REPORT ON THE PHD THESIS OF OISIN FLYNN-CONNOLLY

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Overview

The thesis under review, written by Oisín Flynn-Connolly at the Université Sorbonne Paris Nord under the supervision of Professor Grégory Ginot, is entitled "Higher commutativity in algebra and algebraic topology". It deals with several deep and complicated topics in the field of Higher Algebra with applications to Algebraic Topology.

The introduction is a real pleasure to read. The first part is a historical tour, starting with the ideas behind the origin of topology, from around 200 years ago, and ending up with the emergence of higher algebra. This walk is followed by a summary of the mathematical content of this thesis, stressing the main achievements of each chapter.

There are six other chapters, each of which is intended for publication as a separate article. The first two ones have been submitted, jointly with Moreno-Fernández and Wierstra, and with Moreno-Fernández, respectively. The third one, by Flynn-Connolly alone, is available on `arXiv`. The three remaining ones, yet to appear as preprints, already look very polished to me so I am confident they will be available soon to the general public.

In the following sections of this report, I address the original contributions of each of these chapters. In the last section I derive my conclusions.

1. A recognition principle for iterated suspensions as coalgebras over the little cubes operad

A well-known result of Stasheff from his 1960s thesis characterises loop spaces as *algebras* over the operad A_∞. This milestone marked the birth of Higher Algebra, initially as a subfield of Algebraic Topology which soon afterwards took off as an independent branch, due to its applications in many other contexts. Stasheff's result was later generalised by May to iterated loop spaces, using the little disks operads C_n, $n \geq 1$, whose first iteration C_1 is weakly equivalent to A_∞.

In this first chapter, Flynn-Connolly characterises iterated suspensions as *coalgebras* over the same operad C_n. It might look surprising that this result has taken 60 years after Stasheff's to be discovered. After all, the suspension functor is left adjoint to the loop space functor, so it would not be hare-brained to think that some magic categorical duality would operate and yield the result for suspensions from that for loop spaces. However, there is nothing further from reality. A lot of the category theory needed to deal with algebras depends on the underlying category being well-behaved with respect to certain limits, e.g. being locally presentable. This property is not self-dual to the extent that almost no locally presentable category has a locally presentable dual. That is why the algebra and the homotopy theory of *coalgebras* is usually way more complicated than their counterpart for *algebras*, and depend heavily on the nuances of the underlying category, rather than on general properties.

This chapter's main result is therefore achieved after a sequence of radically new results on coalgebras over unitary operads in the category of topological spaces. Curiously, coalgebras over such operads can be described as modules over the arity 1 monoid satisfying extra properties involving the higher part of the operad. Such extra properties are aparently so cumbersome that some readers might think they would never be met, but iterated suspensions are indeed examples and, as the mail result proves, they are *all* examples up to homotopy.

The author's skilled used of point-set topology in this chapter reminds this referee of the theory of $\mathbb{S}$-modules, developed by May and his collaborators as a symmetric monoidal model for stable homotopy theory, although there is no real mathematical connection between both.

2. Higher-order Massey products for algebras over algebraic operads

Massey discovered in the late 1960s a secondary product for triples of elements in singular cohomology which allows to distinguish between the Borromean rings and the unlink. This secondary product, later known as *Massey product*, is only defined whenever the cup-product of any two consecutive elements of the triple vanish. Moreover, it is not a well defined element of the cohomology but a subset, actually a coset by a subgroup called *indeterminacy*.

Massey products exist in the cohomology of any differential graded associative algebra. Moreover, when we have a sequence of four elements whose consecutive cup-products and triple Massey products vanish, they have a quadruple Massey product, and so forth.

In the 1970s, Allday and Retakh found similar operations in the cohomology of a differential graded Lie algebra and obtained applications in rational homotopy theory. This new branch of homotopy theory had been recently discovered by Quillen. It was one of the achievements that granted him the Fields medal in 1978.

Surprisingly, Massey products for algebras over operads other than the associative and the Lie operad were not introduced until last year, in a paper published by myself. I defined a secondary operation for each relation in the presentation of a Koszul operad. These operations only generalise triple Massey product.

In this second chapter, Flynn-Connolly defines higher order Massey-product-like operations for algebras over Koszul operads. Each of them is associated to an element in a higher syzygy of the presentation. More precisely, if the operad $\mathcal{P}$ has a presentation with generating symmetric sequence E and relations $R \subset E \circ_{(1)} E$, its Koszul dual cooperad $\mathcal{P}^{\text{¡}}$ is weighted with $\mathcal{P}^{\text{¡}} = sE \oplus s^2R \oplus \cdots$. There is a generalized Massey operation for each way of writing a weight n element of $\mathcal{P}^{\text{¡}}$ in terms of a basis of E.

As expected, non-vanishing generalized higher Massey products detect non-formal algebras over operads. Moreover, they compute differentials in the Eilenberg–Moore spectral sequence of operadic algebras. Connections with higher operations in minimal models are also established.

