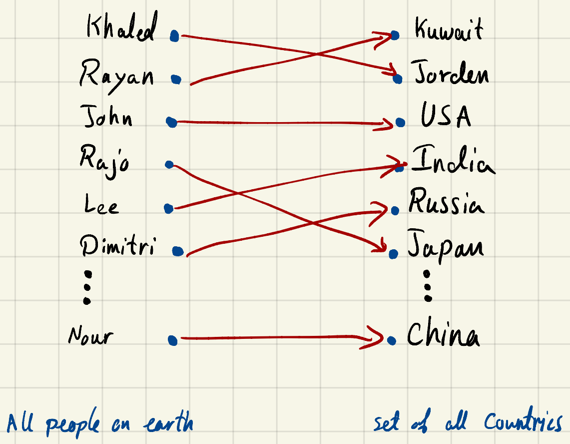
2.3 Functions

- A systematic way to map elements from one set to another set or to itself

Example



Here we mapped People to Countries they belong to.

Can you think of a good Name for such mapping?

Example

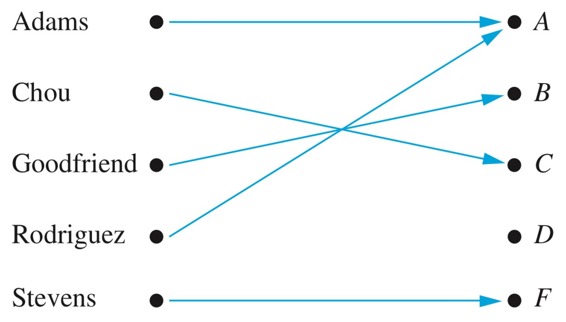


FIGURE 1 **Assignment of Grades in a Discrete Mathematics Class.**

Defi:

Let A and B be nonempty sets. A *function* f from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.

Remark: Functions are sometimes also called **mappings** or **transformations**.

Defi:

If f is a function from A to B, we say that A is the *domain* of f and B is the *codomain* of f. If $f(a) = b$, we say that b is the *image* of a and a is a *preimage* of b. The *range*, or *image*, of f is the set of all images of elements of A. Also, if f is a function from A to B, we say that f *maps* A to B.

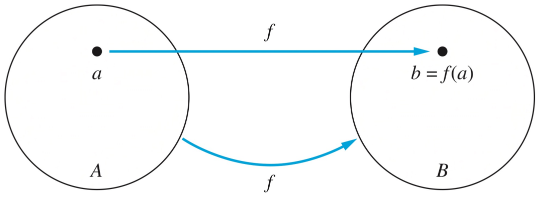
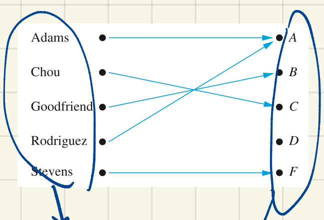


FIGURE 2 **The Function f Maps A to B.**

- A is the domain of f.
- B is the codomain of f.
- b is the image of a.
- a is the preimage of b.
- The range of f is the set all images of elements in A.

Example



function f

Domain = {students}

Codomain = {grades}

Image of Adam = A

Preimage of A = {Addms, Rodi}

preimage of D = { }

range of f = {A, B, C, F}

One-to-One

Def:

A function f is said to be *one-to-one*, or an *injunction*, if and only if $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f. A function is said to be *injective* if it is one-to-one.

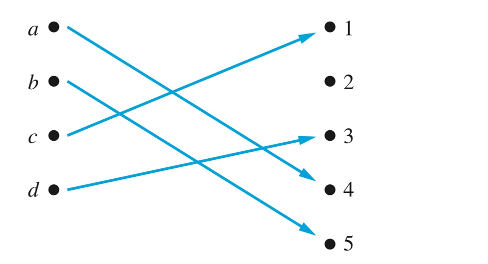
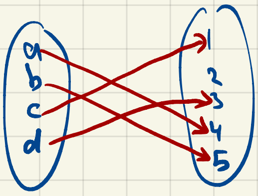


FIGURE 3 **A One-to-One Function.**

if $f(a)=f(b)$
and $a,b \in$ Domain
then $a=b$

EXAMPLE 8 Determine whether the function f from $\{a, b, c, d\}$ to $\{1, 2, 3, 4, 5\}$ with $f(a) = 4$, $f(b) = 5$, $f(c) = 1$, and $f(d) = 3$ is one-to-one.

