The Structure of Decoherence-free Subsystems

Ji Guan, Yuan Feng and Mingsheng Ying

***Abstract*—Decoherence-free subsystems have been successfully developed as a tool to preserve fragile quantum information against noises. In this work, we develop a structure theory for decoherence-free subsystems. Based on it, we present an effective algorithm to find, under perfect initialization of quantum states, an optimal set of decoherence-free subsystems for any given quantum operations such that any other such subsystem is a subspace of one of them. Furthermore, we find an easy way to pick up faulty-tolerant ones from them. Using these mathematical techniques, we propose a simple numerical method to obtain a basis for any tensor, generating a family of matrix product states.**

***Index Terms*—decoherence-free subsystems, continuous coherences, faulty-tolerance, matrix product states, structure theory.**

I. Introduction

To build large scale quantum computers, the obstacles, such as decoherences and noises, must be managed and overcome [1]. One of the effective methods for this purpose is through decoherence-free subspaces proposed by Daniel A. Lidar in [2]. A subspace of the system Hilbert space is said to be decoherence-free if the effect of the noise on it is simply unitary, and thus easily correctable. For this sake, decoherence-free subspaces are important tools in quantum computing, where coherent control of quantum systems is often the desired goal [3]. On the other hand, decoherence-free subspaces can be characterized as a special case of quantum error correcting codes to preserve quantum information against noises [3]. Indeed, we do not even need to restrict the decoherence-free dynamics to a subspace. E. Knill, R. Laflamme, and L. Viola introduced the concept of noiseless subsystems, by extending higher-dimensional irreducible representations of the algebra generating the dynamical symmetry in the system-environment interaction [4]. A subsystem is a factor in a tensor product decomposition of a subspace, and the noiseless subsystem requires the evolution on it to be strict identity.

Noiseless subsystems have been fully characterized and intensely studied in [5], [6], [7], [8], [9]. Remarkably, a structure theory of noiseless subsystems was established in [5], leading to an algorithm which finds all noiseless subsystems for a given quantum operation (i.e. the evolution of an open quantum system, mathematically modeled by a super-operator) [10], [11]. For the more general case of decoherence-free subsystems, however, a structure theory is still lacking, although several conditions for their existence were found in [12], and subsystems with significantly reduced noises were carefully examined in [13]. Without such a structure theory, it is hard to compute all decoherence-free subsystems (subspaces) or the highest-dimensional ones for any given super-operator.

The aim of this paper is to develop a structure theory that shows precisely how a super-operator determines its decoherence-free subsystems, with the structure theory of noiseless subsystems as a special case. As an application, we develop an algorithm (Algorithm 1) to generate, under perfect initialization of quantum states, an optimal set of decoherence-free subsystems for any given super-operator such that any other decoherence-free subsystem is a subspace of one of them. Furthermore, a very simple method to find them is proposed for decoherence-free subsystems with imperfect initialization. Moreover, we use this mathematical tool in the quantum many-body system described by a family of matrix product states generated by a tensor and find a feasible way to numerically derive a basis for the tensor. Such a basis plays an important role in establishing the fundamental theorems of matrix product states [14], [15].

This paper is organized as follows. We recall some basic notions of quantum information theory and, in particular, introduce a central concept, continuous coherence, in Section II. In Section III, we review the structure theory of noiseless subsystems by studying the fixed points of super-operators. We then establish a corresponding structure theory for decoherence-free subsystems in Section IV, which leads to an algorithm for constructing, with the assumption of perfect initialization, an optimal set of such subsystems for any given super-operator. In Section V, we present a procedure for checking whether or not a subsystem with a co-subsystem is decoherence-free under imperfect initialization. Furthermore, in Section VI, we apply the results obtained in the previous sections to find a basis for any tensor, generating a set of matrix product states that represent a quantum many-body system. A brief conclusion is drawn in the last section.

II. Preliminaries

In this section, for convenience of the reader, we review some basic notions and results from quantum information theory; for details we refer to [1]. Recall that given a quantum system S with the associated (finite-dimensional) state Hilbert space $\mathcal{H}$, the evolution of the system can be mathematically modeled by a super-operator, i.e. a completely positive and trace-preserving (CPTP) map $\mathcal{E}$ on $\mathcal{H}$. We say that a quantum system A is a subsystem of S if $\mathcal{H} = (\mathcal{H}_A \otimes \mathcal{H}_B) \oplus \mathcal{K}$ for some co-subsystem B, where $\mathcal{K} = (\mathcal{H}_A \otimes \mathcal{H}_B)^\perp$, $\mathcal{H}_A$ and $\mathcal{H}_B$ are the state spaces of A and B, respectively. For any two Hilbert spaces $\mathcal{H}$ and $\mathcal{H}'$, let $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ be the set of all operators from $\mathcal{H}$ to $\mathcal{H}'$. Simply, we define that $\mathcal{L}(\mathcal{H}) = \mathcal{L}(\mathcal{H}, \mathcal{H})$ and $\mathcal{D}(\mathcal{H})$

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is the set of all quantum states, i.e. density operators with unit trace, on $\mathcal{H}$. The support of a quantum state ρ, denoted by supp(ρ), is the linear span of the eigenvectors corresponding to non-zero eigenvalues of ρ.

Definition 1: Given a super-operator $\mathcal{E}$ on $\mathcal{H}$.

(1) A quantum state ρ is said to be stationary if it is a fixed point of $\mathcal{E}$, i.e. $\mathcal{E}(\rho) = \rho$. Furthermore, ρ is minimal if there is no other stationary state σ such that $\text{supp}(\sigma) \subseteq \text{supp}(\rho)$;

(2) A subspace $\mathcal{H}_1 \subseteq \mathcal{H}$ is minimal if it is a support of some minimal stationary state. Furthermore, if the whole space $\mathcal{H}$ is minimal, then we call it irreducible; otherwise it is reducible.

