The Structure Theorem of Decoherence-free Subsystems

Ji Guan, Yuan Feng and Mingsheng Ying

***Abstract*—Decoherence-free subsystems have been successfully developed as a tool to preserve fragile quantum information against noises. In this work, we develop a structure theory for decoherence-free subsystems. Consequently, we propose an effective algorithm to find an optimal set of decoherence-free subsystems for any given quantum super-operator such that any other such subsystem is a subspace of one of them. Furthermore, we give an easy way to pick up faulty-tolerant ones from them. Using these mathematical techniques, we obtain a simple numerical method to obtain a basis for any tensor, generating a family of matrix product states.**

***Index Terms*—decoherence-free subsystems, continuous coherences, faulty-tolerance, matrix product states, structure theory.**

I. Introduction

To build large scale quantum computers, the obstacles, such as decoherences and noises, must be managed and overcome [1]. One of the effective methods for this purpose is through decoherence-free subspaces proposed by Daniel A. Lidar in [2]. A subspace of the system Hilbert space is said to be decoherence-free if the effect of the noise on it is simply unitary, and thus easily correctable. For this sake, decoherence-free subspaces are important subjects in quantum computing, where coherent control of quantum systems is often the desired goal [3]. On the other hand, decoherence-free subspaces can also be characterized as a special case of quantum error correcting codes to preserve quantum information against noises [3]. However, we do not need to restrict the decoherence-free dynamics to a subspace. E. Knill, R. Laflamme, and L. Viola introduced the concept of noiseless subsystems, by extending higher-dimensional irreducible representations of the algebra generating the dynamical symmetry in the system-environment interaction [4]. A subsystem is a factor in a tensor product decomposition of a subspace and the noiseless subsystem requires the evolution on it to be strict identity. Such noiseless subsystems have been fully characterized and intensely studied in [5], [6], [7], [8], [9]. Remarkably, the structure theory of noiseless subsystems was proposed in [5], leading to an algorithm which finds all noiseless subsystems for a given quantum super-operator [10], [11]. In the meanwhile, the general case of decoherence-free subspaces and noiseless subsystems, called decoherence-free subsystems, was examined and the conditions for their existence were found in [12], and subsystems with significantly reduced noises were considered in [13]. However, a clear picture of the structure of decoherence-free subsystems (subspaces) is still lacking such that we can not compute all decoherence-free subsystems (subspaces) or the highest-dimensional ones for any given quantum super-operator.

In this paper, we develop a structure theory that shows precisely how a super-operator, the evolution of a quantum system, determines its decoherence-free subsystems, generating the existing results for noiseless subsystems. As an application, we develop an algorithm (Algorithm 1) to generate an optimal set of decoherence-free subsystems for any given super-operator such that any other decoherence-free subsystem is a subspace of one of them. We then consider decoherence-free subsystems with error tolerance, the and obtain a necessary and sufficient condition for determining those. Checking this condition is a linear algebra exercise. We further use this mathematical tool in the quantum many-body system described by a family of matrix product states generated by a tensor and propose a simple method to numerically derive a basis for the tensor. The basis plays an important role in fundamental theorems of Matrix product states [19], [20].

This paper is organized as follows. We recall some basic notions of quantum information theory and introduce one central concept, continuous coherences in Section II. In Section III, we review the structure theory of noiseless subsystems by studying the fixed points of super-operators. We then show a similar structure theory of decoherence-free subsystems in Section IV, which leads to an algorithm of constructing an optimal set of such subsystems for a given super-operator. In Section V, we present a necessary and sufficient condition for checking whether or not a decoherence-free subsystem is faulty-tolerant. Furthermore, we apply previous results to find a basis for any tensor, generating a set of matrix product states to represent a quantum many-body system in Section VI. A brief conclusion is drawn in the last section.

II. Preliminaries

In this section, for convenience of the reader, we review some basic notions and results from quantum information theory; for details we refer to [1]. Recall that given a quantum system S with the associated (finite-dimensional) state Hilbert space $\mathcal{H}$, the evolution of the system can be mathematically modeled by a super-operator, i.e., a completely positive and trace-preserving (CPTP) map $\mathcal{E}$ on $\mathcal{H}$. We say that a quantum system A is a subsystem of S if $\mathcal{H} = (\mathcal{H}_A \bigotimes \mathcal{H}_B) \bigoplus \mathcal{K}$ for some co-subsystem B, where $\mathcal{K} = (\mathcal{H}_A \bigotimes \mathcal{H}_B)^{\perp}$, $\mathcal{H}_A$ and

Ji Guan and Yuan Feng are with Centre for Quantum Software and Information, University of Technology Sydney, NSW 2007, Australia.

Mingsheng Ying is with Centre for Quantum Software and Information, University of Technology Sydney, NSW 2007, Australia and Department of Computer Science and Technology, Tsinghua University, Beijing 100084, China.

$\mathcal{H}_B$ are the state spaces of A and B, respectively. Let $B(\mathcal{H})$ be the set of all operators and $D(\mathcal{H})$ the set of all quantum states, i.e. density operators with unit trace, on $\mathcal{H}$. The support of a quantum state ρ, denoted by supp(ρ), is the linear span of the eigenvectors corresponding to non-zero eigenvalues of ρ.

Definition 1: Given a super-operator $\mathcal{E}$ on $\mathcal{H}$,

(1) a quantum state ρ is said to be stationary if it is a fixed point of $\mathcal{E}$, i.e. $\mathcal{E}(\rho) = \rho$. Furthermore, ρ is minimal if there is no other stationary state σ such that supp$(\sigma) \subseteq$ supp(ρ);
(2) a subspace $\mathcal{H}_1 \subseteq \mathcal{H}$ is minimal if it is a support of some minimal stationary state. Furthermore, if the whole space $\mathcal{H}$ is minimal, then we call it irreducible; otherwise it is reducible.
(3) a subspace $\mathcal{K}$ is called transient if for any state $\rho \in D(\mathcal{H})$,
$$\lim_{n\to\infty} \operatorname{tr}(P_{\mathcal{K}}\mathcal{E}^n(\rho)) = 0,$$
where $P_{\mathcal{K}}$ is the projector onto $\mathcal{K}$.

