Strictly commutative dg-algebras in positive characteristic

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2. Introduce and motivate strictly commutative dg-algebras in positive characteristic and their basic properties.
3. Obstruction theory in positive characteristic. Define Massey products, compute the primitive secondary cohomology operations for strictly commutative dg-algebras for and formulate the coherent vanishing of higher Steenrod operations.
4. Introduce an explicit model for the de Rham forms over $\widehat{\mathbb{Z}_p}$ which provides a best approximation to the singular cochains. Study what information can be extracted from it.

Part 0: A crash-course in E_∞-algebras

dg-algebras

Definition

A (commutative) dg-algebra is a chain complex (A, d) equipped with a binary (graded commutative) associative multiplication $m : A^p \otimes A^q \to A^{p+q}$ and such that d is a derivation with respect to m. Alternatively it is an algebra over the operad Assoc (or Com) in dg-modules.

Example

Let X be a topological space. Then the cohomology ring $(H^\bullet(X, R), 0)$ equipped with the cup product forms a commutative dg-algebra.

Problem: the cohomology is not a complete invariant of homotopy type.

Example

Let X be a topological space or simplicial set. Then the singular cochains $(C^\bullet(X, R), d)$ equipped with the cochain level cup product forms a dg-algebra that is generally not graded commutative.

E_∞-algebras

Definition

An E_∞-operad is any operadic resolution $\mathcal{E} \xrightarrow{\sim} \mathrm{Com}$ such that the $\mathbb{S}_k$ action on $\mathcal{E}$ is free.

The singular cochain complex $C^\bullet(X, R)$ is an E_∞-algebra. This is a complete homotopy invariant.

Theorem (Mandell, 2003)

Two finite type nilpotent spaces X and Y are weakly equivalent and only if their E_∞-algebras of singular cochains with integral coefficients are quasi-isomorphic as E_∞-algebras.

The Barratt-Eccles operad

Definition

The Barratt-Eccles operad $\mathcal{E}$ is an operad in simplicial sets given in each arity are of the form

$$\mathcal{E}(r)_n = \{(w_0, \dots, w_n) \in \mathbb{S}_r \times \cdots \times \mathbb{S}_r\}$$

equipped with face and degeneracy maps

$$d_i(w_0, \dots, w_n) = (w_0, \dots, w_{i-1}, \hat{w}_i, w_{i+1}, \dots, w_n)$$
$$s_i(w_0, \dots, w_n) = (w_0, \dots, w_{i-1}, w_i, w_i, w_{i+1}, \dots, w_n).$$

$\mathbb{S}_r$ acts on $\mathcal{E}(n)$ diagonally. Finally the compositions are also defined componentwise via the explicit composition law of

$$\gamma : \mathbb{S}(r) \times \mathbb{S}(n_1) \times \cdots \times \mathbb{S}(n_r) \to \mathbb{S}(n_1 + \cdots + n_r)$$
$$(\sigma, \sigma_1, \dots, \sigma_r) \mapsto \sigma_{n_1 \cdots n_r} \circ (\sigma_1 \times \cdots \times \sigma_r)$$

Steenrod operations

Let $\mathcal{P}$ be an operad and let V be a dg-module. Recall that the free $\mathcal{P}$-algebra on V is

$$\mathcal{P}(V) = \bigoplus_{i=1}^{\infty} \mathcal{P}(i) \otimes^{\mathbb{S}_i} V^{\otimes i}$$

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When working in finite characteristic, the cohomology of the free E_∞-algebra is not the symmetric algebra. Instead one has

$$H^\bullet \mathcal{E}(V) = \mathcal{A}(H^\bullet(V))$$

Here $\mathcal{A}$ is the (unstable) Steenrod algebra which contains $\mathrm{Sym}(H^\bullet(V))$

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When working in finite characteristic, the cohomology of the free E_∞-algebra is not the symmetric algebra. Instead one has

$$H^\bullet \mathcal{E}(V) = \mathcal{A}(H^\bullet(V))$$

Here $\mathcal{A}$ is the (unstable) Steenrod algebra which contains $\mathrm{Sym}(H^\bullet(V))$ but also extra elements like $\mathrm{Sq}^n(v)$. One has a map

$$\mathcal{A}(H^\bullet(V)) \xrightarrow{H^\bullet(\gamma)} H^\bullet(V)$$

This means that the cohomology of an E_∞-algebra is commutative but also acted on by these extra elements in the Steenrod algebra.

