CHAPTER

10 Graphs

10.1 Graphs and Graph Models

DEFINITION 1 A *graph* $G = (V, E)$ consists of V, a nonempty set of *vertices* (or *nodes*) and E, a set of *edges*. Each edge has either one or two vertices associated with it, called its *endpoints*. An edge is said to *connect* its endpoints.

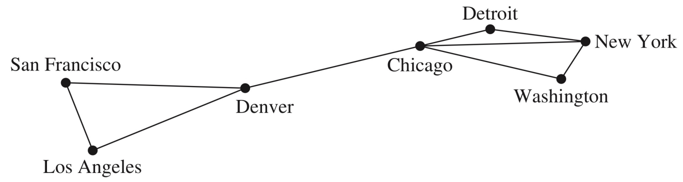


FIGURE 1 **A Computer Network.**

simple graph

loops & multi edges are not allowed.

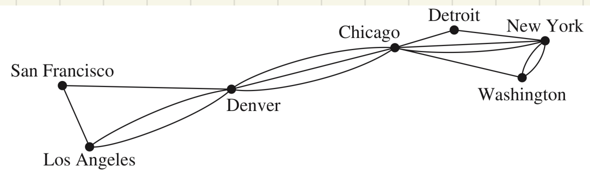


FIGURE 2 **A Computer Network with Multiple Links between Data Centers.**

Multigraph

Multiple edges are allowed but loop are not.

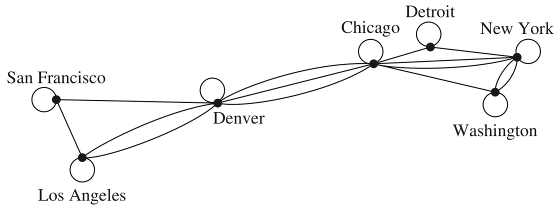


FIGURE 3 **A Computer Network with Diagnostic Links.**

Pseudograph

Both loops and mult edges are allowed.

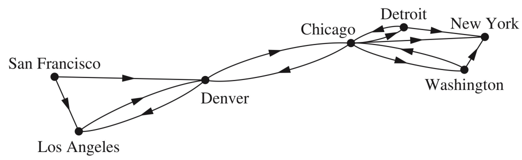
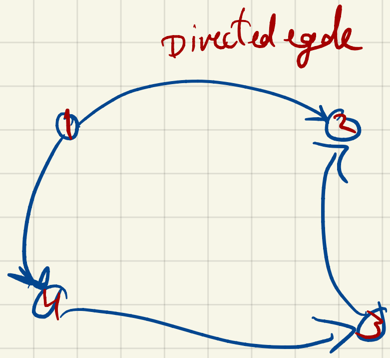


FIGURE 4 **A Communications Network with One-Way Communications Links.**

Directed Multigraph

DEFINITION 2 A *directed graph* (or *digraph*) (V, E) consists of a nonempty set of vertices V and a set of *directed edges* (or *arcs*) E. Each directed edge is associated with an ordered pair of vertices. The directed edge associated with the ordered pair (u, v) is said to *start* at u and *end* at v.

