Poisson's equation is

$$\Delta\varphi = f \tag{1}$$

where Δ is the Laplace operator, and f and φ are real or complex-valued functions. In Euclidean space, the Laplace operator is often denoted as ∇^2 and so Poisson's equation is frequently written as

$$\nabla^2\varphi = f \tag{2}$$

In three-dimensional Cartesian coordinates, it takes the form

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi(x, y, z) = f(x, y, z) \tag{3}$$

When $f = 0$ we have Laplace's Equation.

0.1 Problem Statement

Consider the second-order differential equation

$$\frac{\partial^2}{\partial x^2}\varphi(x) = f(x) \tag{4}$$

Use the finite difference method to approximate the partial differential equation over the interval $x \in (a, b)$ assuming a uniform spatial discretization Δx. φ need not be twice-differentiable over $\mathbb{R}$ but only over the interval $x \in (a, b)$. Although conceptually easy, finite differences are difficult to apply over domains with heterogenous composition?

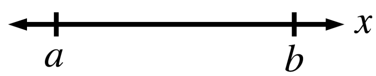
0.2 Finite-Difference Method

The Finite-Difference Method (FDM) is a numerical procedure which solves a PDE by:

- Discretizing the continuous physical domain into a discrete **finite difference grid**
- Approximating the individual exact partial derivatives in the PDE by algebraic **finite difference approximations (FDAs)**
- Substituting the FDAs into the PDE to obtain an algebraic **finite difference equation (FDE)**
- Solving the resulting algebraic FDEs for the dependent variable.

0.2.1 Finite Difference Grid

Consider the following one-dimensional domain D(x),



which must be covered by a one-dimensional grid of lines (figure 0.2.1). Let $x_i = a + (i-1)\Delta x$, $f(x_i) = f_i$, $f_x(x_i) = f_x|_i$ and $f_{xx}(x_i) = f_{xx}|_i$.

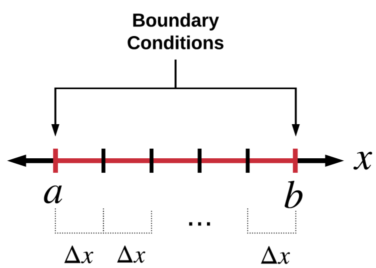


Figure 1: Discrete difference grid with a uniform spatial discretization of Δx

0.2.2 Finite Difference Approximations

Note: For elliptic PDEs containing only second derivatives, there are no preferred physical informative propagation paths. Thus, centered-space finite difference approximations should be used for the second-order spatial derivatives in the Laplace equation and the Poisson equation.

Exact Solution: $\bar{f}(x)$
Approximate Solution: $f(x)$

Consider the Taylor Series expansion of $\bar{f}_{i-1}$ and $\bar{f}_{i-1}$ using i as the base point gives

$$\bar{f}_{i+1} = \bar{f}_i + \bar{f}_x|_i \Delta x + \frac{1}{2}\bar{f}_{xx}|_i (\Delta x)^2 + \cdots \tag{5}$$

$$\bar{f}_{i-1} = \bar{f}_i - \bar{f}_x|_i \Delta x + \frac{1}{2}\bar{f}_{xx}|_i (\Delta x)^2 - \cdots \tag{6}$$

Adding the first three terms of equations (5) and (6) (up to the second partial) we can approximate $\bar{f}_{xx}|_i$,

$$f_{xx}|_i = \frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2} \tag{7}$$

which serves as our second-order centered-difference approximation at i.

0.3 Finite Difference Solution of the Laplace Equation

If $f(x) = 0$ in equation (4) of our problem statement, we have the Laplace Equation

$$\frac{\partial^2}{\partial x^2}\varphi(x) = \bar{f}_{xx} = 0 \tag{8}$$

Substituting equation (7) yields

$$\frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2} = 0 \tag{9}$$

while solving for f_i gives

$$f_i = \frac{f_{i-1} + f_{i+1}}{2} \tag{10}$$

Simply stated, the solution at every point is the arithmetic average of the solutions of the neighboring points. **Note:** This result applies only to the Laplace equation (i.e. no nonhomogeneous term).

