Nautilus: Algorithmic Construction of Nested Polygons

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Abstract

This paper introduces a novel parametric function for constructing nested polygons, a structured sequence of nested regular polygons. Using a trigonometric approach in polar coordinates, the function precisely encodes its progressive transformations and is algorithmically implemented in Python. Beyond reconstruction, a geometric analysis reveals an emergent spiral-like behavior arising from the connection of overlapping segments in successive polygons. Additionally, Bézier curves are employed to refine the interpolation between nested structures, improving continuity and minimizing angular discontinuities.

Introduction

The Bill Picture is a structured sequence of nested regular polygons, where each successive polygon, having fewer sides, is rotated so that its first vertex shifts two units to the right relative to the previous one. This geometric transformation was numerically reconstructed and formalized by Michael Trott in 2004 [1], paying homage to Swiss architect and artist Max Bill, who originally conceived *Quinze variations sur un même thème* in 1938 [2]. The structure was later documented by Eric W. Weisstein [3], further exploring its mathematical properties.

(a) (b)

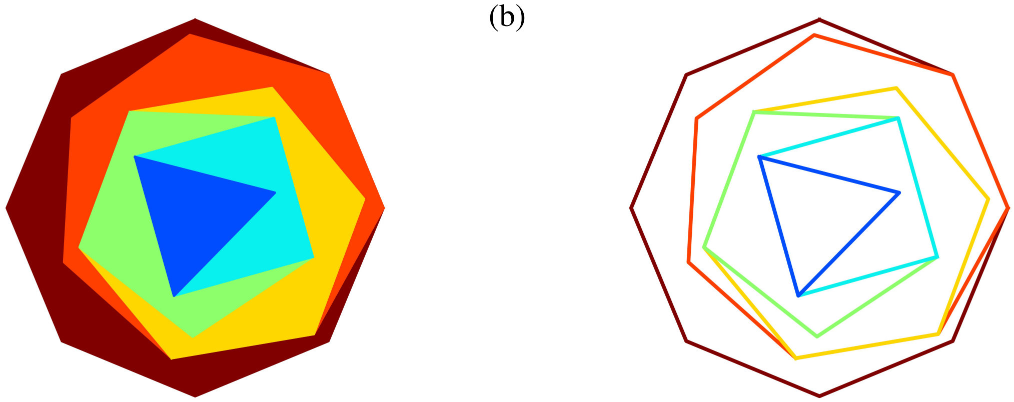

Figure 1. Filled (a) and contour representation (b) of the Bill Picture.

Before Trott's formalization, this generative approach had already influenced computational artists. Notably, Jean-Pierre Hébert created Metagon in 1998 [4], extending Max Bill's concept with a spiral-like dynamic in a generative art framework.

This study develops a mathematical formulation that algorithmically reconstructs the Bill Picture by capturing its progressive transformations. Using a trigonometric framework in polar coordinates, our approach precisely determines vertex positions in each nested polygon, establishing a structured computational model for constructive geometric art. This formulation not only rigorously reproduces Max Bill's composition but also allows for controlled parametric modifications, fostering broader exploration in algorithmic design and computational mathematics.

Methodology

Trigonometric Parameterization

A regular n-sided polygon can be divided into n identical triangles by drawing segments from each vertex to the center. These segments, all of length R_n , form the external angle $\sigma_n(360/n)$, which defines the polygon's structure. Using polar coordinates, the following trigonometric function determines the position of a point $P_{n,i}$ along a segment based on R_n and σ_n, applying rotational transformation :

$$P_{n,i}(R_n,\sigma_n) = \begin{cases} x_i = sin(\sigma_n \cdot (i-1)) \cdot R_n \\ y_i = cos(\sigma_n \cdot (i-1)) \cdot R_n \end{cases}, \quad (1)$$

where i represents the index of a vertex in a given regular polygon.

