild homology and the module of Kähler differentials. Dually, there is an isomorphism between the exterior algebra of derivations and the Hochschild cohomology

$$HH^*((A,A) \cong \Lambda^*(\mathrm{Der}_{\Bbbk}(A,A))$$

The assumption of smoothness is essentially a projectivity assumption on the module of Kähler differentials and can be be removed by working with cotangent complex instead. The Hochschild homology can be computed as the homology of the *Hochschild chain complex* $A \boxtimes S^1$ and cotangent complex can be viewed as the homology of $\mathrm{Sym}(V \otimes H_*(X), d_X)$ where $(\mathrm{Sym}(V), d)$ is a cofibrant resolution of A. The main result of this chapter is therefore following generalised version of the HKR theorem, which exhibits HKR-type statements primarily as consequences of formality.

Theorem 6.1.2. *Let X be a formal simplicial set of finite type in each degree. Let A be a CDGA. Suppose that $(\mathrm{Sym}(V), d)$ is a cofibrant, quasi-free resolution of A. Then there is a natural equivalence of chain complexes*

$$A \boxtimes X \xrightarrow{\sim} \mathrm{Sym}(V \otimes H_*(X), d_X)$$

We call $\mathrm{Sym}(V \otimes H_*(X), d_X)$ the higher X-shaped cotangent complex of $A \cong (\mathrm{Sym}(V), d)$. Moreover this equivalence is functorial with respect to formal maps.

When $X = S^1$, one recovers the classical HKR theorem. In the cases of S^n and wedges of spheres, one recovers the various HKR theorems of [40, Proposition 4.7].

This result has some immediate applications. Let M be a topological space and A be a cdga rational model for M. There is a canonical algebra map $C^*(\mathsf{Map}(X, M)) \to A \boxtimes X$ induced by the Chen iterated integrals, see [39]. This map is an equivalence whenever M is $\dim(X)$- connected. There is a dual map from chains on the mapping space to higher Hochschild cochains as well. Our HKR-theorem therefore shows that the (co)tangent complex is a very simple model for such mapping spaces.

A second motivation for studying the higher HKR quasi-isomorphism comes from derived algebraic geometry and mathematical physics. Indeed, in this context one [71] can define derived n-Poisson (dg-)schemes (or stacks), which is the data for instance provided by the observables of a n-dimensional quantum field theory. The classical Kontsevich theorem is the $n = 1$ case. In this higher context one might aim to deform the sheaf of functions $\mathcal{O}_X$ into an E_1-algebra structure on $\mathcal{O}_X[[\hbar]]$ (in the smooth affine case) or rather to deform its symmetric monoidal category of left modules into an E_{n-1}-monoidal category locally equivalent to modules over an E_n-deformation of $\mathcal{O}_X[[\hbar]]$. The higher Hochschild cochain complexes along with E_{n+1}-structure are the objects controlling those deformations while their cohomologies are precisely the higher analogues of polyvector-fields for higher Poisson structures.

Motivated by this physical intuition, we explain how to define an n-Poisson algebra structure on the homology of an n-fold suspension, generalising the usual cup coproduct on Hochschild homology.

Finally, we conclude by constructing a completely explicit $n+1$-Poisson coalgebra structures on the cotangent complex. By the formality of the little n-discs operad in zero characteristic and by our HKR-theorem, these very simple models are equivalent to the full E_n-coalgebra structure on Hochschild homology and therefore provide a solution to the Deligne conjecture in this context.

In this paper, we found it convenient to work with Hochschild homology rather than cohomology, even though though the latter appears to be more used in practice. Statements about the latter can generally be obtained from our statements by taking the linear dual at the chain level.

6.1.1 Conventions

We will always be working over a base field of characteristic 0, usually taken to be $\mathbb{Q}$. Our model for the E_∞-operad is the singular chains on the Barratt-Eccles operad $\mathcal{E}$. We shall use $*$ to denote the coproduct in the category of associative dg-algebras.

6.1.2 Structure of this chapter

In Section 2, we shall review some preliminaries. In Section 3, we shall explain the connection between $\mathscr{E}$-coalgebras and the Hochschild chain complex. In Section 4, we discuss concrete models for derived tensor products of $\mathscr{E}$-algebras. In Section 5, we prove our version of the HKR-theorem. In Section 6, we construct a n-Poisson algebra structure on the (ordinary) homology of an n-fold suspension $H_*(\Sigma^n X)$. Finally, in Section 7, we apply this theory to show how to construct and compute explicit operations on the higher Hochschild chain complex and cotangent complex generalising the usual cup coproduct on Hochschild homology.

6.2 Preliminaries

In this section, we review some preliminaries about the higher Hochschild homology and the Dold-Kan theorem.

6.2.1 The higher Hochschild chain complex

A higher order version of Hochschild cohomology was introduced by Pirashvili in [74]. In this theory, $\mathscr{E}$-algebras assume the role of commutative algebras. The category of $\mathscr{E}$-algebras is enriched in simplicial sets via

$$\mathrm{Map}_{\mathscr{E}-\mathrm{alg}}((A,B)_n = \mathrm{Hom}\left((A, B \otimes C^*\left(\Delta^n\right)\right),$$

and has all ∞-colimits. In particular, one can therefore define the tensor product $X \boxtimes A \in \mathscr{E}-\mathrm{alg}$ of an $\mathscr{E}$-algebra A and a simplicial set X. Explicitly, $X \boxtimes A$ is defined by the universal property that for every $\mathscr{E}$-algebra B there is a natural equivalence of derived mapping sets

$$\mathrm{Map}_{\mathscr{E}-\mathrm{alg}}(X \boxtimes A, B) \cong \mathrm{Map}_{\mathrm{sSet}}\left(X, \mathrm{Map}_{\mathscr{E}-\mathrm{alg}}(A,B)\right)$$

We briefly remark that the same construction holds for all $(\infty,1)$-categories. Ginot, Tradler and Zeinalian have shown that when A is a CDGA, one can choose a model for the $\mathscr{E}$-algebra $X \boxtimes A$ that is a CDGA and which admits a purely combinatorial construction [38]. They use this to show the following

Proposition 6.2.1. *[39] Let A be a CDGA over a field of characteristic 0. Then $A \boxtimes S^1$ is modelled by the usual Hochschild chain complex $C^*(A,A)$, where S^1 is the usual simplicial model for the unit circle.*

This motivates the following definition.

Definition 6.2.2. Let A be a CDGA or an $\mathscr{E}$-algebra over a field of characteristic 0 and let X be a simplicial set. Then we refer to $X \boxtimes A$ as the *higher X-shaped Hochschild chain complex*. We refer to its homology as the *X-shaped Hochschild homology* and denote it by $HH_*^X(A,A)$.

6.2.2 Dold-Kan for simplicial commutative rings

We will later need the following version of the Dold-Kan correspondence.

