Math Notes

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Contents

1 Matrices

Definition 1 (Positive-Definite)**.** A matrix A is *positive-definite* if for every nonzero vector x,

$$x^T A x > 0 \tag{1}$$

Claim 1 (Positive-Definite $\rightarrow \lambda_i > 0$)**.** If A is a positive-definite matrix, then the eigenvalues of A are positive.

Proof. Let A be a positive-definite matrix. Since A is positive-definite

$$x^T A x = x^T \lambda x = \lambda x^T x > 0$$

but $x^T x = \sum x_i^2$ which is greater than 0, thus λ is also greater than zero. $\square$

Definition 2 (Diagonalizable)**.** A matrix A is diagonalizable if there exists an invertible matrix P such that PAP^{-1} is a diagonal matrix.

For all matrices A, it is not guaranteed a diagonlization exists. If a diagonalization exists, then $P^{-1}AP = D$ where D is some diagonal matrix so $AP = PD \rightarrow A_i\vec{\alpha}_i = \vec{\alpha}_i d_i = d_i\vec{\alpha}_i$ for $P = (\vec{\alpha}_1 \; \dots \; \vec{\alpha}_n)$ so D is merely the lambda values of A. This implies we must have n eigenvalues but do they necessarily need to be distinct?

Claim 2. If A is diagonlizable and has only non-zero eigenvalues, then A is invertible.

Proof. Since D exists and consists of the eigenvalues of A which are non-zero, D^{-1} exists (and is easily calculable) and we have $D^{-1} = (P^{-1}AP)^{-1} = PA^{-1}P^{-1} \rightarrow A^{-1} = P^{-1}D^{-1}P$; therefore, A^{-1} also exists. $\square$

Definition 3. A square matrix U is unitary if $U^{-1} = U^T$.

Theorem 1 (Invertible Matrix Theorem)**.** *A is invertible if and only if any of the following hold:*

- A is row-equivalent to the $n \times n$ identity matrix
- A has n pivot positions
- The equation $A\mathbf{x} = 0$ has only the trival solution $\mathbf{x} = 0$.
- The columns of A form a linearly independent set
- The linear transformation $x \mapsto Ax$ is one-to-one
- For each column vector $b \in \mathbb{R}^n$, the equation $A\mathbf{x} = b$ has a unique solution.
- The columns of A span $\mathbb{R}^n$
- The linear transformation $x \mapsto Ax$ is a surjection (onto)

- There is an $n \times n$ matrix C such that $CA = I_n$ or $AC = I_n$ (**Note:** C can be found by multiplying the elementary operation matrices)
- The transpose matrix A^T is invertible
- The columns of A form a basis for $\mathbb{R}^n$
- The column space of A is equal to $\mathbb{R}^n$
- The rank of A is n
- The null space of A is $\{0\}$
- The dimension of the null space of A is 0.
- 0 fails to be an eigenvalue of A (Seen in the claim above)
- The determinant of A is non-zero This is an if and only if?
- The orthogonal complement of the column space of A is $\{0\}$
- The orthogonal complement of the null space of A is $\mathbb{R}^n$
- The row space of A is $\mathbb{R}^n$
- The matrix A has n non-zero singular values

1.1 Finding Eigenvalues

Eigenvalues are the solutions to the equation

$$Av = \lambda v$$

for $v \neq 0$ or alternatively

$$(A - \lambda I)v = 0$$

Since $\det(AB) = \det(A) \cdot \det(B)$ and $v \neq 0$, $\det((A - \lambda I)v) = 0 \leftrightarrow \det(A - \lambda I) = 0$

2 Finite Poisson

Poisson's equation is

$$\Delta \varphi = f \tag{2}$$

where Δ is the Laplace operator, and f and φ are real or complex-valued functions. In Euclidean space, the Laplace operator is often denoted as ∇^2 and so Poisson's equation is frequently written as

$$\nabla^2 \varphi = f \tag{3}$$

In three-dimensional Cartesian coordinates, it takes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi(x, y, z) = f(x, y, z) \tag{4}$$

When $f = 0$ we have Laplace's Equation.

2.1 Problem Statement

Consider the second-order differential equation

$$\frac{\partial^2}{\partial x^2}\varphi(x) = f(x) \tag{5}$$

Use the finite difference method to approximate the partial differential equation over the interval $x \in (a, b)$ assuming a uniform spatial discretization Δx. φ need not be twice-differentiable over $\mathbb{R}$ but only over the interval $x \in (a, b)$. Although conceptually easy, finite differences are difficult to apply over domains with heterogenous composition?