(3) A subspace $\mathcal{K}$ is called transient if for any state $\rho \in \mathcal{D}(\mathcal{H})$,
$$\lim_{n\to\infty} \text{tr}(P_{\mathcal{K}}\mathcal{E}^n(\rho)) = 0,$$
where $P_{\mathcal{K}}$ is the projector onto $\mathcal{K}$.

Note that if two minimal states ρ and σ have the same support, then $\rho = \sigma$. Thus $\mathcal{H}$ is irreducible if and only if there is only one stationary state and the state is full-rank.

Applying the techniques developed in [16], [17], we can decompose $\mathcal{H}$ into mutually orthogonal minimal subspaces $\mathcal{H}_p$ and the largest transient subspace $\mathcal{K}$:
$$\mathcal{H} = \bigoplus_{p=1}^{m} \mathcal{H}_p \oplus \mathcal{K}. \tag{1}$$
Furthermore, the Kraus operators $\{E_k\}$ of $\mathcal{E}$ have the corresponding block form:
$$E_k = \left[\begin{array}{cccc|c} E_{k,1} & & & & \\ & E_{k,2} & & & \\ & & \ddots & & T_k \\ & & & E_{k,m} & \\ \hline & & 0 & & K_k \end{array}\right]$$
for some operators $E_{k,p} \in \mathcal{L}(\mathcal{H}_p)$, $K_k \in \mathcal{L}(\mathcal{K})$, and $T_k \in \mathcal{L}(\mathcal{K}, \mathcal{K}^{\perp})$. We then define a set of associated maps $\{\mathcal{E}_{p,q} : p, q = 1, \dots, m\}$ of $\mathcal{E}$:
$$\mathcal{E}_{p,q}(\cdot) = \sum_k E_{k,p} \cdot E_{k,q}^{\dagger}. \tag{2}$$
Obviously, for any p and q, $\mathcal{E}_{p,q}$ is a linear map from $\mathcal{L}(\mathcal{H}_q, \mathcal{H}_p)$ to itself. If $p \neq q$, $\mathcal{L}(\mathcal{H}_q, \mathcal{H}_p)$ can be viewed as (outer) coherences from $\mathcal{H}_q$ to $\mathcal{H}_p$, i.e. upper off-diagonal blocks of all matrices restricted in the decomposition $\mathcal{H}_p \oplus \mathcal{H}_q$. Thus the coherence between $\mathcal{H}_p$ and $\mathcal{H}_q$ is $\mathcal{L}(\mathcal{H}_q, \mathcal{H}_p) \oplus \mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$ and $\mathcal{L}(\mathcal{H}_q)$ can be regarded as inner coherences.

For all p and q, the following two properties are easy to observe:

(1) $\mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$ is invariant under $\mathcal{E}$; that is, for all $A \in \mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$, $\mathcal{E}(A) \in \mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$.

(2) $\lambda(\mathcal{E}_{p,q}) \subseteq \lambda(\mathcal{E})$, where $\lambda(\cdot)$ is the set of eigenvalues of a linear map.

Furthermore, the coherence $\mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$ is said to be continuous if there exists $A \in \mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$ such that $\mathcal{E}(A) = e^{i\theta}A$ for some real number θ; that is, $\lambda(\mathcal{E}_{p,q})$ has an element with magnitude one. Specially, if $\theta = 0$, then $\mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$ is stationary. Obviously, $\mathcal{L}(\mathcal{H}_p, \mathcal{H}_p)$ is always stationary because a super-operator has at least one stationary state. Stationary coherences have been intensely studied in [17], where a nice structure of $fix(\mathcal{E})$, the set of fixed points of super-operator $\mathcal{E}$, is discovered. We will restate this result in the next section.

Definition 2: Let $\mathcal{E}$ be a super-operator on $\mathcal{H} = (\mathcal{H}_A \otimes \mathcal{H}_B) \oplus \mathcal{K}$. $\mathcal{H}_A$ is a decoherence-free subsystem if there is a unitary matrix U_A on $\mathcal{H}_A$ such that for any initial state ρ,
$$\text{tr}_B[\mathcal{E}(P_{AB}\rho P_{AB})] = U_A \text{tr}_B[(P_{AB}\rho P_{AB})]U_A^{\dagger}, \tag{3}$$
where P_{AB} is the projector onto $\mathcal{H}_A \otimes \mathcal{H}_B$. Furthermore, if $U_A = I_A$, the identity operator on $\mathcal{H}_A$, then we say that $\mathcal{H}_A$ is noiseless.

Given a decomposition of $\mathcal{H} = (\mathcal{H}_A \otimes \mathcal{H}_B) \oplus \mathcal{K}$ and a initial state $\rho \in D(\mathcal{H})$, ρ has the following block form:
$$\rho = \begin{bmatrix} \rho_{AB} & \rho' \\ \rho'^{\dagger} & \rho_{\mathcal{K}} \end{bmatrix} \tag{4}$$
If $\mathcal{H}_A$ is decoherence-free, then the quantum information $\text{tr}_B(\rho_{AB})$ can be preserved in it, even though $\text{tr}_B(\rho_{AB})$ is not a valid quantum state as its trace might be less than one. Ideally, we hope that ρ is initialized in $\mathcal{H}_A \otimes \mathcal{H}_B$, i.e. $\rho' = 0$ and $\rho_{\mathcal{K}} = 0$; this situation is called perfect initialization. But in experiments, preparing such an initialization might be challenging, so we will face an imperfect initialization, i.e. $\rho' \neq 0$ or $\rho_{\mathcal{K}} \neq 0$.

