Associativity $\cong$ commutativity (mod p)?

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Abstract

In this expository piece, we construct an example in characteristic p of two commutative dg-algebras which are quasi-isomorphic as associative but not commutative dg-algebras.

1 Introduction

The name of this piece is a deliberate riff on Bruno Vallette's excellent introduction to operads: *Algebra+Homotopy=Operad* [5]. Homotopy theory has become somewhat infamous for its inaccessibly long papers and complex constructions, but don't be intimidated. Operads, and the other technical gadgets of the discipline, will not appear today. On the other hand, some interesting combinatorics and very basic number theory will show up. In this short article, I hope to accomplish the impossible - write an easily accessible introduction to a recent problem in homotopy theory and a short explanation on how to solve it using surprisingly elementary tools. Without excessive jargon, if you will.

To commence with, alas, I will need to discuss some theory. But it is theory that almost every mathematician will be somewhat familiar with and more details can be found in a good homological algebra textbook: for example [6].

Definition 1.1. A chain complex over a field is a collection of vector spaces $A^i : i \in \mathbb{Z}$ and maps $d^n : A^n \to A^{n+1}$ such that $d^{n+1} \circ d^n = 0$ for all n.

$$\cdots A^i \xrightarrow{d} A^{i+1} \xrightarrow{d} A^{i+2} \xrightarrow{d} A^{i+3} \xrightarrow{d} A^{i+4} \xrightarrow{d} \cdots$$

A great example of this, and in fact the primordial one, is that if you have a smooth manifold M, the collection of differential forms on M form a chain complex $\Omega^\bullet(M)$ called *the de Rham forms*. The stratification here is the obvious one with the exterior derivative. And, if you know about de Rham forms, you will also know that they can be used to compute some interesting global invariants of manifolds.

The cohomology of a chain complex A is the following set

$$H^i(A) = \ker d^i / \operatorname{Im} d^{i-1}.$$

We refer to elements in $\ker d^i$ as cocycles and elements in $\operatorname{Im} d^{i-1}$ as coboundaries. The remarkable thing about these is that they are a *homotopy invariant* of the manifold. In other words, they are preserved by squashing and squeezing (but not cutting) it.

A quasi-isomorphism between chain complexes A to B is a map that induces an isomorphism on cohomology. Quasi-isomorphisms do not necessarily admit inverses so this is not an equivalence relation. But we can "pretend it is" by saying that chain complexes A and B are quasi-isomorphic if there is a *zig-zag*

$$A \xrightarrow{\sim} C_1 \xleftarrow{\sim} C_2 \xrightarrow{\sim} \cdots \xrightarrow{\sim} C_n \xleftarrow{\sim} B. \tag{1}$$

This is still all a bit dull though, because there is always a quasi-isomorphism

$$H(A) \xrightarrow{\sim} A.$$

So, if A has the same cohomology groups as B, we can always find a quasi-isomorphism

$$B \xleftarrow{\sim} H(B) = H(A) \xrightarrow{\sim} A.$$

Thus, knowing the cohomology groups fully determines the quasi-isomorphism type of the chain complex. The attentive reader probably notices an issue here: not all manifolds with the same cohomology groups are homotopy equivalent. But the de Rham forms actually have a little more structure, a commutative operation called *wedge product* that behaves well with respect to the differential.

Definition 1.2. An associative dg-algebra is a chain complex A equipped with a binary associative multiplication $-\cup- : A^p \otimes A^q \to A^{p+q}$ and d satisfies the Leibniz rule from differentiation ie.

$$d(x \cup y) = d(x) \cup y + (-1)^{|x|} x \cup d(y).$$

Sometimes the product is commutative, in which case we call it a commutative dg-algebra.

So now, a natural question is when are two dg-algebras A and B quasi-equivalent **as algebras**? In other words, we require all the C_i in Diagram (1) to be dg-algebras and maps to all be algebra maps. This means that not only are the cohomology groups of A, B the same but the multiplication induces a graded multiplication

$$H^i(A) \otimes H^j(A) \to H^{i+j}(A)$$

on the cohomology that is also preserved.