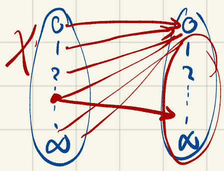


No two elements from the domain point to the same element from Codomain

if $f(a)=f(b)$ the $a=b$

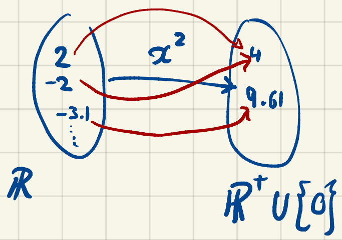
$f(x) = 0$

$f: \mathbb{N} \to \mathbb{N} = \mathbb{N} \to \{0\}$



Side note

EXAMPLE 9 Determine whether the function $f(x) = x^2$ from the set of integers to the set of integers is one-to-one.



Not one-to-one

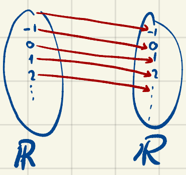
Because $f(2)=f(-2) = 4$

$f(x) = x^2$

$f(-x) = (-x)^2 = x^2$

$x \neq -x$

EXAMPLE 10 Determine whether the function $f(x) = x + 1$ from the set of real numbers to itself is one-to-one.



if $x \neq y$ and $f(x)=f(y)$

$\Rightarrow x+1 = y+1 \Rightarrow x = y$

Onto Functions

Def:

A function f from A to B is called *onto*, or a *surjection*, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$. A function f is called *surjective* if it is onto.

EXAMPLE 12 Let f be the function from $\{a, b, c, d\}$ to $\{1, 2, 3\}$ defined by $f(a) = 3$, $f(b) = 2$, $f(c) = 1$, and $f(d) = 3$. Is f an onto function?

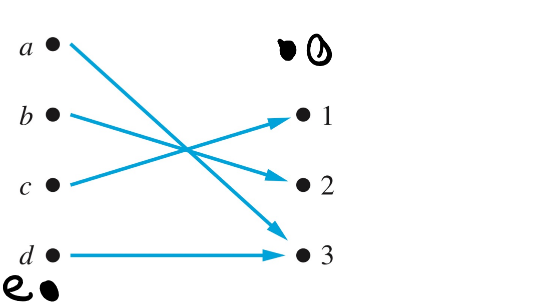
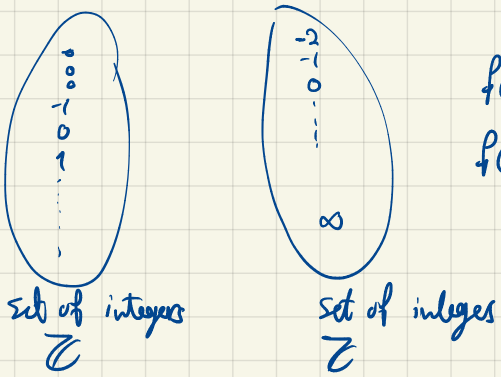


FIGURE 4 **An Onto Function.**

* Every element in the codomain has a preimage.

EXAMPLE 13 Is the function $f(x) = x^2$ from the set of integers (domain) to the set of integers onto?

Solution: The function f is not onto because there is no integer x with $x^2 = -1$, for instance.