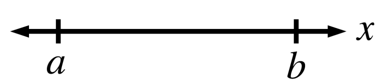
2.2 Finite-Difference Method

The Finite-Difference Method (FDM) is a numerical procedure which solves a PDE by:

- Discretizing the continuous physical domain into a discrete **finite difference grid**
- Approximating the individual exact partial derivatives in the PDE by algebraic **finite difference approximations (FDAs)**
- Substituting the FDAs into the PDE to obtain an algebraic **finite difference equation (FDE)**
- Solving the resulting algebraic FDEs for the dependent variable.

2.2.1 Finite Difference Grid

Consider the following one-dimensional domain D(x),



which must be covered by a one-dimensional grid of lines (figure 2.2.1). Let $x_i = a + (i-1)\Delta x$, $f(x_i) = f_i$, $f_x(x_i) = f_x|_i$ and $f_{xx}(x_i) = f_{xx}|_i$.

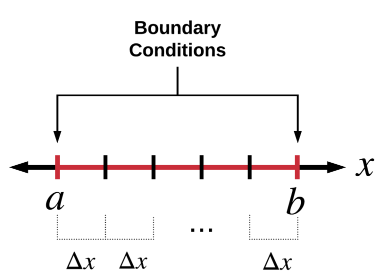


Figure 1: Discrete difference grid with a uniform spatial discretization of Δx

2.2.2 Finite Difference Approximations

Note: For elliptic PDEs containing only second derivatives, there are no preferred physical informative propagation paths. Thus, centered-space finite difference approximations should be used for the second-order spatial derivatives in the Laplace equation and the Poisson equation.

Exact Solution: $\bar{f}(x)$
Approximate Solution: $f(x)$

Consider the Taylor Series expansion of $\bar{f}_{i-1}$ and $\bar{f}_{i-1}$ using i as the base point gives

$$\bar{f}_{i+1} = \bar{f}_i + \bar{f}_x|_i \Delta x + \frac{1}{2}\bar{f}_{xx}|_i (\Delta x)^2 + \cdots \tag{6}$$

$$\bar{f}_{i-1} = \bar{f}_i - \bar{f}_x|_i \Delta x + \frac{1}{2}\bar{f}_{xx}|_i (\Delta x)^2 - \cdots \tag{7}$$

Adding the first three terms of equations (6) and (7) (up to the second partial) we can approximate $\bar{f}_{xx}|_i$,

$$f_{xx}|_i = \frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2} \tag{8}$$

which serves as our second-order centered-difference approximation at i.

2.3 Finite Difference Solution of the Laplace Equation

If $f(x) = 0$ in equation (5) of our problem statement, we have the Laplace Equation

$$\frac{\partial^2}{\partial x^2}\varphi(x) = \bar{f}_{xx} = 0 \tag{9}$$

Substituting equation (8) yields

$$\frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2} = 0 \tag{10}$$

while solving for f_i gives

$$f_i = \frac{f_{i-1} + f_{i+1}}{2} \tag{11}$$

Simply stated, the solution at every point is the arithmetic average of the solutions of the neighboring points. **Note:** This result applies only to the Laplace equation (i.e. no nonhomogeneous term).

Let u_i represent the approximation of $\varphi(x_i)$ in (9). Assuming the uniform spatial discretization yields n partitions, we have the following system of equations,

$$
\begin{aligned}
u_2 &= \frac{a+u_3}{2} \\
u_3 &= \frac{u_2+u_4}{2} \\
&\vdots \\
u_{n-2} &= \frac{u_{n-3}+u_{n-1}}{2} \\
u_{n-1} &= \frac{u_{n-2}+b}{2}
\end{aligned}
\tag{12}
$$

or

$$
\begin{aligned}
(2)u_2 - (1)u_3 + (0)u_4 + \cdots + (0)u_{n-3} + (0)u_{n-2} + (0)u_{n-1} &= a \\
-(1)u_2 + (2)u_3 - (1)u_4 + \cdots + (0)u_{n-3} + (0)u_{n-2} + (0)u_{n-1} &= 0 \\
&\vdots \\
(0)u_2 + (0)u_3 + (0)u_4 + \cdots - (1)u_{n-3} + (2)u_{n-2} - (1)u_{n-1} &= 0 \\
(0)u_2 + (0)u_3 + (0)u_4 + \cdots + (0)u_{n-3} + (1)u_{n-2} + (2)u_{n-1} &= b
\end{aligned}
$$

which when expressed in matrix form

$$
\begin{bmatrix}
2 & -1 & 0 & \cdots & 0 & 0 & 0 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
\ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & -1 & 2 & -1 \\
0 & 0 & 0 & \ldots & 0 & -1 & 2
\end{bmatrix}
\cdot
\begin{bmatrix}
u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1}
\end{bmatrix}
=
\begin{bmatrix}
a \\ 0 \\ \vdots \\ 0 \\ b
\end{bmatrix}
$$

