The nondimensional lee wave equations and the Froude number

or, squeezing stratified flow

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1 Abstract

We nondimensionalize the 2D boussinesq equations describing the flow of infinite depth water over a ridge of height, h_0, and width, $2\pi/k$, with uniform upstream velocity, U, and buoyancy frequency $N^2 = -\frac{g}{\rho_0}\frac{\partial \bar{\rho}}{\partial z}$. We choose U and N as our free parameters and find that the equations have dynamical similarity based on the dimensionless numbers $\epsilon = \frac{Uk}{N}$ and $J = \frac{Nh_0}{U}$. We then use our scaling to show that, although it looks like an inverse Froude number, J is in fact the square of an internal Froude number, defined as $Fr_i^2 = \frac{u_0^2}{g'\delta}$. This curious inversion of one's intuition reflects the fact that the vertical scale of flow over the ridge, δ is not set by the ridge's height, but by the vertical wavelength of the lee wave, N/U.

This relation, however, breaks down if the $h_0 \geq O(N/U)$, at which point the flow becomes hydraulically controlled, with the internal Froude number held constant at 1 (Winters and Armi, 2012). Thereafter, $\frac{Nh_0}{U}$ informs instead the degree of blocking. It is as if these hills larger than a buoyancy wavelength have squeezed all the juice out of the upstream stratified flow. Hence we term this square of an internal Froude number J for Juice.

2 Introduction

In its most generally accepted use, the Froude number represents a ratio of the speed with which two processes, advection and wave propagation, carry information of a disturbance throughout the system. Most of us first meet a Froude number when studying the open channel flow of a single density hydrostatic fluid. Here, the Froude number is given by $Fr = U/\sqrt{gd}$, where d is the local depth of the fluid. When $Fr < 1$, a disturbance in the flow is carried up and downstream by waves, and thus the local state of the flow can be altered by disturbances both up and downstream. Alternatively, when $Fr > 1$, disturbances are rapidly swept downstream by the current, and only those generated upstream can change the local state.

The special case of $F = 1$ demands an alternative view the Froude number as the partitioning of the flow's energy between potential (gd) and kinetic (U^2) energy. A flow with a given total energy E can conceivably partition its energy into a spectrum of configurations from entirely potential ($Fr = 0$) to almost entirely kinetic ($Fr \to \infty$). However, the volume flux of the flow, $q = Ud$, is not constant over this spectrum, as clearly a flow with all potential energy has no flow. Rather, for a given energy, the flow acheives its maximum volume flux when $Fr = 1$. For a more thorough review, see Baines, Chaper 2.

The addition of a topographic feature to the system adds at least two new length scales, namely, the obstacle's height, h_0, and width, L, permitting two more dimensionless numbers that are tempting to call Froudes. Indeed, Froude's work focused on the wake of ships, for which $U/\sqrt{gL}$ is the most relevant dynamical parameter (Baines, 1995). The additional introduction of stratification produces a fourth length scale in the form of the wavelength of a characteristic internal gravity wave, and generates a new set of Froude looking dimensionless numbers. However, not all of these numbers represent a ratio of advective to wave transmition, and thus calling all of these parameters Froude numbers robs the concept of its intuitive dynamic significance. In his seminal text on stratified flow over topography, Baines proposed that, as a solution to this "Froude for everything syndrome," we call only the most obvious extension of open channel flow a Froude number; that is, $Fr = U/ND$, where ND is the speed of the first mode internal gravity wave (Baines, 1995).

For oceanic flows away from continents or mid ocean ridges, however, $Fr = U/ND$ is often very small, and the dynamics of the flow are captured nearly as well by considering the mathematically more simple infinitely deep ocean (e.g. Long (1953)). For example, in the Drake Passage region of the Antarctic Circumpolar Current, where lee waves are predicted to be especially dynamically important (Nikurashin and Ferrari, 2010), typical values for dimensional quantities are $U \approx 0.1$ m s^{-1}, $N \approx 10^{-}3$ rad s^{-1}, and

$D \approx 4000$ m, giving $U/ND \approx 0.025$.

