Joana Cirici
Department of Mathematics and Computer Science
Universitat de Barcelona

Gran Via 585
08007 Barcelona
jcirici@ub.edu

Report on the PhD Thesis:

Higher commutativity in algebra and algebraic topology
by
Oisín Flynn-Connolly

It is my pleasure to report on the excellent thesis of Oisín Flynn-Connolly, submitted to Université Sorbonne Paris Nord, in pursuit of a PhD.

Flynn-Connolly's thesis focuses on the study of algebraic invariants for topological spaces. The thesis contains innovative results, combining classical methods in algebraic topology with the more modern approach to homotopy theory through operadic machinery. The initial questions that motivate the thesis are fundamental and in general very subtle, leading to original constructions with many potential applications that I will try to contextualize and summarize below.

A large part of Flynn-Connolly's thesis lies in between two fundamental theories: Sullivan's approach to rational homotopy theory through commutative dg-algebras and Mandell's approach to p-adic homotopy theory through E_∞-algebras.

Rational homotopy theory offers an extremely pleasant algebraic set-up to study homotopy types of topological spaces: indeed, an important subcategory of the homotopy category of rational spaces is equivalent to a subcategory of the homotopy category of commutative dg-algebras over $\mathbb{Q}$. The functor underlying this equivalence, known as *Sullivan's functor of piece-wise linear forms*, is closely related to the singular cochain functor and generalizes de Rham's real algebra of differential forms defined for manifolds, to topological spaces. Moreover, the theory of minimal models for commutative dg-algebras makes rational homotopy a powerful theory computational-wise. In particular, let us mention that the notion of formality is extremely useful in rational homotopy theory, and is present several times in this thesis. A topological space is said to be *formal* if its Sullivan algebra of piece-wise linear forms is quasi-isomorphic to its cohomology viewed as a commutative dg-algebra with trivial differential. This definition makes sense for any operadic algebra in cochain complexes. A main obstruction to formality is given by *Massey products*. These are algebraic operations defined on the cohomology of associative algebras and generalizing the cup product. They also play an important role in this thesis.

Since the invention of localization and completion of topological spaces, it has proved very useful in homotopy theory to study topological spaces through the lenses of a single prime at a time, landing in the p-adic homotopy category. The non-commutativity of singular cochains over $\mathbb{F}_p$ is visible already on the cohomology level, through the presence of Steenrod operations. Therefore any faithful subcategory of the p-adic homotopy category cannot be equivalent to a category of commutative dg-algebras. Instead, one is led to consider E_∞-algebras. These are cochain complexes with an infinitely coherent homotopy associative and commutative multiplication. They provide a generalization of commutative dg-algebras and are best described using operads. In any case, while E_∞-algebras capture the full homotopy type of a space, do not provide a computational paradise as in the strictly commutative case. This is where the work of Flynn-Connolly comes in.

Chapters 3, 4 and 5 of the thesis under review explore the relationship between commutative algebras and E_∞-algebras in characteristic p and mixed characteristic, tackling very general questions in homotopy theory with elegant constructions. In particular, in Chapter 3, Flynn-Connolly introduces new cohomology operations over $\mathbb{F}_p$, called *cotriple products*, as non-linear generalizations of both Massey products and Steenrod operations. A subclass of such cotriple products, called *higher Steenrod operations*, is then shown to encode the obstructions for an E_∞-algebra to be quasi-isomorphic to a strictly commutative dg-algebra. The author also uses cotriple products to produce interesting examples: commutative dg-algebras that are formal over the rationals but not over $\mathbb{F}_p$, algebras with a divided power structure on cohomology that are not weakly equivalent to a divided powers algebra, and commutative dg-algebras (over positive characteristic fields) that are quasi-isomorphic as associative dg-algebras but not as commutative dg-algebras. As another application of the cotriple products, in Chapter 5, Flynn-Connolly constructs an explicit example of two strictly commutative algebras that are quasi-isomorphic in the category of E_∞-algebras but not in the category of strictly commutative algebras. This exhibits in particular that, in characteristic 2, the homotopy category of strictly commutative dg-algebras does not form a subcategory of the homotopy category of E_∞-algebras.

Connecting the above purely algebraic results with topology, Flynn-Connolly proposes in Chapter 4 a new algebraic approach to study topological spaces via the construction of a functor, called *p-adic de Rham forms*, from topological spaces to strictly commutative algebras over the p-adic numbers $\widehat{\mathbb{Z}_p}$. The construction is analogous to Sullivan's functor of piece-wise linear forms over the rationals, although of course it does not encode the full p-adic homootpy types of spaces. Still, after composing with cohomology, this functor computes the cohomology ring of the space with coefficients in $\widehat{\mathbb{Z}_p}$, and it carries additional homotopical invariants of the space, including Massey products living in the torsion part of the cohomology. The author shows that if a space is formal in the rational homotopy theory sense, then the p-adic de Rham forms are also formal except at possibly finitely many primes. The p-adic de Rham complex is related to other constructions in the literature. Perhaps the oldest one arises in work of Cartan, who defined a strictly commutative dg-algebra over the integers using divided powers. This in turn is related to tame homotopy theory. There also seems to be a surprising connection with the theory of crystalline cohomology for algebraic varieties, through the functor of Berthelot-Ogus-Deligne. Interestingly, Flynn-Connolly shows that his p-adic de Rham functor is quasi-isomorphic to a sub-algebra of the E_∞-algebra of singular cochains with coefficients in $\widehat{\mathbb{Z}_p}$, proving that, in some sense, this is the best approximation of the algebra of cochains of a space by a strictly commutative algebra. I believe this construction opens up many potential avenues for future research, and will surely lead to many interesting applications in algebraic topology and also in algebraic geometry, especially through the study of formality in the p-adic sense of certain algebraic varieties and the construction of explicit p-adic de Rham models using resolution of singularities. In this direction, it would be interesting to develop a sheaf-theoretic approach to the theory. It would also be desirable to determine to what extent, p-adic de Rham forms determine part of the p-adic homotopy types.

I believe the results mentioned above would be more than enough for an excellent PhD thesis. Still, the thesis includes additional work, some of it in collaboration with other young researchers, that I next briefly explain. Chapter 1 is based on a paper written by the candidate jointly with Moreno-Fernández and Wierstra, where the three authors prove a recognition principle for iterated suspensions as coalgebras over the little cubes operad. This result may be understood as the Eckmann–Hilton dual of May's foundational results on iterated loop spaces. Chapter 2 is about higher-order Massey products for algebras over algebraic operads. Recently, Fernando Muro generalized the definition of triple Massey products for associative algebras to more general operadic algebras. In this paper, Flynn-Connolly together with Moreno-Fernandez generalize Muro's ideas to higher order Massey products, showing that such products are obstructions to the formality of operadic algebras. The authors also make precise the relationship between their Massey products and the higher structures appearing when doing homotopy transfer on cohomology. Both results generalize well-known and very useful facts previously known for associative algebras. I find it very surprising that such work had not been developed before, and it is surely a great addition to the literature and of general interest to homotopy theorists. Lastly, in Chapter 6, Flynn-Connolly proves a generalization of the Hochschild–Kostant–Rosenberg theorem that holds for all commutative algebras and formal spaces, with applications to derived algebraic geometry and mathematical physics.

Overall, this is an outstanding and very well written PhD thesis, which is clearly worth the degree of doctor. I have no doubt its contents will be published in high-quality journals and that they will lead to many further developments and advances in the field. I strongly endorse the defense of this thesis.

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Joana Cirici