Differential equations on manifolds

Mathematical Structures in Computer Science

Oisín Flynn-Connolly

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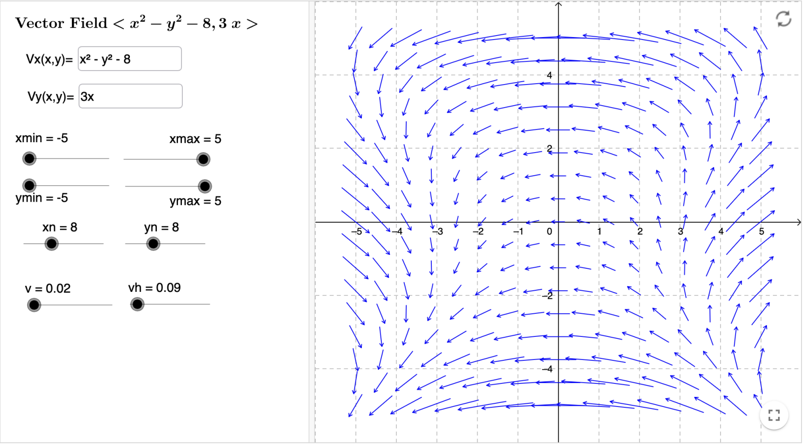
Motivation

Goal: To understand ordinary differential equations on curved surfaces

- **Modelling physical system.** One models the time evolution of physical systems as paths through a manifold. The equations of motion are differential equations.
- **Optimal control** in system design and robotics.
- **Gradient descent** in machine learning can also be seen as a solution to a certain differential equation.

Vector fields visualised



This applet was done thanks to the work of Linda Fahlberg-Stojanovska: https://www.geogebra.org/u/lfs-d

Vector fields

Vector fields as sections

Recall the tangent bundle $(TM, M, \pi_M : TM \to M)$, the functor T^c : **Cart** $\to$ **VBund**, i : **Cart** $\to$ **Man**, and the isomorphism $\alpha : T^c \to T \circ i$.

Definition

A **vector field** on a manifold M is a **smooth** section of π_M. Ie. a map

$$s : M \to TM$$

such that $\pi \circ s = \mathrm{id}_M$. The set of all vector fields on M is denoted $\Gamma(TM)$.

Example

The wind is an example of a vector field in $\mathbb{R}^3$.

A vector field $X(p) = (p, u) \in TM$ For shorthand we write

$$X_p = u.$$

Examples of vector fields

Example 1: The trivial vector field
For all manifolds M, we have a vector field X given by

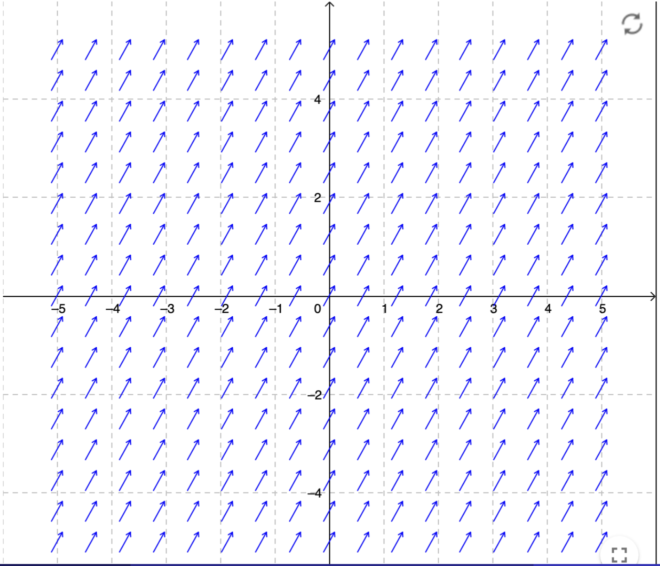
$$X_p = 0$$

Proposition

The trivial vector field is a vector field.

