RESEARCH PROPOSAL

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1. Introduction

The basic theme of my research is the production of *better and more tractable* models for geometric and algebraic phenomena. In the rest of this document, I explain what this means in practice.

2. Koszul duality and Massey products

Over the last 40 years, a school of researchers, most notably Loday and his various disciples, have extended the classical techniques of rational homotopy theory and Koszul duality and developed the very powerful theory of the *operadic calculus* [14] that captures the complete homotopy theory of many important classes of algebras. However, this theory suffers from two key drawbacks.

(1) In almost all cases, homotopical operadic methods are powerful but not *effective*, in the sense they cannot be implemented by a terminating computer algorithm.
(2) It does not generalise as well in several directions, such as in nonzero characteristic or to objects more general than operads.

With the goal of removing these defects, I propose the following research lines.

2.1. **Massey products for Koszul operads**. Both these considerations motivate the introduction of theoretically weaker but computable invariants. The basic idea was introduced by Massey [17] in the 1950s. It is to consider *higher products* in the homology of associative algebras. These correspond to differentials in the naturally associated Eilenberg-Moore spectral sequence, but, critically, can be computed simply, algorithmically and directly by concrete formulae without the need for spectral sequence techniques. In [8], I, together with Moreno-Fernandez, building on work of Muro [18], combine these invariants with operadic calculus to prove that there are similar simple homotopy invariants for all other Koszul operads, and that such formulae fundamentally arise from operadic Koszul duality. These are not complete invariants, but they *almost* are, the only missing piece is some coherence data. In practice, this means they are almost always *good enough* for practical purposes.

2.2. **Cotriple products**. In [4], I define similar invariants called *cotriple products* in positive characteristic. Instead of using the Eilenberg-Moore spectral sequence, these are products associated to differentials in the cotriple spectral sequence. More interestingly, these turn out to work especially well in the homotopy category of strictly commutative algebras. This category lacks a model structure and therefore cannot be studied via any of the typical fibrant-cofibrant replacement or Koszul duality methods present in the literature. There, one can use them to produce an array of interesting counterexamples. In particular, one can produce non-homotopic commutative algebras that are weakly equivalent as associative algebras, settling a question raised in [2, Section 0.3]. More general, E_∞-algebras may not be *rectifiable*, that is to say, be weakly equivalent to a strictly commutative algebra. Cotriple

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products act as obstructions to rectifiability for E_∞-algebras in exactly the same way as traditional Massey products act as obstructions to formality. These observations opens the door to studying a range of properties in characteristic p using the tools developed for formality in characteristic 0.

Intriguingly, I show in [6], that in characteristic 2, cotriple products are finer invariants in the category of strictly commutative algebras than in the category of E_∞-algebras, allowing one to produce the bizarre phenomenon of commutative algebras that are weakly equivalent as E_∞-algebras but not as strictly commutative algebras.

2.3. **Further work**. There are a number of promising future directions of interest which I intend to collaborate with Moreno-Fernandez on.

(1) Firstly, implementing Massey product methods in computer algebra could open up new combinatorial approaches to the study of algebras up to homotopy. Generally, the main constraint to doing this that one must be able to solve the word problem via the Diamond Lemma, Groebner bases, or other such methods. This is frequently possible in practice [22]. I intend to start a concrete Python implementation of several such cases over the summer. In particular, Massey products open a new perspective on the study of Poisson manifolds, or indeed any other manifold equipped with an additional algebraic structure.
(2) The second direction is to further study the relationship between Koszul duality and Massey products in a deeper way to that in [8]. The first step is to observe that one can generalise Massey products to curved algebras via a twisted version of the formulae in [8]. Here, they act as obstructions to weak equivalences. The next step is to relate this directly to the more general phenomenon of (non-operadic) Koszul duality.
(3) The third idea is to study weak equivalences in situations where there is no hope of applying traditional methods. A strong candidate is to use them to study algebras over modular operads (or even modular operads themselves), where one would need to generalise our approach from rooted trees to more general graphs.

3. Approximating spaces with commutative algebras

Mandell's proof [15] that the study of all nilpotent, finite type spaces *integrally* can be reduced to studying E_∞-algebras was a milestone in algebraic topology. The trouble is that E_∞-algebras are large and computationally intractable. In keeping with the philosophy of the previous section, it would therefore be more computationally convenient to have models with a simpler algebraic structure. This is what this line of research aims to achieve.