The results of [12] have shown that in a perfect initialization, $\mathcal{H}_A$ is decoherence-free if and only if the Kraus operators $\{E_k\}_k$ of $\mathcal{E}$ have the matrix representation:
$$E_k = \begin{bmatrix} U \otimes E'_k & T_k \\ 0 & K_k \end{bmatrix} \quad \forall k \tag{5}$$
and in an imperfect initialization, we have
$$E_k = \begin{bmatrix} U \otimes E'_k & 0 \\ 0 & K_k \end{bmatrix} \quad \forall k \tag{6}$$

Obviously, co-subsystem $\mathcal{H}_B$ is important in the later case, but is inessential in the former case as the initial state is prepared in $\mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{H}_B$ can be traced over. A decoherence-free subsystem $\mathcal{H}_A$ under an imperfect initialization is always decoherence-free under a perfect initialization, but the converse is not true.

In the following two sections, we assume a perfect initialization, i.e. the initial states can be prepared exactly in any subspace. In Section V, we will deal with an imperfect initialization.

III. Fixed Points and Noiseless Subsystems

Noiseless subsystems are a special case of decoherence-free subsystems and have been intensely studied in the areas of quantum error correction [18], [7] and quantum memory [19]. As we are going to present a structure theory of decoherence-free subsystems in the upcoming section, we first review the counterpart in noiseless subsystems which is inspired by the structure of fixed points of super-operators.

To characterize $fix(\mathcal{E})$, the main step is to identify stationary coherences of minimal subspaces in the decomposition of Eq. (1) and this can be achieved by the following lemma.

Lemma 1 ([17]): Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$ with the orthogonal decomposition presented in Eq. (1). Then for any $1 \leq p, q \leq m$, $\mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$ is stationary if and only if there is a unitary matrix U such that $E_{k,p} = UE_{k,q}U^\dagger$ for all k. Furthermore, $\mathcal{H}_p \simeq \mathcal{H}_q$.

From Lemma 1, $\mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$ is stationary if and only if so is $\mathcal{L}(\mathcal{H}_q, \mathcal{H}_p)$. Thus in the following, we simply say that there is a stationary coherence between $\mathcal{H}_p$ and $\mathcal{H}_q$ without referring to the direction. Furthermore, we group together minimal subspaces by stationary coherences and obtain a structure of $fix(\mathcal{E})$ as follows.

Theorem 1 ([17]): Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$. Then there is a unique orthogonal decomposition of $\mathcal{H}$

$$\mathcal{H} = \bigoplus_{l=1}^{L} \mathcal{X}_l \oplus \mathcal{K} \tag{7}$$

where:

(1) $\mathcal{K}$ is the largest transient subspace;

(2) each $\mathcal{X}_l$ is either a minimal subspace or can be further decomposed into mutually orthogonal minimal subspaces with stationary coherences between any two of them:

$$\mathcal{X}_l = \bigoplus_{p=1}^{m_l} \mathcal{B}_{l,p} \simeq \mathbb{C}^{m_l} \otimes \mathcal{B}_l, \quad \mathcal{B}_l \simeq \mathcal{B}_{l,p} \ \forall p \tag{8}$$

so that the Kraus operators $\{E_k\}$ of $\mathcal{E}$ have a unitarily equivalent block form:

$$E_k \simeq \left[\begin{array}{ccc|c} I_1 \otimes E_{k,1} & & & \\ & \ddots & & T_k \\ & & I_L \otimes E_{k,L} & \\ \hline & 0 & & K_k \end{array}\right] \tag{9}$$

for some operators $E_{k,l} \in \mathcal{L}(\mathcal{B}_l)$, $K_k \in \mathcal{L}(\mathcal{K})$, and $T_k \in \mathcal{L}(\mathcal{K}, \mathcal{K}^\perp)$. Here I_l is the identity operator on $\mathbb{C}^{m_l}$ and $\mathcal{B}_l$ is irreducible under $\mathcal{E}_l(\cdot) = \sum_k E_{k,l} \cdot E_{k,l}^\dagger$. Furthermore,

$$fix(\mathcal{E}) \simeq \bigoplus_l [\mathcal{L}(\mathbb{C}^{m_l}) \otimes \rho_l] \oplus 0_{\mathcal{K}}$$

where ρ_l is the unique stationary state of $\mathcal{E}_l$, and $0_{\mathcal{K}}$ is the zero operator on $\mathcal{K}$.

(3) there is no stationary coherence between any minimal subspaces $\mathcal{B}_{l,p}$ and $\mathcal{B}_{l',p'}$ whenever $l \neq l'$.

In the following parts of this paper we will, with a slight abuse of notation, write all the formulas related to splitting $\mathcal{H}$ as in Eq. (8) or Kraus operators as in Eq. (9) with "$=$" instead of "$\simeq$".

Actually, as the following theorem shows, all noiseless subsystems have been captured by the above decomposition.

Theorem 2 ([6]): Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$ with its unique orthogonal decomposition

$$\mathcal{H} = \bigoplus_{l=1}^{m} \left[\mathbb{C}^{m_l} \bigotimes \mathcal{B}_l\right] \oplus \mathcal{K},$$

presented in Theorem 1. Then a subsystem $\mathcal{H}_A$ is noiseless if and only if $\mathcal{H}_A \subseteq \mathbb{C}^{m_l}$ for some l.

