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IR Effective Actions from Holography

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Abstract: We present a novel perspective on the low-energy effective actions of confining gauge theories with gravity duals, inspired by the idea of holographic Wilsonian renormalization. By identifying an IR-boundary value of a bulk field that overlaps with the massless mode in the gauge theory, we are able to efficiently derive the IR on-shell effective action upon integrating over the rest of the geometry. We illustrate the details of this formalism by computing the chiral Lagrangian coefficients in a simple AdS/QCD toy model, which conform to previous results. In addition, we obtain new results in that model at higher orders, including a closed formula for the four-pion scattering amplitude to all orders in momenta. Finally, we reformulate our method using a diagrammatic approach, in analogy to Witten diagrams but ones that start and end on the IR boundary.

Contents

1 Introduction

One can think of the Wilsonian approach to quantum field theory (QFT) as an ordered slicing of *field space*. Though it is the full path integral which ultimately determines the values of physical observables, we are free to foliate field space into co-dimension one slices, which we incrementally integrate out to produce a renormalization group (RG) flow. For concreteness, foliating field space with respect to a particular length assignment (l) for each of the field theory modes (M), we can integrate over modes that are assigned lengths $l < \delta$ to find a reduced field space and an effective action S_δ:

$$Z_{QFT} = \int \mathcal{D}M_{l\leq\delta}\mathcal{D}M_{l>\delta}e^{-S_0[M_<,M_>]} \equiv \int \mathcal{D}M_{l>\delta}e^{-S_\delta[M_>]} \ . \tag{1.1}$$

Formally, the bare action S_0 is defined with a regularization ϵ. There are countless ways to slice modes and assign them a length (or energy) scale; each scheme defines a certain RG flow. For example, a Euclidean-momentum sharp cutoff for fundamental fields in the path integral ($k_E^2 < \Lambda^2$) is a common regularization. However, in principle, one can construct any other arbitrarily intricate slicing.[1]

With the advent of AdS/CFT, a renormalization scheme termed *holographic Wilsonian renormalization* (HWR) was defined. For a gauge theory with an asymptotically AdS (aAdS) classical gravity dual, one can order the slicing of modes according to their radial position in the bulk. This idea was inspired by the identification of the radial position in the bulk with an energy scale in the gauge theory, $z \sim 1/\Lambda$. Evolution along the bulk radial direction provides a geometrical picture for Wilsonian-like RG flow in the dual field theory. Initially [1–3], the holographic RG approach related the RG equations to Hamilton-Jacobi equations in the bulk (with the radial direction playing the role of time). This treatment was not Wilsonian in nature, as the flow depended on information at the IR (see [7] for more details). More recently, a truly Wilsonian holographic renormalization was formulated [4, 5] (see also the earlier work of [6] and a recent extension [7]).

In the HWR formalism of [4, 5], the separation of the path integral (1.1) into modes $M_>$ and $M_<$ is identified with a separation of the bulk path integral into integrations of bulk fields (Φ) above and below a radial slice,

$$Z_{bulk} = \int \mathcal{D}\Phi_{z\leq l}\mathcal{D}\Phi_{z>l}e^{-\mathcal{S}_0[\Phi_<,\Phi_>]} \equiv \int \mathcal{D}\Phi_{z>l}e^{-\mathcal{S}_l[\Phi_>]} \ . \tag{1.2}$$

$\mathcal{S}_0$ includes the original bulk action (regularized at $z = \epsilon$), and possibly also an action at the UV boundary. $\mathcal{S}_l$ is the total effective action, which includes the original bulk action on the remaining bulk $z > l$, and an induced boundary action on the new

[1] The only demand of a 'good' slicing is to lead to a *local* effective actions along the flow.

cutoff slice at $z = l$. The new UV-boundary action was suggested to be dual to the induced Wilsonian effective action at scale $\delta \sim l$.[2]

Despite its intuitive appeal, the HWR formalism's most crucial ingredient remains obscure: what is the precise cutoff scheme in QFT that corresponds to the radial cutoff in the bulk? This question is related to a local understanding of the holographic duality: the precise translation of local bulk excitations to boundary modes and the "emergence" of the radial direction. We do not address these fundamental questions here. Instead we solve a simpler problem that we hope may shed new light on the subject.

In this work we explain how to compute low-energy effective actions for strongly-coupled confining gauge theories with gravity duals. The framework we propose is driven by the HWR formalism, which for this particular task can be made completely well-defined. Concretely, the HWR procedure yields a scheme-independent, on-shell, IR effective action, in a more direct and efficient way than previous techniques.

We consider the large N limit of confining gauge theories in d-dimensional flat space, that have a classical gravity dual in an asymptotically AdS_{d+1} space ($aAdS_{d+1}$). For the sake of simplicity we use a "hard-wall" toy model, in which confinement is induced by a sharp cutoff in the bulk geometry at a finite radial value, $z < L$ [20, 21]. We also assume the gauge theory to be IR free but non-trivial, so there is some interesting weakly-coupled description of the physics at the IR. When the $(d + 1)$-dimensional bulk theory terminates on a non-degenerate d-dimensional surface at $z = L$, we can perform the whole path integration along the z-direction in the spirit of HWR (from UV to IR), and reach an effective action in d dimensions. According to the HWR prescription it is natural to identify the resulting action as the IR effective action (IREA) for the lightest mode. While the effective action at finite cutoff is scheme-dependent (and, as mentioned above, we know nothing about the scheme in the HWR formalism), we might expect the IREA obtained from integrating over the whole geometry to be somewhat scheme-*independent*. More precisely, the complete integration of bulk geometry should exclude a mode that is localized on the IR boundary. This mode is not integrated out, and the effective action we compute depicts its dynamics. Indeed, the choice of IR mode is to some extent arbitrary, as it affects only off-shell data. The on-shell action, or the S-matrix, is invariant under general field redefinitions, and as such will be insensitive to the precise choice of the mode that is left unintegrated, as long as it has 'anything to do' with the true light degree of freedom we are after, in a way that we will make more precise below.

While some of the perspective and formalism we present is novel, as in particular the methodology, many of the ideas we describe have been actually floating around in the literature for some time, in various areas of research. For example, in the context of holographic hydrodynamics, a closely related work is that of [9] (see also the

[2] The precise relation is ambiguous and depends on the slicing definition.

review of [8]), who present an effective description for the long wavelength behavior of holographic fluids. Other holographic models have been used to derive effective actions in a variety of contexts, such as [10, 11], who study the low-energy dynamics of the Goldstone boson of broken conformal invariance (the dilaton). [12] describes a holographic framework for spontaneous SUSY breaking, deriving the 4d effective action for Goldstinos by integrating out bulk fields. In a more phenomenological context, a very closely related work is that of [13] (see also the earlier papers of [14, 15]), who define a similar effective action although from a different perspective and following a different procedure. Other related holography-inspired, phenomenological works are that of [16] and subsequent works. To finally mention, for the specific example we use, the Hirn-Sanz model of holographic QCD [22], similar computations were made with older techniques; the four-derivative order result was written already in the original papers [20][21], and recently also the six-derivative order result has been computed [17]. Using our technique we extend those results to infinite orders in derivatives for the cases of four and six external pions (the former to be written in a closed form).

The paper is organized as follows. In section 2 we discuss IR effective actions (IREAs) of confining gauge theories with a classical gravity dual. We first review the "traditional" Kaluza-Klein method of computing the IREA holographically, then describe in greater detail our prescription for the holographic Wilsonian IREA. We give arguments for the robustness of the procedure, and make contact with the HWR formalism. In section 3 we work out an example that illustrates the simplicity of our prescription: computing the on-shell IREA for the pions of a simple AdS/QCD model. We compare to known results in the leading orders of the momentum expansion, and extend them further, in some cases to all orders. In section 4 we reformulate the procedure in terms of Feynman diagrams in the bulk. This provides some additional intuition for the HWR process and simplifies the computations significantly. We conclude in section 5 with a short discussion and some interesting open questions. Technical details are deferred to the Appendices.

2 IR Effective Actions from Holography

Let us firstly review the standard method for deriving IREAs from holography via KK decomposition. Throughout this paper we refer to a general d-dimensional, strongly-coupled confining gauge theory at large N, that admits a $(d+1)$-dimensional gravity dual. We consider an aAdS bulk geometry with radial coordinate z and a generic warp factor,[3]

$$ds^2 = w^2(z)\left(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2\right) \ . \tag{2.1}$$

We will now consider confinement induced by sharply cutting off space at finite radial coordinate $z < L$, leaving the generalization with smooth confining geometry to future work.

2.1 Holographic IR Effective Action via Kaluza-Klein

Gauge-invariant excitations, such as mesons and glueballs, are encoded holographically as normalizable modes of bulk fields. Each bulk field, upon KK reduction along the radial direction, gives rise to an infinite tower of such excitations, all having the same quantum numbers in the d-dimensional theory. For example, a single vector gauge field in five dimensions produces a tower of vector mesons in four dimensions. Consider the simple example of a bulk scalar field with action

$$S = \int d^{d+1}x\sqrt{G}\left[\frac{1}{2}\partial_M\Phi\partial^M\Phi - \frac{1}{2}M^2\Phi^2 - \frac{1}{4!}\lambda\Phi^4\right] \ , \tag{2.2}$$

and equations of motion

$$\left(\partial^2 - w^{-d+1}\partial_z w^{d-1}\partial_z + M^2w^2\right)\Phi = -\frac{1}{3!}\lambda w^2\Phi^3 \ . \tag{2.3}$$

We can separate out the radial dependence by expanding

$$\Phi(x,z) = \sum_{n=1}^{\infty}\phi_n(x)\psi_n(z) \ , \tag{2.4}$$

with the radial wave functions that solve the quadratic equation (and with the appropriate boundary conditions, inherited from those of Φ),

$$\left(w^{-d+1}\partial_z w^{d-1}\partial_z - M^2w^2\right)\psi_n = m_n^2\psi_n \ . \tag{2.5}$$

Plugging the expansion back into the action and explicitly performing the z-integration, one finds an infinite tower of interacting d-dimensional fields with masses m_n,

$$S_{KK} = \int d^dx\left[\sum_n \frac{1}{2}\partial_\mu\phi_n\partial^\mu\phi_n - \frac{1}{2}m_n^2\phi_n^2 - \lambda\sum_{n_1<n_2<n_3<n_4} v_{n_1,n_2,n_3,n_4}\int d^dx\,\phi_{n_1}\phi_{n_2}\phi_{n_3}\phi_{n_4}\right] \ . \tag{2.6}$$

Momenta in the radial direction translate to d-dimensional masses, and the spectrum is discrete due to the effectively finite size of the radial direction. The relative

[3] We simply ignore the extra compact manifold which plays no special role in what follows. Capital Latin letters $M = (\mu, z)$ denote 5d coordinates, while Greek indices denote flat space directions. We use the mainly minus convention, and 5d (4d) indices are lowered with g_{MN} ($\eta_{\mu\nu}$).

couplings between various resonances in the KK tower are given by overlap integrals of the corresponding z-momentum wave functions,

$$v_{n_1,n_2,n_3,n_4} = \int_0^L dz w^{d+1} \psi_{n_1} \psi_{n_2} \psi_{n_3} \psi_{n_4} \ . \tag{2.7}$$

We now have a d-dimensional action for the infinite tower of resonances; this is what we call the *dual resonance model*. If we are primarily interested in the deep IR effective action, defined for the lightest field ϕ_1, we may proceed in the standard (d-dimensional) Wilsonian fashion by integrating out all heavy resonances

$$Z = \int \mathcal{D}\Phi \ e^{iS^{(d+1)}[\Phi]} = \left(\prod_n \int \mathcal{D}\phi_n \right) e^{iS^{(d)}[\{\phi_n\}]} \equiv \int \mathcal{D}\phi_1 e^{iS^{(d)}_{eff}[\phi_1]} \ . \tag{2.8}$$

Since we work at large N there are no energetic modes of ϕ_1 to integrate over in loops, and nor we have integrated over ϕ_1 at tree-level. The resulting effective action is the IR effective action of ϕ_1, and it is expected to be valid and local below m_1.[4]

2.2 A Holographic Wilsonian Approach to IR Effective Actions

In the KK procedure we split up field space modes according to their radial *momenta*, and to obtain the IREA we integrated out all but the lightest. In some sense, this procedure does not take full advantage of the holographic description: the information is translated to a d-dimensional language "too early." In the HWR formalism we instead slice and order modes by their radial *position*. We suggest that one can retrieve the same physical information at the IR by integrating out the whole bulk geometry apart from a single mode on the IR boundary. Keeping manifest the holographic radial dimension both simplifies the computation and adds some geometrical intuition. Concretely, one can take the "IR limit" of the formulation in (1.2), by sliding the separating slice at $z = l$ all the way down to the place where the bulk terminates, $z = l \to L$:

$$Z_{bulk} = \int \mathcal{D}\varphi \int_{\Phi(L)=\varphi} \mathcal{D}\Phi_{z<L} e^{-\mathcal{S}_0[\Phi]} \equiv \int \mathcal{D}\varphi e^{-\mathcal{S}_L[\varphi]} \ . \tag{2.9}$$

In this limit we are left with an effective action $\mathcal{S}_L[\varphi]$ for the unintegrated d-dimensional boundary field,

$$\varphi(x) \equiv \Phi(x, L) \ . \tag{2.10}$$

[4]In fact, this effective action is valid through all energies, since no modes are left un-integrated at the end; but it is still expected to be local only below m_1. In the fortunate case where a model can be solved exactly and re-summed to all derivative orders, for a certain process, it may be written in a closed form and the final result would hold at all energies. This is what we will encounter below, computing the four-pion scattering amplitude exactly, to all orders, and at all energies (for which the original bulk model is well defined to begin with).

