When are Emptiness and Containment Decidable for Probabilistic Automata?

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Kaiserslautern May 2019

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Automata

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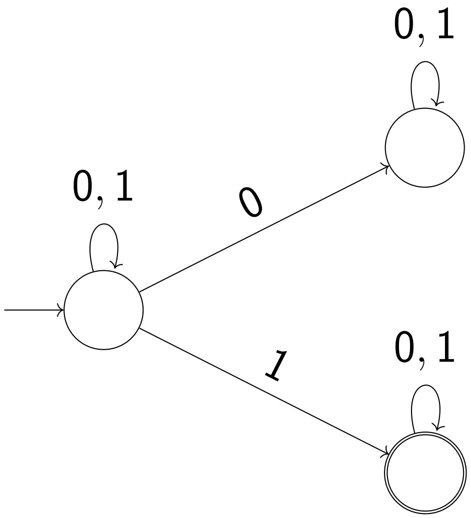
$\mathcal{A}$



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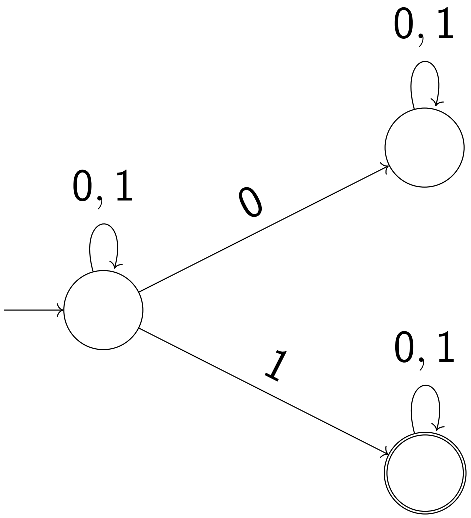

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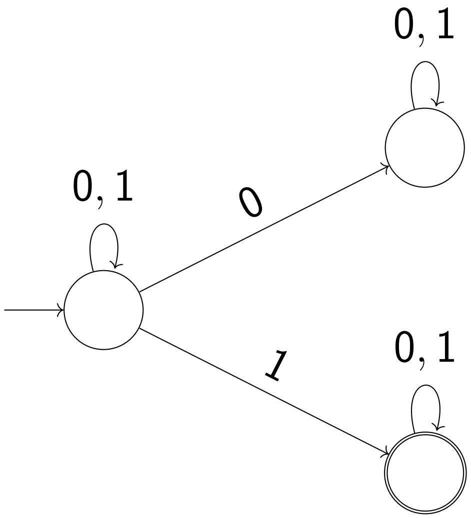

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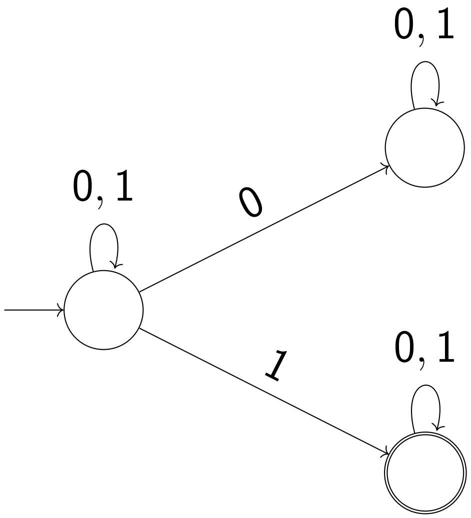

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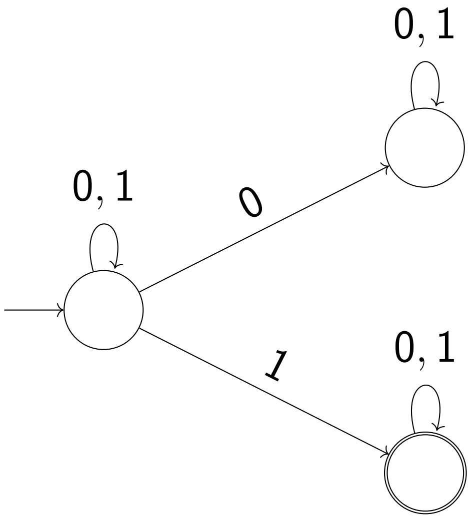

