# Pumping lemmas for weighted automata

Filip Mazowiecki[1] and Cristian Riveros[2]

[1]University of Bordeaux

[2]Pontificia Universidad Católica de Chile

ULB 2018

# Pumping lemmas for weighted automata

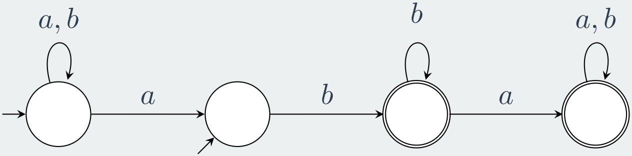
Filip Mazowiecki[1] and Cristian Riveros[2]

[1]University of Bordeaux

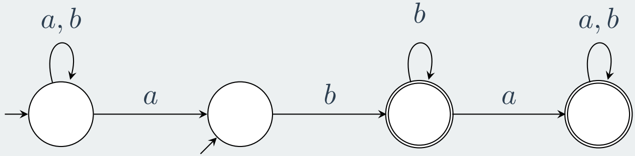
[2]Pontificia Universidad Católica de Chile

ULB 2018

# Automata

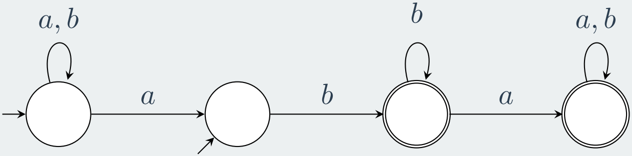


# Automata




$f : \Sigma^* \to \{0, 1\}$

## Automata

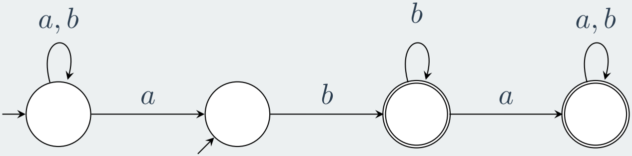



$f : \Sigma^* \to \{0, 1\}$

## Weighted automata

$f : \Sigma^* \to$ "some numbers"?

# Automata




$f : \Sigma^* \to \{0, 1\}$

# Weighted automata

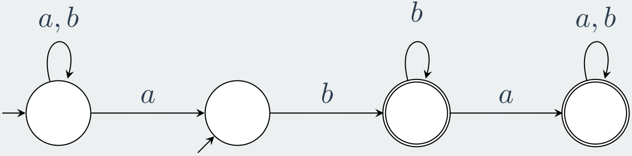
$f : \Sigma^* \to$ "some numbers"? $\mathbb{N}$?

## Weighted automata (WA)

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$

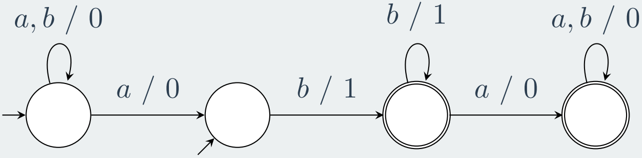
# Weighted automata (WA)

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$



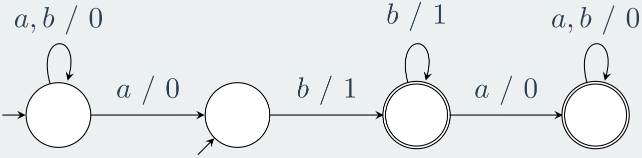
# Weighted automata (WA)

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$



# Weighted automata (WA)

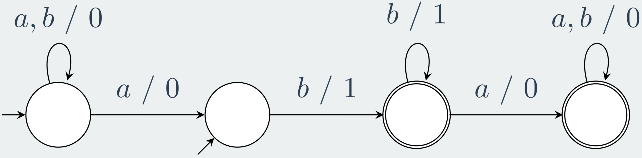
Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$




Consider $w = bbab$

# Weighted automata (WA)

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$

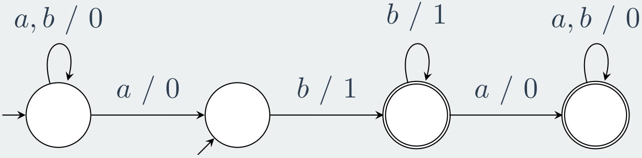



Consider $w = bbab$

| | | | |
|---|---|---|---|
| b | b | a | b |
| 1 | 1 | 0 | 0 |

# Weighted automata (WA)

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$



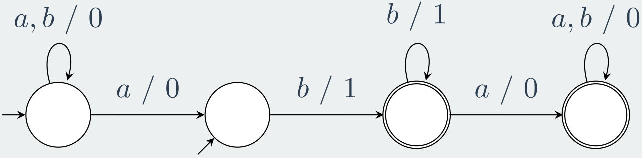

Consider $w = bbab$

b b a b

$1 + 1 + 0 + 0 = 2$

# Weighted automata (WA)

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$

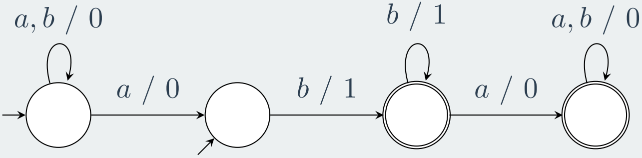



Consider $w = bbab$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| b | b | a | b | | b | b | a | b |
| $1 + 1 + 0 + 0 = 2$ | | | | | 0 | 0 | 0 | 1 |

# Weighted automata (WA)

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$

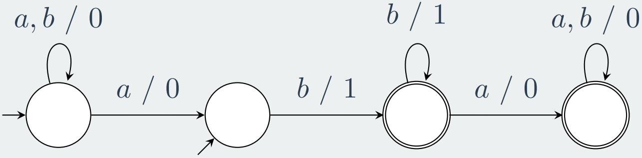



Consider $w = bbab$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| b | b | a | b | | b | b | a | b |
| $1 + 1 + 0 + 0 = 2$ | | | | | $0 + 0 + 0 + 1 = 1$ | | | |

