The Reachability Problem for Petri Nets is Not Elementary

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Rennes 2018

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Introduction

Petri Nets, VASS, programs with no zero tests

Vector addition systems with states (VASS)

(d, Q, T), where $T \subseteq Q \times \mathbb{Z}^d \times Q$

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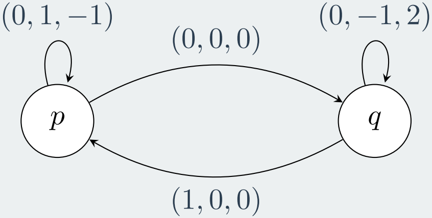
Example: $d = 3$, $Q = \{p, q\}$



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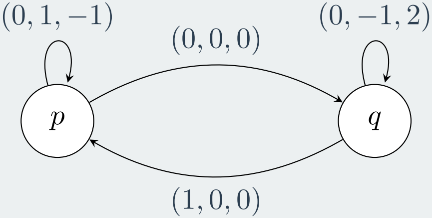

Configurations $p(\mathbf{v}) = (p, \mathbf{v}) \in Q \times \mathbb{N}^d$

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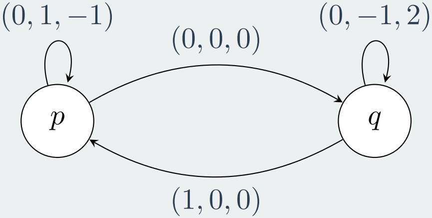

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Example run:

$p(0, 0, 1) \to p(0, 1, 0) \to q(0, 1, 0) \to q(0, 0, 2) \to p(1, 0, 2)$

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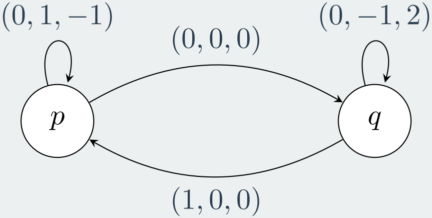

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$p(0,0,1) \rightarrow p(0,1,0) \rightarrow q(0,1,0) \rightarrow q(0,0,2) \rightarrow p(1,0,2)$

Notation: $p(0,0,1) \rightarrow^* p(1,0,2)$

Decision problems

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Reachability problem:

GIVEN: VASS (d, Q, T) and configurations $p(\mathbf{u}), q(\mathbf{v})$

DECIDE: whether $p(\mathbf{u}) \to^* q(\mathbf{v})$?

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Coverability problem:

GIVEN: VASS (d, Q, T) and configurations $p(\mathbf{u}), q(\mathbf{v})$

DECIDE: whether exists $\mathbf{v}'$ s.t. $p(\mathbf{u}) \to^* q(\mathbf{v}')$ and $\mathbf{v}' \geq \mathbf{v}$?

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- Coverability can be reduced to reachability
- We can assume $\mathbf{u} = \mathbf{v} = \mathbf{0}$

Counter programs (with or without zero tests)

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$\mathsf{x} \mathrel{+}= \mathsf{m}$	(add m to variable x)
$\mathsf{x} \mathrel{-}= \mathsf{m}$	(subtract m from variable x)
goto L **or** L'	(jump to either line L or line L')
test $\mathsf{x} = 0$	(continue if variable x is zero)
halt if $\mathsf{x}_1, \ldots, \mathsf{x}_l = 0$	(terminate if listed variables are zero).

Counter programs (with or without zero tests)

x += m	(add m to variable x)
x −= m	(subtract m from variable x)
goto L **or** L'	(jump to either line L or line L')
~~**test** x $= 0$~~	~~(continue if variable x is zero)~~
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1: $\mathsf{x}' \mathrel{+}= B$
2: **goto** 6 **or** 3
3: $\mathsf{x} \mathrel{+}= 1 \quad \mathsf{x}' \mathrel{-}= 1$
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A complete run ends with $\mathsf{x} = B$, $\mathsf{y} = 2B$

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GIVEN: A counter program with no zero tests.

