Pumping lemmas for weighted automata

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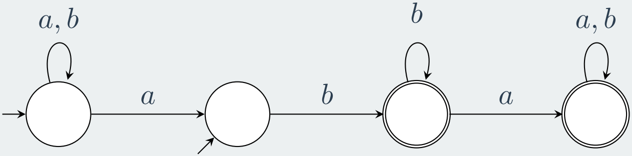
[2]Pontificia Universidad Católica de Chile

Oxford verification seminar 2018

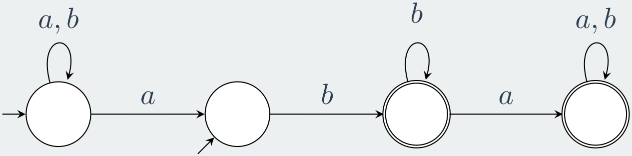
Introduction

Weighted automata

Automata



Automata



$f : \Sigma^* \to \{0, 1\}$

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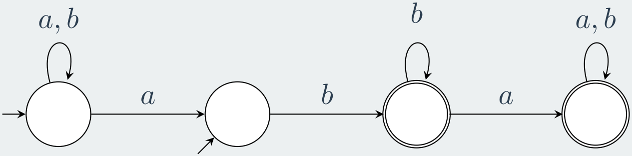

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Weighted automata

$f : \Sigma^* \to$ “some numbers”?

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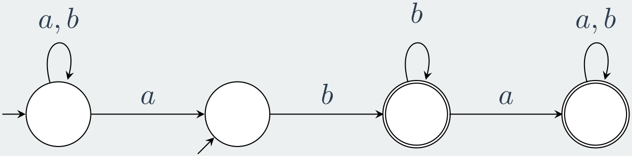

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$f : \Sigma^* \to$ “some numbers”? $\mathbb{N}$?

Commutative semirings

$\mathbb{S}(\oplus, \odot, \mathbb{0}, \mathbb{1})$ with some axioms $s \oplus \mathbb{0} = s,\ s \odot \mathbb{1} = s,\ \ldots$

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$\oplus = \min,\ \odot = +,\ \mathbb{0} = \infty, \mathbb{1} = 0$

$n \oplus \mathbb{0} = n$ becomes $\min(n, \infty) = n$

$n \odot \mathbb{1} = n$ becomes $n + 0 = n$

Weighted automata (WA)

Semiring: $\mathbb{N}_{\infty}(\min, +, \infty, 0)$

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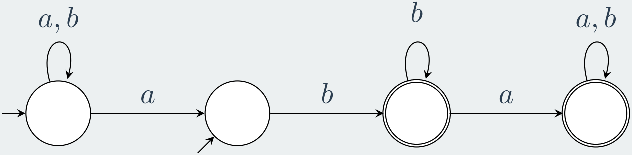

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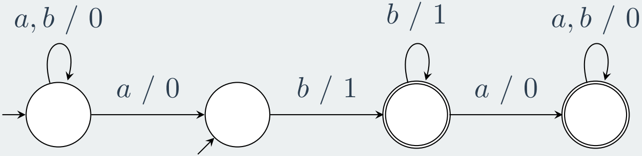

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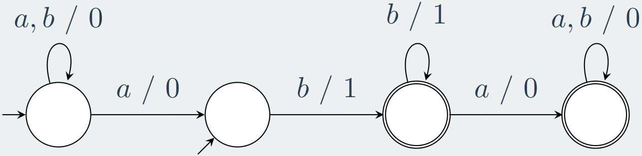

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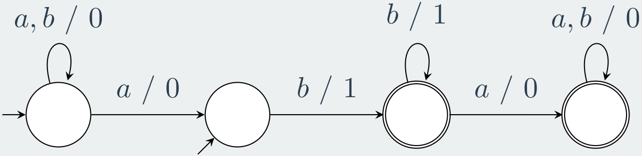