Let u_i represent the approximation of $\varphi(x_i)$ in (8). Assuming the uniform spatial discretization yields n partitions, we have the following system of equations,

$$\begin{aligned}
u_2 &= \frac{a + u_3}{2} \\
u_3 &= \frac{u_2 + u_4}{2} \\
&\vdots \\
u_{n-2} &= \frac{u_{n-3} + u_{n-1}}{2} \\
u_{n-1} &= \frac{u_{n-2} + b}{2}
\end{aligned} \tag{11}$$

or

$$\begin{aligned}
(2)u_2 - (1)u_3 + (0)u_4 + \cdots + (0)u_{n-3} + (0)u_{n-2} + (0)u_{n-1} &= a \\
-(1)u_2 + (2)u_3 - (1)u_4 + \cdots + (0)u_{n-3} + (0)u_{n-2} + (0)u_{n-1} &= 0 \\
&\vdots \\
(0)u_2 + (0)u_3 + (0)u_4 + \cdots - (1)u_{n-3} + (2)u_{n-2} - (1)u_{n-1} &= 0 \\
(0)u_2 + (0)u_3 + (0)u_4 + \cdots + (0)u_{n-3} + (1)u_{n-2} + (2)u_{n-1} &= b
\end{aligned}$$

which when expressed in matrix form

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix}$$

0.4 Finite Difference Solution of the Poisson Equation

Similar to equation (8) except that $f(x) \neq 0$. Let $f(x) = g(x)$, then we have

$$\frac{\partial^2}{\partial x^2}\varphi(x) = \bar{f}_{xx} = g(x) \tag{12}$$

For $g(x) \neq 0$, equations (9) and (10) become

$$\frac{f_{i+1} - 2f_i + f_{i-1}}{(\Delta x)^2} = g(x) \tag{13}$$

$$f_i = \frac{f_{i-1} + f_{i+1} - g_i(\Delta x)^2}{2} \tag{14}$$

which yields a similar system of equations when represented in matrix form gives

$$\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ldots & -1 & 2 & -1 \\ 0 & 0 & 0 & \ldots & 0 & -1 & 2 \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ u_3 \\ \vdots \\ u_{n-2} \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} a - g_2(\Delta x)^2 \\ -g_3(\Delta x)^2 \\ \vdots \\ -g_{n-2}(\Delta x)^2 \\ b - g_{n-1}(\Delta x)^2 \end{bmatrix}$$

0.5 Derivative Boundary Conditions

Consider equation (4) in our problem statement with unknown boundary point conditions but known derivatives. We approximate these unknown boundary points using their derivatives and adjacent interior point. Recall the approximation for f_I,

$$f_{I+1} + f_{I-1} - 2f_I = g_I(\Delta x)^2 \tag{15}$$

for either f_{I-1} or f_{I+1} unknown (depending on which boundary point we are considering). Using the difference formula

$$f_x|_I = \frac{\bar{f}_{I+1} - \bar{f}_{I-1}}{2\Delta x} \tag{16}$$

and solving for either f_{I-1} or f_{I+1} (again depending on which boundary point we are considering) we arrive at

$$2f_{I-1} - 2f_I = -2\bar{f}_x|_I\Delta x + g_I(\Delta x)^2 \tag{17}$$

$$2f_{I+1} - 2f_I = -2\bar{f}_x|_I\Delta x + g_I(\Delta x)^2 \tag{18}$$

which can be explicitly solved for f_{I-1} or f_{I+1}. **Note:** $\bar{f}_x|_I$ and g_I in these equations are not necessarily the same. The original form of equation (16) is

$$f_x|_i = \frac{f_{i+1} - f_{i-1}}{2\Delta x} - \frac{1}{6}f_{xxx}(\zeta)(\Delta x)^2 \tag{19}$$

I easily see how equation (16) was derived but not equation (19). Is this derived from the Taylor Series?

0.6 Iterative Methods of Solution

0.6.1 Jacobi Iteration

0.6.2 Gauss-Seidel Iteration

0.6.3 Successive-over-relaxation (SOR)