DECIDE: Does it have a complete run (executing **halt**)?

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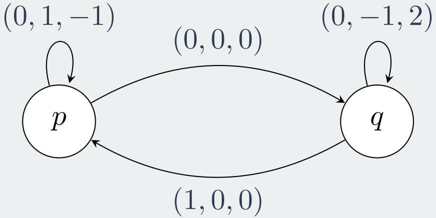

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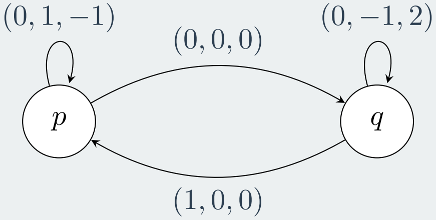

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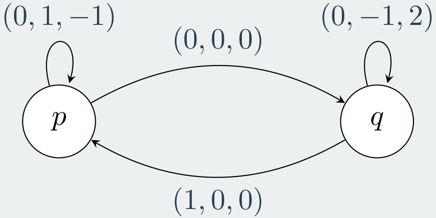

$p(0,0,1) \to^* p(1,0,2)$?

z += 1
loop
    loop
        y += 1   z −= 1
    loop
        y −= 1   z += 2
    x += 1
x −= 1   z −= 2
halt if x, y, z = 0.

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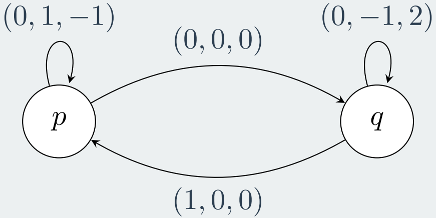

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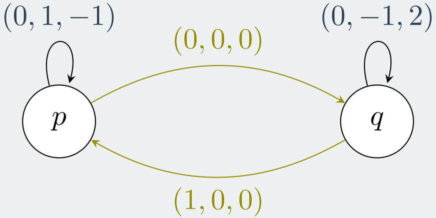

$p(0,0,1) \rightarrow^* p(1,0,2)$?

```
z += 1
loop
    loop
        y += 1    z -= 1
    loop
        y -= 1    z += 2
    x += 1
x -= 1    z -= 2
halt if x, y, z = 0.
```

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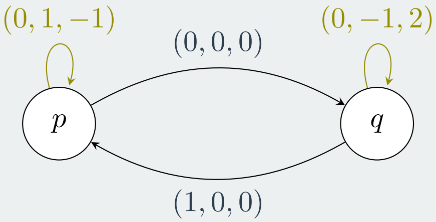

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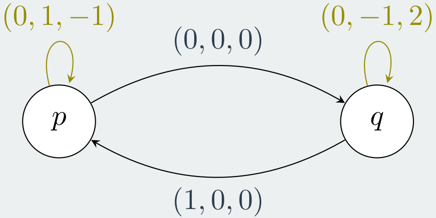

$p(0, 0, 1) \to^* p(1, 0, 2)$?

Coverability if **halt** is empty

z $+= 1$
loop
    loop
        y $+= 1$ z $-= 1$
    loop
        y $-= 1$ z $+= 2$
    x $+= 1$
x $-= 1$ z $-= 2$
halt if x, y, z $= 0$.

Reachability state of art

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Coverability
Rackoff 1978
EXPSPACE-complete

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Coverability
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So **halt** is important

Outline

- High level idea of the proof
- Key construction

Programs with zero tests

Additional command: **test** $\mathrm{x} = 0$

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Reachability becomes undecidable

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Let k – size of input

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Programs with zero tests

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Let k – size of input

Suppose counters are bounded by $B = f(k)$

If f is n-EXP, i.e., $f(k) = 2^{\cdot^{\cdot^{\cdot^{2^k}}}} \Big\} n$ times.

