Pumping lemmas for weighted automata

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STACS 2018

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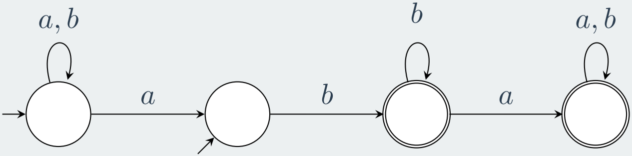
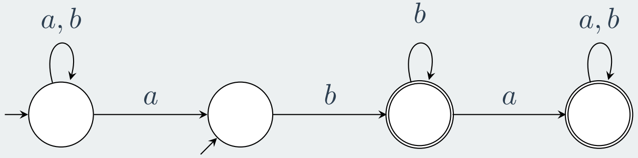
Automata

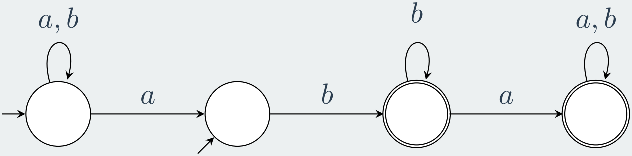


Automata



$f : \Sigma^* \to \{0, 1\}$

Automata



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Weighted automata

$f : \Sigma^* \to$ "some numbers"?

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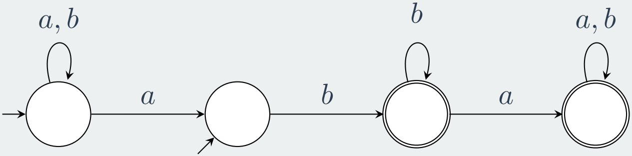

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$f : \Sigma^* \to$ "some numbers"? $\mathbb{N}$?

Weighted automata (WA)

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$

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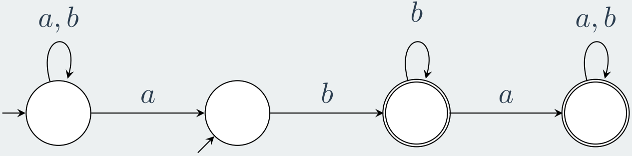

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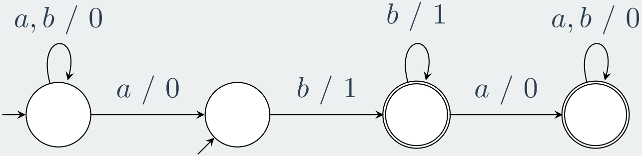

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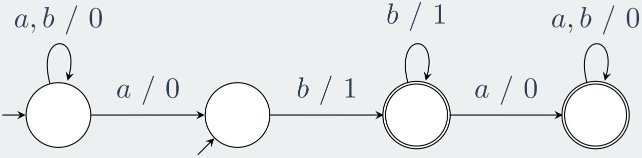

Consider $w = bbab$

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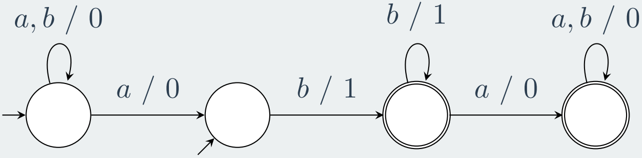

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b	b	a	b
1	1	0	0

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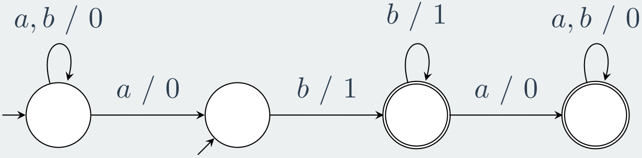

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b b a b

$1 + 1 + 0 + 0 = 2$

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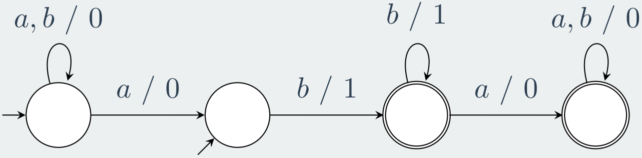

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b	b	a	b		b	b	a	b
1 +	1 +	0 +	0 = 2		0	0	0	1

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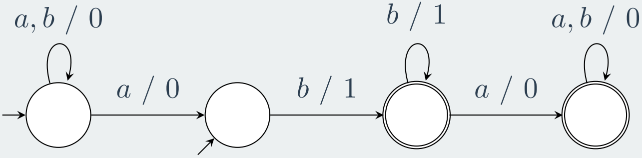

