Financial Mathematics for Actuaries (Third Edition)

Chapter 10

Stochastic Interest Rates

Learning Objectives

- Deterministic scenarios of interest rates

- Random-scenario model

- Independent lognormal model

- Autoregressive lognormal model

- Dynamic term structure model

- Illustration of stochastic interest rate approach via an application

10.1 Deterministic Scenarios of Interest Rates

- We denote i_t, for $t = 2, 3, \cdots$, as the 1-period future spot rate applicable from time $t-1$ to time t. For completeness we denote $i_1 = i_1^S$, which is known at time 0.

- Unlike the forward rates i_t^F, which are determined by the current term structure, i_t for $t > 1$ is unknown at time 0. However, scenarios may be constructed to model the possible values of i_t.

- A deterministic interest rate scenario is a sequence of pre-specified 1-period rates i_t applicable in the future.

- The US 1990 Standard Valuation Law requires cash-flow testing under seven prescribed interest rate scenarios. These scenarios are

similar to those used in the New York Regulation 126 and they are now commonly known as the NY7 scenarios. See Table 10.1.

Table 10.1: The NY7 interest rate scenarios

	Scenario	Interest rate movement
1	Base scenario	Level
2	Gradual increase	Uniformly increasing over 10 years at 0.5% per year, and then level
3	Up-down	Uniformly increasing over 5 years at 1% per year, then uniformly decreasing over 5 years at 1% per year to the original level at the end of 10 years, and then level
4	Pop-up	An immediate increase of 3%, and then level
5	Gradual decrease	Uniformly decreasing over 10 years at 0.5% per year, and then level
6	Down-up	Uniformly decreasing over 5 years at 1% per year, then uniformly increasing over 5 years at 1% per year to the original level at the end of 10 years, and then level
7	Pop-down	An immediate decrease of 3%, and then level

Example 10.1: Given that the current 1-year spot rate is 6%, (a) find i_t for $t = 1, \cdots, 12$ under the "up-down" scenario of NY7, and (b) compute $a_{\overline{12}|}$ under this interest rate scenario.

Solution: (a) The interest rates under the "up-down" scenario are given in the following table:

t	1	2	3	4	5	6	7	8	9	10	11	12
i_t	6%	7%	8%	9%	10%	11%	10%	9%	8%	7%	6%	6%

(b) Using equation (3.12) (with i_t^F replaced by i_t) and the above interest rate scenario, we obtain $a_{\overline{12}|} = 7.48$. □

10.2 Random-Scenario Model

- A random-scenario model is a collection of specified plausible interest rate scenarios.

- The modeler, however, needs to state the probability distribution of these scenarios. Both the choice of scenarios and the corresponding probability assignment require personal judgement, which should reflect the modeler's view on the future interest-rate environment of the economy.

Example 10.2: Consider the following random interest rate scenario model.

Scenario	Probability	i_1	i_2	i_3	i_4	i_5
1	0.1	3.0%	2.0%	2.0%	1.5%	1.0%
2	0.6	3.0%	3.0%	3.0%	3.5%	4.0%
3	0.3	3.0%	4.0%	5.0%	5.0%	5.0%

Find the mean, the variance and the standard deviation of $a(5)$, $1/a(5)$, $a_{\overline{5}|}, \ddot{a}_{\overline{5}|}, s_{\overline{5}|}$ and $\ddot{s}_{\overline{5}|}$ under this model.

Solution: Note that

$$
\begin{aligned}
\ddot{a}_{\overline{n}|} &= \sum_{t=0}^{n-1} \frac{1}{a(t)} \\
&= 1 + \sum_{t=1}^{n-1} \frac{1}{a(t)} \\
&= 1 + \sum_{t=1}^{n-1} \frac{1}{\prod_{j=1}^{t}(1+i_j)}.
\end{aligned} \tag{10.1}
$$