Part 1: Strictly commutative dg-algebras

Motivations

The geometric motivation The starting observation of rational homotopy theory is that, in zero characteristic, every E_∞-algebra is weakly equivalent to a commutative dg-algebra. This viewpoint allows us to "completely" understand spaces rationally.

- A natural question: when does an E_∞-algebra admit a commutative model over $\mathbb{F}_p$ or $\widehat{\mathbb{Z}_p}$?

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- A natural question: when does an E_∞-algebra admit a commutative model over $\mathbb{F}_p$ or $\widehat{\mathbb{Z}_p}$?
- In situations where you cannot give such a model, what is the best model that you can give? What information can we extract from it?

The algebraic motivation Studying E_∞-algebras is hard. There are still being papers written on the primary Steenrod operations and the secondary Steenrod operations are incredibly complicated. Studying commutative dg-algebras gives us insight into this difficult structure in a baby case.

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- Firstly, one can take coinvariants: $\mathcal{P}(A) = \bigoplus_{k=1}^{\infty}(\mathcal{P}(k) \otimes A^{\otimes k})_{\mathbb{S}_k}$. Algebras over this monad are dg-modules A equipped with a binary multiplication $m : A^\bullet \otimes A^\bullet \to A^\bullet$.

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- Secondly, one can take invariants $\Gamma\mathcal{P}(A) = \bigoplus_{k=1}^{\infty}(\mathcal{P}(k) \otimes A^{\otimes k})^{\mathbb{S}_k}$. Algebras over this monad are *divided power algebras*: dg-modules A equipped with a binary multiplication $m : A^\bullet \otimes A^\bullet \to A^\bullet$ and extra operations γ_k which behave like $\frac{x^k}{k!}$. Over $\mathbb{F}_p$, this implies that $x^p = 0$.

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- Finally one has a monad $\mathcal{P}(A) \to \Lambda\mathcal{P}(A) \to \Gamma\mathcal{P}(A)$ given by the image of the norm map.

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When we are working over a field of characteristic 0 (the classical theory of Loday-Vallette) or the action of $\mathbb{S}_k$ on $\mathcal{P}(k)$ is free (theory of quasi-planar operads of Le Grignou-Roca Lucio), invariants coincide with coinvariants and the three notions above coincide (subject to certain finiteness assumptions).

Model structures

Theorem (Hinich, 1997)

Let $\mathcal{P}$ be a cofibrant (or $\mathbb{S}$-split) operad over a commutative ring R. Then the category of $\mathcal{P}$-algebras over R is a closed model category with quasi-isomorphisms as the weak equivalences and surjective maps as fibrations.

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Example

Consider $M = \mathbb{F}_p[x \to dx]$. One has $H^\bullet(\mathrm{Sym}(M)) \neq 0$ because 1) x^{p^n} is a cocycle 2) $x^{p^n-1}dx$ is not closed.

Part 2: Obstruction theory over $\mathbb{F}_p$

A crash-course in Massey products 1

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Definition

Let A be a dg-algebra. Let $a, b, c \in H^\bullet(A)$ by such that $ab = 0$ and $bc = 0$. Let x, y, z be cocycles representing a, b, c and suppose $du = xy$ and $dv = yz$. Then $uz - xv$ is a cocycle that we call the (primitive, secondary) Massey product, it represents a well-defined class of

$$\frac{H^{|a|+|b|+|c|-1}(A)}{aH^{|b|+|c|-1}(A) + H^{|a|+|b|-1}(A)c}$$

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Proposition (Massey, 1958)

If for some $a, b, c \in H^\bullet(A)$, the class above is nonzero, then A is not formal

Higher Massey products and coherent vanishing

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- These can be packaged together as the differentials in the *Eilenberg-Moore spectral sequence* which computes $\mathrm{Tor}^A(\Bbbk, \Bbbk)$ from $\mathrm{Tor}^{H(A)}(\Bbbk, \Bbbk)$.
- More recently, this machinery has been extended to general quadratic operads. (Muro, 2023)

Coherent vanishing of Massey products

Unfortunately, it is not enough for Eilenberg-Moore spectral sequence to collapse on E_2-page, in other words, for all Massey products to vanish. Formality turns out to be equivalent to all of these vanishing in a *coherent* way.