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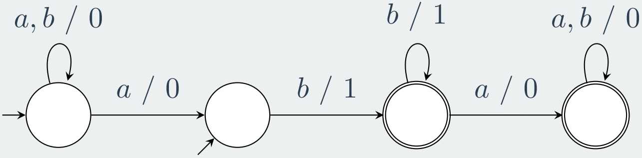

Consider $w = bbab$

b b a b	b b a b
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Output: $\min\{2, 1\} = 1$

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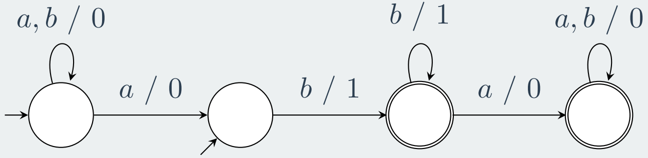

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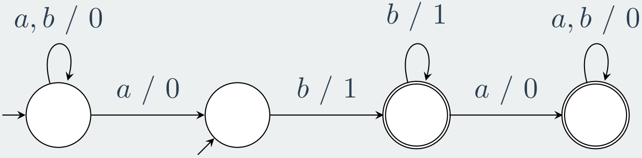

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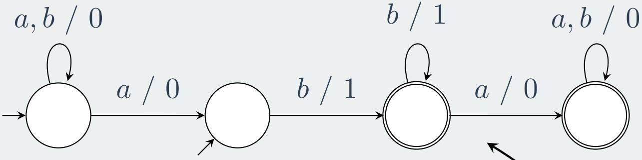

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"smallest block of b's"

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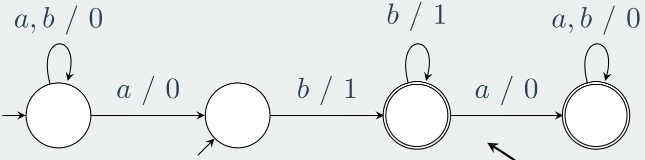

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"smallest block of b's"

(∞ if there is no b)

WA subclasses

Number of accepting runs?

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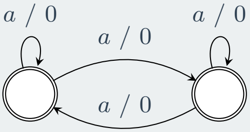
- could be exponential

accepting runs: 2^n (for a^n)



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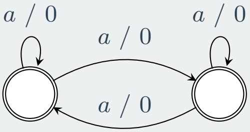

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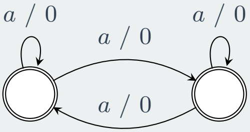

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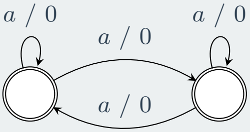

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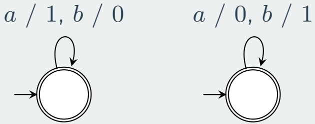
accepting runs: $|\Sigma|$ (constant)



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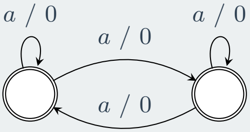

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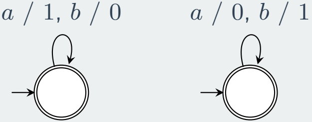
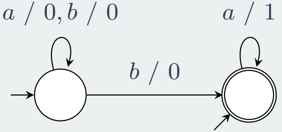


- f longest suffix of a's; $f(abaa) = 2$

accepting runs: 1



Hierarchy of subclasses

Fix $\mathbb{N}_\infty(\min, +, \infty, 0)$

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State of art

$$\begin{array}{c} \text{WA} \\ \cup\!\!\!\!\not\; \\ \text{polynomially ambiguous WA} \\ \cup\!\!\!\!\not\; \\ \text{finitely ambiguous WA} \\ \cup\!\!\!\!\not\; \\ \text{unambiguous WA} \\ \cup\!\!\!\!\not\; \\ \text{deterministic WA} \end{array}$$

Hierarchy of subclasses

Fix $\mathbb{N}_\infty(\min, +, \infty, 0)$

State of art

WA

$\cup\!\!\!\!\not\;$

polynomially ambiguous WA

$\cup\!\!\!\!\not\;$

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unambiguous WA [Klimann et al., 2004]

$\cup\!\!\!\!\not\;$

deterministic WA

Hierarchy of subclasses

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$\supsetneq$

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$\supsetneq$ [Kirsten, 2008]