Consider $w = bbab$

b	b	a	b
1	1	0	0

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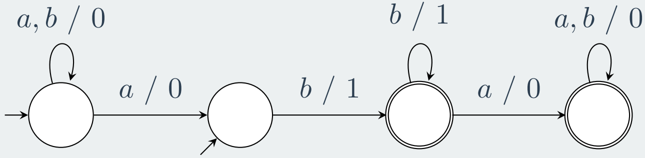

Consider $w = bbab$

b b a b

$1 + 1 + 0 + 0 = 2$

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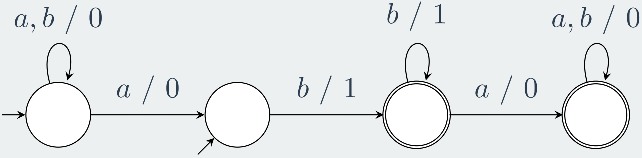

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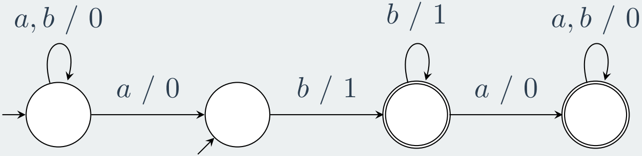

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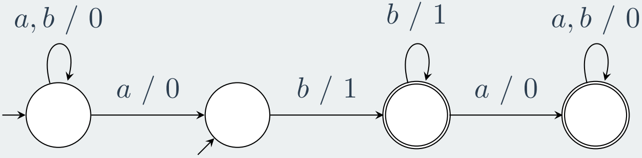

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$1 + 1 + 0 + 0 = 2$						$0 + 0 + 0 + 1 = 1$			

Output: $\min\{2, 1\} = 1$

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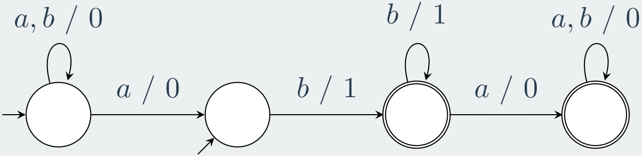

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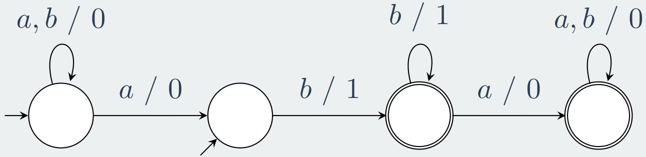

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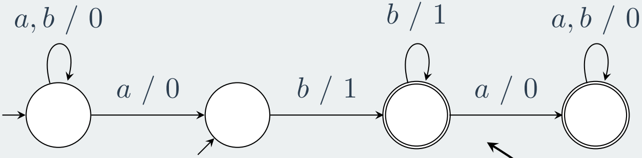

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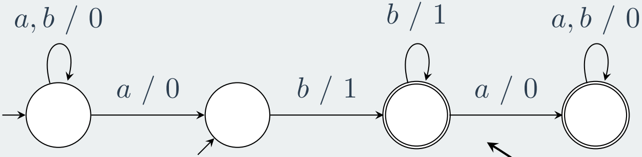

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(∞ if there is no b)

Weighted automata over unary alphabet

$\Sigma = \{a\}, \quad \mathbb{S} = \mathbb{N}(+,\cdot)$ or $\mathbb{Z}(+,\cdot)$

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WA over unary alphabet = Linear Recurrence Sequences

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$f(n+2) = f(n+1) + f(n)$, $f(0) = f(1) = 1$

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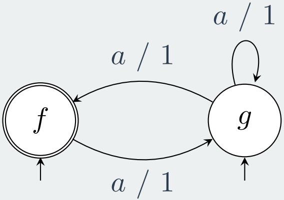

$\mathcal{A}(a^n) = F_n$

WA subclasses

What is the number of accepting runs?