In the spirit of HWR it is natural to identify φ with a low-energy mode, and $\mathcal{S}_L[\varphi]$ with its IR effective action. We now examine this idea more closely.

What effective action? Since φ is not being integrated over in (2.9), one might naively guess that $\mathcal{S}_L[\varphi]$ is an off-shell effective action. However, the limit we have taken is "singular" in a sense that the path-integral in (2.9) *before the integration over* φ is classically over-constrained. It already contains the two boundary conditions inherited from the bulk theory, together with the additional assignment of (2.10). It is therefore consistent only if the assignment of φ already sits on the classical solution of the path-integral with the original boundary conditions, which then translates to φ being a solution of its own equation of motion. We thus identify $\mathcal{S}_L[\varphi]$ with the *on-shell* effective action of φ.

Secondly, in the HWR formalism it is *assumed* that the resulting effective action is Wilsonian,[5] and the specific cutoff scale goes inversely with z. We would like to make no such assumptions here, and thus should ask: to what extent is $\mathcal{S}_L$ really a low-energy action? To be precise, it is the effective action that has already accounted for all fluctuations of all fields apart from the φ mode. It is then a low-energy action as much as φ is a low-energy mode.

Effective action of what? Although we have identified the on-shell effective action of an IR boundary mode φ, eventually we are interested in the on-shell IREA of the lightest mode in the gauge theory, ϕ_1.[6] The two modes, φ and ϕ_1, are not identical, nor are their off-shell effective actions. Their on-shell effective actions, however, might be the same. This is due to the "universality of the S-matrix", a theorem stating that the S-matrix of arbitrary scalars is invariant under field redefinitions of the form

$$\phi' = c_1\phi + c_2\phi^2 + \dots \tag{2.11}$$

This holds even if ϕ, ϕ' denote multiple fields (and the c_i matrices), or if the c_i's depend on momentum.[7] A similar statement holds for the on-shell effective actions of ϕ and ϕ', in which the overall normalization c_1 needs to be accounted for separately. For our purpose, we simply conclude that as long as we choose the IR mode φ so that it overlaps with the lightest mode ϕ_1 in the way defined by (2.11), the off-shell effective actions of the two modes may differ, but their on-shell actions will be the same. We can then identify

$$e^{-\mathcal{S}_{IREA}^{(on-shell)}[\phi_1]} = \int_{\Phi(L)=\phi_1} \mathcal{D}\Phi_{z<L} e^{-\mathcal{S}_0[\Phi]} \ . \tag{2.12}$$

[5] Its locality is the only real assumption.

[6] At low enough energies this is the only propagating mode.

[7] The exact statement is that the scattering amplitudes of ϕ' particles with Lagrangian $\mathcal{L}_{\phi'}$ equal the corresponding scattering amplitudes of ϕ particles with Lagrangian $\mathcal{L}_{\phi'(\phi)}$. It stems directly from a simple pole analysis (see for example[18, 19]).

More generally, in addition to its overlap with ϕ_1, φ can also mix with heavier fields. We can still study ϕ_1's scattering amplitudes using those of φ' in such cases; if we put all incoming φ states on the mass-shell of ϕ_1, we isolate the leading singularity that corresponds to the ϕ_1's S-matrix. There will often be some freedom in our choice of which IR mode to leave unintegrated. Some options may simplify computations considerably, but roughly speaking any choice of bulk mode that has some coupling with the lightest mode of the gauge theory will work.

In a general holographic setup, the IR mode φ will mix with the whole tower of KK excitations ϕ_n, but there exists a simpler case where it can be set to mix only with the lightest one. This happens when the lightest mode is a massless Nambu-Goldstone boson (NGB), coming from a spontaneous symmetry breaking in the gauge theory. We will explore this example in section 3. Equation (2.12) is the main conceptual message of our work, and we now explain how to use it in broad strokes. When we work out an specific example in section 3, we will explain the technical details of the procedure.

The prescription in practice: To compute the effective action in (2.12) we must: (a) solve the equations of motion of Φ with the assignment (2.10) in addition to the original two boundary conditions, and then (b) plug the solution written in terms of φ back into the original action. As explained earlier, the requirement for having a solution will enforce a constraining equation on $\varphi(x)$, which corresponds to its equation of motion in the effective theory. Note that this (off-shell) information is *complementary* to the obtained on-shell effective action.

From now on, we concentrate on the IREAs of massless particles, ϕ_0, which we can probe at arbitrarily low energies. The effective action then has a useful derivative expansion, as derivatives scale with the low-energy momentum transfer p/m_1, where m_1 is the lowest mass that was integrated out. We also assume the massless field itself to scale with p/m_1, so that we can also expand order by order in the IR mode itself. On dimensional grounds, we can see that this would generally hold true if the theory has a single dimensionful scale and no dimensionless parameters, as for example in the case of QCD. The prescription is simple:

- First, expand the equations of motion of the various fields order by order in φ. Note that the bulk field of which φ is an IR value, say $\Phi(x,z)$, has non-zero contribution at leading order in φ. Other fields, denote them $\Psi(x,z)$, are sourced by φ through the equations of motion, and have their first non-zero coefficient at $O(\varphi^2)$ if their mixed two-point function with Φ vanishes, $\langle\Psi\Phi\rangle = 0$. Note that we compute φ's IREA *in the vacuum*, and thus the solution must vanish when taking $\varphi \to 0$. In particular, no field in our solution can have a term at $O(\varphi^0)$.

- Solve the equations order by order in φ. At each order we will encounter free equations of motion, added with a source term obtained from lower-order solutions.

- Finally, plug the solution into the action and integrate over z to result with the on-shell effective action for φ.

Note that in (2.9) we have kept the UV and IR boundary conditions on bulk fields implicit. One might also raise the concern that $\Phi(L) = \varphi$ explicitly contradicts the IR boundary condition if it is explicitly Dirichlet, say $\Phi(L) = 0$. Because of the universality of the S-matrix described above, we have the freedom in such a case to choose e.g. $\partial_z \Phi(L) = \varphi$. Whatever the IR boundary conditions, we simply choose φ in a way that does not conflict the boundary conditions on bulk fields which fundamentally define the dual gauge theory.

In section 4 we describe an equivalent procedure in terms of Feynman diagrams which allows one to compute $\mathcal{S}_{on-shell}[\varphi]$ more directly. First, however, we demonstrate a pedestrian version of the procedure on a concrete example.

3 An Example: AdS/QCD

In order to illustrate how to apply our prescription, we now work out the IR effective action for the massless modes of a well-studied example – hard wall AdS/QCD. We emphasize that our goal here is not to discuss AdS/QCD or this specific model (and thus we exclude any motivations for it), but rather to demonstrate the details of our prescription in a concrete example. Towards the end of the section we will reproduce known results for this model as well as extend them to higher (and in some cases infinite) orders, to exhibit the efficiency of this framework.

3.1 The Model of Hirn and Sanz

We briefly review the AdS/QCD model of Hirn and Sanz [22], which we use to demonstrate the procedure described above. The bulk geometry is simply a slice of AdS_5, in Poincaré coordinates, with an IR boundary at $z = L$ and the usual UV conformal boundary at $z = 0$.[8] The metric is

$$ds^2 = \left(\frac{R}{z}\right)^2 \left(\eta_{\mu\nu} dx^\mu dx^\nu - dz^2\right) \ , \tag{3.1}$$

where R is the AdS radius. We ignore metric fluctuations. The global flavor symmetry currents of (massless) QCD are dual to bulk gauge fields of $SU(N_f)_L \times SU(N_f)_R$, labeled $L_M(x,z), R_M(x,z)$, with field strengths

$$\begin{aligned} L_{MN} &= \partial_M L_N - \partial_N L_M - i[L_M, L_N] \\ R_{MN} &= \partial_M R_N - \partial_N R_M - i[R_M, R_N] \ , \end{aligned} \tag{3.2}$$

[8] Due to UV divergences, in traditional techniques the UV boundary needs to be regulated, put at $z = \epsilon$, eventually taking the conformal limit $\epsilon \to 0$ at the end of the computation. Our analysis is insensitive to those divergences, and thus we can ignore this subtlety and already take the limit.

and Yang-Mills action[9],

$$\mathcal{S}_{5d} = -\frac{1}{4g_5^2} \int d^5x \sqrt{g} \; \mathrm{Tr} \left(L_{MN} L^{MN} + R_{MN} R^{MN} \right) \; . \tag{3.3}$$

Under 5d gauge transformations, with gauge group elements $L(x,z)$ and $R(x,z)$, the gauge fields transform as usual, $L_M \equiv L_M^a \frac{T^a}{\sqrt{2}} \to L L_M L^\dagger + iL\partial_M L^\dagger$,with the generators of $SU(N_f)$ normalized by $\mathrm{Tr}(T^a T^b) = 2\delta^{ab}$. It is natural to work with the vector and axial gauge fields

$$V_M = \frac{1}{2} \left(L_M + R_M \right) \; , \; A_M = \frac{1}{2} \left(L_M - R_M \right) \; , \tag{3.4}$$

and with the corresponding field strengths

$$\begin{aligned} V_{MN} &= \partial_{[M} V_{N]} - i[V_M, V_N] - i[A_M, A_N] \; , \\ A_{MN} &= \partial_{[M} A_{N]} - i[V_M, A_N] - i[A_M, V_N] \; , \end{aligned} \tag{3.5}$$

with which the bulk action reads

$$\mathcal{S}_{5d} = -\frac{1}{2g_5^2} \int d^5x \sqrt{g} \; \mathrm{Tr} \left(V_{MN} V^{MN} + A_{MN} A^{MN} \right) \; . \tag{3.6}$$

We will consider only normalizable modes (i.e. no background sources in the gauge theory), and accordingly set vanishing UV boundary conditions (BCs),[10]

$$V_\mu^a(x,0) = A_\mu^a(x,0) = 0 \; . \tag{3.7}$$

At the IR, non-gauge-invariant BCs are imposed on the axial gauge field (only)

$$V_{\mu z}^a(x,L) = A_\mu^a(x,L) = 0 \; , \tag{3.8}$$

in order to realize chiral symmetry breaking. This is the only field content of the model, minimally capturing the global symmetries of (massless) QCD and their breaking pattern.

We will work in the axial gauge for the vector gauge field

$$V_z = 0 \; , \tag{3.9}$$

which is compatible with the boundary conditions. We cannot fix a similar gauge for the axial gauge field, due to the non-trivial gauge holonomy along the radial direction, induced by the symmetry-breaking BCs. The closest we can expect is having

$$A_z = z f(x) \; , \tag{3.10}$$

[9]We ignore here the Chern-Simons term, as it will not affect the results we will focus on.
[10]The generalization to the presence of sources is immediate.

for some arbitrary function $f(x)$. For the time being we will keep A_z general, and later on we will see how this gauge can be consistently fixed *on-shell* (i.e. under the equations of motion).

