# The Reachability Problem for Petri Nets is Not Elementary

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DIMAP seminar
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# Introduction

Petri Nets, VASS, programs with no zero tests

## Vector addition systems with states (VASS)

$(d, Q, T)$, where $T \subseteq Q \times \mathbb{Z}^d \times Q$

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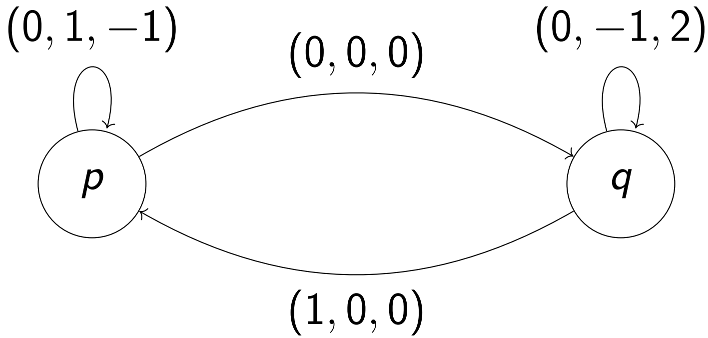
Example: $d = 3$, $Q = \{p, q\}$



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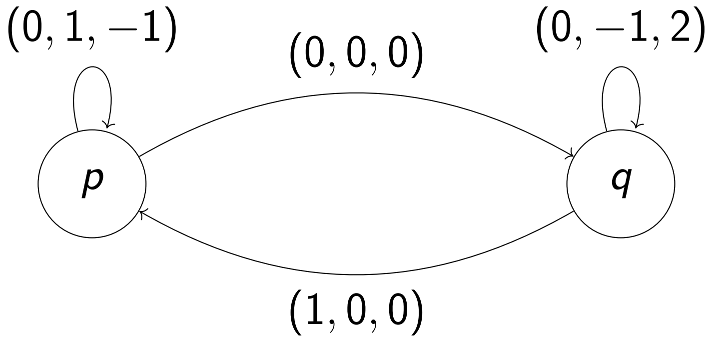


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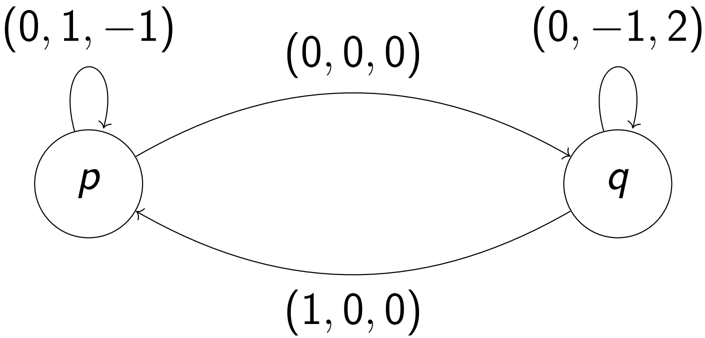


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Example run:

$p(0, 0, 1) \rightarrow p(0, 1, 0) \rightarrow q(0, 1, 0) \rightarrow q(0, 0, 2) \rightarrow p(1, 0, 2)$

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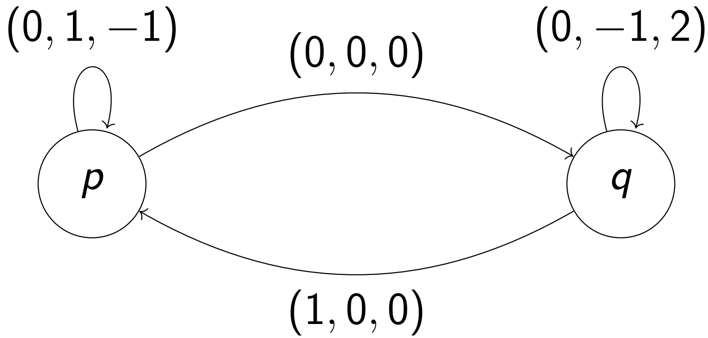


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Example run:

$p(0, 0, 1) \to p(0, 1, 0) \to q(0, 1, 0) \to q(0, 0, 2) \to p(1, 0, 2)$

Notation: $p(0, 0, 1) \to^* p(1, 0, 2)$

# Decision problems

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Reachability problem:

$\textsc{Given}$: VASS $(d, Q, T)$ and configurations $p(\mathbf{u}), q(\mathbf{v})$

$\textsc{Decide}$: whether $p(\mathbf{u}) \rightarrow^* q(\mathbf{v})$?