3.1. **A p-adic de Rham complex**. In 1977, Sullivan [20] introduced a functor $A_{PL}(X)$, similar to the de Rham complex, which captures the complete *rational* homotopy theory of any simplicial set X . In [7], I have generalised Sullivan's A_{PL}-functor to the p-adic numbers. This produces a p-adic version of the de Rham complex which computes the cohomology of X. As an E_∞-algebra, this complex turns out to be equivalent to the Berthelot-Ogus-Deligne *décalage* of the singular cochains complex with respect to the p-adic filtration. This provides new, more explicit models for, and extends to arbitary simplicial sets, the work of Bhatt-Lurie-Mathew on the de-Rham Witt complex on smooth varieties [1]. I further prove that our de Rham complex is, in a precise sense, the best commutative approximation to the singular cochains complex. Homotopy invariants such as Massey products can potentially be extracted from it.

3.2. **Further work**. While the previous project was in progress, Geoffrey Horel showed us lecture slides [16] where Mandell conjectures an alternative approach to approximating E_∞-algebras. The idea is that, for a n-connected space X, one could truncate the E_∞ structure on the cochain complex to an E_n-algebra and then search for a commutative model for it. Mandell conjectures that such an approximation may be possible at all but finitely many primes. I have two possible ideas for proving such a conjecture, both involving deformation theory.

(1) Using methods similar to the construction of the p-adic de Rham complex, compute a convenient, well-behaved choice of E_n-truncation. I have already completed this step. *Deform* this truncation and use the cotriple products as an obstruction theory to commutativity.
(2) A second, much more speculative but also more interesting, approach is to generalise simplicial sets to obtain a new larger presheaf category whose homotopy category is isomorphic (at least on objects) to the category of E_∞-algebras. Then such deformation can be done geometrically.

4. The homotopy theory of iterated suspensions

In 1955, James [13] introduced the *James construction*, a free associative algebra in topological spaces that acts as a model for $\Omega\Sigma X$, where Σ is the suspension functor and Ω is the based loop space functor, a perspective that unlocks a lot of its hidden homotopical properties. In 1972, May introduced operads to generalise this and thus to prove his celebrated *recognition principle* for iterated loop spaces. This complete characterisation of iterated loop spaces in terms of their E_n-structure has had many applications since, opening the door to stable homotopy theory, and most importantly for our purposes, opened the way for the pioneering work of Cohen-Lada-May [3]. Their work led to a complete classification of the *Dyer-Lashoff homology operations*, which generate the algebra $H_*(\Omega^n\Sigma^n X)$ in terms of $H_*(X)$ and,in general, which permit the computation of the (co)homology of any iterated loop spaces.

It has long been suspected[1] that May's work on iterated loop spaces should have a parallel Eckmann-Hilton dual theory. But even a derived dual to the James construction was not known. It is here that our work enters the picture.

4.1. **The geometry of iterated suspensions**. In [9], Moreno-Fernandez, Wierstra and I develop a theory of counital coalgebras over an operad in topological spaces with respect to the wedge sum $\vee$. There is an intrinsic definition of such coalgebras in terms of *coendomorphism operads*. However, it also turns out that, given an operad $\mathscr{P}$, one can produce a comonad $C_{\mathscr{P}}$ such the coalgebras over it define the same notion. In particular, the arity 1 component $\mathscr{P}(1)$ of the operad plays a critical role in this theory, and one can use it to show that there are no nontrivial coassociative coalgebras in topological spaces. This, along with some subtleties involving quasi-fibrations, explains why May's original proof fail to dualise.

Armed with this new perspective, we are able to prove the following results. Firstly, one has an analogue of May's *approximation theorem*

Theorem A. For every $n \geq 1$, there is a natural morphism of comonads

$$\alpha_n : \Sigma^n\Omega^n \longrightarrow C_n.$$

[1] see the comments on this MathOverflow question [12] by John Klein for a more complete history.

Furthermore, for every pointed space X, there is an explicit natural homotopy retract of pointed spaces

$$\begin{tikzcd} \Sigma^n\Omega^n X \arrow[r, shift left] & C_n(X) \arrow[l, shift left] \arrow[loop above] \end{tikzcd}$$

In particular, $\alpha_n(X)$ is a weak equivalence.

This result is actually slightly stronger than in the loop space case, as it exerts the existence of a retract, not just a weak homotopy equivalence. Following this, via categorical arguments, we are able to supply a proof of the following dual of the recognition principle.