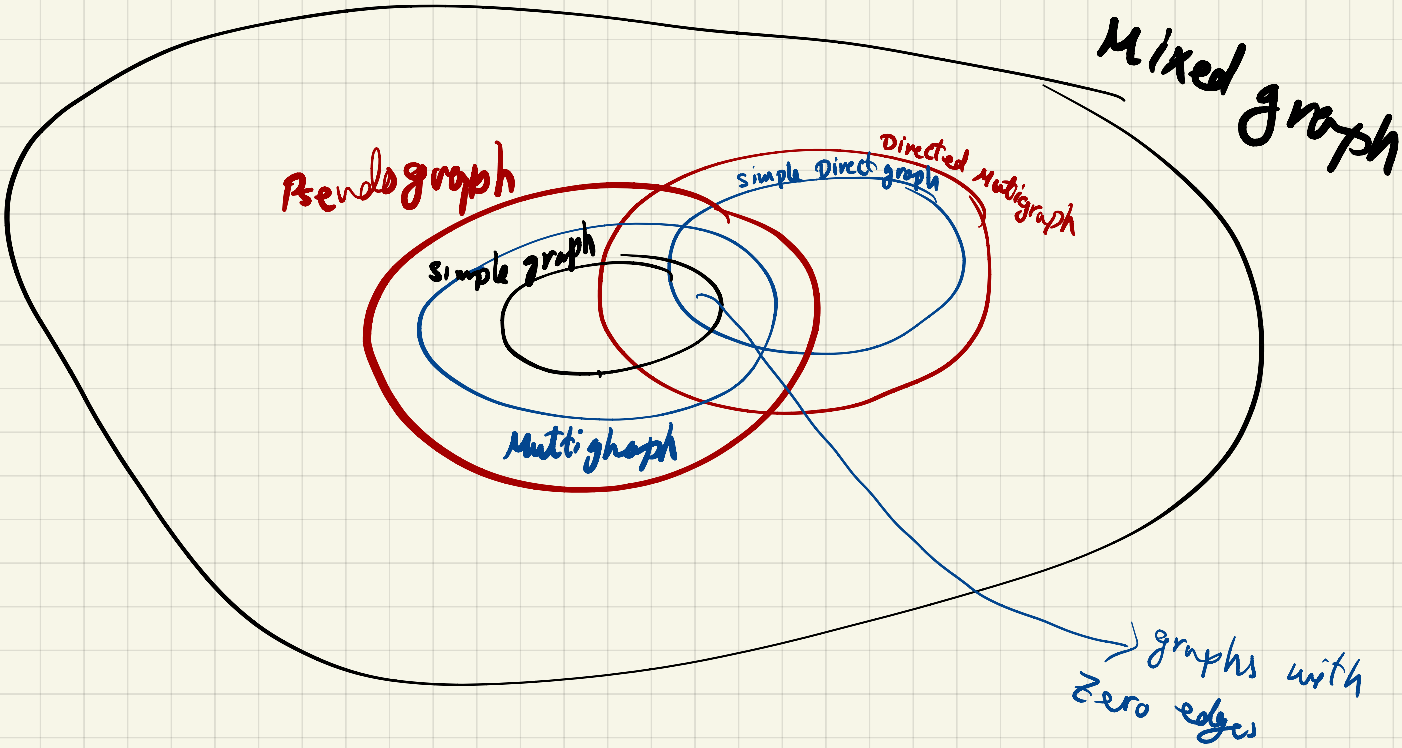


(1,2)

start at 1 and ends at 2

TABLE 1 **Graph Terminology.**

Type	*Edges*	*Multiple Edges Allowed?*	*Loops Allowed?*
Simple graph	Undirected	No	No
Multigraph	Undirected	Yes	No
Pseudograph	Undirected	Yes	Yes
Simple directed graph	Directed	No	No
Directed multigraph	Directed	Yes	Yes
Mixed graph	Directed and undirected	Yes	Yes