From (3.6), the equations of motion for the gauge fields are

$$\begin{aligned}\frac{1}{\sqrt{g}}\partial_M\sqrt{g}V^{MN} &= i[V_M,V^{MN}]+i[A_M,A^{MN}]\ ,\\ \frac{1}{\sqrt{g}}\partial_M\sqrt{g}A^{MN} &= i[V_M,A^{MN}]+i[A_M,V^{MN}]\ ,\end{aligned} \tag{3.11}$$

and in appendix B we write them in components. Ultimately we will be interested in the on-shell action, so it is useful to plug (3.11) into (3.6) to get an on-shell bulk action

$$\mathcal{S}_{5d}^{(on-shell)} = \frac{-i}{2g_5^2}\int d^5x\sqrt{g}\mathrm{Tr}\Big\{\Big([V_M,V_N]+[A_M,A_N]\Big)V^{MN}+2[V_M,A_N]A^{MN}\Big\}\ , \tag{3.12}$$

where the boundary terms vanish with our BCs (3.7)(3.8). Finally, note that from here on we set $L=1$ and will restore it at the end of each computation by dimensional analysis; and that since g_5^2 is dimensionful (having the dimension of length), it is convenient to define $g_5^2\equiv Rg_4^2$. The dimensionless coupling g_4^2 is the effective coupling in the four-dimensional theory, which at the IR eventually relates to the pion decay constant $g_4^2\sim 1/f_\pi$, as we will see.

3.2 Applying the Prescription

Chiral symmetry is spontaneously broken in the gauge theory, and accordingly we expect to find a weakly interacting theory of massless pions. We turn now to use our holographic prescription to compute the low-energy effective action of the pions, the (on-shell) effective chiral Lagrangian. First, we need to identify a mode that has some overlap with the pion. As the chiral symmetry breaking is embedded in the bulk axial gauge field, it is natural to identify the pion with the IR boundary value of the axial mode[11]

$$\pi(x)\equiv A_z(x,L)\ . \tag{3.13}$$

We will denote this $\pi(x)$ as *our* pion, which overlaps with the "true" pion, defined via the non-linear transformation of the broken flavor symmetry as in (3.36). Our pion identification is not unique, and is in fact quite arbitrary.[12] We will see below

[11]This assignment is not gauge-invariant, even when restricting to gauge transformation that respect the BCs (3.7)(3.8) and gauge fixing (3.9). Gauge freedom that modifies the boundary value of A_z can either be fixed, or will remain as a gauge redundancy in the effective theory of $\pi(x)$.

[12]For example, we could have also defined $\pi=\partial_zA_z|_{z=L}$, or even setting a non-linear relation. In addition, it could have also simultaneously resided in A_μ via a relation of the form $A_\mu(x,L)\sim\partial_\mu\pi(x)$.

that there is some gauge freedom left (after the symmetry-breaking BCs and gauge fixing of V_z) that let us "reshuffle" the pion between the boundary values of A_z and A_μ. With it, we can adhere to the choice made in (3.13), while not having the pion to overlap with $A_\mu(x,L)$ (so that $A_\mu = O(\pi^3)$). We will also find this choice to be very useful for computations. In fact, we will see below that the same gauge freedom can be used order by order in $\pi(x)$ to set a complete gauge fixing (on-shell, under the equations of motion) of the form of (3.10). To all orders we will then have

$$A_z(x,z) = z\pi(x) \ . \tag{3.14}$$

We should note that in this simple model the true pion by itself can be easily identified; it is simply the gauge holonomy $\pi \sim \int_0^1 dz A_z$, which then agrees with our pion up to a normalization. We will come back to this later, but for the purpose of practicing our formalism, we simply ignore this and go on working with $\pi(x)$ as defined in (3.13). With this definition the pion serves as an IR source for bulk fields. We now solve for these fields in terms of $\pi(x)$.

Leading order solution: Considering parity, we see that V_μ can only contain even powers of π, while A_z, A_μ contain only odd powers. At leading order $O(\pi)$, we have the free equations of motion,

$$\partial^\mu\left(\partial_\mu A_z^{(1)} - \partial_z A_\mu^{(1)}\right) = 0 \ , \tag{3.15a}$$

$$z\partial_z \frac{1}{z}\left(\partial_\mu A_z^{(1)} - \partial_z A_\mu^{(1)}\right) - \partial^\nu\left(\partial_\mu A_\nu^{(1)} - \partial_\nu A_\mu^{(1)}\right) = 0 \ , \tag{3.15b}$$

where the superscript signals the order in π. At this point we need to choose the physical state we want to perturb around. As explained in section 2, computing the pion effective in the vacuum means demanding that that solution vanish when turning off $\pi \to 0$; this implies that *all fields must begin at order $O(\pi)$ or higher.* The general solution to (3.15) is, for generic $\pi(x)$,

$$\begin{aligned} A_z^{(1)}(x,z) &= a'(z)\pi(x) \ , \\ A_\mu^{(1)}(x,z) &= a(z)\partial_\mu\pi(x) \ . \end{aligned} \tag{3.16}$$

At this order (3.16) is a pure gauge, and it can be shown that for an unconstraint $\pi(x)$ the solution remains a pure gauge also at higher orders, and so is trivial. Thus, for a non-trivial solution we are forced to consider functions $\pi(x)$ that obey some constraining equation. Allowing for

$$\partial^2\pi(x) = O(\pi^3) \ , \tag{3.17}$$

the general solution is now of the form

$$\begin{aligned} A_z^{(1)}(x,z) &= (a'(z) + cz)\pi(x) \ , \\ A_\mu^{(1)}(x,z) &= a(z)\partial_\mu\pi(x) \ . \end{aligned} \tag{3.18}$$

Exhausting our gauge freedom and applying the BCs, we find

$$A_z^{(1)}(x,z) = z\pi(x) \quad , \quad A_\mu^{(1)} = 0 \; . \tag{3.19}$$

As explained earlier, (3.17) is to be identified with the pion's leading-order equation of motion in the effective theory. Note that (3.19) is always a solution to the A_μ equation of motion (3.15b) at this order, and that it becomes also a solution to the A_z equation (3.15a) with the pion's equation (3.17). This would be a recurring element.

Second order solution: We now expand the equations of motion (B.1) to second order in $\pi(x)$,

$$\begin{aligned} \partial_z \left(\partial \cdot V^{(2)}\right) &= 0 \; , \\ z\partial_z \frac{1}{z}\partial_z V_\mu^{(2)} - \partial^\nu \left(\partial_\nu V_\mu^{(2)} - \partial_\mu V_\nu^{(2)}\right) &= -i\Big[A_z^{(1)}, \partial_\mu A_z^{(1)}\Big] \; . \end{aligned} \tag{3.20}$$

Considering only normalizable modes and plugging in the first order solution (3.19), these become

$$\partial \cdot V^{(2)} = 0 \; , \tag{3.21a}$$

$$z\partial_z \frac{1}{z}\partial_z V_\mu^{(2)} - \partial^2 V_\mu^{(2)} = -iz^2\Big[\pi, \partial_\mu \pi\Big] \; . \tag{3.21b}$$

In appendix C we solve (3.21) with separation of variables and find

$$\begin{aligned} V_\mu^{(2)}(x,z) &= \frac{iz^2}{\partial^2}\left(1 - \frac{2}{z\partial}\frac{J_1(z\partial)}{J_0(\partial)}\right)\left[\pi(x), \partial_\mu \pi(x)\right] \\ &= -iz^2\left\{\frac{1}{8}\left(z^2-2\right) + \frac{1}{192}\left(z^2-3\right)^2\partial^2 + O(\partial^4)\right\}\left[\pi, \partial_\mu \pi\right] \; , \end{aligned} \tag{3.22}$$

where the J's are Bessel functions and $\partial \equiv \sqrt{\partial^2}$. Note again that (3.22) is the unique solution to (3.21b) (with BCs (3.7-3.8)), and it immediately becomes also a solution of (3.21a), up to higher orders in $\pi(x)$, provided the pion's equation of motion (3.17).

We can also expand already the equations of motion (3.21) in (4d) derivatives:

$$\partial \cdot V^{(2,1)} = 0 \; , \tag{3.23a}$$

$$z\partial_z \frac{1}{z}\partial_z V_\mu^{(2,1)} = -iz^2\Big[\pi, \partial_\mu \pi\Big] \; , \tag{3.23b}$$

where now the superscript (m,n) identifies a term with m pions and n flat-space derivatives. With BCs (3.7-3.8), (3.23b) immediately solves to

$$V_\mu^{(2,1)} = iv_{2,1}[\pi, \partial_\mu \pi] \quad , \quad v_{2,1} = -\frac{1}{8}z^2(z^2-2) \; , \tag{3.24}$$

which is also a solution to (3.23a) under (3.17). Iterating the procedure we can expand (3.21) to next-to-leading order in derivatives

$$\partial \cdot V^{(2,3)} = 0 \ , \tag{3.25a}$$

$$z\partial_z \frac{1}{z}\left(\partial_z V_\mu^{(2,3)}\right) = \partial^\nu\left(\partial_\nu V_\mu^{(2,1)} - \partial_\mu V_\mu^{(2,1)}\right) \ , \tag{3.25b}$$

where again the r.h.s. is written in terms of previous solution (3.24). Again, (3.25) instantly gives

$$V_\mu^{(2,3)} = i v_{2,3}\Big[\partial^\nu \pi, \partial_\nu \partial_\mu \pi\Big] \quad , \quad v_{2,3} = -\frac{1}{96} z^2 (z^2-3)^2 \ , \tag{3.26}$$

which, together with (3.24), agrees with (3.22).

Third order solution: Expanding (B.1) *naively* to $O(\pi^3)$ we find

$$\partial^\mu\left(\partial_\mu A_z^{(3)} - \partial_z A_\mu^{(3)}\right) = -2i\partial^\mu\Big[A_z^{(1)}, V_\mu^{(2)}\Big] \ , \tag{3.27a}$$

$$z\partial_z \frac{1}{z}\left(\partial_\mu A_z^{(3)} - \partial_z A_\mu^{(3)}\right) + \partial^\nu\left(\partial_\nu A_\mu^{(3)} - \partial_\mu A_\nu^{(3)}\right) = -2iz\partial_z \frac{1}{z}\Big[A_z^{(1)}, V_\mu^{(2)}\Big] \ . \tag{3.27b}$$

Firstly notice that A_z and A_μ only enter the l.h.s. through the combination $\partial_\mu A_z - \partial_z A_\mu$. For $n \geq 1$, we can use the remaining gauge freedom to set $A_z^{(2n+1)} = 0$ at each order, and shift the entire contribution to $A_\mu^{(2n+1)}$. This amounts to setting (3.14) exactly as a gauge fix, as proclaimed earlier. Expanding (3.27) in derivatives and solving first the A_μ equation we find at leading order

$$A_\mu^{(3,1)} = a_{3,1}(z)\Big[\pi, \big[\pi, \partial_\mu \pi\big]\Big] \quad , \quad a_{3,1}(z) = \frac{1}{24} z^2 (z^2-1)(z^2-2) \ , \tag{3.28}$$

and at next-to-leading order

$$\begin{aligned} A_\mu^{(3,3)} &= a_{3,3}^{(1)}(z)\Big[\partial^\nu \pi, \big[\partial_\nu \pi, \partial_\mu \pi\big]\Big] + a_{3,3}^{(2)}(z)\Big[\pi, \big[\partial^\nu \pi, \partial_\nu \partial_\mu \pi\big]\Big] \ , \\ a_{3,3}^{(1)} &= \frac{1}{384} z^2 (z^2-1)(z^4 - 5z^2 + 7) \quad , \quad a_{3,3}^{(2)} = \frac{1}{192} z^2 (z^2-1)(z^2-3)^2 \ . \end{aligned} \tag{3.29}$$

Plugging the solution into the A_z equation leads to a contradiction! This is because we have forgotten a term in (3.27). Indeed, when expanded to first order in $\pi(x)$, we used (3.17) to neglect the term $\partial^2 A_z = z\partial^2 \pi = O(\pi^3)$. But we should then include this term at higher orders. The expansion in the presence of π's equation of motion should be treated with care, since the equation itself (3.17) shuffles the various orders. Adding the forgotten term $\left(\partial^2 A_z^{(1)}\right)^{(3)}$ into the l.h.s. of (3.27a) we find that (3.28) and (3.29) are also solutions of (3.27a), provided a unique correction to π's equation of motion at $O(\pi^3)$,

$$\partial^2 \pi = \frac{1}{6}\Big[\partial^\mu \pi, \big[\pi, \partial_\mu \pi\big]\Big] + \frac{11}{192}\Big[\partial^\nu \pi, \big[\partial^\mu \pi, \partial_\nu \partial_\mu \pi\big]\Big] + O(\pi^5, \partial^6 \pi^3) \ . \tag{3.30}$$

It is straightforward to iterate this procedure and obtain the solution up to any finite order in derivatives and in pions. (When expanding only in pions, the solution at any given order, containing all orders in derivatives, can also be obtained systematically, but in a less simple manner.) The theme outlined above is also maintained to all orders. At any even order in π there is a unique solution to V_μ's equations of motion (and BCs), which is then automatically a solution to V_z's equation of motion, when using π's equation of motion at lower orders. At each odd order in π there is a unique solution to the A_μ equation of motion. That solution also solves the A_z equation of motion, provided a unique correction to the equation of motion of π. The role of the A_z and V_z equations of motion is thus only to enforce the pion's equation of motion in the effective theory.