Instead, much of the literature focuses on the dynamical significance of the number Nh_0/U, and chooses to call it a variety of creative names. Miles calls it the Russel number, Ru, principally so that the dimensionless number describing geophysical flows may be remembered by the vowel ordered set Ra, Re, Ri, Ro, Ru (Miles, 1969). Aguilar and Sutherland call it the Long number, Lo, in honor of Robert Long's pioneering work on the lee wave problem (Aguilar and Sutherland, 2006; Long, 1953). Nikurashin and Ferrari call it a steepness parameter and use the symbol ϵ, after showing that in the hydrostatic limit, it is identical to the ratio of topographic slope to wave ray slope, a parameter popular in the internal tides literature (Nikurashin and Ferrari, 2010). And of course, much of the literature simply calls Nh/U and inverse Froude number (Legg and Klymak, 2008; Klymnak et al., 2010; Eckermann et al., 2010; Winters and Armi, 2012).

By nondimensionalizing the equations describing this flow, however, we will show that Nh/U is in fact the square of what Winters and Armi call the flow's "layer" Froude number, provided that the height of the topography remains smaller than the wavelength of an internal gravity wave.

3 Nondimensional equations

The flow of an unbounded fluid over an isolated hill of height, h_0, and width, $2\pi/k$ is characterized by the dimensional quantities U, k, N, and h_0, where U and N are the constant horizontal velocity and buoyancy upstream of the hill. Choosing U and N to nondimensionalize k and h_0, the governing nondimensional parameters are $J = Nh_0/U$ and $\epsilon = kU/N$.

Let $\mathbf{u}_{\text{total}} = U\mathbf{e}_x + \mathbf{u}'$, $\rho_{\text{total}} = \overline{\rho}(z) + \rho'$, and $p_{\text{total}} = \rho_0\overline{p}(z) + \rho_0 p'$. Making the Boussinesq approximation, the steady momentum and density transport equations are given by

$$\begin{aligned}
U\frac{\partial u'}{\partial x} + \mathbf{u}' \cdot \nabla u' &= -\frac{\partial p'}{\partial x}\,, \\
U\frac{\partial w'}{\partial x} + \mathbf{u}' \cdot \nabla w' &= -\frac{\partial p'}{\partial z} - \frac{\rho'}{\rho_0}g\,, \\
U\frac{\partial \rho'}{\partial x} + \mathbf{u}' \cdot \nabla \rho' &= \frac{\rho_0 N^2}{g}w\,,
\end{aligned}$$

where $N^2 = -g/\rho_0 \partial\overline{\rho}/\partial z$, subject to continuity $\nabla \cdot \mathbf{u}' = 0$, and the kinematic bottom boundary condition

$$U\frac{\partial h}{\partial x} + u'\frac{\partial h}{\partial x} = w' \,.$$

Nondimensionalize with

$$\begin{aligned}
u' &= u_0 u^* \,, \\
w' &= w_0 w^* \,, \\
\rho' &= R\rho^* \,, \\
p' &= P p^* \,, \\
x &= k^{-1} x^* \,, \\
z &= \delta z^* \,.
\end{aligned}$$

Nondimensionalizing the kinematic bottom boundary condition gives (omitting the $*$ on nondimensional variables)

$$kUh_0\frac{\partial h}{\partial x} + ku_0h_0\frac{\partial h}{\partial x} = w_0 w \,.$$

If we require a balance between the linear terms, this implies $w_0 = kh_0U$, so that

$$\frac{\partial h}{\partial x} + \frac{u_0}{U}\frac{\partial h}{\partial x} = w \,.$$

Now, the vertical scale of the flow as given by δ is not the same as the hill height h_0, since δ must be finite as $h_0 \to 0$ (the linear limit). The vertical scale is thus dictated by continuity, which requires

$$ku_0\frac{\partial u}{\partial x} + \frac{w_0}{\delta}\frac{\partial w}{\partial z} = 0 \,,$$

or, since this implies $ku_0 = w_0/\delta$, then we must have $\delta = w_0/(ku_0) = kh_0U/(ku_0) = h_0U/u_0$. Nondimensionalizing the u-momentum equation,

$$ku_0U\frac{\partial u}{\partial x} + ku_0^2\mathbf{u}\cdot\nabla u \;=\; -kP\frac{\partial p}{\partial x} \,.$$

Since we require a leading-order balance between the pressure gradient and the linear momentum advection term, we must have $P = u_0U$, which gives

$$\frac{\partial u}{\partial x} + \frac{u_0}{U}\mathbf{u}\cdot\nabla u \;=\; -\frac{\partial p}{\partial x} \,.$$