Then reachability is $(n-1)$-EXPSPACE-complete

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Lipton encoded programs for $f = 2$-EXP

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Lipton encoded programs for $f = 2$-EXP

We can do it for any $f = n$-EXP

Encoding programs with zero tests and bounded counters

B – bound on the counters

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We encode this into programs with no zero tests

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Suppose we get (magically) three counters $\mathsf{b}, \mathsf{c}, \mathsf{d}$

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loop

$$\mathsf{x}_1' \mathrel{+}= 1 \quad \cdots \quad \mathsf{x}_l' \mathrel{+}= 1$$

$$\mathsf{b} \mathrel{-}= 1$$

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Encoding: for every x_i add x'_i

Intuitively $\mathsf{x}_i + \mathsf{x}'_i = B$, so start with:

loop

$$\mathsf{x}'_1 \mathrel{+}= 1 \quad \cdots \quad \mathsf{x}'_l \mathrel{+}= 1$$
$$\mathsf{b} \mathrel{-}= 1$$

Replace $\mathsf{x}_i \mathrel{+}= m$ with $\mathsf{x}_i \mathrel{+}= m \quad \mathsf{x}'_i \mathrel{-}= m$

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Encoding (continued)

B – bound on the counters

$\mathsf{b} = B, \ \mathsf{c} \geq 0, \ \mathsf{d} = \mathsf{c} \cdot \mathsf{b}$

$\mathsf{x}'_i = B - \mathsf{x}_i$

Encoding (continued)

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Replace **test** $\mathsf{x}_i = 0$ with

loop

$\quad \mathsf{x}_i \mathrel{+}= 1 \quad \mathsf{x}_i' \mathrel{-}= 1$

$\quad \mathsf{d} \mathrel{-}= 1$

$\mathsf{c} \mathrel{-}= 1$

loop

$\quad \mathsf{x}_i \mathrel{-}= 1 \quad \mathsf{x}_i' \mathrel{+}= 1$

$\quad \mathsf{d} \mathrel{-}= 1$

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Extend **halt** with $\mathsf{b}, \mathsf{d} = 0$

Encoding (continued)

B – bound on the counters

$\mathsf{b} = B,\ \mathsf{c} \geq 0,\ \mathsf{d} = \mathsf{c} \cdot \mathsf{b}$ ← c is "number of zero tests" $\cdot\, 2$

$\mathsf{x}_i' = B - \mathsf{x}_i$ ← holds because $\mathsf{b} = 0$

Replace **test** $\mathsf{x}_i = 0$ with

loop
    $\mathsf{x}_i \mathrel{+}= 1 \quad \mathsf{x}_i' \mathrel{-}= 1$
    $\mathsf{d} \mathrel{-}= 1$
$\mathsf{c} \mathrel{-}= 1$
loop
    $\mathsf{x}_i \mathrel{-}= 1 \quad \mathsf{x}_i' \mathrel{+}= 1$
    $\mathsf{d} \mathrel{-}= 1$
$\mathsf{c} \mathrel{-}= 1$

Extend **halt** with $\mathsf{b}, \mathsf{d} = 0$

Encoding (continued)

B – bound on the counters

$\mathsf{b} = B,\ \mathsf{c} \geq 0,\ \mathsf{d} = \mathsf{c} \cdot \mathsf{b}$ ← c is "number of zero tests" $\cdot\, 2$

$\mathsf{x}_i' = B - \mathsf{x}_i$ ← holds because $\mathsf{b} = 0$

Replace **test** $\mathsf{x}_i = 0$ with

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This is the challenge

Extend **halt** with $\mathsf{b}, \mathsf{d} = 0$

The main construction

to obtain b, c and d

Recall what we wanted

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$\mathsf{b} = B, \ \mathsf{c} \geq 0, \ \mathsf{d} = \mathsf{c} \cdot \mathsf{b}$

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If B is fixed, just start the program with:

$\mathsf{b} \mathrel{+}= B$
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$\mathsf{b} \mathrel{+}= B$ ← "gadget for ratio B"

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b $+=$ B ← "gadget for ratio B"
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    c $+=$ 1 d $+=$ B

But in general we want $B = \left.2^{\cdot^{\cdot^{\cdot^{2^k}}}}\right\}n$ times.