On-shell effective action: We now plug the solutions obtained above back into the bulk action, and explicitly perform the integration in z. This leave us with a four-dimensional action for $\pi(x)$. As explained before, this is identified with the low-energy on-shell effective action for the pions. There is no quadratic term in the expansion of (3.12), as expected for an on-shell action. At fourth order it gives

$$\mathcal{S}_{\pi^4}^{(5d,o.s.)} = \frac{i}{g_4^2}\int_0^1 \frac{dz}{z}\int d^4x \mathrm{Tr}\left\{\left[V_\mu^{(2)}, A_z^{(1)}\right]\partial^\mu A_z^{(1)}\right\} . \tag{3.31}$$

Plugging (3.19)(3.24)(3.26) and integrating over z we find

$$\mathcal{S}_{\partial^2\pi^4} = -\frac{L^2}{24 g_4^2}\int d^4x \mathrm{Tr}\left[\pi, \partial^\mu\pi\right]\left[\pi, \partial_\mu\pi\right] , \tag{3.32a}$$

$$\mathcal{S}_{\partial^4\pi^4} = -\frac{11 L^4}{768 g_4^2}\int d^4x \mathrm{Tr}\left[\partial_\mu\pi, \partial_\nu\pi\right]\left[\partial^\mu\pi, \partial^\nu\pi\right] , \tag{3.32b}$$

where we have reinstated the IR cutoff scale L.[13] We can also insert the all-order-in-derivative result for V_μ (3.22) to (3.31) in order to find

$$\mathcal{S}_{\pi^4} = -\frac{1}{g_4^2}\int d^4x \left[\pi, \partial_\mu\pi\right]\frac{1}{4\partial^2}\left(1 - \frac{8}{(L\partial)^2}\frac{J_2(L\partial)}{J_0(L\partial)}\right)\left[\pi, \partial^\mu\pi\right] , \tag{3.33}$$

which agrees with (3.32) at first orders. This is the exact on-shell effective action controlling the dynamics of four external pions through all energies! Similarly, we derive the six-pion, two-derivative term,

$$\mathcal{S}_{\partial^2\pi^6} = -\frac{L^4}{360 g_4^2}\int d^4x \mathrm{Tr}\left[\pi, \left[\pi, \partial_\mu\pi\right]\right]\left[\pi, \left[\pi, \partial^\mu\pi\right]\right] . \tag{3.34}$$

3.3 Comparing with Hirn and Sanz

We have derived above the on-shell effective action for the pions in the Hirn-Sanz model [22]. Our result conceptually differ from those of Hirn and Sanz [22] in three

[13] Notice that this scale controls the derivative expansion.

ways. Our effective action is written in terms of the NGB field, π, whereas [22] provide their result in terms of $U = e^{i\pi}$. We have an on-shell effective action, instead of the off-shell action of [22]. Finally, we differ in our definitions for π, so we should only expect the two modes to overlap, and thus the results to agree on-shell. In the next section we will also compare our results at the off-shell level, from which we will derive the exact relation between the two modes. (This will agree with our expectations; as explained earlier, in this simple model we can directly identify the true pion and find the precise relation to our pion.)

As in (2.11), our pion mode, for which we were computing the effective action, $\pi(x)$, is related to the *true* pion $\Pi(x)$ via

$$\pi = c\Pi + O(\Pi^3) \ , \tag{3.35}$$

with some yet unknown c. The true pion (i.e. that of Hirn and Sanz) is defined through its exponential,

$$U \equiv \exp\left[i\Pi/f_\pi\right] \tag{3.36}$$

(f_π is the pion decay constant), which transforms *covariantly* under the spontaneously broken flavor symmetry,

$$U \to LUR^\dagger \ . \tag{3.37}$$

To make the comparison we first convert the chiral Lagrangian from the U language to the π language, and then turn it to its on-shell version by deriving its equations of motion and plugging them back into the action.

The chiral effective Lagrangian is written at lower orders in terms of derivatives of U.[14] The unique two-derivative term is

$$\mathcal{L}_2 = \frac{f_\pi^2}{4}\mathrm{Tr}\partial_\mu U^\dagger \partial^\mu U \ , \tag{3.38}$$

by which the pion Π obtains a canonical kinetic term (given the trace conventions in A and the definition of (3.36)). Expanding this term up to six-pion order and putting it on shell we find

$$\mathcal{L}_{2,4}^{(o.s.)} + \mathcal{L}_{2,6}^{(o.s.)} = \mathrm{Tr}\left\{ -\frac{1}{48 f_\pi^2}\left[\Pi, \partial_\mu\Pi\right]\left[\Pi, \partial^\mu\Pi\right] - \frac{1}{720 f_\pi^4}\Big[\Pi, \left[\Pi, \partial_\mu\Pi\right]\Big]\Big[\Pi, \left[\Pi, \partial^\mu\Pi\right]\Big]\right\} . \tag{3.39}$$

Matching with our result at the two-derivative order (3.32), and remembering the overall normalization between the two modes, we find

$$\frac{L_1^2 c^4}{24 g_4^2} = \frac{1}{48 f_\pi^2} \quad , \quad \frac{L^4 c^6}{360 g_4^2} = \frac{1}{720 f_\pi^4} \ . \tag{3.40}$$

[14] Both ∂ and π scale with E/f_π for a process at energy E. This is the small parameter in the expansion.

From this f_π and c read

$$f_\pi^2 = \frac{2}{(g_4 L)^2} \quad , \quad c = \frac{g_4}{\sqrt{2}} = \frac{1}{f_\pi L} \ , \tag{3.41}$$

in a perfect agreement with the pion decay constant of Hirn and Sanz. At the four-derivative order, the chiral Lagrangian for $N_f = 3$ (a condition specifically used in [22]) and with no background fields turned on consists of three independent terms [25, 26],

$$\mathcal{L}_4 = L_1 \langle \partial_\mu U^\dagger \partial^\mu U \rangle^2 + L_2 \langle \partial_\mu U^\dagger \partial_\nu U \rangle \langle \partial^\mu U^\dagger \partial^\nu U \rangle + L_3 \langle \partial_\mu U^\dagger \partial^\mu U \partial_\nu U^\dagger \partial^\nu U \rangle \ , \tag{3.42}$$

where $\langle ... \rangle$ stands for the flavor trace. Expanding in Π's and turning it on-shell we find at the four-pion order

$$\mathcal{L}_{4,4}^{(o.s.)} = -\frac{L_1}{f_\pi^4} \langle \partial_\mu \Pi \partial^\mu \Pi \rangle^2 - \frac{L_2}{f_\pi^4} \langle \partial_\mu \Pi \partial_\nu \Pi \rangle \langle \partial^\mu \Pi \partial^\nu \Pi \rangle - \frac{L_3}{f_\pi^4} \langle \partial_\mu \Pi \partial^\mu \Pi \partial_\nu \Pi \partial^\nu \Pi \rangle \ . \tag{3.43}$$

We can now rewrite our result (3.32b) in the form of (3.43). For flavor group $SU(3)$ we have the following identify (for example, see [19]),

$$\begin{aligned} \langle \Big[\partial_\mu \pi, \partial_\nu \pi \Big] \Big[\partial^\mu \pi, \partial^\nu \pi \Big] \rangle &= 2 \langle \partial_\mu \pi \partial_\nu \pi \partial^\mu \pi \partial^\nu \pi \rangle - 2 \langle \partial_\mu \pi \partial^\mu \pi \partial_\nu \pi \partial^\nu \pi \rangle \\ &= \langle \partial_\mu \pi \partial^\mu \pi \rangle^2 + 2 \langle \partial_\mu \pi \partial_\nu \pi \rangle \langle \partial^\mu \pi \partial^\nu \pi \rangle - 6 \langle \partial_\mu \pi \partial^\mu \pi \partial_\nu \pi \partial^\nu \pi \rangle \ . \end{aligned} \tag{3.44}$$

Using (3.44) and (3.41) in (3.32b) our on-shell action reads

$$\mathcal{S}_{4d}^{(\partial^4 \pi^4)} = \frac{11 R}{768 g_5^2 f_\pi^4} \int d^4 x \Big(6 \langle \partial_\mu \pi \partial^\mu \pi \partial_\nu \pi \partial^\nu \pi \rangle - \langle \partial_\mu \pi \partial^\mu \pi \rangle^2 - 2 \langle \partial_\mu \pi \partial_\nu \pi \rangle \langle \partial^\mu \pi \partial^\nu \pi \rangle \Big) \ . \tag{3.45}$$

Matched with (3.43), we get

$$L_2 = 2 L_1 \quad , \quad L_3 = -6 L_1 \quad , \quad L_1 = \frac{11}{768 g_4^2} \ , \tag{3.46}$$

again in a perfect agreement with Hirn and Sanz. We should point out one implicit (and unimportant) difference from the results of Hirn and Sanz. Whereas we have defined $g_4^2 \equiv g_5^2/R$, they make a slightly different definition $g_4^2 \equiv g_5^2/l_0$, where l_0 is their regularization parameter, cutting off z at the UV bulk geometry, that is taken eventually to zero. In any event, this is only a redefinition of the scale f_π, which is anyhow arbitrary. In other words, we could have absorbed this into our c, or to the definition of our pion, and match also that scale (but that, again, is unimportant).

In sum, using our formalism we have computed the chiral Lagrangian coefficients at order ∂^4 in the AdS/QCD model of Hirn and Sanz, and found perfect agreement with previous results. It is now a simple matter to extend our results to higher orders and compare with those in the chiral Lagrangian expansion, in order to compute higher order coefficients. (In fact, (3.33) already contains an infinite number of independent chiral Lagrangian coefficients, an information equivalent to the exact four-pion scattering amplitude for this model which we obtain below.)

4 Diagrammatics

The procedure described above provides a straightforward method for deriving the pion's effective action: we define the pion to be the IR-boundary value of the radial component of the axial bulk gauge field; we solve the classical equations of motion for all bulk fields in terms of the pion, and order-by-order in its magnitude; and finally, we plug these solutions back into the action and integrate over the radial direction. Instead of all that, we can equivalently derive the same on-shell effective action (or the S-matrix) directly using tree-level Feynman diagrams with pions sitting on external legs. Since the pions are *defined to live* on the IR boundary, pictorially we will be drawing 'mirrored Witten diagrams': bulk Feynman diagrams that start and end on the IR boundary.[15] This method would be more economical than the first one described, as it obviates the need to solve for A_μ, V_μ as a middle step. More concretely, considering the bulk partition function

$$Z_{bulk} = \int D\pi \int_{A_z(x,L)=\pi(x)} DA_z DA_\mu DV_\mu e^{iS[A_z,V_\mu,A_\mu]} \ , \tag{4.1}$$

the main statement of our initial prescription identifies the pion effective action as

$$e^{iS_{eff}^{(on-shell)}[\pi]} = \int_{A_z(x,L)=\pi(x)} DA_z DA_\mu DV_\mu e^{iS[A_z,V_\mu,A_\mu]} \ . \tag{4.2}$$

This is the analog of (2.12) for the AdS/QCD example of section 3. The bulk fields are also subject to the boundary conditions (3.7), (3.8). We have already imposed the gauge condition $V_z = 0$ in writing (4.1). The exclusive identification of A_z's boundary value with the pion mode is also not gauge-invariant, and we consider the particular splitting we use to be part of the gauge choice. (Note also that because we work strictly at tree level, we can safely neglect ghosts.) There is still too much gauge freedom left that need to be fixed. Given the gauge-symmetry-breaking boundary condition for A_μ (3.8) and the boundary-value assignment for A_z (2.10), a rigorous gauge fixing procedure is somewhat tricky. However, under the classical equations of motion we have shown that $A_z = z\pi$ constitutes a legal gauge choice. Since we only compute eventually on-shell observables (such as scattering amplitudes), at tree level, we may guess that using this gauge fixing inside the path integral might give the correct result. For now, we will take this as an ansatz and see later that it works, up to a fine subtlety which arises at $O(\pi^6)$ and which we will describe in more detail below. A simplified form of the partition function is then,

$$e^{iS_{eff}^{(off-shell)}[\pi]} = \int DA_\mu DV_\mu e^{iS[A_z=z\pi(x),V_\mu,A_\mu]} \ . \tag{4.3}$$

[15] Note, though, that our gauge fixing $A_z(x,z) \to z\pi(x)$ will qualitatively change this picture. Instead, after gauge fixing, the pions will be practically smeared along the radial direction.