Using (3.10) through (3.12), (10.1), (3.14) and (3.15) (with i_t^F replaced by i_t), we can compute the values of $a(5)$, $1/a(5)$, $a_{\overline{5}|}, \ddot{a}_{\overline{5}|}, s_{\overline{5}|}$ and $\ddot{s}_{\overline{5}|}$ under each scenario. For example, under the first scenario, we have

$$a(5) = (1.03)(1.02)(1.02)(1.015)(1.01) = 1.0986,$$

$$\frac{1}{a(5)} = \frac{1}{1.0986} = 0.9103,$$

$$a_{\overline{5}|} = \frac{1}{(1.03)} + \frac{1}{(1.03)(1.02)} + \cdots + \frac{1}{(1.03)(1.02)(1.02)(1.015)(1.01)} = 4.6855,$$

$$\ddot{a}_{\overline{5}|} = 1 + \frac{1}{(1.03)} + \cdots + \frac{1}{(1.03)(1.02)(1.02)(1.015)} = 4.7753,$$

$$s_{\overline{5}|} = 1 + (1.01) + \cdots + (1.01)(1.015)(1.02)(1.02) = 5.1474,$$

and

$$\ddot{s}_{\overline{5}|} = (1.01) + (1.01)(1.015) + \cdots + (1.01)(1.015)(1.02)(1.02)(1.03) = 5.2459.$$

We repeat the calculations for the other two scenarios and summarize the results in the following table.

Scenario	Probability	$a(5)$	$1/a(5)$	$a_{\overline{5}|}$	$\ddot{a}_{\overline{5}|}$	$s_{\overline{5}|}$	$\ddot{s}_{\overline{5}|}$
1	0.1	1.0986	0.9103	4.6855	4.7753	5.1474	5.2459
2	0.6	1.1762	0.8502	4.5630	4.7128	5.3670	5.5433
3	0.3	1.2400	0.8064	4.4466	4.6402	5.5141	5.7541

Therefore,

$$\mathrm{E}\left[\frac{1}{a(5)}\right] = (0.1 \times 0.9103) + (0.6 \times 0.8502) + (0.3 \times 0.8064) = 0.8431,$$

and

$$\mathrm{Var}\left[\frac{1}{a(5)}\right] = 0.1 \times (0.9103 - 0.8431)^2 + 0.6 \times (0.8502 - 0.8431)$$

$$+ \ 0.3 \times (0.8064 - 0.8431)^2$$

$$= 0.00089.$$

The mean, the variance and the standard deviation of other variables can be computed similarly. The results are given in the following table.

	$a(5)$	$1/a(5)$	$a_{\overline{5}|}$	$\ddot{a}_{\overline{5}|}$	$s_{\overline{5}|}$	$\ddot{s}_{\overline{5}|}$
Mean	1.1876	0.8431	4.5403	4.6973	5.3892	5.5768
Variance	0.00170	0.00089	0.00505	0.00173	0.01082	0.02105
Std Dev	0.0412	0.0298	0.0711	0.0416	0.1040	0.1451

$\square$

10.3 Independent Lognormal Model

- The **Independent Lognormal Model** assumes that $1 + i_t$ are independently lognormally distributed with parameters μ and σ^2. In other words, $\ln(1 + i_t)$ follows a normal distribution with mean μ and variance σ^2 (see Appendix A.14).

- Thus,

$$\ln(1 + i_t) \sim N(\mu, \sigma^2). \tag{10.2}$$

- The mean and variance of the lognormal random variable $(1 + i_t)$ are, respectively,

$$\mathrm{E}\left(1 + i_t\right) = e^{\mu + \frac{1}{2}\sigma^2}, \tag{10.3}$$

and

$$\mathrm{Var}(1 + i_t) = \left(e^{2\mu + \sigma^2}\right)\left(e^{\sigma^2} - 1\right). \tag{10.4}$$

- We now consider the mean and variance of $a(n)$, $1/a(n)$, $\ddot{s}_{\overline{n}|}$, $a_{\overline{n}|}$, $s_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$ under the independent lognormal interest rate model.

- We shall assume all 1-period spot rates to be random, including the rate for the first period i_1.

- From (3.10), we have

$$a(n) = \prod_{t=1}^{n} (1 + i_t).$$

- Thus,

$$\ln\left[a(n)\right] = \sum_{t=1}^{n} \ln(1 + i_t). \tag{10.5}$$

- Therefore, $\ln\left[a(n)\right]$ is normal with mean $n\mu$ and variance $n\sigma^2$, implying $a(n)$ is lognormal with parameters $n\mu$ and $n\sigma^2$.