Note that if two minimal states ρ and σ have the same support, then $\rho = \sigma$. Thus $\mathcal{H}$ is irreducible if and only if there is only one stationary state.

Applying the techniques developed in [14], [15], we can decompose $\mathcal{H}$ into mutually orthogonal minimal subspaces and the largest transient subspace:
$$\mathcal{H} = \bigoplus_{p=1}^{m} \mathcal{H}_p \bigoplus \mathcal{K}. \tag{1}$$
Furthermore, the Kraus operators $\{E_k\}$ of $\mathcal{E}$ have the corresponding block form:
$$E_k = \left[\begin{array}{cccc|c} E_{k,1} & & & & \\ & E_{k,2} & & & \\ & & \ddots & & T_k \\ & & & E_{k,m} & \\ \hline & & 0 & & K_k \end{array}\right]$$
for some operators $E_{k,p} \in B(\mathcal{H}_p)$, $K_k \in B(\mathcal{K})$, and T_k from $\mathcal{K}$ to $\mathcal{K}^\perp$. We then define a set of associated maps $\{\mathcal{E}_{p,q} : p, q = 1, \ldots, m\}$ of $\mathcal{E}$:
$$\mathcal{E}_{p,q}(\cdot) = \sum_k E_{k,p} \cdot E_{k,q}^\dagger. \tag{2}$$
Obviously, for any p and q, $\mathcal{E}_{p,q}$ is a linear map from $C(\mathcal{H}_p, \mathcal{H}_q)$ to itself, where
$$C(\mathcal{H}_p, \mathcal{H}_q) = \text{lin.span}\{|\psi_p\rangle\langle\psi_q| : |\psi_p\rangle \in \mathcal{H}_p, |\psi_q\rangle \in \mathcal{H}_q\}$$
is the coherence from $\mathcal{H}_q$ to $\mathcal{H}_p$. If $p = q$, then $\mathcal{E}_{p,q}$ is a super-operator on $\mathcal{H}_p$ and $C(\mathcal{H}_p, \mathcal{H}_q)$ is said to be inner; otherwise it is outer.

For all p and q, the following two properties are easy to observe:

(1) $C(\mathcal{H}_p, \mathcal{H}_q)$ is invariant under $\mathcal{E}$; that is, for all $A \in C(\mathcal{H}_p, \mathcal{H}_q)$, $\mathcal{E}(A) \in C(\mathcal{H}_p, \mathcal{H}_q)$.
(2) $\lambda(\mathcal{E}_{p,q}) \subseteq \lambda(\mathcal{E})$, where $\lambda(\cdot)$ is the set of eigenvalues of a linear map.

Furthermore, a coherence $C(\mathcal{H}_p, \mathcal{H}_q)$ is said to be continuous if there exists $A \in C(\mathcal{H}_p, \mathcal{H}_q)$ such that $\mathcal{E}(A) = e^{i\theta}A$ for some real number θ; that is, $\lambda(\mathcal{E}_{p,q})$ has an element with magnitude one. Specially, if $\theta = 0$, then the coherence is stationary. Obviously, inner coherences are always stationary because a super-operator has at least one stationary state. Stationary coherences have been intensely studies in [15], where a nice structure of $fix(\mathcal{E})$, the set of fixed points of super-operator $\mathcal{E}$, is discovered. We will restate this result in the next section.

Definition 2: Given a super-operator $\mathcal{E}$ on $\mathcal{H}$ with $\mathcal{H} = \mathcal{H}_A \bigotimes \mathcal{H}_B \bigoplus \mathcal{K}$, subsystem $\mathcal{H}_A$ is decoherence-free if there exists an unitary matrix U_A on $\mathcal{H}_A$ such that for any $\rho_A \in D(\mathcal{H}_A)$ and $\rho_B \in D(\mathcal{H}_B)$,
$$\mathcal{E}(\rho_A \otimes \rho_B) = U_A \rho_A U_A^\dagger \otimes \sigma_B \text{ for some } \sigma_B \in D(\mathcal{H}_B). \tag{3}$$
Furthermore, if $U_A = I_A$, the identity operator on $\mathcal{H}_A$, then we say that $\mathcal{H}_A$ is noiseless.

In an open quantum system $\mathcal{H}$ which is under the dynamic $\mathcal{E}$, decoherence-free subsystem $\mathcal{H}_A$ is a closed quantum system, i.e. the evolution on it is completely unitary. Thus quantum information can be stored faithfully in $\mathcal{H}_A$.

III. Fixed Points and Noiseless Subsystems

Noiseless subsystems are a special case of decoherence-free subsystems and have been intensely studied in quantum error correction [16], [7] and quantum memory [17]. As we are going to show a structure theory of decoherence-free subsystems in the upcoming section, we first review the counterpart in noiseless subsystems which is inspired by the structure of fixed points of super-operators.

To characterize $fix(\mathcal{E})$, the main step is to identify stationary coherences of minimal subspaces in the decomposition of Eq.(1) and this can be achieved by the following lemma.

Lemma 1 ([15]): Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$. For any two orthogonal minimal subspaces $\mathcal{H}_p$ and $\mathcal{H}_q$, $C(\mathcal{H}_p, \mathcal{H}_q)$ is stationary if and only if there is a unitary matrix U such that $E_{k,p} = UE_{k,q}U^\dagger$ for all k, where $\{E_{k,p}\}_k$ and $\{E_{k,q}\}_k$ are the restriction of Kraus operators of $\mathcal{E}$ onto $\mathcal{H}_p$ and $\mathcal{H}_q$, respectively. Furthermore, $\mathcal{H}_p \simeq \mathcal{H}_q$.

From Lemma 1, $C(\mathcal{H}_p, \mathcal{H}_q)$ is stationary if and only if so is $C(\mathcal{H}_q, \mathcal{H}_p)$. Thus in the following, we simply say that there is a stationary coherence between $\mathcal{H}_p$ and $\mathcal{H}_q$ without referring to the direction. Furthermore, we group together minimal subspaces by stationary coherences and obtain a structure of $fix(\mathcal{E})$ as follows.

