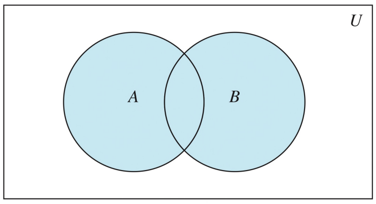
2.2 Set Operations

Definition of union $\cup$

Let A and B be sets. The *union* of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.



$A \cup B$ is shaded.

Examples:

$$\{1,2,3\} \cup \{1,\{2\},\{3\}\} = \{1,2,3,\{2\},\{3\}\}$$

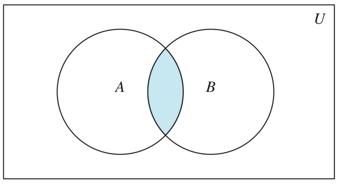
$$\{\ \} \cup \{1\} = \{\cancel{\{\},}\{1\}\} = \{1\}$$

Can you define union of sets using set builder notation?

$$A \cup B = \{x \mid x \in A \vee x \in B\}$$

Definition of intersection $\cap$

Let A and B be sets. The *intersection* of the sets A and B, denoted by $A \cap B$, is the set containing those elements in both A and B.



$A \cap B$ is shaded.

Examples:

$$\{1,2,3\} \cap \{\{1\},\{2\},3\} = \{\cancel{1},\cancel{2},3\}$$
$$= \{3\}$$

$$\{x \in \mathbb{N} \mid 1 \leq x \leq 10\} \cap \{x \in \mathbb{Z} \mid x \geq 10\} = \{10\}$$

$$\{x \in \mathbb{N} \mid 1 \leq x \leq 10\} \cap \{x \in \mathbb{Z} \mid x > 10\} = \{\ \} = \emptyset$$

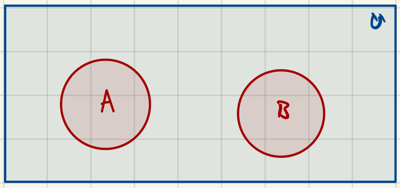
Can you define intersection of sets using set builder notation?

$$A \cap B = \{x \mid x \in A \wedge x \in B\}$$

Disjoint Sets

Def:

Two sets are called *disjoint* if their intersection is the empty set.



A & B are disjoint

if $A \cap B = \emptyset \rightarrow$ A and B are disjoint

EXAMPLE 5 Let $A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 4, 6, 8, 10\}$. Because $A \cap B = \emptyset$, A and B are disjoint.

Difference of sets

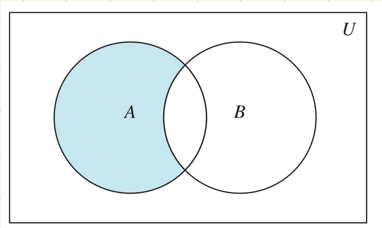
Def:

Let A and B be sets. The *difference* of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B. The difference of A and B is also called the *complement of B with respect to A*.

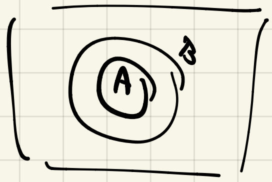
$$A - B = \{x \mid x \in A \wedge x \notin B\}$$

$$B - A = \{x \mid x \in B \wedge x \notin A\}$$

$$A - B \neq B - A$$



$A - B$ is shaded.



$A - B$

* Some books denote difference by $A \setminus B$.

Example

$A = \{1, 2, 3, 10\}$, $B = \{5, 1, 6, x, a\}$

$A - B = \{2, 3, 10\}$

$B - A = \{6, 5, x, a\}$

Example

$A = \{x \mid x \in \mathbb{N}\}$, $B = \{x \mid x \in \mathbb{Z}\}$, $C = \{x \mid x \in \mathbb{R}^+\}$

Positive real numbers

$A - B = \{\ \} = \emptyset$

$B - A = \{x \mid x \in \mathbb{Z}^-\}$

$C - A = \{x \mid x \notin \mathbb{N} \wedge x \in \mathbb{R}^+\} = \{x \mid \exists a_{\in \mathbb{N}}\ x \in (a, a+1) \}$

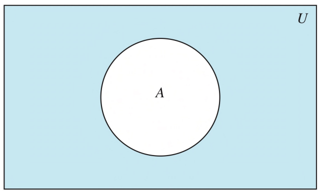
open interval $\neq$ order pairs

Complement

Def:

Let U be the universal set. The *complement* of the set A, denoted by $\overline{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is $U - A$.

$$\overline{A} = \{x \mid x \in U \wedge x \notin A\}$$



$\overline{A}$ is shaded.

Example

$A = \{x \mid 0 \leq x \leq 100\}$ where $U = \mathbb{Z}^+$

$\overline{A} = \{x \mid x > 100 \wedge x \in \mathbb{Z}^+\}$

Set Identities

From Chapter 1

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

EXAMPLE 11 Use set builder notation and logical equivalences to establish the first De Morgan law $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Solution: We can prove this identity with the following steps.

$$
\begin{aligned}
\overline{A \cap B} &= \{x \mid x \notin A \cap B\} && \text{by definition of complement}\\
&= \{x \mid \neg(x \in (A \cap B))\} && \text{by definition of does not belong symbol}\\
&= \{x \mid \neg(x \in A \wedge x \in B)\} && \text{by definition of intersection}\\
&= \{x \mid \neg(x \in A) \vee \neg(x \in B)\} && \text{by the first De Morgan law for logical equivalences}\\
&= \{x \mid x \notin A \vee x \notin B\} && \text{by definition of does not belong symbol}\\
&= \{x \mid x \in \overline{A} \vee x \in \overline{B}\} && \text{by definition of complement}\\
&= \{x \mid x \in \overline{A} \cup \overline{B}\} && \text{by definition of union}\\
&= \overline{A} \cup \overline{B} && \text{by meaning of set builder notation}
\end{aligned}
$$

EXAMPLE 14 Let A, B, and C be sets. Show that

$$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}.$$

$\underline{A \cup B \cup C}$

Solution: We have

$$
\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap \overline{(B \cap C)} && \text{by the first De Morgan law}\\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law}\\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections}\\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}
$$