The nondimensional density transport equation is given by

$$kUR\frac{\partial \rho}{\partial x} + ku_0R\mathbf{u}\cdot\nabla\rho = \frac{k\rho_0 h_0 N^2 U}{g}w\,.$$

If we require a balance between the linear advection terms, then the scale for the density perturbation is

$$R = \frac{\rho_0 N^2 h_0}{g}\,,$$

so that the nondimensional density equation is

$$\frac{\partial \rho}{\partial x} + \frac{u_0}{U}\mathbf{u}\cdot\nabla\rho = w\,.$$

Nondimensionalizing the vertical momentum equation, we have

$$k^2h_0U^2\frac{\partial w}{\partial x} + k^2h_0u_0^2\mathbf{u}\cdot\nabla w = -\frac{P}{\delta}\frac{\partial p}{\partial z} - \frac{gR}{\rho_0}\rho\,.$$

If we require a vertical hydrostatic balance to leading order, then we must have

$$\frac{P}{\delta} = \frac{gR}{\rho_0} = N^2h_0\,,$$

and

$$P = \delta N^2 h_0 = \frac{N^2h_0^2U}{u_0}.$$

which gives

$$\epsilon^2\left(\frac{\partial w}{\partial x} + \frac{u_0}{U}\mathbf{u}\cdot\nabla w\right) = -\frac{\partial p}{\partial z} - \rho\,,$$

where

$$\epsilon = \frac{Uk}{N}$$

is the nonhydrostatic parameter, and represents a ratio of the frequency with which the flow over the hill excites a wave, Uk, to the frequency of buoyancy's response, N. In analogy to a boxer at a speed-bag, a propagating wave is only possible if the excitation frequency allows buoyancy enough time to bounce back ($\epsilon < 1$). Within this propagating regime, one can also think of ϵ as a ratio of the wavelength of the wave to the width of the hill. For waves much smaller than the hill is long ($\epsilon << 1$), the wave is approximately hydrostatic and the group velocity of the wave (in the reference frame of the hill) is oriented vertically.

Now, returning to the pressure, since from the vertical momentum equation we require $P = N^2 h_0^2 U u_0^{-1}$ and from the horizontal momentum equation we require $P = u_0 U$, then equating the two implies that $u_0 = N h_0$ and thus

$$\frac{u_0}{U} = \frac{N h_0}{U} \equiv J \,.$$

Therefore, in terms of J, the governing nondimensional equations are given by

$$\begin{aligned} \frac{\partial u}{\partial x} + J\mathbf{u}\cdot\nabla u &= -\frac{\partial p}{\partial x}\,, \\ \epsilon^2\left(\frac{\partial w}{\partial x} + J\mathbf{u}\cdot\nabla w\right) &= -\frac{\partial p}{\partial z} - \rho\,, \\ \frac{\partial \rho}{\partial x} + J\mathbf{u}\cdot\nabla \rho &= w\,, \end{aligned}$$

subject to $\nabla\cdot\mathbf{u} = 0$ and the kinematic bottom boundary condition

$$(1+J)\,\frac{\partial h}{\partial x} = w\,.$$

These nondimensional equations are consistent with the original nondimensionalization which implied that the problem is uniquely characterized by ϵ and J. The relevant scales (nondimensionalized by N and U) are given by

$$\begin{aligned} \frac{u_0}{U} &= J\,, \\ \frac{w_0}{U} &= \epsilon J\,, \\ \frac{gR}{\rho_0 U N} &= J\,, \\ \frac{P}{U^2} &= J\,, \\ \frac{\delta N}{U} &= 1\,. \end{aligned}$$

Finally, if we define the internal Froude number as $Fr_\delta = u_0/\sqrt{g'\delta}$, where $g'\delta = g(R/\rho_0)\delta = JU^2$, this gives $Fr_\delta = J^{1/2}$. Thus although it looks like an inverse Froude number when expressed in outer variables, this scaling shows that it is in fact appropriate to refer to J as the square of an internal Froude number

4 Discussion

That the outer and inner variable representations of J should present velocity and gravity inversely highlights a unique element of this problem's physics. In the limit of subcritical bathymetry, for which the bottom boundary condition becomes linear, there is no vertical scale imposed upon the vertical perturbation from any boundary condition. Rather, the vertical scale of the wave must come from properties of the fluid itself, hence the scaling $\delta = U/N$.