Theorem 1 ([15]): Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$. There is a unique orthogonal decomposition of $\mathcal{H}$
$$\mathcal{H} = \bigoplus_{l=1}^{L} \mathcal{X}_l \bigoplus \mathcal{K}. \tag{4}$$
where

(1) $\mathcal{K}$ is the largest transient subspace;

(2) each $\mathcal{X}_l$ is either a minimal subspace or can be further decomposed into mutually orthogonal minimal subspaces with stationary coherences between any two of them:

$$\mathcal{X}_l = \bigoplus_{p=1}^{m_l} \mathcal{B}_{l,p} \simeq \mathbb{C}^{m_l} \bigotimes \mathcal{B}_l, \tag{5}$$

where for all p, $\mathcal{B}_l \simeq \mathcal{B}_{l,p}$. Furthermore, the Kraus operators $\{E_k\}$ of $\mathcal{E}$ have a corresponding block form:

$$E_k = \left[\begin{array}{ccc|c} I_1 \otimes E_{k,1} & & & \\ & \ddots & & T_k \\ & & I_L \otimes E_{k,L} & \\ \hline & 0 & & K_k \end{array}\right] \tag{6}$$

for some operators $E_{k,l} \in B(\mathcal{B}_l)$, $K_k \in B(\mathcal{K})$, and T_k from $\mathcal{K}$ to $\mathcal{K}^\perp$. Here I_l is the identity operator on $\mathbb{C}^{m_l}$. Moreover, $\mathcal{B}_l$ is irreducible under $\mathcal{E}_l(\cdot) = E_{k,l} \cdot E_{k,l}^\dagger$. In this decomposition,

$$fix(\mathcal{E}) = \oplus_l [B(\mathbb{C}^{m_l}) \otimes \rho_l] \oplus 0_{\mathcal{K}}$$

where ρ_l is the unique stationary state of $\mathcal{E}_l$, and $0_{\mathcal{K}}$ is the zero operator on $\mathcal{K}$.

(3) there are no stationary coherences between minimal subspaces $\mathcal{X}_{l_1}$ and $\mathcal{X}_{l_2}$ whenever $l_1 \neq l_2$.

Actually, as the following theorem shows, all noiseless subsystems have been captured by the above decomposition.

Theorem 2 ([6]): Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$ with its unique orthogonal decomposition

$$\mathcal{H} = \bigoplus_{l=1}^{m} \left[\mathbb{C}^{m_l} \bigotimes \mathcal{B}_l\right] \bigoplus \mathcal{K},$$

presented in Theorem 1. Then a subsystem $\mathcal{H}_A$ is noiseless if and only if $\mathcal{H}_A \subseteq \mathbb{C}^{m_l}$ for some l.

The decomposition in Theorem 1 can also be obtained by applying the structure of C^*-algebra generated by the Kraus operators of $\mathcal{E}$; see [5] for details. Subsequently, some algorithms for implementing the above decompositions in Eqs.(4-6) were developed from the structure of the C^*-algebra or $fix(\mathcal{E})$ [18], [10], [11]. Such algorithms will be referred as NSDecompose($\mathcal{H}, \mathcal{E}$) in this paper and the time complexity is $O(n^8)$, where $\dim(\mathcal{H}) = n$.

Example 1: Given Hilbert space $\mathcal{H} = \mathbb{C}^4 \bigotimes \mathbb{C}^3$ with associated basis $\{|k\rangle_1\}_{k=0}^3$ and $\{|k\rangle_2\}_{k=0}^2$, let $\mathcal{E}$ be super-operator on it and the Kraus operators are:

$$\begin{aligned} E_1 &= |00\rangle\langle 01| + |10\rangle\langle 11| - |20\rangle\langle 21| - |30\rangle\langle 31| \\ E_2 &= |01\rangle\langle 00| + |11\rangle\langle 10| - |21\rangle\langle 20| - |31\rangle\langle 30| \\ E_3 &= |00\rangle\langle 02| + |10\rangle\langle 12| - |20\rangle\langle 22| - |30\rangle\langle 32| \end{aligned}$$

where $|kl\rangle = |k\rangle_1 \otimes |l\rangle_2$. Obviously, the unique decomposition in Theorem 1 is

$$\mathcal{H} = \bigoplus_{l=1}^{2} (\mathcal{H}_l \bigotimes \mathcal{H}') \bigoplus \mathcal{K}$$

where $\mathcal{H}_1 = \text{lin.span}\{|0\rangle_1, |1\rangle_1\}$, $\mathcal{H}_2 = \text{lin.span}\{|2\rangle_1, |3\rangle_1\}$, $\mathcal{H}' = \text{lin.span}\{|0\rangle_2, |1\rangle_2\}$ and $\mathcal{K} = (\bigoplus_{l=1}^2 (\mathcal{H}_l \bigotimes \mathcal{H}'))^\perp$. Then we can store 2-qubit quantum information in $\mathcal{H}_1$ or $\mathcal{H}_2$.

IV. Decoherence-free Subsystems

In this section, a generalized decomposition of Theorem 1 will be shown, which also indicates a structure theorem of decoherence-free subsystems. Using these results, we can develop an efficient algorithm to find all such subsystems for any given super-operator.

By the definition, a decoherence-free subsystem $\mathcal{H}_A$ with co-subsystem $\mathcal{H}_B$ is an closed quantum system (the evolution is a unitary super-operator) which can not be disturbed by quantum noises; that is there is no decoherence.

The restriction of $\mathcal{E}$ onto $\mathcal{H}_A \bigotimes \mathcal{H}_B$ can be rewritten as

$$\mathcal{E}_{AB} = \mathcal{U}_A \otimes \mathcal{E}_B \tag{7}$$

where $\mathcal{U}_A$ is a unitary super-operator on $\mathcal{H}_A$ and $\mathcal{E}_B$ is a super-operator on $\mathcal{H}_B$. By the decomposition Eq.(1), $\mathcal{H}_B$ can be chosen to be irreducible. From now on, we assume that the co-subsystem of a decoherence-free subsystem is always irreducible.

