Lecture 3

Ambiguity of automata

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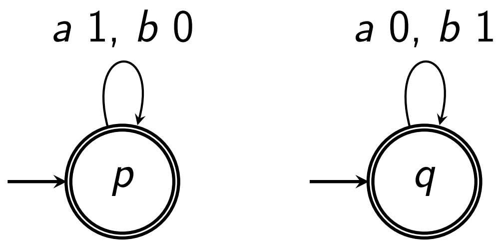
How many accepting runs are there for each word?

- Maximum of number of a's and number of b's



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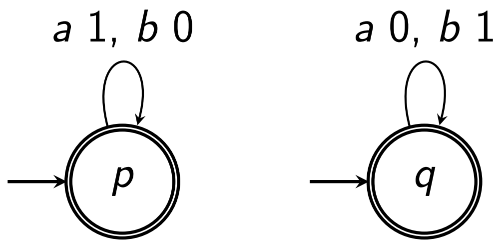

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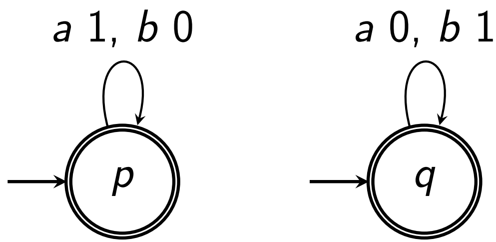

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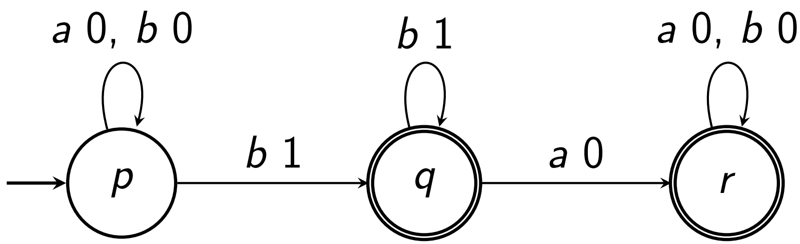

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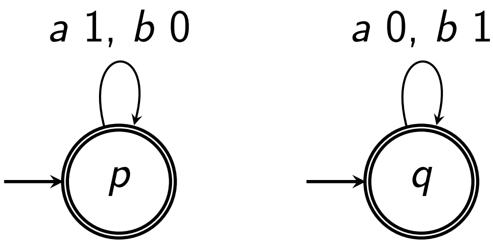

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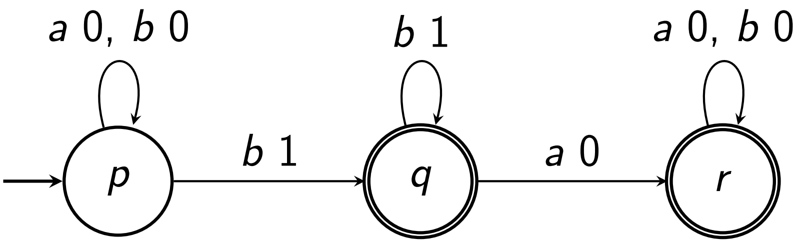
  $\mathcal{O}(|w|)$ runs



Ambiguity subclasses

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Hierarchy of classes for weighted automata

Weighted automata (WA)

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- We will focus mostly on two classes

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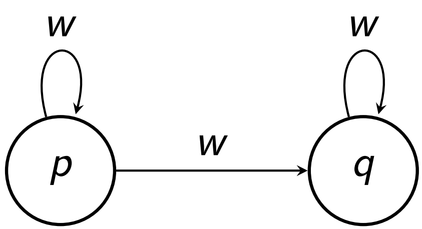

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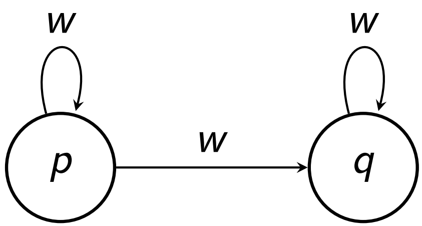

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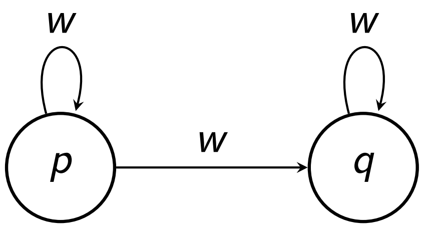

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Then $|Acc(v_a w^n v_b)| \geqslant n - 1$.