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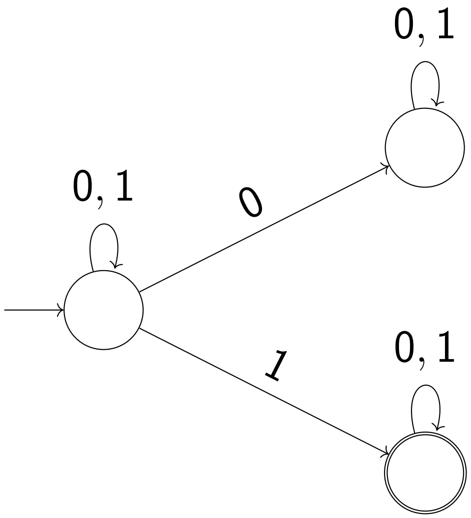

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artificial intelligence, verification of probabilistic systems...

Probabilistic automata (PA)

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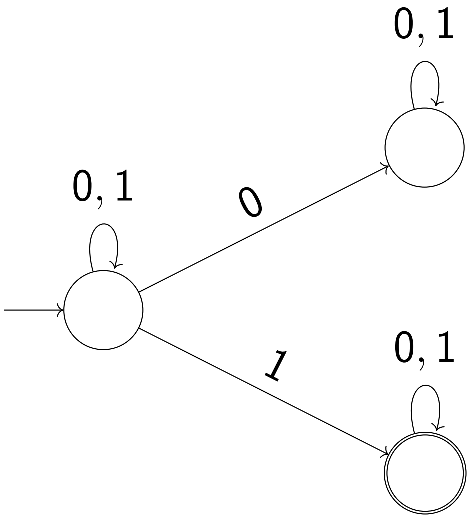

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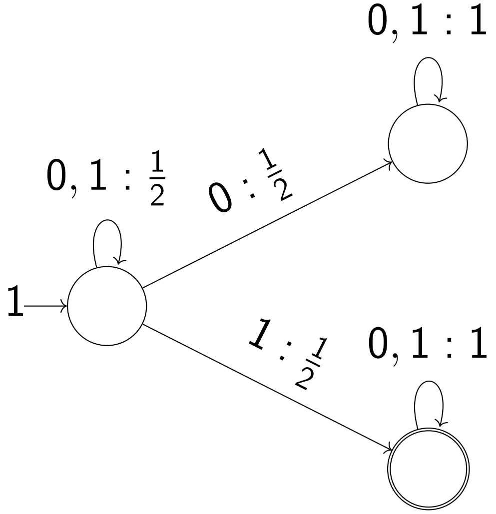

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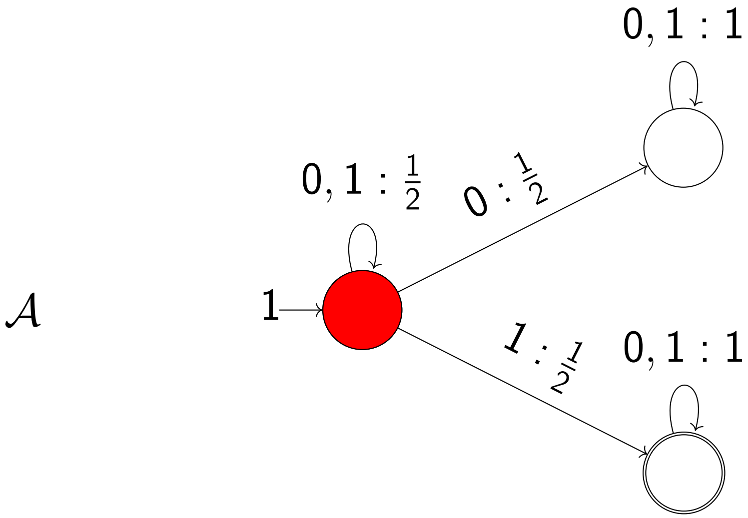