# Weighted automata (WA)

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$




Consider $w = bbab$

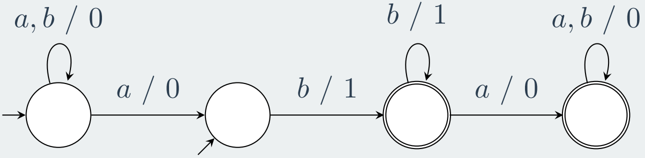
| | | | | | | | |
|---|---|---|---|---|---|---|---|
| b | b | a | b | b | b | a | b |

$1 + 1 + 0 + 0 = 2$ $\qquad$ $0 + 0 + 0 + 1 = 1$

Output: $\min\{2, 1\} = 1$

# Weighted automata (WA)

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$




Consider $w = bbab$

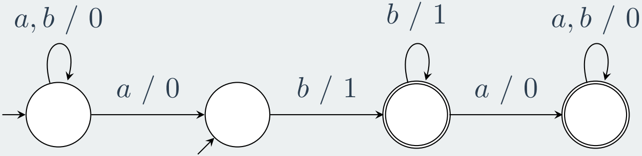
| b | b | a | b | | b | b | a | b |
|---|---|---|---|---|---|---|---|---|
| $1 + 1 + 0 + 0 = 2$ | | | | | $0 + 0 + 0 + 1 = 1$ | | | |

Output: $\min\{2, 1\} = 1$

In general: $\odot$ transitions, $\oplus$ accepting runs

# Weighted automata (WA)

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$




Consider $w = bbab$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| b | b | a | b | | b | b | a | b |
| 1 + 1 + 0 + 0 = 2 | | | | | 0 + 0 + 0 + 1 = 1 | | | |

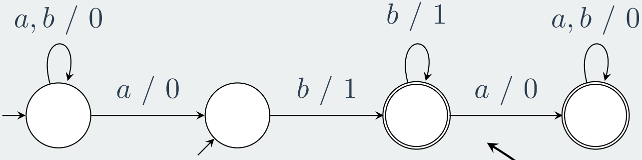
Output: $\min\{2, 1\} = 1$

In general: $\odot$ transitions, $\oplus$ accepting runs

$\mathbb{0}$ if there is no accepting run

## Weighted automata (WA)

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$




Consider $w = bbab$

| b | b | a | b | | b | b | a | b |
|---|---|---|---|---|---|---|---|---|

$1 + 1 + 0 + 0 = 2$ $\qquad$ $0 + 0 + 0 + 1 = 1$

Output: $\min\{2, 1\} = 1$

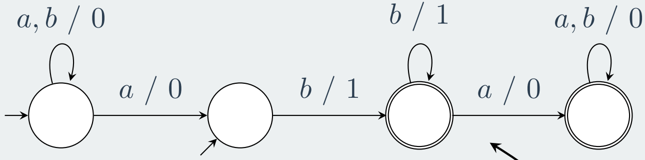
In general: $\odot$ transitions, $\oplus$ accepting runs

$\mathbb{0}$ if there is no accepting run

"smallest block of b's"

## Weighted automata (WA)

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$




Consider $w = bbab$

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| b | b | a | b | | b | b | a | b |
| $1 + 1 + 0 + 0 = 2$ | | | | | $0 + 0 + 0 + 1 = 1$ | | | |

Output: $\min\{2, 1\} = 1$

In general: $\odot$ transitions, $\oplus$ accepting runs

$\mathbb{0}$ if there is no accepting run

"smallest block of b's"

($\infty$ if there is no $b$)

# WA subclasses

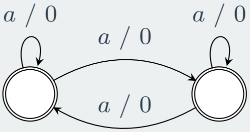
Number of accepting runs?

## WA subclasses

Number of accepting runs?

- could be exponential

accepting runs: $2^n$ (for $a^n$)

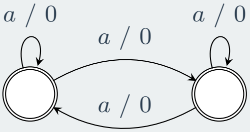


## WA subclasses

Number of accepting runs?

- could be exponential

accepting runs: $2^n$ (for $a^n$)

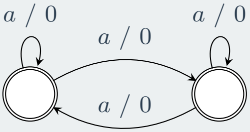



- "smallest block of $b$'s"

accepting runs: blocks of $b$'s (linear)

## WA subclasses

Number of accepting runs?




- could be exponential

accepting runs: $2^n$ (for $a^n$)

- "smallest block of $b$'s"

accepting runs: blocks of $b$'s (linear)

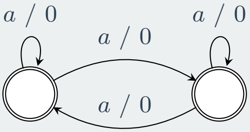
- $\min_{a \in \Sigma}\{$ number of $a$'s $\}$?

## WA subclasses

Number of accepting runs?

- could be exponential

accepting runs: $2^n$ (for $a^n$)

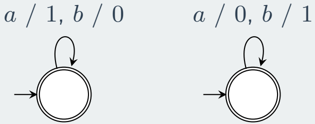



- "smallest block of $b$'s"

accepting runs: blocks of $b$'s (linear)

- $\min_{a \in \Sigma}\{$ number of $a$'s $\}$?

accepting runs: $|\Sigma|$ (constant)

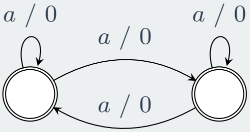


## WA subclasses

Number of accepting runs?

