# Financial Mathematics for Actuaries (Third Edition)

## Chapter 8

## Bond Management

# Learning Objectives

1. Macaulay duration and modified duration

2. Duration and interest-rate sensitivity

3. Convexity

4. Some rules for duration calculation

5. Asset-liability matching and immunization strategies

6. Target-date immunization and duration matching

7. Redington immunization and full immunization

8. Cases of nonflat term structure

# 8.1 Macaulay Duration and Modified Duration

- Suppose an investor purchases a $n$-year semiannual coupon bond for $P_0$ at time 0 and holds it until maturity.

- As the amounts of the payments she receives are different at different times, one way to summarize the horizon is to consider the *weighted average* of the time of the cash flows.

- We use the present values of the cash flows (not their nominal values) to compute the weights.

- Consider an investment that generates cash flows of amount $C_t$ at time $t = 1, \cdots, n$, measured in payment periods. Suppose the rate of interest is $i$ per payment period and the initial investment is $P$.

- We denote the present value of $C_t$ by $\text{PV}(C_t)$, which is given by

$$\text{PV}(C_t) = \frac{C_t}{(1+i)^t}. \tag{8.1}$$

and we have

$$P = \sum_{t=1}^{n} \text{PV}(C_t). \tag{8.2}$$

- Using $\text{PV}(C_t)$ as the factor of proportion, we define the weighted average of the time of the cash flows, denoted by $D$, as

$$D = \sum_{t=1}^{n} t \left[ \frac{\text{PV}(C_t)}{P} \right]$$

$$= \sum_{t=1}^{n} t w_t, \tag{8.3}$$

where

$$w_t = \frac{\text{PV}(C_t)}{P}. \tag{8.4}$$

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- As $w_t \geq 0$ for all $t$ and $\sum_{t=1}^{n} w_t = 1$, $w_t$ are properly defined weights and $D$ is the weighted average of $t = 1, \cdots, n$.

- We call $D$ the **Macaulay duration**, which measures the *average period* of the investment.

- The value computed from (8.3) gives the Macaulay duration in terms of the *number of payment periods.*

- If there are $k$ payments per year and we desire to express the duration in years, we replace $t$ in (8.3) by $t/k$. The resulting value of $D$ is then the Macaulay duration in years.

**Example 8.1:** Calculate the Macaulay duration of a 4-year annual coupon bond with 6% coupon and a yield to maturity of 5.5%.

**Solution:**    The present values of the cash flows can be calculated using (8.1) with $i = 5.5\%$. The computation of the Macaulay duration is presented in Table 8.1.

**Table 8.1:**    Computation for Example 8.1

| $t$ | $C_t$ | $\text{PV}(C_t)$ | $w_t$ | $tw_t$ |
|-----|-------|------------------|-------|--------|
| 1 | 6 | 5.6872 | 0.0559 | 0.0559 |
| 2 | 6 | 5.3907 | 0.0530 | 0.1060 |
| 3 | 6 | 5.1097 | 0.0502 | 0.1506 |
| 4 | 106 | 85.5650 | 0.8409 | 3.3636 |
| Total | | 101.7526 | 1.0000 | 3.6761 |

The price of the bond $P$ is equal to the sum of the third column, namely 101.7526. Note that the entries in the fourth column are all positive and sum up to 1. The Macaulay duration is the sum of the last column, which

is 3.6761 years. Thus, the Macaulay duration of the bond is less than its time to maturity of 4 years. □

**Example 8.2:** Calculate the Macaulay duration of a 2-year semiannual coupon bond with 4% coupon per annum and a yield to maturity of 4.8% compounded semiannually.

**Solution:** The cash flows of the bond occur at time 1, 2, 3 and 4 half-years. The present values of the cash flows can be calculated using (8.1) with $i = 2.4\%$ per payment period (i.e., half-year). The computation of the Macaulay duration is presented in Table 8.2.

**Table 8.2:**    Computation for Example 8.2

| $t$ | $C_t$ | PV($C_t$) | $w_t$ | $tw_t$ |
|---|---|---|---|---|
| 1 | 2 | 1.953 | 0.0198 | 0.0198 |
| 2 | 2 | 1.907 | 0.0194 | 0.0388 |
| 3 | 2 | 1.863 | 0.0189 | 0.0568 |
| 4 | 102 | 92.768 | 0.9419 | 3.7676 |
| Total | | 98.491 | 1.0000 | 3.8830 |

The price of the bond is equal to the sum of the third column, namely 98.491. The Macaulay duration is the sum of the last column, namely 3.8830 half-years, which again is less than the time to maturity of the bond of 4 half-years. The Macaulay duration of the bond can also be stated as $3.8830/2 = 1.9415$ years.    □

- Consider a bond with face value (also the redemption value) $F$, coupon rate $r$ per payment, and time to maturity of $n$ payment periods. The rate of interest $i$ applicable to (8.3) is the yield to maturity per coupon-payment period.

- Now $C_t$ is equal to $Fr$ for $t = 1, \cdots, n-1$ and $C_n = Fr + F$. Thus, from (8.3) we have

$$
\begin{aligned}
D &= \frac{1}{P} \left[ \sum_{t=1}^{n} \frac{tFr}{(1+i)^t} + \frac{nF}{(1+i)^n} \right] \\
&= \frac{1}{P} \left[ Fr \sum_{t=1}^{n} \text{PV}(t) + Fnv^n \right].
\end{aligned}
\tag{8.5}
$$

- From (6.1) we have $P = (Fr)a_{\overline{n}|} + Fv^n$. Hence, the Macaulay

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duration of the bond is (in terms of the number of payment periods)

$$D = \frac{Fr \sum_{t=1}^{n} \mathrm{PV}(t) + Fnv^n}{(Fr)a_{\overline{n}|} + Fv^n}$$

$$= \frac{r(Ia)_{\overline{n}|} + n\,v^n}{r\,a_{\overline{n}|} + v^n}. \tag{8.6}$$

**Example 8.3:** Calculate the Macaulay duration of the bonds in Examples 8.1 and 8.2 using equation (8.6).

**Solution:** For Example 8.1, $r = 6\%$, $i = 5.5\%$ and $n = 4$. Thus, $a_{\overline{4}|} = 3.5052$ and we use (2.36) to obtain $(Ia)_{\overline{4}|} = 8.5285$. Now using (8.6) we have

$$D = \frac{0.06 \times 8.5294 + 4(1.055)^{-4}}{0.06 \times 3.5052 + (1.055)^{-4}} = 3.6761 \text{ years},$$

which is the answer in Example 8.1. For Example 8.2, we have $r = 2\%$, $i = 2.4\%$ and $n = 4$. Thus, $a_{\overline{4}|} = 3.7711$ and $(Ia)_{\overline{4}|} = 9.3159$. Hence, we

have

$$D = \frac{0.02 \times 9.3159 + 4(1.024)^{-4}}{0.02 \times 3.7711 + (1.024)^{-4}} = 3.8830 \text{ half-years.}$$

$\square$

- We introduce the Macaulay duration in (8.3) assuming the cash flows occur at regular intervals. The formula can be extended to any fixed income security with cash flows occurring at irregular intervals.

- Let $i$ be the effective rate of interest per year, $t_j$ be the times of occurrence (in years) of the cash flows and $C_j$ be the amounts of cash flows, for $j = 1, \cdots, n$. The Macaulay duration is then defined as

$$D = \sum_{j=1}^{n} t_j \left[ \frac{\mathrm{PV}(C_j)}{P} \right] = \sum_{j=1}^{n} t_j \left[ \frac{C_j}{P(1+i)^{t_j}} \right] \qquad (8.7)$$

where

$$P = \sum_{j=1}^{n} \frac{C_j}{(1+i)^{t_j}}. \tag{8.8}$$

- While the Macaulay duration was originally proposed to measure the average horizon of an investment, it turns out that it can be used to measure the *price sensitivity* of the investment with respect to interest-rate changes.

- To measure this sensitivity we consider the derivative $dP/di$. As the price of the investment $P$ drops when interest rate $i$ increases, $dP/di < 0$.

- We consider (the negative of) the percentage change in the price of the investment per unit change in the rate of interest, i.e., $-(dP/di)/P$.