2.4 Finite Difference Solution of the Poisson Equation

Similar to equation (9) except that $f(x) \neq 0$. Let $f(x) = g(x)$, then we have

$$\frac{\partial^2}{\partial x^2}\varphi(x) = \bar{f}_{xx} = g(x) \tag{13}$$

For $g(x) \neq 0$, equations (10) and (11) become

$$\frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2} = g(x) \tag{14}$$

$$f_i = \frac{f_{i-1} + f_{i+1} - g_i(\Delta x)^2}{2} \tag{15}$$

which yields a similar system of equations when represented in matrix form gives

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} a - g_2(\Delta x)^2 \\ -g_3(\Delta x)^2 \\ \vdots \\ -g_{n-2}(\Delta x)^2 \\ b - g_{n-1}(\Delta x)^2 \end{bmatrix}$$

2.5 Derivative Boundary Conditions

Consider equation (5) in our problem statement with unknown boundary point conditions but known derivatives. We approximate these unknown boundary points using their derivatives and adjacent interior point. Recall the approximation for f_I,

$$f_{I+1} + f_{I-1} - 2f_I = g_I(\Delta x)^2 \tag{16}$$

for either f_{I-1} or f_{I+1} unknown (depending on which boundary point we are considering). Using the difference formula

$$f_x|_I = \frac{\bar{f}_{I+1} - \bar{f}_{I-1}}{2\Delta x} \tag{17}$$

and solving for either f_{I-1} or f_{I+1} (again depending on which boundary point we are considering) we arrive at

$$2f_{I-1} - 2f_I = -2\bar{f}_x|_I \Delta x + g_I(\Delta x)^2 \tag{18}$$

$$2f_{I+1} - 2f_I = -2\bar{f}_x|_I \Delta x + g_I(\Delta x)^2 \tag{19}$$

which can be explicitly solved for f_{I-1} or f_{I+1}. **Note:** $\bar{f}_x|_I$ and g_I in these equations are not necessarily the same. The original form of equation (17) is

$$f_x|_i = \frac{f_{i+1} - f_{i-1}}{2\Delta x} - \frac{1}{6} f_{xxx}(\zeta)(\Delta x)^2 \tag{20}$$

I easily see how equation (17) was derived but not equation (20). Is this derived from the Taylor Series?

2.6 Iterative Methods of Solution

2.6.1 Jacobi Iteration

2.6.2 Gauss-Seidel Iteration

2.6.3 Successive-over-relaxation (SOR)

3 Conjugate Gradient

The Conjugate Gradient Method is an algorithm for finding a numerical solution of a system of linear equations of the form

$$Ax = b \tag{21}$$

where A is a symmetric, positive-definite matrix.

3.1 Quadratic Form

Finding conflicting definitions for a *quadratic form* when considering whether the function is homogeneous or not.

Definition 4. A *quadratic form* is a scalar quadratic function of a vector with the form

$$f(x) = x^T Ax - b^T x + c \tag{22}$$

where A is a matrix, x and b are vectors, and c is a scalar constant.

3.1.1 Derivative of a Quadratic Form

Let $f(x) = x^T Ax$ over the real numbers. Let $y(x) = Ax$ so that $f(x, y(x)) = x^T y(x)$, then

$$\begin{aligned} f'(x, y(x)) &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot y'(x) \\ &= y(x)^T + x^T \cdot A \\ &= (Ax)^T + x^T A \\ &= x^T A^T + x^T A \end{aligned}$$

Let $g(x) = \frac{1}{2} f(x)$ from (21) then

$$g'(x) = x^T A^T - b^T \tag{23}$$

Suppose $g'(x) = 0$ and A is symmetric, then solving the linear system $Ax = b$ is equivalent to finding the minimum/maximum of the quadratic form $g(x)$. If A is positive-definite, then this will be a minimum.

Claim 3. Let A be a symmetric, positive-definite matrix, then x is a solution for $Ax = b$ if and only if $g(x) = \frac{1}{2} x^T Ax - b^T x + c$ is minimized at x.