- could be exponential

accepting runs: $2^n$ (for $a^n$)

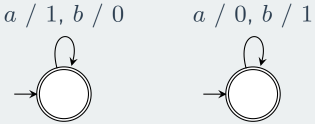



- "smallest block of $b$'s"

accepting runs: blocks of $b$'s (linear)

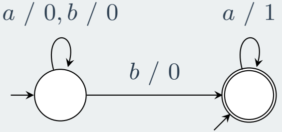
- $\min_{a \in \Sigma}\{$ number of $a$'s $\}$?

accepting runs: $|\Sigma|$ (constant)




- $f$ longest suffix of $a$'s; $f(abaa) = 2$

accepting runs: 1



## Hierarchy of subclasses

Fix $\mathbb{N}_\infty(\min, +, \infty, 0)$

## Hierarchy of subclasses

Fix $\mathbb{N}_\infty(\min, +, \infty, 0)$

State of art

$$\begin{array}{c}\text{WA}\\ \subsetneq\\ \text{polynomially ambiguous WA}\\ \subsetneq\\ \text{finitely ambiguous WA}\\ \subsetneq\\ \text{unambiguous WA}\\ \subsetneq\\ \text{deterministic WA}\end{array}$$

## Hierarchy of subclasses

Fix $\mathbb{N}_\infty(\min, +, \infty, 0)$

State of art

$$
\begin{array}{c}
\text{WA} \\
\cup\!\!\!\!\!\not\; \\
\text{polynomially ambiguous WA} \\
\cup\!\!\!\!\!\not\; \\
\text{finitely ambiguous WA} \\
\cup\!\!\!\!\!\not\; \\
\text{unambiguous WA} \\
\cup\!\!\!\!\!\not\; \\
\text{deterministic WA}
\end{array}
$$

[Klimann et al., 2004]

## Hierarchy of subclasses

Fix $\mathbb{N}_\infty(\min, +, \infty, 0)$

State of art

WA

$\supsetneq$

polynomially ambiguous WA

$\supsetneq$ [Kirsten, 2008]

finitely ambiguous WA

$\supsetneq$

unambiguous WA [Klimann et al., 2004]

$\supsetneq$

deterministic WA

## Hierarchy of subclasses

Fix $\mathbb{N}_\infty(\min, +, \infty, 0)$

State of art

WA

$\cup\!\!\!\!\nmid$ ??? 

polynomially ambiguous WA

$\cup\!\!\!\!\nmid$ [Kirsten, 2008]

finitely ambiguous WA

$\cup\!\!\!\!\nmid$

unambiguous WA [Klimann et al., 2004]

$\cup\!\!\!\!\nmid$

deterministic WA

## Hierarchy of subclasses

Fix $\mathbb{N}_\infty(\min, +, \infty, 0)$

State of art

WA

$\cup\!\!\!\not\;$ $\longleftarrow$ ???

polynomially ambiguous WA

$\cup\!\!\!\not\;$ $\longleftarrow$ [Kirsten, 2008]

finitely ambiguous WA

$\cup\!\!\!\not\;$

unambiguous WA [Klimann et al., 2004]

$\cup\!\!\!\not\;$

deterministic WA

- Strictness shown by examples

## Hierarchy of subclasses

Fix $\mathbb{N}_\infty(\min, +, \infty, 0)$

State of art

WA

$\supsetneq \longleftarrow$ ???

polynomially ambiguous WA

$\supsetneq \longleftarrow$ [Kirsten, 2008]

finitely ambiguous WA

$\supsetneq$

unambiguous WA [Klimann et al., 2004]

$\supsetneq$

deterministic WA

- Strictness shown by examples
- Papers are about determinization

# Meanwhile other formalisms

Boolean world

## Meanwhile other formalisms

Boolean world

- Finite automata

Show that $L = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

## Meanwhile other formalisms

Boolean world

- Finite automata

Show that $L = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

Solution: pumping lemma

## Meanwhile other formalisms

Boolean world

- Finite automata

Show that $L = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

Solution: pumping lemma

Take $w \in L$ big enough

## Meanwhile other formalisms

Boolean world

- Finite automata

Show that $L = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

Solution: pumping lemma

Take $w \in L$ big enough

exists a decomposition $w = xyz$, $|y| > 0$

s.t. $xy^n z \in L$ for all $n$

## Meanwhile other formalisms

Boolean world

- Finite automata

Show that $L = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

Solution: pumping lemma

Take $w \in L$ big enough

exists a decomposition $w = xyz$, $|y| > 0$

s.t. $xy^n z \in L$ for all $n$

quick case analysis

## Meanwhile other formalisms

Boolean world

- Finite automata

Show that $L = \{a^n b^n \mid n \in \mathbb{N}\}$ is not regular.

Solution: pumping lemma

Take $w \in L$ big enough

exists a decomposition $w = xyz$, $|y| > 0$

s.t. $xy^n z \in L$ for all $n$

quick case analysis

- Context-free languages – pumping lemmas
- First order logic – Ehrenfeucht-Fraïssé games

## On which fragments we work?

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$

# On which fragments we work?

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$

WA

$\cup\!\!\nmid$

polynomially ambiguous WA

$\cup\!\!\nmid$

finitely ambiguous WA

$\cup\!\!\nmid$

unambiguous WA

## On which fragments we work?

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$

WA

$\supsetneq$

polynomially ambiguous WA

$\supsetneq$

finitely ambiguous WA

$\supsetneq$

unambiguous WA

No $\min$ fragment

## On which fragments we work?

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$

WA

$\supsetneq$

polynomially ambiguous WA

$\supsetneq$

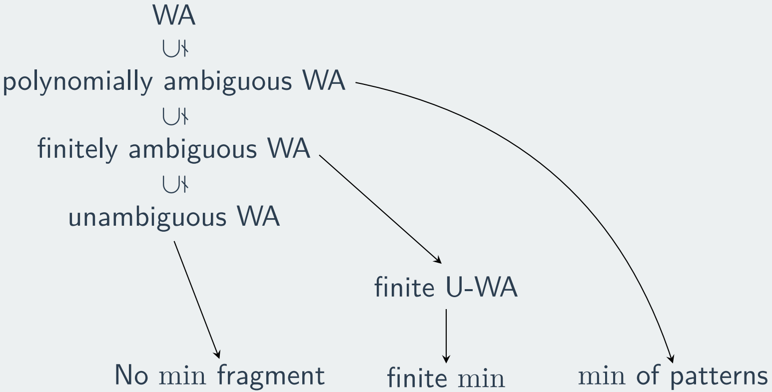
finitely ambiguous WA $\longrightarrow$ finite U-WA $\longrightarrow$ finite $\min$