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Reachability problem:

GIVEN: VASS $(d, Q, T)$ and configurations $p(\mathbf{u}), q(\mathbf{v})$

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Coverability problem:

GIVEN: VASS $(d, Q, T)$ and configurations $p(\mathbf{u}), q(\mathbf{v})$

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- Coverability can be reduced to reachability

# Counter programs (with or without zero tests)

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| $\mathrm{x} \mathrel{+}= \mathrm{m}$ | (add $m$ to variable x) |
| $\mathrm{x} \mathrel{-}= \mathrm{m}$ | (subtract $m$ from variable x) |
| **goto** $L$ **or** $L'$ | (jump to either line $L$ or line $L'$) |
| **test** $\mathrm{x} = 0$ | (continue if variable x is zero) |
| **halt if** $\mathrm{x}_1, \ldots, \mathrm{x}_l = 0$ | (terminate if listed variables are zero). |

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1: $\mathsf{x}' \mathrel{+}= B$
2: **goto** 6 **or** 3
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| 5: **goto** 2 | |
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A complete run ends with $\mathrm{x} = B$, $\mathrm{y} = 2B$

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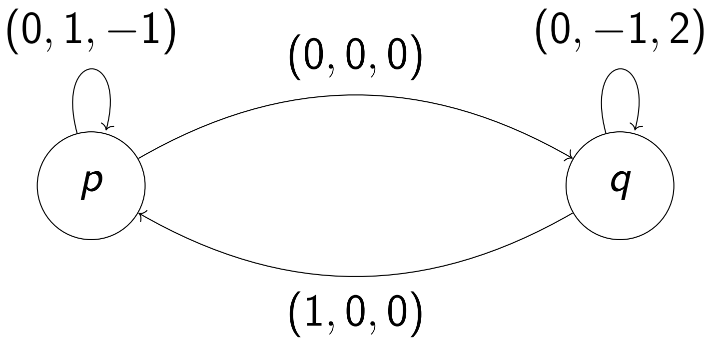

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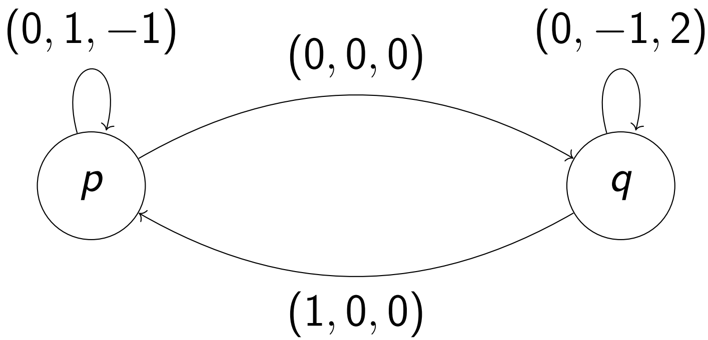


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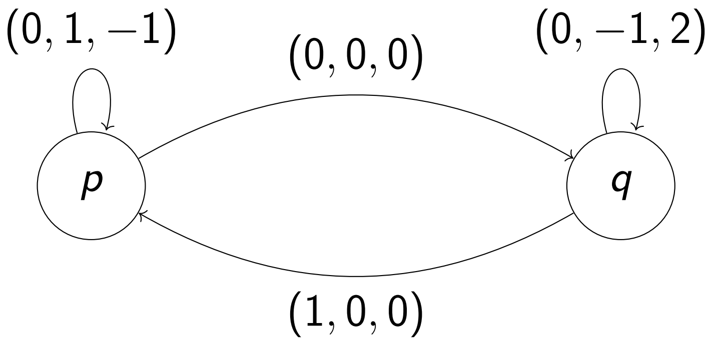