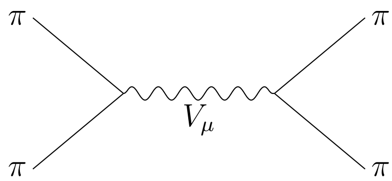


Figure 4.1. The only 5d diagram contributing to four-pion scattering (up to exchange of external pion lines).

Since we apply the gauge fixing $A_z = z\pi$ and ignore the path integration over A_z, we are actually computing now an *off-shell* effective action for π. We can obtain the on-shell action either by deriving the resulting EOM and plugging them back in, as shown in the previous section, or by also including tree-level diagrams of propagating pions, governed by the obtained off-shell effective action (including the kinetic term this time). Alternatively, by taking the pions to be in asymptotic states, we can directly compute their S-matrix.

First, we find the kinetic term in the off-shell effective action $S_0[\pi]$ by evaluating A_z's kinetic term subject to the gauge fixing $S_0[A_z = z\pi]$:

$$S_0[\pi] = \frac{1}{2g_4^2}\int d^4x \mathrm{Tr}\partial_\mu\pi\partial^\mu\pi \ . \tag{4.4}$$

We can obtain the n-pion interaction term, meanwhile, by summing all connected tree-level diagrams with propagating V_μ's and A_μ's and with n external π legs (see for instance figure 4.1). The 5d Feynman rules and Green's functions are straightforward to compute, and are summarized in Appendix E. Below, we show explicitly how to obtain the four-pion and six-pion vertices. We compare the four-pion term directly to results derived in previous sections, and derive the six-pion term in the effective action to all orders in derivatives, a novel result.

4.1 Four-Pion Effective Action

There is only one diagram that contributes to the four-pion term, shown in Figure 4.1. Using the Feynman rules described in Appendix E we find[16]

$$
\begin{aligned}
S_{\pi^4} = -\frac{1}{24 g_4^2 L^4} \int \frac{d^4k_1 d^4k_2 d^4k_3 d^4k_4}{(2\pi)^{16}} \delta^{(4)}(k_1+k_2+k_3+k_4) \prod_{i=1}^{4} \pi^{a_i}(k_i) \\
\Big[f^{a_1 a_2 e} f^{a_3 a_4 e} I_4(k_{12})(k_1-k_2)\cdot(k_3-k_4) \\
+ f^{a_1 a_3 e} f^{a_2 a_4 e} I_4(k_{13})(k_1-k_3)\cdot(k_2-k_4) \\
+ f^{a_1 a_4 e} f^{a_2 a_3 e} I_4(k_{14})(k_1-k_4)\cdot(k_2-k_3) \Big] \qquad (4.5) \\
= -\frac{1}{g_4^2 L^2} \int d^4x \mathrm{Tr}[\pi, \partial^\mu \pi] I_4(i\partial)[\pi, \partial_\mu \pi] \qquad (4.6)
\end{aligned}
$$

where $\partial \equiv \sqrt{\partial^2}$ and we define

$$
k_{ij} = \sqrt{(k_i + k_j)^2}\ . \tag{4.7}
$$

The vector Green's function is defined in equation (E.4), and the integral I_4 is

$$
I_4(k) \equiv \int_0^1 dz \int_0^1 dz' z z' G_V(k, z, z') = \frac{1}{4k^2}\left[1 - \frac{8J_2(k)}{k^2 J_0(k)}\right] \tag{4.8}
$$

$$
= -\frac{1}{24} - \frac{11}{1536}k^2 - \frac{19}{15360}k^4 + \mathcal{O}(k^6)\ . \tag{4.9}
$$

The function I_4 has poles at the zeroes of J_0, coinciding (as they should) with the masses of vector meson states. We thus have the off-shell action up to fourth order in pions and to all orders in derivatives,

$$
S_4^{(off-shell)} = \frac{1}{2g_4^2} \int d^4x \mathrm{Tr}\Big\{ \partial_\mu \pi \partial^\mu \pi - 2[\pi, \partial^\mu \pi] I_4(i\partial)[\pi, \partial_\mu \pi] \Big\} \tag{4.10}
$$

It is straightforward to check that this action produces the same on-shell action we derived in the previous section (3.33), by writing the pion's equation of motion

$$
\partial^2 \pi = -4\left[\partial_\mu \pi, I_4(i\partial)[\pi, \partial^\mu \pi]\right]\ , \tag{4.11}
$$

and plugging it back in. By using the Feynman rules derived from (4.10) we also find the four-pion scattering amplitude to all orders in derivatives:

$$
\begin{aligned}
\mathcal{M}_{\pi^4} = -\frac{g_4^2}{L^4} \delta^{(4)}(\sum_i k_i) \Big[f^{a_1 a_2 c} f^{a_3 a_4 c} I_4(\sqrt{s})(t-u) \\
+ f^{a_1 a_3 c} f^{a_2 a_4 c} I_4(\sqrt{t})(u-s) \\
+ f^{a_1 a_4 c} f^{a_2 a_3 c} I_4(\sqrt{u})(s-t) \Big]\ , \qquad (4.12)
\end{aligned}
$$

in terms of the standard Mandelstam variables: $s = k_{12}^2\ , t = k_{13}^2\ , u = k_{14}^2$.

[16] Since we work at most to sixth order in pions, we are allowed to put the external pions on-shell at leading order in π. In other words, we can set $k_1^2 = k_2^2 = 0$, which in turn implies that the longitudinal piece of the vector propagator does not contribute at this order. Such terms may contribute at order π^8 and higher, and should be taken into account in those cases.

4.2 Six-Pion Effective Action

The power of this method becomes evident when computing higher order terms. Here we outline the computation of the six-pion term, which we can easily compute to arbitrary order in derivatives, though we are unable to write it in a closed analytic form. Details are relegated to Appendix F. The six-pion term in the effective action is obtained from tree-level diagrams with six external pions; there are only three such diagrams (up to relabeling of external pion legs). These diagrams are shown in Figure F.

Each diagram is characterized by some 4d Lorentz structure and some integral in z-space. The integrals corresponding to the diagrams in Figure F are given by

$$I_{6,1}(k_a,k_b,k_c) = \int \frac{dz_d}{z_d} dz_a z_a dz_b z_b dz_c z_c G_V(k_a,z_a,z_d) G_V(k_b,z_b,z_d) G_V(k_c,z_c,z_d) \tag{4.13}$$

$$I_{6,2}(k_a,k_b,q_c) = \int dz_a z_a dz_b z_b dz_c dz_d \partial_{z_c} G_V(k_a,z_a,z_c) \partial_{z_d} G_V(k_b,z_b,z_d) G_A(q_c,z_c,z_d) \tag{4.14}$$

$$I_{6,3}(k_a,k_b) = \int dz_a z_a dz_b z_b dz_c z_c G_V(k_a,z_a,z_c) G_V(k_b,z_b,z_c) \tag{4.15}$$

Summing the contributions from all three diagrams, we find

$$\begin{aligned} S_6^{(off-shell)} = -\frac{R}{g_5^2} \int \prod_{i=1}^{6} dk_i \ \mathrm{Tr}\Big\{ \big[\pi(k_5),[\pi(k_1),\pi(k_2)]\big] \big[\pi(k_6),[\pi(k_3),\pi(k_4)]\big] \Big\} \\ \Big\{ (k_1-k_2)\cdot(k_3-k_4) \left[\frac{1}{4} I_{6,3}(k_{12},k_{34}) + I_{6,2}(k_{12},k_{34},|k_1+k_2+k_5|) \right] \\ - (k_5\cdot(k_1-k_2))(k_6\cdot(k_3-k_4)) \frac{I_{6,2}(k_{12},k_{34},|k_1+k_2+k_5|) - I_{6,2}(k_{12},k_{34},0)}{(k_1+k_2+k_5)^2} \\ + \frac{1}{6} I_{6,1}(k_{12},k_{34},k_{56}) \left[((k_1+k_2)\cdot(k_5-k_6))((k_1-k_2)\cdot(k_3-k_4)) \right. \\ + ((k_3+k_4)\cdot(k_1-k_2))((k_3-k_4)\cdot(k_5-k_6)) \\ \left. + ((k_5+k_6)\cdot(k_3-k_4))((k_5-k_6)\cdot(k_1-k_2)) \right] \Big\} \end{aligned} \tag{4.16}$$

This result is exact to all orders in derivatives. To verify our results from section 3, we can expand order-by-order in derivatives, and rewrite the interaction term in position space. The leading term is

$$S_{\partial^2\pi^6} = -\frac{L^4}{720 g_4^2} \int d^4x \mathrm{Tr}\Big[\pi,\big[\pi,\partial_\mu\pi\big]\Big]\Big[\pi,\big[\pi,\partial^\mu\pi\big]\Big] , \tag{4.17}$$

which agrees with (3.34) on shell.

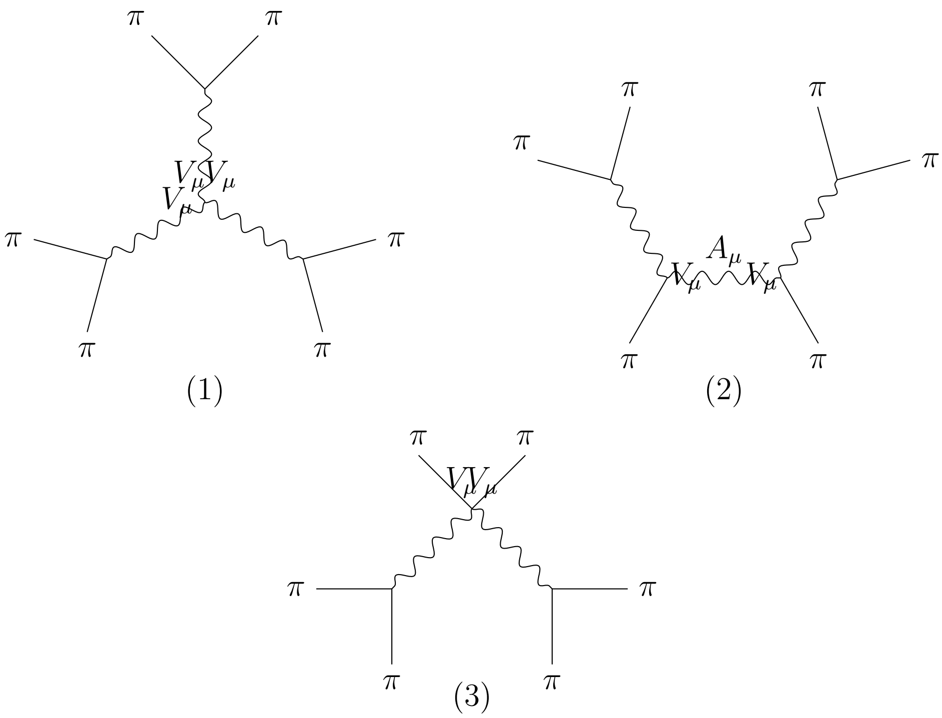


Figure 4.2. The only three diagrams from the 5d action that contribute to the six-pion interaction term (up to exchanges of external pion legs). Diagrams (2) and (3) contribute at $O(\partial^2)$ in the derivative expansion, while Diagram (1) only begins to contribute at $O(\partial^4)$.

There is, however, the subtlety mentioned earlier about this result. Diagram (2) in Figure F involves a vertex coupling of the form $\int d^4xdz/z\ A_zA_\mu\partial_zV^\mu$ which turns into $\int d^4xdz\ \pi A_\mu\partial_zV^\mu$ after our gauge fixing. Naively, one could integrate by parts this to arrive at $-\int d^4xdz\ \pi\partial_zA_\mu V^\mu$, which should then give the same final result. (Remember that $\pi(x)$ is strictly four-dimensional.) This does not turn out to be the case, though, and the correct result is only obtained with the first form of the vertex above. The subtlety is due to the heuristic *on-shell* gauge fixing we have made, $A_z = z\pi$, inside the path integral. If we first integrate by parts in the action, and then use the same (strictly illegal) gauge fixing procedure, we get different (and wrong) results. In other words, while it is found that replacing everywhere in the path integral $A_z \to z\pi$ is most of the time consistent, we evidently find that replacing $\partial_z(A_z/z) \to 0$ in the path integral is *inconsistent*. To recap, the heuristic gauge fixing process we have made does not commute with integration by parts, and the exact form of the action needs to be unambiguously 'chosen' to obtain the correct results. We leave the making of an honest gauge fixing procedure and the full exploration of this subtlety to future work.