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For this we need an iterative construction

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But in general we want $B = \left.2^{\cdot^{\cdot^{2^k}}}\right\} n$ times.

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Some variables will be bounded and allowed to be 0-tested

Gadget for ratio $B = n$-EXP

$\mathsf{b} = B,\ \mathsf{c} \geq 0,\ \mathsf{d} = \mathsf{c} \cdot \mathsf{b}$ allows for 0-tests on variables bounded by B

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A program proving the lemma is what's left

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loop
    x $+= 1$ i $-= 1$ i$' += 1$
test i $= 0$
loop
    i $+= 1$ i$' -= 1$
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    x += 1    i −= 1    i′ += 1
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    i += 1    i′ −= 1
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Let $i \leq B$ stored in i, and i' auxiliary (guaranteed to be 0)

- We want e.g.: $\mathsf{x} \mathrel{+}= \boxed{\mathsf{i}}$ or $\mathsf{x} \mathrel{-}= \boxed{\mathsf{i}}$

```
loop
    x += 1    i -= 1    i' += 1
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- Or **loop at most** b **times** $<\mathit{body}>$

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loop
  b$'$ $-= 1$ b $+= 1$
  $<\!\mathit{body}\!>$

High level description of the program

B – previous bound

Output: $\mathsf{b} = B!,\ \mathsf{c} \geq 0,\ \mathsf{d} = \mathsf{c} \cdot \mathsf{b}$

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$\mathsf{b} \mathrel{+}= 1, \quad \mathsf{k} \mathrel{+}= B$

loop

  $\mathsf{c} \mathrel{+}= 1 \quad \mathsf{d} \mathrel{+}= 1 \quad \mathsf{x} \mathrel{+}= 1 \quad \mathsf{y} \mathrel{+}= 1$

$\mathsf{i} \mathrel{+}= 1 \quad \mathsf{k} \mathrel{-}= 1$

<*main loop*>

loop

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$\mathsf{i} \mathrel{+}= 1 \quad \mathsf{k} \mathrel{-}= 1$

$<$*main loop*$>$

loop

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<*main loop*> ⟵ $\mathsf{c} := \mathsf{c}/(B-1)!, \quad \mathsf{d}, \mathsf{x} := d \cdot B, \quad \mathsf{b} := b \cdot B!, \quad \mathsf{k} = 0, \quad \mathsf{i} = B$

loop

$\quad \mathsf{x} \mathrel{-}= \mathsf{i} \quad \mathsf{y} \mathrel{-}= 1$

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Invariants
$\mathsf{i} + \mathsf{k} = B, \quad \mathsf{b} \cdot \mathsf{c} = \mathsf{d}$

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$$\prod_{i=1}^{k-1} \frac{i+1}{i} = k$$

The main loop

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loop
    loop
        $\mathsf{c} \mathrel{-}= \boxed{\mathsf{i}} \quad \mathsf{c}' \mathrel{+}= 1$
        loop at most b **times**
            $\mathsf{d} \mathrel{-}= \boxed{\mathsf{i}} \quad \mathsf{d}' \mathrel{+}= \boxed{\mathsf{i}+1} \quad \mathsf{x} \mathrel{-}= \boxed{\mathsf{i}} \quad \mathsf{x}' \mathrel{+}= \boxed{\mathsf{i}+1}$