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z += 1
loop
    loop
        y += 1    z −= 1
    loop
        y −= 1    z += 2
    x += 1
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halt if x, y, z = 0.
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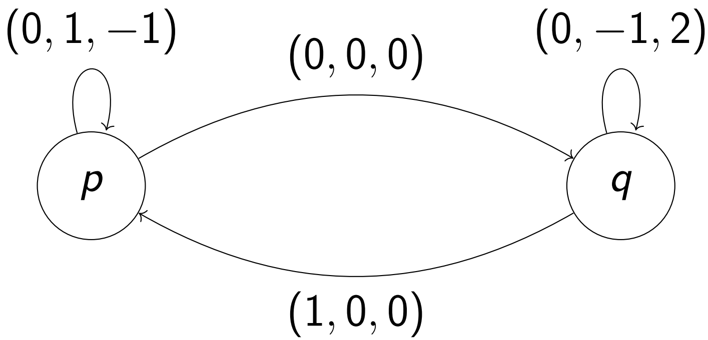


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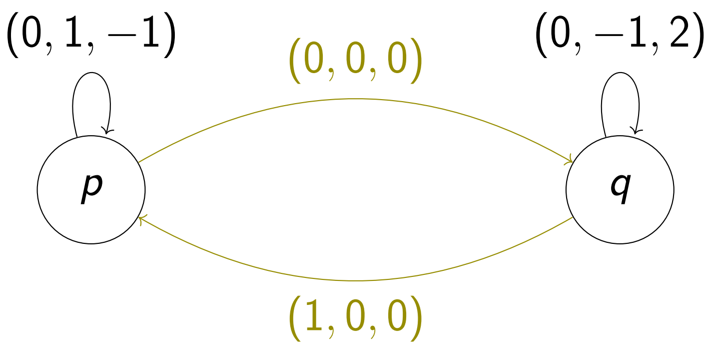


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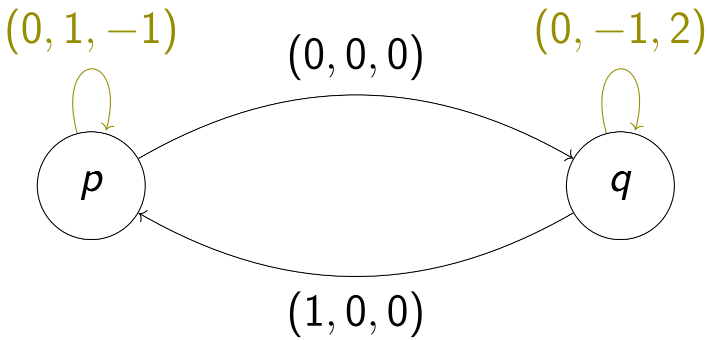


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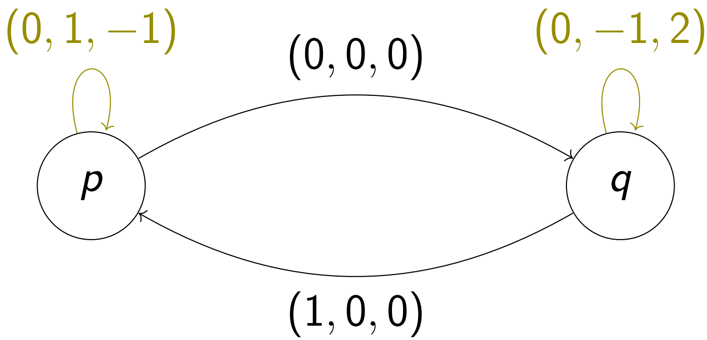