Graph Models

SOCIAL NETWORKS

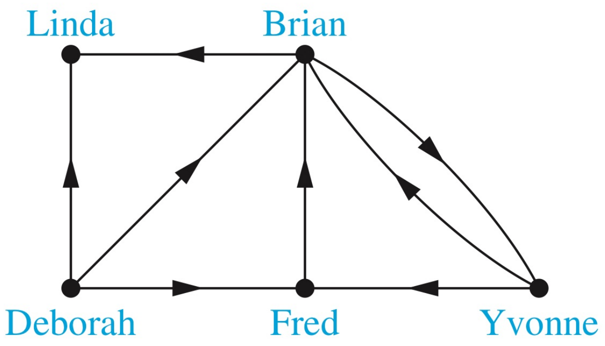


FIGURE 7 **An Influence Graph.**

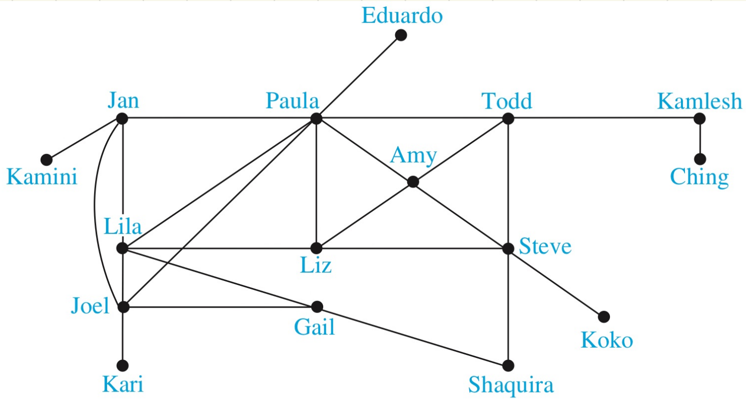


FIGURE 6 **An Acquaintanceship Graph.**

TOURNAMENTS

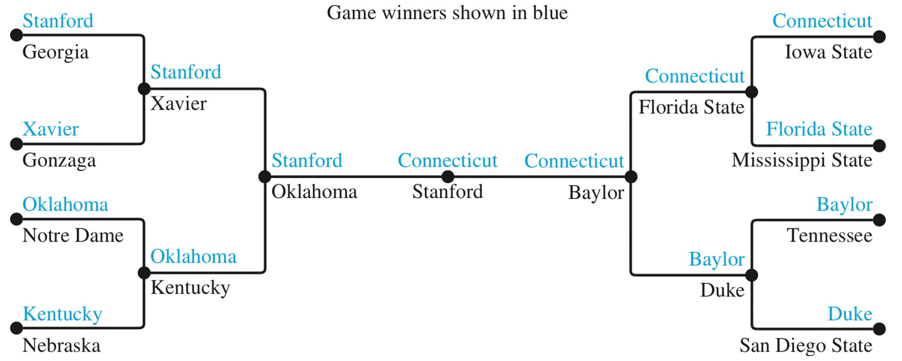


FIGURE 14 **A Single-Elimination Tournament.**

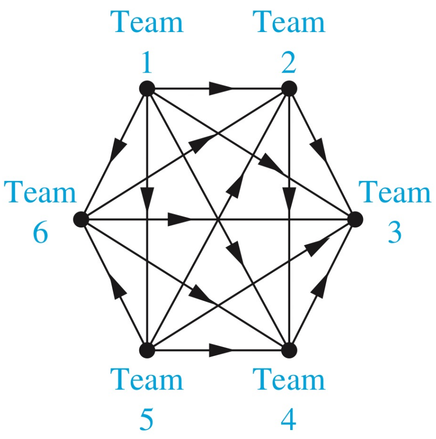
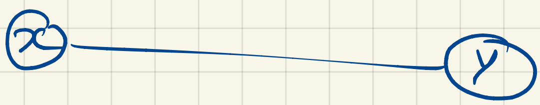


FIGURE 13 **A Graph Model of a Round-Robin Tournament.**

10.2 Graph Terminology and Special Types of Graphs

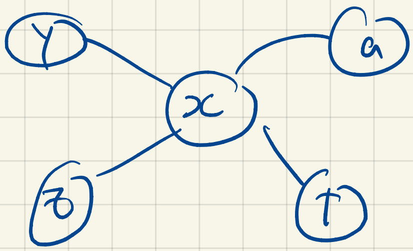
Basic Terminology

DEFINITION 1 Two vertices u and v in an undirected graph G are called *adjacent* (or *neighbors*) in G if u and v are endpoints of an edge e of G. Such an edge e is called *incident with* the vertices u and v and e is said to *connect* u and v.



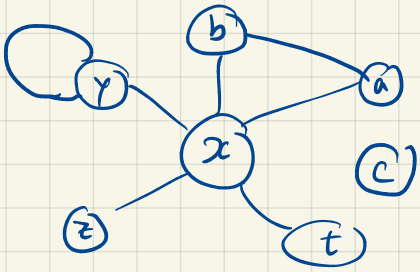
x is adjacent to node y
y is adjacent to node x

DEFINITION 2 The set of all neighbors of a vertex v of $G = (V, E)$, denoted by $N(v)$, is called the *neighborhood* of v. If A is a subset of V, we denote by $N(A)$ the set of all vertices in G that are adjacent to at least one vertex in A. So, $N(A) = \bigcup_{v \in A} N(v)$.



$N(x) = \{y, z, t, a\}$

DEFINITION 3 The *degree of a vertex in an undirected graph* is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex v is denoted by $\deg(v)$.