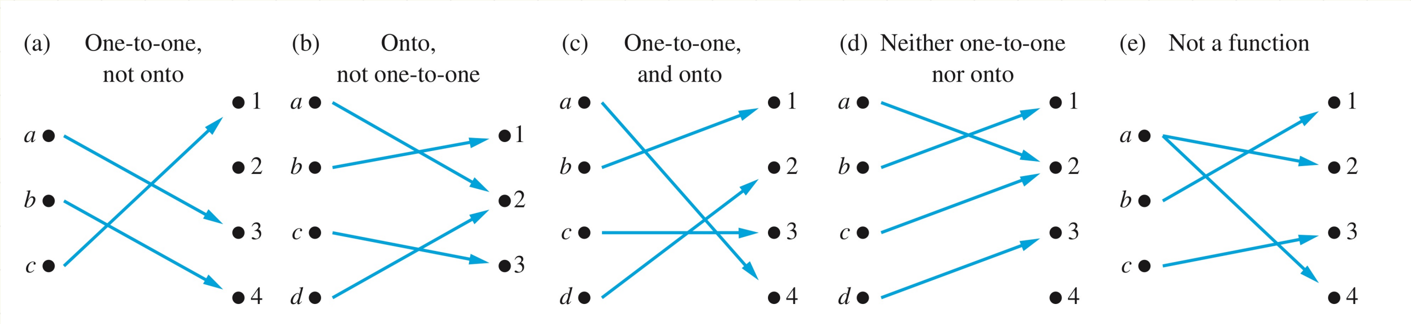


$f(a) = a^2 = -2$

$f(b) = b^2 = 3$

This function is not onto.

Example of functions



Def:

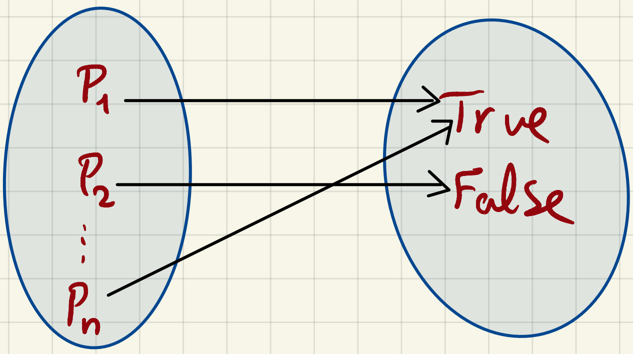
The function f is a *one-to-one correspondence,* or a *bijection,* if it is both one-to-one and onto. We also say that such a function is *bijective*.

$f(x) = x^2 \Rightarrow$ one single element

$f(x) = x^2 + 2x + 1 =$

$f(\text{Adam}) = A$

Example



set of Propositions possible values of propositions

* Functions help us to deepen our understanding of how proposition are mapped to truth values.

$f: A \to B \Rightarrow f$ is one-to-one Correspondence

$|A| \leqq |B|$

$\Rightarrow f$ is one-to-one $\Rightarrow |A| \leqq |B|$

$\Rightarrow f$ is onto $\Rightarrow |A| \geqq |B|$

Suppose that $f : A \to B$.

To show that f is injective Show that if $f(x) = f(y)$ for arbitrary $x, y \in A$ with $x \neq y$, then $x = y$.

To show that f is not injective Find particular elements $x, y \in A$ such that $x \neq y$ and $f(x) = f(y)$.

To show that f is surjective Consider an arbitrary element $y \in B$ and find an element $x \in A$ such that $f(x) = y$.

To show that f is not surjective Find a particular $y \in B$ such that $f(x) \neq y$ for all $x \in A$.

Inverse Functions

Def:

Let f be a one-to-one correspondence from the set A to the set B. The *inverse function* of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$. The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.

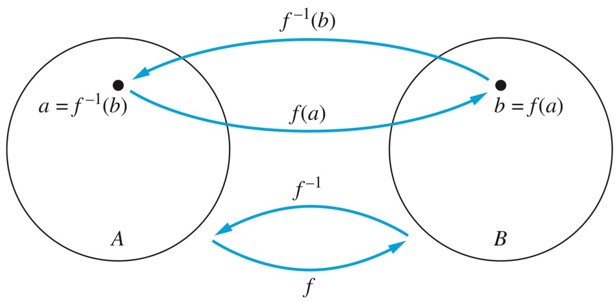


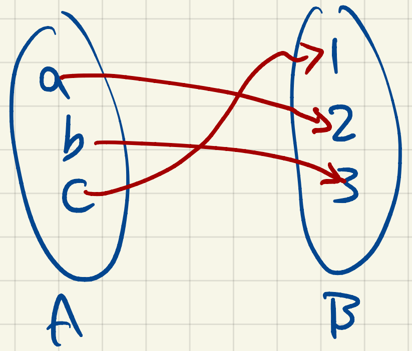
FIGURE 6 The Function f^{-1} Is the Inverse of Function f.