- Using (8.8), this quantity is given by

$$
\begin{aligned}
-\frac{1}{P}\frac{dP}{di} &= \frac{1}{P}\sum_{j=1}^{n}\frac{t_j C_j}{(1+i)^{t_j+1}} \\
&= \frac{1}{P(1+i)}\sum_{j=1}^{n}\frac{t_j C_j}{(1+i)^{t_j}} \\
&= \frac{1}{1+i}\sum_{j=1}^{n}t_j\left[\frac{\mathrm{PV}(C_j)}{P}\right] \\
&= \frac{D}{1+i}.
\end{aligned}
\tag{8.9}
$$

- We define

$$
D^* = \frac{D}{1+i},
\tag{8.10}
$$

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and call it the **modified duration**, which is always positive and measures the percentage decrease of the value of the investment per unit increase in the rate of interest.

**Example 8.4:** Calculate the modified duration of the bonds in Examples 8.1 and 8.2.

**Solution:** For Example 8.1, we have

$$D^* = \frac{3.6761}{1.055} = 3.4845 \text{ years.}$$

Thus, the bond drops in value by 3.4845% per 1 percentage point increase (*not* percentage increase) in interest rate per year. However, as the bond price and interest rate relationship is nonlinear, this statement is only correct approximately and applies to the current rate of interest of 5.5%.

For Example 8.2, we have $i = (1.024)^2 - 1 = 0.04858$ so that

$$D^* = \frac{1.9415}{1.04858} = 1.8515 \text{ years.}$$

Thus, the bond drops in value by 1.8516% per 1 percentage point increase in the annual effective rate of interest. $\square$

# 8.2 Duration for Price Correction

- We now consider the use of the modified duration to approximate the price change of a bond when the rate of interest changes.

- We denote $P(i)$ as the price of a bond at the annual effective rate of $i$.

- When the rate of interest changes to $i + \Delta i$, the bond price is revised to $P(i + \Delta i)$.

- While the bond price can be re-calculated at the rate of interest $i + \Delta i$ using one of the pricing formulas in Chapter 6, an approximation is available using the modified duration.

- For a continuous function $f(x)$ with first- and second-order derivatives, the function evaluated at $x + \Delta x$, i.e., $f(x + \Delta x)$, can be

approximated by Taylor's expansion as follows (see Appendix A.8):

$$f(x + \Delta x) \approx f(x) + \frac{df(x)}{dx} \Delta x + \frac{1}{2} \frac{d^2 f(x)}{dx^2} (\Delta x)^2. \qquad (8.11)$$

- Thus, if we expand the bond price $P(i+\Delta i)$ using Taylor's expansion up to the first-order derivative, we obtain

$$
\begin{aligned}
P(i + \Delta i) &\approx P(i) + \frac{dP(i)}{di} \Delta i \\
&= P(i) \left[ 1 - \left( -\frac{1}{P(i)} \frac{dP(i)}{di} \right) \Delta i \right] \\
&= P(i) \left( 1 - D^* \Delta i \right). \qquad (8.12)
\end{aligned}
$$

- Hence, we can use the modified duration $D^*$ to obtain a linear approximation to the revised bond price with respect to a change in the rate of interest.

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**Example 8.5:** A 10-year semiannual coupon bond with coupon rate of 7% is selling to yield 6.5% per year compounded semiannually. What is the bond price if the yield changes to (i) 5.5%, and (ii) 7.5%, compounded semiannually? You should use (a) the exact method, (b) the first-order linear approximation based on the modified duration, and (c) the first-order approximation based on the Macaulay duration.

**Solution:** The annual effective rates corresponding to 5.5%, 6.5% and 7.5% compounded semiannually are, respectively, 5.5756%, 6.6056% and 7.6406%. For (a), using the basic price formula (6.1), the exact prices at these rates are

$$P(0.055756) = 111.4204, \quad P(0.066056) = 103.6348, \quad P(0.076406) = 96.5259.$$

For (b) we compute the Macaulay duration using (8.6) to obtain $D =$

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7.4083 so that the modified duration is

$$D^* = \frac{7.4083}{1.066056} = 6.9492.$$

For (i) the annual effective rate changes by $\Delta i = -0.01030$, and for (ii) the change is $\Delta i = 0.01035$. Thus, the approximate prices using modified duration are:

$$P(0.055756) \approx 103.6348[1 - 6.9492(-0.01030)] = 111.0528 < 111.4204,$$

and

$$P(0.076406) \approx 103.6348[1 - 6.9492(0.01035)] = 96.1808 < 96.5259.$$

For (c), we use (8.13) to obtain

$$P(0.055756) \approx 103.6348 \left[ \frac{1.066056}{1.055756} \right]^{7.4083} = 111.3634 < 111.4204$$
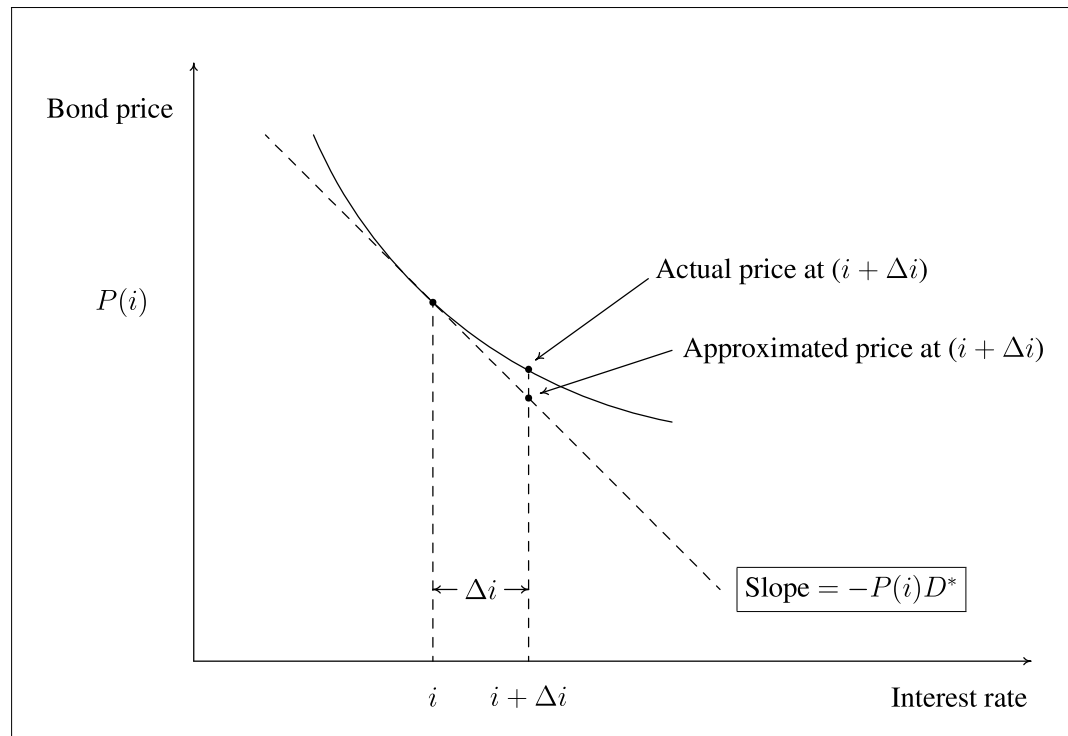
and

$$P(0.076406) \approx 103.6348 \left[ \frac{1.066056}{1.076406} \right]^{7.4083} = 96.4761 < 96.5259.$$

Results in (b) and (c) are quite close to the exact results obtained in (a), although they are less than the exact values in both cases. Also, the approximation (8.13) using the Macaulay duration provides better results.

$\square$

- Figure 8.2 below illustrates the application of (8.12). The relationship between the bond price and the rate of interest is given by the curve, which is *convex* to the origin.

# Figure 8.2: Bond-price approximation



- Equation (8.12) approximates the bond price using the straight line which is tangent to the point $(i, P(i))$ with a negative slope of $-P(i)D^*$.

- Note that due to the convexity of the relationship between the interest rate and the bond price, the correction based on the modified duration always under-approximates the exact price.

- This point is illustrated by Example 8.5. To improve the approximation, we may take into account the *convexity* of the relationship.