Not finitely ambiguous $\implies$ pattern (1)

- A strongly connected component is $Q_i \subseteq Q$ s.t. for all $p, q \in Q$ there are v_1, v_2: $p \xrightarrow{v_1} q$ and $q \xrightarrow{v_2} p$

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So sets G is a DAG and Q_a can be topologically sorted

■

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\begin{tikzcd}
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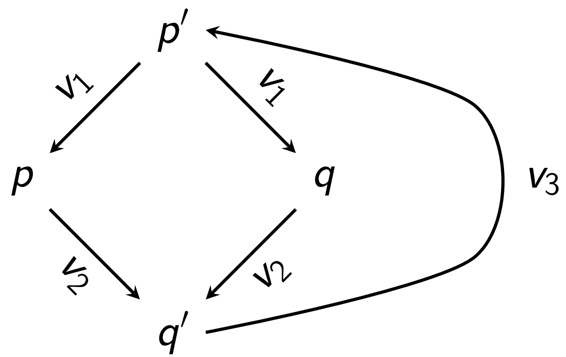

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- Let $w = v_2 v_3 v_1$

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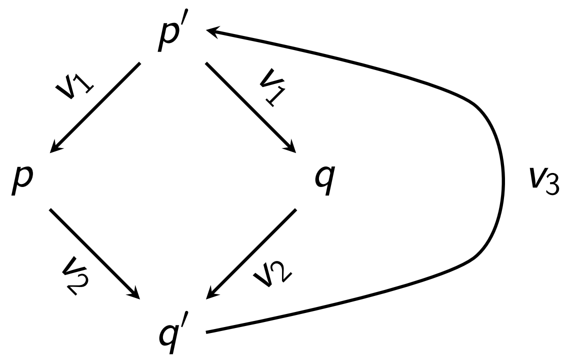

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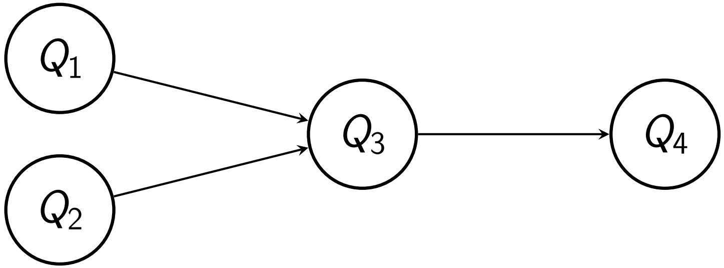

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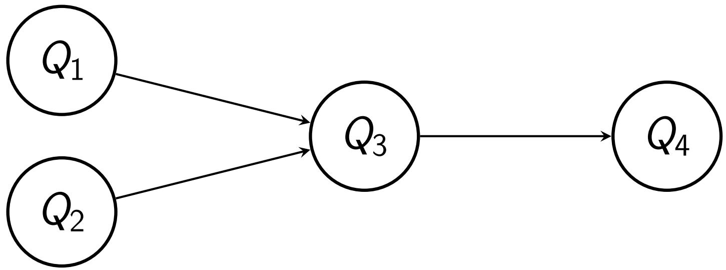

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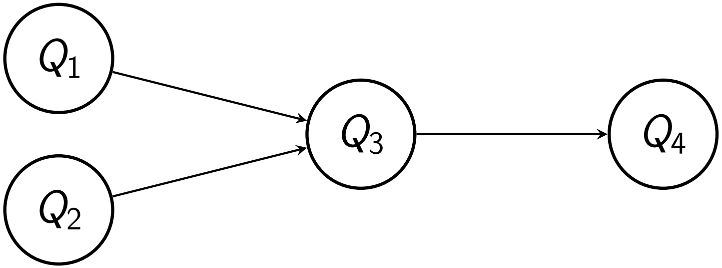