* If f^{-1} exist, then we call f invertable. otherwise f is not invertable.

* f is invertable if and only if f is one-to-one correspondance.

EXAMPLE 18 Let f be the function from $\{a, b, c\}$ to $\{1, 2, 3\}$ such that $f(a) = 2$, $f(b) = 3$, and $f(c) = 1$. Is f invertible, and if it is, what is its inverse?

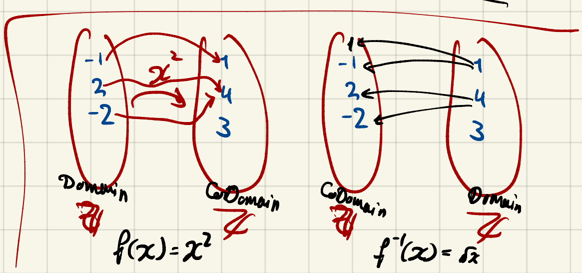
Solution: The function f is invertible because it is a one-to-one correspondence. The inverse function f^{-1} reverses the correspondence given by f, so $f^{-1}(1) = c$, $f^{-1}(2) = a$, and $f^{-1}(3) = b$. ◀



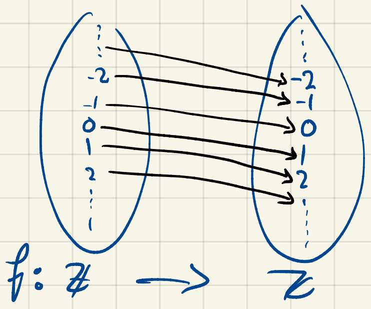
$f: A \to B$, $f^{-1}: B \to A$

To check if f is invertable

1. Is f one-to-one? Yes
2. Is f onto? Yes
3. Is f one-to-one correspondance? Yes



EXAMPLE 19 Let $f : \mathbf{Z} \to \mathbf{Z}$ be such that $f(x) = x + 1$. Is f invertible, and if it is, what is its inverse?



$f^{-1}(x) = x-1$

$f(x) = x+1$

$f^{-1}(x+1) = x+1-1 = x$

EXAMPLE 20 Let f be the function from $\mathbf{R}$ to $\mathbf{R}$ with $f(x) = x^2$. Is f invertible?

$f(2) = f(-2) = 4$ is not one-to-one Correspondence.

Compositions of Functions

Let g be a function from the set A to the set B and let f be a function from the set B to the set C. The *composition* of the functions f and g, denoted for all $a \in A$ by $f \circ g$, is defined by