# 8.3 Convexity

- To obtain a better approximation for the bond price, we apply Taylor's expansion in (8.11) to the second order, giving

$$
\begin{aligned}
P(i + \Delta i) &\approx P(i) + \frac{dP(i)}{di}\Delta i + \frac{1}{2}\frac{d^2 P(i)}{di^2}(\Delta i)^2 \\
&= P(i)\left[1 - \left(-\frac{1}{P(i)}\frac{dP(i)}{di}\right)\Delta i + \frac{1}{2P(i)}\left(\frac{d^2 P(i)}{di^2}\right)(\Delta i)^2\right].
\end{aligned}
$$

$$(8.14)$$

- Now we define the **convexity of the bond** as

$$
C = \frac{1}{P(i)} \times \frac{d^2 P(i)}{di^2}, \tag{8.15}
$$

so that (8.14) becomes

$$P(i + \Delta i) \approx P(i) \left[ 1 - D^* \Delta i + \frac{1}{2} C (\Delta i)^2 \right]. \qquad (8.16)$$

- For a fixed income security with possibly payments at irregular intervals, we have

$$\frac{d^2 P(i)}{di^2} = \sum_{j=1}^{n} \frac{(t_j + 1) t_j C_j}{(1 + i)^{t_j + 2}}, \qquad (8.17)$$

- Hence, the convexity is

$$C = \frac{1}{P(i)} \times \frac{d^2 P(i)}{di^2}$$

$$= \frac{1}{P(i)(1 + i)^2} \sum_{j=1}^{n} (t_j + 1) t_j \, \mathrm{PV}(C_j). \qquad (8.18)$$

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- For a bond investment, $C_j \geq 0$ for all $t_j$, so that $C > 0$, verifying the convexity relationship in Figure 8.2. Thus, the correction term $\frac{C(\Delta i)^2}{2}$ in (8.16) is always positive, which compensates for the under-approximation in (8.12).

- Note that the definitions in (8.15) and (8.18) are sometimes referred to as the **modified convexity**. In contrast, some authors define the **Macaulay convexity** as

$$\frac{1}{P(i)} \sum_{j=1}^{n} \frac{t_j^2 C_j}{(1+i)^{t_j}}. \tag{8.19}$$

- We shall always use the term convexity to refer to the modified convexity, unless otherwise stated.

**Example 8.6:** Revisit Example 8.5(b) and approximate the bond prices with convexity correction.

**Solution:** We calculate the convexity using (8.18) to obtain

$$C = \frac{1}{(103.6348)(1.066056)^2} \left[ \frac{1.5(0.5)(3.5)}{1.0325} + \frac{2(1)(3.5)}{(1.0325)^2} + \cdots + \frac{11(10)(103.5)}{(1.0325)^{20}} \right]$$
$$= 73.2528.$$

Thus, the approximate bond prices are

$$P(0.055756) \approx 111.0528 + (103.6348)(0.5)(73.2528)(-0.01030)^2 = 111.4559,$$

and

$$P(0.076406) \approx 96.1808 + (103.6348)(0.5)(73.2528)(0.01035)^2 = 96.5874.$$

We can see that the approximation is now improved.

$\square$

# 8.4 Some Rules for Duration

- We summarize some useful rules for duration.

**Rule 1:** *The Macaulay duration D of a bond is always less than or equal to its time to maturity n. Equality holds only for a zero-coupon bond.*

**Rule 2:** *Holding the time to maturity n of a bond constant, when the coupon rate of interest r decreases, the Macaulay duration D increases.*

**Rule 3:** *Other things being equal, when the yield to maturity i decreases, the Macaulay duration D increases.*

**Rule 4:** *For a level perpetuity-immediate, the modified duration $D^*$ is equal to $1/i$.*

**Rule 5:** *For a level annuity-immediate of n payments, the modified duration is*

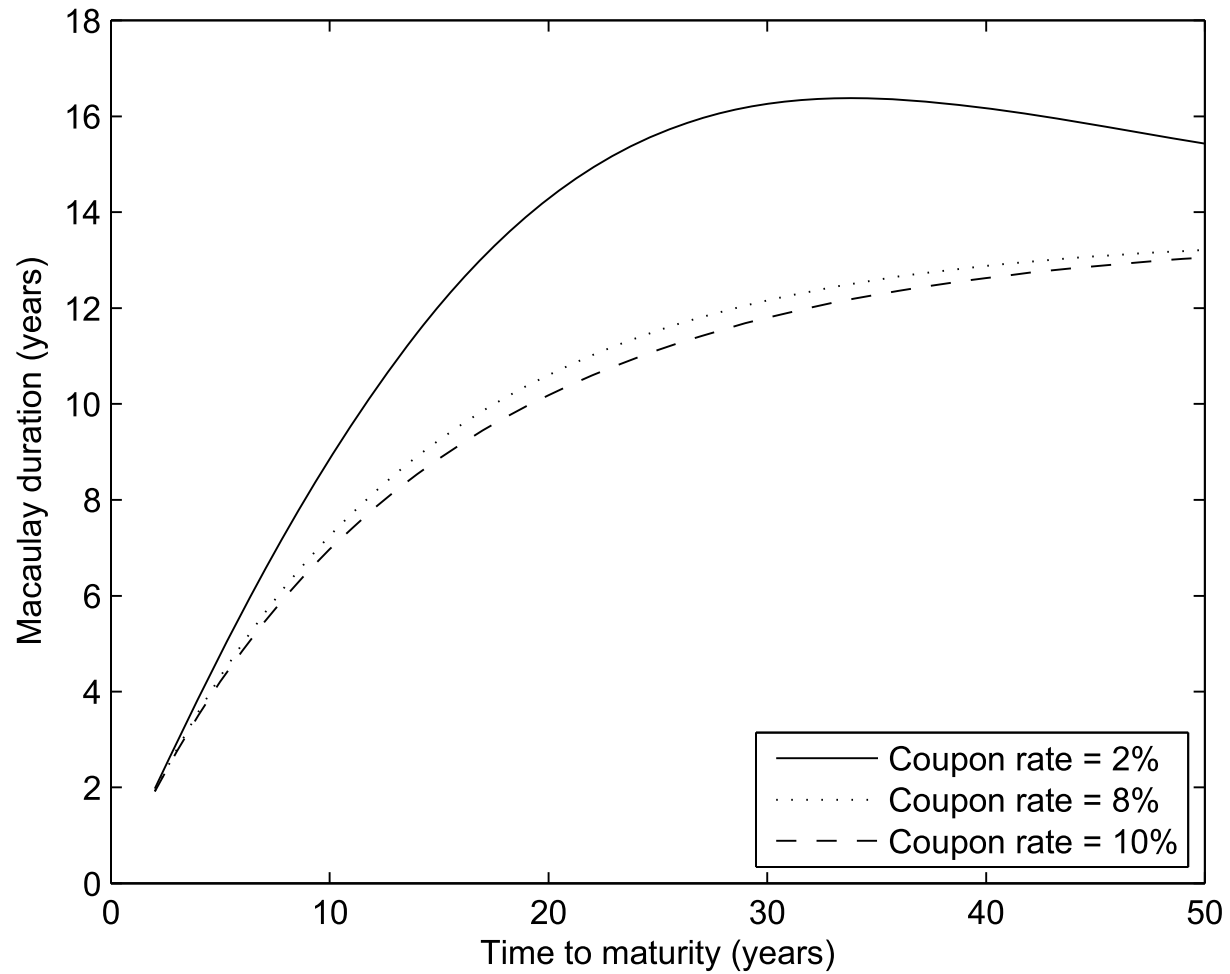$$D^* = \frac{1}{i} - \frac{n}{(1+i)\left[(1+i)^n - 1\right]}. \tag{8.20}$$

This rule can be proved by direct differentiation of the price of the annuity, which is

$$P(i) = a_{\overline{n}|} = \frac{1 - v^n}{i}.$$

**Rule 6:** *Holding other things constant, a bond's duration $D$ usually increases with its time to maturity $n$.*

- For premium and par bonds, the relationship is monotonic so that $D$ *always* increases with $n$. For deep-discount bonds, however, $D$ may increase with $n$ for bonds with short maturity and then decreases with increases in maturity. Figure 8.3 illustrates this phenomenon, where the prevailing yield curve is flat at 8%.

# Figure 8.3: Duration and time to maturity

- Suppose a portfolio of bonds is constructed from $M$ bonds, with durations $D_1, \cdots, D_M$.

- Let the bond values be $P_1, \cdots, P_M$, so that their total is $P = \sum_{j=1}^{M} P_j$.