Given state (●) and letter (0)

transitions induce a distribution $(\frac{1}{2}, \frac{1}{2})$

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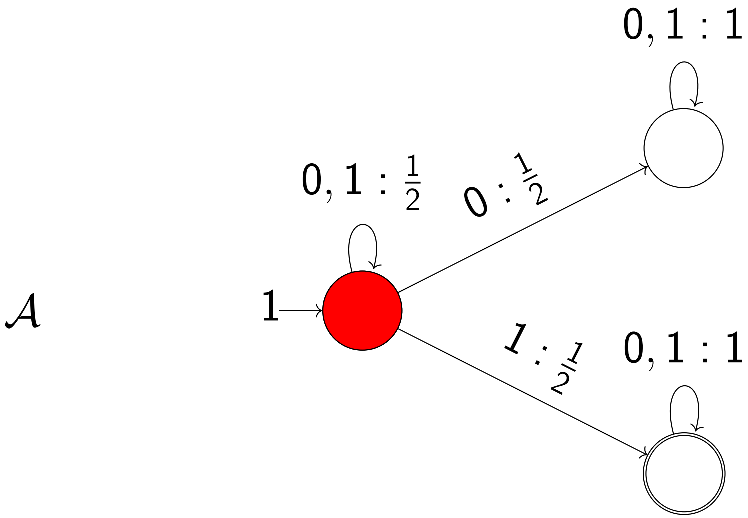

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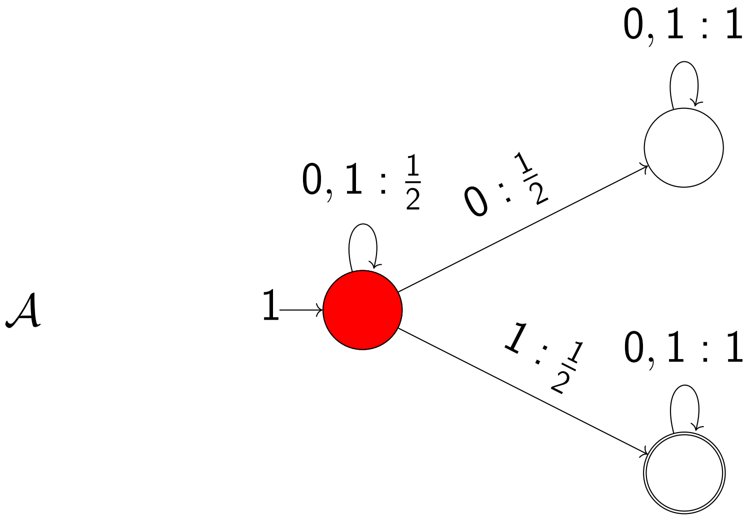

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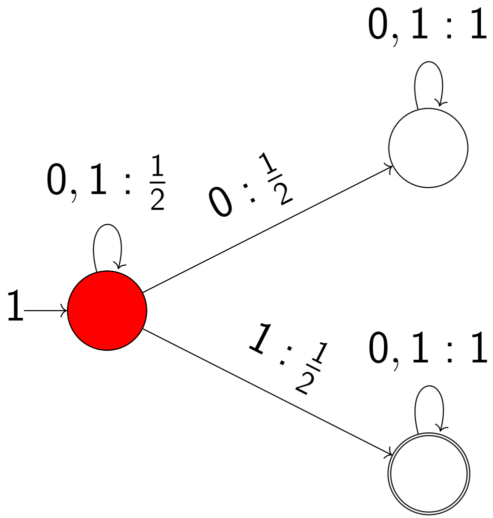

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E.g. $[\![\mathcal{A}]\!](1010) = 2^{-1} + 2^{-3}, \quad [\![\mathcal{A}]\!](w) = \mathrm{bin}(0.w) = \sum_{i | w(i)=1} 2^{-i}$

Decision problems

What do we study?

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Emptiness problem

GIVEN: a PA automaton $\mathcal{A}$

DECIDE: does $[\![\mathcal{A}]\!](w) \leq \frac{1}{2}$ hold for all w?

($[\![\mathcal{A}]\!] \leq \frac{1}{2}$ in short)

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ϵ-gap emptiness problem

GIVEN: ϵ, a PA automaton $\mathcal{A}$ s.t. either $[\![\mathcal{A}]\!] \leq \frac{1}{2}$ or $[\![\mathcal{A}]\!](w) > \frac{1}{2} + \epsilon$

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Remark

Emptiness reduces to containment (define $[\![\mathcal{B}]\!] = \frac{1}{2}$)

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tl;dr: it's all almost always undecidable

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- But equivalence is in PTIME [Schützenberger, 1961]
- Many subclasses with some decidability results:
  hierarchical, leaktight, bounded ambiguity