3. An obstruction theory for strictly commutative algebras in positive characteristic

Higher algebra works remarkably well over fields of characteristic zero, e.g. Koszul duality. However, things get more complicated when working in positive characteristic, let alone integrally, to the extent that sometimes the homotopy theory of algebras is not governed by a Quillen model category. This is due to the fact that the vector space parametrising operations of n variables in an algebra is a representation of the corresponding symmetric group, and not all representations are projective in positive characteristic. The simplest non-projective representation in the trivial one, which occurs as the space of arity n operations for the commutative operad, for all n.

The failure of Koszul duality in positive characteristic makes the homotopy theory of algebras over the commutative operad different from the that of its Koszul resolution, the operad E_∞. Fortunately, the former is governed by a Quillen model category in this particular case, and it is somehow contained in the latter, but still it is not as well behaved.

The traditional approach has been neglecting (differential graded) commutative algebras in positive characteristic, arguing that one should consider E_∞-algebras instead, by the aforementioned reasons and many others which could also have been mentioned. Nevertheless, the fact is that commutative algebras in positive characteristic do occur in nature and we should be able to tell them apart from general E_∞-algebras.

This goal is achieved in the third chapter of this thesis. It is well known that the cohomology of an E_∞-algebra in characteristic p carries an action of the mod p Steenrod algebra (notice that the mod p singular cochain complex is an example of such an algebra). Such an action vanishes for commutative algebras but the vanishing is not enough to detect commutativity. Flynn-Connolly defines here a wise notion of *coherent vanishing* which characterises commutativity.

4. A p-adic de Rham complex

This chapter deals with algebraic topology at a prime p, but with a perspective different from Section 3. While that section dealt with the mod p case, this one is about the p-adic context.

Sullivan's take on Quillen's rational homotopy theory asserts that the homotopy theory of rational spaces meeting certain finiteness conditions is equivalent to that of certain rational (differential graded) commutative algebras. The equivalence is given by a functor sending each space to its *Sullivan model*, which is a strictly commutative version of the rational singular cochains E_∞-algebra.

In this chapter, Flynn-Connolly defines a Sullivan-like model over the p-adics, that he calls the *p-adic forms* on the space. It is a strictly commutative version of the p-adic singular cochains E_∞-algebra. He uses it to prove a very surprising formality result. It says that a rationally formal space is also p-adically formal for almost all primes. Formality is a very important topic in the topology of manifolds after the Deligne–Griifiths–Morgan–Sullivan theorem establishing the formality of Kähler manifolds over the reals. It is also relevant in higher algebra. Indeed, Deligne's conjecture for Hochschild cohomology was solved over the rational by Kontsevich using the formality of the little disks operad C_2.

5. Homotopically, E_∞-algebras do not generalise commutative algebras

When reviewing Section 3, I mentioned that the homotopy theory of (differential graded) commutative algebras in positive characteristic is *somehow* contained in that of E_∞-algebras. This chapter deals with the meaning of *somehow* in this sentence. If the inclusion were full, two commutative algebras which are quasi-isomorphic as E_∞-algebras would also be quasi-isomorphic as commutative algebras, i.e. if A, B are commutative algebras and we have a zig-zag of quasi-isomorphisms in the category of E_∞-algebras

$$A \xleftarrow{\sim} C \xrightarrow{\sim} B,$$

we should be able to find a similar zig-zag of commutative algebras.

This turns out to be impossible in general and a very explicit counterexample is presented in this chapter. This chapter is rather brief and the example is as simple as it can be, although following the computations requires quite a bit of effort. The technical notion of cup-1-algebra, introduced in Section 3, plays also an important role here.

The existence of this counterexample comes as a surprise because, if we replace commutative with associative and E_∞ with A_∞, there is none, and its actually a recent and important theorem that the inclusion is indeed full in this case.

6. A higher Hochschild–Kostant–Rosenberg theorem and associated operations

The Hochschild–Konstant–Rosenberg theorem establishes an isomorphism between the Hochschild homology of a commutative algebra over a field of characteristic zero and its de Rham complex (Kähler differentials).

After Pirashvili, Hochschild cohomology can be understood via Higher Algebra as the derived tensor product of a commutative algebra and the E_∞-coalgebra of singular chains on the circle. Such a construction can be generalised, taking as inputs any E_∞-algebra and the E_∞-coalgebra of singular chains on any space.

In this thesis's final chapter, Flynn-Connolly extends the Hochschild–Konstant–Rosenberg theorem to this more general context. The analogue of the complex of Kähler differentials in this setting is a sort of cotangent complex built from a minimal model of the commutative algebra and the singular homology of the space. This complex is shown to be an E_{n+1}-algebra when the space is an n-fold suspension. This result is very relevant because it generalises the Deligne conjecture (now Kontsevich's theorem) recalled in Section 4 (it is the case $n = 1$).

7. Conclusions

The thesis under review contains several relevant results in the fields of Higher Algebra and Algebraic Topology. This not-so-short report has only been able to show the tip of the iceberg. Notwithstanding, I hope that the great relevance and novelty of the mathematics contained in this thesis is properly reflected.

I am positive that the different chapters of this thesis will be published in prestigious journals and will have a significant impact in the community. A mathematical thesis with six potentially important papers as an outcome is an exceptionally good one. I feel very proud to have been able to take part in the judgement of this thesis.

Needless to say at this point, I do consider this thesis deserves to be defended.

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