First, we observe that decoherence-free subsystems consist of minimal subspaces with continuous coherences:

Theorem 3: Given a super-operator $\mathcal{E}$ on $\mathcal{H}$ with $\mathcal{H}_A \bigotimes \mathcal{H}_B \bigoplus \mathcal{K}$, let $\mathcal{H}_A$ be a decoherence-free subsystem and U_A the corresponding unitary matrix in Eq.(3). Let $\{|p\rangle\}_{p=1}^m$ be a set of mutually orthogonal eigenvectors of U_A and $\mathcal{H}_p = \text{supp}(|p\rangle\langle p|) \bigotimes \mathcal{H}_B$. Then for all $1 \le p, q \le m$, $\mathcal{H}_p$ and $\mathcal{H}_q$ are both minimal subspaces and $C(\mathcal{H}_p, \mathcal{H}_q)$ is continuous.

Proof. Let ρ be the stationary state of $\mathcal{E}_B$. As $\mathcal{H}_B$ is irreducible, ρ is the only stationary state and $|p\rangle\langle p| \otimes \rho$ is a minimal stationary state of $\mathcal{E}$ for all p, i.e. $\mathcal{H}_p$ is minimal. Furthermore, for each p, $U_A|p\rangle = e^{i\theta_p}|p\rangle$ for some θ_p, so $\mathcal{E}(|p\rangle\langle q| \otimes \rho) = e^{i(\theta_p - \theta_q)}|p\rangle\langle q| \otimes \rho$ for all p, q. □

Therefore, studying continuous coherences are an interesting problem. The essential issue is to verify continuous coherences for a given mutually orthogonal minimal subspaces. Then such minimal subspaces are candidates for constructing a decoherence-free subsystem.

Lemma 2: Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$. For any twoorthogonal minimal subspaces $\mathcal{H}_p$ and $\mathcal{H}_q$, $C(\mathcal{H}_p, \mathcal{H}_q)$ is continuous if and only if there is a unitary matrix U and a real number θ such that

$$E_{k,p} = e^{i\theta} U E_{k,q} U^\dagger \ \forall k,$$

where $\{E_{k,p}\}_k$ and $\{E_{k,q}\}_k$ are the restriction of Kraus operators of $\mathcal{E}$ onto $\mathcal{H}_p$ and $\mathcal{H}_q$, respectively. Furthermore, $\mathcal{H}_p \simeq \mathcal{H}_q$.

Proof. Assume that $C(\mathcal{H}_p, \mathcal{H}_q)$ is continuous; that is there is a matrix $A \in C(\mathcal{H}_p, \mathcal{H}_q)$ such that $\mathcal{E}(A) = e^{i\theta} A$ for some real number θ. Then let $V = e^{-i\theta} P_p + I - P_p$, where P_p is the projector onto $\mathcal{H}_p$, and $\mathcal{V} \circ \mathcal{E}(A) = A$, where $\mathcal{V}(\cdot) = V \cdot V^\dagger$. Moreover, it is obvious that $\mathcal{H}_p$ and $\mathcal{H}_q$ are also two orthogonal minimal subspaces under $\mathcal{V} \circ \mathcal{E}$ by the decomposition Eq.(1). Therefore, there is a stationary coherence from $\mathcal{H}_q$ to $\mathcal{H}_p$ under $\mathcal{V} \circ \mathcal{E}$. Then from lemma 1 and the minimal decomposition of $\mathcal{V} \circ \mathcal{E}$, we have $\mathcal{H}_p \simeq \mathcal{H}_q$ and

$$E_k^p = e^{i\theta} U E_k^q U^\dagger \ \forall k$$

for some unitary matrix U.

Conversely,

$$\mathcal{E}_{p,q}(\cdot) = \sum_k E_{k,p} \cdot E_{k,q}^\dagger = \sum_k e^{i\theta} U E_{k,q} U^\dagger \cdot E_{k,q}^\dagger$$

Transforming $\mathcal{E}_{p,q}$ to its matrix representation in [18], we have:

$$\begin{aligned} M_{p,q} &= \sum_k e^{i\theta} U E_{k,q} U^\dagger \otimes E_{k,q}^* \\ &= e^{i\theta}(U \otimes I)(\sum_k E_{k,q} \otimes E_{k,q}^*)(U^\dagger \otimes I) \end{aligned}$$

As $\lambda(M_{p,q}) = \lambda(\mathcal{E}_{p,q})$ and $(\sum_k E_{k,q} \otimes E_{k,q}^*)$ is the matrix presentation of $\mathcal{E}_{q,q}$ which is a super-operator and has 1 as its eigenvalues, $e^{i\theta} \in \lambda(\mathcal{E}_{p,q})$. □

The above lemma is a generalization of Lemma 2 in which stationary coherences need the Kraus operators are one-by-one unitarily equivalent.

Corollary 1: Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$. Given a set of mutually orthogonal minimal subspaces $\{\mathcal{H}_p\}_{p=1}^m$, the existence of continuous coherence is an equivalence relation; that is

(1) (reflexivity) For any p, $C(\mathcal{H}_p, \mathcal{H}_p)$ is continuous.
(2) (symmetry) For any p and q, if $C(\mathcal{H}_p, \mathcal{H}_q)$ is continuous, then $C(\mathcal{H}_q, \mathcal{H}_p)$ is continuous.
(3) (transitivity) For any p, q and l, if $C(\mathcal{H}_p, \mathcal{H}_q)$ and $C(\mathcal{H}_q, \mathcal{H}_l)$ are continuous, then $C(\mathcal{H}_p, \mathcal{H}_l)$ is continuous.

Furthermore, we can decompose the whole Hilbert space into finite equivalent classes with an additional largest transient subspace.