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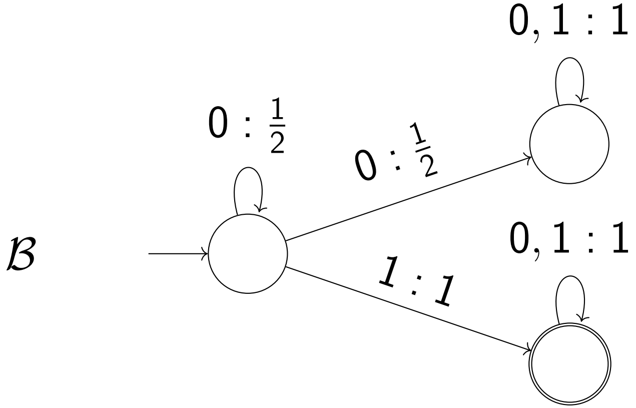

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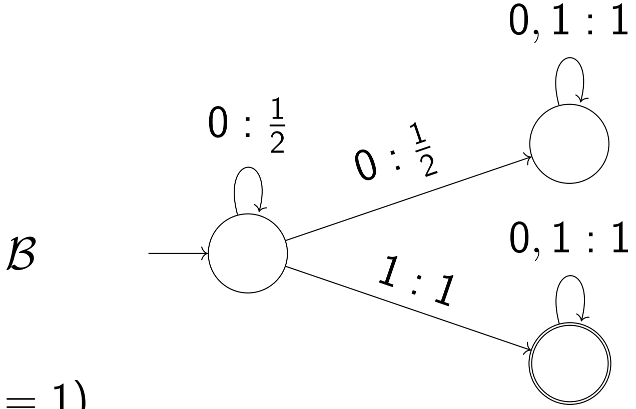

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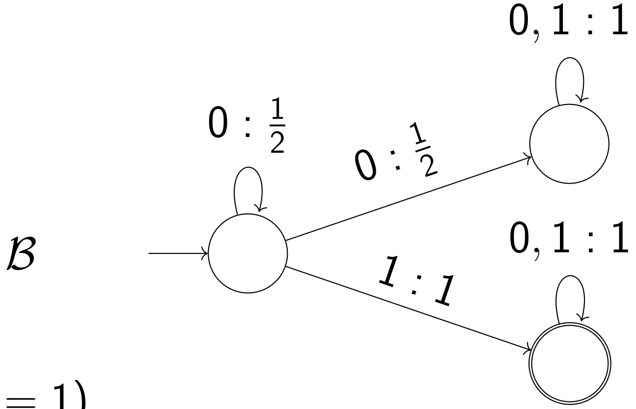

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Remark

Nothing between finitely ambiguous and linearly ambiguous PA

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(only in the submitted journal version)

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rest of the talk

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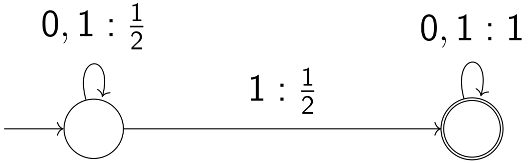

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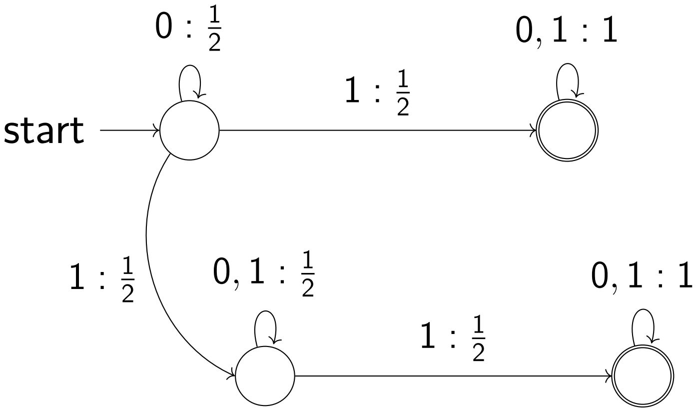