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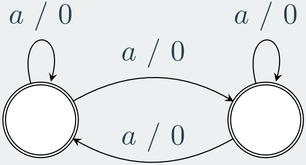

2^n accepting runs for a^n

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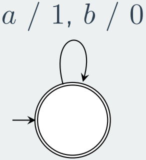

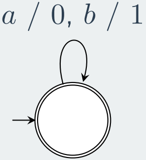

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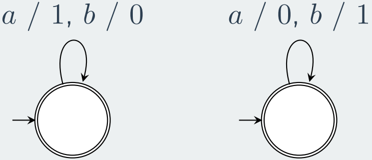

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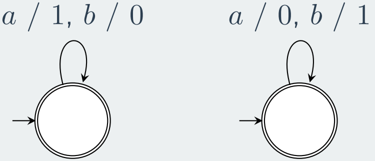

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WA
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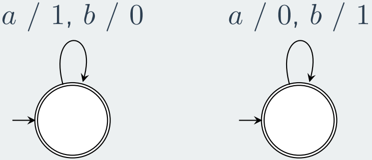
polynomially ambiguous WA



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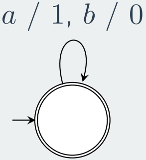

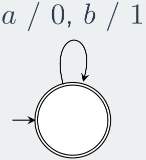

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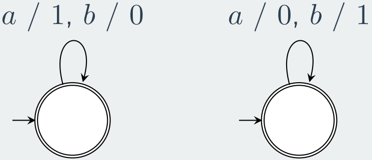

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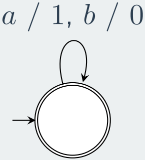
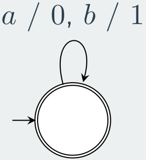
deterministic WA





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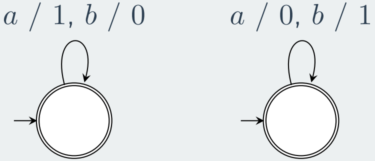

$$\begin{array}{c}\text{WA}\\ \rotatebox{90}{$\subsetneq$}\\ \left.\begin{array}{c}\text{polynomially ambiguous WA}\\ \rotatebox{90}{$\subsetneq$}\\ \text{finitely ambiguous WA}\end{array}\right\}\ \text{decidability border}\\ \rotatebox{90}{$\subsetneq$}\\ \text{unambiguous WA}\\ \rotatebox{90}{$\subsetneq$}\\ \text{deterministic WA}\end{array}$$

What is this talk about?

Separating fragments of weighted automata

Fragments of WA revisited

Fix $\mathbb{N}_\infty(\min, +, \infty, 0)$

Fragments of WA revisited

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Recall

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$\rotatebox{90}{$\subsetneq$}$
polynomially ambiguous WA
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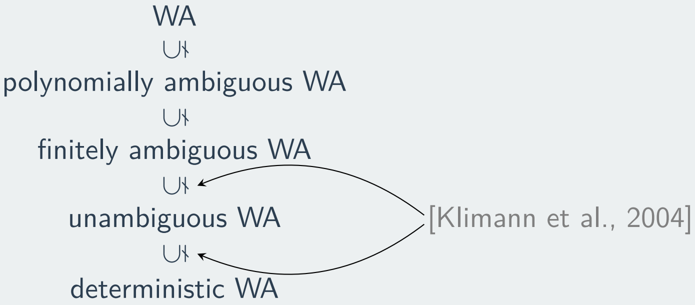

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[Klimann et al., 2004]

Fragments of WA revisited

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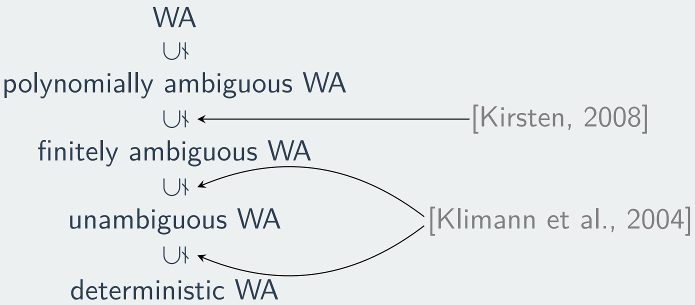