$\deg : V \to \mathbb{N}$

$\deg(x) = 5$ $\quad \deg(c) = 0$

$\deg(a) = 2$ $\quad \deg(y) = 3$

$\deg(z) = 1$

EXAMPLE 1 What are the degrees and what are the neighborhoods of the vertices in the graphs G and H displayed in Figure 1?

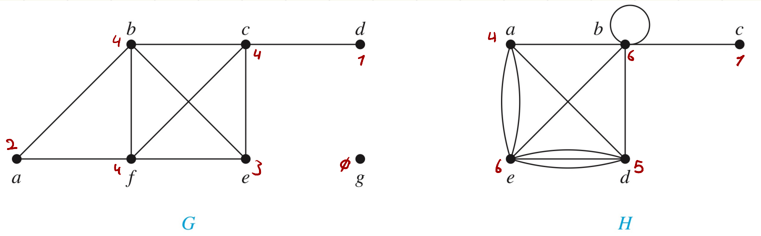


FIGURE 1 **The Undirected Graphs G and H.**

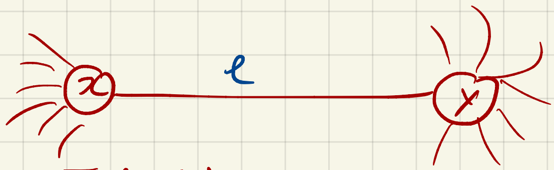
$m = 9$

$\sum deg(v) = 2+2+4+3+4+1+0$
$= 18 = \underline{2 * M}$

THEOREM 1 **THE HANDSHAKING THEOREM** Let $G = (V, E)$ be an undirected graph with m edges. Then

$$2m = \sum_{v \in V} \deg(v).$$

(Note that this applies even if multiple edges and loops are present.)



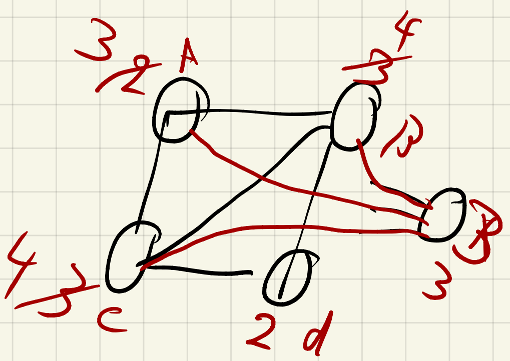
$\sum_{v \in V} deg(v) = \cdots + e + \cdots + e + \cdots$

EXAMPLE 3 How many edges are there in a graph with 10 vertices each of degree six?

Solution: Because the sum of the degrees of the vertices is $6 \cdot 10 = 60$, it follows that $2m = 60$ where m is the number of edges. Therefore, $m = 30$. ◀

$\sum_{v \in V} deg(v) = \sum 6 = 6 \cdot 10 = \underline{60} \implies$ # of edges $\frac{60}{2}$
$= 30$

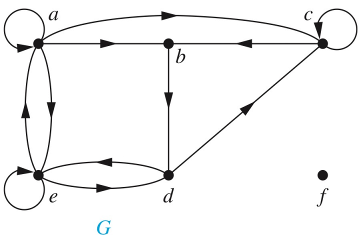
THEOREM 2 An undirected graph has an even number of vertices of odd degree.



$\underline{9 + 4 = 13}$

DEFINITION 5 In a graph with directed edges the *in-degree of a vertex v*, denoted by $\deg^-(v)$, is the number of edges with v as their terminal vertex. The *out-degree of v*, denoted by $\deg^+(v)$, is the number of edges with v as their initial vertex. (Note that a loop at a vertex contributes 1 to both the in-degree and the out-degree of this vertex.)

EXAMPLE 4 Find the in-degree and out-degree of each vertex in the graph G with directed edges shown in Figure 2.



$\deg^+(a) = 4$ $\deg^-(a) = 2$

$\deg^+(d) = 2$ $\deg^-(d) = 2$

$\deg^+(f) = 0$ $\deg^-(f) = 0$

FIGURE 2 The Directed Graph *G*.