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$\supsetneq$

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deterministic WA

- Strictness shown by examples

Hierarchy of subclasses

Fix $\mathbb{N}_\infty(\min, +, \infty, 0)$

State of art

WA

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finitely ambiguous WA

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unambiguous WA

$\rotatebox{90}{$\supsetneq$}$ ← [Klimann et al., 2004]

deterministic WA

- Strictness shown by examples
- Papers are about determinization

Meanwhile other formalisms

Boolean world

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Boolean world

- Finite automata

Show that $L = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

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Take $w \in L$ big enough

exists a decomposition $w = xyz$, $|y| > 0$

s.t. $xy^n z \in L$ for all n

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quick case analysis

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quick case analysis

- Context-free languages – pumping lemmas
- First order logic – Ehrenfeucht-Fraïssé games

On which fragments we work?

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$

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$\supsetneq$

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No $\min$ fragment

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Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$

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finitely ambiguous WA $\longrightarrow$ finite U-WA $\longrightarrow$ finite $\min$

$\supsetneq$

unambiguous WA $\longrightarrow$ No $\min$ fragment

On which fragments we work?

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$

$$\begin{tikzcd}[column sep=small]
\text{WA} & & \\
\text{polynomially ambiguous WA} \arrow[u, phantom, "\supsetneq" sloped] \arrow[ddddrr, bend left] & & \\
\text{finitely ambiguous WA} \arrow[u, phantom, "\supsetneq" sloped] \arrow[ddr] & & \\
\text{unambiguous WA} \arrow[u, phantom, "\supsetneq" sloped] \arrow[dd] & & \\
& \text{finite U-WA} \arrow[d] & \\
\text{No } \min \text{ fragment} & \text{finite } \min & \min \text{ of patterns}
\end{tikzcd}$$

The no $\min$ fragment

Unambiguous WA (U-WA) over $\mathbb{N}_\infty(\min, +, \infty, 0)$

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Theorem (Pumping Lemma 1)

Let f recognizable by an U-WA over $(\min, +)$

there exists N s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

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- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \; = \; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$;
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Example: f – longest suffix of a's

Take any $u \cdot v \cdot w$, $N = 1$, and trivial refinement

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- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \;<\; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$. $\longleftarrow$ $v, w \in a^*$

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A refinement is $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w} = u \cdot \underline{v} \cdot w$ s.t. $\underline{\hat{v}}$ is a fragment of $\underline{v}$

e.g. $aaa\underline{bbbb}aa$ is refined by $aaab\underline{bb}baa$

Theorem (Pumping Lemma 1) ⟵ works for a broader class

Let f recognizable by an U-WA over $(\min, +)$

there exists N s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \; = \; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$; ⟵ otherwise
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \; < \; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$. ⟵ $v, w \in a^*$

Example: f – longest suffix of a's

Take any $u \cdot v \cdot w$, $N = 1$, and trivial refinement

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Corollary: U-WA $\subsetneq$ FA-WA over $(\min, +)$

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Word n-representation: $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n$

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Example, a $(3, 2)$-representation

$w = a\underline{b^3}aa\underline{b^2}a\underline{b^2}aa$

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$w = a\underline{b^3}aa\underline{b^2}a\underline{b^2}aa$

$w(\{1, 3\}, 3) = a\underline{b^9}aa\underline{b^2}a\underline{b^6}aa$

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Corollary: FA-WA $\subsetneq$ PA-WA over $(\min, +)$

Polynomially-ambiguous over $(\min, +)$

over $\mathbb{N}_\infty(\min, +, \infty, 0)$

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Negative example: let $w = w_0\#w_1\#\ldots\#w_m$, where $w_i \in \{a,b\}^*$

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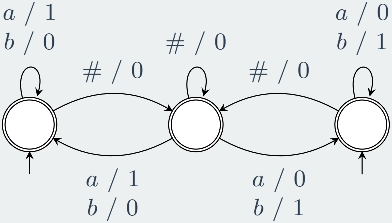

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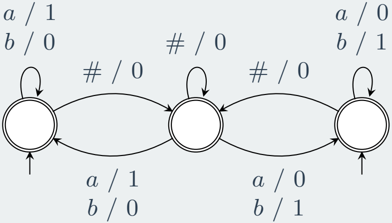

Number of runs: 2^m

Pumping Lemma 3

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Theorem (Pumping lemma 3)

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