Proof. . . . □

3.2 Method of Steepest Descent

Very intuitive. $\nabla f(x)$ points in the direction of steepest ascent at $(x, f(x))$. By equation (22), the steepest descent is then

$$-f'(x) = b - Ax. \tag{24}$$

What size of step α should we take in direction $-f'(x)$? Clearly the α which minimizes f along the line,

$$x_{(i+1)} = x_{(i)} + \alpha r_{(i)} \tag{25}$$

where $r_{(i)} = -f'(x_i)$ or the error transformed by A into the same space as b. α minimizes f when $\frac{d}{d\alpha}(f(x_i)) = 0$

$$\frac{d}{d\alpha} f(x_{(i)}) = r_{(i)} \cdot f'(x_{(i+1)})^T = 0$$

when $r_{(i)}$ and $f'(x_{(i+1)})$ are orthogonal. Solving for α,

$$\begin{aligned} r_{(i)} \cdot f'(x_{(i+1)})^T &= 0 \\ r_{(i)} \cdot (b - Ax_{(i+1)})^T &= 0 \\ r_{(i)} \cdot (b - A(x_{(i)} + \alpha r_i))^T &= 0 \\ &\dots \\ \alpha &= \frac{r_{(i)}^T r_{(0)}}{r_{(i)}^T A r_{(0)}} \end{aligned} \tag{26}$$

4 Eigenvalue Decomposition

Definition 5. Let A be a square $n \times n$ matrix. The eigendecomposition of A is

$$A = Q \Lambda Q^{-1} \tag{27}$$

where A is the square $n \times n$ matrix whose ith column is the eigenvector q_i of A and Λ is the diagonal matrix whose diagonal elements are the corresponding eigenvalues.

4.1 Computing Power Series

Eigendecomposition allows for the easy computation of the power series of a matrix. Let

$$f(x) = \sum_{i=0} a_i x^i$$

then

$$f(A) = Q f(\Lambda) Q^{-1}$$

which reduces to calculating f on each of the eigenvalues.

5 Orthogonal Matrices

Definition 6. An orthogonal matrix is a square matrix with real entries whose columns and rows are orthogonal unit vectors, or

$$QQ^T = I \tag{28}$$

5.1 Properties

- $det(Q) = \pm 1$
- Unitary Transformation (i.e. $u \cdot v = Qu \cdot Qv$)

5.2 Important Orthogonal Decompositions

- QR: $M = QR$ (Q orthogonal, R upper-triangular)
- SVD: $M = U\Sigma V^T$ (U and V orthogonal, Σ diagonal)
- Eigendecomposition: $S = Q\Lambda Q^T$ (S symmetric, Q orthogonal, Λ diagonal)
- Polar Decomposition:...

6 Vector Fields

In general, a vector field is a function whose domain is a set of points in $\mathbb{R}^2$ or $\mathbb{R}^3$ and whose range is a set of vectors in V_2 or V_3.

Definition 7. Let D be a set in $\mathbb{R}^2$. A **vector field** on $\mathbb{R}^2$ is a function $\mathbf{F}$ that assigns to each point (x, y) in D a two-dimensional vector $\mathbf{F}(x, y)$.

Since $\mathbf{F}(x, y)$ is a two-dimensional vector, we can write it in terms of its **component functions** P and Q as follows:

$$\mathbf{F}(x, y) = P(x, y)\ \mathbf{i} + Q(x, y)\ \mathbf{j}$$

P and Q are scalar functions and sometimes called **scalar fields** to distinguish them from vector fields.

Definition 8. A vector field $\mathbf{F}$ is called a **conservative vector field** if it is the gradient of some scalar function, that is, if $\mathbf{F} = \nabla f$. f is called a **potential function** for $\mathbf{F}$.

7 Line Integrals

Let C be a smooth curve (continuous and non-zero derivative) given by the parametric equations

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$$

Dividing C into n subarcs with lengths Δs_i, choosing any point $P_i(x_i, y_i)$ on the ith subarc, evaluating $f(x_i, y_i)$ and multiplying by the length of Δs_i yields

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta s_i$$

Imagine a finite sequence of rectangles positioned on C with height $f(x_i, y_i)$ and length Δs_i.

Definition 9. If f is defined on a smooth curve C given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, then the **line integral of f along C** is

$$\int_C f(x, y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta s_i$$

Since the length of C is

$$L = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\,dt$$

and if f is continuous, we have

$$\int_C f(x, y)\,ds = \int_a^b f\big(x(t), y(t)\big)\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\,dt$$

The value of the line integral does not depend on the parameterization of the curve provided that the curve is traversed exactly once as t increases from a to b.