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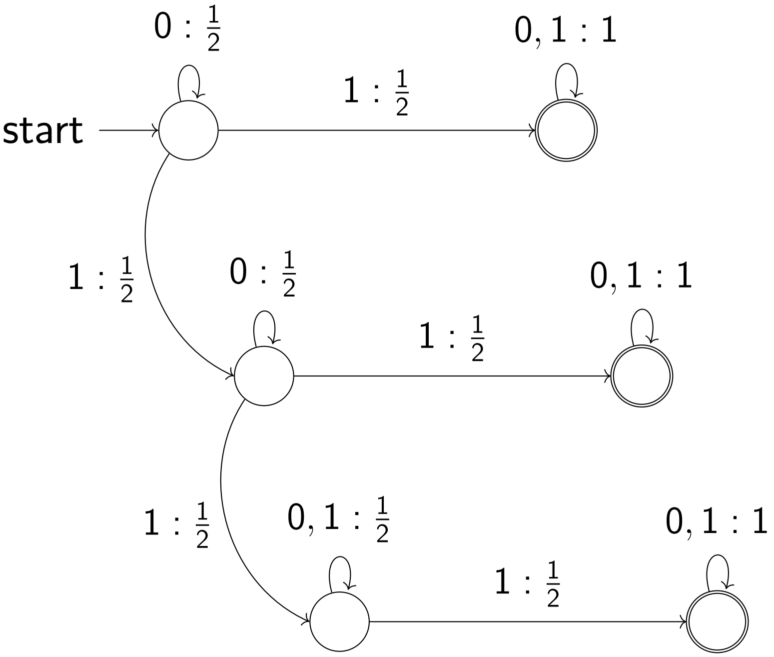

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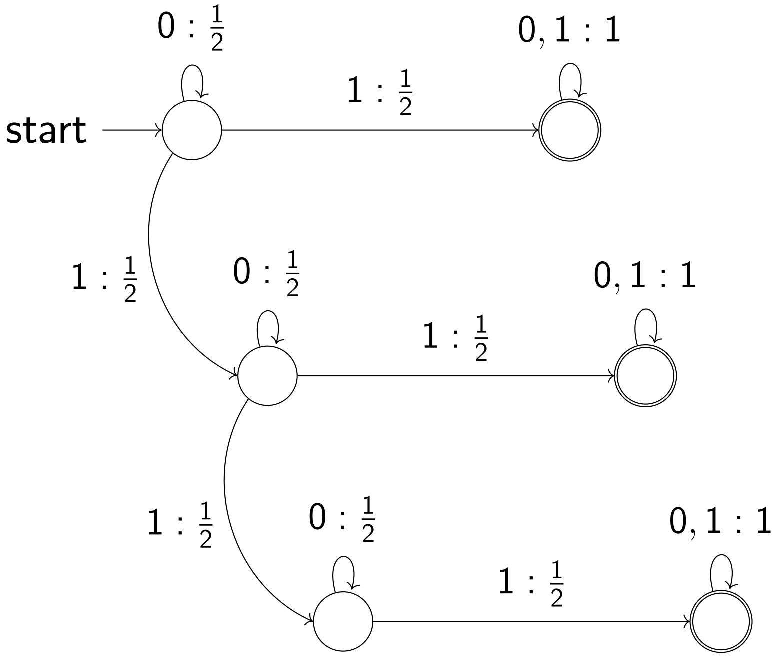

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P is polynomial because of polynomial ambiguity

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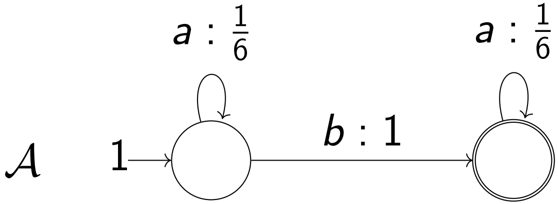

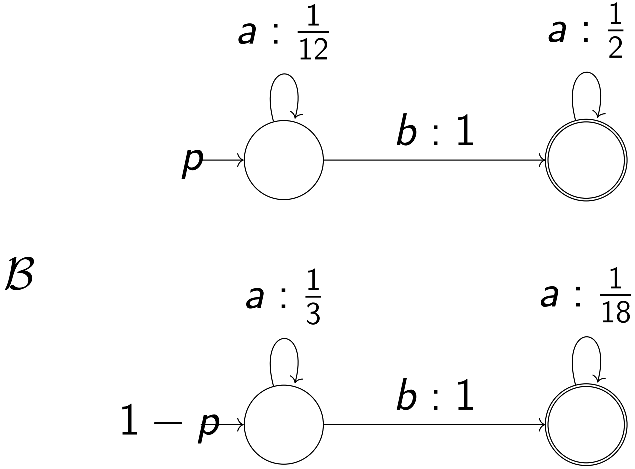

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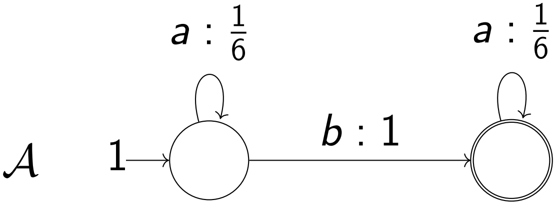