Polygon Alignment

To determine the vertex coordinates of each nested regular polygon, the process begins by rotating the segment connecting the origin to a vertex. However, this alone does not ensure proper alignment among polygons. To achieve this, the polygon with the highest number (N) of sides is chosen as a reference, with one of its segments serving as the alignment baseline.

Each nested polygon with $n = N - k$ sides (where k varies from 1 to $N - 3$) is then rotated to compensate for the accumulated internal angle $\rho\,(180 - \sigma)$ differences, ensuring consistent relative orientation as the number of sides decreases. Additionally, to maintain the structural pattern characteristic of the Bill Picture, each new polygon is rotated so that its first vertex aligns two positions ahead of the corresponding vertex in the previous polygon.

At this point, since each nested polygon already shares a parallel side with the previous polygon, a final translation $(\Delta x_n, \Delta y_n)$ is applied to ensure proper overlap of these segments. This step preserves geometric coherence while maintaining the integrity of the composition. Figure 1 illustrates the sequential transformations applied to the nested polygons, from the triangle to the hexagon, at each of the described steps.

(a) (b) (c) (d)

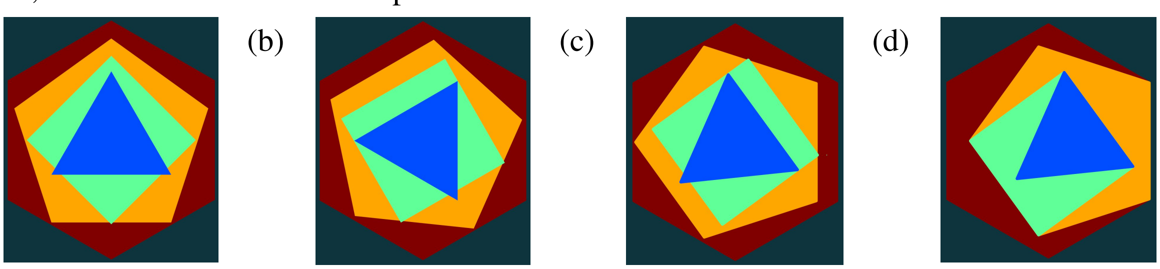

Figure 2: (a) Initial state from Equation (1), (b) Alignment of nested regular polygons to the outermost polygon, (c) Adjustment according to Bill Picture (two-vertex shift), and (d) Translation of parallel segments. Steps (b) to (d) represent the modifications leading to Equation (2)

Ultimately, the following trigonometric function allows for the precise reconstruction of the Bill Picture (as illustrated in Figure 2) by accurately determining the vertex coordinates of the nested regular polygons :

$$P_{n,i}\left(R_n,\sigma_n,\rho_n\right)=\begin{cases} x_i = sin\left(\sigma_n\cdot(i-1)+\sum_{t=n}^{N-1}\frac{|\rho_t-\rho_{t+1}|}{2}+k\sum_{t=n}^{N-1}\sigma_{t+1}\right)\cdot R_n+\sum_{t=n}^{N-1}\Delta x_k \\ y_i = cos\left(\sigma_n\cdot(i-1)+\sum_{t=n}^{N-1}\frac{|\rho_t-\rho_{t+1}|}{2}+k\sum_{t=n}^{N-1}\sigma_{t+1}\right)\cdot R_n+\sum_{t=n}^{N-1}\Delta y_k \end{cases}, \quad (2)$$

where k determines the number of offsets; $k=2$ corresponds to Figure 1(a).

Based on this equation, it is possible to reconstruct the Bill Picture while also varying the number of applied offsets, as illustrated in the following figure.