Theorem 6.2.3. *For $\Bbbk$ a field of characteristic 0 there is a Quillen equivalence*

$$N : \mathrm{sCom}\text{-}alg_{\Bbbk} \leftrightarrows \mathscr{E}-\mathrm{coAlg}_{\Bbbk} : Q$$

between connected commutative connective $\mathscr{E}$-algebras over $\Bbbk$ and connected commutative simplicial algebras over $\Bbbk$

6.2.3 The little n-discs operad and Poisson algebras

We recall the little n-discs operad $\mathbb{D}_n$ from the first chapter. In characteristic zero there are two main theorems about it that we shall use in this chapter. The first is the well known computation of its homology.

Theorem 6.2.4. *[80] The homology of the little n-discs operad $H_*(\mathbb{D}_n)$ is isomorphic to the n-Poisson operad Pois_n.*

The second is its formality.

Theorem 6.2.5. *[86] In characteristic 0, the homology of the little n-discs operad is formal. In other words, the homology $H_*(\mathbb{D}_n)$ is weakly equivalent, in the model category of operads in chain complexes, to the singular chains $C_*(\mathbb{D}_n)$.*

6.3 The Hochschild chain complex and $\mathscr{E}$-coalgebras

The purpose of this section is to prove the following theorem.

Theorem 6.3.1. *Let $X \in \mathsf{sSet}$ and $A \in \mathscr{E}-\mathbf{alg}$. Then there exists an is a weak equivalence of $\mathscr{E}$-algebras between $X \boxtimes A$ and $C_*(X) \otimes^{\mathbb{L}}_{\mathscr{E}} A$. In other words, the diagram*

$$\begin{tikzcd} \mathsf{sSet} \times \mathscr{E}-\mathbf{alg} \arrow[rr, "\boxtimes"] \arrow[dr, "C_* \times id"'] & & \mathscr{E}-\mathbf{alg} \\ & \mathscr{E}-\mathbf{coalg} \times \mathscr{E}-\mathbf{alg} \arrow[ur, "\otimes^{\mathbb{L}}_{\mathscr{E}}"'] & \end{tikzcd}$$

commutes up to homotopy.

The derived tensor product of an $\mathscr{E}$-algebra and an $\mathscr{E}$-coalgebra. The derived tensor product $\otimes^{\mathbb{L}}_{\mathscr{E}}$ in the statement of the theorem is defined as follows. Via the Dold-Kan correspondence for simplicial commutative rings, the category of $\mathscr{E}$-algebras is enriched in $\mathscr{E}$-coalgebras via $N\left(\mathrm{Map}_{\mathscr{E}-\mathrm{alg}}((A,B)\right)$, and has all ∞-colimits. The tensor product of an $\mathscr{E}$-algebra with an $\mathscr{E}$-coalgebra is $\mathscr{E}$-algebra $A \otimes^{\mathbb{L}} B$ is therefore defined by the universal property that for all $\mathscr{E}$-algebras C there is an equivalence of derived mapping sets.

$$\mathrm{Map}_{\mathscr{E}-\mathrm{alg}}\left(A \otimes^{\mathbb{L}} B, C\right) \cong \mathrm{Map}_{\mathscr{E}-\mathrm{coalg}}\left(A, N\left(\mathrm{Map}_{\mathscr{E}-\mathrm{alg}}(B,C)\right)\right)$$

To prove the Theorem 6.3.1, we shall need the following lemma.

Lemma 6.3.2. *Let X be a simplicial set and let G be a simplicial commutative ring. Then there is an isomorphism of mapping sets*

$$F : \mathrm{Map}_{\mathsf{sSet}}(X, G) \to \mathrm{Map}_{\mathscr{E}-\mathbf{coalg}}(C_*(X), N(G)).$$

Proof. The singular chains functor $C_* : \mathsf{sSet} \to \mathscr{E}-\mathrm{coalg}$, along with the Eilenberg-Zilber map, induces a map

$$G : \mathrm{Map}_{\mathsf{sSet}}(X, G) \xrightarrow{C_*(-)} \mathrm{Map}_{\mathscr{E}-\mathbf{coalg}}(C_*(X), C_*(G)).$$

Now, there is a map $C_*(G) \to N(G)$, given explicitly by sending the formal sums of chains in $C_*(G)$ to the internal addition in $N(G)$. Postcomposing with this map gives the desired map

$$F : \mathrm{Map}_{\mathsf{sSet}}(X, G) \to \mathrm{Map}_{\mathscr{E}-\mathbf{coalg}}(C_*(X), N(G)).$$

Now this map is an isomorphism, because it admits an explicit inverse, given by restricting maps $f : C_*(X) \to N(G)$ to the simplicial set of generators X. This gives a map $X \to N(G)$ and $N(G)$ is isomorphic to G as a simplicial set. □

Now we can prove the theorem.

Proof of Theorem 6.3.1. Let $B \in \mathscr{E}-\mathbf{alg}$. We have the following chain of adjunctions. Firstly, the definition of $X \boxtimes A$ tells us that

$$\mathrm{Map}_{\mathscr{E}-\mathbf{alg}}(X \boxtimes A, B) \xrightarrow{\sim} \mathrm{Map}_{\mathsf{sSet}}\left(X, \mathrm{Map}_{\mathscr{E}-\mathbf{alg}}(A, B)\right)$$

where $\mathrm{Map}_{\mathscr{E}-\mathbf{alg}}(A, B)$ is the simplicial enrichment of the mapping space.

By Lemma 6.3.2, there is an isomorphism

$$\mathrm{Map}_{\mathsf{sSet}}\left(X, \mathrm{Map}_{\mathscr{E}-\mathbf{alg}}(A, B)\right) \xrightarrow{F} \mathrm{Map}_{\mathscr{E}-\mathbf{coalg}}\left(C_*(X), N\left(\mathrm{Map}_{\mathscr{E}-\mathbf{alg}}(A, B)\right)\right).$$

Finally by the definition of $C_*(X) \otimes^{\mathbb{L}}_{\mathscr{E}} A$ we showed that there is an isomorphism

$$\mathrm{Map}_{\mathscr{E}-\mathbf{coalg}}\left(C_*(X), Ch\left(\mathrm{Map}_{\mathscr{E}-\mathbf{alg}}(A, B)\right)\right) \cong \mathrm{Map}_{\mathscr{E}-\mathbf{alg}}\left(C_*(X) \otimes^{\mathbb{L}}_{\mathscr{E}} A, B\right).$$

concluding the proof. □

6.4 Computing the derived tensor product

The purpose of this section is to show that concrete descriptions of the derived tensor product for CDGAs and explain how it may be computed simply. The main result is the following.

