Problem 1

a) Please find C file attached.
b)

Worst case:
In the worst case the array would be a sorted list, giving a recursion of:

$$T(n) = T(n-2) + 2n$$

as the complexity of partitioning is $2n$ and the array would be divided into 2 subarrays of length 0 and $n-2$. This gives an overall complexity of:

$$T(n) = \Theta(n)$$

Best case:
The best case would be an array that is split into 3 subarrays each of length $\frac{n}{3}$.

$$T(n) = 3T\left(\frac{n}{3}\right) + 2n$$

Using the master method we can derive the time complexity such that:

$$f(n) = \Theta\left(n^{\log_3 3}\right) = \Theta(n)$$

$$T(n) = \Theta(nlogn)$$

c) Please find attached C file.

Problem 2

a) Using induction prove that:

$$\sum_{k=2}^{n-1} klogk \leq \frac{1}{2}n^2logn - \frac{1}{8}n^2$$

Base case:
Let $n = 3$.

$$2log2 \leq \frac{1}{2}9log3 - \frac{1}{8}9$$
$$\rightarrow 0.6020599913 \leq 1.022045646$$

$\therefore$ true for $n = 3$.

Induction step:
Assume the hypothesis to be true for all n.

$$\sum_{k=2}^{n-1} klogk \leq \frac{1}{2}n^2logn - \frac{1}{8}n^2$$

Proof:
Show that the hypothesis for $n + 1$.

$$\sum_{k=2}^{n} klogk \leq \frac{1}{2}(n+1)^2\log(n+1) - \frac{1}{8}(n+1)^2$$
$$\sum_{k=2}^{n-1} klogk + nlogn \leq \frac{1}{2}(n+1)^2\log(n+1) - \frac{1}{8}(n+1)^2$$

From the assumption in the induction step we know that $\sum_{k=2}^{n-1} klogk \leq \frac{1}{2}n^2logn - \frac{1}{8}n^2$.

$$\therefore \frac{1}{2}n^2\log n - \frac{1}{8}n^2 + n\log n \leq \frac{1}{2}n^2\log(n+1) + n\log(n+1) + \frac{1}{2}\log(n+1) - \frac{1}{8}n^2 - \frac{1}{4}n - \frac{1}{8}$$

$$0 \leq \frac{1}{2}n^2(\log(n+1) - \log n) + n(\log(n+1) - \log n) + \frac{1}{2}\log(n+1) - \frac{1}{4} - \frac{1}{8}$$

Since we know that this is true for all $k \geq 3$, we have:

$$0 \leq \frac{1}{2}9\left(\log\left(1+\frac{1}{3}\right) + 3\log\left(1+\frac{1}{3}\right)\right) + \frac{1}{2}\log 4 - \frac{1}{4} - \frac{1}{8}$$

$$0 \leq 0.36304 \ldots$$

$$Q.E.D.$$

Problem 3

Show that $\lg n! = \Theta(n \lg n)$.

$$\lg n! = \lg 1 + \lg 2 + \lg 3 + \cdots + \lg n$$

Upper bound:

$$\lg n! < \lg 1 + \lg 2 + \lg 3 + \cdots + \lg(n-1) + \lg n = n \lg n$$

Lower bound:

$$\lg n! = \lg 1 + \lg 2 + \lg 3 + \cdots + \lg \frac{n}{2} + \cdots + \lg n$$

We have:

$$\lg n! > \lg \frac{n}{2} + \lg \frac{n}{2+1} + \cdots + \lg(n-1) + \lg n > \lg \frac{n}{2} + \cdots + \lg \frac{n}{2} > \frac{n}{2} \lg \frac{n}{2}$$

$$\therefore \frac{n}{2} \lg \frac{n}{2} < \lg n! < n \lg n$$