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Theorem (Deligne, Griffiths, Morgan, Sullivan, 1975)

Let A be a commutative dg-algebra in $\mathbb{Q}$-vector spaces. Let $\mathfrak{m} = (\mathrm{Sym}(\bigoplus_{i=0}^{\infty} V_i), d)$ be the minimal model for A. Then A is formal if and only if, there is in each V_i a complement B_i to the cocycles Z_i, $V_i = Z_i \oplus B_i$, such that any closed form, a, in the ideal, $I(\bigoplus_{i=0}^{\infty} B_i)$, is exact.

Massey products in positive characteristic

Over $\mathbb{F}_p$ there are more secondary operations.

Definition (F. C.)

Let A be a commutative dg-algebra over $\mathbb{F}_p$. Let $x, y \in H^\bullet(A)$ be such that $xy = 0$. Choose cocycles $a, b \in A$ representing x, y respectively. Then there exists $c \in A$ such that $dc = xy$. Then c^p is a cocycle which we call the *type 1 secondary commutative product* of x and y. This represents a well defined element of

$$\frac{H^{p(|x|+|y|-1)}(A)}{H^{(|x|+|y|-1)}(A)^p + x^p H^{p(|y|-1)}(A) + y^p H^{p(|x|-1)}(A)}$$

where the term $x^p H^{p(|y|-1)}(A) + y^p H^{p(|x|-1)}(A)$ in the denominator accounts for the choice of representatives x and y.

Type 2 commutative products

Definition (F. C.)

Let p be an odd prime. Then there is a *type 2 secondary commutative product* defined for $x, y \in H^*(A)$ such that $xy = 0$ we choose cocycles $a, b \in A$ representing x, y respectively. Then there exists $c \in A$ such that $dc = xy$. Then $c^{p^n-1}ab$ is a cocycle which we call the *type 2 secondary commutative product* of x and y. In this case, the operation represents a well-defined element of

$$\frac{H^{p^n(|x|+|y|-1)+|x|+|y|}(A)}{H^{(|x|+|y|-1)}(A)^{p^n-1}}$$

Observe that $d(\frac{1}{p}c^p) = c^{p-1}ab$. Therefore type 2 secondary commutative products vanish on divided power algebras. Therefore this kind of operation provides an obstruction for a commutative algebra A to be weakly equivalent to a divided power algebra.

Completeness of secondary operations

Definition

We call a Massey product *primitive* if it arises from monomial relations in cohomology.

Proposition

All secondary primitive Massey products on a commutative dg-algebra A over $\mathbb{F}_p$ are linear combinations of

- *classical Massey products.*
- *Type 1 secondary commutative operations*
- *Type 2 secondary commutative operations.*

Counterexamples

Example

The following dg algebras over $\mathbb{Z}$ are quasi-isomorphic over $\mathbb{Q}$ but not $\mathbb{F}_p$.

$$A = \mathrm{Sym}(x, y, z)/(xy, xz, yz)$$

$$B = \mathrm{Sym}(x, y, z, t)/(xy - dt, t^p - z, xz, yz, t^{p+1}, t^{p-1}xy)$$

where $|x| = |t| = 2$, $|y| = 1$ and $|z| = 2p$.

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Example

The following dg-algebra has a divided power structure on its cohomology is not quasi-isomorphic to a divided power algebra

$$\mathrm{Sym}(\mathbb{F}_p\langle x, y, z\rangle, t)/(dt - xy, t^p, t^{p-1}xy - z)$$

where the degrees $|x|, |t|$ are even and $|y|, |z|$ are odd. This is a divided power algebra with cohomology given by $\mathbb{F}_p\langle x, y, z\rangle/(xy)$. Nonetheless, the type 2 commutative product of x, y is z.