The decomposition in Theorem 1 can also be obtained by applying the structure of C^*-algebra generated by the Kraus operators of $\mathcal{E}$; see [5] for details. Subsequently, some algorithms for implementing the above decompositions in Eqs. (7-9) were developed from the structure of the C^*-algebra or $fix(\mathcal{E})$ [20], [10], [11]. Such algorithms will be referred as NSDecompose$(\mathcal{H}, \mathcal{E})$ in this paper and the time complexity is $O(n^8)$, where $\dim(\mathcal{H}) = n$.

Example 1: Given $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$, and $\{|k\rangle_A\}_{k=0}^3$ and $\{|k\rangle_B\}_{k=0}^2$ are orthonormal bases of $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively, let $\mathcal{E}$ be a super-operator on $\mathcal{H}$ with the Kraus operators:

$$\begin{aligned} E_1 &= |00\rangle\langle 01| + |10\rangle\langle 11| - |20\rangle\langle 21| - |30\rangle\langle 31| \\ E_2 &= |01\rangle\langle 00| + |11\rangle\langle 10| - |21\rangle\langle 20| - |31\rangle\langle 30| \\ E_3 &= |00\rangle\langle 02| + |10\rangle\langle 12| - |20\rangle\langle 22| - |30\rangle\langle 32| \end{aligned}$$

where $|kl\rangle = |k\rangle_A \otimes |l\rangle_B$. It is easy to calculate the unique decomposition of $\mathcal{H}$ in Theorem 1 as

$$\mathcal{H} = \bigoplus_{l=1}^{2} [\mathcal{H}_l \otimes \mathcal{H}'] \oplus \mathcal{K}$$

where $\mathcal{H}_1 = \text{lin.span}\{|0\rangle_A, |1\rangle_A\}$, $\mathcal{H}_2 = \text{lin.span}\{|2\rangle_A, |3\rangle_A\}$, $\mathcal{H}' = \text{lin.span}\{|0\rangle_B, |1\rangle_B\}$, and $\mathcal{K} = \mathcal{H}_A \otimes \text{lin.span}\{|2\rangle_B\}$. Then we can store 2-qubit quantum information in $\mathcal{H}_1$ or $\mathcal{H}_2$.

IV. Decoherence-free Subsystems with a Perfect Initialization

In this section, a similar orthogonal decomposition as that in Theorem 1 is presented for decoherence-free subsystems. Employing it, we then develop an efficient algorithm to find all such subsystems for any given super-operator.

By the definition, a decoherence-free subsystem $\mathcal{H}_A$ is a small section of Hilbert space $\mathcal{H}$ with a unitary evolution under the quantum noise, modeled by a super-operator $\mathcal{E}$. From Eq. (5), the restriction of $\mathcal{E}$ onto $\mathcal{H}_A \otimes \mathcal{H}_B$, where $\mathcal{H}_B$ is the co-subsystem of $\mathcal{H}_A$, can be written as

$$\mathcal{E}_{AB} = \mathcal{U}_A \otimes \mathcal{E}_B \tag{10}$$

where $\mathcal{U}_A$ is a unitary super-operator on $\mathcal{H}_A$ and $\mathcal{E}_B$ is a super-operator on $\mathcal{H}_B$. By the decomposition Eq. (1), $\mathcal{H}_B$ can be chosen to be irreducible. In this section, we assume that the co-subsystem of a decoherence-free subsystem is always irreducible.

First, we observe that the joint systems of decoherence-free subsystems and irreducible co-subsystems consist of minimal subspaces with continuous coherences.

Theorem 3: Given a super-operator $\mathcal{E}$ on

$$\mathcal{H} = (\mathcal{H}_A \otimes \mathcal{H}_B) \oplus \mathcal{K}$$

Let $\mathcal{H}_A$ be a decoherence-free subsystem and U_A the corresponding unitary matrix in Eq. (3). If $\{|p\rangle\}_{p=1}^m$ is a set of mutually orthogonal eigenvectors of U_A and $\mathcal{H}_p = \text{lin.span}\{|p\rangle\} \otimes \mathcal{H}_B$, then for all $1 \leq p, q \leq m$, $\mathcal{H}_p$ is a minimal subspace and $\mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$ is continuous.

Proof. Note that we assume $\mathcal{H}_B$ is irreducible. Let ρ be the unique stationary state of $\mathcal{E}_B$. Then for any p, $|p\rangle\langle p| \otimes \rho$ is a minimal stationary state of $\mathcal{E}$, and hence $\mathcal{H}_p$ is minimal. Furthermore, note that $U_A|p\rangle = e^{i\theta_p}|p\rangle$ for some θ_p. Thus

$$\mathcal{E}(|p\rangle\langle q| \otimes \rho) = e^{i(\theta_p - \theta_q)}|p\rangle\langle q| \otimes \rho$$

for all p and q. □

Theorem 3 indicates that minimal subspaces with continuous coherences play an important role in determining decoherence-free subsystems. To check if two orthogonal minimal subspaces have a continuous coherence, we present the following lemma which is similar to Lemma 1 for stationary coherences.

Lemma 2: Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$ with the orthogonal decomposition presented in Eq. (1). Then for any $1 \leq p, q \leq m$, $\mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$ is continuous if and only if there is a unitary matrix U and a real number θ such that $E_{k,p} = e^{i\theta} U E_{k,q} U^\dagger$ for all k. Furthermore, $\mathcal{H}_p \simeq \mathcal{H}_q$.

