Course: Computer Graphics
Assignment 1

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20 March 2019

Problem 1.1

1. False
2. True
3. False
4. True
5. False

Problem 1.2

a) 1) Derive the transformation matrix of each transformation step in homogeneous coordinates:

i. Scaling of a factor of 3 on the x axis:

$$S_x(3) = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ii. Scaling of factor -1 on y axis:

$$S_y(-1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

iii. Rotation 90 degrees (clockwise) around the z axis, $\theta = -\frac{\pi}{2}$:

$$R_y(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore R_y(-\frac{\pi}{2}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

iv. Translate by $b_x = 2$:

$$T_x(b_x) = \begin{bmatrix} 1 & 0 & 0 & b_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore T_x(2) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2) Compute the combined transformation matrix in homogeneous coordinates in the given order:

$$T_x(2)R_y(-\frac{\pi}{2})S_y(-1)S_x(3) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 0 & -1 & 0 & 2 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3) Given a triangle with vertices: $p_1 = (3, 0, 2)$, $p_2 = (2, 0, 2)$, $p_3 = (1, 1, 2)$, apply the combined transformation matrix to the triangle:

$$\begin{bmatrix} 0 & -1 & 0 & 2 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 2 & 1 \\ 0 & 0 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2 & 2 & 1 \\ -9 & -6 & -3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

The coordinates are the columns of the above matrix.

b) The triangle is projected using perspective projection.

1) Compute the projection matrix given that $a = b = 1$.

$$P = \begin{bmatrix} h & 0 & 0 & 0 \\ 0 & h & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$h = 1$ as it is the distance from the camera, located at $(0, 0, 0)$ to the centre of the screen, located at $(0, 0, 1)$.

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

2) Apply the projection matrix P to the triangle.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ -9 & -6 & -3 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\implies \begin{bmatrix} 2 & 2 & 1 \\ -9 & -6 & -3 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

Now we have to normalise the vertices by dividing by 2.

$$1/2 \begin{bmatrix} 2 & 2 & 1 \\ -9 & -6 & -3 \\ 3 & 3 & 3 \\ 2 & 2 & 2 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 1 & 0.5 \\ -4.5 & -3 & -1.5 \\ 1.5 & 1.5 & 1.5 \\ 1 & 1 & 1 \end{bmatrix}$$

The resulting coordinates projected on the screen are:

$$p_1 = (1, -4.5)$$

$$p_2 = (1, -3)$$

$$p_3 = (0.5, -1.5)$$

These coordinates indicate that the triangle is not visible on the defined screen.

Problem 1.3

Taking the vertices of the 3D cube in the order:

$$\langle(-1, 1, -1), (1, 1, -1), (1, -1, -1), (-1, -1, -1), (-1, 1, 1), (1, 1, 1), (1, -1, 1), (-1, -1, 1)\rangle$$

we get a homogeneous matrix representation of:

$$C = \begin{bmatrix} -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

1. Compute H_{ry} which rotates about the y axis by $\theta = 30^\circ = \pi/6$:

$$H_{ry}(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore H_{ry}(\pi/6) = \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ -1/2 & 0 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Compute T_x which translates by along the x axis by $b_x = 0.5$:

$$T_x = \begin{bmatrix} 1 & 0 & 0 & b_x \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Apply the combined transformation matrix to the cube.

$$[T_x][h_{ry}] = \begin{bmatrix} 1 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 & 0 \\ 0 & 1 & 0 & 0 \\ -1/2 & 0 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 \\ -1/2 & 0 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore [T_x][H_{ry}][C] = \begin{bmatrix} \sqrt{3}/2 & 0 & 1/2 & 1/2 \\ 0 & 1 & 0 & 0 \\ -1/2 & 0 & \sqrt{3}/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & -1 & -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

This gives new vertex coordinates:

$$\begin{bmatrix} -0.866 & 0.866 & -0.866 & -1.366 & 0.134 & 1.866 & 1.866 & -0.134 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -0.366 & -1.366 & -1.366 & -0.366 & 1.366 & 0.366 & 0.366 & 1.366 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

3. Given a unit cube with vertices $x, y, z \in \{-1, 1\}$ and assuming $r = -l$ and $t = -b$ pick appropriate camera frustum parameters r, l, t, b, f, and n to ensure all points of the cube are in the frustum.
Orthographic projection is defined as:

$$P_\perp = \begin{bmatrix} \frac{2}{r-1} & 0 & 0 & -\frac{r+1}{r-1} \\ 0 & \frac{2}{t-b} & 0 & -\frac{t+b}{t-b} \\ 0 & 0 & -\frac{2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{r} & 0 & 0 & 0 \\ 0 & \frac{1}{t} & 0 & 0 \\ 0 & 0 & -\frac{2}{f-n} & -\frac{f+n}{f-n} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The conditions for the minimum parameters of the frustum are:
r is the farthest point right on the x axis $\Rightarrow 1$
l is the farthest point left on the x axis $\Rightarrow -1$
t is the farthest point north on the y axis $\Rightarrow 1$
b is the farthest point south on the y axis $\Rightarrow -1$
n is the distance to the near clipping plane $\Rightarrow -1$
f is the distance to the far clipping plane $\Rightarrow 1$

Therefore, we can pick parameters above these minimum values. Let's take:
$r = 2, t = 2, n = -2, f = 2$

This gives the projection matrix:

$$P_\perp = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. Carry out the projection on the vertices:

$$[P_\perp][C] = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.866 & 0.866 & -0.866 & -1.366 & 0.134 & 1.866 & 1.866 & -0.134 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ -0.366 & -1.366 & -1.366 & -0.366 & 1.366 & 0.366 & 0.366 & 1.366 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} -0.433 & 0.433 & -0.433 & -0.683 & -0.067 & 0.933 & 0.933 & -0.067 \\ 0.5 & 0.5 & -0.5 & -0.5 & 0.5 & 0.5 & -0.5 & -0.5 \\ 0.183 & 0.683 & 0.683 & 0.183 & -0.683 & -0.183 & -0.183 & -0.683 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

5. In this case the cube remains in the frustum, therefore the parameters do not need to be updated. In the case where $n \leqslant 2$ the frustum would need to be updated because the cube would lie outside the near plane.