5 Discussion and Future Directions

We have described a way to derive IR effective actions for large N confining gauge theories. Our approach isolates the lightest mode of the field theory and integrates out the remaining degrees of freedom. Though inspired by holographic Wilsonian RG, we have isolated a situation in which the hWRG procedure yields scheme-independent results. Though the approach can be applied to massive particles, it is particularly interesting for the case of Goldstone bosons, where one needs not impose any external on-shell conditions on the massless mode, but in fact derives its equation of motion directly from the bulk physics. We have demonstrated the mechanics of the procedure using the well-studied case of the Hirn-Sanz AdS/QCD model, where we could straightforwardly generate S-matrix elements for arbitrary numbers of pions to arbitrary orders in derivatives.

Our methods would find fruitful application in a variety of areas, from more complicated versions of holographic QCD or holographic technicolor, to duals of condensed matter systems. For example, this framework renders the study of additional (e.g. $(F_{\mu\nu})^n$) interaction terms quite straightforward, allowing one to concretely estimate the error introduced by neglecting such terms in bottom-up AdS/QCD frameworks. One might also use such techniques to define the low energy effective action of fluctuations around holographic realizations of spatially inhomogeneous vacua (or "striped phases") recently identified in holographic QCD and AdS/CMT systems ***[[CITE: first Ooguri+Park paper; first one by Gauntlett + Donos]]***, especially in the confining phase [[CITE Bayona, Zamaklar, Peeters 2011]].

We have developed this method for a very simple case: a truncated AdS, with easily identified Goldstone modes. It should not, however, depend on the truncation of the spacetime. It would be interesting, therefore, to explore the extension of our techniques to spacetimes in which a smooth gravitational potential induces confining behavior (such as the soft wall model of AdS/QCD ***[[CITE Karch, Katz, Son, Stephanov hep-ph/0602229]], or other confining geometries, like the warped deformed conifold of ***[[CITE Klebanov, Strassler hep-th/0007191]] . The latter may lead to new insights into Seiberg duality. ***[[CITE Zohar 1010.4105 ?? Say more??]]

Acknowledgments

We are indebted to Zohar Komargodski for his collaboration in the early stages of this work, and for useful discussions and comments later on. We are also very grateful to Ofer Aharony for many helpful discussions and detailed suggestions. MF would also like to thank Rajesh Gopakumar for a useful discussion, and the Weizmann Institute for a support throughout most of this project.

A Conventions

In order to facilitate comparison to [22] we adopt their conventions,

$$L_M = L_M^a \frac{T^a}{\sqrt{2}} \qquad \text{and} \qquad \mathrm{Tr}(T^a T^b) = 2\delta^{ab} \ . \tag{A.1}$$

The action

$$\mathcal{S}_{5d} = -\frac{1}{4g_5^2} \int d^5x \sqrt{g}\ \mathrm{Tr}\left(L_{MN}L^{MN} + R_{MN}R^{MN}\right) \tag{A.2}$$

thus gives canonically normalized fields in 5d. We work with the vector and axial-vector combinations,

$$V_M = \frac{1}{2}\left(L_M + R_M\right) \qquad \text{and} \qquad A_M = \frac{1}{2}\left(L_M - R_M\right) \ , \tag{A.3}$$

and maintain the same normalization convention $A_M = \frac{1}{\sqrt{2}} A_M^a T^a$, and similarly for the pion, $\pi = \frac{1}{\sqrt{2}} \pi^a T^a$.

B Equations of Motion

The equations of motion in component form in the model of Hirn and Sanz, having applied the gauge condition $V_z = 0$, are

$$A_z : \quad \partial^\mu \Big(\partial_\mu A_z - \partial_z A_\mu\Big) = i\partial^\mu \Big[V_\mu, A_z\Big] + i\Big[V^\mu, A_{\mu z}\Big] + i\Big[A^\mu, V_{\mu z}\Big] \ , \tag{B.1a}$$

$$\begin{aligned} A_\mu : \quad & z\partial_z \frac{1}{z}\Big(\partial_\mu A_z - \partial_z A_\mu\Big) + \partial^\nu\Big(\partial_\nu A_\mu - \partial_\mu A_\nu\Big) = iz\partial_z \frac{1}{z}\Big[V_\mu, A_z\Big] + i\Big[A_z, V_{\mu z}\Big] \\ & + i\partial^\nu\Big([A_\nu, V_\mu] + [V_\nu, A_\mu]\Big) + i\Big[A^\nu, V_{\nu\mu}\Big] + i\Big[V^\nu, A_{\nu\mu}\Big] \ , \end{aligned} \tag{B.1b}$$

$$V_z : \quad \partial^\mu\Big(\partial_z V_\mu\Big) = i\partial^\mu\Big[A_z, A_\mu\Big] + i\Big[A^\mu, A_{z\mu}\Big] + i\Big[V^\mu, V_{z\mu}\Big] \ , \tag{B.1c}$$

$$\begin{aligned} V_\mu : \quad & z\partial_z \frac{1}{z}\partial_z V_\mu - \partial^\nu\Big(\partial_\nu V_\mu - \partial_\mu V_\nu\Big) = iz\partial_z \frac{1}{z}\Big[A_z, A_\mu\Big] + i\Big[A_z, A_{z\mu}\Big] \\ & - i\partial^\nu\Big([A_\nu, A_\mu] + [V_\nu, V_\mu]\Big) - i\Big[A^\nu, A_{\nu\mu}\Big] - i\Big[V^\nu, V_{\nu\mu}\Big] \ . \end{aligned} \tag{B.1d}$$

As described in subsection 3.2, we solve these equations order-by-order in pion fields.

C All-Order Solution

Expanding (B.1c),(B.1d) at $O(\pi^2)$ and using the $O(\pi)$ solution, we have

$$\begin{aligned} & \partial^\mu\left(\partial_z V_\mu^{(2)}\right) = 0 \ , \\ & z\partial_z \frac{1}{z}\partial_z V_\mu^{(2)} - \partial^\nu\left(\partial_\nu V_\mu^{(2)} - \partial_\mu V_\nu^{(2)}\right) = -i\Big[A_z^{(1)}, \partial_\mu A_z^{(1)}\Big] \ . \end{aligned} \tag{C.1}$$

More generally, having no non-normalizable modes in V_μ, we find

$$\partial^\mu V_\mu^{(2)} = 0 \ , \tag{C.2}$$

at this order. The full equation at $O(\pi^2)$ is

$$z\partial_z \frac{1}{z}\partial_z V_\mu^{(2)} - \partial^2 V_\mu^{(2)} = -iz^2\Big[\pi, \partial_\mu \pi\Big] \ . \tag{C.3}$$

We begin at $O(\pi^2)$ with the $V_\mu^{(2)}$ equation (C.3) and first solve for the homogenous part,

$$z\partial_z \frac{1}{z}\partial_z \bar{V}_\mu - \partial^2 \bar{V}_\mu = 0 \ . \tag{C.4}$$

Using separation of variables, we find the solution in terms of Bessel functions,

$$V_\mu(k,z) = \left[z\left(c_1 J_1(kz) + c_2 Y_1(kz)\right)\right] e^{ik\cdot x}\epsilon_\mu(k) \ . \tag{C.5}$$

Since we work in the absence of external vector sources, we keep only the normalizable mode J_1. Fourier-transforming we find the net homogeneous solution

$$\bar{V}_\mu(x,z) = \int \frac{d^4k}{(2\pi)^4} z J_1(kz)\epsilon_\mu(k) e^{ik\cdot x} \ , \tag{C.6}$$

where for reality of $\bar{V}_\mu$ we have $\epsilon_\mu(-k) = \epsilon_\mu^*(k)$.

We now need to find a particular solution for the original equation. We again solve by separation of variables, $\hat{V}_\mu = \xi(z) w_\mu(x)$:

$$\xi(z)\partial^2 w_\mu(x) - z\partial_z \frac{1}{z}\partial_z \xi(z) w_\mu(x) = iz^2\left[\pi(x), \partial_\mu \pi(x)\right] \ . \tag{C.7}$$

Taking $\xi(z) = z^2$ we find

$$w_\mu(x) = \frac{i}{\partial^2}\left[\pi, \partial_\mu \pi\right] \ , \tag{C.8}$$

or, in Fourier space,

$$w_\mu(x) = \int \frac{d^4k}{(2\pi)^4}\frac{e^{ik\cdot x}}{k^2}\int \frac{d^4k'}{(2\pi)^4} k'_\mu [\pi(k-k'), \pi(k')] \ . \tag{C.9}$$

The most general solution to equation (C.3) is then $V_\mu^{(2)} = \bar{V}_\mu + \hat{V}_\mu$,

$$V_\mu^{(2)}(x,z) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x}\left(zJ_1(kz)\epsilon_\mu(k) + \frac{z^2}{k^2}\int \frac{d^4k'}{(2\pi)^4} k'_\mu \left[\pi(k-k'), \pi(k')\right]\right) \ . \tag{C.10}$$

Imposing the boundary condition at $z = 1$ we have

$$\partial_z V_\mu^{(2)}(x,1) = \int \frac{d^4k}{(2\pi)^4} e^{ik\cdot x} \left(k J_0(k) \epsilon_\mu(k) + \frac{2}{k^2} \int \frac{d^4k'}{(2\pi)^4} k'_\mu \left[\pi(k-k'), \pi(k')\right] \right) = 0 \ . \tag{C.11}$$

(The boundary condition at $z = 0$ is already satisfied.) This uniquely sets the solution

$$\epsilon_\mu(k) = \frac{-2}{k^3 J_0(k)} \int \frac{d^4k'}{(2\pi)^4} k'_\mu \left[\pi(k-k'), \pi(k')\right] \ , \tag{C.12}$$

where we have used the identity

$$\partial_z \left(z J_1(kz) \right) = kz J_0(kz) \ . \tag{C.13}$$

The full solution is

$$V_\mu^{(2)}(x,z) = z^2 \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} k'_\mu \left[\pi(k-k'), \pi(k')\right] \frac{1}{k^2} \left(1 - \frac{2}{kz} \frac{J_1(kz)}{J_0(k)} \right) e^{ik\cdot x} \ . \tag{C.14}$$

It is convenient to expand the Bessel functions in k so we can find an expression in terms of spacetime derivatives:

$$\frac{1}{k^2} \left(1 - \frac{2}{kz} \frac{J_1(kz)}{J_0(k)} \right) = \frac{(z^2-2)}{8} - \frac{\left(z^2-3\right)^2}{192} k^2 + \frac{(z^6 - 12z^4 + 54z^2 - 76)}{9216} k^4 + O(k^6) \ . \tag{C.15}$$

Looking, for example, at the first two derivative orders,

$$\begin{aligned} V_\mu^{(2)}(x,z) &= \frac{z^2}{8} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4k'}{(2\pi)^4} k'_\mu \left[\pi(k-k'), \pi(k')\right] \left\{ \left(z^2-2\right) - \frac{1}{24} \left(z^2-3\right)^2 k^2 + O(k^4) \right\} e^{ik\cdot x} \\ &= -i \frac{z^2}{8} \left\{ \left(z^2-2\right) - \frac{1}{24} \left(z^2-3\right)^2 \partial^2 + O(\partial^4) \right\} \left[\pi, \partial_\mu \pi\right] \ , \end{aligned} \tag{C.16}$$

we see they coincide with previous results. Note that for this simple case we could just guess the particular solution, but in general one can solve in the nonlinear equations of motion in terms of the Green's functions developed in Appendix E.2.