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Coverability if **halt** is empty

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Rackoff 1978
EXPSPACE-complete

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Coverability
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So **halt** is important

# Outline

- High level idea of the proof

- Key construction

# Programs with zero tests

Additional command: **test** $x = 0$

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Suppose counters are bounded by $B = f(k)$

If $f$ is $n$-EXP, i.e., $f(k) = 2^{\left.2^{\cdot^{\cdot^{\cdot^{2^k}}}}\right\} n \text{ times.}}$

Then reachability is $(n-1)$-EXPSPACE-complete

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We can do it for any $f = n$-EXP

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$\quad \mathsf{x}'_1 \mathrel{+}= 1 \quad \cdots \quad \mathsf{x}'_l \mathrel{+}= 1$

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Replace $\mathsf{x}_i \mathrel{+}= m$ with $\mathsf{x}_i \mathrel{+}= m \quad \mathsf{x}'_i \mathrel{-}= m$

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# Encoding (continued)

$B$ – bound on the counters

$\mathsf{b} = B, \ \mathsf{c} \geq 0, \ \mathsf{d} = \mathsf{c} \cdot \mathsf{b}$

$\mathsf{x}'_i = B - \mathsf{x}_i$

# Encoding (continued)

$B$ – bound on the counters

$\mathrm{b} = B, \ \mathrm{c} \geq 0, \ \mathrm{d} = \mathrm{c} \cdot \mathrm{b}$ $\longleftarrow$ c is "number of zero tests" $\cdot\, 2$

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Replace **test** $\mathsf{x}_i = 0$ with

**loop**
  $\mathsf{x}_i \mathrel{+}= 1 \quad \mathsf{x}_i' \mathrel{-}= 1$
  $\mathsf{d} \mathrel{-}= 1$
$\mathsf{c} \mathrel{-}= 1$
**loop**
  $\mathsf{x}_i \mathrel{-}= 1 \quad \mathsf{x}_i' \mathrel{+}= 1$
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## Encoding (continued)

$B$ – bound on the counters

$\mathsf{b} = B,\ \mathsf{c} \geq 0,\ \mathsf{d} = \mathsf{c} \cdot \mathsf{b}$ ← c is "number of zero tests" $\cdot\, 2$

$\mathsf{x}'_i = B - \mathsf{x}_i$

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\end{array}\right\} \text{c decreased by 2 and d by at most } 2B$$

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so a false zero test implies $\mathsf{d} \neq 0$

Extend **halt** with $\mathsf{b}, \mathsf{d} = 0$

# Encoding (continued)

$B$ – bound on the counters

$\boxed{b = B,\ c \geq 0,\ d = c \cdot b}$ ← c is "number of zero tests" $\cdot\ 2$

$x'_i = B - x_i$ ← holds because $b = 0$

Replace **test** $x_i = 0$ with

$$\left.\begin{array}{l}
\textbf{loop} \\
\quad x_i \mathrel{+}= 1 \quad x'_i \mathrel{-}= 1 \\
\quad d \mathrel{-}= 1 \\
c \mathrel{-}= 1 \\
\textbf{loop} \\
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\quad d \mathrel{-}= 1 \\
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\end{array}\right\} \text{c decreased by 2 and d by at most } 2B$$

so a false zero test implies $d \neq 0$

This is the challenge

Extend **halt** with $b, d = 0$

# The main construction

to obtain b, c and d

# Recall what we wanted

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$\mathsf{b} = B,\ \mathsf{c} \geq 0,\ \mathsf{d} = \mathsf{c} \cdot \mathsf{b}$

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For this we need an iterative construction

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For this we need an iterative construction

Some variables will be bounded and allowed to be 0-tested

# Gadget for ratio $B = n$-EXP

$\mathsf{b} = B,\ \mathsf{c} \geq 0,\ \mathsf{d} = \mathsf{c} \cdot \mathsf{b}$ allows for 0-tests on variables bounded by $B$

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A program proving the lemma is what's left