$\supsetneq$

unambiguous WA $\longrightarrow$ No $\min$ fragment

## On which fragments we work?

Semiring: $\mathbb{N}_\infty(\min, +, \infty, 0)$



## The no $\min$ fragment

Unambiguous WA (U-WA) over $\mathbb{N}_\infty(\min, +, \infty, 0)$

## The no $\min$ fragment

Unambiguous WA (U-WA) over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Let $u \cdot v \cdot w \in \Sigma^*$

## The no $\min$ fragment

Unambiguous WA (U-WA) over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Let $u \cdot \underline{v} \cdot w \in \Sigma^*$

## The no $\min$ fragment

Unambiguous WA (U-WA) over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Let $u \cdot \underline{v} \cdot w \in \Sigma^*$

A refinement is $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w} = u \cdot \underline{v} \cdot w$ s.t. $\underline{\hat{v}} \neq \epsilon$ is a fragment of $\underline{v}$

## The no $\min$ fragment

Unambiguous WA (U-WA) over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Let $u \cdot \underline{v} \cdot w \in \Sigma^*$

A refinement is $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w} = u \cdot \underline{v} \cdot w$ s.t. $\underline{\hat{v}} \neq \epsilon$ is a fragment of $\underline{v}$

e.g. $aaa\underline{bbbb}aa$ is refined by $aaab\underline{bb}baa$

## The no $\min$ fragment

Unambiguous WA (U-WA) over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Let $u \cdot \underline{v} \cdot w \in \Sigma^*$

A refinement is $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w} = u \cdot \underline{v} \cdot w$ s.t. $\underline{\hat{v}} \neq \epsilon$ is a fragment of $\underline{v}$

e.g. $aaa\underline{bbbb}aa$ is refined by $aaab\underline{bb}baa$

**Theorem** (Pumping Lemma 1)

Let $f$ recognizable by an U-WA over $(\min, +)$

there exists $N$ s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

## The no $\min$ fragment

Unambiguous WA (U-WA) over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Let $u \cdot \underline{v} \cdot w \in \Sigma^*$

A refinement is $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w} = u \cdot \underline{v} \cdot w$ s.t. $\underline{\hat{v}} \neq \epsilon$ is a fragment of $\underline{v}$

e.g. $aaa\underline{bbbb}aa$ is refined by $aaab\underline{bb}baa$

**Theorem** (Pumping Lemma 1)

Let $f$ recognizable by an U-WA over $(\min, +)$

there exists $N$ s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \ = \ f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$;
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \ < \ f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.

## The no $\min$ fragment

Unambiguous WA (U-WA) over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Let $u \cdot \underline{v} \cdot w \in \Sigma^*$

A refinement is $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w} = u \cdot \underline{v} \cdot w$ s.t. $\underline{\hat{v}} \neq \epsilon$ is a fragment of $\underline{v}$

e.g. $aaa\underline{bbbb}aa$ is refined by $aaab\underline{bb}baa$

**Theorem** (Pumping Lemma 1)

Let $f$ recognizable by an U-WA over $(\min, +)$

there exists $N$ s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \; = \; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$;
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \; < \; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.

Example: $f$ – longest suffix of $a$'s

Take any $u \cdot v \cdot w$, $N = 1$, and trivial refinement

## The no $\min$ fragment

Unambiguous WA (U-WA) over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Let $u \cdot \underline{v} \cdot w \in \Sigma^*$

A refinement is $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w} = u \cdot \underline{v} \cdot w$ s.t. $\underline{\hat{v}} \neq \epsilon$ is a fragment of $\underline{v}$

e.g. $aaa\underline{bbbb}aa$ is refined by $aaab\underline{bb}baa$

**Theorem** (Pumping Lemma 1)

Let $f$ recognizable by an U-WA over $(\min, +)$

there exists $N$ s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) = f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$;
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) < f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$. $\longleftarrow$ $v, w \in a^*$

Example: $f$ – longest suffix of $a$'s

Take any $u \cdot v \cdot w$, $N = 1$, and trivial refinement

## The no $\min$ fragment

Unambiguous WA (U-WA) over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Let $u \cdot \underline{v} \cdot w \in \Sigma^*$

A refinement is $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w} = u \cdot \underline{v} \cdot w$ s.t. $\underline{\hat{v}} \neq \epsilon$ is a fragment of $\underline{v}$

e.g. $aaa\underline{bbbb}aa$ is refined by $aaab\underline{bb}baa$

**Theorem** (Pumping Lemma 1)

Let $f$ recognizable by an U-WA over $(\min, +)$

there exists $N$ s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) = f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$; $\longleftarrow$ otherwise
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) < f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$. $\longleftarrow$ $v, w \in a^*$

Example: $f$ – longest suffix of $a$'s

Take any $u \cdot v \cdot w$, $N = 1$, and trivial refinement

## The no $\min$ fragment

Unambiguous WA (U-WA) over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Let $u \cdot \underline{v} \cdot w \in \Sigma^*$

A refinement is $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w} = u \cdot \underline{v} \cdot w$ s.t. $\underline{\hat{v}} \neq \epsilon$ is a fragment of $\underline{v}$

e.g. $aaa\underline{bbbb}aa$ is refined by $aaab\underline{bb}baa$

**Theorem** (Pumping Lemma 1) works for a broader class

Let $f$ recognizable by an U-WA over $(\min, +)$

there exists $N$ s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \;=\; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$; otherwise
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \;<\; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$. $v, w \in a^*$

Example: $f$ – longest suffix of $a$'s

Take any $u \cdot v \cdot w$, $N = 1$, and trivial refinement

**Theorem** (Pumping lemma 1)

Let $f$ be a WA over $\mathbb{N}_\infty(\min, +, \infty, 0)$

there exists $N$ s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \; = \; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \; < \; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.