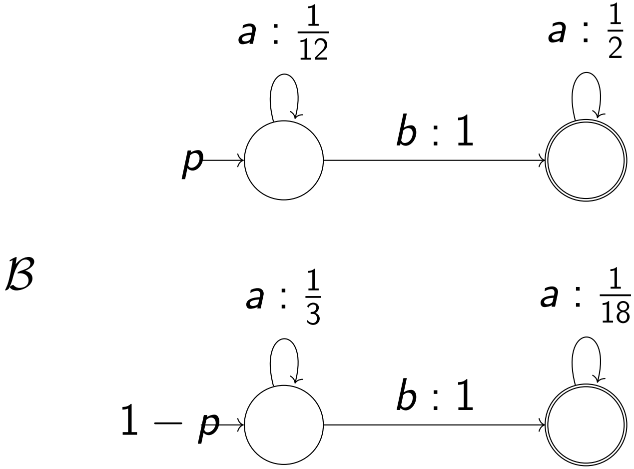

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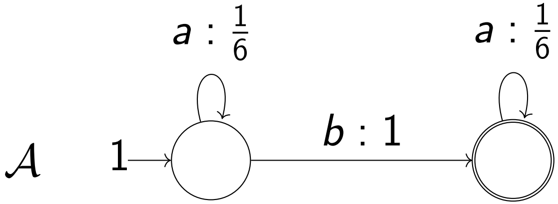

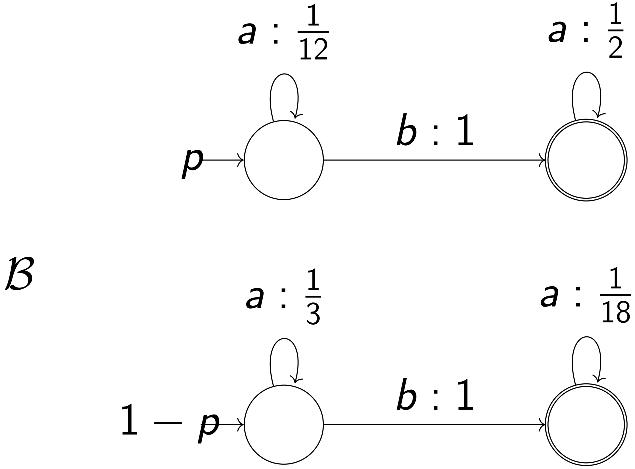

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$$\left(\tfrac{1}{6}\right)^{x_1} \left(\tfrac{1}{6}\right)^{x_2} \;\leq\; p \left(\tfrac{1}{12}\right)^{x_1} \left(\tfrac{1}{2}\right)^{x_2} \;+\; (1-p) \left(\tfrac{1}{3}\right)^{x_1} \left(\tfrac{1}{18}\right)^{x_2}$$

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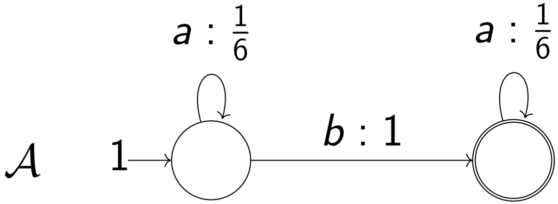