Definition 6.4.1. Let A be a $\mathscr{E}$-algebra and C be a $\mathscr{E}$-coalgebra. Their tensor product is

$$C \otimes_{\mathscr{E}} A = coeq\left(\bigoplus_{f:\{1,\ldots p\}\to\{1,\ldots q\}} C^{\otimes p} \otimes \mathscr{E}(p,q) \otimes A^{\otimes q} \rightrightarrows \bigoplus_n C^{\otimes n} \otimes A^{\otimes n}\right)$$

The upper map in the coequalizer is induced by the unique maps $f^* : C^{\otimes p} \otimes \mathscr{E}(p,q) \otimes A^{\otimes q} \to C^{\otimes q} \otimes A^{\otimes q}$ obtained from the $\mathscr{E}$-coalgebra structure of C and the lower map is induced by the maps $f_* : C^{\otimes p} \otimes \mathscr{E}(p,q) \otimes A^{\otimes q} \to C^{\otimes p} \otimes A^{\otimes p}$ induced by the $\mathscr{E}$-algebra structure on A.

Proposition 6.4.2. *Let A be a fibrant $\mathscr{E}$-coalgebra and C be a cofibrant $\mathscr{E}$-algebra. Then $C \otimes^{\mathbb{L}}_{\mathscr{E}} A$ is weakly equivalent to $C \otimes_{\mathscr{E}} A$.*

To prove this proposition, we first introduce a concrete model $B(C, \mathscr{E}, A)$ for $C \otimes^{\mathbb{L}}_{\mathscr{E}} A$. Then our proof shall follow by constructing an explicit retraction on this model.

6.4.0.1 Modelling the derived tensor product by a bar construction

In this subsubsection, we shall given an explicit model for the tensor product of a $\mathscr{E}$-algebra with a $\mathscr{E}$-coalgebra, via the bar construction. First, we recall the notion of the properad associated to an operad.

Definition 6.4.3. Let $\mathscr{P}(-)$ be an operad. The properad $\mathscr{P}(-,-)$ associated to $\mathscr{P}$ is defined componentwise to be

$$\mathscr{P}(p,q) = \bigoplus_{n_1+\cdots n_p=q}\left(\bigotimes_{i=1}^{p}\mathscr{P}(n_i)\right).$$

Now, $C\otimes^{\mathbb{L}}_{\mathscr{E}} A$ is a left adjoint, and therefore may be computed as homotopy colimit.

Proposition 6.4.4. *Let A be a $\mathscr{P}$-algebra and C be a $\mathscr{P}$-coalgebra. Then their derived tensor product $C\otimes^{\mathbb{L}}_{\mathscr{P}} A$ may be computed as an homotopy coequalizer in the category of $\mathscr{P}$-algebras*

$$C\otimes^{\mathbb{L}}_{\mathscr{P}} A = hocoeq\left(\bigoplus_{f:\{1,\dots p\}\to\{1,\dots q\}} C^{\otimes p}\otimes\mathscr{P}\left(q,p\right)\otimes A^{\otimes q}\rightrightarrows\bigoplus_n C^{\otimes n}\otimes A^{\otimes n}\right)$$

where the maps $f:\{1,\dots p\}\to\{1,\dots q\}$ are maps of sets. The upper map in the coequalizer is induced by the maps $f^*: C^{\otimes p}\otimes\mathscr{P}\left(q,p\right)\otimes A^{\otimes q}\to C^{\otimes q}\otimes A^{\otimes q}$ obtained from the coalgebra structure of C and the lower map is induced by the maps $f_*: C^{\otimes p}\otimes\mathscr{P}\left(q,p\right)\otimes A^{\otimes q}\to C^{\otimes p}\otimes A^{\otimes p}$ induced by the algebra structure. The homotopy coequalizer is computed as realizations of a bar construction, and therefore inherits an $\mathscr{E}$-algebra structure.

Proof. It suffices to show that the homotopy colimit defined in the statement satisfies the universal property of the derived tensor product, ie. for all $\mathscr{E}$-algebras E, one has an isomorphism of derived mapping spaces

$$\mathrm{Map}_{\mathscr{E}-\mathrm{alg}}\left(C\otimes^{\mathbb{L}}_{\mathscr{E}} A, E\right)\cong\mathrm{Map}_{\mathscr{E}-\mathrm{coalg}}\left(A, N\left(\mathrm{Map}_{\mathscr{E}-\mathrm{alg}}(A,E)\right)\right)$$

The homotopy left adjoint has an explicit formula, which is the homotopy colimit formula appearing above. □

As usual, the homotopy coequalizer above has an explicit model given by the two-sided bar construction

$$B(C,\mathscr{P},A).$$

To be precise, this is defined as the totalization of the following simplicial chain complex.

- The r-dimensional component is

$$B(C,\mathscr{E},A)_r = \bigoplus_{\substack{f:\{1,\dots p\}\to\{1,\dots q\}\\ k_1,\dots k_{r-1}}} C^{\otimes p}\otimes\mathscr{E}\left(q,k_1\right)\otimes\mathscr{E}(k_1,k_2)\otimes\mathscr{E}\left(k_{r-1},p\right)\otimes A^{\otimes q}\text{ for } r>0.$$

$$B(C,\mathscr{E},A)_0 = \bigoplus_n C^{\otimes n}\otimes A^{\otimes n}$$

- The face map d_0 is induced by the coalgebra structure on C, the map d_n is induced by the algebra structure on A, and the map d_i for $0<i<n$ come from composition in the properad.
- Let $\mathrm{id}\in\mathscr{E}(1)$ be the operadic identity. The degeneracy maps are all given by inserting the n-fold tensor product of identity map $\mathrm{id}^{\otimes n}\in\mathscr{E}(n,n)$.

Proof of Proposition 6.4.2. We prove this using an extra degeneracy argument. Let $A = (\mathscr{E}(V), d_1)$ and $C = (\mathscr{E}^c(W), d_2)$, then there is a canonical map

$$v_{-1} : \mathscr{E}(V)^{\otimes n} \to \mathscr{E}(r, n) \otimes V^{\otimes r}$$

$$\gamma_1(x_{1,1}, \dots x_{1,k_1}) \otimes \cdots \otimes \gamma_n(x_{n,1}, \dots x_{n,k_n}) \mapsto \left(\gamma_1 \otimes \cdots \otimes \gamma_n\right) \otimes x_{1,1} \otimes \cdots \otimes x_{1,k_1} \otimes \cdots \otimes x_{n,k_n}$$

where $r = k_1 + \cdots + k_n$. This defines a map

$$s_{-1} = \mathrm{id} \otimes v_{-1} : B\left(\mathscr{E}^c(W), \mathscr{E}, \mathscr{E}(V)\right)_l \to B\left(\mathscr{E}^c(W), \mathscr{E}, \mathscr{E}(V)\right)_{l+1}$$

One can check that this defines an extra degeneracy. By the standard argument, this defines a contraction between $B(\mathscr{E}^c(W), \mathscr{E}, \mathscr{E}(V))$ and the simple tensor product $\mathscr{E}^c(W) \otimes_{\mathscr{E}} \mathscr{E}(V)$. $\square$

6.5 The higher Hochschild-Konstant-Rosenberg theorem

The main result of this chapter is the following generalised version of the HKR theorem.