- Using (10.3) and (10.4), we have

$$\mathrm{E}\left[a(n)\right] = e^{n\mu + \frac{n}{2}\sigma^2}, \tag{10.6}$$

and

$$\mathrm{Var}\left[a(n)\right] = \left(e^{2n\mu + n\sigma^2}\right)\left(e^{n\sigma^2} - 1\right). \tag{10.7}$$

- For the distribution of $(1+i_t)^{-1}$, under the lognormal model we have

$$\ln\left[(1 + i_t)^{-1}\right] = -\ln(1 + i_t),$$

so that

$$\ln\left[(1 + i_t)^{-1}\right] \sim N(-\mu, \sigma^2), \tag{10.8}$$

and $(1 + i_t)^{-1}$ follows a lognormal distribution with parameters $-\mu$ and σ^2.

- From (3.11), we have

$$\frac{1}{a(n)} = \prod_{t=1}^{n}(1 + i_t)^{-1},$$

so that

$$E\left[\frac{1}{a(n)}\right] = e^{-n\mu + \frac{n}{2}\sigma^2}, \qquad (10.9)$$

and

$$\text{Var}\left[\frac{1}{a(n)}\right] = \left(e^{-2n\mu + n\sigma^2}\right)\left(e^{n\sigma^2} - 1\right). \qquad (10.10)$$

- The statistical properties of annuities (say, $\ddot{s}_{\overline{n}|}$, $a_{\overline{n}|}$, $s_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$) under the lognormal interest rate assumption are fairly complex.

- We present and discuss some applications of their mean and variance formulas without going through the proof.

- We first define r_s such that $1 + r_s = \mathrm{E}(1 + i_t) = e^{\mu + \frac{1}{2}\sigma^2}$, implying

$$r_s = e^{\mu + \frac{1}{2}\sigma^2} - 1. \qquad (10.11)$$

- Furthermore, we let

$$j_s = 2r_s + r_s^2 + v_s^2, \qquad (10.12)$$

where

$$v_s^2 = \mathrm{Var}(1 + i_t) = \left(e^{2\mu + \sigma^2}\right)\left(e^{\sigma^2} - 1\right). \qquad (10.13)$$

- The mean of $\ddot{s}_{\overline{n}|}$ is

$$\mathrm{E}\left(\ddot{s}_{\overline{n}|}\right) = \mathrm{E}\left[\sum_{t=1}^{n}\prod_{j=1}^{t}(1 + i_{n-j+1})\right]$$

$$= \sum_{t=1}^{n}\prod_{j=1}^{t}\mathrm{E}(1 + i_{n-j+1})$$

$$= \sum_{t=1}^{n}(1 + r_s)^t$$

$$= \ddot{s}_{\overline{n}|r_s}, \tag{10.14}$$

where $\ddot{s}_{\overline{n}|r_s}$ is $\ddot{s}_{\overline{n}|}$ evaluated at the rate r_s.

- Similarly, the variance of $\ddot{s}_{\overline{n}|}$ is

$$\mathrm{Var}\left(\ddot{s}_{\overline{n}|}\right) = \left(\frac{\dot{j}_s + r_s + 2}{j_s - r_s}\right)\ddot{s}_{\overline{n}|j_s} - \left(\frac{2j_s + 2}{j_s - r_s}\right)\ddot{s}_{\overline{n}|r_s} - \left(\ddot{s}_{\overline{n}|r_s}\right)^2.$$

$$(10.15)$$

- Next, we consider $a_{\overline{n}|}$. We define r_a such that $(1 + r_a)^{-1} = \mathrm{E}(1 + i_t)^{-1} = e^{-\mu + \frac{1}{2}\sigma^2}$, implying

$$r_a = e^{\mu - \frac{1}{2}\sigma^2} - 1.$$

$$(10.16)$$

- Furthermore, we let

$$j_a = e^{2(\mu - \sigma^2)} - 1.$$

$$(10.17)$$

- Similar to (10.14) and (10.15), we can show that the mean of $a_{\overline{n}|}$ is

$$\mathrm{E}\left(a_{\overline{n}|}\right) = a_{\overline{n}|r_a} \quad \text{and}$$

$$\mathrm{Var}\left(a_{\overline{n}|}\right) = \left(\frac{j_a + r_a + 2}{r_a - j_a}\right) a_{\overline{n}|j_a} - \left(\frac{2r_a + 2}{r_a - j_a}\right) a_{\overline{n}|r_a} - \left(a_{\overline{n}|r_a}\right)^2. \tag{10.19}$$