$$(f \circ g)(a) = f(g(a)).$$

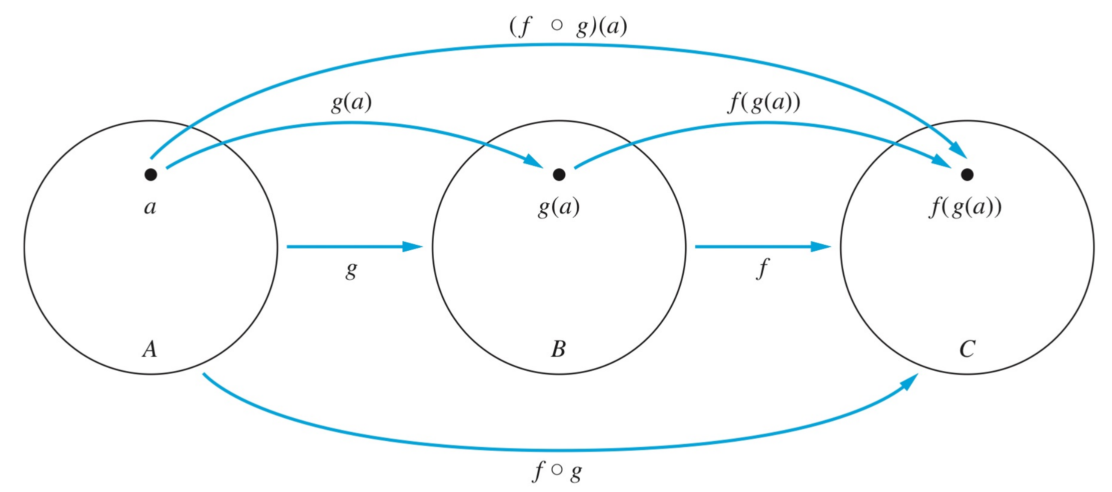


FIGURE 7 **The Composition of the Functions f and g.**

Example

$g: \mathbb{N} \to \mathbb{Z} \Rightarrow g(x) = -x$

$g(10) = -10$, $g(0) = -0 = 0$

$f: \mathbb{Z} \to \mathbb{Z} \Rightarrow f(x) = 2x^2 + x$

$f(10) = 2 \cdot 100 + 10 = 210$

$$f(g(10)) = 2 \cdot g(10)^2 + g(10)$$
$$= 2 \cdot (-10)^2 + -10$$
$$= 2 \cdot 100 - 10 = 190$$

Example 2

$f(x) = x + 1$, $f^{-1}(x) = x - 1$

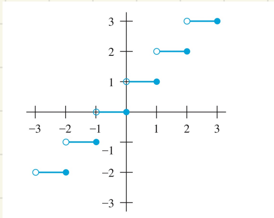
$$f(f^{-1}(x)) = f^{-1}(x) + 1 = x - 1 + 1 = x$$

$$f^{-1}(f(x)) = f(x) - 1 = x + 1 - 1 = x$$

Floor & Cieling Functions

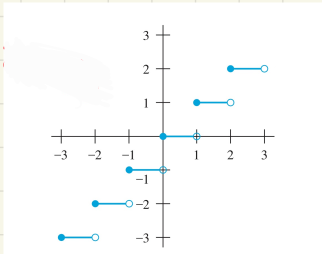
The *floor function* assigns to the real number x the largest integer that is less than or equal to x. The value of the floor function at x is denoted by $\lfloor x \rfloor$. The *ceiling function* assigns to the real number x the smallest integer that is greater than or equal to x. The value of the ceiling function at x is denoted by $\lceil x \rceil$.

Ceiling = $\overbrace{\lceil \underset{\text{real number}}{x} \rceil}^{\text{an Integer}} \equiv$ The smallest integer that is greater than or equal to real number x.



(b) $y = \lceil x \rceil$

floor = $\overbrace{\lfloor \underset{\text{real number}}{x} \rfloor}^{\text{an Integer}} \equiv$ The ~~smallest~~ largest integer that is ~~greater~~ less than or equal to real number x.



(a) $y = \lfloor x \rfloor$

Examples

$\lceil 3.5 \rceil = 4 \qquad \lfloor 3.5 \rfloor = 3$

$\lceil 3.0 \rceil = 3 \qquad \lfloor 3.0 \rfloor = 3$

$\lceil -2.1 \rceil = -2 \qquad \lfloor -2.1 \rfloor = -3$

$\lceil 2.1 \rceil = 3 \qquad \lfloor 2.1 \rfloor = 2$

$\lceil 0.1 \rceil = 0 \qquad \lfloor 0.1 \rfloor = 0$

TABLE 1 Useful Properties of the Floor and Ceiling Functions.
(n is an integer, x is a real number)

(1a)	$\lfloor x \rfloor = n$ if and only if $n \le x < n+1$
(1b)	$\lceil x \rceil = n$ if and only if $n-1 < x \le n$
(1c)	$\lfloor x \rfloor = n$ if and only if $x-1 < n \le x$
(1d)	$\lceil x \rceil = n$ if and only if $x \le n < x+1$
(2)	$x - 1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x + 1$
(3a)	$\lfloor -x \rfloor = -\lceil x \rceil$
(3b)	$\lceil -x \rceil = -\lfloor x \rfloor$
(4a)	$\lfloor x + n \rfloor = \lfloor x \rfloor + n$
(4b)	$\lceil x + n \rceil = \lceil x \rceil + n$

EXAMPLE 29 Prove that if x is a real number, then $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$.

EXAMPLE 30 Prove or disprove that $\lceil x + y \rceil = \lceil x \rceil + \lceil y \rceil$ for all real numbers x and y.