Theorem 6.5.1. *Let X be a simplicial set of finite type in each degree and suppose that X is formal. Let A be a CDGA. Suppose that $\left(\mathrm{Sym}(V), d\right)$ is a cofibrant, quasi-free resolution of A. Then there is a natural equivalence*

$$A \boxtimes X \xrightarrow{\sim} \mathrm{Sym}(V \otimes H_*(X), d_X)$$

We call $\mathrm{Sym}(V \otimes H_*(X), d_X)$ the higher X-shaped cotangent complex of $A \cong \left(\mathrm{Sym}(V), d\right)$. Moreover, if $f : X \to Y$ is a formal map, we have a homotopy commutative diagram

$$\begin{tikzcd}
A \boxtimes X \arrow[r, "\sim"] \arrow[d, "id \boxtimes f"] & \left(\mathrm{Sym}(V \otimes H_*(X)), d_X\right) \arrow[d, "\mathrm{Sym}(id \otimes H(f_*))"] \\
A \boxtimes Y \arrow[r, "\sim"] & \left(\mathrm{Sym}(V \otimes H_*(Y)), d_Y\right)
\end{tikzcd}$$

The differential d_X is defined as follows. Let $\Delta^{(n-1)}$ be the iterated coproduct on $H_*(X)$. Then, using Sweedler notation, define

$$\Delta^{(n-1)}(\alpha) = \Sigma \alpha_{(1)} \otimes \cdots \otimes \alpha_{(n)}$$

and, for $v \in V$ let

$$d(v) = \Sigma v_{(1)} \cdots v_{(n)}$$

where $v_{(1)}, \dots v_{(n)} \in V$. Then d_X is the unique derivation extending the product

$$d_X(v \otimes \alpha) = \Sigma\left(v_{(1)} \otimes \alpha_{(1)}\right) \cdots \left(v_{(n)} \otimes \alpha_{(n)}\right)$$

The above statements may be dualized.

The remainder of this section will be devoted to proving the above theorem. First, we shall prove a lemma that gives a description of the differentials

Lemma 6.5.2. *Let A be a cocommutative coalgebra and let $(\mathscr{E}(V), d)$ be a quasi-free commutative algebra. Then there is an isomorphism of commutative algebras, which is functorial in A*

$$A \otimes_{\mathsf{Com}} \mathrm{Sym}(V) \cong \left(\mathrm{Sym}(A \otimes V), d_X\right)$$

where the differential d_X is defined as follows. Let $\Delta^{(n-1)}$ be the iterated coproduct on A. Then, using Sweedler notation, define

$$\Delta^{(n-1)}(\alpha) = \Sigma\alpha_{(1)} \otimes \cdots \otimes \alpha_{(n)}$$

and, for $v \in V$ let

$$d(v) = \Sigma v_{(1)} \cdots v_{(n)}$$

where $v_{(1)}, \ldots v_{(n)} \in V$. Then d_X is the unique derivation extending the product

$$d_X(v \otimes \alpha) = d_A(v) \otimes \alpha + \Sigma\left(v_{(1)} \otimes \alpha_{(1)}\right) \cdots \left(v_{(n)} \otimes \alpha_{(n)}\right)$$

Proof. An element of $u \in A \otimes_{\mathrm{Comm}} \mathrm{Sym}(V)$ may be expressed, non-uniquely, as

$$(a_1 \otimes a_2 \otimes \cdots a_k) \otimes \left(v_{1,1} v_{1,2} \cdots v_{1,l_1} \otimes v_{2,1} v_{2,2} \cdots v_{2,l_2} \otimes \cdots \otimes v_{k,1} v_{k,2} \cdots v_{k,l_k}\right)$$

where the $a_i \in A$ and the $v_{i,j} \in V$. The *canonical form* of u may be defined as follows. Let $\Delta^{(l_i-1)}(a_i) = a_{i,1} \otimes \cdots \otimes a_{i,l_i}$. Then u can be written as

$$\left((a_{1,1} \otimes v_{1,1}\right) \otimes \left((a_{1,2} \otimes v_{1,2}\right) \otimes \cdots \otimes \left((a_{1,l_1} \otimes v_{1,l_1}\right) \otimes \cdots \otimes \left((a_{k,l_k} \otimes v_{k,l_k}\right).$$

It is clear that this is a unique way to express u. The tensor product of two elements in canonical form is still in canonical form. This proves that $A \otimes_{\mathrm{Comm}} \mathrm{Sym}(V)$ is a free commutative algebra on the basis $A \otimes V$. Next we need to compute the differential. We see that

$$d(\alpha \otimes v) = d(\alpha) \otimes v + \alpha \otimes d(v) = \alpha \otimes d(v) = \alpha \otimes \Sigma v_{(1)} \cdots v_{(n)}.$$

Put in canonical form, we see that this is exactly the desired differential. □

We now proceed to the proof of the main result.

Proof of Theorem 6.5.1. By Theorem 6.3.1 there is a weak equivalence of $\mathscr{E}$-algebras

$$X \boxtimes A \cong C_*(X) \otimes^{\mathbb{L}}_{\mathscr{E}} A.$$

The $\mathscr{E}$ coalgebra $C_*(X)$ has a minimal model $\left(\mathscr{E}^c(W), d'\right)$ and $A \cong (\mathscr{E}(V), d)$, so it follows that

$$C_*(X) \otimes^{\mathbb{L}}_{\mathscr{E}} A \cong \left(\mathscr{E}^c(W), d'\right) \otimes^{\mathbb{L}}_{\mathscr{E}} (\mathscr{E}(V), d).$$

By Proposition 6.4.2, this can now be computed as the ordinary tensor product

$$\left(\mathscr{E}^c(W), d'\right) \otimes^{\mathbb{L}}_{\mathscr{E}} (\mathscr{E}(V), d) \cong \left(\mathscr{E}^c(W), d'\right) \otimes_{\mathscr{E}} (\mathscr{E}(V), d).$$

Since $\mathscr{E} \to \mathsf{Com}$ is a weak equivalence, one has

$$\left(\mathscr{E}^c(W), d'\right) \otimes_{\mathscr{E}} (\mathscr{E}(V), d) \cong \left(\mathrm{Sym}^c(W), d'\right) \otimes_{\mathsf{Com}} \left(\mathrm{Sym}(V), d\right)$$