## How to use $B$-bounded $0$-tests?

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```
loop
    x += 1    i −= 1    i′ += 1
test i = 0
loop
    i += 1    i′ −= 1
test i′ = 0
```

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Let $i \leq B$ stored in i,  and i′ auxiliary (guaranteed to be 0)

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# How to use $B$-bounded $0$-tests?

Let $i \leq B$ stored in $\mathsf{i}$, and $\mathsf{i}'$ auxiliary (guaranteed to be 0)

- We want e.g.: $\mathsf{x} \mathrel{+}= \boxed{\mathsf{i}}$ or $\mathsf{x} \mathrel{-}= \boxed{\mathsf{i}}$

**loop**
$\quad \mathsf{x} \mathrel{+}= 1 \quad \mathsf{i} \mathrel{-}= 1 \quad \mathsf{i}' \mathrel{+}= 1$
**test** $\mathsf{i} = 0$
**loop**
$\quad \mathsf{i} \mathrel{+}= 1 \quad \mathsf{i}' \mathrel{-}= 1$
**test** $\mathsf{i}' = 0$

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**test** $\mathsf{i} = 0$
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- Or **loop at most** b **times** $<\mathit{body}>$

(b has no bound)

## How to use $B$-bounded 0-tests?

Let $i \leq B$ stored in i, and i$'$ auxiliary (guaranteed to be 0)

- We want e.g.: x $+=$ $\boxed{\text{i}}$ or x $-=$ $\boxed{\text{i}}$

**loop**
    x $+=$ 1    i $-=$ 1    i$'$ $+=$ 1
**test** i $=$ 0
**loop**
    i $+=$ 1    i$'$ $-=$ 1
**test** i$'$ $=$ 0

- Or **loop at most** b **times** $<$*body*$>$

(b has no bound)

**loop**
    b $-=$ 1    b$'$ $+=$ 1
**loop**
    b$'$ $-=$ 1    b $+=$ 1
    $<$*body*$>$

# High level description of the program

$B$ – previous bound

Output: $\mathsf{b} = B!,\ \mathsf{c} \geq 0,\ \mathsf{d} = \mathsf{c} \cdot \mathsf{b}$

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$\mathsf{b} \mathrel{+}= 1, \quad \mathsf{k} \mathrel{+}= B$

**loop**

$\quad \mathsf{c} \mathrel{+}= 1 \quad \mathsf{d} \mathrel{+}= 1 \quad \mathsf{x} \mathrel{+}= 1 \quad \mathsf{y} \mathrel{+}= 1$

$\mathsf{i} \mathrel{+}= 1 \quad \mathsf{k} \mathrel{-}= 1$

$<$*main loop*$>$

**loop**

$\quad \mathsf{x} \mathrel{-}= \mathsf{i} \quad \mathsf{y} \mathrel{-}= 1$

**halt if** $\mathsf{y}, \mathsf{k} = 0$

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$\mathrm{i} \mathrel{+}= 1 \quad \mathrm{k} \mathrel{-}= 1$

<*main loop*> ⟵ $\mathrm{c} := \mathrm{c}/(B-1)!, \quad \mathrm{d}, \mathrm{x} := d \cdot B, \quad \mathrm{b} := b \cdot B!, \quad \mathrm{k} = 0, \quad \mathrm{i} = B$

**loop**

$\quad \mathrm{x} \mathrel{-}= \mathrm{i} \quad \mathrm{y} \mathrel{-}= 1$

**halt if** $\mathrm{y}, \mathrm{k} = 0$

# High level description of the program

$B$ – previous bound

Output: $\mathsf{b} = B!,\ \mathsf{c} \geq 0,\ \mathsf{d} = \mathsf{c} \cdot \mathsf{b}$