Example 2: Every smooth endomorphism on $\mathbb{R}^n$
Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function. Then $X_p = f(p)$ defines a vector field via the isomorphism $\alpha : \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\sim} T\mathbb{R}^n$

Plot of constant field $(1, 2)$



Example 3: Every real valued function on $\mathbb{R}^n$
Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is a smooth function. Then $Df : \mathbb{R}^n \to \mathrm{Vect}(\mathbb{R}^n, \mathbb{R})$ assigns a linear map $Df_x = (a_1, \ldots, a_n)$ to each point $x \in \mathbb{R}^n$. Take the dual of Df_x ie. the transpose of $(a_1, \ldots, a_n)$ to get a vector $\nabla(f)(x)$ in $\mathbb{R}^n$.
This can also be seen as $T(f$
So $X_p = \nabla(f)(p)$ defines a vector field via $\alpha : \mathbb{R}^n \times \mathbb{R}^n \to T\mathbb{R}^n$

This example can be extended to any sufficiently well behaved manifold, but one must take care with the dualisation operation $\Rightarrow$ **Riemannian geometry** (HW3!)

Algebraic structure of vector fields

1. **Vector space structure**. There is a vector space structure on $\Gamma(TM)$. Given two vector fields $X, Y : M \to TM$, this is given by

$$(aX + bY)_p = aX_p + bY_p.$$

2. **Multiplication by smooth functions** Let $f \in C^\infty(M)$ ie. $f : M \to \mathbb{R}$ is a smooth map. Then fX is the vector field

$$fx_p = f(p)X_p$$

Frames

Definition

Let M be an n-manifold. Let $X_1, X_2, \dots X_n$ be vector fields. They are said to be **a local frame** on some open $U \subset M$, if for all $p \in U$: the vectors $(X_1)_p, \dots, (X_n)_p \in \{p\} \times \mathbb{R}^n$ are linearly independent. If $U = M$, they are said to be a **global frame.**

Given a coordinate chart $\psi : U \subset \mathbb{R}^n \to M$, we can consider the basis vector $\hat{e}_i : \mathbb{R} \xrightarrow{e_i} U \to M$. Then we can define a local frame on the tangent bundle by to $L_U(\hat{e}_i)$ for $i \in \{1, \dots, n\}$.

Diffeomorphisms

Definition

A smooth map $f : M \to N$ is a **diffeomorphism of manifolds** if it has a smooth inverse f^{-1}.

Proposition

Suppose that $f : M \to N$ is a diffeomorphism. Then $Df_x : T_x \to T_{f(x)}$ is invertible at all $x \in M$.

In fact a converse is true:

Theorem (Hadamard theorem)

Suppose that M, N are n-manifolds and N is **simply connected***. Then $f : M \to N$ is a diffeomorphism if and only if 1) the preimage of any compact set is compact; 2) Df_x is invertible for all $x \in M$.*

Simply connected informally means every loop can be contracted to a point.

Pushforwards and pullbacks of vector fields

Definition

Let $f : M \to N$ be a **diffeomorphism** of manifolds and $\Phi_M : M \to TM$ and $\Phi_N : N \to TN$ is a vector field. Then

1. the **pushforward** of Φ_M is the vector field $f_*\Phi_M$

$$N \xrightarrow{f^{-1}} M \xrightarrow{\Phi_M} TM \xrightarrow{Tf} TN$$

$$f_*\Phi_M(x) = Tf_{f^{-1}(x)} \circ \Phi \circ f^{-1}(x)$$

2. the **pullback** of Φ_N is the vector field $f^*\Phi_N$

$$M \xrightarrow{f} N \xrightarrow{\Phi_N} TN \xrightarrow{(Tf)^{-1}} TM$$

$$f^*\Phi_N = T(f^{-1})_{f(x)} \circ \Phi_N \circ f(x)$$

Pushforwards and pullbacks

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\Phi_M"] & N \arrow[d, "f_*\Phi_M"] \\ TM \arrow[r, "Tf"] & TN \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "f"] \arrow[d, "f^*\Phi_N"] & N \arrow[d, "\Phi_N"] \\ TM \arrow[r, "Tf"] & TN \end{tikzcd}$$

Differential equations

Differential equations as vector fields on $\mathbb{R}^n$

Any system of ordinary differential equation of the form

$$\frac{d\vec{x}}{dt} = f(\vec{x})$$

where $\vec{x} \in \mathbb{R}^n$ and f is a C^k-function can be interpreted as a C^k-vector field on $\mathbb{R}^n$

$$X_p = f(\vec{p})$$

Moral: You should think of vector fields on a manifold M **as ODEs**.
We can make this correspondence very precise.