This result allows us to compute the four-pion on-shell action to all derivative orders directly by plugging (C.14) into (3.31) and performing the z-integration before expanding the Bessel functions,

$$\int_0^1 dz \frac{z^3}{k^2} \left(1 - \frac{2}{kz} \frac{J_1(kz)}{J_0(k)} \right) = \frac{1}{4k^2} \left(1 - \frac{8}{k^2} \frac{J_2(k)}{J_0(k)} \right) \ . \tag{C.17}$$

The formal solution is

$$\mathcal{S}_{\pi^4} = -\frac{R}{g_5^2}\int d^4x\left[\pi,\partial_\mu\pi\right]\frac{1}{4\partial^2}\left(1-\frac{8}{\partial^2}\frac{J_2(\partial)}{J_0(\partial)}\right)\left[\pi,\partial^\mu\pi\right] =$$
$$= \frac{R}{g_5^2}\int d^4x\left[\pi,\partial_\mu\pi\right]\left\{-\frac{1}{24}+\frac{11}{1536}\partial^2-\frac{19}{15360}\partial^4+O(\partial^6)\right\}\left[\pi,\partial^\mu\pi\right] \ , \qquad \text{(C.18)}$$

which coincides with previous results, and the result using Feynman diagrams described in section 4. Note that when using the equations of motion for π this term will also contribute at higher orders in π, starting with $\partial^4\pi^6$.

D Two-derivative terms to all orders in pions

The spontaneously broken flavor symmetry is realized nonlinearly in the chiral Lagrangian. A term is made invariant by a completion to an infinite series of terms, of all pion orders and with the same derivative order as the original one. Since the on-shell action should be invariant under the full symmetry, at any derivative order we can start with the term at leading order in pions and systematically complete it to the invariant action to all orders in pions. For example, at leading order in derivatives this would simply be the completion from the π to the $U \equiv \exp\left(i\pi/f_\pi\right)$ language. For a consistency check we would like to solve for the two-derivatives term in the action, to all orders in pions, and see that it is consistent with the non-linear flavor symmetry. We have already checked this above at each order. We will now examine the all-orders solution at once.

To leading order in derivatives, the equations of motion can be recast as

$$\begin{aligned}\partial_z\frac{1}{z}A_{\mu z} &= i[\pi, V_{\mu z}] \ ,\\ \partial_z\frac{1}{z}V_{\mu z} &= i[\pi, A_{\mu z}] \ .\end{aligned} \qquad \text{(D.1)}$$

Let us define first,

$$Y_\mu^\pm \equiv e^{\pm\frac{i}{2}\left(z^2-1\right)\pi}X_\mu e^{\mp\frac{i}{2}\left(z^2-1\right)\pi} \ , \qquad \text{(D.2)}$$

for an arbitrary algebra-valued, z-independent X_μ. Then it is easily seen to satisfy

$$\partial_z Y_\mu^\pm = \pm iz\left[\pi, Y_\mu^\pm\right] \ , \qquad \text{(D.3)}$$

and

$$Y_\mu^\pm(z=1) = X_\mu \ . \qquad \text{(D.4)}$$

We immediately see that

$$\begin{aligned}A_{\mu z} &= \frac{z}{2}\left(Y_\mu^+ + Y_\mu^-\right)\\ V_{\mu z} &= \frac{z}{2}\left(Y_\mu^+ - Y_\mu^-\right)\end{aligned} \qquad \text{(D.5)}$$

solves the equations of motion (D.1), and

$$V_{\mu z}(z=1) = 0 \quad , \quad A_{\mu z}(z=1) = X_\mu \quad . \tag{D.6}$$

More explicitly it is,

$$\begin{aligned} A_{\mu z} &= \frac{z}{2}\left(e^{+\frac{i}{2}(z^2-1)\pi} X_\mu e^{-\frac{i}{2}(z^2-1)\pi} + e^{-\frac{i}{2}(z^2-1)\pi} X_\mu e^{+\frac{i}{2}(z^2-1)\pi}\right) \ , \\ V_{\mu z} &= \frac{z}{2}\left(e^{+\frac{i}{2}(z^2-1)\pi} X_\mu e^{-\frac{i}{2}(z^2-1)\pi} - e^{-\frac{i}{2}(z^2-1)\pi} X_\mu e^{+\frac{i}{2}(z^2-1)\pi}\right) \end{aligned} \tag{D.7}$$

Note that still $X\mu$ is left arbitrary in (D.5), since that, at the level of $V_{\mu z}$ and $A_{\mu z}$ we are missing one boundary condition. However, in order to have $V_{\mu z}$ ($A_{\mu z}$) π-even (π-odd), we like X_μ to be π-odd. For example, expanding $A_{\mu z}$ to leading order in π,

$$A_{\mu z} = z X_\mu + O(\pi^2) \ , \tag{D.8}$$

fixes $X_\mu = \partial_\mu \pi + O(\pi^2)$ when compared to our solution above.

E 5d Green's functions and Feynman Rules

We collect here the Green's functions of the propagating 5d fields in the gauge used throughout the paper. The vector and axial vector propagators are written as piece-wise functions along the z direction. This form lends itself most straightforwardly to obtaining results to arbitrary order in momentum, but it is simple to verify that the poles of these propagators correspond to the masses of the vector and axial-vector states predicted by the Hirn-Sanz model.

We begin by finding the Green's functions of the Abelian part of the equations of motion.

E.1 A_z wavefunction and pion propagator

As described in the body of the text, we can choose a gauge where

$$A_z(x,z) = z\pi(x) \tag{E.1}$$

to all orders in π. (As usual we set $L = 1$ for convenience and restore it using dimension counting in the final result.) A_z does not propagate in 5d, and does not run on internal legs.

The quadratic order action for pions comes from

$$S_{\pi^2} = -2\frac{R}{4g_5^2}\int dz d^4x \sqrt{g} A^a_{MN} A^{aMN} \supset \frac{R}{2g_5^2}\int d^4x \ \partial_\mu \pi^a \partial^\mu \pi^a \tag{E.2}$$

where we used $A_z = \pi z$ and integrated over z. The pion propagator in momentum space thus takes the form

$$\langle \pi^a(k)\pi^b(0)\rangle = \left(\frac{g_5^2}{R}\right) \ \frac{i}{k^2}\delta^{ab} \tag{E.3}$$

E.2 V_μ propagator

In the $V_z = 0$ gauge there are no ghosts in the 5d theory (and anyway we consider only tree-level diagrams) so we need only consider the Green's function of the fields V_μ. The two-point function takes the form

$$\langle V_\mu^a(k,z)V_\nu^b(q,z')\rangle = -i\frac{g_5^2}{2R}\,(2\pi)^4\delta(k+q)\delta^{ab}\,\left[G_V(k,z,z')\left(\eta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+G_V(0,z,z')\frac{k_\mu k_\nu}{k^2}\right] \tag{E.4}$$

where the k-momentum and zero-momentum Green's functions satisfy

$$z\partial_z\frac{1}{z}\partial_z G_V(k,z,z') + k^2 G_V(k,z,z') = z\delta(z-z') \tag{E.5}$$

$$z\partial_z\frac{1}{z}\partial_z G_V(0,z,z') = z\delta(z-z')\ . \tag{E.6}$$

The vector mode has to satisfy the boundary conditions,

$$\partial_z G_V(1,z') = G_V(0,z') = 0\ . \tag{E.7}$$

One can solve for the Green's function piecewise and show that

$$G_V(k|z,z') = -\frac{\pi z z'}{2}\left(\frac{Y_0(k)}{J_0(k)}J_1(kz)J_1(kz') - J_1(kz_-)Y_1(kz_+)\right)\ , \tag{E.8}$$

when z_+ (z_-) is the larger (resp. smaller) of z, z'. In particular, we have then

$$G_V(0|z,z') = -\frac{z_-^2}{2}\ . \tag{E.9}$$

This form guarantees that there is no pole at $k=0$ as can be seen by expanding order-by-order in momentum:

$$G_V(k,z,z') = \begin{cases} -\frac{z'^2}{2} + k^2\frac{1}{16}\left(-2z^2z'^2 + z'^4 + 4z^2z'^2\log(z)\right) + \dots & \text{for } z<z' \\ -\frac{z^2}{2} + k^2\frac{1}{16}\left(-2z'^2z^2 + z^4 + 4z'^2z^2\log(z')\right) + \dots & \text{for } z>z' \end{cases} \tag{E.10}$$

E.3 A_μ propagator

The story is almost identical for the A_μ propagator, except that the boundary condition at $z=1$ is now $A_\mu(z=1)=0$. The two-point function takes the form

$$\langle A_\mu^a(k,z)A_\nu^b(q,z')\rangle = -i\frac{g_5^2}{2R}\,(2\pi)^4\delta(k+q)\delta^{ab}\,\left[G_A(k,z,z')\left(\eta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)+G_A(0,z,z')\frac{k_\mu k_\nu}{k^2}\right] \tag{E.11}$$

where the k-momentum and zero-momentum Green's functions satisfy the same equations of motion as the vector. (Note that there are no quadratic order cross terms between A_μ and A_z. This is a direct result of the gauge choice $A_z = z\pi$.) The A_μ Green's function thus take the form

$$G_A(k,z,z') = \begin{cases} G_{A<}(k,z,z') \text{ for } z < z' \\ G_{A>}(k,z,z') \text{ for } z > z' \end{cases} \tag{E.12}$$

where

$$G_{A<}(k,z,z') = \frac{\pi}{2} z z' J_1(kz) \frac{J_1(k)Y_1(kz') - J_1(kz')Y_1(k)}{J_1(k)} \tag{E.13}$$

$$G_{A>}(k,z,z') = \frac{\pi}{2} z z' J_1(kz') \frac{J_1(k)Y_1(kz) - J_1(kz)Y_1(k)}{J_1(k)} \tag{E.14}$$

and

$$G_{A<}(0,z,z') = -\frac{z^2}{2}(1 - z'^2) \tag{E.15}$$

$$G_{A>}(0,z,z') = -\frac{z'^2}{2}(1 - z'^2) \ . \tag{E.16}$$

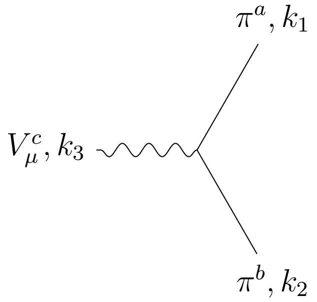
E.4 Feynman Rules for 5d theory

Here we collect the Feynman rules from the bulk action, written in momentum space along the flat space directions, and position space for the radial direction. All momenta are assumed to be incoming. Each n-point vertex should be accompanied by the integral

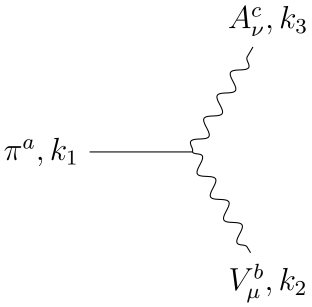
$$\frac{R}{g_s^2} \int \frac{dz}{z} \frac{d^4k_1 \ldots d^4k_n}{(2\pi)^{4n}} (2\pi)^4 \delta(k_1 + k_2 + \cdots + k_n) \ . \tag{E.17}$$

At each order

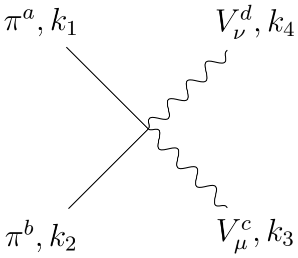
When we compute the 4d off-shell action, the A_z lines are external only. They are allowed to propagate (in 4d) only when we compute the S-matrix directly. V_μ and A_μ lines are always internal.