Proof. Assume that $\mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$ is continuous; that is, there is a matrix $A \in \mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$ such that $\mathcal{E}(A) = e^{i\theta} A$ for some real number θ. Let $V = e^{-i\theta} P_q + I - P_q$, where P_p is the projector onto $\mathcal{H}_p$, and $\mathcal{V} \circ \mathcal{E}(A) = A$ with $\mathcal{V}(\cdot) = V \cdot V^\dagger$. Moreover, it is obvious that $\mathcal{H}_p$ and $\mathcal{H}_q$ are also orthogonal minimal subspaces under $\mathcal{V} \circ \mathcal{E}$ by the decomposition Eq.(1). Therefore, there is a stationary coherence from $\mathcal{H}_p$ to $\mathcal{H}_q$ under $\mathcal{V} \circ \mathcal{E}$. From Lemma 1 and the minimal decomposition of $\mathcal{V} \circ \mathcal{E}$, we have $\mathcal{H}_p \simeq \mathcal{H}_q$, and there exists some unitary matrix U such that that for any k,

$$E_{k,p} = e^{i\theta} U E_{k,q} U^\dagger.$$

Conversely, for any p and q let

$$\mathcal{E}_{p,q}(\cdot) = \sum_k E_{k,p} \cdot E_{k,q}^\dagger = \sum_k e^{i\theta} U E_{k,q} U^\dagger \cdot E_{k,q}^\dagger.$$

Its matrix representation [20] reads

$$\begin{aligned} M_{p,q} &= \sum_k e^{i\theta} U E_{k,q} U^\dagger \otimes E_{k,q}^* \\ &= e^{i\theta} (U \otimes I) \left(\sum_k E_{k,q} \otimes E_{k,q}^* \right) (U^\dagger \otimes I) \end{aligned}$$

As $\lambda(M_{p,q}) = \lambda(\mathcal{E}_{p,q})$ and $\sum_k E_{k,q} \otimes E_{k,q}^*$ is the matrix representation of $\mathcal{E}_{q,q}$ which is a super-operator and has 1 as one of its eigenvalues, we have $e^{i\theta} \in \lambda(\mathcal{E}_{p,q})$. □

Corollary 1: Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$ with the orthogonal decomposition presented in Eq. (1). Then the relation

$$\{(p,q) : 1 \leq p, q \leq m, \mathcal{L}(\mathcal{H}_p, \mathcal{H}_q) \text{ is continuous}\}$$

is an equivalence relation. That is, for any p, q, and r,

(1) (reflexivity) $\mathcal{L}(\mathcal{H}_p, \mathcal{H}_p)$ is continuous;
(2) (symmetry) if $\mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$ is continuous, then so is $\mathcal{L}(\mathcal{H}_q, \mathcal{H}_p)$;
(3) (transitivity) if $\mathcal{L}(\mathcal{H}_p, \mathcal{H}_q)$ and $\mathcal{L}(\mathcal{H}_q, \mathcal{H}_r)$ are both continuous, then so is $\mathcal{L}(\mathcal{H}_p, \mathcal{H}_r)$.

With Corollary 1, we can group together minimal subspaces by continuous coherences and obtain a structure of $fix(\mathcal{E})$, in a similar way to Theorem 1 for stationary coherences.

Theorem 4: Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$. There is a unique orthogonal decomposition of $\mathcal{H}$

$$\mathcal{H} = \bigoplus_{l=1}^{L} \mathcal{X}_l \oplus \mathcal{K}. \tag{11}$$

where

(1) $\mathcal{K}$ is the largest transient subspace;
(2) each $\mathcal{X}_l$ is either a minimal subspace or can be further decomposed into mutually orthogonal minimal subspaces with continuous coherences between any two of them:

$$\mathcal{X}_l = \bigoplus_{p=1}^{m_l} \mathcal{B}_{l,p} = \mathbb{C}^{m_l} \otimes \mathcal{B}_l, \quad \mathcal{B}_l \simeq \mathcal{B}_{l,p} \quad \forall p \tag{12}$$

such that the Kraus operators $\{E_k\}$ of $\mathcal{E}$ have a corresponding block form:

$$E_k = \left[\begin{array}{ccc|c} U_1 \otimes E_{k,1} & & & \\ & \ddots & & T_k \\ & & U_L \otimes E_{k,L} & \\ \hline & 0 & & K_k \end{array} \right] \tag{13}$$

for some operators $E_{k,l} \in \mathcal{L}(\mathcal{B}_l)$, $K_k \in \mathcal{L}(\mathcal{K})$, $T_k \in \mathcal{L}(\mathcal{K}, \mathcal{K}^\perp)$, and unitary matrix $U_l = diag(e^{i\theta_{l,1}}, \cdots, e^{i\theta_{l,m_l}})$ for some real numbers $\{\theta_{l,p}\}_{p=1}^{m_l}$ on $\mathbb{C}^{m_l}$. Moreover, $\mathcal{B}_l$ is irreducible under $\mathcal{E}_l = \sum_k E_{k,l} \cdot E_{k,l}^\dagger$. Furthermore,

$$fix(\mathcal{E}) = \bigoplus_l [fix(\mathcal{U}_l) \otimes \rho_l] \oplus 0_{\mathcal{K}}$$

where $\mathcal{U}_l(\cdot) = U_l \cdot U_l^\dagger$.

(3) there is no continuous coherence between any minimal subspaces $\mathcal{B}_{l,p}$ and $\mathcal{B}_{l',p'}$ whenever $l \neq l'$.