When A and B are **commutative** and the zig-zag consists of **commutative** algebras, this turns out to be a very subtle question, which was elegantly answered by Sullivan [4] with his theory of minimal models, which completely answers the classification problem *rationally*. This is an amazing result, because it shows that the rational homotopy type of a topological space is completely determined by a (relatively!) small commutative dg-algebra.

This led Ricardo Campos, Dan Petersen, Daniel Robert-Nicoud, and Felix Wierstra to ask the following question [1]:

Question 1.3. If two commutative algebras A and B are quasi-isomorphic as **associative** algebras are they quasi-isomorphic as **commutative** algebras?

At first glance, the answer appears to be obviously no. There are many more potential zig-zags in the larger category of associative dg algebras. However, if you work over a field of characteristic 0, they proved that the answer is actually yes [1]. The proof, as one might expect, is very technical and makes a lot of use of a collection of methods informally referred to as the *operadic calculus*.

However, this still leaves open the case of what happens when one works over a field of characteristic p. Sadly and not completely unexpectedly, I should warn you that the answer here is also no, but how to construct a counterexample? The construction actually ends up being mainly number theoretic and combinatorial in nature rather than homotopic.

The strategy here would be familiar to anyone who has ever sat a maths contest - one wants to look for homotopy invariants of commutative algebras that are not homotopy invariants of associative algebras. There is something very weird going on with the Frobenius map in characteristic p. Observe that, the Leibniz rule for differentiating tells us that in a commutative algebra, we have:

$$d(x^p) = px^{p-1}dx = 0.$$

In other words, x^p is always a cocycle. But this relation does not hold in a general associative algebra. Instead, one has

$$d(x^p) = (dx)x^{p-1} + x(dx)x^{p-2} + \cdots + x^{p-1}dx.$$

So this may not be a cocycle. This suggests the following general strategy. You know that, if two commutative algebras A and B are associative quasi-isomorphic, they have the same cohomology ring and that this ring is even commutative. But the ring may still have some relation $ab = 0$ for some $a, b \in H(A)$. Interpreting this on the cochain level, we can take lifts $\bar{a}, \bar{b} \in A$ ie. cocycles representatives for $a, b \in H(A)$ in cohomology. The product $\bar{a}\bar{b}$ is not necessarily 0, it just needs to be a coboundary, so

$$\bar{a}\bar{b} = d\bar{c}$$

for some c. There are some choices being made here at each point, but these are being made in a very controlled way. For example, you can add any cocycle σ to c and one still has

$$d(\bar{c}+\sigma) = \bar{a}\bar{b}$$

but things break if you add an element such that $d\sigma \neq 0$. Cocycles determine elements of the cohomology, so everything is well-defined in a quotient ring of the cohomology.

Now, you can just take the p^{th} power $\bar{c}^p$ and you get a cocycle and therefore an element in the cohomology. This is is not perfectly well defined, but, by keeping track of all our previous choices, you can show, without much difficulty, that it is well defined as an element in

$$\frac{H(A)}{H(A)^p + a^p H(A) + b^p H(A)}.$$

So now our strategy becomes clear. We just need to find a pair of commutative algebras A and B with the same cohomology, but where this invariant differs. Then we need to find an associative algebra C such that there is a zig-zag of associative weak equivalences

$$A \xleftarrow{\sim} C \xrightarrow{\sim} B.$$

Building C, and ensuring it has the same cohomology as A and B, involves a little combinatorial trickery. But it's fundamentally just linear algebra - the full details of which can be found in the paper [2, Section 4.2.4].

This method of finding *higher invariants* that live just above the cohomology to solve very concrete problems has a rich history, going back to Massey [3], who used his eponymous products, constructed using similar vanishing arguments, to show that the Borromean rings were pairwise unlinked but cannot be separated.

References

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