Definitions for the line integral with respect to x or y exist, but what is the significance of these definitions? How to visualize...

A vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1$$

7.1 Line Integrals in Space

Similar to definition 7 (line integrals in a plane) the line integral of C a smooth curve in space given by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ where $a \le t \le b$ with respect to length is

$$\int_C f(x, y, z)\,ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|$$

7.2 Line Integrals of Vector Fields

Definition 10. Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t), a \le t \le b$. Then the *line integral of F along C* is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)dt = \int_C \mathbf{F} \cdot \mathbf{T}ds$$

Integrals with respect to arc length are independent of orientation.

8 Fundamental Theorem for Line Integrals

Recall the Fundamental Theorem of Calculus

$$\int_a^b F'(x)dx = F(b) - F(a)$$

where $F'(x)$ is continuous on $[a, b]$. Considering ∇f of a function f of two or three variables as a sort of derivative of f, then FTC can be extending to line integrals.

Theorem 2. Let C be a smooth curve given by the vector function $\mathbf{r}(t), a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

The line integral of a conservative vector field can be evaluated by simply knowing the value of f at the endpoints of C.

Proof.

$$\begin{aligned}
\int_C \nabla f \cdot dr &= \int_a^b \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \\
&= \int_a^b (\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt})dt \\
&= \int_a^b \frac{d}{dt} f(\mathbf{r}(t))dt \\
&= f(\mathbf{r}(b)) - f(\mathbf{r}(a))
\end{aligned}$$

□

8.1 Independence of Path

Theorem 3. $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D

Clearly it follows that $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for any closed path.

Proof. ... □

Theorem 4 (Independent Path → Conservative)**.** Let $\mathbf{F}$ be a continuous vector field on an open connected region D. If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field on D; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

Proof. ... □

Definition 11. A *simple curve* is a curve that does not intersect itself any where between its endpoints.

Theorem 5 (Conservative $\to$ Equal Partials)**.** If $\mathbf{F}(x,y) = P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D, then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Converse is true if D is a simply-connected region (i.e. simple closed curve with no holes).

Proof. ... $\square$

9 Curl

Curl is an operation on a vector field that represents the rotation at a point (x,y,z) about the axis that points in the direction of $\mathrm{curl}(\mathbf{F}(x,y,z))$.

Definition 12 (Curl)**.** If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derives of P, Q, and R all exist, then the **curl** of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\mathrm{curl}\ \mathbf{F} = \Big(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\Big)\mathbf{i} + \Big(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\Big)\mathbf{j} + \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\mathbf{k}$$

Looks like a vector component version of Green's Theorem. The Latex command for ∇ is *nabla* but it is called *del*?

This definition is easier to remember as

$$\mathrm{curl}\ \mathbf{F} = \nabla \times \mathbf{F}$$

Theorem 6 ($\mathbf{F}$ conservative $\to$ curl is 0)**.** If f is a function of three variables that has continuous second-order partial derivatives, then

$$\mathrm{curl}(\nabla f) = 0 \tag{29}$$

$\mathbf{F}$ conservative implies $\mathbf{F} = \nabla f$, then $\mathbf{F}$ conservative implies $\mathrm{curl}(\mathbf{F}) = 0$.

Theorem 7 (Restrictive converse)**.** If $\mathbf{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = 0$, then $\mathbf{F}$ is a conservative vector field.

10 Divergence

The divergence represents the net rate of change at a point.

Definition 13 (Divergence)**.** If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and $\nabla\mathbf{F}$ exists, then the **divergence** of $\mathbf{F}$ is the function of three variables defined by

$$\text{div }\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \tag{30}$$

This definition is easy to remember as

$$\text{div }\mathbf{F} = \nabla \cdot \mathbf{F} \tag{31}$$

Theorem 8 (No flow through curl?)**.** If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and T have continuous second-order partial derivatives, then

$$\text{div curl }\mathbf{F} = 0 \tag{32}$$

Note the scalar triple analogy $a \cdot (a \times b) = 0$.

11 Green's Theorem

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C. By convention, **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C.