THEOREM 3 Let $G = (V, E)$ be a graph with directed edges. Then

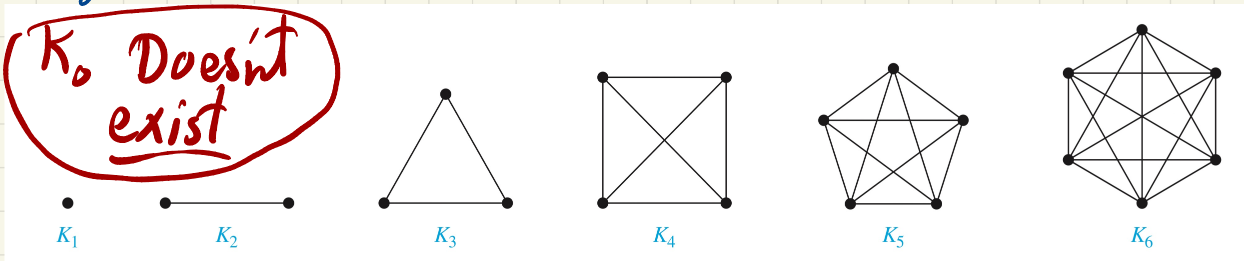
$$\sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v) = |E|.$$

Some Special Simple Graphs

Complete Graphs

In a Complete graph K_n on n nodes, every pair of nodes in K_n share an edge. In other words, $E = \{(a,b) \mid a \neq b \wedge a,b \in V\}$, $V = \{1,2,\cdots,n\}$, $K_n = (V,E)$.

K_0 Doesn't exist



# of edges in $K_n = \sum_{i=1}^{n} deg(i) / 2 = \frac{n(n-1)}{2}$.

K_n

$\sum deg(V) = n(n-1)$

$deg(V) = n-1$

$V \in K_n$

Cycles

Cycles A **cycle** C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}$, $\{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}$, and $\{v_n, v_1\}$. The cycles C_3, C_4, C_5, and C_6 are displayed in Figure 4. ◀

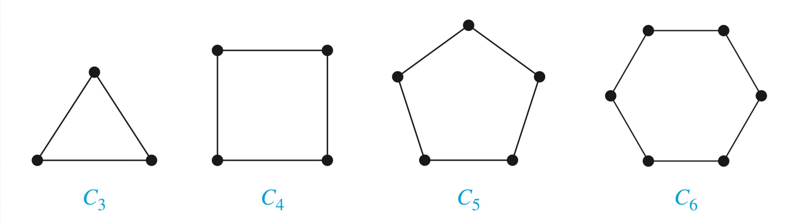
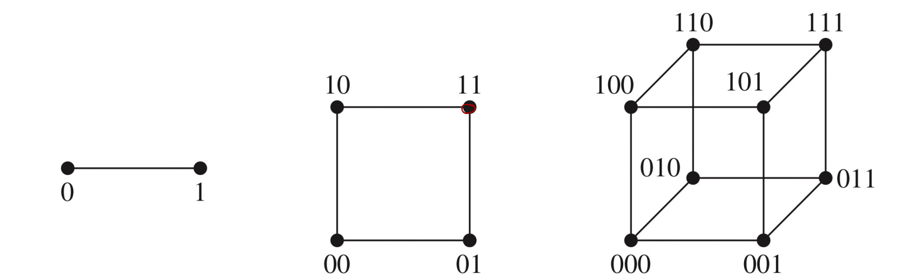


FIGURE 4 **The Cycles C_3, C_4, C_5, and C_6.**

An *n*-dimensional hypercube, or *n*-cube, denoted by Q_n



2

$2^2 = 2 \times 2$

$2 \times 2^2 = 2^3$

FIGURE 6 **The *n*-cube Q_n, $n = 1, 2, 3$.**

Q_n is a simple graph with 2^n. Nodes i, j are adjacent if and only if the bit string representation of i & j differ by one position only.

$T(1) = 1$

$T(2) = 2 * T(1) + 2^{2-1} = 4$

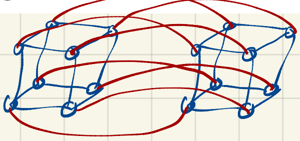
$T(3) = 2 * T(2) + 2^{3-1} = 12$

$T(4) = 2 * T(3) + 2^{4-1} = 32$

$T(n) = 2T(n-1) + 2^{n-1}$

$T(n) = \frac{n * 2^n}{2}$

$Q_4 \Rightarrow$



4-Cube

Bipartite Graphs

DEFINITION 6 A simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2). When this condition holds, we call the pair (V_1, V_2) a *bipartition* of the vertex set V of G.