- Define $w_j = P_j/P$ as the weight of Bond $j$ in the portfolio, then the duration $D_P$ of the portfolio is the weighted average of the bond durations, i.e.,

$$D_P = \sum_{j=1}^{M} w_j D_j. \qquad (8.21)$$

- This result is very useful for bond portfolio management when a portfolio with a certain duration is required.

**Example 8.7:** A bond manager has a choice of two bonds, A and B.

Bond A is a 4-year annual coupon bond with coupon rate of 6%. Bond B is a 2-year annual coupon bond with coupon rate of 4%. The current yield to maturity in the market is 5.5% per annum for all maturities. How does the manager construct a portfolio of $100 million, consisting of bonds A and B, with a Macaulay duration of 2.5 years?

**Solution:** From Example 8.1, we know that Bond A has a duration of 3.6761 years. We compute the duration of Bond B as 1.9610 years. Let $w$ be the proportion of investment in Bond A. Thus, from (8.19) we have

$$3.6761w + 1.9610(1 - w) = 2.5,$$

so that

$$w = \frac{2.5 - 1.9610}{3.6761 - 1.9610} = 31.43\%.$$

Hence the portfolio should consist of $31.43 million of Bond A and $68.57 million of Bond B. □

# 8.5 Immunization Strategies

- Financial institutions are often faced with the problem of meeting a liability of a given amount some time in the future.

- We consider a liability of amount $V$ to be paid $T$ periods later.

- A simple strategy to meet this obligation is to purchase a zero-coupon bond with face value $V$, which matures at time $T$.

- This strategy is called **cash-flow matching**.

- When cash-flow matching is adopted, the obligation is always met, even if there is fluctuation in the rate of interest.

- However, zero-coupon bonds of the required maturity may not be available in the market.

- **Immunization** is a strategy of managing a portfolio of assets such that the business is *immune* to interest-rate fluctuations.

- For the simple situation above, the **target-date immunization** strategy may be adopted.

- This involves holding a portfolio of bonds that will accumulate in value to $V$ at time $T$ at the current market rate of interest.

- The portfolio, however, should be constructed in such a way that its Macaulay duration $D$ is equal to the targeted date of the liability $T$.

- Suppose the current yield rate is $i$, the current value of the portfolio of bonds, denoted by $P(i)$, must be

$$P(i) = \frac{V}{(1 + i)^T}. \qquad (8.22)$$

- If the interest rate remains unchanged until time $T$, this bond portfolio will accumulate in value to $V$ at the maturity date of the liability.

- If interest rate increases, the bond portfolio will drop in value. However, the coupon payments will generate higher interest and compensate for this.

- On the other hand, if interest rate drops, the bond portfolio value goes up, with subsequent slow-down in accumulation of interest.

- Under either situation, as we shall see, the bond portfolio value will finally accumulate to $V$ at time $T$, provided the portfolio's Macaulay duration $D$ is equal to $T$.

- We consider the bond value for a *one-time small change* in the rate of interest.

- If interest rate changes to $i + \Delta i$ immediately after the purchase of the bond, the bond price becomes $P(i + \Delta i)$ which, at time $T$, accumulates to $P(i + \Delta i)(1 + i + \Delta i)^T$ if the rate of interest remains at $i + \Delta i$.

- We approximate $(1 + i + \Delta i)^T$ to the first order in $\Delta i$ to obtain (apply Taylor's expansion to $f(i) = (1 + i)^T$)

$$(1 + i + \Delta i)^T \approx (1 + i)^T + T(1 + i)^{T-1}\Delta i. \qquad (8.23)$$
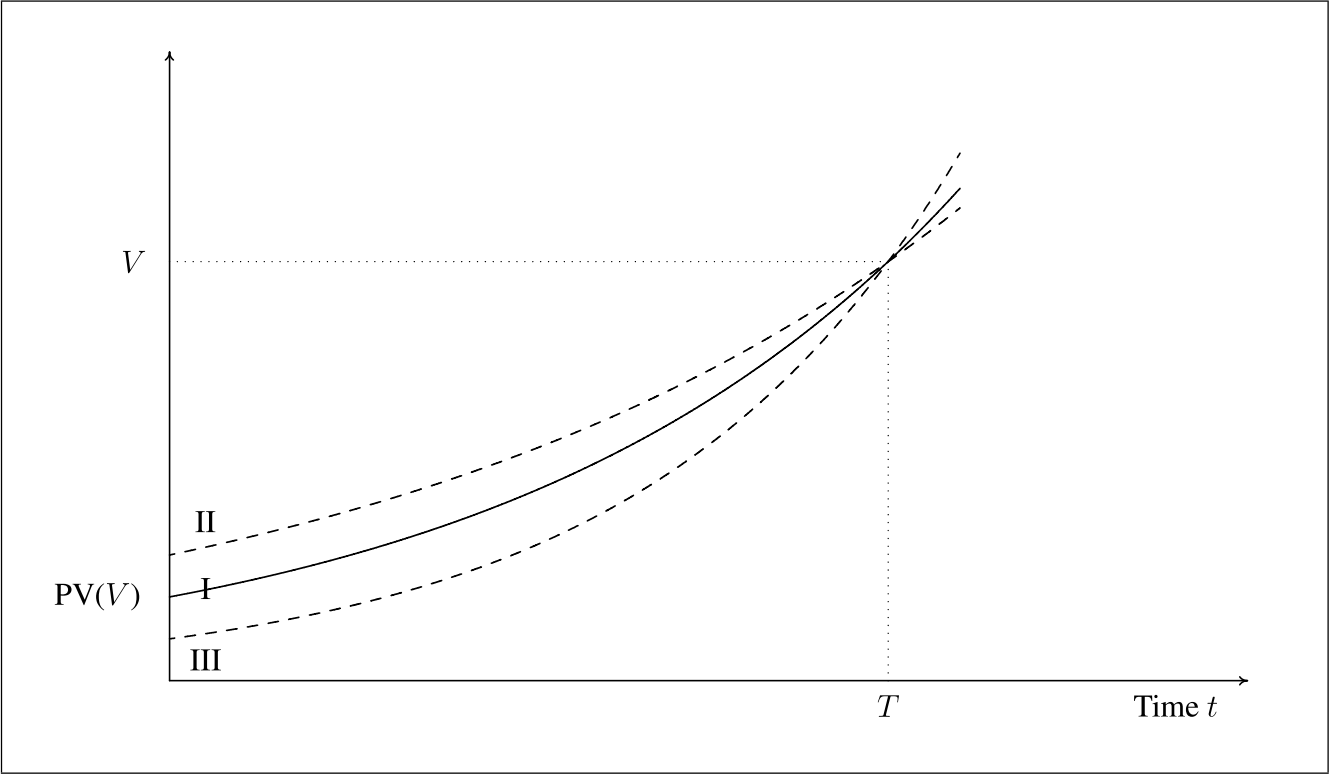
- Using (8.12) and (8.23) we have

$$P(i + \Delta i)(1 + i + \Delta i)^T \approx P(i)(1 - D^*\Delta i)\left[(1 + i)^T + T(1 + i)^{T-1}\Delta i\right].$$

- However, as $D^* = D/(1 + i)$ and $T = D$, the above equation becomes

$$P(i + \Delta i)(1 + i + \Delta i)^T \approx P(i) \left[ (1+i)^D - D^* \Delta i (1+i)^D + D(1+i)^{D-1} \Delta i \right]$$

$$= P(i)(1+i)^D$$

$$= V. \tag{8.24}$$

- The working of the target-date immunization strategy is illustrated by Figure 8.4 below.

# Figure 8.4: Target-date immunization

**Example 8.8:** A company has to pay \$100 million 3.6761 years from now. The current market rate of interest is 5.5%. Demonstrate the funding strategy the company should adopt with the 6% annual coupon bond in Example 8.1. Consider the scenarios when there is an immediate one-time change in interest rate to (a) 5%, and (b) 6%.