It follows from Lemma 6.5.2 that one has

$$\left(\mathrm{Sym}^c(W), d'\right) \otimes_{\mathsf{Com}} \left(\mathrm{Sym}(V), d\right) \xrightarrow{\sim} \left(\mathrm{Sym}\left(\mathrm{Sym}^c(W) \otimes V\right), d_W\right).$$

Since $C_*(X)$ is formal, there is a quasi-isomorphism of coalgebras $H_*(X) \to \mathrm{Sym}^c(W)$. It remains to show that this induces a quasi-isomorphism of chain complexes

$$\left(\mathrm{Sym}(H_*(X) \otimes V), d_X\right) \xrightarrow{\sim} \left(\mathrm{Sym}\left(\mathrm{Sym}^c(W) \otimes V\right), d_W\right).$$

To prove this, filter both complexes by weight in the outermost copy of Sym. As, by Kunneth's theorem

$$H_*\left((H_*(X) \otimes V)^{\otimes k}\right) \cong H_*\left(\left(\mathrm{Sym}^c(W) \otimes V\right)^{\otimes k}\right),$$

it is obvious that the associated spectral sequences have isomorphic E^1-pages. Since there is a map of chain complexes between them, they therefore converge to the same object. The result follows.

Functoriality follows from the functoriality in Lemma 6.5.2. □

6.6 Transferring coalgebra structures from topological spaces to cohomology

We explain two approaches to transferring the E_n-coalgebra structure from the first chapter to chain complexes. The second approach will be used later to extend the E_n-structure on Hochschild chain complex to an E_{n+1}-structure.

6.6.0.1 Applying the singular chains functor directly

The simplest way to obtain an E_n-coalgebra structure in chain complexes from that is spaces is to apply the singular chains functor. One then sends wedge product is sent to the direct sum via the quasi-isomorphism of chain complexes $C_*(X \vee Y) \twoheadrightarrow C_*(X) \oplus C_*(Y)$.

Proposition 6.6.1. *Let $\Sigma^n X$ be the n-fold suspension of a space X. Then the singular chains $C_*(\Sigma^n X)$ form a coalgebra over $C_*(\mathbb{D}_n)$, the singular chains on the little n-discs operad with respect to the direct sum.*

Proof. Consider the following composite map

$$C_*(\mathbb{D}_n(k)) \otimes C_*\left(\Sigma^n X\right) \xrightarrow{EZ} C_*\left(\mathbb{D}_n(k) \times \Sigma^n X\right) \xrightarrow{C_*(\triangle_k)} C_*\left(\left(\Sigma^n X\right)^{\vee k}\right) \to C_*\left(\Sigma^n X\right)^{\oplus k} \tag{6.1}$$

where EZ is the Eilenberg-Zilber map. This defines an E_n-coalgebra structure on $C_*(\Sigma^n X)$ with respect to the direct sum. □

Example 6.6.2. We can take the homology of the map in (6.1). So we obtain maps

$$H_*(\mathbb{D}_n(k)) \otimes H_*\left(S^n\right) \to H_*\left(\left(S^n\right)^{\vee k}\right) \cong H_*\left(S^n\right)^{\oplus k}$$

The homology of S^n is concentrated in degrees 0 and n. The homology of $H_*(\mathbb{D}_n(k))$ is concentrated in degrees $i(n-1)$ for $k-1 \geq i \geq 0$. So, for degree reasons, there is only one element of $\mu \in H_*(\mathbb{D}_n(k))$ for which the map

$$\mu \times H_*\left(S^n\right) \to H_*\left(S^n\right)^{\oplus k}$$

is nonzero, that is, for $|\mu| = 0$. In this case, the resulting map is $x \mapsto x_1 + \cdots + x_k$. In particular the Poisson bracket vanishes.

Remark 6.6.3. A problem that we shall later face is that the map $C_*(X \vee Y) \twoheadrightarrow C_*(X) \oplus C_*(Y)$ is not a map of associative coalgebras and loses some structure that existed on the topological level. Indeed, on the chain level, the quasi-isomorphism $C_*\left((\Sigma^n X)^{\vee k}\right) \to C_*(\Sigma^n X)^{\oplus k}$ introduce relations that kill a lot of the higher structure. The image of the Whitehead product in $C_*(S^n)^{\oplus 2}$ can be expressed as the sum of two simplices, one of the form $(\sigma_1, 0)$ and the other of the form $(0, \sigma_2)$. No such relation exists in $C_*\left((\Sigma^n X)^{\vee 2}\right)$. The coalgebra structure of Lemma 6.6.1 is therefore the forced strictification of the E_n-coalgebra structure to a Com-coalgebra.

When working rationally, the key to solving this problem is taking into account the associative structure of the diagonal map.

6.6.1 Coalgebras in chain complexes

In this section, we define of the coendomorphism operad that takes into account the A_∞-structure. This will be important because when constructing the Poisson operad structure.

Definition 6.6.4. Let C be a conilpotent, coassociative coalgebra in chain complexes with zero differential. Then the A_∞-*coendomorphism operad* $\mathrm{CoEnd}_{A_\infty}(C)$ of C, in arity k is the following chain complex

$$\mathrm{CoEnd}_{A_\infty}(C)(k) = \mathsf{Map}_{\mathsf{Ass}-alg}\left(\Omega(C), \Omega(C)^{*k}\right).$$

Here $\Omega(-)$ is the cobar construction. The symmetric action is given by permuting the coproduct and the composition is determined by the function composition.

It is straightforward to verify that that above is an operad. In the concrete cases we look at in this chapter, C will also have primitive multiplication and therefore one has

$$\mathrm{CoEnd}_{A_\infty}(C)(k) = \mathsf{Map}_{Ch}\left(s^{-1}C, T\left(s^{-1}C\right)^{*k}\right).$$

Remark 6.6.5. This is a simplified version of the coendomorphism operad that we discuss in the appendix. This may shed more light on the geometric meaning of the coendomorphism operad construction.

6.6.2 Transferring E_n-coalgebra structures to chain complexes

In this section, we shall transfer the E_n-coalgebra structure on iterated suspensions from the first chapter to chains and homology.

6.6.2.1 Compatibility with the diagonal map

The diagram

$$\begin{tikzcd}
\mathbb{D}_n(k) \times S^n \arrow[r, "\triangle_k"] \arrow[d, "id \times d_l"] & (S^n)^{\vee k} \arrow[d, "d_l"] \\
\mathbb{D}_n(k) \times (S^n)^{\times l} \arrow[r, "(\triangle_k)^{\times l}"] & \left((S^n)^{\vee k}\right)^{\times l}
\end{tikzcd}$$

where $d_l : X \to X^{\times l}$ is the diagonal, commutes. Therefore, for every $\mu \in \mathbb{D}_n(k)$ the map $C_*\left(\triangle_k\left(\mu \times -\right)\right)$ is also a map of $\mathscr{E}$-coalgebras, where the $\mathscr{E}$-coalgebra structures on S^n and $(S^n)^{\vee k}$ are given by the diagonal maps. One can therefore think of the associative structure on singular chains as being codistributive over the E_n-structure.