- Furthermore, we have

$$\mathrm{E}\left(\ddot{a}_{\overline{n}|}\right) = 1 + \mathrm{E}\left(a_{\overline{n-1}|}\right), \tag{10.20}$$

$$\mathrm{E}\left(s_{\overline{n}|}\right) = 1 + \mathrm{E}\left(\ddot{s}_{\overline{n-1}|}\right). \tag{10.21}$$

$$\mathrm{Var}\left(\ddot{a}_{\overline{n}|}\right) = \mathrm{Var}\left(a_{\overline{n-1}|}\right), \tag{10.22}$$

and

$$\mathrm{Var}\left(s_{\overline{n}|}\right) = \mathrm{Var}\left(\ddot{s}_{\overline{n-1}|}\right). \tag{10.23}$$

Example 10.3: Consider a lognormal interest rate model with parameters $\mu = 0.04$ and $\sigma^2 = 0.016$. Find the mean and variance of $a(5)$, $1/a(5)$, $\ddot{s}_{\overline{5}|}$, $a_{\overline{5}|}$, $s_{\overline{5}|}$ and $\ddot{a}_{\overline{5}|}$ under this model.

Solution: We apply equations (10.6) through (10.10) to obtain

$$\mathrm{E}\left[a(5)\right] = 1.27125,$$

$$\mathrm{Var}\left[a(5)\right] = 0.13460,$$

$$\mathrm{E}\left[\frac{1}{a(5)}\right] = 0.85214,$$

$$\mathrm{Var}\left[\frac{1}{a(5)}\right] = 0.06058.$$

For $\ddot{s}_{\overline{5}|}$, using expressions (10.11) through (10.13), we have

$$r_s = e^{0.04 + \frac{1}{2}(0.016)} - 1 = 0.04917,$$

$$v_s^2 = \left(e^{2(0.04)+0.016}\right)\left(e^{0.016} - 1\right) = 0.01775,$$

$$j_s = 2r_s + r_s^2 + v_s^2 = 0.11851,$$

$$\ddot{s}_{\overline{5}|r_s} = 5.78773,$$

$$\ddot{s}_{\overline{5}|j_s} = 7.08477.$$

Applying equations (10.14) and (10.15), we obtain

$$\mathrm{E}\left(\ddot{s}_{\overline{5}|}\right) = 5.78773,$$

$$\mathrm{Var}\left(\ddot{s}_{\overline{5}|}\right) = 1.26076.$$

For $a_{\overline{5}|}$, using expressions (10.16) and (10.17), we have

$$r_a = e^{0.04 - \frac{1}{2}(0.016)} - 1 = 0.03252,$$

$$j_a = e^{2(0.04 - 0.016)} - 1 = 0.04917,$$

$$a_{\overline{5}| r_a} = 4.54697,$$

$$a_{\overline{5}| j_a} = 4.33942.$$

Applying equations (10.18) and (10.19), we obtain

$$\mathrm{E}\left(a_{\overline{5}|}\right) = 4.54697,$$

$$\mathrm{Var}\left(a_{\overline{5}|}\right) = 0.72268.$$

Finally, employing formulas (10.20) through (10.23), we obtain

$$\mathrm{E}\left(\ddot{a}_{\overline{5}|}\right) = 1 + \mathrm{E}\left(a_{\overline{4}|}\right) = 1 + 3.69483 = 4.69483,$$

$$\mathrm{Var}\left(\ddot{a}_{\overline{5}|}\right) = \mathrm{Var}\left(a_{\overline{4}|}\right) = 0.40836,$$

$$\mathrm{E}\left(s_{\overline{5}|}\right) = 1 + \mathrm{E}\left(\ddot{s}_{\overline{4}|}\right) = 1 + 4.51648 = 5.51648,$$

$$\mathrm{Var}\left(s_{\overline{5}|}\right) = \mathrm{Var}\left(\ddot{s}_{\overline{4}|}\right) = 0.64414.$$