WA

$\supsetneq$

polynomially ambiguous WA

$\supsetneq$ ← [Kirsten, 2008]

finitely ambiguous WA

$\supsetneq$

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$\supsetneq$

deterministic WA

Fragments of WA revisited

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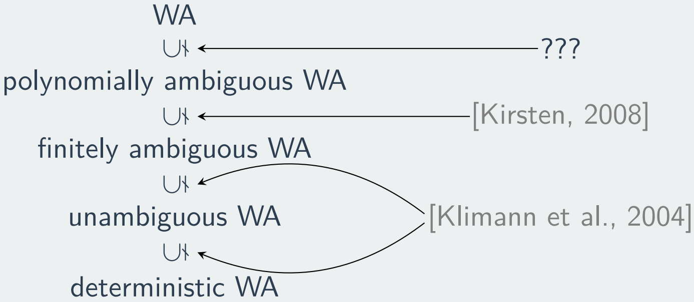

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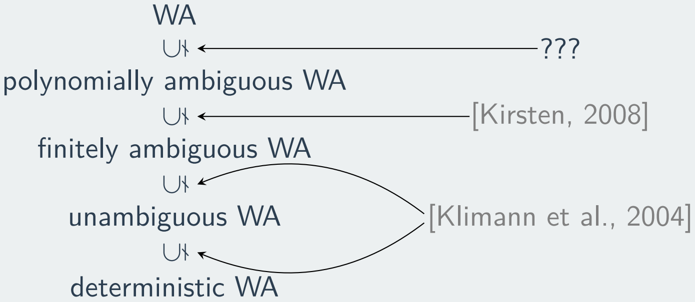

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deterministic WA

- Strictness shown by examples

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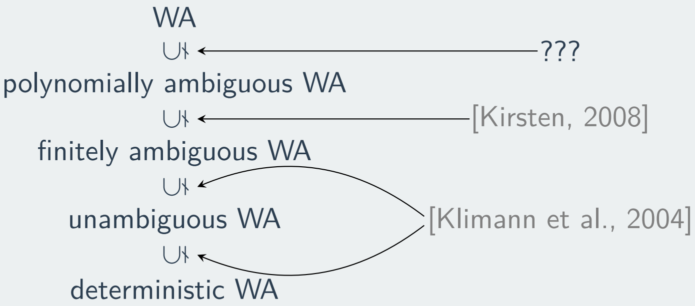

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deterministic WA

- Strictness shown by examples
- Papers are about determinization

Meanwhile other formalisms

Boolean world

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Boolean world

- Finite automata

Show that $L = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

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- Finite automata

Show that $L = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

Solution: pumping lemma

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- Context-free languages – pumping lemmas
- First order logic – Ehrenfeucht-Fraïssé games

On which fragments we work?

Semiring: $\mathbb{N}_{\infty}(\min, +, \infty, 0)$

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Three fragments

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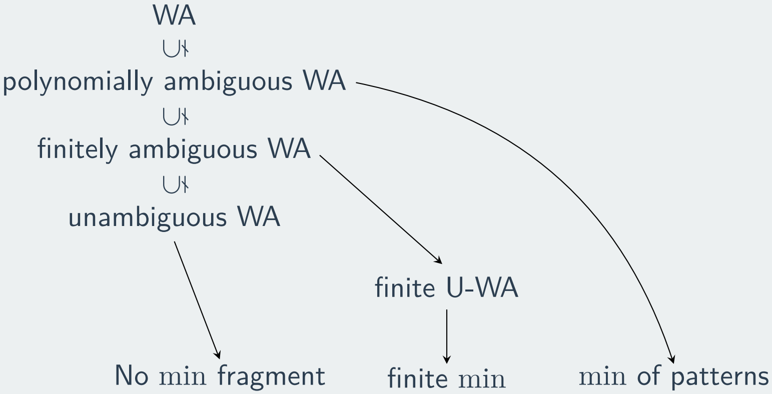

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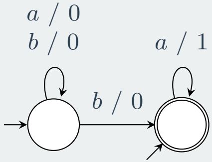

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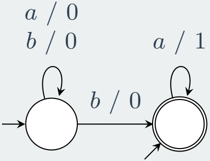