6.6.3 The homological level

Consider the E_n-algebra map from the first chapter with a slight twist. Instead of studying the E_n-coalgebra structure on S^n, we apply the Moore loop space functor $\mathbf{\Omega}$ to all the maps obtained this way. Taking homology, one obtains

$$H_*(\mathbb{D}_n(k)) \times H_*\left(\mathbf{\Omega}S^n\right) \xrightarrow{\alpha} H_*\left(\mathbb{D}_n(k) \times \mathbf{\Omega}S^n\right) \xrightarrow{H_*(p)} H_*\left(\mathbf{\Omega}\left(\left(S^n\right)^{\vee k}\right)\right) \tag{6.2}$$

Here, p is the map coming from the following factorisation

$$\begin{tikzcd}
\mathbb{D}_n(k) \times \mathbf{\Omega} X \arrow[r, "i_1 \times i_2"] \arrow[drrr, dotted, "p"'] & \mathscr{L}(\mathbb{D}_n(k)) \times \mathscr{L}(X) \arrow[r] & \mathscr{L}(\mathbb{D}_n(k) \times X) \arrow[r] & \mathscr{L}\left(X^{\vee k}\right) \\
 & & & \mathbf{\Omega}\left(X^{\vee k}\right) \arrow[u, hook]
\end{tikzcd}$$

Here $i_1 : \mathbb{D}_n(k) \to \mathscr{L}(\mathbb{D}_n(k))$ is the inclusion of constant loops and $i_2 : \mathbf{\Omega} X \to \mathscr{L}(X)$ is the inclusion of the based loop space into the free loop space. The reader may check that the coproduct of $\mathbf{\Omega} X$ and $\mathbf{\Omega} Y$ in the category of associative topological monoids is $\mathbf{\Omega}(X \vee Y)$. The reader may verify that loop composition is preserved in the above diagram. The following proposition therefore immediately follows from [33, Theorem 2.22].

Proposition 6.6.6. *The Moore loop space $\mathbf{\Omega}\Sigma^n X$ of an n-fold suspension $\Sigma^n X$ is a coalgebra over the little n-discs operad $\mathbb{D}_n$ in the category of associative monoids in topological spaces.*

Example 6.6.7. More generally, we perform this construction for any n-fold suspension $\Sigma^n X$.

$$H_*(\mathbb{D}_n(k)) \otimes H_*\left(\mathbf{\Omega}\Sigma^n X\right) \xrightarrow{EZ} H_*\left(\mathbb{D}_n(k) \times \mathbf{\Omega}\Sigma^n X\right) \xrightarrow{H_*(\mathbf{\Omega}\Delta_k)} H_*\left(\mathbf{\Omega}\left(\left(\Sigma^n X\right)^{\vee k}\right)\right). \tag{6.3}$$

By Adams' theorem [1], $H_*(\mathbf{\Omega}\Sigma^n X)$ is a Hopf algebra and is isomorphic to $\Omega H_*(\Sigma^n X)$. Then, we have the following lemma.

Proposition 6.6.8. *Let X be a simply connected topological space. Then there is an isomorphism of Hopf algebras*

$$H_*\left(\mathbf{\Omega}\left((X)^{\vee k}\right)\right) \cong \Omega(H_*(X))^{*k}.$$

Proof. By Adams' theorem $H_*\left(\mathbf{\Omega}\left(X^{\vee k}\right)\right)$ is isomorphic to $\Omega H_*\left(X^{\vee k}\right)$. It is well known that reduced cohomology ring of $\bar{H}_*\left(X^{\vee k}\right)$ is the direct product of rings $\prod_{i=1}^k \bar{H}_*(X)$. The result follows either by direct computation or by observing that the functor Ω is a left adjoint and therefore preserves coproducts. □

So, we can rewrite (6.3) as a map

$$H_*(\mathbb{D}_n(k)) \otimes \Omega\left(H_*\left(\Sigma^n X\right)\right) \to \Omega\left(H_*\left(\left(\Sigma^n X\right)\right)\right)^{*k}$$

By adjunction, this is a map

$$\mathsf{Pois}_n(k) = H_*(\mathbb{D}_n(k)) \to \mathrm{Map}_{Ch}\left(\Omega\left(H_*\left(\Sigma^n X\right)\right), \Omega\left(H_*\left(\left(\Sigma^n X\right)\right)\right)^{*k}\right) \tag{6.4}$$

Our next theorem states that this map can be restricted to produce a coalgebra map.

Proposition 6.6.9. *For $\Sigma^n X$ an n-fold suspension, the map (6.4) admits a factorisation*

$$\begin{tikzcd}
H_*(\mathbb{D}_n(k)) \arrow[r] \arrow[dr, dotted] & \mathrm{Map}_{Ch}\left(\Omega(H_*(\Sigma^n X)), \Omega(H_*((\Sigma^n X)))^{*k}\right) \\
 & \mathrm{CoEnd}_{A_\infty}(H_*(\Sigma^n X)) \arrow[u]
\end{tikzcd}$$

which is a map of operads. It follows that $H_*(\Sigma^n X)$ is equipped with a Pois_n-coalgebra structure. The n-Poisson operad is generated by a symmetric operation $\mu \in \mathsf{Pois}_n(2)_0$ and an anti-symmetric operation $\tau \in \mathsf{Pois}_n(2)_n$. The coalgebra structure on $H_*(\Sigma^n X)$

$$\triangle_\mu : \Omega H_*(\Sigma^n X) \to \Omega H_*(\Sigma^n X)^{*2}$$

is generated by the map

$$s^{-1}x \mapsto s^{-1}x_l + s^{-1}x_r$$

where x_l *is in the first copy of* $\bar{H}_*(\Sigma^n X)$ *contained in* $\Omega H_*(\Sigma^n X)^{*2}$ *and* x_r *is in the second. The degree* $n-1$ *map*

$$\triangle_\tau : \Omega H_*(\Sigma^n X) \to \Omega H_*(\Sigma^n X)^{*2}$$

is generated by the Whitehead bracket.

Proof. The products on $\Omega(H_*(\Sigma^n X))$ and $H_*(\mathbf{\Omega} X)$ coincide and in the latter case is induced by the associative loop composition. Examining the diagram 6.6.3, we see that, for each $\mu \in \mathbb{D}_n(k)$, the induced map is a map of associative algebras. Since the homology of the loop space precisely coincides with the cobar construction of the homology, and the same is true for the relationship between the coproduct of associative algebras and the loop space of the wedge sum, we deduce the desired factorisation and the fact that it is a morphism of operads follows from Proposition 6.6.6.