    loop
        $\mathsf{b} \mathrel{-}= 1 \quad \mathsf{b}' \mathrel{+}= \boxed{\mathsf{i}+1}$
    loop
        $\mathsf{b}' \mathrel{-}= 1 \quad \mathsf{b} \mathrel{+}= 1$
    loop
        $\mathsf{c}' \mathrel{-}= 1 \quad \mathsf{c} \mathrel{+}= 1$
        loop at most b **times**
            $\mathsf{d}' \mathrel{-}= 1 \quad \mathsf{d} \mathrel{+}= 1 \quad \mathsf{x}' \mathrel{-}= 1 \quad \mathsf{x} \mathrel{+}= 1$
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$\mathsf{c}' := \mathsf{c} \cdot \frac{1}{\mathsf{i}},\ \mathsf{d}' := \mathsf{d} \cdot \frac{\mathsf{i}+1}{\mathsf{i}}$

loop

  loop

    $\mathsf{c} \mathrel{-}= \boxed{\mathsf{i}} \quad \mathsf{c}' \mathrel{+}= 1$

    loop at most b **times**

      $\mathsf{d} \mathrel{-}= \boxed{\mathsf{i}} \quad \mathsf{d}' \mathrel{+}= \boxed{\mathsf{i}+1} \quad \mathsf{x} \mathrel{-}= \boxed{\mathsf{i}} \quad \mathsf{x}' \mathrel{+}= \boxed{\mathsf{i}+1}$

  loop

    $\mathsf{b} \mathrel{-}= 1 \quad \mathsf{b}' \mathrel{+}= \boxed{\mathsf{i}+1}$

  loop

    $\mathsf{b}' \mathrel{-}= 1 \quad \mathsf{b} \mathrel{+}= 1$

  loop

    $\mathsf{c}' \mathrel{-}= 1 \quad \mathsf{c} \mathrel{+}= 1$

    loop at most b **times**

      $\mathsf{d}' \mathrel{-}= 1 \quad \mathsf{d} \mathrel{+}= 1 \quad \mathsf{x}' \mathrel{-}= 1 \quad \mathsf{x} \mathrel{+}= 1$

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  loop at most b **times**

   $\mathsf{d} \mathrel{-}= \boxed{\mathsf{i}} \quad \mathsf{d}' \mathrel{+}= \boxed{\mathsf{i}+1} \quad \mathsf{x} \mathrel{-}= \boxed{\mathsf{i}} \quad \mathsf{x}' \mathrel{+}= \boxed{\mathsf{i}+1}$

 loop

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   $\mathsf{d}' \mathrel{-}= 1 \quad \mathsf{d} \mathrel{+}= 1 \quad \mathsf{x}' \mathrel{-}= 1 \quad \mathsf{x} \mathrel{+}= 1$

 $\mathsf{k} \mathrel{-}= 1 \quad \mathsf{i} \mathrel{+}= 1$

$\mathsf{c}' := \mathsf{c} \cdot \frac{1}{\mathsf{i}}, \ \mathsf{d}' := \mathsf{d} \cdot \frac{\mathsf{i}+1}{\mathsf{i}}$

$\mathsf{b}' := \mathsf{b} \cdot (\mathsf{i}+1)$

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Invariants

$\mathsf{i} + \mathsf{k} = B, \quad \mathsf{b} \cdot \mathsf{c} = \mathsf{d}$

$\prod_{i=1}^{k-1} \frac{i+1}{i} \;=\; k$

loop

  loop

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    loop at most b **times**

      $\mathsf{d} \mathrel{-}= \boxed{\mathsf{i}} \quad \mathsf{d}' \mathrel{+}= \boxed{\mathsf{i}+1} \quad \mathsf{x} \mathrel{-}= \boxed{\mathsf{i}} \quad \mathsf{x}' \mathrel{+}= \boxed{\mathsf{i}+1}$

  loop

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if any **loop** not maximal then $\mathsf{x} < \mathsf{y} \cdot B$

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So maybe it's good to study restrictions of generalizations of etc. . .