Theorem 4: Given a super-operator $\mathcal{E}$ on $\mathcal{H}$, there is an unique orthogonal decomposition of $\mathcal{H}$ as

$$\mathcal{H} = \bigoplus_{l=1}^{L} \mathcal{X}_l \bigoplus \mathcal{K} \tag{8}$$

where

(1) $\mathcal{K}$ is the largest transient subspace;
(2) each $\mathcal{X}_l$ is either a minimal subspace or can be further decomposed into mutually orthogonal minimal subspaces, which have continuous coherences between any two of them:

$$\mathcal{X}_l = \bigoplus_{p=1}^{m_l} \mathcal{B}_{l,p} \simeq \mathbb{C}^{m_l} \bigotimes \mathcal{B}_l, \quad \mathcal{B}_l \simeq \mathcal{B}_{l,p} \ \forall p \tag{9}$$

such that the Kraus operators $\{E_k\}$ have a corresponding block form of :

$$E_k = \left[\begin{array}{ccc|c} U_1 \otimes E_{k,1} & & & \\ & \ddots & & T_k \\ & & U_L \otimes E_{k,L} & \\ \hline & 0 & & K_k \end{array}\right] \tag{10}$$

for some operators $E_{k,l} \in B(\mathcal{B}_l)$, $K_k \in B(\mathcal{K})$, T_k from $\mathcal{K}$ to $\mathcal{K}^\perp$, and unitary matrix $U_l = diag(e^{i\theta_{l,1}}, \cdots, e^{i\theta_{l,m_l}})$ for some real numbers $\{\theta_{l,p}\}_{p=1}^{m_l}$ on $\mathbb{C}^{m_l}$. Moreover, $\mathcal{B}_l$ is irreducible under $\mathcal{E}_l := \sum_k E_{k,l} \cdot E_{k,l}^\dagger$ for all l. In this decomposition,

$$fix(\mathcal{E}) = \oplus_l [fix(\mathcal{U}_l) \otimes \rho_l] \oplus 0_{\mathcal{K}}$$

where $\mathcal{U}_l(\cdot) = U_l \cdot U_l^\dagger$.

(3) there are no continuous coherences between minimal subspaces in $\mathcal{X}_p$ and $\mathcal{X}_q$ if $p \neq q$.

Proof. By Theorem 1, there is an unique orthogonal decomposition of $\mathcal{H}$ as

$$\mathcal{H} = \bigoplus_{l=1}^{L'} \mathcal{X}_l' \bigoplus \mathcal{K}$$

where

(1) $\mathcal{K}$ is the largest transient subspace;
(2) each $\mathcal{X}_l^{'}$ is either a minimal subspace or can be further decomposed into mutually orthogonal minimal subspaces:

$$\mathcal{X}_l^{'} = \bigoplus_{p=1}^{m_l'} \mathcal{B}_{l,p}^{'}$$

such that there is a stationary coherence between $\mathcal{B}_{l,p}^{'}$ and $\mathcal{B}_{l,q}^{'}$;

(3) there are no stationary coherences between minimal subspaces in $\mathcal{X}_{l_1}^{'}$ and $\mathcal{X}_{l_2}^{'}$ if $l_1 \neq l_2$.

Then we divide $\{\mathcal{X}_l^{'}\}$ into finite disjoint subsets by continuous coherences; that is for any $l_1 \neq l_2$, if there is a continuous coherence between any minimal subspaces in $\mathcal{X}_{l_1}^{'}$ and $\mathcal{X}_{l_2}^{'}$, then they are in the same subset. This can be done as the existence of continuous coherences is an equivalence relation by Corollary 1. Then we define $\{\mathcal{X}_l\}_{l=1}^L$ be the set of the direct sum of all elements in each subset. Therefore, $\mathcal{H}$ can be uniquely decomposed as $\mathcal{H} = \bigoplus_l \mathcal{X}_l \bigoplus \mathcal{K}$. Obviously, for any two orthogonal minimal subspaces $\mathcal{B}_{l_1} \in \mathcal{X}_{l_1}$ and $\mathcal{B}_{l_2} \in \mathcal{X}_{l_2}$, $C(\mathcal{B}_{l_1}, \mathcal{B}_{l_2})$ is continuous if and only if $l_1 = l_2$.

Furthermore, for each l, $\mathcal{X}_l$ can be further decomposed to mutually orthogonal minimal subspaces:

$$\mathcal{X}_l = \bigoplus_{p=1}^{m_l} \mathcal{B}_{l,p}$$

By Lemma 2, in an appropriate decomposition of $\mathcal{X}_l = \bigoplus_{p=1}^{m_l} \mathcal{B}_{l,p} \simeq \mathbb{C}^{m_l} \bigotimes \mathcal{B}_l$ and $\mathcal{B}_l \simeq \mathcal{B}_{l,p}$ for all p:

$$E_k = \left[\begin{array}{ccc|c} U_1 \otimes E_{k,1} & & & \\ & \ddots & & T_k \\ & & U_L \otimes E_{k,L} & \\ \hline & 0 & & K_k \end{array}\right] \tag{11}$$

and $\mathcal{B}_l$ is irreducible under $\mathcal{E}_l := \sum_k E_{k,l} \cdot E_{k,l}^\dagger$ for all l, where $U_l = diag(e^{i\theta_{l,1}}, \cdots, e^{i\theta_{l,m_l}})$ for a set of real numbers $\{\theta_{l,p}\}_{p=1}^{m_l}$. From Theorem 1 and noting that stationary coherences is continuous, we have that

$$fix(\mathcal{E}) = \oplus_l [fix(\mathcal{U}_l) \otimes \rho_l] \oplus 0_{\mathcal{K}}.$$

where ρ_l is the unique stationary state of $\mathcal{E}_l$. □

Corollary 2: Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$ with unique decomposition:

$$\mathcal{H} = \bigoplus_l (\mathbb{C}^{m_l} \bigotimes \mathcal{B}_l) \bigoplus \mathcal{K}$$

presented in Theorem 4, for any minimal subspace $\mathcal{H}'$, there is a pure state $|\psi\rangle \in \mathbb{C}^{m_l}$ for some l such that $\mathcal{H}' = \mathrm{supp}(|\psi\rangle\langle\psi|) \otimes \mathcal{B}_l$.

The above theorem shows minimal subspaces with continuous coherences can construct decoherence-free subsystem $\mathbb{C}^{m_l}$. Fortunately, we can further show that others are subspaces of the decoherence-free subsystems constructed in Eq.(9).

Theorem 5: Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$ with the unique decomposition:

$$\mathcal{H} = \bigoplus_l (\mathbb{C}^{m_l} \bigotimes \mathcal{B}_l) \bigoplus \mathcal{K},$$

presented in Theorem 4. Then subsystem $\mathcal{H}_A$ is decoherence-free if and only if $\mathcal{H}_A \subseteq \mathbb{C}^{m_l}$ for some l and $\mathcal{H}_A$ is a decoherence-free subspace under $\mathcal{U}_l(\cdot) = U_l \cdot U_l^\dagger$, where U_l is the corresponding unitary matrix on $\mathbb{C}^{m_l}$ in the decomposition Eq.(10).