This does not, however, indicate that the height of the topography is irrelevant to the potential energy of the flow, only that it plays a higher order role in the kinetic energy. Holding $\delta = U/N$ constant, the scaling shows that $u_0 = JU$, implying that the perturbation velocity above the sill increases in step with increasing J, and thus with increasing h_0. However, the gravitational force resulting from the perturbation is given by $\sqrt{g'\delta} = J^{1/2}U$, which grows with half the power of h_0 as u_0. This is because h_0 can only enter into the first term of the potential energy side of the Froude number, g'.

Conceptually, we can picture a column of water headed for an isolated hill. As it approaches the hill, it enters the wave field, and some elements are lifted in preparation for a race across the crest. The height of this lift must be enough to overtop the hill, and scales with h_0. Then buoyancy acts on these lifted parcels, translating the wave's potential energy into kinetic energy. This is the source of the perturbation velocity over the hill, as indicated by the scaling $u_0 = JU = Nh_0$. As the column reaches the crest of the hill, its elements are all h_0 higher than they were, thus the density perturbation, and the resulting reduced gravity to work against for potential energy, also scales with h_0. We see this in the scaling $g' = g(R/\rho_0) = JUN = N^2h_0$. However, the length scale of this work against gravity is the wavelength, δ, which is oblivious to h_0. Hence the scaling $\sqrt{g'\delta} = J^{1/2}U = \sqrt{Nh_0U}$.

This identification of J as the internal froude number squared appears to have gone without notice in the 65 years of studying Long's model. However, an inquiry into the upper limit of this relationship recently emerged from consideration of blocked flow past a half cylinder (Winters and Armi, 2012).

In the blocked regime, that is, for flow in which $J > 1$, the lowest fluid elements upstream of the obstacle lack sufficient kinetic energy to overtop the obstacle, and thus form a pool of stagnant fluid at the obstacle's base (Baines, 1995). As a result, the obstacle takes on the apparent height to the unblocked flow of U/N, that is, exactly the height of the flow's kinetic hill-climbing capacity, giving $J_{unblocked} = 1$. In this case, Winter's and Armi show that the internal Froude number above the hill is exactly 1, and in analogy to hydraulic control of an unstratified river, the flow exhibits a transition from subcritical flow upstream to a supercritical jet downstream followed by a dissipative hydraulic jump. In other words, the relation between J and Fr_δ holds only up to $J = Fr_\delta^2$=1. Above this limit, Fr_δ^2=1, and J informs the depth of the stagnant layer as well as the thickness and speed of the accelerated jet resulting from hydraulic control (Winters and Armi, 2012).

That J as a Froude number only holds up until the critical limit $J = 1$ has perhaps prevented its general interpretation as a Froude number. Nonetheless, in the subcritical regime, $J^{1/2} = (Nh_0/U)^{1/2} = u_0/\sqrt{g'\delta}$ carries the true meaning of a Froude number, relating the speed with which the competing processes of advective non-linearities and gravity waves respond to the introduction of bathymetry into the flow. When J is small, waves accommodate the disturbance adiabatically, and carry it away from the site of generation, just as in an unstratified river flowing over a sub-critical sill. As J approaches 1, the two processes come into balance, and the volume flux above the hill approaches its maximum potential.

The unique element of this system is that the volume flux never backs off from this maximum with $J > 1$. That is, so long as the height of the hill is still significantly shorter than the depth of the fluid (so that the assumption of infinite depth remains valid), the flow will always adjust such that the apparent height is not greater than the flux maximizing $\delta = U/N$. In this sense, the upstream characteristics of the flow present the system with a wave making capacity, and it is up top the hill to squeeze the wave into existence. But there is only so much juice in the fruit.

5 Appendix: Rotation

Including rotation in the nondimensional equations requires only slight modification. Because rotational effects necessarily involve a span wise direction, we must now include an equation for the span wise momentum.

We begin by aligning the x-direction with lines of latitude, and posit that the background currents are in geostrophic balance

$$
\begin{aligned}
0 &= -\frac{\partial P_G}{\partial x} + fV \, , \\
0 &= -\frac{\partial P_G}{\partial y} - fU \, ,
\end{aligned}
$$

where P_G is a geostrophic pressure field that is decoupled from the perturbation pressure due the lee wave.