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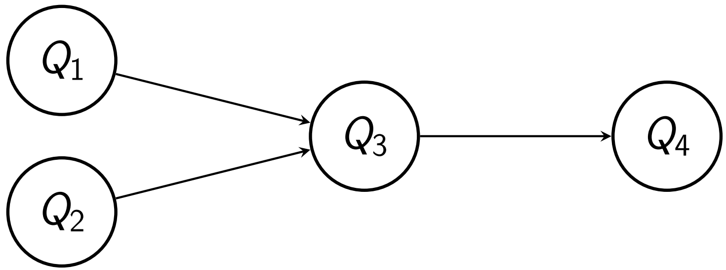

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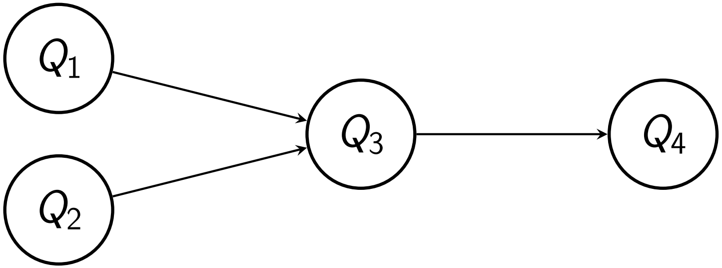

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- Note: the number of such M is bounded by $2^{|Q|}$.

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$V = \{(q, j) \in (Q_1 \cup \ldots \cup Q_k) \times \{0, \ldots, s\} \mid$

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If $\mathcal{A}$ is not finitely ambiguous there is a word $x_1 \ldots x_s$ s.t. $Acc(x) \geqslant N$

For any N (in the end we choose N big enough)

Not finitely ambiguous $\implies$ pattern (4)

$Q = Q_1 \cup Q_2 \cup \ldots \cup Q_m$

When $M = \{i_1, \ldots, i_k\}$ write $Q_1, \ldots Q_k$ instead of $Q_{i_1}, \ldots Q_{i_k}$

Definition

Let $x = x_1 \ldots x_s \in \Sigma^*$. The graph $G_M(x) = (V, E)$ is defined as

$V = \{(q,j) \in (Q_1 \cup \ldots \cup Q_k) \times \{0, \ldots, s\} \mid$

$\exists q_I \in I \cap Q_1, q_F \in F \cap Q_k : q_I \xrightarrow{x_1 \ldots x_j} q, q \xrightarrow{x_{j+1} \ldots x_s} q_F\}$

$E = \{(p, j-1) \to (q,j) \mid p \xrightarrow{x_j} q\}$

- Case 2: for all i if $p', q' \in Q_i$ then for every v at most one run $p' \xrightarrow{v} q'$

If $\mathcal{A}$ is not finitely ambiguous there is a word $x_1 \ldots x_s$ s.t. $Acc(x) \geqslant N$

For any N (in the end we choose N big enough)

Then there is an M s.t. the number of accepting runs by M is at least $\frac{N}{2^{|Q|}}$

Not finitely ambiguous $\implies$ pattern (5)

$M = \{i_1, \dots i_k\}$, $\quad x = x_1 \dots x_s$, $\quad Acc(x)$ from Q_1 to Q_k big

$$Q_1 \longrightarrow Q_2 \longrightarrow \cdots \longrightarrow Q_k$$

Not finitely ambiguous $\implies$ pattern (5)

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- for all $l = 1, \dots, k-1$ let $D_l(x) \subseteq E$ s.t.