**Solution:** From equation (8.20), the current value of the bond should be

$$\frac{100}{(1.055)^{3.6761}} = \$82.1338 \text{ million}.$$

From Example 8.1, the bond price is 101.7526% of the face value and the Macaulay duration is 3.6761 years, which is the target date for the payment. Hence, the bond purchased should have a face value of

$$\frac{82.13375}{1.017526} = \$80.7191 \text{ million}.$$

At the end of year 3, the accumulated value of the coupon payments is

$$80.7191 \times 0.06 s_{\overline{3}|0.055} = \$15.3432 \text{ million},$$

and the bond price is (the bond will mature in 1 year with a 6% coupon payment and redemption payment of 80.7191)

$$\frac{80.7191 \times 0.06 + 80.7191}{1.055} = \$81.1017 \text{ million}.$$

Thus, the bond price plus the accumulated coupon values at time 3.6761 years is

$$(81.1017 + 15.3432)(1.055)^{0.6761} = \$100 \text{ million}.$$

Suppose interest rate drops to 5% immediately after the purchase of the bond, the accumulated coupon value 3 years later is

$$80.7191 \times 0.06 s_{\overline{3}|0.05} = \$15.2680 \text{ million},$$

and the bond price at year 3 is

$$\frac{80.7191(1.06)}{1.05} = \$81.4879 \text{ million.}$$

The total of the bond value and the accumulated coupon payments at time 3.6761 years is

$$(81.4879 + 15.2680)(1.05)^{0.6761} = \$100 \text{ million.}$$

On the other hand, if the interest rate increases to 6% immediately after the purchase of the bond, the accumulated coupon value 3 years later is

$$80.7191 \times 0.06 s_{\overline{3}|0.06} = \$15.4186 \text{ million,}$$

and the bond price at year 3 is 80.7191 (this is a par bond with yield rate equal to coupon rate). Thus, the total of the bond value and the accumulated coupon payments at time 3.6761 years is

$$(80.7191 + 15.4186)(1.06)^{0.6761} = \$100 \text{ million.}$$

Thus, for an immediate one-time small change in interest rate, the bond accumulates to the targeted value of $100 million at 3.6761 years, and the business is immunized. □

**Example 8.9:** A company has to pay $100 million 4 years from now. The current market rate of interest is 5.5%. The company uses the 6% annual coupon bond in Example 8.1 to fund this liability. Is the bond sufficient to meet the liability when there is an immediate one-time change in interest rate to (a) 5%, and (b) 6%?

**Solution:** As the target date of the liability is 4 years and the Macaulay duration of the bond is 3.6761 years, there is a mismatch in the durations and the business is not immunized. To fund the liability in 4 years, the value of the bond purchased at time 0 is

$$\frac{100}{(1.055)^4} = \$80.7217 \text{ million },$$

and the face value of the bond is

$$\frac{80.7217}{1.017526} = \$79.3313 \text{ million.}$$

If interest rate drops to 5%, the asset value at year 4 is

$$79.3313 \times 0.06 s_{\overline{4}|0.05} + 79.3313 = \$99.8470 \text{ million,}$$

so that the liability is under-funded. On the other hand, if the interest rate increases to 6%, the asset value at year 4 is

$$79.3313 \times 0.06 s_{\overline{4}|0.06} + 79.3313 = \$100.1539 \text{ million,}$$

so that the liability is over-funded. $\qquad\square$

- If a financial institution has multiple liability obligations to meet, the manager may adopt cash-flow matching to each obligation.

- This is a **dedication strategy** in which the manager selects a portfolio of bonds (zero-coupon or coupon bonds) to provide total cash flows in each period to match the required obligations.

- The manager may also consider the liability obligations as a whole and construct a portfolio to fund these obligations with the objective of controlling for the interest-rate risk. A commonly adopted strategy is **duration matching**.

- We assume a financial institution has a stream of liabilities $L_1, L_2, \cdots, L_N$ to be paid out at various times in the future.

- It will fund these liabilities with assets generating cash flows $A_1, A_2, \cdots, A_M$ at various times in the future.

- We assume that the rate of interest $i$ is flat for cash flows of all maturities and applies to both assets and liabilities.

- We denote

$$\text{PV(assets)} = \sum_{j=1}^{M} \text{PV}(A_j) = V_A, \qquad (8.25)$$

and

$$\text{PV(liabilities)} = \sum_{j=1}^{N} \text{PV}(L_j) = V_L. \qquad (8.26)$$

- We denote the Macaulay durations of the assets and liabilities by $D_A$ and $D_L$, respectively.

- The duration matching strategy involves constructing a portfolio of assets such that the following conditions hold:

1. $V_A \geq V_L$

2. $D_A = D_L$.

- Condition 2 ensures that, to the first-order approximation, the asset-liability ratio will not drop when interest rate changes. This result can be deduced as follows:

$$
\begin{aligned}
\frac{d}{di}\left(\frac{V_A}{V_L}\right) &= \frac{V_L \dfrac{dV_A}{di} - V_A \dfrac{dV_L}{di}}{V_L^2} \\
&= \frac{V_A}{V_L}\left(\frac{1}{V_A}\frac{dV_A}{di} - \frac{1}{V_L}\frac{dV_L}{di}\right) \\
&= \frac{V_A}{V_L(1+i)}(D_L - D_A) \\
&= 0.
\end{aligned}
\tag{8.27}
$$

**Example 8.10:** A financial institution has to pay \$1,000 after 2 years and \$2,000 after 4 years. The current market interest rate is 10%, and the

yield curve is assumed to be flat at any time. The institution wishes to immunize the interest rate risk by purchasing zero-coupon bonds which mature after 1, 3 and 5 years. One member in the risk management team of the institution, Alan, devised the following strategy:

- Purchase a 1-year zero-coupon bond with a face value of $44.74,

- Purchase a 3-year zero-coupon bond with a face value of $2,450.83,

- Purchase a 5-year zero-coupon bond with a face value of $500.00.

(a) Find the present value of the liability. (b) Show that Alan's portfolio satisfies the conditions of the duration matching strategy. (c) Define surplus $S = V_A - V_L$, calculate $S$ when there is an immediate one-time change of interest rate from 10% to (i) 9%, (ii) 11%, (iii) 15%, (iv) 30% and (v) 80%. (d) Find the convexity of the portfolio of assets and the portfolio of liabilities at $i = 10\%$.

**Solution:** (a) The present value of the liabilities is

$$V_L = 1{,}000 \, (1.1)^{-2} + 2{,}000 \, (1.1)^{-4} = \$2{,}192.47.$$

For (b), the present value of Alan's asset portfolio is

$$V_A = 44.74 \, (1.1)^{-1} + 2{,}450.83 \, (1.1)^{-3} + 500.00 \, (1.1)^{-5} = \$2{,}192.47.$$

The Macaulay duration of the assets and liabilities can be calculated using equation (8.3) to give

$$D_A = \left[ 1 \times 44.74 \, (1.1)^{-1} + 3 \times 2{,}450.83 \, (1.1)^{-3} + 5 \times 500.00 \, (1.1)^{-5} \right] / 2{,}192.47$$
$$= 3.2461 \text{ years},$$

$$D_L = \left[ 2 \times 1{,}000 \, (1.1)^{-2} + 4 \times 2{,}000 \, (1.1)^{-4} \right] / 2{,}192.47$$
$$= 3.2461 \text{ years}.$$

Since $V_A = V_L$ and $D_A = D_L$, the conditions of the duration matching strategy are met for Alan's portfolio.

For (c), when there is an immediate one-time shift in interest rate from 10% to 9%, using equation (8.2), we have

$$V_A = 44.74 \, (1.09)^{-1} + 2{,}450.83 \, (1.09)^{-3} + 500.00 \, (1.09)^{-5} = \$2{,}258.50,$$

$$V_L = 1{,}000 \, (1.09)^{-2} + 2{,}000 \, (1.09)^{-4} = \$2{,}258.53,$$

$$S = 2{,}258.50 - 2{,}258.53 = -\$0.03.$$

We repeat the above calculations for interest rate of 11%, 15%, 30% and 80%. The results are summarized as follows:

| $i$ | $V_A$ | $V_L$ | $S$ |
|------|-----------|-----------|---------|
| 0.09 | 2,258.50 | 2,258.53 | $-0.03$ |
| 0.10 | 2,192.47 | 2,192.47 | 0.00 |
| 0.11 | 2,129.05 | 2,129.08 | $-0.03$ |
| 0.15 | 1,898.95 | 1,899.65 | $-0.70$ |
| 0.30 | 1,284.61 | 1,291.97 | $-7.36$ |
| 0.80 | 471.55 | 499.16 | $-27.61$ |

For (d), using equation (8.15), the convexity of the assets is

$$C_A = \frac{2 \times 1 \times 44.74\,(1.1)^{-1} + 4 \times 3 \times 2{,}450.83\,(1.1)^{-3} + 6 \times 5 \times 500.00\,(1.1)^{-5}}{(1.1)^2 \times 2{,}192.47}$$

$$= 11.8706,$$

and the convexity of the liabilities is

$$C_L = \frac{[3 \times 2 \times 1{,}000\,(1.1)^{-2} + 5 \times 4 \times 2{,}000\,(1.1)^{-4}]}{[(1.1)^2 \times 2{,}192.47]}$$

$$= 12.1676.$$

□

- It should be noted that the duration matching strategy is based on the first-order approximation. To improve the strategy, we may take into account the convexity of the asset and liability portfolios.