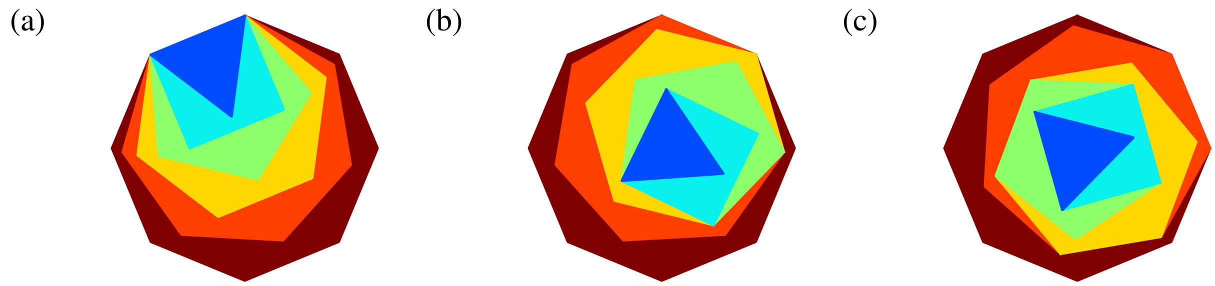

Figure 3: (a) No shift, (b) one-segment and (c) two-segment shift.

Polygonal Spiral Derived from the Bill Picture

A spiral-like evolution emerges in Metagon, the work of Jean-Pierre Hébert, demonstrating progressive geometric transformations. Similarly, an intriguing spiral-like behavior is observed in our reconstructed version of the Bill Picture, where overlapping segments of successive regular polygons create a continuously evolving configuration, as illustrated in Figure 4. This suggests an underlying geometric structure that warrants further investigation.

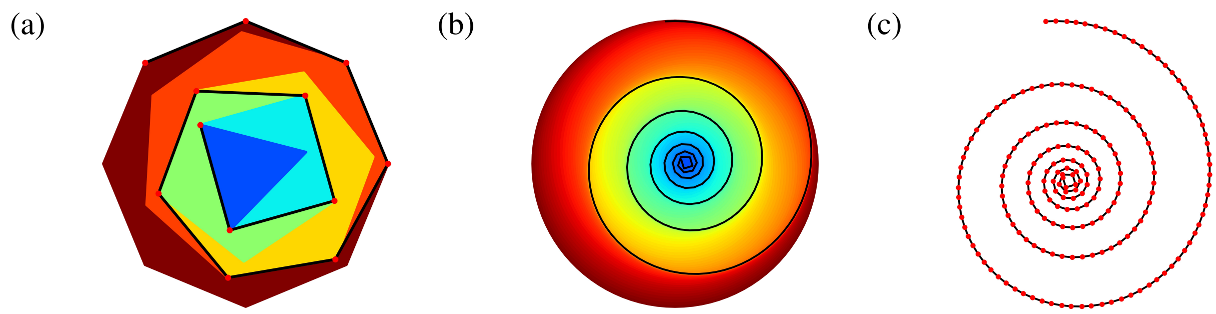

Figure 4: Nested regular polygons and emerging polygonal spiral: (a) from triangle to octagon, (b) from triangle to hectogon, and (c) highlighting the spiral-like evolution.

Application of Bézier Curves

As shown in Figure 4, increasing the number of sides in the Bill Picture reduces misalignment but does not fully eliminate it. To achieve a more fluid transition, Bézier curves, which are widely used in computational geometry and graphic modeling, provide an effective solution. In this study, De Casteljau's recursive algorithm [5] was applied to the midpoints of segments connecting nested regular polygons, reducing discontinuities and ensuring a smoother transition. Figure 5 illustrates the achieved refinement, enhancing continuity between successive segments and producing a more fluid interpolation of the polygonal spiral. As shown in Figure 5, the one-segment shift (a) results in an asymmetric convergence pattern whereas the two-segment shift (b) exhibits a more balanced and symmetric alignment.