Locally extracting ODEs from vector bundles

Given a vector bundle $\Phi : M \to TM$ and a point $x \in M$, there is a chart $\psi : U \to M$. This induces a local diffeomorphism $U \xrightarrow{\sim} \psi(U)$. So we can pullback the vector field to

$$\psi^* \Phi : U \to U \times \mathbb{R}^n.$$

So we have a differential equation

$$\frac{dx}{dt} = (\psi^* \Phi)_x$$

with initial condition $x(0) = 0$ and where π_2 is the projection onto the second coordinate. By existence-uniqueness theorem for differential equations, we can solve find a unique solution $x(t)$ for locally about 0. So we take $\psi(x(t))$ to be a local solution about x.

Integral curves

Definition

Let $\Phi : M \to TM$ be a vector field. An **integral curve** for the vector field is a differentiable map

$$\gamma : (a, b) \to M$$

where $0 \in (a, b)$ (the interval (a, b) is possibly unbounded) such that

$$\gamma(0) = x_0 \text{ and } \gamma'(t) = \Phi(\gamma(t)).$$

Flows

Definition

A flow domain is a open subset

$$W \subseteq \mathbb{R} \times M$$

such that

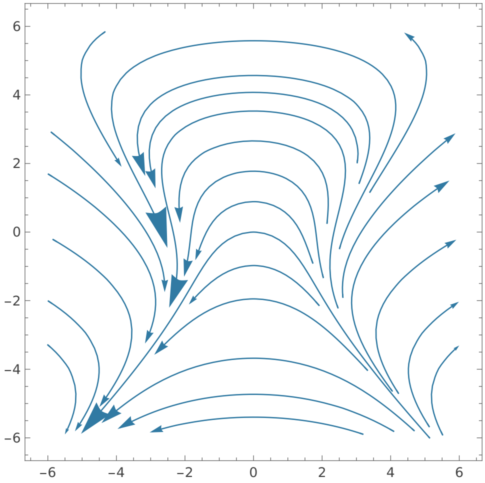
$$W^{(p)} = \{(t, p) : t \in \mathbb{R}\}$$

is an open interval containing 0. A **flow** is a continuous map

$$\theta : W \subseteq \mathbb{R} \to M$$

where $\theta(0, p) = p$ for all $p \in M$ and $\theta(t, \theta(s, p)) = \theta(t + s, p)$.
If $W = \mathbb{R} \times M$, we say that θ is a **global flow.**

Example of flow with generator $(x^2 - y^2 - 4, 3x)$



For each $p \in M$; define a curve $\theta^{(p)} : (a, b) \to M$ by

$$\theta^{(p)}(t) = \theta(t, p).$$

Recall that you can lift this curve to the tangent bundle $\overline{\theta(t, p)} : (a, b) \to TM$

Definition

There is a unique vector field associated to every flow θ. This is given by

$$V_p = \left(v, \overline{\theta(t, p)}(0)\right).$$

ie. the tangent vector to $\theta^{(p)}$ at 0. Note that $\theta^{(p)}(0) = p$ so this lives in the correct fibre! We call this the **infinitesimal generator** of the flow.

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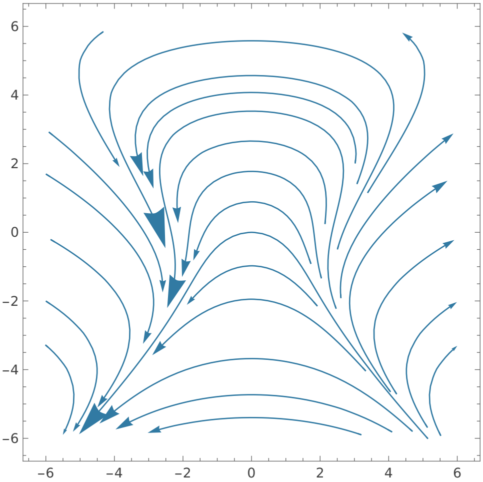