- To protect the asset-liability ratio from dropping when interest rate changes, the **Redington immunization** strategy, named after the British actuary Frank Redington, imposes the following three conditions for constructing a portfolio of assets:

   1. $V_A \geq V_L$

   2. $D_A = D_L$

   3. $C_A \geq C_L$.

50

**Example 8.11:** For the financial institution in Example 8.10, a risk consultant, Alfred, recommended the following strategy:

- Purchase a 1-year zero-coupon bond with a face value of $154.16,

- Purchase a 3-year zero-coupon bond with a face value of $2,186.04,

- Purchase a 5-year zero-coupon bond with a face value of $660.18.

(a) Show that Alfred's portfolio satisfies the three conditions of the Redington immunization strategy. (b) Define surplus $S = V_A - V_L$, calculate $S$ when there is an immediate one-time change of interest rate from 10% to (i) 9%, (ii) 11%, (iii) 15%, (iv) 30% and (v) 80%.

**Solution:** (a) The present value of Alfred's asset portfolio is

$$V_A = 154.16 \, (1.1)^{-1} + 2{,}186.04 \, (1.1)^{-3} + 660.18 \, (1.1)^{-5} = \$2{,}192.47.$$

The Macaulay duration of the assets and liabilities can be calculated using equation (8.3) to give

$$D_A = \left[ 1 \times 154.16 \, (1.1)^{-1} + 3 \times 2{,}186.04 \, (1.1)^{-3} + 5 \times 660.18 \, (1.1)^{-5} \right] / 2{,}192.47$$

$$= 3.2461 \text{ years,}$$

$$D_L = \left[ 2 \times 1{,}000 \, (1.1)^{-2} + 4 \times 2{,}000 \, (1.1)^{-4} \right] / 2{,}192.47$$

$$= 3.2461 \text{ years.}$$

Furthermore, using equation (8.15), we get

$$C_A = \frac{2 \times 1 \times 154.16 \ (1.1)^{-1} + 4 \times 3 \times 2{,}186.04 \ (1.1)^{-3} + 6 \times 5 \times 660.18 \ (1.1)^{-5}}{(1.1)^2 \times 2{,}192.47}$$

$$= 12.1704,$$

$$C_L = \frac{\left[3 \times 2 \times 1{,}000 \ (1.1)^{-2} + 5 \times 4 \times 2{,}000 \ (1.1)^{-4}\right]}{\left[(1.1)^2 \times 2{,}192.47\right]}$$

$$= 12.1676.$$

Since $V_A = V_L$, $D_A = D_L$ and $C_A = C_L$, the conditions of the Redington immunization strategy are met for Alfred's strategy.

For (b), when there is an immediate one-time shift in interest rate from

10% to 9%, using equation (8.2), we have

$$V_A = 154.16\,(1.09)^{-1} + 2{,}186.04\,(1.09)^{-3} + 660.18\,(1.09)^{-5} = \$2{,}258.53,$$

$$V_L = 1{,}000\,(1.09)^{-2} + 2{,}000\,(1.09)^{-4} = \$2{,}258.53,$$

$$S = 2{,}258.53 - 2{,}258.53 = 0.$$

We repeat the above calculations for interest rate of 11%, 15%, 30% and 80%. The results are summarized as follows:

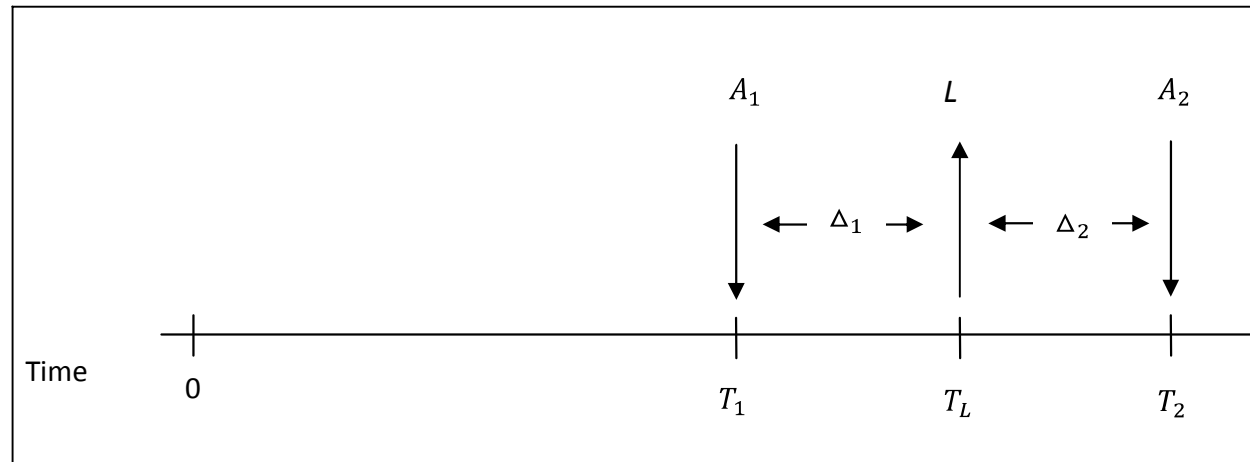| $i$ | $V_A$ | $V_L$ | $S$ |
|------|----------|----------|-------|
| 0.09 | 2,258.53 | 2,258.53 | 0.00 |
| 0.10 | 2,192.47 | 2,192.47 | 0.00 |
| 0.11 | 2,129.08 | 2,129.08 | 0.00 |
| 0.15 | 1,899.64 | 1,899.65 | −0.02 |
| 0.30 | 1,291.40 | 1,291.97 | −0.57 |
| 0.80 | 495.42 | 499.16 | −3.74 |

□

- Under certain conditions, it is possible to construct a portfolio of assets such that the net-worth position of the financial institution is guaranteed to be non-negative in any positive interest rate environment.

- A **full immunization** strategy is said to be achieved if under any one-time shift of interest rate from $i_0$ to $i$,

$$S(i) = V_A(i) - V_L(i) \geq 0, \quad \text{for } i > 0.$$

- We consider the example of a single liability of amount $L$ to be paid $T_L$ periods later.

- Full immunization strategy involves funding the liability by a portfolio of assets which will produce two cash inflows. The first inflow of amount $A_1$ is located at time $T_1$, which is $\Delta_1$ periods *before* time $T_L$. The second inflow of amount $A_2$ is at time $T_2$, which is $\Delta_2$ periods *after* time $T_L$.

- Figure 8.5 illustrates these three cashflows. It should be noted that all the values of $i_0, i, A_1, A_2, L, \Delta_1, \Delta_2, T_0, T_1$ and $T_2$ are positive, and $\Delta_1$ is not necessarily equal to $\Delta_2$.

# Figure 8.5: A full immunization strategy



- The conditions for constructing a portfolio of assets under the full immunization strategy are:

1. $V_A = V_L$

2. $D_A = D_L$.

- The above conditions can be rewritten as

1. $A_1(1 + i_0)^{-T_1} + A_2(1 + i_0)^{-T_2} = L(1 + i_0)^{-T_L}$

2. $T_1 A_1(1 + i_0)^{-T_1} + T_2 A_2(1 + i_0)^{-T_2} = T_L L(1 + i_0)^{-T_L}$.

**Example 8.12:** For the financial institution in Examples 8.10 and 8.11, an actuary, Albert, constructed the following strategy:

- Purchase a 1-year zero-coupon bond with a face value of $454.55,

- Purchase a 3-year zero-coupon bond with a face value of $1,459.09,

- Purchase a 5-year zero-coupon bond with a face value of $1,100.00.