To keep the system as simple as possible, we further assume: it is in steady state; the bathymytery varies only in the x-direction; and rotation has a constant rate $f = \Omega sin(\bar{\phi})$, where Ω is the earth's rate of rotation, and $\bar{\phi}$ is the average lattitude of the domain. The assumption of steady state filters out inertial oscillations, and may be invalid in regions of the ocean where rotation is strong, such as the ACC (Nikurashin and Ferrari, 2010). However, in regions closer to the equator, such as Palau, this assumption is quite good, as the following scaling analysis will demonstrate. In combination, these three assumptions allow us to neglect all span wise gradients in the perturbation fields because the hill only perturbs the flow in the x-direction, there are no inertial oscillations to deflect the flow from its hill-perturbed state, and rotation remains constant at all locations in the domain. Thus, again making the Boussinesq approximation, the steady momentum and density transport equations that include (some representation of) rotation are given by

$$
\begin{aligned}
U\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x} + w'\frac{\partial u'}{\partial z} &= -\frac{\partial p'}{\partial x} + fv' \, , \\
U\frac{\partial v'}{\partial x} + u'\frac{\partial v'}{\partial x} + w'\frac{\partial v'}{\partial z} &= -fu' \, , \\
U\frac{\partial w'}{\partial x} + u'\frac{\partial w'}{\partial x} + w'\frac{\partial w'}{\partial z} &= -\frac{\partial p}{\partial z} - \frac{\rho}{\rho_0}g \, , \\
U\frac{\partial \rho'}{\partial x} + u'\frac{\partial \rho'}{\partial x} + w'\frac{\partial \rho'}{\partial z} &= \frac{\rho_0 N^2}{g}w \, ,
\end{aligned}
$$

where $N^2 = -g/\rho_0 \partial\overline{\rho}/\partial z$, subject to continuity $\nabla \cdot \mathbf{u}' = 0$, and the kinematic bottom boundary condition

$$
U\frac{\partial h}{\partial x} + u'\frac{\partial h}{\partial x} = w' \, .
$$

Note that these equations are unchanged from those in the irrotational case except for the addition of $+fv'$ to the x-momentum equation, and of course the presence of the y-momentum equation. Thus the scalings resulting from our irrotational work above hold in all cases except for these two equations.

Nondimensionalizing as above, with the addition of $v = v_0 v*$, the y-momentum equation becomes

$$kUv_0\frac{\partial v}{\partial x} + ku_0v_0(u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z}) \;\; = \;\; -fu_0u\,.$$

Requiring a balance of lowest order terms, this gives $v_0 = \frac{fu_0}{Uk} = u_0/Ro$, where $Ro = Uk/f$ is the Rossby number, and

$$\frac{\partial v}{\partial x} + J(u\frac{\partial v}{\partial x} + w\frac{\partial v}{\partial z}) \;\; = \;\; -u\,.$$

Turning next to the x-momentum equation, we have

$$ku_0U\frac{\partial u}{\partial x} + ku_0^2(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}) \;\; = \;\; -kP\frac{\partial p}{\partial x} + fv_0v\,.$$

Again balancing lowest order terms, we have $P = Uu_0$, and the x-momentum equation becomes

$$\frac{\partial u}{\partial x} + J(u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}) \;\; = \;\; -\frac{\partial p}{\partial x} + Ro^{-2}v\,.$$

From this result, we can diagnose the frailty of our assumptions about rotational effects. Using characteristic values of the ACC, $U \approx 0.1$ m s^{-1}, $N \approx 10^-3$ rad s^{-1}, $k \approx 2\pi/2$ km, and $f \approx 10^{-4}$ rad s^{-1}, and $h_0 \approx 60$ m, we have $Ro \approx 0.5$, $Ro^{-2} \approx 4$ and $J \approx 0.6$ (Nikurashin and Ferrari, 2010). In other words, in the ACC, rotational effects on the scale of lee waves are zeroth order, and our assumptions simplifying them were likely misguided. However, if we focus instead on a more equatorial region with equally strong geostrophic currents, such as Palau, f becomes an order of magnitude smaller, and $Ro \approx 5$ for the lee wave system, giving $Ro^{-2} \approx 0.04$. Here, then, is a part of the ocean where rotation might only enter the equations as meridional jets squirting out of the lee waves.

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