$\square$

10.4 Autoregressive Model

- Let $Y_t = \ln(1 + i_t)$. Under the independent lognormal model, we have

$$Y_t = \ln(1 + i_t) \sim N(\mu, \sigma^2), \qquad (10.24)$$

where there are no correlations between Y_t and Y_{t-k} for all $k \geq 1$.

- We now consider interest rate models that allow some dependence among Y_t's, say,

$$\mathrm{Corr}(Y_t, Y_{t-k}) \neq 0, \qquad \text{for some } k \geq 1, \qquad (10.25)$$

while maintaining the lognormal assumption.

- We consider the class of first-order autoregressive models, denoted

by AR(1). The AR(1) model has the form

$$Y_t = c + \phi Y_{t-1} + e_t, \tag{10.26}$$

where c is the constant intercept and ϕ is the autoregressive parameter and e_t are independently and identically distributed normal random variates each with mean zero and variance σ^2.

- For $|\phi| < 1$, the correlation structure of the interest-rate process $\{Y_t\}$ is

$$\text{Corr}(Y_t, Y_{t-k}) = \phi^k, \qquad \text{for } k = 1, 2, \cdots, \tag{10.27}$$

which does not vary with t.

- The mean and variance expressions of $a(n)$, $1/a(n)$, $a_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $s_{\overline{n}|}$ and $\ddot{s}_{\overline{n}|}$ under an AR(1) interest rate model are very complex.

- Thus, we consider a stochastic simulation approach to obtain the empirical distributions illustrated in Table 10.2.

Table 10.2: A simulation procedure of $a_{\overline{n}\rceil}$ for the AR(1) process