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- Number of accepting runs on x is bounded by

$$|D_1| \cdot |D_2| \cdot \ldots \cdot |D_{k-1}| \cdot |Q_k|$$

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- We choose N s.t. $|D_l| > 2^{|Q|}$

Not finitely ambiguous $\implies$ pattern (6)

$(p, j-1) \to (q, j) \in D_l(x)$ if $p \in Q_l$ and $q \in Q_{l+1}$,, $\quad |D_l| > 2^{|Q|}$, $\quad x = x_1 \ldots x_s$

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- There are: $q_l \in Q_l, p_{l+1} \in Q_{l+1}$, $\quad$ s.t. $q_l \xrightarrow{a_1} p_{l+1}$ $\quad$ and $r \in A$ s.t. $r \xrightarrow{y_1} q_l$

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- There is a sequence r_i s.t. $r_1 \xrightarrow{y_1} q_l$ and $r_i \xrightarrow{y} r_{i-1}$

  For some i_1, i_2 we get $r_{i_1} = r_{i_1+i_2} = p$

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- Similarly a sequence s_i: $\quad s_0 = p_{l+1}$ and $s_{i-1} \xrightarrow{y} s_i$, $\quad$ so $s_{i_3} = s_{i_3+i_4} = q$

Not finitely ambiguous $\implies$ pattern (7)

$y = y_1 a_1$

$$\begin{tikzcd} p \arrow[loop above, "y^{i_2}"] \arrow[r, "y^{i_1 - 1}"] & q_l \arrow[r, "a_1"] & p_{l+1} \arrow[r, "y^{i_3}"] & q \arrow[loop above, "y^{i_4}"] \end{tikzcd}$$

- $p \xrightarrow{y^{i_2}} p, \quad p \xrightarrow{y^{i_1} y_1} q_l, \quad q_l \xrightarrow{a_1} p_{l+1}, \quad p_{l+1} \xrightarrow{y^{i_3}} q, \quad q \xrightarrow{y^{i_4}} q$

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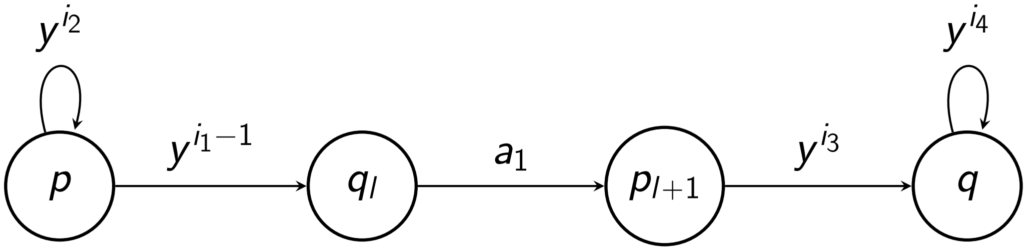

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- Given i_1, i_2 we can choose i_3 and i_4 s.t. $i_1 + i_3 \equiv 0 \mod i_2 \cdot i_4$
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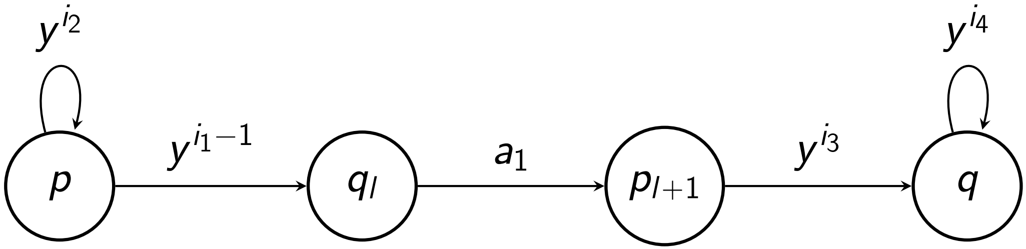

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- $p \neq q, \quad$ let $w = y^{j \cdot i_2 \cdot i_4}$

  $p \xrightarrow{w} p,\ p \xrightarrow{w} q,\ q \xrightarrow{w} q$

$\blacksquare$

Polynomially ambiguous

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Theorem (Weber, Seidl 1991)

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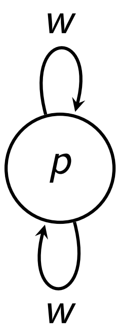

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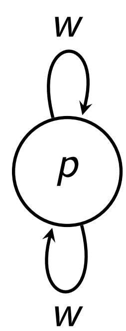

Proof. **(2) $\Longrightarrow$ (1)**

There is $a \in I$ and v_1 s.t. $a \xrightarrow{v_a} p$ and $b \in F$ and v_2 s.t. $p \xrightarrow{v_b} b$

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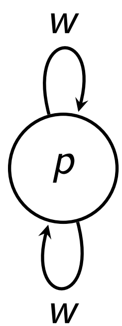

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Then $|Acc(v_a w^n v_b)| \geqslant 2^n$.