(a) (b)

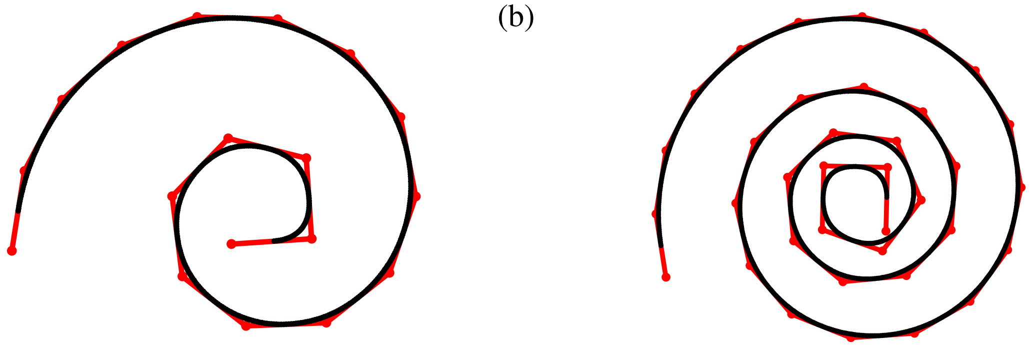

Figure 5: Interpolation of polygonal spirals, via De Casteljau's algorithm, derived from the Bill Picture. Comparison between a one-segment (a) and two-segment shift (b), with nested regular polygons from triangle to icosagon.

Variation of The Figure

Building on the data we previously calculated, including the internal and external radii of the polygons and the segments forming the spiral, we can explore several geometric configurations. These include drawing inscribed or circumscribed circles in place of the polygons, placing circles with a radius equal to half the side length at each vertex, drawing segments from the center to each vertex, and connecting each point of the spiral to every other point on the same polygon.

(a) (b) (c)

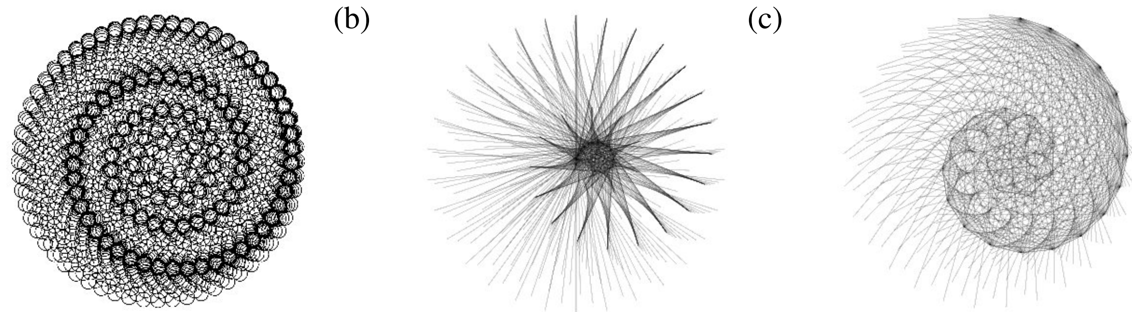

Figure 6. The circles placed at each vertex of the polygons (a), the segments connecting the center to the vertices (b), and those connecting each point of the spiral to the other points of the same polygon (c).

Furthermore, by modifying the equation, which allows for infinite possibilities, such as reversing the vertices of a polygon relative to a segment, we can generate stunning figures while preserving the overall spiral shape.

(a) (b) (c)

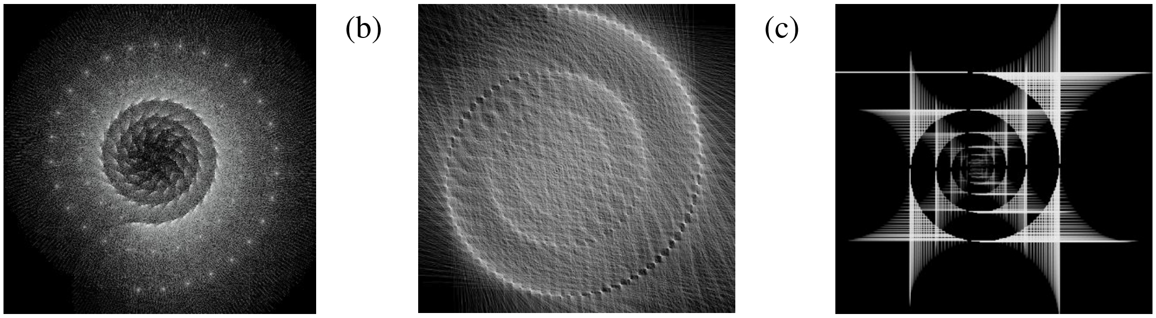

Figure 7. Example of a modified spiral generated through equation variation.