Theorem B. Every n-fold suspension is a $\mathscr{C}_n$-coalgebra, and if a pointed space is a $\mathscr{C}_n$-coalgebra then it is homotopy equivalent to an n-fold suspension.

4.2. **Application to homotopy operations.** In the previously mentioned [3], the little n-cubes operad is used to classify all cohomology operations and compute the homology of iterated loop spaces. Eckmann-Hilton duality suggests that one should be able to produce similar operations on homotopy groups of iterated suspensions. In particular, if one recalls that $\Sigma\Omega K(\mathbb{Z}, 2) = S^2$, a strict Eckmann-Hilton dualisation would permit the computation of homotopy groups of spheres in terms of such operations. This is not as crazy as it sounds, the work of [21] already hints that operations arise from some kind of underlying algebraic structure.

Moreno-Fernandez, Wierstra and I have a proof that one can extract both the Whitehead brackets and the Hopf fibrations from the E_n-structure on n-spheres. It follows therefore that the rational homotopy operations are totally determined by this program. Unlocking similar results for the p-local structure is work in progress.

5. HKR theorems in higher Hochschild homology

The original Hochschild-Kostant-Rosenberg theorem computes the Hochschild homology of smooth commutative algebras in terms of the Kahler differentials. This is a cornerstone theorem in terms of any practical computations of the Hochschild homology associated to smooth geometric objects. The smoothness assumption is effectively a projectivity assumption and may be removed by considering the derived version, ie. the full tangent complex. The Hochschild chain complex $C_*(A, A)$ is intuitively 'circle'-shaped. Pirashvili [19] has generalised this to more general complex $A \boxtimes X$ for any simplicial set X. This is a CDGA version of factorisation homology [10].

5.1. **The higher HKR theorem.** For higher Hochschild homology over a sphere, Pirashvili proved a version of the HKR theorem. A natural question is to try to understand how to generalize HKR to other spaces and what understand what is specific to spheres. It is in this context, that I have proven the following result [5].

Theorem C. Let X be a formal simplicial set. Let A be a CDGA. Suppose that $(\mathrm{Sym}(V), d)$ is a cofibrant, quasi-free resolution of A. Then there is a natural equivalence

$$A \boxtimes X \xrightarrow{\sim} \mathrm{Sym}(V \otimes H_\bullet(X), d_X),$$

where d_X depends on the coalgebra structure of $H_\bullet(X)$[2] and the original differential.

[2]The triviality of this coalgebra for spheres explain the very simple shape of HKR in this context.

We call $\mathrm{Sym}(V \otimes H_\bullet(X), d_X)$ the *higher X-shaped tangent complex* of $A \cong (\mathrm{Sym}(V), d)$ In the case $X = S^1$, one recovers precisely the classical HKR theorem. Theorem C has an immediate application. There is a canonical algebra map $C^*(\mathsf{Map}(X, M)) \to A \boxtimes X$ induced by the Chen iterated integrals, see [11]. This map is an equivalence whenever M is $\dim(X)$-connected. It follows that the X-shaped (co)tangent complex is a small model for many rational mapping spaces.

5.2. **The higher Deligne structure**: If X is an n-fold suspension, for example an n-sphere, the Hochschild homology of $A \boxtimes X$ possesses an E_n-structure induced by the pinch map on X. The natural generalisation of the Deligne conjecture in this context is to ask is if this structure can be promoted to an E_{n+1}-structure. I have shown that the X-shaped tangent complex possesses a natural Pois_{n+1} structure, and by the formality of the E_n operad in zero characteristic, this proves the conjecture.

5.3. **Further work.** I shall attempt next to prove similar results in the context of *factorization algebras*. The conjecture would be the following.

Conjecture 5.1. Let X be a formal n-manifold. Let A be a E_n-algebra. Suppose that $(\mathscr{E}_n(V), d)$ is a cofibrant, quasi-free resolution of A in the category of E_n-algebras. Then there is a natural equivalence

$$\int_X A \xrightarrow{\sim} \mathscr{E}_n(V \otimes H_\bullet(X), d_X),$$

where $\int_X A$ is the factorisation algebra of X with coefficents in A and d_X depends on the coalgebra structure of $H_\bullet(X)$.

Many important classes of manifolds are known to be formal, for example, Kähler manifolds.

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