Proof. By Theorem 1, there is a unique orthogonal decomposition of $\mathcal{H}$ as

$$\mathcal{H} = \bigoplus_{l=1}^{L'} \mathcal{X}'_l \oplus \mathcal{K}$$

such that for any orthogonal minimal subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$, they have stationary coherences if and only if $\mathcal{H}_1 \oplus \mathcal{H}_2 \in \mathcal{X}'_l$ for some l. Then we divide $\{\mathcal{X}'_l\}$ into a finite number of disjoint subsets by continuous coherences; that is for any $l_1 \neq l_2$, if there is a continuous coherence between any minimal subspaces in $\mathcal{X}'_{l_1}$ and $\mathcal{X}'_{l_2}$, then they are in the same subset. This can be done as the existence of continuous coherences is an equivalence relation by Corollary 1. Then we define $\{\mathcal{X}_l\}_{l=1}^L$ to be the set of the direct sum of all elements in each subset. Therefore, $\mathcal{H}$ can be uniquely decomposed as $\mathcal{H} = \bigoplus_l \mathcal{X}_l \oplus \mathcal{K}$. Obviously, for any two orthogonal minimal subspaces $\mathcal{B}_{l_1} \in \mathcal{X}_{l_1}$ and $\mathcal{B}_{l_2} \in \mathcal{X}_{l_2}$, $\mathcal{L}(\mathcal{B}_{l_1}, \mathcal{B}_{l_2})$ is continuous if and only if $l_1 = l_2$.

Furthermore, for each l, $\mathcal{X}_l$ can be further decomposed to mutually orthogonal minimal subspaces:

$$\mathcal{X}_l = \bigoplus_{p=1}^{m_l} \mathcal{B}_{l,p}$$

By Lemma 2, in an appropriate decomposition of $\mathcal{X}_l = \bigoplus_{p=1}^{m_l} \mathcal{B}_{l,p} = \mathbb{C}^{m_l} \otimes \mathcal{B}_l$ and $\mathcal{B}_l \simeq \mathcal{B}_{l,p}$ for all p:

$$E_k = \left[\begin{array}{ccc|c} U_1 \otimes E_{k,1} & & & \\ & \ddots & & T_k \\ & & U_L \otimes E_{k,L} & \\ \hline & 0 & & K_k \end{array}\right] \tag{14}$$

and $\mathcal{B}_l$ is irreducible under $\mathcal{E}_l := \sum_k E_{k,l} \cdot E_{k,l}^\dagger$ for all l, where $U_l = diag(e^{i\theta_{l,1}}, \cdots, e^{i\theta_{l,m_l}})$ for a set of real numbers $\{\theta_{l,p}\}_{p=1}^{m_l}$. From Theorem 1 and noting that stationary coherences is continuous, we have

$$fix(\mathcal{E}) = \bigoplus_l [fix(\mathcal{U}_l) \otimes \rho_l] \oplus 0_{\mathcal{K}}.$$

where ρ_l is the unique stationary state of $\mathcal{E}_l$. □

Corollary 2: Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$ with the unique decomposition:

$$\mathcal{H} = \bigoplus_l (\mathbb{C}^{m_l} \otimes \mathcal{B}_l) \oplus \mathcal{K},$$

presented in Theorem 4. For any minimal subspace $\mathcal{H}'$, there is a pure state $|\psi\rangle \in \mathbb{C}^{m_l}$ for some l such that $\mathcal{H}' = \text{supp}(|\psi\rangle\langle\psi|) \otimes \mathcal{B}_l$.

The above theorem shows that minimal subspaces with continuous coherences can construct decoherence-free subsystem $\mathbb{C}^{m_l}$. Fortunately, we can further show that others are subspaces of the decoherence-free subsystems constructed in Eq. (12).

Theorem 5: Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$ with the unique decomposition:

$$\mathcal{H} = \bigoplus_l (\mathbb{C}^{m_l} \otimes \mathcal{B}_l) \oplus \mathcal{K},$$

presented in Theorem 4. Then subsystem $\mathcal{H}_A$ is decoherence-free if and only if

(1) $\mathcal{H}_A \subseteq \mathbb{C}^{m_l}$ for some l;
(2) $\mathcal{H}_A$ is a support of some stationary state of $\mathcal{U}_l(\cdot) = U_l \cdot U_l^\dagger$,

where U_l is the corresponding unitary matrix on $\mathbb{C}^{m_l}$ in the decomposition Eq.(13).

Proof. Assume that $\mathcal{H}_A$ is decoherence-free. By Theorem 4 and Corollary 2, $\mathcal{H}_A \subseteq \mathbb{C}^{m_l}$ for some l. From the definition of decoherence-free subsystems and the restriction of $\mathcal{E}$ onto $\mathbb{C}^{m_l}$ being $\mathcal{U}_l$, $\mathcal{H}_A$ is a decoherence-free subspace under $\mathcal{U}_l$ and $\mathcal{U}(P_A) = P_A$, where P_A be the projector onto $\mathcal{H}_A$.

To prove the other direction, we observe that if $\mathcal{H}_A$ is a support of some stationary state of $\mathcal{U}_l$, then $P_A U_l P_A = P_A U_l = U_l P_A$. Thus $\mathcal{H}_A$ is a decoherence-free subspace under $\mathcal{U}_l$. The rest of the proof is direct from Theorem 4. □

This theorem confirms that the set of decoherence-free subsystems $\{\mathbb{C}^{m_l}\}_l$ identified in Theorem 4 is optimal; that is any other decoherence-free subsystem is a subspace of one of them. So we only need to implement the decompositions in Theorem 4 and all decoherence-free subsystems can be easily found by Theorem 5.

One easy way of achieving this is to first transform all continuous coherences to stationary ones, and then use NSDecompose($\mathcal{H}, \mathcal{E}$) proposed already in the literature.