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Finitely ambiguous class more details

Lemma (tutorials)

If $\mathcal{A}$ is finitely ambiguous then it is k-ambiguous for some k bounded exponentially in $|Q|$.

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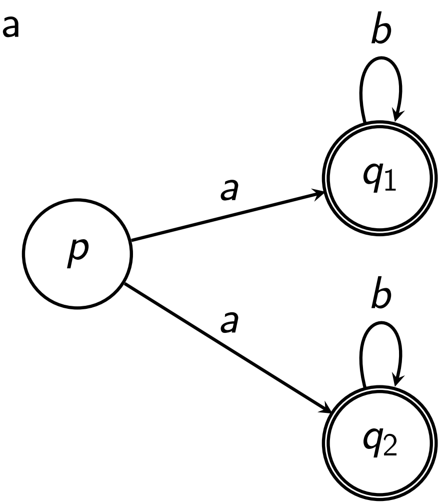

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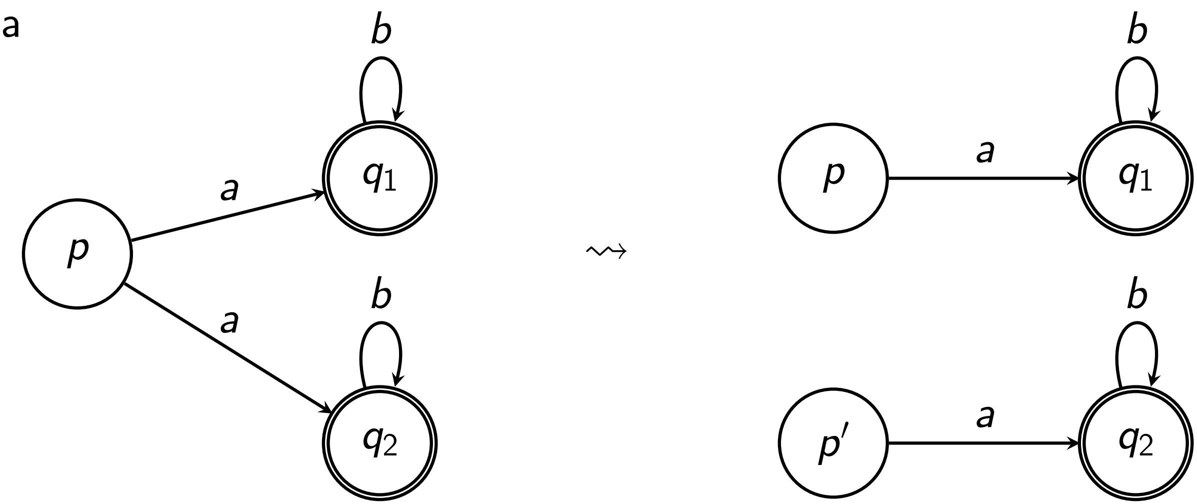

Construction of the union of unambiguous automata

Proof (sketch).

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- $\mathcal{B}$ keeps track of all active runs in $\mathcal{A}$

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- $\mathcal{B}$ states are partial functions $f : \{1, 2, \ldots, kn\} \to Q$

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- Then $\mathcal{A} \times \mathcal{B}$ has the same accepting runs as $\mathcal{A}$ but with extra information

  Accepting states etc are when they are accepting in the $\mathcal{A}$ component

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- This adds the information, which $\{1, \dots, kn\}$ has the current run in $\mathcal{A}$
  It can be extracted when updating $\mathcal{B}$
- The final automata are divided into kn unambiguous automata
  Restricting the accepting states to accepting in $\mathcal{A}$
  and $i \in \{1, \dots, kn\}$ in the final component