We directly compute the map (6.3). The cup coproduct on the homology of suspensions is primitive , so it follows that $\Omega(H_*(\Sigma^n X)) = T\left(\bar{H}_*\left(\Sigma^{n-1}X\right)\right)$ where $T(-)$ is the tensor algebra. The homology $H_*(\mathbb{D}_n(k))$ along representatives for each class are known and have been computed in [80]. We therefore need only compute the map on these representatives. We therefore need only compute the map on the two generators in $H_*(\mathbb{D}_n(2)) = H_*\left(S^{n-1}\right)$. It is trivial to check that $\mu \in H_*(\mathbb{D}_n(2))_0$ coincides with the cup product. Finally to compute the map corresponding to $\tau \in H_*(\mathbb{D}_n(2))_{n-1}$ we apply the Milnor-Moore theorem to the following result from [32] *The degree* $n-1$ *map*

$$\pi_i\left(S^n\right) \to \pi_{i+n-1}\left(S^n \vee S^n\right)$$

induced by the fundamental class of $\mathbb{D}_n(2) = S^{n-1}$ *is the Whitehead bracket.* □

We remark at this point that the delooping can be iterated n-times on n-fold suspensions. We conclude this section by illustrating with an example of the n-sphere.

Example 6.6.10. Using the Serre spectral sequence, it is straightforward to show that $H_*\left(\mathbf{\Omega}\left((S^n)^{\vee k}\right)\right)$ is the free associative algebra generated by k variables

$$H_*\left(\mathbf{\Omega}\left(\left(S^n\right)^{\vee k}\right)\right) = \mathbb{Q}\langle x_1, \ldots, x_k\rangle$$

where $|x_i| = n-1$. and where each generator is the desuspension of the fundamental class of one of the copies of S^n. We describe the Poisson coalgebra structure on the degree 0 coassociative comultiplication $\triangle_\mu$ and the degree $n-1$ coLie cobracket $\triangle_\tau$. It follows from Proposition 6.6.9 that these are as follows:

$$\triangle_\mu : \mathbb{Q}\langle x\rangle = H_*\left(\mathbf{\Omega} S^n\right) \to H_*\left(\mathbf{\Omega}\left(S^n \vee S^n\right)\right) = \mathbb{Q}\langle a, b\rangle$$

$$x \mapsto a+b$$

and

$$\triangle_\tau : \mathbb{Q}\langle x\rangle = H_*\left(\mathbf{\Omega} S^n\right) \to H_*\left(\mathbf{\Omega}\left(S^n \vee S^n\right)\right) = \mathbb{Q}\langle a, b\rangle$$

$$x \mapsto ab - ba.$$

6.7 Application: operations on the Hochschild homology and cotangent complex

In this section, as an application of the theory from the previous sections, we construct operations solving the Deligne conjecture on the Hochschild chain complex.

6.7.1 The E_n-structure on the Hochschild homology

When $X = \Sigma^n Y$ is a n-fold suspension, the Hochschild chains carry an E_n-structure that generalises the usual cup coproduct on Hochschild homology.

Theorem 6.7.1. *Let $X = \Sigma^n Y$. Then for each $\mathscr{E}$-algebra A, the Hochschild chain complex is an E_n-coalgebra and the Hochschild homology $HH_*^X(A, A)$ is an Pois_n-coalgebra with trivial n-Poisson bracket.*

Proof. First we observe that X. Using the HKR theorem, it suffices to exhibit a coalgebra structure on $\left(\mathrm{Sym}\left(H_*(X) \otimes V\right), d_X\right)$. First, there is an obvious map

$$\mu : \left(\mathrm{Sym}\left(H_*\left(X^{\vee k}\right) \otimes V\right), d_X\right) = \left(\mathrm{Sym}\left(H_*(X)^{\oplus k} \otimes V\right), d_X\right) \to \left(\mathrm{Sym}\left(H_*(X) \otimes V\right), d_X\right)^{\otimes k}$$

The coalgebra structure from Proposition 6.6.1 therefore extends to an E_n-structure on the cotangent complex as desired. The triviality of the Poisson bracket on $H^*(X)$ observed in Example 6.6.2 then implies the triviality of the Poisson cobracket on the Hochschild homology. □

6.7.2 The E_{n+1}-structure and the higher Deligne conjecture

The Deligne conjecture states that the natural E_n-structure on the Hochschild chain complex can be extended to an E_{n+1}-structure. Indeed this is strongly suggested by the vanishing of the Poisson bracket in the last theorem. The methods of Section (6.6) enables us to concretely write this down on the cotangent complex $\left(\mathrm{Sym}\left(H_*(\Sigma^n Y) \otimes V\right), d_X\right)$. Indeed it suffices to construct a homotopy Poisson structure, since E_n is formal.

Theorem 6.7.2. *Let $X = \Sigma^n Y$ be an n-fold suspension. Then the cotangent complex*

$$\left(\mathrm{Sym}\left(H_*(X) \otimes V\right), d_X\right)$$

is, up to homotopy, a coalgebra over Pois_{n+1}. The associative structure is given by the map

$$\triangle_\mu : \left(\mathrm{Sym}\left(H_*(X) \otimes V\right), d_X\right) \to \left(\mathrm{Sym}\left(H_*(X) \otimes V\right), d_X\right) \otimes \left(\mathrm{Sym}\left(H_*(X) \otimes V\right), d_X\right)$$

which is given on the generators $H_*(X) \otimes v$ by

$$x \otimes v \mapsto (x_l \otimes v) \otimes 1 + 1 \otimes (x_r \otimes v)$$

and is extended as an algebra map.

Proof. We first do the arity 2 case for illustrative purposes. Recall from the proof of Lemma 6.5.2 that the cotangent complex is a quotient of $\bigoplus_{k=1}^{\infty} H_*(X)^{\otimes k} \otimes \left(\mathrm{Sym}(V), d\right)^{\otimes k}$. Then by Proposition 6.6.9 for each k, there is a degree n map

$$f_k : H_*(X)^{\otimes k} \to \left(H_*(X)^{\oplus 2}\right)^{\otimes (k+1)}$$

given by restricting the map $\mathsf{Pois}_n(2) \otimes T\left(s^{-1}H_*(X)\right) \to T\left(s^{-1}H_*(X)^{\oplus 2}\right)$ to to the k-fold tensor product in the free associative algebra and forgetting the desuspensions. There is also a map s_k : $\left(\mathrm{Sym}(V), d\right)^{\otimes k} \to \left(\mathrm{Sym}(V), d\right)^{\otimes(k+1)}$ given by inserting the unit 1 in the $(k+1)^{th}$ place. Consider

$$F_k : H_*(X)^{\otimes k} \otimes \left(\mathrm{Sym}(V), d\right)^{\otimes k} \xrightarrow{f_k \otimes s_k} \left(H_*(X)^{\oplus 2}\right)^{\otimes(k+1)} \otimes \left(\mathrm{Sym}(V), d\right)^{\otimes(k+1)}.$$