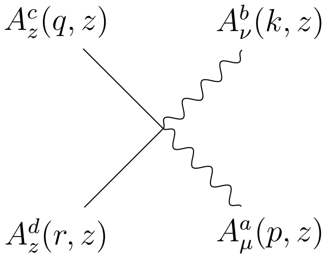
$$\sqrt{2} f^{abc} z^2 \pi^a(k_1)\pi^b(k_2)(k_2 - k_1)_\mu$$



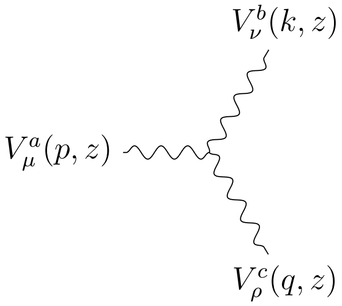
$$-2\sqrt{2} i f^{abc} z \pi^a(k_1) \eta_{\mu\nu} \partial_z^{(V)}$$



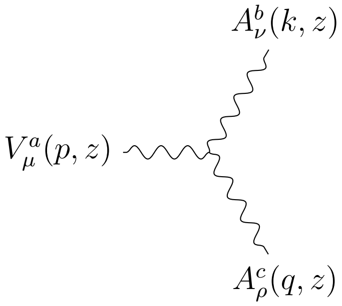
$$i(f^{ace} f^{bde} + f^{adf} f^{bcf}) z^2 \pi^a(k_1) \pi^b(k_2) \eta_{\mu\nu}$$



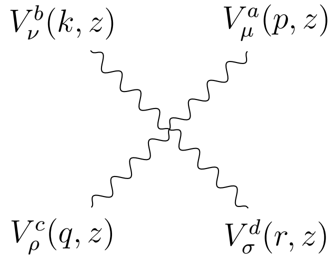
$$\frac{iR}{g_5^2}(f^{acf} f^{bdf} + f^{adf} f^{bcf}) \frac{z^2}{L^2} \pi^c(q) \pi^d(r) \eta_{\mu\nu}$$



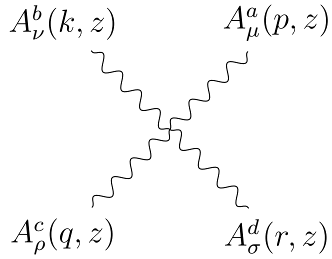
$$\frac{\sqrt{2}R}{g_5^2} f^{abc} \left[(p_\rho - k_\rho)\eta_{\mu\nu} + (q_\nu - p_\nu)\eta_{\rho\mu} + (k_\mu - q_\mu)\eta_{\nu\rho}\right]$$



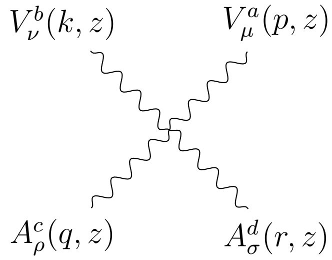
$$-\frac{\sqrt{2}R}{g_5^2} f^{abc} \left[(q_\mu - k_\mu)\eta_{\rho\nu} + (p_\nu - q_\nu)\eta_{\rho\mu} + (k_\rho - p_\rho)\eta_{\mu\nu}\right]$$



$$-\tfrac{iR}{g_5^2}\Big[f^{abf}f^{cdf}(\eta_{\mu\rho}\eta_{\nu\sigma}-\eta_{\nu\rho}\eta_{\mu\sigma}) \\ +f^{acf}f^{bdf}(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\nu\rho}\eta_{\mu\sigma}) \\ +f^{bcf}f^{adf}(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\mu\rho}\eta_{\nu\sigma})\Big]$$



$$-\tfrac{iR}{g_5^2}\Big[f^{abf}f^{cdf}(\eta_{\mu\rho}\eta_{\nu\sigma}-\eta_{\nu\rho}\eta_{\mu\sigma}) \\ +f^{acf}f^{bdf}(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\nu\rho}\eta_{\mu\sigma}) \\ +f^{bcf}f^{adf}(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\mu\rho}\eta_{\nu\sigma})\Big]$$



$$-\tfrac{iR}{g_5^2}\Big[f^{abf}f^{cdf}(\eta_{\mu\rho}\eta_{\nu\sigma}-\eta_{\nu\rho}\eta_{\mu\sigma}) \\ +f^{acf}f^{bdf}(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\nu\rho}\eta_{\mu\sigma}) \\ +f^{bcf}f^{adf}(\eta_{\mu\nu}\eta_{\rho\sigma}-\eta_{\mu\rho}\eta_{\nu\sigma})\Big]$$

F Six-pion interaction

We present here a more detailed derivation of the six-pion interaction term in the effective action. The three Feynman diagrams which contribute to this interaction are shown in Figure , labelled (1), (2), and (3). For each figure, we have to sum over all possible numbering of the external pion legs. Since not all of the 6! possible combinations are independent, we have to divide the net result in each case by a symmetry factor, as indicated in each case below.

In Diagram (1), we must divide by a symmetry factor of 48. Noting that only the $\eta_{\mu\nu}$ pieces of the vector propagators contribute at this order in π, and using momentum conservation at all vertices, we find that the contribution of Diagram (1) is

$$\begin{aligned} S_{6,1} = -\frac{R}{12g_5^2}\int dQ_1 I_{6,1}(k_{12},k_{34},k_{56})\,&[((k_1+k_2)\cdot(k_5-k_6))((k_1-k_2)\cdot(k_3-k_4)) \\ &+((k_3+k_4)\cdot(k_1-k_2))((k_3-k_4)\cdot(k_5-k_6)) \\ &+((k_5+k_6)\cdot(k_3-k_4))((k_5-k_6)\cdot(k_1-k_2))] \\ &\mathrm{Tr}\,\{[[\pi(k_1),\pi(k_2)],[\pi(k_3),\pi(k_4)]]\,[\pi(k_5),\pi(k_6)]\} \end{aligned} \tag{F.1}$$

with some manipulation, using the fact that the expression involving the momenta is antisymmetric under exchanging k_5 and k_6, we have

$$\begin{aligned}S_{6,1} = -\frac{R}{6g_5^2}\int dQ_1 I_{6,1}(k_{12},k_{34},k_{56})\,&[((k_1+k_2)\cdot(k_5-k_6))((k_1-k_2)\cdot(k_3-k_4))\\&+((k_3+k_4)\cdot(k_1-k_2))((k_3-k_4)\cdot(k_5-k_6))\\&+((k_5+k_6)\cdot(k_3-k_4))((k_5-k_6)\cdot(k_1-k_2))]\\&\mathrm{Tr}\left\{[\pi(k_5),[\pi(k_1),\pi(k_2)]][\pi(k_6),[\pi(k_3),\pi(k_4)]]\right\}\ .\end{aligned} \tag{F.2}$$

where the integral $I_{6,1}$ (and $I_{6,2}$, $I_{6,3}$ for the other diagrams) are defined in equation 4.13.

In Diagram (2), the symmetry factor is 8. Both the $\eta_{\mu\nu}$ and $k_\mu k_\nu$ tensor structures from the axial propagator contribute, while we can again neglect the latter in the vector propagators. The diagram (2) contribution thus amounts to

$$\begin{aligned}S_{6,2} = -\frac{R}{g_5^2}\int dQ_2\Big[&(k_1-k_2)\cdot(k_3-k_4)I_{6,2}(k_{12},k_{34},|k_1+k_2+k_5|)\\&-(k_5\cdot(k_1-k_2))(k_6\cdot(k_3-k_4))\frac{I_{6,2}(k_{12},k_{34},|k_1+k_2+k_5|)-I_{6,2}(k_{12},k_{34},0)}{(k_1+k_2+k_5)^2}\Big]\\&\mathrm{Tr}\left\{[\pi(k_5),[\pi(k_1),\pi(k_2)]][\pi(k_6),[\pi(k_3),\pi(k_4)]]\right\}\ .\end{aligned} \tag{F.3}$$

Finally, Diagram (3) (with symmetry factor 16) yields

$$S_{6,3} = -\frac{R}{4g_5^2}\int dQ_3 I_{6,3}(k_{12},k_{34})(k_1-k_2)\cdot(k_3-k_4)\mathrm{Tr}\left\{[\pi(k_5),[\pi(k_1),\pi(k_2)]][\pi(k_6),[\pi(k_3),\pi(k_4)]]\right\}\ . \tag{F.4}$$

Summing these three contributions yields the full 6-pion term in the off-shell effective action, equation (4.16).

G Off-Shell Identifications

We have repeatedly emphasized that our pion that we define, π, is not the same as the NG pion, Π, but is only expected to have an overlap with it. Accordingly, only the on-shell quantities of the two need to agree. Indeed, we have successfully matched the on-shell effective action of our pion with that of Hirn and Sanz, defined for the NG pion, finding full agreement. On the other hand, we can also compare computed quantities of the two pions at the *off-shell* level, in order to extract their precise relation (in the spirit of (2.11)). In this appendix we separately compare the two pions' equation of motion and off-shell effective action, and in both cases find the same relation between π and Π. Finally, we show that this is the precise relation that should have been anticipated a priori, by identifying the NG pion (having the standard transformation laws under the flavor symmetry) in the model under considerations.

Considering first the chiral Lagrangian. From (3.36) and (3.38) we read,

$$\mathcal{L}_\Pi = \frac{1}{4}\mathrm{Tr}\left(\partial_\mu \Pi \partial^\mu \Pi + \frac{1}{12 f_\Pi^2}\left[\Pi, \partial_\mu \Pi\right]\left[\Pi, \partial^\mu \Pi\right] + \frac{1}{360 f_\Pi^4}\Big[\Pi, \left[\Pi, \partial_\mu \Pi\right]\Big]\Big[\Pi, \left[\Pi, \partial^\mu \Pi\right]\Big] \right.$$
$$\left. + \frac{4L_1}{f_\Pi^4}\left[\partial_\mu \Pi, \partial_\nu \Pi\right]\left[\partial^\mu \Pi, \partial^\nu \Pi\right] + ...\right) \ . \tag{G.1}$$

Matching with our off-shell Lagrangian derived in section 4,

$$\mathcal{L}_\pi = \frac{1}{2g_4^2}\mathrm{Tr}\left(\partial_\mu \pi \partial^\mu \pi + \frac{L^2}{12}\left[\pi, \partial_\mu \pi\right]\left[\pi, \partial^\mu \pi\right] + \frac{L^4}{360}\Big[\pi, \left[\pi, \partial_\mu \pi\right]\Big]\Big[\pi, \left[\pi, \partial^\mu \pi\right]\Big] \right.$$
$$\left. + \frac{11 L^4}{384}\left[\partial_\mu \pi, \partial_\nu \pi\right]\left[\partial^\mu \pi, \partial^\nu \pi\right] + ...\right) \ . \tag{G.2}$$

we rederive our previous results (3.41)(3.46), but this time we also find the full relation between the pions (up to our working order),

$$\pi = c\Pi + O(\Pi^7, \partial^2 \Pi^5) \ , \tag{G.3}$$

with $c = g_4/\sqrt{2}$ as in (3.41).

The same off-shell information can be extracted from a comparison of the pions' equations of motion. For our pion we found it in (3.30) to be

$$\partial^2 \pi = \frac{L^2}{6}\Big[\partial^\mu \pi, \left[\pi, \partial_\mu \pi\right]\Big] + \frac{11 L^4}{192}\Big[\partial^\nu \pi, \left[\partial^\mu \pi, \partial_\nu \partial_\mu \pi\right]\Big] + O(\partial^2 \pi^5, \partial^6 \pi^3) \ . \tag{G.4}$$

For the NG pion, the equations of motion follow directly from (G.1),

$$\partial^2 \Pi = \frac{1}{6 f_\Pi^2}\left[\partial^\mu \Pi, \left[\Pi, \partial_\mu \Pi\right]\right] - \frac{8L_1}{f_\Pi^4}\left[\partial^\nu \Pi, \left[\partial^\mu \Pi, \partial_\mu \partial_\nu \Pi\right]\right] + O(\partial^2 \Pi^5, \partial^6 \Pi^3) \ , \tag{G.5}$$

and a comparison with (G.4) immediately leads to the same relation (G.3).
CHECK THE SIGN OF THE SECOND TERM ABOVE*

It should be noted that, while here we have found a very simple relation between the IR-boundary pion and then Nambu-Goldstone pion, in the general case the relation may be more complicated, with infinite-order corrections. The method described here can be used generically to derive this relation, order by order. We would like now to explain why do we get such a simple relation in our specific case.

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To-do list

1. Add the 2-derivative, all-order-in-π solution.
2. Edit all appendices.
3. Edit section 4.
4. Last edit of secftions 1-3.
5. Agree (between ourselves) on the coefficient in the six-pion result.
6. Correct the $\partial^2\pi^6$ result in diagrams to agree with previous result.
7. Add/edit references.
8. Add a short description of overlapping works in the literature.
9. Add off-shell comparison and separate that from diagrammatic approach: the diagrammatic approach naturally gives the off-shell IREA, but this last one can also be obtained in the previous procedure, just by not imposing the A_z EOM. Emphasize that, and can also write these results already then. Then perhaps can have a short section, or subsection, or appendix, for the off-shell comparison – both at the level of effective actions and EOMs. See that they agree, giving the same relation between π and Π, and see also that this agrees with the identification of Π as the holonomy.
10. Re-order (perhaps) the appendices; also decide what really should be in the appendices, what should be in the text, and what should not be at all.
11. Decide about capitalization in (sub)sections titles.
12. Add figures/cartoons/diagrams for the computation of off-shell 1PI effective action (i.e. the vertices) and S-matrix of pions.
13. Go over all title of (sub)sections/appendices once again.
14. Improve the notations of the 5d Feynman rules (in particular for the bulk propagators), and the 6-pion results.
15. Add the infi. gauge transformation of bulk fields to the appendix (?)