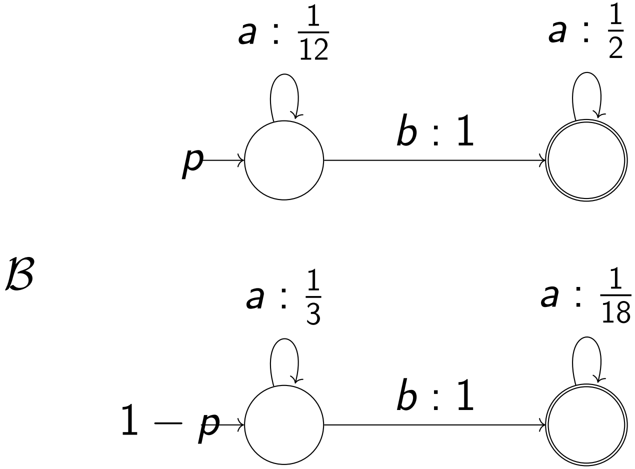

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$$\left(\tfrac{1}{6}\right)^{x_1} \left(\tfrac{1}{6}\right)^{x_2} \;\leq\; p \left(\tfrac{1}{12}\right)^{x_1} \left(\tfrac{1}{2}\right)^{x_2} \;+\; (1-p) \left(\tfrac{1}{3}\right)^{x_1} \left(\tfrac{1}{18}\right)^{x_2}$$

$$1 \;\leq\; p \left(\tfrac{1}{2}\right)^{x_1} (3)^{x_2} \;+\; (1-p)\, (2)^{x_1} \left(\tfrac{1}{3}\right)^{x_2}$$

Reformulating containment of finitely ambiguous PA

$\mathcal{A} \leq \mathcal{B}$?

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Exponential inequalities

GIVEN: $k, l, n > 0$, and vectors $\boldsymbol{p} \in \mathbb{Q}^k_{>0}$ and $\boldsymbol{q}_1, \ldots, \boldsymbol{q}_k \in \mathbb{Q}^n_{>0}$

$\boldsymbol{r} \in \mathbb{Q}^l_{>0}$ and $\boldsymbol{s}_1, \ldots, \boldsymbol{s}_l \in \mathbb{Q}^n_{>0}$

DECIDE: for every $x_1, \ldots, x_n \in \mathbb{N}$ does it hold

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Proposition

Containment of finitely ambiguous PA and exponential inequalities are effectively equi-decidable

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Proposition

Containment of finitely ambiguous PA and exponential inequalities are effectively equi-decidable

- k, l ambiguity of $\mathcal{A}$ and $\mathcal{B}$
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(without
Shanuel)

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$$\begin{matrix} \exp(\log(p) - x_1 \log(2) + x_2 \log(3)) \\ + \exp(\log(1-p) + x_1 \log(2) - x_2 \log(3)) \end{matrix} \; < \; 1$$

Geometric presentation

$$\begin{array}{r}\exp(\log(p) - x_1 \log(2) + x_2 \log(3)) \\ + \exp(\log(1-p) + x_1 \log(2) - x_2 \log(3))\end{array} \quad < \; 1$$

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$$\begin{array}{l} \exp\overbrace{(\log(p) - x_1 \log(2) + x_2 \log(3))}^{x} \\ + \exp\underbrace{(\log(1-p) + x_1 \log(2) - x_2 \log(3))}_{y} \end{array} < 1$$

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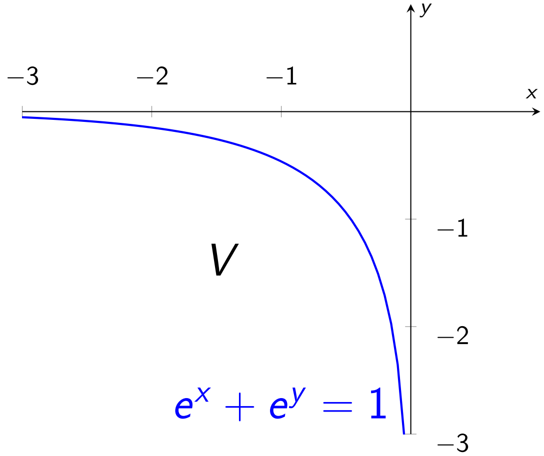
$$V = \{(x,y) \in \mathbb{R}^2 \mid e^x + e^y < 1\} \qquad (\log = \log_e)$$



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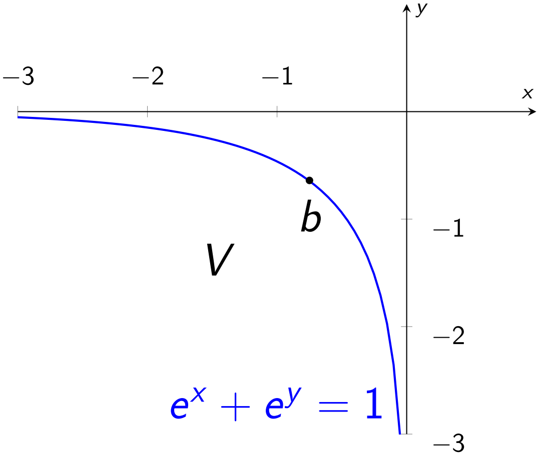