Obstructions to commutativity

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The E_∞-algebra $C^\bullet(X, \mathbb{F}_p)$ is rectifiable iff X is the disjoint union of contractible spaces.

Obstructions to commutativity

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Proposition (Mandell, 2009)

The E_∞-algebra $C^\bullet(X, \mathbb{F}_p)$ is rectifiable iff X is the disjoint union of contractible spaces.

There are less obvious obstructions given by secondary operations.

Conjecture (Mandell, 2009)

Let X be a finite n-connected simplicial set. Then, after inverting finitely many primes $C^\bullet(X, \mathbb{Z})$ has a commutative model as an E_n-algebra. If X is formal, then, after possibly inverting more primes, this commutative model is formal.

Sullivan algebras

Definition

Let $\mathcal{P}$ be an operad over a field and A is a $\mathcal{P}$-algebra. A *Sullivan model* for A is a semi-free algebra $f : (\mathcal{P}(\bigoplus_{i=0}^{\infty} V_i), d) \xrightarrow{\sim} A$ such that

- the map $f|_{V_0} : V_0 \to A$ is a weak equivalence of dg-vector spaces. In particular $V_0 = H^\bullet(A)$.
- the differential satisfies $d(V_k) \subseteq (\mathcal{P}(\bigoplus_{i=0}^{k-1} V_i), d)$.
- We require that $V_k \oplus (\mathcal{P}(\bigoplus_{i=0}^{k-1} V_i) \to A$ is a weak equivalence for each k.

The intuition is that a Sullivan model captures the idea of building a quasi-free resolution in stages, starting with a map $H \to A$ by killing cocycles.

$\mathcal{P}$-Massey products

We use a similar idea to define Massey products in this context.

Definition

A *N-step Sullivan model* for A is a semi-free algebra $f : (\mathcal{P}(\bigoplus_{i=0}^{N} V_i), d) \to A$ such that

- the differential satisfies $d(V_k) \subseteq (\mathcal{P}(\bigoplus_{i=0}^{k-1} V_i), d)$
- We require that $V_k \oplus (\mathcal{P}(\bigoplus_{i=0}^{k-1} V_i) \to A$ is a weak equivalence for each k.

Let $I(\mathcal{P}(\bigoplus_{i=1}^{N} V_i), d))$ be the ideal generated by $\mathcal{P}(\bigoplus_{i=1}^{k-1} V_i), d)$. We call nonzero $\sigma \in H^\bullet(I(\mathcal{P}(\bigoplus_{i=1}^{k-1} V_i), d))$ an N^{th} order Massey product with value $H^\bullet(f)(\sigma)$

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Remark: cocycles σ can be identified with differentials in a spectral sequence where the E_1-page is homotopy invariant for sufficiently nice $\mathcal{P}$. This shows that this notion of Massey product is homotopy invariant for such $\mathcal{P}$.

Coherent vanishing of higher Steenrod operations

Definition

Let A be an E_∞-algebra over $\mathbb{F}_p$. Then the higher Steenrod operations *vanish coherently* if one can find a Sullivan resolution $(\mathcal{E}(\bigoplus_{i=0}^{\infty} V_i), d)$ for A, such that there exists a splitting $V_i = X_i \bigoplus Y_i$, with $X_0 = V_0$ and $Y_0 = 0$. We further require that the nonexact cocycles $Z(\mathcal{E}(V_0))$ admit a splitting of vector spaces $Z(\mathcal{E}(V_0)) = \mathrm{Sym}(V_0) \oplus K_0$. and that $d(Y_1) = K_0$. Inductively, for every $k > 0$, we assume one has a choice of splitting

$$H^\bullet(I(\overline{\mathcal{E}}(\bigoplus_{i=1}^{k-1} V_i), d))) := H^\bullet(I(\mathrm{Sym}(\bigoplus_{i=1}^{k-1} X_i), d)) \oplus K_{k-1}$$

and we require that for some choice of cocycles $\overline{K_{k-1}}$ representing K_{k-1} we have $d(Y_k) = \overline{K_{k-1}}$.

Coherent vanishing

Conjecture

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Part 2: de Rham forms in positive characteristic

de Rham forms in positive characteristic

Motivation: we want to find a best commutative approximation to the E_∞ algebra $C^\bullet(X, \widehat{\mathbb{Z}_p})$.