Proof. Assume that $\mathcal{H}_A$ is decoherence-free. By Theorem 4 and Corollary 2, $\mathcal{H}_A \subseteq \mathbb{C}^{m_l}$ for some l. From the definition of decoherence-free subsystems and the restriction of $\mathcal{E}$ onto $\mathbb{C}^{m_l}$ being $\mathcal{U}_l$, $\mathcal{H}_A$ is a decoherence-free subspace under $\mathcal{U}_l$.

On the other hand, it is direct from Theorem 4. □

These theorem confirms that the set of decoherence-free subsystems identified in Theorem 4 is the largest ones; that is any other is a subspace of one of them. So we only need to implement the decompositions in Theorem 4 and all decoherence-free subsystems can be easily found by Theorem 5.

The easy way to do this is transforming all continuous coherences to be stationary and we can use NSDecompose$(\mathcal{H}, \mathcal{E})$ to get the results in Theorem 4.

For any two operators $E_{k,p}$ and $E_{k,q}$ in Eq.(10) of Theorem 4, if they are unitarily equivalent with a phase θ, i.e. $E_{k,p} \simeq e^{i\theta} E_{k,q}$, then let $E'_{k,q} = e^{i\theta} E_{k,q}$ and $E'_{k,p} = E_{k,p}$. Then $E'_{k,p}$ is unitarily equivalent to $E'_{k,q}$; a continuous coherence is transformed to be stationary. Using this method, we develop Algorithm 1 to implement decompositions in Theorem 4 and the time complexity is $O(n^8)$.

Now we back to see Example 1. By Algorithm 1, we can confirm that the first subsystem $\mathbb{C}^4$ is decoherence-free and further compute that the evolution on it is $|0\rangle\langle 0| + |1\rangle\langle 1| - |2\rangle\langle 2| - |3\rangle\langle 3|$. Then we can store 4-qubit information in this subsystem, which doubles the capacity of noiseless subsystems.

Algorithm 1 Decompose$(\mathcal{H}, \mathcal{E})$

Input: A Hilbert space $\mathcal{H}$ and a super-operator $\mathcal{E}$ with Kraus operators $\{E_k\}_{k=1}^d$ on it
Output: The two-level decomposition of $\mathcal{H}$ in the form of Eqs.(8) and (9).

```
NSDecompose(H, {E_k}_{k=1}^d):
H = ⊕_{l=1}^L (C^{m_l} ⊗ B_l) ⊕ K in Eq.(5) and E_k = [⊕_{l=1}^L I_l ⊗ E_{k,l}  T_k ; 0  K_k] for all k in Eq.(6)
L ← {1, 2, ··· , L}
for each p ∈ {1, 2, ··· , L} do
  if p ∈ L then
    for each q > p and q ∈ L do
      M ← Σ_k E_{k,p} ⊗ E*_{k,q}
      if λ(M) has one element with magnitude one then
        θ ← (Σ_{a∈λ(E_{k,p})} a) / (Σ_{b∈λ(E_{k,q})} b)
        E_{k,q} ← e^{iθ} E_{k,q}
        L ← L \ {q}
      end if
    end for
  end if
end for
for each 1 ≤ k ≤ d do
  E_k ← [⊕_{l=1}^L I_l ⊗ E_{k,l}  T_k ; 0  K_k]
end for
return NSDecompose(H, {E_k}_{k=1}^d)
```

V. Perfect and Imperfect Initialization

Given a decomposition of $\mathcal{H} = \mathcal{H}_A \bigotimes \mathcal{H}_B \bigoplus \mathcal{K}$, for any $\rho \in D(\mathcal{H})$, ρ has the following block form:

$$\rho = \begin{bmatrix} \rho_{AB} & \rho' \\ \rho'^\dagger & \rho_{\mathcal{K}} \end{bmatrix} \tag{12}$$

If $\mathcal{H}_A$ is decoherence-free, then the required initial state ρ in Definition 2 should satisfy: $\rho_{AB} = \rho_A \otimes \rho_B$, $\rho' = 0$ and $\rho_{\mathcal{K}} = 0$. But in many cases, specially experiments, such initialization might be challenging, so we discuss weaker cases in this section.

Following [12], we relax the definition of decoherence-free subsystems as

Definition 3: Let $\mathcal{E}$ be a super-operator on $\mathcal{H} = \mathcal{H}_A \bigotimes \mathcal{H}_B \bigoplus \mathcal{K}$. $\mathcal{H}_A$ is a decoherence-free subsystem if for any $\rho \in D(\mathcal{H})$ with $P_{AB}\rho P_{AB} \neq 0$,

$$\mathrm{tr}_B[\mathcal{E}(P_{AB}\rho P_{AB})] = U \mathrm{tr}_B[(P_{AB}\rho P_{AB})] U^\dagger,$$

where P_{AB} is the projector onto $\mathcal{H}_A \bigotimes \mathcal{H}_B$.

In the new definition, we do not need the reduced state ρ_{AB} is a valid state, since its trance might be less than one. But the quantum information stored in $\mathcal{H}_A$ is preserved, i.e. the evolution on $\mathcal{H}_A$ is unitary. We say that perfect initialization occurs when $\rho' = 0$ and $\rho_{\mathcal{K}} = 0$; otherwise imperfect initialization. The results of [12] have shown that in perfect initialization, $\mathcal{H}_A$ is decoherence-free if and only if the Kraus operators $\{E_k\}_k$ have the matrix representation:

$$E_k = \begin{bmatrix} U \otimes E'_k & T_k \\ 0 & K_k \end{bmatrix}$$

; in imperfect initialization, we further need $T_k = 0$ for all k. Obviously, co-subsystem $\mathcal{H}_B$ is important in later case, but is inessential in former case as the initial state is prepared in $\mathcal{H}_A \bigotimes \mathcal{H}_B$ and $\mathcal{H}_B$ can be tranced over. Therefore, for fault-tolerant computation, we should choose decoherence-free subsystem under imperfect initialization.