Auxiliary variables $\mathsf{x}, \mathsf{y}, \mathsf{k}, \mathsf{i}$ (to check correctness)

b += 1,  k += $B$

**loop**

  c += 1  d += 1  x += 1  y += 1 ⟵ $\mathsf{c}, \mathsf{d}, \mathsf{x}, \mathsf{y} := c \cdot (B-1)!$

i += 1  k −= 1

<*main loop*> ⟵ $\mathsf{c} := \mathsf{c}/(B-1)!,\quad \mathsf{d}, \mathsf{x} := d \cdot B,\quad \mathsf{b} := b \cdot B!,\quad \mathsf{k} = 0,\quad \mathsf{i} = B$

**loop**

  x −= i  y −= 1

**halt if** $\mathsf{y}, \mathsf{k} = 0$

Invariants

$\mathsf{i} + \mathsf{k} = B,\quad \mathsf{b} \cdot \mathsf{c} = \mathsf{d}$

# High level description of the program

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Output: $\mathsf{b} = B!,\ \mathsf{c} \geq 0,\ \mathsf{d} = \mathsf{c} \cdot \mathsf{b}$

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$\mathsf{b} \mathrel{+}= 1, \quad \mathsf{k} \mathrel{+}= B$

**loop**

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**loop**

$\quad \mathsf{x} \mathrel{-}= \mathsf{i} \quad \mathsf{y} \mathrel{-}= 1$

**halt if** $\mathsf{y}, \mathsf{k} = 0$

Invariants

$\mathsf{i} + \mathsf{k} = B, \quad \mathsf{b} \cdot \mathsf{c} = \mathsf{d}$

$$\prod_{i=1}^{k-1} \frac{i+1}{i} = k$$

# The main loop

**Invariants**

$$\mathsf{i} + \mathsf{k} = B, \quad \mathsf{b} \cdot \mathsf{c} = \mathsf{d}$$

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$\mathrm{i} + \mathrm{k} = B, \quad \mathrm{b} \cdot \mathrm{c} = \mathrm{d}$

$$\prod_{i=1}^{k-1} \frac{i+1}{i} = k$$

**loop**
    **loop**
        $\mathrm{c} \mathrel{-}= \boxed{\mathrm{i}} \quad \mathrm{c}' \mathrel{+}= 1$
        **loop at most** $\mathrm{b}$ **times**
            $\mathrm{d} \mathrel{-}= \boxed{\mathrm{i}} \quad \mathrm{d}' \mathrel{+}= \boxed{\mathrm{i}+1} \quad \mathrm{x} \mathrel{-}= \boxed{\mathrm{i}} \quad \mathrm{x}' \mathrel{+}= \boxed{\mathrm{i}+1}$
    **loop**
        $\mathrm{b} \mathrel{-}= 1 \quad \mathrm{b}' \mathrel{+}= \boxed{\mathrm{i}+1}$
    **loop**
        $\mathrm{b}' \mathrel{-}= 1 \quad \mathrm{b} \mathrel{+}= 1$
    **loop**
        $\mathrm{c}' \mathrel{-}= 1 \quad \mathrm{c} \mathrel{+}= 1$
        **loop at most** $\mathrm{b}$ **times**
            $\mathrm{d}' \mathrel{-}= 1 \quad \mathrm{d} \mathrel{+}= 1 \quad \mathrm{x}' \mathrel{-}= 1 \quad \mathrm{x} \mathrel{+}= 1$
    $\mathrm{k} \mathrel{-}= 1 \quad \mathrm{i} \mathrel{+}= 1$

# The main loop

Invariants

$i + k = B, \quad b \cdot c = d$

$$\prod_{i=1}^{k-1} \frac{i+1}{i} = k$$

$c' := c \cdot \frac{1}{i}, \; d' := d \cdot \frac{i+1}{i}$

**loop**

  **loop**

    $c \mathrel{-}= \boxed{i} \quad c' \mathrel{+}= 1$

    **loop at most** $b$ **times**

      $d \mathrel{-}= \boxed{i} \quad d' \mathrel{+}= \boxed{i+1} \quad x \mathrel{-}= \boxed{i} \quad x' \mathrel{+}= \boxed{i+1}$