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Motivation: we want to find a best commutative approximation to the E_∞ algebra $C^\bullet(X, \widehat{\mathbb{Z}_p})$.
Key idea: We want to imitate Sullivan's approach to rational homotopy theory.

Theorem (Sullivan, 1978)

Suppose one has a functor $A_{PL} : \triangle^\bullet \to \mathrm{CDGA}_{\mathbb{Q}}$ that satisfies the Poincaré Lemma: $H^0(\triangle^n, \mathbb{Q}) = \mathbb{Q}$ and $H^i(\triangle^n, \mathbb{Q}) = 0$ for $i > 0$; and which is extendable $\pi_k(A^k_{PL}(\triangle^\bullet)) = 0$ for all $k \geq 0$. Then the left Kan extension along $\triangle^\bullet \to \mathrm{Set}_\triangle$

$$A_{PL} : \mathrm{Set}_\triangle \to \mathrm{CDGA}_{\mathbb{Q}}$$

is such that there is a zig-zag of E_∞-algebras

$$A^\bullet_{PL}(X) \xrightarrow{\sim} (A_{PL} \otimes C)^\bullet(X) \xleftarrow{\sim} C^\bullet(X, \mathbb{Q})$$

Part 2: de Rham forms in positive characteristic

Definition

The simplicial cochain coalgebra $\Omega^*_\bullet$ has for n-simplices

$$\Omega^*_n = \frac{\widehat{\mathbb{Z}_p}\langle x_0, \ldots x_n\rangle \otimes \Lambda(dx_0, \ldots, dx_n)}{(x_0 + \cdots + x_n - p, dx_0 + \cdots dx_n)}, \ |x_i| = 0, \ |dx_i| = 1.$$

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The differential $d : \Omega^*_n \to \Omega^{*+1}_n$ is determined by the formula

$$d(f) = \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} dx_i$$

for $f \in \Gamma_p(x_0, \ldots, x_n)/(x_0 + \cdots + x_n - p)$ and then extended by the Leibniz rul

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$$d_i^n : \Omega^*_n \to \Omega^*_{n+1} : x_k \mapsto \begin{cases} x_k & \text{for } k < i. \\ 0 & \text{for } k = i. \\ x_{k-1} & \text{for } k > i. \end{cases}$$

and

$$s_i^n : \Omega^*_n \to \Omega^*_{n-1} : x_k \mapsto \begin{cases} x_k & \text{for } k < i. \\ x_k + x_{k+1} & \text{for } k = i. \\ x_{k+1} & \text{for } k > i. \end{cases}$$

The cohomology of de Rham forms

Cartan considered a similar construction except over $\mathbb{Z}\langle t\rangle$. The functor

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Theorem (Cartan, F.C)

Consider the left Kan extension along $\triangle^\bullet \to \mathsf{Set}_\triangle$

$$\Omega : \mathsf{Set}_\triangle \to \mathsf{CDGA}_{\widehat{\mathbb{Z}_p}}$$

Then there is an isomorphism of cohomology algebras

$$H^\bullet(X, \widehat{\mathbb{Z}_p}) = H^\bullet(\Omega(X)).$$

The homotopy type of the de Rham forms

The previous result can be upgraded to the E_∞-homotopy type.

Definition

Let X be a simplicial set. We define the *altered singular cochain algebra* $\mathcal{C}^\bullet(X)$ to be the following subalgebra of the singular cochains $C^\bullet(X)$.

$$\mathcal{C}^n(X) = \left\langle p^i\sigma : \text{ for } \sigma \in C^n(X, \widehat{\mathbb{Z}_p}) \text{ and } \begin{cases} i = n & \text{if } d\sigma = 0. \\ i = n+1 & \text{otherwise.} \end{cases} \right\rangle$$

The differential and the E_∞ structure are that induced by those on $C^\bullet(X, \widehat{\mathbb{Z}_p})$.

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Theorem (F.C.)

As an E_∞-algebra, $\Omega(X)$ is quasi-isomorphic to $\mathcal{C}(X)$.