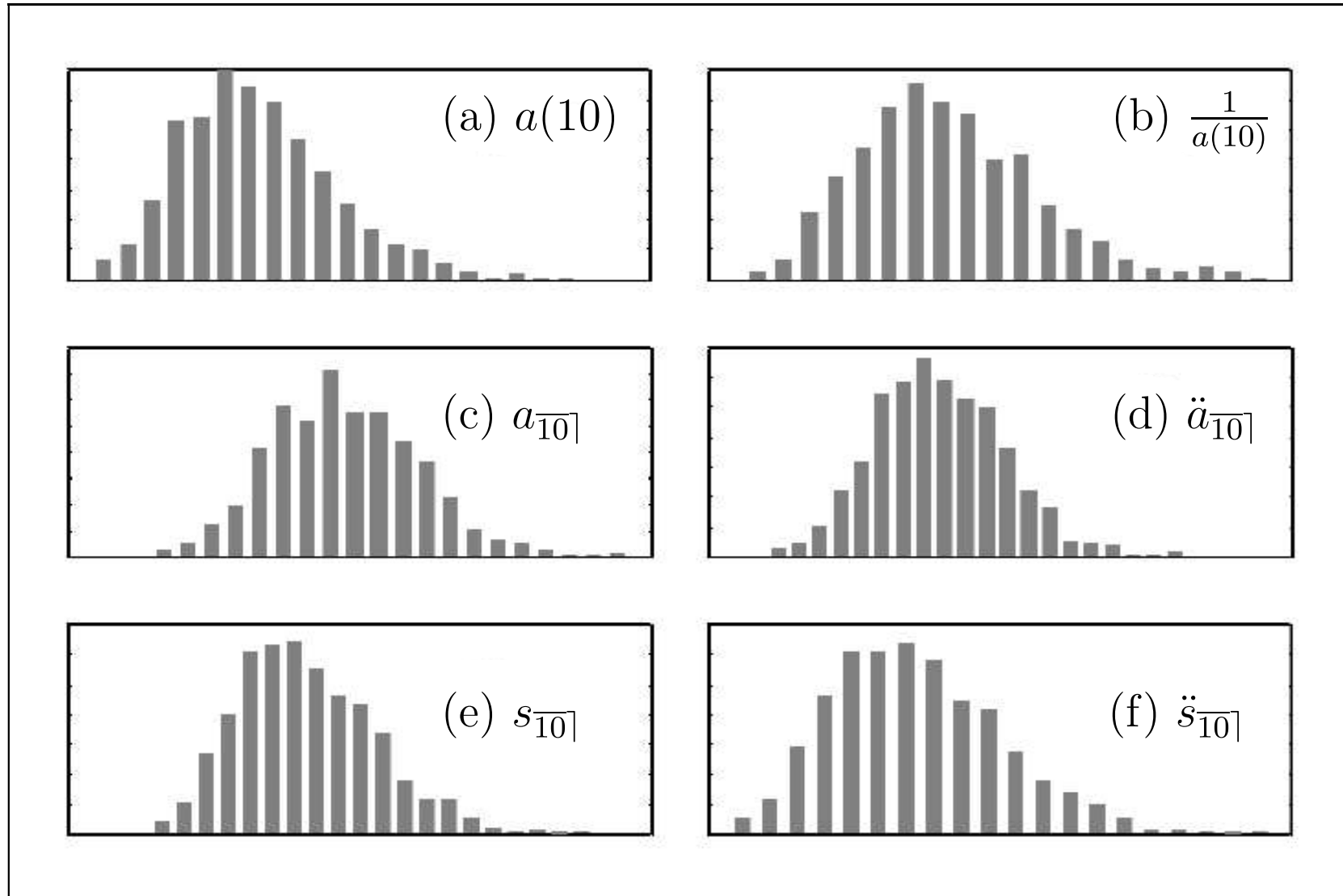| Step	Procedure
1	Using historical interest rate data, estimate the parameters (c, ϕ, σ^2) in the AR(1) model.
2	Draw random normal numbers $e_1, e_2, \cdots, e_n$ (with mean zero and variance σ^2) from a random number generator.
3	Compute $Y_1, Y_2, \cdots, Y_n$ using the AR(1) equation (10.26) and setting the initial value $Y_0 = \overline{Y}$, where $\overline{Y}$ is the sample average of the observed data.
4	Convert Y_t's to an interest rate path $(i_1, i_2, \cdots, i_n)$ using the relationship $i_t = e^{Y_t} - 1$.
5	Compute the $a_{\overline{n}\rceil}$ function under the simulated interest rate path in Step 4.
6	Repeat Steps 2 to 4 m times. Note that the random normal variates in Step 2 are redrawn each time and we have a different simulated interest rate path in each replication.

Example 10.4: Consider an AR(1) interest-rate model with parameters $c = 0.03$, $\phi = 0.6$ and $\sigma^2 = 0.001$. Using the stochastic simulation method described in Table 10.2 (with $m = 1,000$ and $Y_0 = 0.06$), find the mean and variance of $a(10)$, $1/a(10)$, $a_{\overline{10|}}$, $\ddot{a}_{\overline{10|}}$, $s_{\overline{10|}}$ and $\ddot{s}_{\overline{10|}}$ under this model.

Solution: We follow the simulation steps in Table 10.2 and obtain the empirical distributions (histograms) of the six functions, which are plotted in Figure 10.1 below. The empirical means and variances are as follows:

	$a(10)$	$1/a(10)$	$a_{\overline{10|}}$	$\ddot{a}_{\overline{10|}}$	$s_{\overline{10|}}$	$\ddot{s}_{\overline{10|}}$
Mean	2.1656	0.4836	6.8382	7.3546	14.5018	15.6674
Variance	0.2224	0.0110	0.5678	0.4383	3.5101	5.3524

$\square$

Figure 10.1: Results for Example 10.4

(a) $a(10)$

(b) $\frac{1}{a(10)}$

(c) $a_{\overline{10|}}$

(d) $\ddot{a}_{\overline{10|}}$

(e) $s_{\overline{10|}}$

(f) $\ddot{s}_{\overline{10|}}$

10.5 Dynamic Term Structure Model

- There are two popular approaches to model the dynamic term structure: **no-arbitrage approach** and **equilibrium modeling approach**.