(a) Show that Albert's portfolio satisfies the conditions of the full immunization strategy. (b) Define surplus $S = V_A - V_L$, calculate $S$ when there

is an immediate one-time change of interest rate from 10% to (i) 9%, (ii) 11%, (iii) 15%, (iv) 30% and (v) 80%.

**Solution:** (a) The present value of Albert's asset portfolio is

$$V_A = 454.55 \, (1.1)^{-1} + 1{,}459.09 \, (1.1)^{-3} + 1{,}100.00 \, (1.1)^{-5} = \$2{,}192.47.$$

Let $A_1$ and $A_2$ be the amount of 1-year and 3-year zero-coupon bonds that are needed to fully immunize the first liability of $L = 1{,}000$. Note that $T_1 = 1, T_L = 2$ and $T_2 = 3$. The two conditions for the full immunization strategy require

$$A_1(1.1)^{-1} + A_2(1.1)^{-3} = 1{,}000(1.1)^{-2}$$
$$(1)A_1(1.1)^{-1} + (3)A_2(1.1)^{-3} = (2)1{,}000(1.1)^{-2}.$$

Solving the above system of equations, we get $A_1 = 454.55$ and $A_2 = 550.00$. Next, let $A_1^*$ and $A_2^*$ be the amounts of 3-year and 5-year zero-coupon bonds that would be needed to fully immunize the second liability

$L = 2{,}000$. Note that now $T_1 = 3, T_L = 4$ and $T_2 = 5$. The two conditions for the full immunization strategy require

$$A_1^*(1.1)^{-3} + A_2^*(1.1)^{-5} = 2{,}000(1.1)^{-4}$$

$$(3)A_1^*(1.1)^{-3} + (5)A_2^*(1.1)^{-5} = (4)1{,}000(1.1)^{-4}.$$

Solving the above system of equations, we get $A_1^* = 909.09$ and $A_2^* = 1{,}100.00$. The combined asset portfolio consists of a 1-year zero-coupon bond with a face value of \$454.55, a 3-year zero-coupon bond with a face value of (\$550.00 + \$909.09) = \$1,459.09 and a 5-year zero-coupon bond with a face value of \$1,100.00. This is indeed Albert's asset portfolio, which satisfies the full immunization conditions.

For (b), when there is an immediate one-time shift in interest rate from

10% to 9%, using equation (8.2), we have

$$V_A = 454.55\,(1.09)^{-1} + 1{,}459.09\,(1.09)^{-3} + 1{,}100.00\,(1.09)^{-5} = \$2{,}258.62,$$

$$V_L = 1{,}000\,(1.09)^{-2} + 2{,}000\,(1.09)^{-4} = \$2{,}258.53,$$
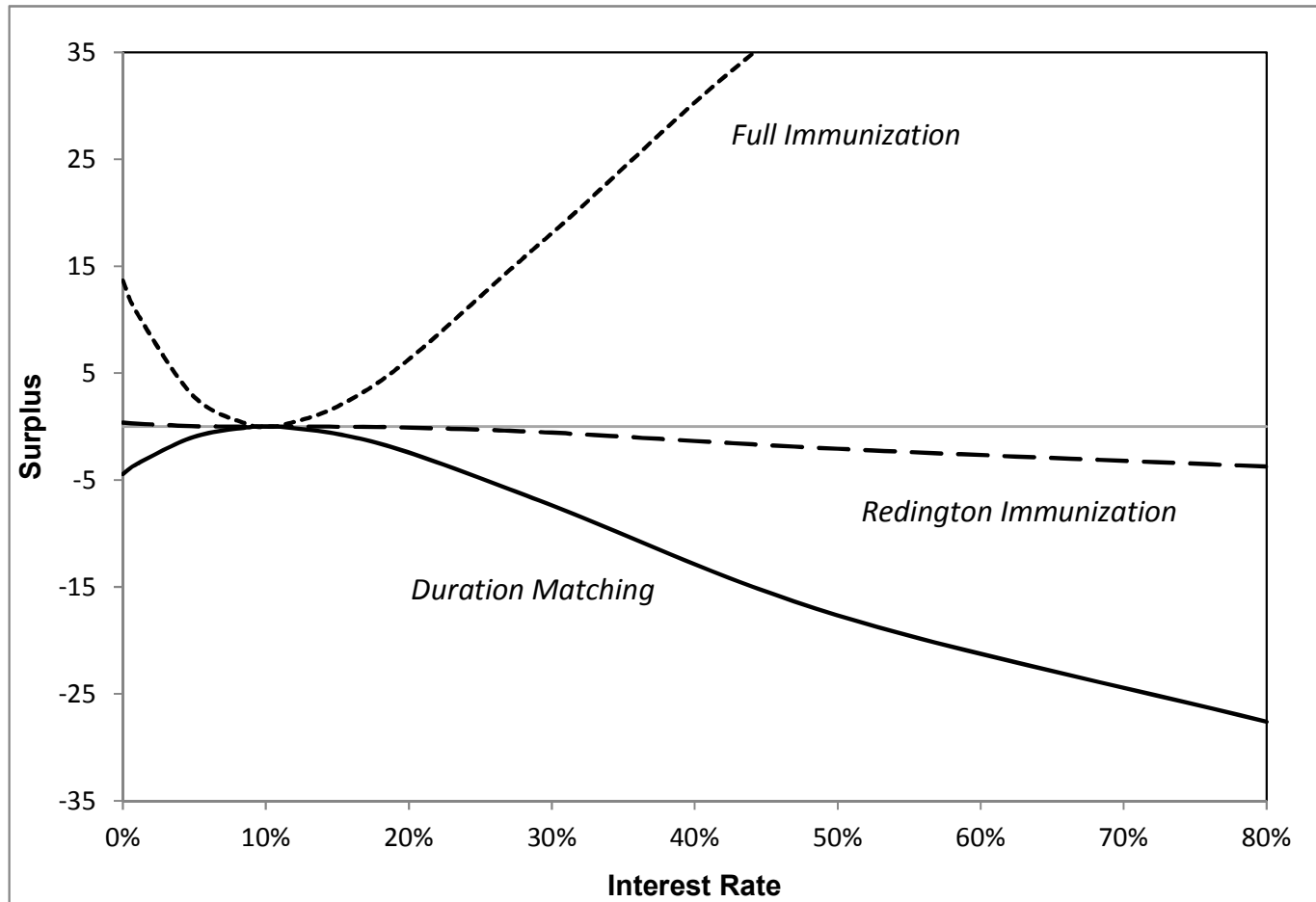
$$S = 2{,}258.62 - 2{,}258.53 = 0.09.$$

We repeat the above calculations for interest rate of 11%, 15%, 30% and 80%. The results are summarized as follows:

| $i$ | $V_A$ | $V_L$ | $S$ |
|---|---|---|---|
| 0.09 | 2,258.62 | 2,258.53 | 0.09 |
| 0.10 | 2,192.47 | 2,192.47 | 0.00 |
| 0.11 | 2,129.17 | 2,129.08 | 0.09 |
| 0.15 | 1,901.53 | 1,899.65 | 1.88 |
| 0.30 | 1,310.04 | 1,291.97 | 18.07 |
| 0.80 | 560.93 | 499.16 | 61.76 |

$\square$

- To compare the duration matching, Redington and full immunization strategies, the results of Examples 8.10, 8.11 and 8.12 are plotted in Figure 8.6 below.

# Figure 8.6: Illustration of Examples 8.10, 8.11 and 8.12

# 8.6 Some Shortcomings of Duration Matching

- In classical duration matching, the term structure is assumed to be flat.

- This limitation can be relaxed to allow for a term structure that is not flat, resulting in a more general definition of duration.

- Furthermore, we may consider different changes in the interest rate depending on the maturity of the cash flow. In other words, we may allow the shift in the term structure to be non-parallel.

- The characteristics of the assets and liabilities change through time. Even if the prevailing rate of interest remains unchanged, the durations of the assets and liabilities change over time due to time decay.

- Thus, the portfolio has to be re-balanced periodically to keep the durations matched.

- We have assumed that the cash flows of the assets and liabilities are *fixed*, and there is no uncertainty in their timing and value. For some cases, however, cash flows may be contingent on some events.

- For cash flows that are contingent on some uncertain events, the traditional duration measure will not be applicable. Improved methods such as the **option-adjusted duration** (also called **effective duration** or **stochastic duration**) should be considered.