Theorem 9 (Green's Theorem)**.** Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C. If P and Q have continuous partial derivatives on an open region that contains D, then

$$\int_C P\ dx + Q\ dy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dA$$

in vector form,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_D (\text{curl }\mathbf{F}) \cdot \mathbf{k}\ dA \tag{33}$$

which expresses the line integral of the tangential component of $\mathbf{F}$ along C as the double integral of the vertical component of curl $\mathbf{F}$ over the region D enclosed by C and in the other vector form

$$\oint_C \mathbf{F} \cdot \mathbf{n}\ ds = \iint_D \text{div}\,\mathbf{F}(x, y)\ dA \tag{34}$$

which expresses the line integral of the normal component of $\mathbf{F}$ along C as the double integral of the divergence of $\mathbf{F}$ over the region D enclosed by C.

Note the similarities between Green's Theorem and the fundamental theorem of calculus

$$\int_a^b F'(x)\ dx = F(b) - F(a)$$

Why do we differentiate between types of regions (i.e. Type I, Type II, and Type III)? Isn't each type simply an integration over an area projected onto either the xy-plane, xz-plane, or yz-plane?

Proof. Green's Theorem is true if and only if

$$\int_C P\ dx = -\iint_D \frac{\partial P}{\partial y} dA \tag{35}$$

and

$$\int_C Q\ dy = -\iint_D \frac{\partial Q}{\partial x} dA \tag{36}$$

Let D be a type I region and consider equation (34):

$$D = \{(x, y) : a \leq x \leq b \text{ and } g_1(x) \leq y \leq g_2(x)\}$$

$$-\int_a^b \int_{g_1(x)}^{g_2(x)} \frac{\partial P}{\partial y}\ dy\ dx = -\int_a^b P(g_2(x)) - P(g_1(x))\ dx$$

Let $C = C_1 \cup C_2 \cup C_3 \cup C_4$ and the (LHS) of equation (34) will achieve the same result. Showing equation (35) uses similar steps. □

11.1 Applications of Green's Theorem

- Calculating $\iint_D \frac{\partial Q}{\partial y} - \frac{\partial P}{\partial x}\ dA$ may be easier then $\int_C P\ dx + Q\ dy$
- When calculating areas (i.e. $\iint_D 1\ dA$), we can choose any P and Q such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$ which can yield the following equality:

$$A = \oint_C x\ dy = -\oint_C y\ dx = \frac{1}{2}\oint_C x\ dy - y\ dx$$

12 Surface Integrals

The relationship between surface integrals and surface area is similar to the relationship between line integrals and arc length.

12.1 Parametric Surfaces

To evaluate the surface integrals, we approximate path area ΔS_{ij} by the area of the an approximating parallelogram in the tangent plane. If the components $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel in D, then

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\,dA \tag{37}$$

where

$$\mathbf{r}_u = \frac{\partial x}{\partial u}\,\mathbf{i} + \frac{\partial y}{\partial u}\,\mathbf{j} + \frac{\partial z}{\partial u}\,\mathbf{k} \qquad \mathbf{r}_v = \frac{\partial x}{\partial v}\,\mathbf{i} + \frac{\partial v}{\partial v}\,\mathbf{j} + \frac{\partial z}{\partial v}\,\mathbf{k}$$

which is comparable to the formula for a line integral:

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|\,dt$$

Formula (36) allows to compute the surface integral of S be converting into a double integral over the parameter domain D.

12.2 Graphs

Let surface S be represented by the equation $z = g(x,y)$, then equation (36) can be used with the following parameterization:

$$x = x(u,v) \qquad y = y(u,v) \qquad z = g(u,v)$$

which yields

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Therefore, equation (36 becomes

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}$$

Similar formulas exist if it is convenient to project S onto the yz-plane or xz-plane.

13 Stoke's Theorem

Stoke's Theorem is a higher-dimensional version of Green's Theorem and relates a surface integral over a surface S to a line integral around the boundary curve of S.

Theorem 10 (Stoke's Theorem)**.** Let S be an oriented piecewise-smooth surface that is bounded by a simple closed, piecewise-smooth boundary curve C with positive orientation. LET $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region in $\mathbb{R}^3$ that contains S, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \mathrm{curl}\mathbf{F} \cdot d\mathbf{S}$$

Since

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds \qquad \text{and} \qquad \iint_S \mathrm{curl}\mathbf{F} \cdot d\mathbf{S} = \iint_S \mathrm{curl}\mathbf{F} \cdot \mathbf{n}\, d\mathbf{S}$$

Difficult to visualizewhich says that the line integral around the boundary curve of S of the tangential component of $\mathbf{F}$ is equal to the surface integral of the normal component of the curl of $\mathbf{F}$