- The **no-arbitrage approach** focuses on perfectly fitting the term structure at a point in time to ensure that no arbitrage opportunities exist in the financial markets. This approach is important for pricing derivatives.

- The **equilibrium modeling approach** focuses on modeling the instantaneous spot rate of interest, called the **short rate**, and use **affine models** to derive the yields at other maturities under various assumptions of the risk premium.

10.6 An Application: Guaranteed Investment Income

- We consider a hypothetical **guaranteed income fund** offered by a bank with the following features:

  - It is a **closed-end** fund. The bank sells only a fixed number of units (say, 10,000 units with $1,000 face amount per unit) in the initial public offering (IPO). No more units will be issued by the bank after the IPO. Redemption prior to the maturity date is not allowed.

  - Investment period is 9 years.

  - At maturity, the bank will return 100% of the face amount to the investors.

- At the end of each of the nine years, earnings from the fund will be distributed to investors. In addition, the bank provides a guarantee on the minimum rate of return each year according to a fixed schedule:

At the end of	Guaranteed rate (as % of initial investment)
Year 1	3%
Year 2	4%
Year 3	4%
Year 4	5%
Year 5	5%
Year 6	5%
Year 7	5%
Year 8	5%
Year 9	5%

Thus, if the fund earns less than the guaranteed rate in any

year, the bank has to top up the difference.

Example 10.5: Consider the above guaranteed income fund. Assume that the rates of return for the fund in the next nine years are: 2%, 3%, 4%, 5%, 6%, 5%, 4%, 3%, 2%, respectively. Compute the cost of the guarantee to the bank (i.e., the present value of the top-up amounts).

Solution: Let G_t be the guaranteed rate for year t offered by the bank, i_t be the rate of return earned by the fund for year t, and U_t be the top-up amount (per \$1,000 face amount) by the bank to honor the guarantee payable at the end of year t. Note that

$$U_t = \begin{cases} 0, & \text{if } i_t \geq G_t, \\ 1{,}000(G_t - i_t), & \text{if } i_t < G_t. \end{cases} \tag{10.28}$$

Using equation (10.28), we compute U_t for each year. The results are given as follows.

t	1	2	3	4	5	6	7	8	9
i_t	2%	3%	4%	5%	6%	5%	4%	3%	2%
G_t	3%	4%	4%	5%	5%	5%	5%	5%	5%
U_t	10	10	0	0	0	0	10	20	30

The cost of the guarantee C is the present value of the U_t's. Thus,

$$
\begin{aligned}
C &= \sum_{t=1}^{9} \frac{U_t}{\prod_{j=1}^{t}(1+i_j)} \\
&= \frac{10}{1.02} + \frac{10}{(1.02)(1.03)} + \frac{0}{(1.02)(1.03)(1.04)} + \cdots \\
&= 62.98.
\end{aligned}
\tag{10.30}
$$

$\square$

- The bank needs to charge an up-front premium to cover the cost of the guarantee. We now apply the stochastic interest rate approach

to examine the distribution of C. This information is useful to the bank's management for setting the premium.

- For illustration, we assume that the fund invests only in government bonds and the returns of the fund follow an independent lognormal interest rate model with parameters $\mu = 0.06$ and $\sigma^2 = 0.0009$, i.e.,

$$\ln(1 + i_t) \sim N(0.06, 0.0009). \tag{10.30}$$

- We use stochastic simulation to produce a series of future interest rate movements. For each of the simulated path, using equations (10.28) and (10.29), a realization of C can be computed. We repeat the simulation experiment m times (say, $m = 5{,}000$) and the empirical distribution (histogram) of C can be obtained.

- The time lag between the bank's filing of the required fund documents to the regulator for approval and the actual IPO is often

33

lengthy. The bank has to determine the premium rate well before the IPO date. Therefore, in the simulation, the first-period rate of return i_1 is not known at time 0.

- The simulation study is carried out and the empirical distribution (histogram) of C is plotted in Figure 10.2 below.

Figure 10.2: Results for Example 10.5