For any two operators $E_{k,p}$ and $E_{k,q}$ in Eq. (13) of Theorem 4, if they are unitarily equivalent with a phase θ, i.e. $E_{k,p} \simeq e^{i\theta} E_{k,q}$, then let $E'_{k,q} = e^{i\theta} E_{k,q}$ and $E'_{k,p} = E_{k,p}$. Then $E'_{k,p}$ is unitarily equivalent to $E'_{k,q}$; a continuous coherence is transformed to be stationary. Using this method, we develop Algorithm 1 to implement decompositions in Theorem 4. Its time complexity is $O(n^8)$, where $\dim(\mathcal{H}) = n$.

Algorithm 1 Decompose($\mathcal{H}, \mathcal{E}$)

Input: A Hilbert space $\mathcal{H}$ and a super-operator $\mathcal{E}$ with Kraus operators $\{E_k\}_{k=1}^d$ on it
Output: The two-level decomposition of $\mathcal{H}$ in the form of Eqs.(11) and (12).

$\{\mathbb{C}^{m_l}\}_{l=1}^L, \{\mathcal{B}_l\}_{l=1}^L, \mathcal{K} \leftarrow$ NSDecompose($\mathcal{H}, \mathcal{E}$)
$\left\{\begin{bmatrix} \bigoplus_{l=1}^L I_l \otimes E_{k,l} & T_k \\ 0 & K_k \end{bmatrix}\right\}_{k=1}^d \leftarrow$ NSDecompose($\mathcal{H}, \mathcal{E}$)
$\mathcal{L} \leftarrow \{1, 2, \cdots, L\}$
for each $p \leftarrow 1$ to L **do**
  if $p \in \mathcal{L}$ **then**
    for each $q \leftarrow p+1$ to L with $q \in \mathcal{L}$ **do**
      $M \leftarrow \sum_k E_{k,p} \otimes E_{k,q}^*$
      if $\lambda(M)$ has one element with magnitude one **then**
        $\eta \leftarrow \text{tr}(E_{k,p})/\text{tr}(E_{k,q})$
        $E_{k,q} \leftarrow \eta E_{k,q}$
        $\mathcal{L} \leftarrow \mathcal{L} \setminus \{q\}$
      end if
    end for
  end if
end for
for each $k \leftarrow 1$ to d **do**
  $E_k \leftarrow \begin{bmatrix} \bigoplus_{l=1}^L I_l \otimes E_{k,l} & T_k \\ 0 & K_k \end{bmatrix}$
end for
return NSDecompose($\mathcal{H}, \{E_k\}_{k=1}^d$)

Now we return back to see Example 1. By Algorithm 1, we can confirm that the first subsystem $\mathcal{H}_A \simeq \mathbb{C}^4$ is decoherence-free and further show that the evolution on it is $|0\rangle\langle 0| + |1\rangle\langle 1| - |2\rangle\langle 2| - |3\rangle\langle 3|$. Thus we can store 4-qubit information in this subsystem, which doubles the capacity of noiseless subsystems.

V. Imperfect Initialization

In the last section, we have developed techniques to find all decoherence-free subsystems under a perfect initialization. In this section, we plan to find counterparts under an imperfect initialization. These subsystems are more useful in experiments.

In a perfect initialization, the co-subsystem of a decoherence-free subsystem is not important, but it plays an essential role when we allow an imperfect initialization. Choosing an appropriate co-subsystem for a candidate of decoherence-free subsystems is a natural problem.

Theorem 6: Assume an imperfect initialization. Let $\mathcal{E}$ be a super-operator on $\mathcal{H}$ with the unique decomposition:

$$\mathcal{H} = \bigoplus_l (\mathbb{C}^{m_l} \otimes \mathcal{B}_l) \oplus \mathcal{K},$$

presented in Theorem 4. For any l and $\mathcal{H}_A \subseteq \mathbb{C}^{m_l}$, if $\mathcal{H}_A$ with co-subsystem $\mathcal{B}_l$ is not decoherence-free, then there are no non-trial co-subsystems for $\mathcal{H}_A$ such that $\mathcal{H}_A$ is decoherence-free, where $\mathcal{H}_B$ a trial co-subsystem if $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$.

Proof. It is easy to observe that any co-subsystem $\mathcal{H}_B$ (which can not to be irreducible) of $\mathcal{H}_A$ must satisfy $\mathcal{B}_l \subseteq \mathcal{H}_B$ from Corollary 2.

Obviously, if $\mathcal{H}_A$ is decoherence-free in $\mathcal{H} = (\mathcal{H}_A \otimes \mathcal{H}_B) \oplus \mathcal{Y}$ with $\mathcal{Y} = (\mathcal{H}_A \otimes \mathcal{H}_B)^\perp \neq \emptyset$, then $\mathcal{H}_A$ is also decoherence-free in $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{B}_l \oplus \mathcal{K}'$, as $\mathcal{H}_A \otimes \mathcal{B}_l \subseteq \mathcal{H}_A \otimes \mathcal{H}_B$. □

The above theorem tells us a fact that the co-subsystems identified in Theorem 4 are sufficient to determine whether a subsystem is decoherence-free or not under imperfect initialization. For example, in Example 1, we can get a decoherence-free subsystem $\mathcal{H}_A$ and the co-subsystem $\mathcal{H}'$ under a perfect initialization from Algorithm 1. Then through Theorem 6, we can use the co-subsystem $\mathcal{H}'$ to verify that $\mathcal{H}_A$ is decoherence-free only with trial co-subsystem $\mathcal{H}_B$ under an imperfect initialization.

VI. An application to Matrix Product States

The techniques for describing quantum many body systems are usually not scalable due to the exponential growth of the Hilbert space dimension with the number of subsystems. Matrix Product States (MPS), a special case of tensor networks (a theoretical and numerical tool describing quantum many-body systems), have proven to be a useful family of quantum states for the description of ground states of one-dimensional quantum many-body systems [14].