Definition 4: Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$. Subsystem $\mathcal{H}_A$ is a fault-tolerant decoherence-free subsystem if there exist a co-subsystem $\mathcal{H}_B$, i.e. $\mathcal{H} = \mathcal{H}_A \bigotimes \mathcal{H}_B \bigoplus \mathcal{K}$, such that $\mathcal{H}_A$ is a decoherence-free subsystem under imperfect initialization.

In the last section, we have obtained a way to compute all decoherence-free subsystem under perfect initialization. Now we need choose faulty-tolerant ones from them.

Theorem 6: Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$. In its unique decomposition:

$$\mathcal{H} = \bigoplus_l (\mathbb{C}^{m_l} \bigotimes \mathcal{B}_l) \bigoplus \mathcal{K}$$

presented in Theorem 4, $\mathcal{H}_A$ is a fault-tolerant decoherence-free subsystem if and only if $\mathcal{H}_A \subseteq \mathbb{C}^{m_l}$ for some l and $\mathcal{H}_A$ is a decoherence-free subsystem under imperfect initialization of $\mathcal{H} = \mathcal{H}_A \bigotimes \mathcal{B}_l \bigoplus \mathcal{K}'$, where $\mathcal{K}' = (\mathcal{H}_A \bigotimes \mathcal{B}_l)^\perp$.

Proof. The sufficient part is direct from Definition 4 and Theorem 5.

For necessary part, assume that $\mathcal{H}_A$ is decoherence-free under perfect initialization. Then $\mathcal{H}_A \subseteq \mathbb{C}^{m_l}$ for some l by Theorem 5. Furthermore, it is easy to observe that any co-subsystem $\mathcal{H}_B$ (can not to be irreducible) of $\mathcal{H}_A$ must satisfy $\mathcal{B}_l \subseteq \mathcal{H}_B$ from Corollary 2.

Obviously, if $\mathcal{H}_A$ is decoherence-free under imperfect initialization of $\mathcal{H} = \mathcal{H}_A \bigotimes \mathcal{H}_B \bigoplus \mathcal{Y}$ with $\mathcal{Y} = (\mathcal{H}_A \bigotimes \mathcal{H}_B)^\perp$, then $\mathcal{H}_A$ is also decoherence-free under imperfect initialization of $\mathcal{H} = \mathcal{H}_A \bigotimes \mathcal{B}_l \bigoplus \mathcal{K}'$, as $\mathcal{H}_A \bigotimes \mathcal{B}_l \subseteq \mathcal{H}_A \bigotimes \mathcal{H}_B$. □

The above theorem tells us a fact that the co-subsystems identified in Eq.(9) are sufficient to determine whether a subsystem is a faulty-tolerant decoherence-free subsystem or not. For example, in Example 1, we can get a decoherence-free subsystem $\mathbb{C}^4$ and the co-subsystem $\mathcal{H}'$ under perfect initialization from Algorithm 1. Then through Theorem 6, we can use the co-subsystem $\mathcal{H}'$ to verify that $\mathbb{C}^4$ is not faulty-tolerant.

VI. Basis of Matrix Product States

Matrix Product States (MPS), a special case of tensor networks (a theoretical and numerical tool describing quantum many-body systems), have proven to be an useful family of quantum states for describing ground states of one-dimensional quantum many-body systems, which is not scalable due to the exponential growth of the Hilbert space dimension with the number of subsystems [19].

Given a tenor $\mathcal{A} = \{A_k \in \mathcal{M}_D\}_{k=1}^d$, where $\mathcal{M}_D$ denotes $D \times D$ complex matrices, d is the physical dimension of a Hilbert space $\mathcal{H}_d$ described by a set of mutually orthogonal pure states $\{|k\rangle\}_{k=1}^d$, and D is the bond dimension, it generates a family of translationally invariant MPS, namely

$$V(\mathcal{A}) = \{|V_n(\mathcal{A})\rangle\}_{n\in\mathbb{N}^+},$$

where

$$|V_n(\mathcal{A})\rangle = \sum_{k_1,\cdots,k_n}^d \mathrm{tr}(A_{k_1}\cdots A_{k_n})|k_1\cdots k_n\rangle \in \mathcal{H}_d^{\otimes n}$$

Then we can get a associated completely positive map $\mathcal{E}_{\mathcal{A}}(\cdot) = \sum_{k=1}^d A_k \cdot A_k^\dagger$.

By [20], we can always find a set of irreducible tensors $\{\mathcal{A}_j\}_{j=1}^m$ with the same physical dimension d, and a set of complex number $\{\mu_j\}_{j=1}^m$ such that for any $n \in \mathbb{N}^+$

$$|V_n(\mathcal{A})\rangle = \sum_{j=1}^m \mu_j^n |V_n(\mathcal{A}_j)\rangle \tag{13}$$

where a tenor is called irreducible if the associated map is CPTP and irreducible. That is for any tensor $\mathcal{A}$, the generated MPS can be linearly represented by MPS of a set of irreducible tensors. Therefore, studying irreducible tensors is an interesting problem, especially the conditions allowing two tensors describing the same family of MPS. First, we can group irreducible tensors that are essentially the same in the following sense.

Definition 5 ([20]): We say that two irreducible tensors with the same physical dimension d, say $\mathcal{A} = \{A_k\}_{k=1}^d$ and $\mathcal{B} = \{B_k\}_{k=1}^d$, are repeated if there exist a phase θ and an unitary matrix U so that

$$A_k = e^{i\theta} U B_k U^\dagger \ \forall k$$

By the definition, $\mathcal{A}$ and $\mathcal{B}$ are repeated, then $|\mathcal{A}\rangle_n = e^{in\theta}|\mathcal{B}\rangle_n$ for all $n \in \mathbb{N}^+$. Therefore, for any tensor $\mathcal{A}$, we can reduce the set of irreducible tensors in Eq.(13) to be non-repeated. Such simplified set is called a basis of $\mathcal{A}$. The fundamental problem, which relates different tensors rising the same MPS, can be answered by the basis. That is if two tensors have the same MPS, then their basis must be related by a unitary transform [20]. Therefore, determining whether two irreducible tensors are repeated or not is a key problem. Even though, such repeatability relation can be verified by Jordan decompositions of matrices by the definition, Jordan decompositions is sensitive to errors and should be avoided in numerical analysis of many body systems. Here we propose a feasible method to finish this by the results of continuous coherences in previous sections.