Fibonacci Spiral

The Fibonacci spiral is a geometric form constructed using squares whose sides follow the Fibonacci sequence (1, 1, 2, 3, 5, 8, etc.). By connecting quarter-circle arcs drawn within each square, a smooth, continuously expanding spiral emerges, closely related to the golden ratio (approximately 1.618).

This spiral is frequently observed in nature, appearing prominently in sunflower heads, pinecones, certain plant growth patterns, and leaf arrangements, reflecting an optimal spatial arrangement and aesthetic harmony.

Continuing our previous analysis, we initially considered a direct connection between the Fibonacci spiral, and the spiral depicted in our figure, given its resemblance to the Nautilus shell. However, after thorough research, we found—contrary to popular belief—that the Nautilus shell actually follows a logarithmic spiral distinct from the Fibonacci spiral, with a ratio ranging from approximately 1.261 to 1.348 [6].

To investigate these observations further, we revisited the classical formulation of the Fibonacci spiral using a logarithmic spiral approximation [7] but replaced the golden ratio φ with its square root $\sqrt{\varphi} \approx 1.27$, as illustrated in the following equation.

$$\begin{cases} x(\theta) = exp\left(\frac{ln(\sqrt{\varphi}}{2\pi}\theta\right) \cdot cos(\theta) \\ y(\theta) = exp\left(\frac{ln(\sqrt{\varphi}}{2\pi}\theta\right) \cdot sin(\theta) \end{cases}. \tag{3}$$

This adjustment allowed us to superimpose the two curves and revealed a striking similarity, suggesting that this variant of the Fibonacci spiral provides a noteworthy approximation of the observed curve.

(a) (b) (c)

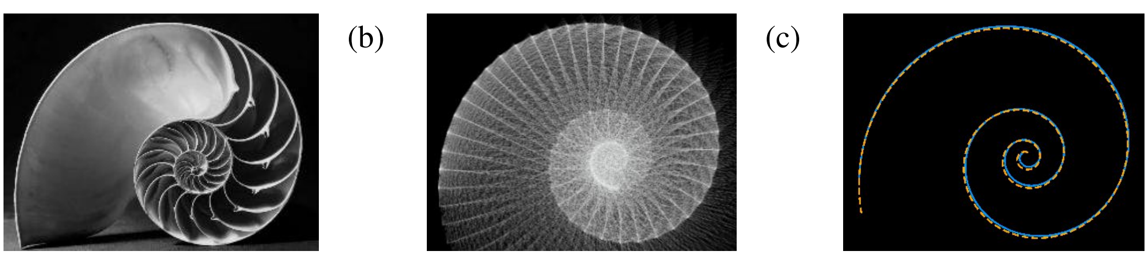

Figure 8. Comparison between the Nautilus shell and the figure obtained by combining the parameters from Figures 12.b and 12.c. An overlay of the logarithmic spiral with a ratio $\sqrt{\varphi} \approx 1.27$ (in orange) and the Bézier curves (in blue).

Conclusion

Equation (2), developed in this study, enables independent segment shifts, including the one segment shift proposed by Eric W. Weisstein, no shift, or other variations. This extends the original two-segment shift used by Max Bill and Michael Trott. Figure 3 illustrates these variations.

It is striking that a spiral generated through this discrete, polygonal stepping process aligns so closely with a Fibonacci-based spiral model. This unexpected correspondence underscores a deep connection between simple polygonal constructions and the continuous growth principles embodied by Fibonacci geometry.

Further investigation is required to better understand the properties of the polygonal spiral introduced in this study and to explore the potential applications of Equation (2). The developed framework provides a versatile basis that can be adapted to various design applications according to specific needs.

References

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