**Theorem** (Pumping lemma 1)

Let $f$ be a WA over $\mathbb{N}_\infty(\min, +, \infty, 0)$

there exists $N$ s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \ = \ f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \ < \ f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.

Let $f(w) = \min(\#_a(w), \#_b(w))$ $\quad$ ($f \in$ FA-WA)

**Theorem** (Pumping lemma 1)

Let $f$ be a WA over $\mathbb{N}_\infty(\min, +, \infty, 0)$

there exists $N$ s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \; = \; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \; < \; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.

Let $f(w) = \min(\#_a(w), \#_b(w))$ $\quad$ ($f \in$ FA-WA)

Let $u \cdot v \cdot w = a^{(N+1)^2}\underline{b^N}$, $\quad f(u \cdot v \cdot w) = N$

**Theorem** (Pumping lemma 1)

Let $f$ be a WA over $\mathbb{N}_\infty(\min, +, \infty, 0)$

there exists $N$ s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \ = \ f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \ < \ f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.

Let $f(w) = \min(\#_a(w), \#_b(w)) \quad (f \in$ FA-WA)

Let $u \cdot v \cdot w = a^{(N+1)^2}\underline{b^N}, \quad f(u \cdot v \cdot w) = N$

Any refinement $\hat{u} \cdot \hat{v} \cdot \hat{w} = a^{(N+1)^2} \cdot b^n \underline{b^m} b^l, \quad 1 \leq m \leq N$

**Theorem** (Pumping lemma 1)

Let $f$ be a WA over $\mathbb{N}_\infty(\min, +, \infty, 0)$

there exists $N$ s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \; = \; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \; < \; f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.

Let $f(w) = \min(\#_a(w), \#_b(w)) \quad (f \in$ FA-WA)

Let $u \cdot v \cdot w = a^{(N+1)^2}\underline{b^N}, \quad f(u \cdot v \cdot w) = N$

Any refinement $\hat{u} \cdot \hat{v} \cdot \hat{w} = a^{(N+1)^2} \cdot b^n \underline{b^m} b^l, \quad 1 \leq m \leq N$

$f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) = (N+1)^2$ for $i$ big enough

**Theorem** (Pumping lemma 1)

Let $f$ be a WA over $\mathbb{N}_\infty(\min, +, \infty, 0)$

there exists $N$ s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \ = \ f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \ < \ f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.

Let $f(w) = \min(\#_a(w), \#_b(w)) \quad (f \in \text{FA-WA})$

Let $u \cdot v \cdot w = a^{(N+1)^2}\underline{b^N}, \quad f(u \cdot v \cdot w) = N$

Any refinement $\hat{u} \cdot \hat{v} \cdot \hat{w} = a^{(N+1)^2} \cdot b^n \underline{b^m} b^l, \quad 1 \leq m \leq N$

$f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) = (N+1)^2$ for $i$ big enough

but take $i = N$, then $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \leq N + mN \leq N + N^2 < (N+1)^2$

**Theorem** (Pumping lemma 1)

Let $f$ be a WA over $\mathbb{N}_\infty(\min, +, \infty, 0)$

there exists $N$ s.t. for every $u \cdot \underline{v} \cdot w$ with $|v| \geq N$

there is a refinement $\hat{u} \cdot \underline{\hat{v}} \cdot \hat{w}$ and either:

- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \ = \ f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.
- $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \ < \ f(\hat{u} \cdot \underline{\hat{v}}^{i+1} \cdot \hat{w})$ for every $i \geq N$.

Let $f(w) = \min(\#_a(w), \#_b(w))$ $\quad$ ($f \in$ FA-WA)

Let $u \cdot v \cdot w = a^{(N+1)^2}\underline{b^N}, \quad f(u \cdot v \cdot w) = N$

Any refinement $\hat{u} \cdot \hat{v} \cdot \hat{w} = a^{(N+1)^2} \cdot b^n \underline{b^m} b^l, \quad 1 \leq m \leq N$

$f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) = (N+1)^2$ for $i$ big enough

but take $i = N$, then $f(\hat{u} \cdot \underline{\hat{v}}^i \cdot \hat{w}) \leq N + mN \leq N + N^2 < (N+1)^2$

Corollary: U-WA $\subsetneq$ FA-WA over $(\min, +)$

# Preparing for the next lemma

Word $n$-representation: $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n$

# Preparing for the next lemma

Word $n$-representation: $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n$

$(n, N)$-representation: $|v_i| \geq N$ for all $i$

## Preparing for the next lemma

Word $n$-representation: $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n$

$(n, N)$-representation: $|v_i| \geq N$ for all $i$

A refinement is $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$
if $y_i$ refine $v_i$

## Preparing for the next lemma

Word $n$-representation: $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n$

$(n, N)$-representation: $|v_i| \geq N$ for all $i$

A refinement is $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$
if $y_i$ refine $v_i$

Let $S \subseteq \{1, \ldots, n\}$, $\underline{y_k}(S, i) = y_k^i$ if $k \in S$ and $\underline{y_k}(S, i) = y_k$ otherwise.

