Let C be a smooth curve (continuous and non-zero derivative) given by the parametric equations

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$$

Dividing C into n subarcs with lengths Δs_i, choosing any point $P_i(x_i, y_i)$ on the ith subarc, evaluating $f(x_i, y_i)$ and multiplying by the length of Δs_i yields

$$\sum_{i=1}^{n} f(x_i, y_i)\Delta s_i$$

Imagine a finite sequence of rectangles positioned on C with height $f(x_i, y_i)$ and length Δs_i.

Definition 1. If f is defined on a smooth curve C given by $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, then the **line integral of f along C** is

$$\int_C f(x, y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i, y_i)\Delta s_i$$

Since the length of C is

$$L = \int_a^b \sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\,dt$$

and if f is continuous, we have

$$\int_C f(x, y)\,ds = \int_a^b f\big(x(t), y(t)\big)\sqrt{\frac{dx}{dt}^2 + \frac{dy}{dt}^2}\,dt$$

The value of the line integral does not depend on the parameterization of the curve provided that the curve is traversed exactly once as t increases from a to b.

Definitions for the line integral with respect to x or y exist, but what is the significance of these definitions? How to visualize...

A vector representation of the line segment that starts at $\mathbf{r}_0$ and ends at $\mathbf{r}_1$ is given by

$$\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1 \qquad 0 \le t \le 1$$

0.1 Line Integrals in Space

Similar to definition **??** (line integrals in a plane) the line integral of C a smooth curve in space given by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ where $a \le t \le b$ with respect to length is

$$\int_C f(x, y, z)\,ds = \int_a^b f(\mathbf{r}(t))|\mathbf{r}'(t)|$$

0.2 Line Integrals of Vector Fields

Definition 2. Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t), a \leq t \leq b$. Then the *line integral of F along C* is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

Integrals with respect to arc length are independent of orientation.