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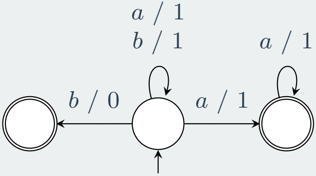

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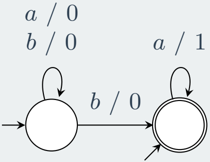

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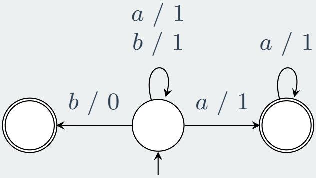

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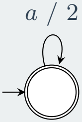

$g(a^n) = 2^n$

The no $\min$ fragment

WA over $\mathbb{N}_\infty(+, \cdot, 0, 1)$

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Corollary: U-WA $\subsetneq$ FA-WA over $(\min, +)$

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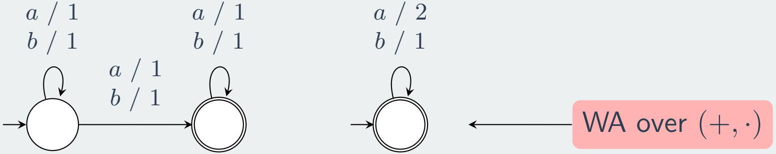

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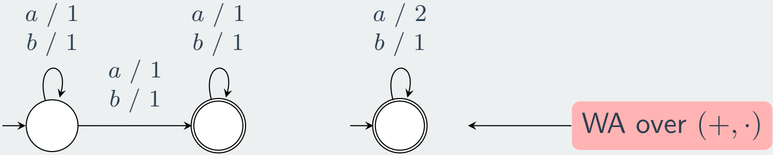

Negative examples: • “smallest block of b’s”,

The finite-$\min$ fragment

$f(w) = \min\{f_1(w), \ldots, f_m(w)\}$, f_i in WA over $\mathbb{N}_\infty(+, \cdot, 0, 1)$

Example: $f(w) = \min(\#_a(w), \#_b(w))$

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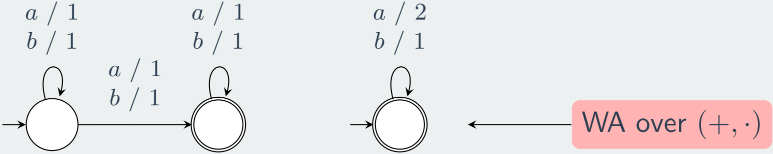

WA over $(+, \cdot)$

Negative examples:
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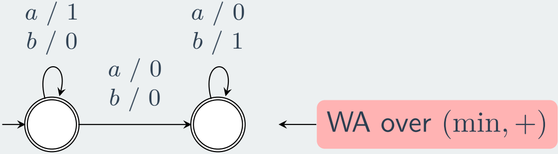

WA over $(\min, +)$

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Word n-representation: $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n$

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Corollary: FA-WA $\subsetneq$ PA-WA over $(\min, +)$

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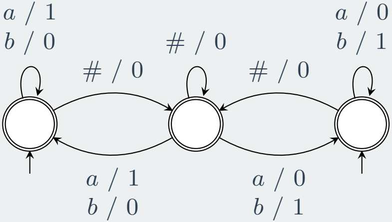

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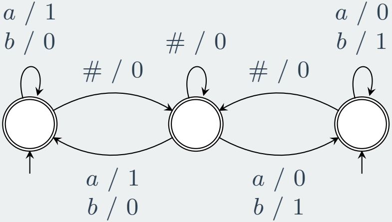

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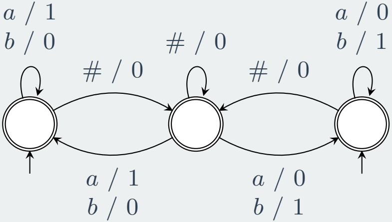

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Another example: $f(w) = \sum_{i=0}^{m} g(w_i)$, where g – “smallest block of b’s”

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What next?

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Candidate for WL $\not\subseteq$ CRA is $f(n) = n^n$

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