## Preparing for the next lemma

Word $n$-representation: $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n$

$(n, N)$-representation: $|v_i| \geq N$ for all $i$

A refinement is $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$
if $y_i$ refine $v_i$

Let $S \subseteq \{1, \ldots, n\}$, $\underline{y_k}(S, i) = y_k^i$ if $k \in S$ and $\underline{y_k}(S, i) = y_k$ otherwise.
$w(S, i) = u_0 \cdot \underline{y_1}(S, i) \cdot u_1 \cdot \underline{y_2}(S, i) \cdot \ldots u_{n-1} \cdot \underline{y_n}(S, i) \cdot u_n$

## Preparing for the next lemma

Word $n$-representation: $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n$

$(n, N)$-representation: $|v_i| \geq N$ for all $i$

A refinement is $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$
if $y_i$ refine $v_i$

Let $S \subseteq \{1, \ldots, n\}$, $\underline{y_k}(S, i) = y_k^i$ if $k \in S$ and $\underline{y_k}(S, i) = y_k$ otherwise.
$w(S, i) = u_0 \cdot \underline{y_1}(S, i) \cdot u_1 \cdot \underline{y_2}(S, i) \cdot \ldots u_{n-1} \cdot \underline{y_n}(S, i) \cdot u_n$

Example, a $(3, 2)$-representation

$w = a\underline{b^3}aa\underline{b^2}a\underline{b^2}aa$

## Preparing for the next lemma

Word $n$-representation: $w = u_0 \cdot \underline{v_1} \cdot u_1 \cdot \underline{v_2} \cdot \ldots u_{n-1} \cdot \underline{v_n} \cdot u_n$

$(n, N)$-representation: $|v_i| \geq N$ for all $i$

A refinement is $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$
if $y_i$ refine $v_i$

Let $S \subseteq \{1, \ldots, n\}$, $\underline{y_k}(S, i) = y_k^i$ if $k \in S$ and $\underline{y_k}(S, i) = y_k$ otherwise.
$w(S, i) = u_0 \cdot \underline{y_1}(S, i) \cdot u_1 \cdot \underline{y_2}(S, i) \cdot \ldots u_{n-1} \cdot \underline{y_n}(S, i) \cdot u_n$

Example, a $(3, 2)$-representation

$w = a\underline{b^3}aa\underline{b^2}a\underline{b^2}aa$

$w(\{1, 3\}, 3) = a\underline{b^9}aa\underline{b^2}a\underline{b^6}aa$

**Theorem** (Pumping Lemma 2)

**Theorem** (Pumping Lemma 2)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by FA-WA over $(\min, +)$

**Theorem** (Pumping Lemma 2)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by FA-WA over $(\min, +)$

there exists $N$ s.t. for all $(n, N)$-representations $(n \geq N)$

**Theorem** (Pumping Lemma 2)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by FA-WA over $(\min, +)$
there exists $N$ s.t. for all $(n, N)$-representations $(n \geq N)$
exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$
s.t. given $S_1, \ldots, S_N \subseteq \{1 \ldots n\}$ (nonempty, different) either:

**Theorem** (Pumping Lemma 2)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by FA-WA over $(\min, +)$
there exists $N$ s.t. for all $(n, N)$-representations $(n \geq N)$
exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$
s.t. given $S_1, \ldots, S_N \subseteq \{1 \ldots n\}$ (nonempty, different) either:

- exists $j$ s.t. $f(w(S_j, i)) < f(w(S_j, i+1))$;
- exists $j_1 \neq j_2$ s.t. $f(w(S_{j_1} \cup S_{j_2}, i)) = f(w(S_{j_1} \cup S_{j_2}, i+1))$.

**Theorem** (Pumping Lemma 2)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by FA-WA over $(\min, +)$ there exists $N$ s.t. for all $(n, N)$-representations $(n \geq N)$ exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$ s.t. given $S_1, \ldots, S_N \subseteq \{1 \ldots n\}$ (nonempty, different) either:

- exists $j$ s.t. $f(w(S_j, i)) < f(w(S_j, i+1))$;
- exists $j_1 \neq j_2$ s.t. $f(w(S_{j_1} \cup S_{j_2}, i)) = f(w(S_{j_1} \cup S_{j_2}, i+1))$.

Example: $f$ – "smallest block of $b's$"

**Theorem** (Pumping Lemma 2)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by FA-WA over $(\min, +)$

there exists $N$ s.t. for all $(n, N)$-representations ($n \geq N$)

exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$

s.t. given $S_1, \ldots, S_N \subseteq \{1 \ldots n\}$ (nonempty, different) either:

- exists $j$ s.t. $f(w(S_j, i)) < f(w(S_j, i+1))$;
- exists $j_1 \neq j_2$ s.t. $f(w(S_{j_1} \cup S_{j_2}, i)) = f(w(S_{j_1} \cup S_{j_2}, i+1))$.

Example: $f$ – "smallest block of $b's$"

Let $w = (\underline{b^N} a^N)^N \quad (n = N)$

**Theorem** (Pumping Lemma 2)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by FA-WA over $(\min, +)$

there exists $N$ s.t. for all $(n, N)$-representations $(n \geq N)$

exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$

s.t. given $S_1, \ldots, S_N \subseteq \{1 \ldots n\}$ (nonempty, different) either:

- exists $j$ s.t. $f(w(S_j, i)) < f(w(S_j, i+1))$;
- exists $j_1 \neq j_2$ s.t. $f(w(S_{j_1} \cup S_{j_2}, i)) = f(w(S_{j_1} \cup S_{j_2}, i+1))$.

Example: $f$ – "smallest block of $b's$"

Let $w = (\underline{b^N} a^N)^N \quad (n = N)$

Let $S_j = \{1, \ldots, N\} \setminus \{j\}, \quad f(w(S_j, i)) = N$ for all $i, j$

**Theorem** (Pumping Lemma 2)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by FA-WA over $(\min, +)$
there exists $N$ s.t. for all $(n, N)$-representations $(n \geq N)$
exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$
s.t. given $S_1, \ldots, S_N \subseteq \{1 \ldots n\}$ (nonempty, different) either:

- exists $j$ s.t. $f(w(S_j, i)) < f(w(S_j, i+1))$;
- exists $j_1 \neq j_2$ s.t. $f(w(S_{j_1} \cup S_{j_2}, i)) = f(w(S_{j_1} \cup S_{j_2}, i+1))$.