- Conceptually, the effective duration is defined as the percentage drop in the price of the asset per unit increase in the rate of interest, i.e., $-(dP/di)/P$.

- When the cash flows are contingent on some random events, however, equation (8.7) cannot be used. The effective duration can be numerically estimated using the formula

$$- \left[ \frac{P(i + \Delta i) - P(i - \Delta i)}{2(\Delta i) P(i)} \right], \qquad (8.27)$$

where the prices at perturbed interest rates $i + \Delta i$ and $i - \Delta i$ are calculated numerically taking account of the embedded options of the asset.

# 8.7 Duration under a Nonflat Term Structure

- We consider a nonflat term structure defined by the sequence of spot rates $i_t^S$, for $t = 1, \cdots, n$.

- Denoting $\boldsymbol{i} = (i_1^S, \cdots, i_n^S)'$ as the vector of the spot rates and $P(\boldsymbol{i})$ as the price of the asset under the current term structure, we have

$$P(\boldsymbol{i}) = \sum_{t=1}^{n} \frac{C_t}{(1 + i_t^S)^t}. \tag{8.30}$$

- Let $\boldsymbol{\Delta} = (\Delta_1, \cdots, \Delta_n)'$ denote the vector of shifts in the spot rates so that the new term structure is

$$\boldsymbol{i} + \boldsymbol{\Delta} = (i_1^S + \Delta_1, \cdots, i_n^S + \Delta_n)', \tag{8.31}$$

and the price of the asset under the new term structure is

$$P(\boldsymbol{i} + \boldsymbol{\Delta}) = \sum_{t=1}^{n} \frac{C_t}{\left(1 + i_t^S + \Delta_t\right)^t}. \tag{8.32}$$

- If, however, $\Delta_t = \Delta$ for $t = 1, \cdots, n$, then the term structure has a *parallel shift,* and (8.32) becomes

$$P(\boldsymbol{i} + \boldsymbol{\Delta}) = \sum_{t=1}^{n} \frac{C_t}{\left(1 + i_t^S + \Delta\right)^t}. \tag{8.33}$$

- Using the first-order approximation in Taylor's expansion

$$\frac{1}{\left(1 + i_t^S + \Delta\right)^t} \approx \frac{1}{\left(1 + i_t^S\right)^t} - \frac{t\Delta}{\left(1 + i_t^S\right)^{t+1}},$$

we re-write (8.33) as

$$P(\boldsymbol{i} + \boldsymbol{\Delta}) \approx \sum_{t=1}^{n} \frac{C_t}{\left(1 + i_t^S\right)^t} - \Delta \sum_{t=1}^{n} \frac{tC_t}{\left(1 + i_t^S\right)^{t+1}},$$

which implies

$$P(\boldsymbol{i} + \boldsymbol{\Delta}) - P(\boldsymbol{i}) \approx -\Delta \sum_{t=1}^{n} \frac{tC_t}{\left(1 + i_t^S\right)^{t+1}}$$

$$= -\Delta \sum_{t=1}^{n} \left[\frac{t}{1 + i_t^S}\right] \text{PV}(C_t), \qquad (8.34)$$

where

$$\text{PV}(C_t) = \frac{C_t}{\left(1 + i_t^S\right)^{t}}. \qquad (8.35)$$

- Thus, we conclude

$$-\frac{1}{P(\boldsymbol{i})} \lim_{\Delta \to 0} \left[\frac{P(\boldsymbol{i} + \boldsymbol{\Delta}) - P(\boldsymbol{i})}{\Delta}\right] = \sum_{t=1}^{n} \frac{t}{1 + i_t^S} \left[\frac{\text{PV}(C_t)}{P(\boldsymbol{i})}\right]$$

$$= \sum_{t=1}^{n} W_t, \qquad (8.36)$$

69

where

$$W_t = \frac{t}{1 + i_t^S} \left[ \frac{\text{PV}(C_t)}{P(\boldsymbol{i})} \right].$$

- The **Fisher-Weil duration**, denoted by $D_F$, is defined as

$$D_F = \sum_{t=1}^{n} t \left[ \frac{\text{PV}(C_t)}{P(\boldsymbol{i})} \right] = \sum_{t=1}^{n} t w_t, \qquad (8.37)$$

where

$$w_t = \frac{\text{PV}(C_t)}{P(\boldsymbol{i})}.$$

- The Fisher-Weil duration is a generalization of the Macaulay duration, with the present values of cash flows computed using a nonflat term structure.

- Equation (8.36) is a generalization of the modified duration of equation (8.10).

**Example 8.13:** Bond A is a 2-year annual coupon bond with coupon rate of 3%. Bond B is a 5-year annual coupon bond with coupon rate of 5.5%. You are given that $i_t^S$ = 4.2%, 4.2%, 4.5%, 4.7% and 4.8%, for $t = 1, \cdots, 5$, respectively. Compute the Fisher-Weil duration of the two bonds, as well as the price sensitivity measure in (8.34). Also, calculate the Fisher-Weil duration of a portfolio with equal weights in the two bonds.

**Solution:** For Bond A, we have cash flows $C_1 = 3$ and $C_2 = 103$. Using the given term structure, we obtain $\text{PV}(C_1) = 2.879$ and $\text{PV}(C_2) = 94.864$, so that the price of Bond A is $2.879 + 94.864 = 97.743$. Consequently, we obtain $W_1 = 0.028$ and $W_2 = 1.863$, so that the price sensitivity measure of Bond A as given in equation (8.34) is 1.891 (i.e., Bond A drops in value by 1.891% per 1 percentage point parallel increase in the term structure). Similarly, the Fisher-Weil duration of the bond is computed as 1.971 years.

Similar calculations can be performed for Bond B, and the results are

shown in Table 8.3, with a Fisher-Weil duration of 4.510 years and a price sensitivity of 4.305 (i.e., Bond B drops in value by 4.305% per 1 percentage point parallel increase in the term structure). To compute the Fisher-Weil duration of the portfolio, we take the weighted average of the Fisher-Weil durations of the bonds to obtain $0.5(1.971 + 4.510) = 3.241$ years.

| | A | | | B | | |
|-----|----------|--------|--------|----------|--------|--------|
| $t$ | PV($C_t$) | $W_t$ | $tw_t$ | PV($C_t$) | $W_t$ | $tw_t$ |
| 1 | 2.879 | 0.028 | 0.029 | 5.278 | 0.049 | 0.051 |
| 2 | 94.864 | 1.863 | 1.941 | 5.066 | 0.094 | 0.098 |
| 3 | | | | 4.820 | 0.134 | 0.140 |
| 4 | | | | 4.577 | 0.169 | 0.177 |
| 5 | | | | 83.454 | 3.858 | 4.044 |
| Sum | 97.743 | 1.891 | 1.971 | 103.194 | 4.305 | 4.510 |

$\square$

**Example 8.14:** Compute the prices of the bonds in Example 8.13 under the following term structures (with the years to maturity of the bonds remaining unchanged):

| $t$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| $i_t^S$ of Case 1 | 4.6% | 4.8% | 5.5% | 6.1% | 6.4% |
| $i_t^S$ of Case 2 | 3.7% | 3.7% | 4.0% | 4.2% | 4.3% |

Comment on the use of equation (8.34) for the price changes of these bonds.

**Solution:** We note that the spot rate of interest $i_t^S$ in Case 1 increases by the amount of 0.004, 0.006, 0.01, 0.014 and 0.016, for $t = 1, 2, \cdots, 5$, respectively. Thus, the shift in the term structure is not parallel, but the average increase is 1 percentage point. Using equation (6.10), the computed prices of Bonds A and B are, respectively, 96.649 and 96.655. Thus, the price of Bond A drops 1.119% and that of Bond B drops 6.337%. These

figures contrast the values of 1.891% and 4.305%, respectively, predicted by equation (8.34). The discrepancy exists due to the fact that the shift in the term structure is not parallel.

Case 2 represents a parallel shift in the spot rates of $-0.5$ percentage point for all time to maturity. Using equation (6.10), the prices of Bonds A and B are found to be 98.674 and 105.447, respectively. Thus, Bonds A and B increase in value by 0.952% and 2.183%, respectively. These values are quite close to the changes predicted by equation (8.34) (i.e., half of the tabulated values, namely, 0.946% and 2.153%, respectively). $\square$