Universal properties 1

In general, $\Omega(X)$ doesn't seem to possess any universal properties.

Definition

As a free $\widehat{\mathbb{Z}_p}$-module $\Omega^\bullet(\triangle^n)$ admits a linear basis consisting of two kinds of monomials. Define

$$\Omega_B^\bullet(\triangle^n) = \langle x_{i_1} \cdots x_{i_n} dx_{j_1} \wedge \cdots \wedge dx_{j_m} \in \Omega^\bullet(\triangle^n) : n \geq 1 \rangle$$

$$\Omega_Z^\bullet(\triangle^n) = \langle dx_{j_1} \wedge \cdots \wedge dx_{j_m} \in \Omega^\bullet(\triangle^n) : m \geq 1 \rangle$$

Therefore, we have, as a sum of graded modules

$$\Omega^\bullet(\triangle^n) = \Omega_B^\bullet(\triangle^n) \oplus \Omega_Z^\bullet(\triangle^n).$$

Definition

We define a commutative algebra

$$\mathcal{R}^k(\triangle^n) = \frac{1}{p^k}\Omega_B^k(\triangle^n) \oplus \frac{1}{p^{k-1}}\Omega_Z^k(\triangle^n)$$

where $\Omega_B^\bullet$ and $\Omega_Z^\bullet$ are as before. The commutative algebra structure on $\Omega^\bullet(X)$ then extends to $\mathcal{R}^\bullet(X)$.

Universal properties 2

Definition

Let X be a simplicial set. We define the $\mathcal{E}^*(X)$ to be the following subalgebra of the singular cochains $C^*(X)$.

$$\mathcal{E}^n(X) = \left\langle p^i\sigma : \text{ for } \sigma \in C^n(X, \widehat{\mathbb{Z}_p}) \text{ and } \begin{cases} i = 1 & \text{if } n > 0 \text{ or } d\sigma \neq 0. \\ i = 0 & \text{if } n = 0 \text{ and } d\sigma = 0 \end{cases} \right\rangle$$

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Theorem

Let $A \in \mathrm{Com}-\mathrm{alg}$, $X \in \mathrm{Set}_\triangle$ *and* $\mathbf{i} : \mathrm{Com}-\mathrm{alg} \to E_\infty - \mathrm{alg}$ *be the inclusion functor. Then there is an equivalence of mapping spaces*

$$\mathrm{Map}_{\mathrm{Com}-\mathrm{alg}}(A, \mathcal{R}^\bullet(X)) \xrightarrow{\sim} \mathrm{Map}_{E_\infty-\mathrm{alg}}(A, C^\bullet(X)).$$

We can therefore think of $\mathcal{R}$ as a partially defined right adjoint to i.

Formality of $\Omega(X)$

The model can be used to compute Massey products.

Proposition (F.C.)

Suppose that $\sigma \in H^\bullet(X, \mathbb{Q})$ *be the higher Massey product of* $\langle x_1, x_2, \ldots, x_n \rangle \in H^\bullet(A_{PL}(X), \mathbb{Q})$*. Then there exists an* $n > 0$ *such that* $p^n\sigma \in H^\bullet(X, \widehat{\mathbb{Z}_p})$ *is the higher Massey product of* $\langle p^n x_1, p^n x_2, \ldots, p^n x_n \rangle \in H^\bullet(A_{PL}(X), \widehat{\mathbb{Z}_p})$ *computed in* $\Omega^\bullet(X)$*.*

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It can also be used to define Massey products in the torsion part of the cohomology. Finally, we have this theorem which is inspired by Mandell's conjecture.

Theorem (F.C.)

Let X *be a finite simplicial set such that* $A_{PL}(X)$ *is formal over* $\mathbb{Q}$. *For all but finitely many primes,* $\Omega^\bullet(X)$ *is formal over* $\widehat{\mathbb{Z}_p}$ *as a dg-commutative algebra.*

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- Extend to other operads.
- There should be a coherent vanishing result for Type 2 operations that determines when a commutative algebra is a divided powers algebras.

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- Koszul duality between restricted Lie algebras and divided powers algebras.
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- Prove Mandell's conjecture.
- Is there a version of Mandell's theorem for $\Omega(X)$?