Example: $f$ – "smallest block of $b's$"

Let $w = (\underline{b^N} a^N)^N \quad (n = N)$

Let $S_j = \{1, \ldots, N\} \setminus \{j\}, \quad f(w(S_j, i)) = N$ for all $i, j$

But $S_{j_1} \cup S_{j_2} = \{1, \ldots, N\}$ for $j_1 \neq j_2$

**Theorem** (Pumping Lemma 2)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by FA-WA over $(\min, +)$

there exists $N$ s.t. for all $(n, N)$-representations $(n \geq N)$

exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$

s.t. given $S_1, \ldots, S_N \subseteq \{1 \ldots n\}$ (nonempty, different) either:

- exists $j$ s.t. $f(w(S_j, i)) < f(w(S_j, i+1))$;
- exists $j_1 \neq j_2$ s.t. $f(w(S_{j_1} \cup S_{j_2}, i)) = f(w(S_{j_1} \cup S_{j_2}, i+1))$.

Example: $f$ – "smallest block of $b's$"

Let $w = (\underline{b^N} a^N)^N \quad (n = N)$

Let $S_j = \{1, \ldots, N\} \setminus \{j\}, \quad f(w(S_j, i)) = N$ for all $i, j$

But $S_{j_1} \cup S_{j_2} = \{1, \ldots, N\}$ for $j_1 \neq j_2$

Hence $f(w(S_{j_1} \cup S_{j_2}, i)) < f(w(S_{j_1} \cup S_{j_2}, i+1))$

**Theorem** (Pumping Lemma 2)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by FA-WA over $(\min, +)$
there exists $N$ s.t. for all $(n, N)$-representations ($n \geq N$)
exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$
s.t. given $S_1, \ldots, S_N \subseteq \{1 \ldots n\}$ (nonempty, different) either:

- exists $j$ s.t. $f(w(S_j, i)) < f(w(S_j, i+1))$;
- exists $j_1 \neq j_2$ s.t. $f(w(S_{j_1} \cup S_{j_2}, i)) = f(w(S_{j_1} \cup S_{j_2}, i+1))$.

Example: $f$ – "smallest block of $b's$"
Let $w = (\underline{b^N} a^N)^N \quad (n = N)$
Let $S_j = \{1, \ldots, N\} \setminus \{j\}, \quad f(w(S_j, i)) = N$ for all $i, j$
But $S_{j_1} \cup S_{j_2} = \{1, \ldots, N\}$ for $j_1 \neq j_2$
Hence $f(w(S_{j_1} \cup S_{j_2}, i)) < f(w(S_{j_1} \cup S_{j_2}, i+1))$

Corollary: FA-WA $\subsetneq$ PA-WA over $(\min, +)$

# Polynomially-ambiguous over $(\min, +)$

over $\mathbb{N}_\infty(\min, +, \infty, 0)$

## Polynomially-ambiguous over $(\min, +)$

over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Example: "smallest block of $b$'s"

## Polynomially-ambiguous over $(\min, +)$

over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Example: "smallest block of $b$'s"

Negative example: let $w = w_0 \# w_1 \# \ldots \# w_m$, where $w_i \in \{a, b\}^*$

$$f(w) = \sum_{i=0}^{k} \min(\#_a(w_i), \#_b(w_i))$$

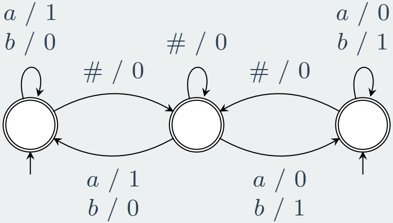
## Polynomially-ambiguous over $(\min, +)$

over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Example: "smallest block of $b$'s"

Negative example: let $w = w_0\#w_1\#\ldots\#w_m$, where $w_i \in \{a,b\}^*$

$f(w) = \sum_{i=0}^{k} \min(\#_a(w_i), \#_b(w_i))$

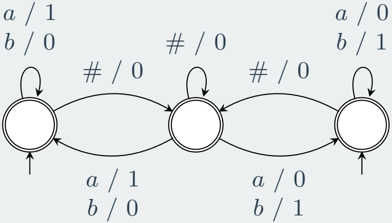


## Polynomially-ambiguous over $(\min, +)$

over $\mathbb{N}_\infty(\min, +, \infty, 0)$

Example: “smallest block of $b$’s”

Negative example: let $w = w_0\#w_1\#\ldots\#w_m$, where $w_i \in \{a,b\}^*$
$f(w) = \sum_{i=0}^{k} \min(\#_a(w_i), \#_b(w_i))$




Number of runs: $2^k$

## Pumping Lemma 3

First, some notation

## Pumping Lemma 3

First, some notation

Let $S_1, \ldots, S_m \subseteq \{1, \ldots, n\}$

## Pumping Lemma 3

First, some notation

Let $S_1, \ldots, S_m \subseteq \{1, \ldots, n\}$

A partition if $\bigcup_{j=1}^{m} S_j = \{1, \ldots, n\}, \quad S_j$ nonempty, $S_{j_1} \cap S_{j_2} = \emptyset$

## Pumping Lemma 3

First, some notation

Let $S_1, \ldots, S_m \subseteq \{1, \ldots, n\}$

A partition if $\bigcup_{j=1}^{m} S_j = \{1, \ldots, n\}$, $\quad S_j$ nonempty, $S_{j_1} \cap S_{j_2} = \emptyset$

$S \subseteq \{1, \ldots, n\}$ is a selector if $|S \cap S_j| = 1$ for all $j$

## Pumping Lemma 3

First, some notation

Let $S_1, \ldots, S_m \subseteq \{1, \ldots, n\}$

A partition if $\bigcup_{j=1}^{m} S_j = \{1, \ldots, n\}$, $S_j$ nonempty, $S_{j_1} \cap S_{j_2} = \emptyset$