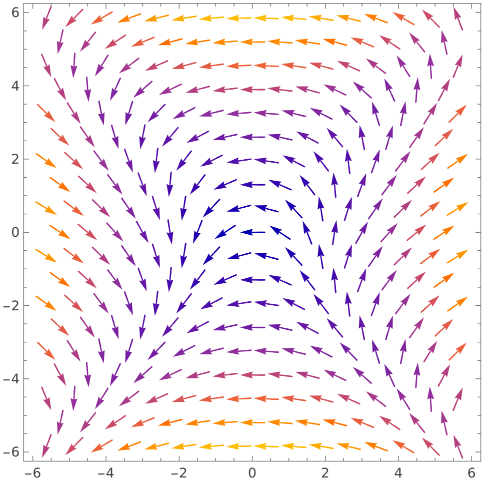

Other examples of flow

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corresponds to

$$\theta(t,(x,y)) = (x+\sin(t),\cos(t)+y-1)$$

Maximal flows

We can build a unique maximal flow by piecing together integral curves.

Theorem (Fundamental Theorem on Flows)

Let Φ be a smooth vector field on a smooth manifold M. There exists a unique smooth maximal flow $\theta : W \subset \mathbb{R} \times M \to M$ whose infinitesimal generator is V. This flow satisfies:

(a) *For each $p \in M$, the curve $\theta^{(p)}$ is the unique maximal integral curve of V starting at p.*

(b) *If $s \in W^{(p)}$, then*

$$W^{(\theta(s,p))} = \{t - s \mid t \in W^{(p)}\}.$$

(c) *For each $t \in \mathbb{R}$, the set*

$$M_t = \{p \in M : (t,p) \in W\}$$

is open in M, and the map $\theta_t : M_t \to M_{-t}$, with $\theta_t(x) = \theta(t,x)$, is a diffeomorphism with inverse θ_{-t}.

If the flow is global, ie. $W = \mathbb{R} \times M$ we say that the vector field is **complete**.

Complete vector fields

There are examples of flows that are not complete. [Examples on board]
We have the following very useful theorem

Theorem

Every smooth vector field on a Cartesian manifold supported on a closed and bounded subset is complete.

Gradient flow on Cartesian manifolds

Problem: Given a **cost function** $f : M \to \mathbb{R}$, find its (local) minima.

1. Take its differential $Df : M \to \text{Vect}(\mathbb{R}^n, \mathbb{R})$. This is a covector.
2. Convert this into a vector field. There is a canonical way to do this on a Cartesian manifold. Take the image under the dual isomorphism of $Df(x)$. In other words

   $$Df(x)^t = \nabla f(x)$$

   Since M is Cartesian this is equivalent to looking at the covector $(a_1, \ldots, a_n)$ representing $Df(x)$ and transposing it: so $Y_p = \nabla f(x)$.
3. You want to move in the opposite direction to the direction of maximum increase so take the vector field $X = -Y$ ie.

   $$X_p = -\nabla f(x)$$

   . This is a vector field.
4. Look at the vector flow for this manifold. The stationary points of the vector flow are the local extrema.

Completeness of gradient flow

Theorem

Let $f : M \to \mathbb{R}$ be a smooth function on a Cartesian manifold U that is supported on a closed and bounded subset $K \subset U$. Then it has global maxima and minima. Moreover for every $p \in M$

$$\lim_{t \to \infty} (\theta(t, p))$$

is a local extrema of f where θ is the global flow associated to the gradient vector field.

A numerical approach to optimisation

Gradient descent in $\mathbb{R}^n$ is a numerical approximation to gradient flow.

1. Start at a point $x_0 \in M$ and define a sequence recursively as

$$x_{n+1} = x_n - \eta \nabla(f)(x_n)$$

for $\eta \in \mathbb{R}^+$. The constant η is called the **learning rate.**

Theorem

Let $\mathbb{R}^n \to \mathbb{R}$ be convex and L-smooth. Suppose its minimiser is x^* and that $|\eta| < 1$, then

$$|f(x_t) - f(x)| \leq \frac{|x_* - x_0|_2^2}{2t\eta}$$

L-smooth means that the gradient of f is a L-Lipschitz function.