Derivation of the nondimensional lee wave equations

Oliver Fringer

March 30, 2017

1 Dimensional analysis

The form drag F (per unit width) due to a hill of height h_0 in an infinitely deep fluid is characterized by the dimensional quantities U, k, N, and h_0. Choosing U and N to nondimensionalize k and h_0, the governing nondimensional parameters are $J = Nh_0/U$ and $\epsilon = kU/N$. Since the units of the form drag are length^3 time^{-2}, then a scale for F in terms of U and N is $\rho_0 U^3 N^{-1}$. Therefore, the form drag must satisfy

$$\frac{F}{\rho_0 U^3 N^{-1}} = f(J,\epsilon)\,.$$

2 Nondimensional equations

Let $\mathbf{u}_{\text{total}} = U\mathbf{e}_x + \mathbf{u}$, $\rho_{\text{total}} = \overline{\rho}(z) + \rho$, and $p_{\text{total}} = \rho_0\overline{p}(z) + \rho_0 p$ (the total notation is to avoid carrying around the prime, but in a paper you should carry around the prime). The steady momentum and density transport equations are given by

$$\begin{aligned}
U\frac{\partial u}{\partial x} + \mathbf{u}\cdot\nabla u &= -\frac{\partial p}{\partial x}\,, && (1)\\
U\frac{\partial w}{\partial x} + \mathbf{u}\cdot\nabla w &= -\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0}g\,, && (2)\\
U\frac{\partial \rho}{\partial x} + \mathbf{u}\cdot\nabla \rho &= \frac{\rho_0 N^2}{g}w\,, && (3)
\end{aligned}$$

where $N^2 = -g/\rho_0 \partial\overline{\rho}/\partial z$, subject to continuity $\nabla \cdot \mathbf{u} = 0$ and the kinematic bottom boundary condition

$$U\frac{\partial h}{\partial x} + u\frac{\partial h}{\partial x} = w\,.$$

Nondimensionalize with

$$\begin{aligned} u &= u_0 u^*\,, \\ w &= w_0 w^*\,, \\ \rho &= R\rho^*\,, \\ p &= Pp^*\,, \\ x &= k^{-1}x^*\,, \\ z &= \delta z^*\,. \end{aligned}$$

Nondimensionalizing the kinematic bottom boundary condition gives (after ignoring the $*$)

$$kUh_0\frac{\partial h}{\partial x} + ku_0h_0\frac{\partial h}{\partial x} = w_0 w\,.$$

If we require a balance between the linear terms, this implies $w_0 = kh_0U$, so that

$$\frac{\partial h}{\partial x} + F\frac{\partial h}{\partial x} = w\,,$$

where $F = u_0/U$ is a Froude number. Now, the vertical scale of the flow as given by δ is not the same as the hill height h_0, since δ must be finite as $h_0 \to 0$ (the linear limit). The vertical scale is thus dictated by continuity, which requires

$$ku_0\frac{\partial u}{\partial x} + \frac{w_0}{\delta}\frac{\partial w}{\partial z} = 0\,,$$

or, since this implies $ku_0 = w_0/\delta$, then we must have $\delta = w_0/(ku_0) = kh_0U/(ku_0) = F^{-1}h_0$. Nondimensionalizing the u-momentum equation,

$$ku_0U\frac{\partial u}{\partial x} + ku_0^2\mathbf{u}\cdot\nabla u = -kP\frac{\partial p}{\partial x}\,. \tag{4}$$

Since we require a leading-order balance between the pressure gradient and the linear momentum advection term, we must have $P = u_0U$, which gives

$$\frac{\partial u}{\partial x} + F\mathbf{u}\cdot\nabla u = -\frac{\partial p}{\partial x}\,. \tag{5}$$

The nondimensional density transport equation is given by

$$kUR\frac{\partial \rho}{\partial x} + ku_0 R\mathbf{u} \cdot \nabla \rho = \frac{k\rho_0 h_0 N^2 U}{g} w\,.$$

If we require a balance between the linear advection terms, then the scale for the density perturbation is

$$R = \frac{\rho_0 N^2 h_0}{g}\,,$$

so that the nondimensional density equation is

$$\frac{\partial \rho}{\partial x} + F\mathbf{u} \cdot \nabla \rho = w\,.$$

Nondimensionalizing the vertical momentum equation, we have

$$k^2 h_0 U^2 \frac{\partial w}{\partial x} + k^2 h_0 u_0^2 \mathbf{u} \cdot \nabla w = -\frac{P}{\delta}\frac{\partial p}{\partial z} - \frac{gR}{\rho_0}\rho\,.$$