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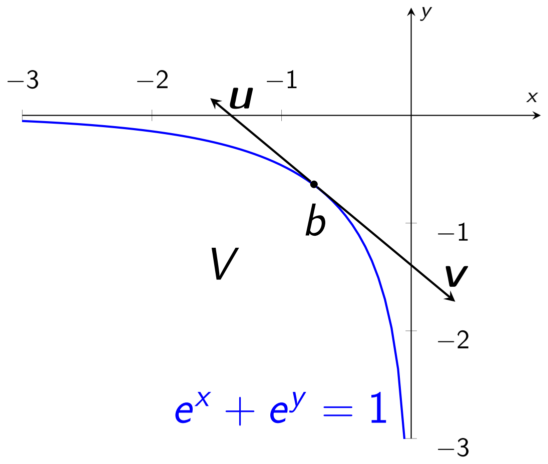

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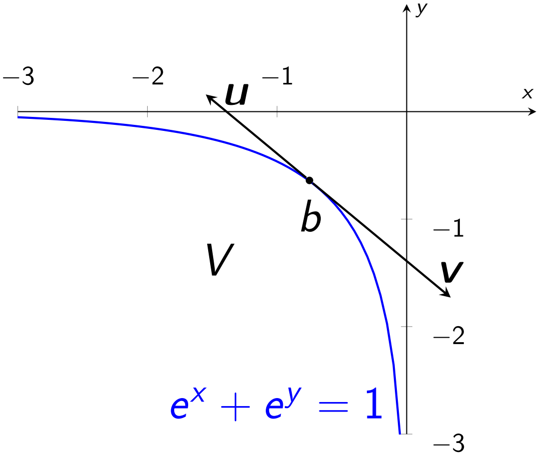

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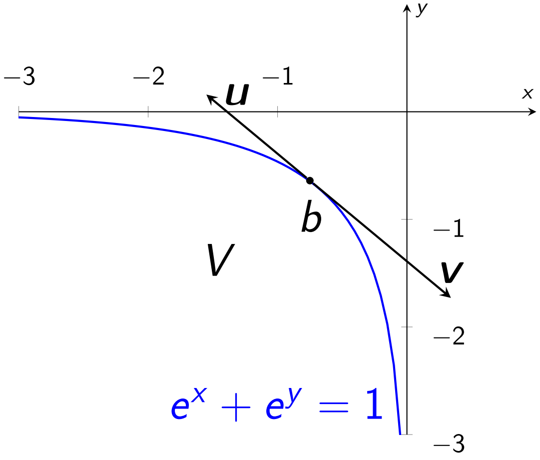

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$b + x_1\boldsymbol{u} + x_2\boldsymbol{v} \in V$? yes iff $p \neq \frac{1}{2}$

Back to the problem

Is there $\boldsymbol{x} \in \mathbb{N}^n$ such that

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Corollary

(1) + (2) give decidability

Solving the complement problem

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Key Lemma (I'm not proving it)

If $X \cap \mathbb{Z}^n = \emptyset$ then exist $\boldsymbol{d} \in \mathbb{Z}^n$, $a, b \in \mathbb{Z}$ s. t. $\{\boldsymbol{d}^\top \boldsymbol{x} \mid \boldsymbol{x} \in X\} \subseteq [a, b]$

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Satisfiability decidable assuming Schanuel [Macintyre, Wilkie; 1996]

Solving the complement problem (continued)

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consider $\boldsymbol{d}^\top \boldsymbol{x} = d_1 x_1 + \ldots + d_n x_n = i$

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recursively start over with dimension reduced by 1 (or output NO)

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If there is no solution: try another i

If you tried all i then go up in the recursion (or output YES)

Solving the complement problem (continued)

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recursively start over with dimension reduced by 1 (or output NO)

If there is no solution: try another i

If you tried all i then go up in the recursion (or output YES)

Previous lemma guarantees semi-decidability ($\mathbb{Z}$ to $\mathbb{N}$ is a technicality)

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Nice, convex etc.

$V' = \{(x_1, \ldots, x_k, y_1, \ldots, y_l) \mid e^{x_1} + \ldots + e^{x_k} < e^{y_1} + \ldots + e^{y_l}\}$???