Given a tensor $\mathcal{A} = \{A_k \in \mathcal{M}_D\}_{k=1}^d$ with a Hilbert space $\mathcal{H}_d = \text{lin.span}\{|k\rangle\}_{k=1}^d$, where $\mathcal{M}_D$ denotes $D \times D$ complex matrices, it generates a family of translationally invariant MPS, namely

$$V(\mathcal{A}) = \{|V_n(\mathcal{A})\rangle\}_{n\in\mathbb{N}^+},$$

where

$$|V_n(\mathcal{A})\rangle = \sum_{k_1,\cdots,k_n=1}^{d} \text{tr}(A_{k_1}\cdots A_{k_n})|k_1\cdots k_n\rangle \in \mathcal{H}_d^{\otimes n}$$

Here, each $|V_n(\mathcal{A})\rangle$ corresponds to a state of n spins of physical dimension d. Then we can define an associated completely positive map $\mathcal{E}_\mathcal{A}(\cdot) = \sum_{k=1}^d A_k \cdot A_k^\dagger$.

By [15], we can always find a set of irreducible tensors $\{\mathcal{A}_j\}_{j=1}^m$ with the same Hilbert space $\mathcal{H}_d$, and a set of complex number $\{\mu_j\}_{j=1}^m$ such that for any $n \in \mathbb{N}^+$

$$|V_n(\mathcal{A})\rangle = \sum_{j=1}^{m} \mu_j^n |V_n(\mathcal{A}_j)\rangle \tag{15}$$

where a tenor is called irreducible if the associated map is CPTP and irreducible. That is, for any tensor $\mathcal{A}$, the generated MPS can be linearly represented by MPS of a set of irreducible tensors. Therefore, studying irreducible tensors, especially identifying the conditions allowing two tensors describing the same family of MPS, is an interesting problem.

First, we can group irreducible tensors that are essentially the same in the following sense.

Definition 3 ([15]): We say that two irreducible tensors with the same Hilbert space $\mathcal{H}_d$, say $\mathcal{A} = \{A_k\}_{k=1}^d$ and $\mathcal{B} = \{B_k\}_{k=1}^d$, are repeated if there exist a phase θ and a unitary matrix U so that

$$A_k = e^{i\theta} U B_k U^\dagger \ \forall k$$

By the definition, $\mathcal{A}$ and $\mathcal{B}$ are repeated, and then $|V_n(\mathcal{A})\rangle = e^{in\theta}|V_n(\mathcal{B})\rangle$ for all $n \in \mathbb{N}^+$. Therefore, for any tensor $\mathcal{A}$, we can reduce the set of irreducible tensors in Eq. (15) to be non-repeated. Such a simplified set is called a basis of $\mathcal{A}$.

Many interesting results obtained for MPS rely on the basis. The fundamental problem of MPS is to relate different tensors rising the same MPS: for any two different tensors $\mathcal{A}$ and $\mathcal{B}$, they can generate the same MPS, i.e. $V(\mathcal{A}) = V(\mathcal{B})$, which introduces an ambiguity for analyzing many-body states by MPS generated by tensors. This problem can be answered by the basis. That is, if $\mathcal{A}$ and $\mathcal{B}$ have the same MPS, then their basis must be related by a unitary transform [15]. Therefore, determining whether two irreducible tensors are repeated or not is a key problem. Even though such repeatability relation can be verified by Jordan decomposition of matrices by the definition, Jordan decomposition is sensitive to errors and should be avoided in numerical analysis. Here we propose a feasible method to achieve this by the results of continuous coherences in previous sections.

Theorem 7: Given two irreducible tensors with the same Hilbert space $\mathcal{H}_d$, $\mathcal{A} = \{A_k\}_{k=1}^d$ and $\mathcal{B} = \{B_k\}_{k=1}^d$. Then they are repeated if and only if $\lambda(\mathcal{E}_{\mathcal{A},\mathcal{B}})$ has an element with magnitude one, where $\mathcal{E}_{\mathcal{A},\mathcal{B}} = \sum_{k=1}^d A_k \cdot B_k^\dagger$.

Proof. Let $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ be the corresponding Hilbert spaces of tensors $\mathcal{A}$ and $\mathcal{B}$, respectively; that is, $A_k \in \mathcal{L}(\mathcal{H}_\mathcal{A})$ and $B_k \in \mathcal{L}(\mathcal{H}_\mathcal{B})$ for all k. Then the Hilbert space of $\mathcal{E}$ is $\mathcal{H}_\mathcal{A} \oplus \mathcal{H}_\mathcal{B}$, where $\mathcal{E}$ is a super-operator with Kraus operators $\{diag(A_k, B_k)\}_{k=1}^d$. Obviously, $\mathcal{H}_\mathcal{A}$ and $\mathcal{H}_\mathcal{B}$ are both minimal subspaces under $\mathcal{E}$. Then the result is direct from Lemma 2. □

By the above theorem, repeatability can be easily checked by computing the eigenvalues of a linear map, which is a linear algebra exercise.

VII. Conclusion

In this paper, we established a structure theory for decoherence-free subsystems. Consequently, an algorithm for finding an optimal set of decoherence-free subsystems under a perfect initialization has been developed. Then we gave a simple way to pick up the faulty-tolerant decoherence-free subsystems. After that, these results helped us find a basis for any tensor by computing the eigenvalues of some constructed linear maps.

For future studies, an immediate topic is to generalize our results to continuous-time quantum systems. In [21], it was studied in the quantum control setting and expected to obtain a linear-algebraic approach for finding all decoherence-free subsystems for a given generator, describing the system.

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