  **loop**

    $b \mathrel{-}= 1 \quad b' \mathrel{+}= \boxed{i+1}$

  **loop**

    $b' \mathrel{-}= 1 \quad b \mathrel{+}= 1$

  **loop**

    $c' \mathrel{-}= 1 \quad c \mathrel{+}= 1$

    **loop at most** $b$ **times**

      $d' \mathrel{-}= 1 \quad d \mathrel{+}= 1 \quad x' \mathrel{-}= 1 \quad x \mathrel{+}= 1$

  $k \mathrel{-}= 1 \quad i \mathrel{+}= 1$

## The main loop

Invariants

$\mathrm{i} + \mathrm{k} = B, \quad \mathrm{b} \cdot \mathrm{c} = \mathrm{d}$

$$\prod_{i=1}^{k-1} \frac{i+1}{i} = k$$

**loop**

    **loop**

        $\mathrm{c} \mathrel{-}= \boxed{\mathrm{i}} \quad \mathrm{c}' \mathrel{+}= 1$

        **loop at most** b **times**

            $\mathrm{d} \mathrel{-}= \boxed{\mathrm{i}} \quad \mathrm{d}' \mathrel{+}= \boxed{\mathrm{i}+1} \quad \mathrm{x} \mathrel{-}= \boxed{\mathrm{i}} \quad \mathrm{x}' \mathrel{+}= \boxed{\mathrm{i}+1}$

$\mathrm{c}' := \mathrm{c} \cdot \frac{1}{\mathrm{i}}, \mathrm{d}' := \mathrm{d} \cdot \frac{\mathrm{i}+1}{\mathrm{i}}$

    **loop**

        $\mathrm{b} \mathrel{-}= 1 \quad \mathrm{b}' \mathrel{+}= \boxed{\mathrm{i}+1}$

$\mathrm{b}' := \mathrm{b} \cdot (\mathrm{i}+1)$

    **loop**

        $\mathrm{b}' \mathrel{-}= 1 \quad \mathrm{b} \mathrel{+}= 1$

    **loop**

        $\mathrm{c}' \mathrel{-}= 1 \quad \mathrm{c} \mathrel{+}= 1$

        **loop at most** b **times**

            $\mathrm{d}' \mathrel{-}= 1 \quad \mathrm{d} \mathrel{+}= 1 \quad \mathrm{x}' \mathrel{-}= 1 \quad \mathrm{x} \mathrel{+}= 1$

    $\mathrm{k} \mathrel{-}= 1 \quad \mathrm{i} \mathrel{+}= 1$

# The main loop

Invariants
$i + k = B, \quad b \cdot c = d$

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**loop**

  **loop**

    $c \mathrel{-}= \boxed{i} \quad c' \mathrel{+}= 1$

    **loop at most** $b$ **times**

      $d \mathrel{-}= \boxed{i} \quad d' \mathrel{+}= \boxed{i+1} \quad x \mathrel{-}= \boxed{i} \quad x' \mathrel{+}= \boxed{i+1}$

  **loop**

    $b \mathrel{-}= 1 \quad b' \mathrel{+}= \boxed{i+1}$

  **loop**

    $b' \mathrel{-}= 1 \quad b \mathrel{+}= 1$

  **loop**

    $c' \mathrel{-}= 1 \quad c \mathrel{+}= 1$

    **loop at most** $b$ **times**

      $d' \mathrel{-}= 1 \quad d \mathrel{+}= 1 \quad x' \mathrel{-}= 1 \quad x \mathrel{+}= 1$

  $k \mathrel{-}= 1 \quad i \mathrel{+}= 1$

$c' := c \cdot \frac{1}{i}$, $d' := d \cdot \frac{i+1}{i}$

$b' := b \cdot (i+1)$

if any **loop** not maximal
then $x < y \cdot B$

# Conclusion

## Conclusion

- Several applications and corollaries
  - coverability is different from reachability

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So maybe it's good to study restrictions of generalizations of etc...