Theorem 7: Given two irreducible tensors with the same physical dimension d, $\mathcal{A} = \{A_k\}_{k=1}^d$ and $\mathcal{B} = \{B_k\}_{k=1}^d$, they are repeated if and only if $\lambda(\mathcal{E}_{\mathcal{A},\mathcal{B}})$ has an element with magnitude one, where $\mathcal{E}_{\mathcal{A},\mathcal{B}} = \sum_{k=1}^d A_k \cdot B_k^\dagger$.

Proof. Let $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ be the corresponding Hilbert spaces of tensors $\mathcal{A}$ and $\mathcal{B}$, respectively; That is $A_k \in B(\mathcal{H}_\mathcal{A})$ and $B_k \in B(\mathcal{H}_\mathcal{B})$ for all k. Then the Hilbert space of $\mathcal{E}$ is $\mathcal{H}_\mathcal{A} \bigoplus \mathcal{H}_\mathcal{B}$, where $\mathcal{E}$ is a super-operator with Kraus operator $\{diag(A_k, B_k)\}_{k=1}^d$. Obviously, $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ are both minimal subspaces under $\mathcal{E}$. Then the result is direct from Lemma 2. □

By the above theorem, repeatability can be easily checked by computing eigenvalues of a linear map, which is a linear algebra exercise.

VII. Conclusion

In this paper, we established a structure theory for decoherence-free subsystems. Consequently, an algorithm for finding an optimal set of decoherence-free subsystems had been developed. Then we gave a simple way to pick up the faulty-tolerant decoherence-free subsystems. After that, these results helped us find a basis for any tensor by computing the eigenvalues for some constructed linear maps.

For future studies, an immediate topic is to generalize our results to continuous-time quantum systems. In [21], authors studied this in quantum control setting and expected to obtain a linear-algebraic approach for finding all decoherence-free subsystems for a given generator, describing the system.

Acknowledgment

The authors would like to thank...

References

[1] M. A. Nielsen and I. L. Chuang, *Quantum computation and quantum information*. Cambridge university press, 2010.

[2] D. A. Lidar, I. L. Chuang, and K. B. Whaley, "Decoherence-free subspaces for quantum computation," *Physical Review Letters*, vol. 81, no. 12, p. 2594, 1998.

[3] D. A. Lidar, "Review of decoherence free subspaces, noiseless subsystems, and dynamical decoupling," *arXiv preprint arXiv:1208.5791*, 2012.

[4] E. Knill, R. Laflamme, and L. Viola, "Theory of quantum error correction for general noise," *Physical Review Letters*, vol. 84, no. 11, p. 2525, 2000.

[5] M.-D. Choi and D. W. Kribs, "Method to find quantum noiseless subsystems," *Physical review letters*, vol. 96, no. 5, p. 050501, 2006.

[6] R. Blume-Kohout, H. K. Ng, D. Poulin, and L. Viola, "Information-preserving structures: A general framework for quantum zero-error information," *Physical Review A*, vol. 82, no. 6, p. 062306, 2010.

[7] C. Bény, A. Kempf, and D. W. Kribs, "Generalization of quantum error correction via the heisenberg picture," *Physical review letters*, vol. 98, no. 10, p. 100502, 2007.

[8] D. W. Kribs and R. W. Spekkens, "Quantum error-correcting subsystems are unitarily recoverable subsystems," *Physical Review A*, vol. 74, no. 4, p. 042329, 2006.

[9] D. Kribs, R. Laflamme, and D. Poulin, "Unified and generalized approach to quantum error correction," *Physical review letters*, vol. 94, no. 18, p. 180501, 2005.

[10] E. Knill, "Protected realizations of quantum information," *Physical Review A*, vol. 74, no. 4, p. 042301, 2006.

[11] X. Wang, M. Byrd, and K. Jacobs, "Numerical method for finding decoherence-free subspaces and its applications," *Physical Review A*, vol. 87, no. 1, p. 012338, 2013.

[12] A. Shabani and D. A. Lidar, "Theory of initialization-free decoherence-free subspaces and subsystems," *Physical Review A*, vol. 72, no. 4, p. 042303, 2005.

[13] X. Wang, M. Byrd, and K. Jacobs, "Minimal noise subsystems," *Physical review letters*, vol. 116, no. 9, p. 090404, 2016.

[14] S. Ying, Y. Feng, N. Yu, and M. Ying, "Reachability probabilities of quantum markov chains," in *International Conference on Concurrency Theory*. Springer, 2013, pp. 334–348.

[15] B. Baumgartner and H. Narnhofer, "The structures of state space concerning quantum dynamical semigroups," *Reviews in Mathematical Physics*, vol. 24, no. 02, p. 1250001, 2012.

[16] D. W. Kribs, R. Laflamme, D. Poulin, and M. Lesosky, "Operator quantum error correction," *arXiv preprint quant-ph/0504189*, 2005.

[17] G. Kuperberg, "The capacity of hybrid quantum memory," *IEEE Transactions on Information Theory*, vol. 49, no. 6, pp. 1465–1473, 2003.

[18] J. Guan, Y. Feng, and M. Ying, "Decomposition of quantum markov chains and zero-error capacity," *arXiv preprint arXiv:1608.06024*, 2016.

[19] J. I. Cirac, D. Perez-Garcia, N. Schuch, and F. Verstraete, "Matrix product density operators: Renormalization fixed points and boundary theories," *Annals of Physics*, vol. 378, pp. 100–149, 2017.

[20] G. D. l. Cuevas, J. I. Cirac, N. Schuch, and D. Perez-Garcia, "Irreducible forms of matrix product states: Theory and applications," *arXiv preprint arXiv:1708.00029*, 2017.

[21] F. Ticozzi and L. Viola, "Quantum markovian subsystems: invariance, attractivity, and control," *IEEE Transactions on Automatic Control*, vol. 53, no. 9, pp. 2048–2063, 2008.