One can check that, since its image is symmetric in each factor of $H_*(X)$, that the direct sum $\bigoplus_{i=1}^{\infty} F_i$ descends to cotangent complexes on both sides. The coalgebra map then follows from postcomposing with the map

$$\mu_2 : \left(\mathrm{Sym}\left(H_*(X)^{\oplus 2} \otimes V\right), d_X\right) \to \left(\mathrm{Sym}(H_*(X) \otimes V), d_X\right)^{\otimes 2}$$

The arity i case, for $i > 2$ is constructed similarly. The difference is that the map f_k will now be the following

$$f_k : H_*(X)^{\otimes k} \to \left(H_*(X)^{\oplus i}\right)^{\otimes(k+i-1)}$$

and the map s_k will be given by inserting the identity i times. □

6.8 Appendix: Coendomorphism operads in the ∞-category of coassociative coalgebras

In this appendix, we define a version of the coendomorphism operad that takes into account ∞-morphisms.

6.8.0.1 Motivation

The usual coendomorphism operad $\mathrm{CoEnd}(X)$ of a chain complex X is essentially a book-keeping method. Roughly, it remembers every chain morphism between X and its various powers $X^{\otimes n}$, and every 'operadic gluing' of such maps. Then an operadic morphism from some operad $\mp$ to $\mathrm{CoEnd}(X)$ picks out a subcollection of these maps that glue together to form a $\mp$-algebra on X. In this appendic, we define a very similar object $\mathrm{CoEnd}_{\mathsf{Ass}_\infty}(X)$. There are three key differences. Firstly, instead of remembering maps between X and $X^{\otimes n}$, we are going to remember maps between X and the coproduct $X^{\oplus n}$. Secondly, we are only going to remember maps that are compatible with some pre-existing coassociative-coalgebra structure on X. Lastly, it will not only encode the maps themselves, but also the higher structure - those homotopies between them. It is worth noting that this approach only works rationally because of the dependence on a strictly commutative model for the cochains on the n-simplex.

6.8.0.2 Construction of the coendomorphism operad

Observation 6.8.1. There is always a tensor product

$$\otimes : \mathsf{Com}\text{-alg} \times \mathsf{Ass}\text{-alg} \to \mathsf{Ass}\text{-alg}$$

and, for all $X, Y \in \mathsf{Ass}\text{-alg}$ and $X \in \mathsf{Com}\text{-alg}$, there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Ass}\text{-alg}}(A, X \otimes B) \cong \mathrm{Hom}_{X-mod,\mathsf{Ass}\text{-alg}}(X \otimes A, X \otimes B)$$

Here, $\mathrm{Hom}_{X-mod,\mathsf{Ass}\text{-alg}}(-,-)$ is the space of Ass-alg morphisms that are also left X-module morphisms.

Our next step is to describe the $\mathbb{S}$-module structure of $\mathrm{CoEnd}_{\mathsf{Ass}_\infty}(X)$. Here we use the standard language of Koszul duality to describe Ass_∞-coalgebra morphisms.

Definition 6.8.2. Let C be a chain complex equipped with the trivial compatible Ass_∞-coalgebra structure. The Ass_∞*-coendomorphism operad of* C, $\mathrm{CoEnd}_{\mathsf{Ass}_\infty}(C)$ is the simplicial set with arity k component

$$\mathrm{CoEnd}_{\mathsf{Ass}_\infty}(C)(k)_* := \mathrm{Map}_{\mathsf{Ass}\text{-alg}}\left(\Omega(C), A_{PL}\left(\triangle^*\right)\otimes\Omega(C)^{*k}\right)$$

Here $\Omega(-)$ is the usual cobar construction. The symmetric action is given by permuting factors in the wedge product.

Our final step is to describe the operad structure on the simplicial set $\mathrm{CoEnd}_{\mathsf{Ass}_\infty}(C)$.

Definition 6.8.3. The composition maps in $\mathrm{CoEnd}_{\mathsf{Ass}_\infty}(C)$ are defined as follows. We have

$$\circ : \mathrm{CoEnd}_{\mathsf{Ass}_\infty}(C)(k)_n \times \mathrm{CoEnd}_{\mathsf{Ass}_\infty}(C)(i_1)_n \times\cdots\times \mathrm{CoEnd}_{\mathsf{Ass}_\infty}(C)(i_k)_n \to \mathrm{CoEnd}_{\mathsf{Ass}_\infty}(C)(i_1+\cdots+i_k)_n$$

$$\left(f, f_1, \ldots f_k\right) \mapsto F$$

where F is defined to be the composition

$$\Omega(C) \xrightarrow{f} A_{PL}\left(\triangle^n\right)\otimes\Omega(C)^{*k} \xrightarrow{\overline{f_1 * f_2 * \cdots * f_k}} A_{PL}\left(\triangle^n\right)\otimes\Omega(C)^{*(i_1+\cdots+i_k)}$$

Here

$$\overline{f_1 * f_2 * \cdots * f_k} : A_{PL}\left(\triangle^n\right)\otimes\Omega(C)^{*k} \to A_{PL}\left(\triangle^n\right)\otimes\Omega(C)^{*(i_1+\cdots+i_k)}$$

is the extension of the following associative algebra morphism

$$\Omega(C)^{*k} \xrightarrow{f_1\otimes f_2\otimes\cdots\otimes f_k} \bigotimes_{l=1}^{k}\left(A_{PL}\left(\triangle^n\right)\otimes\Omega\left(C^{\oplus i_l}\right)\right) \to A_{PL}\left(\triangle^n\right)\otimes\Omega(C)^{*(i_1+\cdots+i_k)}$$

to $A_{PL}(\triangle^n) - mod, \mp^!$-alg morphism. The second arrow, which is an inclusion, comes from the inclusion of the tensor product into the coproduct.

Proposition 6.8.4. *The object* $\mathrm{CoEnd}_{\mathsf{Ass}_\infty}$ *is an operad in simplicial sets.*

Proof. The only non-trivial verification is that the operadic composition maps are associative. This follows from the commutativity of A_{PL} along with the associativity of the coproduct. □

Remark 6.8.5. As we did previously with the Barratt-Eccles operad, we regard $\mathrm{CoEnd}_{\mathsf{Ass}_\infty}(C)$ as an operad in chain complexes by applying the singular chains functor.

Remark 6.8.6. Our definition of a coendomorphism operad in the ∞-category of coassociative coalgebras extends to a definition of a coendomorphism operad in the ∞-category of $\mathscr{P}$-coalgebras for any Koszul operad $\mathscr{P}$. This construction is functorial in the operad.

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