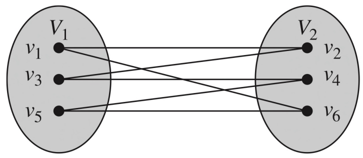


FIGURE 7 **Showing That C_6 Is Bipartite.**

all C_n are bipartite grahp only if n is even.

THEOREM 4 A simple graph is bipartite if and only if it is possible to assign one of two different colors to each vertex of the graph so that no two adjacent vertices are assigned the same color.

Are the below graphs bipartite?

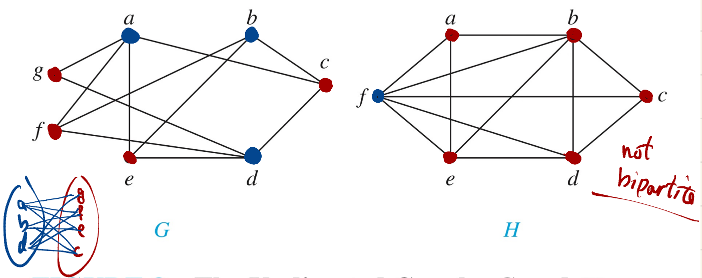
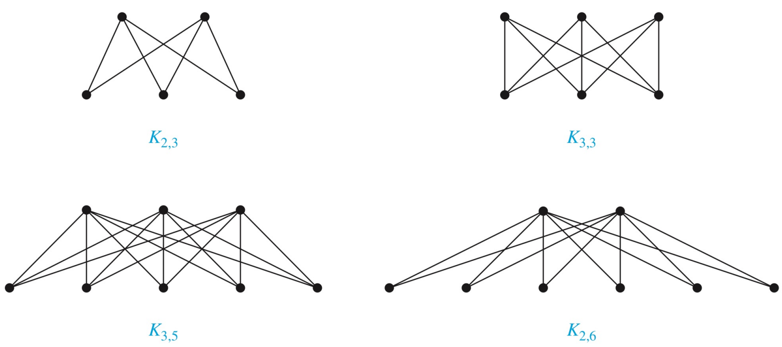


FIGURE 8 **The Undirected Graphs G and H.**

Complete bipartite graphs $K_{n,m}$

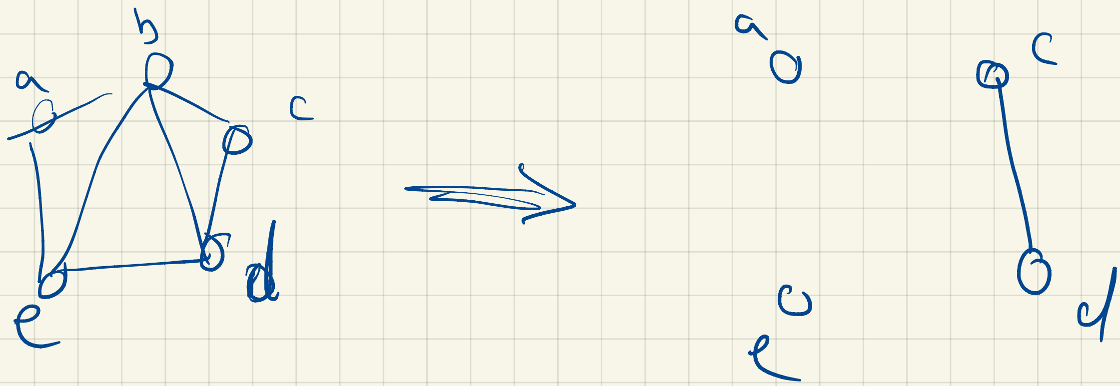


New Graphs from Old

Given a graph $G=(V,E)$, since V & E are sets, then we can directly use set operations to define new graphs.

A is proper subset of B is a subset s.t. $A \neq B$, $A \subset B$

DEFINITION 7 A *subgraph of a graph* $G = (V, E)$ is a graph $H = (W, F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a *proper subgraph* of G if $H \neq G$.



DEFINITION 8 Let $G = (V, E)$ be a simple graph. The **subgraph induced** by a subset W of the vertex set V is the graph (W, F), where the edge set F contains an edge in E if and only if both endpoints of this edge are in W.