If we require a vertical hydrostatic balance to leading order, then we must have

$$\frac{P}{\delta} = \frac{gR\delta}{\rho_0} = N^2 h_0\,,$$

and

$$P = \frac{gR\delta}{\rho_0} = \frac{g}{\rho_0}\frac{\rho_0 N^2 h_0}{g}\frac{h_0}{F} = \frac{g}{\rho_0}\frac{\rho_0 N^2 h_0}{g}\frac{U}{N} = UNh_0\,.$$

which gives

$$\epsilon^2 \left(\frac{\partial w}{\partial x} + F\mathbf{u} \cdot \nabla w \right) = -\frac{\partial p}{\partial z} - \rho\,,$$

where

$$\epsilon = \frac{kU}{N}$$

is the nonhydrostatic parameter. Now, returning to the pressure, since from the vertical momentum equation we require $P = UNh_0$ and from the horizontal momentum equation we require $P = u_0 U$, then equating the two implies that $u_0 = Nh_0$ and the Froude number is given by

$$F = \frac{u_0}{U} = \frac{Nh_0}{U} = J\,,$$

where

$$J = \frac{Nh_0}{U}\,.$$

Therefore, in terms of J, the governing nondimensional equations are given by

$$\begin{aligned}
\frac{\partial u}{\partial x} + J\mathbf{u}\cdot\nabla u &= -\frac{\partial p}{\partial x}\,,\\
\epsilon^2\left(\frac{\partial w}{\partial x} + J\mathbf{u}\cdot\nabla w\right) &= -\frac{\partial p}{\partial z} - \rho\,,\\
\frac{\partial \rho}{\partial x} + J\mathbf{u}\cdot\nabla \rho &= w\,,
\end{aligned}$$

subject to $\nabla\cdot\mathbf{u} = 0$ and the kinematic bottom boundary condition

$$(1 + Ju)\,\frac{\partial h}{\partial x} = w\,.$$

These nondimensional equations are consistent with the original nondimensionalization which implied that the problem is uniquely characterized by ϵ and J. The relevant scales (nondimensionalized by N and U) are given by

$$\begin{aligned}
\frac{u_0}{U} &= J\,,\\
\frac{w_0}{U} &= \epsilon J\,,\\
\frac{gR}{\rho_0 UN} &= J\,,\\
\frac{P}{U^2} &= J\,,\\
\frac{\delta N}{U} &= 1\,.
\end{aligned}$$

Contrary to what is stated in the literature, we can show that it is in fact appropriate to refer to J as an internal Froude number. If we define $Fr_\delta = u_0/\sqrt{g'\delta}$, where $g'\delta = gR/\rho_0\delta = JU^2$, this gives $Fr_\delta = J^{1/2}$. Therefore, $J^{1/2}$ can be thought of as the ratio of the inertial to gravitational forces arising from the perturbed flow above the hill. Although we would expect a larger N to block the flow and reduce the magnitude of the perturbation above the hill, the scaling shows that $u_0 = JU$, implying that the perturbation velocity above the sill increases with increasing J. However, the gravitational force resulting from the perturbation is given by $\sqrt{g'\delta} = J^{1/2}U$, which grows more slowly than u_0 with increasing J.

3 Linear, nonhydrostatic equations

The governing equations in the linear limit $J \to 0$ are given by

$$\begin{aligned} \frac{\partial u}{\partial x} &= -\frac{\partial p}{\partial x}\,, \\ \epsilon^2 \frac{\partial w}{\partial x} &= -\frac{\partial p}{\partial z} - \rho\,, \\ \frac{\partial \rho}{\partial x} &= w\,. \end{aligned}$$

The governing equation for w is then given by

$$\frac{\partial^2 w}{\partial z^2} + \epsilon^2 \frac{\partial^2 w}{\partial x^2} + w = 0\,.$$

Assume a sinusoidal topography such that $h(x) = \sin(x)$.

3.1 Propagating solution

When $\epsilon < 1$, the hydrostatic solution is of the form $w(x, z) = \cos(x + mz)$, which implies $m = (1 - \epsilon^2)^{1/2}$. Substitution into the governing linear equations gives

$$\begin{aligned} u(x,z) &= -m\cos(x + mz)\,, \\ w(x,z) &= \cos(x + mz)\,, \\ \rho(x,z) &= \sin(x + mz)\,, \\ p(x,z) &= m\cos(x + mz)\,. \end{aligned}$$

The dimensional form drag (per unit width) over one wavelength is given by (here, $*$ implies dimensional quantities)

$$F^* = \int_0^{2\pi/k^*} p^*(x^*, z^* = 0) \frac{\partial h^*}{\partial x^*}\, dx^*\,.$$

Nondimensionalizing gives

$$\frac{F^*}{\rho_0 U^3 N^{-1}} = J^2 \int_0^{2\pi} p(x, z = 0) \frac{\partial h}{\partial x}\, dx\,.$$

Substitution of p and h then gives

$$\frac{F^*}{\rho_0 U^3 N^{-1}} = \pi J^2 \left(1 - \epsilon^2\right)^{1/2}\,.$$

3.2 Evanescent solution

When $\epsilon > 1$, the nonhydrostatic solution is of the form $w(x, z) = \cos(x)\exp(-mz)$, which implies $m = (\epsilon^2 - 1)^{1/2}$. Substitution into the governing linear equations gives

$$\begin{aligned} u(x,z) &= m\sin(x)\exp(-mz)\,, \\ w(x,z) &= \cos(x)\exp(-mz)\,, \\ \rho(x,z) &= \sin(x)\exp(-mz)\,, \\ p(x,z) &= -m\sin(x)\exp(-mz)\,. \end{aligned}$$

Since p and w are $\pi/2$ out of phase, the form drag is identically zero.