Definition 1 (Positive-Definite)**.** A matrix A is *positive-definite* if for every nonzero vector x,

$$x^T Ax > 0 \tag{1}$$

Claim 1 (Positive-Definite $\rightarrow \lambda_i > 0$)**.** If A is a positive-definite matrix, then the eigenvalues of A are positive.

Proof. Let A be a positive-definite matrix. Since A is positive-definite

$$x^T Ax = x^T \lambda x = \lambda x^T x > 0$$

but $x^T x = \sum x_i^2$ which is greater than 0, thus λ is also greater than zero. □

Definition 2 (Diagonalizable)**.** A matrix A is diagonalizable if there exists an invertible matrix P such that PAP^{-1} is a diagonal matrix.

For all matrices A, it is not guaranteed a diagonlization exists. If a diagonalization exists, then $P^{-1}AP = D$ where D is some diagonal matrix so $AP = PD \rightarrow A_i\vec{\alpha}_i = \vec{\alpha}_i d_i = d_i\vec{\alpha}_i$ for $P = (\vec{\alpha}_1 \ \ldots \ \vec{\alpha}_n)$ so D is merely the lambda values of A. This implies we must have n eigenvalues but do they necessarily need to be distinct?

Claim 2. If A is diagonlizable and has only non-zero eigenvalues, then A is invertible.

Proof. Since D exists and consists of the eigenvalues of A which are non-zero, D^{-1} exists (and is easily calculable) and we have $D^{-1} = (P^{-1}AP)^{-1} = PA^{-1}P^{-1} \rightarrow A^{-1} = P^{-1}D^{-1}P$; therefore, A^{-1} also exists. □

Definition 3. A square matrix U is unitary if $U^{-1} = U^T$.

Theorem 1 (Invertible Matrix Theorem)**.** A is invertible if and only if any of the following hold:

- A is row-equivalent to the $n \times n$ identity matrix
- A has n pivot positions
- The equation $A\mathbf{x} = 0$ has only the trival solution $\mathbf{x} = 0$.
- The columns of A form a linearly independent set
- The linear transformation $x \mapsto Ax$ is one-to-one
- For each column vector $b \in \mathbb{R}^n$, the equation $A\mathbf{x} = b$ has a unique solution.
- The columns of A span $\mathbb{R}^n$
- The linear transformation $x \mapsto Ax$ is a surjection (onto)
- There is an $n \times n$ matrix C such that $CA = I_n$ or $AC = I_n$ (**Note:** C can be found by multiplying the elementary operation matrices)

- The transpose matrix A^T is invertible
- The columns of A form a basis for $\mathbb{R}^n$
- The column space of A is equal to $\mathbb{R}^n$
- The rank of A is n
- The null space of A is $\{0\}$
- The dimension of the null space of A is 0.
- 0 fails to be an eigenvalue of A (Seen in the claim above)
- The determinant of A is non-zero This is an if and only if?
- The orthogonal complement of the column space of A is $\{0\}$
- The orthogonal complement of the null space of A is $\mathbb{R}^n$
- The row space of A is $\mathbb{R}^n$
- The matrix A has n non-zero singular values

0.1 Finding Eigenvalues

Eigenvalues are the solutions to the equation

$$Av = \lambda v$$

for $v \neq 0$ or alternatively

$$(A - \lambda I)v = 0$$

Since $\det(AB) = \det(A) \cdot \det(B)$ and $v \neq 0$, $\det((A - \lambda I)v) = 0 \leftrightarrow \det(A - \lambda I) = 0$