$S \subseteq \{1, \ldots, n\}$ is a selector if $|S \cap S_j| = 1$ for all $j$

**Theorem** (Pumping lemma 3)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by PA-WA over $\mathbb{N}_\infty(\min, +)$

there exists $N$ s.t. for all $(n, N)$-representations

## Pumping Lemma 3

First, some notation

Let $S_1, \ldots, S_m \subseteq \{1, \ldots, n\}$

A partition if $\bigcup_{j=1}^{m} S_j = \{1, \ldots, n\}$, $\quad S_j$ nonempty, $S_{j_1} \cap S_{j_2} = \emptyset$

$S \subseteq \{1, \ldots, n\}$ is a selector if $|S \cap S_j| = 1$ for all $j$

**Theorem** (Pumping lemma 3)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by PA-WA over $\mathbb{N}_\infty(\min, +)$

there exists $N$ s.t. for all $(n, N)$-representations

there exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$

## Pumping Lemma 3

First, some notation

Let $S_1, \ldots, S_m \subseteq \{1, \ldots, n\}$

A partition if $\bigcup_{j=1}^{m} S_j = \{1, \ldots, n\}$, $\quad S_j$ nonempty, $S_{j_1} \cap S_{j_2} = \emptyset$

$S \subseteq \{1, \ldots, n\}$ is a selector if $|S \cap S_j| = 1$ for all $j$

**Theorem** (Pumping lemma 3)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by PA-WA over $\mathbb{N}_\infty(\min, +)$

there exists $N$ s.t. for all $(n, N)$-representations

there exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$

s.t. for every partition $S_1, \ldots, S_m$ of $\{1, \ldots, n\}$ either:

## Pumping Lemma 3

First, some notation

Let $S_1, \ldots, S_m \subseteq \{1, \ldots, n\}$

A partition if $\bigcup_{j=1}^{m} S_j = \{1, \ldots, n\}$, $\quad S_j$ nonempty, $S_{j_1} \cap S_{j_2} = \emptyset$

$S \subseteq \{1, \ldots, n\}$ is a selector if $|S \cap S_j| = 1$ for all $j$

**Theorem** (Pumping lemma 3)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by PA-WA over $\mathbb{N}_\infty(\min, +)$

there exists $N$ s.t. for all $(n, N)$-representations

there exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$

s.t. for every partition $S_1, \ldots, S_m$ of $\{1, \ldots, n\}$ either:

- there exists $j$ s.t. $f(w(S_j, i)) = f(w(S_j, i+1))$

## Pumping Lemma 3

First, some notation

Let $S_1, \ldots, S_m \subseteq \{1, \ldots, n\}$

A partition if $\bigcup_{j=1}^{m} S_j = \{1, \ldots, n\}$, $\quad S_j$ nonempty, $S_{j_1} \cap S_{j_2} = \emptyset$

$S \subseteq \{1, \ldots, n\}$ is a selector if $|S \cap S_j| = 1$ for all $j$

**Theorem** (Pumping lemma 3)

Let $f : \Sigma^* \to \mathbb{N}_\infty$ be recognizable by PA-WA over $\mathbb{N}_\infty(\min, +)$

there exists $N$ s.t. for all $(n, N)$-representations

there exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$

s.t. for every partition $S_1, \ldots, S_m$ of $\{1, \ldots, n\}$ either:

- there exists $j$ s.t. $f(w(S_j, i)) = f(w(S_j, i + 1))$
- there exists a selector $S$ s.t. $f(w(S, i)) < f(w(S, i + 1))$

**Theorem** (Pumping lemma 3)

$(\dots)$ there exists $N$ s.t. for all $(n, N)$-representations
there exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$
s.t. for every partition $S_1, \ldots, S_m$ of $\{1, \ldots, n\}$ either:

- there exists $j$ s.t. $f(w(S_j, i)) = f(w(S_j, i + 1))$
- there exists a selector $S$ s.t. $f(w(S, i)) < f(w(S, i + 1))$

**Theorem** (Pumping lemma 3)

$(\ldots)$ there exists $N$ s.t. for all $(n, N)$-representations

there exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$

s.t. for every partition $S_1, \ldots, S_m$ of $\{1, \ldots, n\}$ either:

- there exists $j$ s.t. $f(w(S_j, i)) = f(w(S_j, i+1))$
- there exists a selector $S$ s.t. $f(w(S, i)) < f(w(S, i+1))$

Example: $f(w) = \sum_{i=0}^{k} \min(\#_a(w_i), \#_b(w_i))$

**Theorem** (Pumping lemma 3)

$(\dots)$ there exists $N$ s.t. for all $(n, N)$-representations

there exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \dots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$

s.t. for every partition $S_1, \dots, S_m$ of $\{1, \dots, n\}$ either:

- there exists $j$ s.t. $f(w(S_j, i)) = f(w(S_j, i+1))$
- there exists a selector $S$ s.t. $f(w(S, i)) < f(w(S, i+1))$

Example: $f(w) = \sum_{i=0}^{k} \min(\#_a(w_i), \#_b(w_i))$

$w = (\underline{a^N}\ \underline{b^N}\#)^m$

$\{1 \dots n\} = \{(1,1), (2,1) \dots (1,m), (2,m)\}$

**Theorem** (Pumping lemma 3)

$(\dots)$ there exists $N$ s.t. for all $(n, N)$-representations

there exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \dots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$

s.t. for every partition $S_1, \dots, S_m$ of $\{1, \dots, n\}$ either:

- there exists $j$ s.t. $f(w(S_j, i)) = f(w(S_j, i+1))$
- there exists a selector $S$ s.t. $f(w(S, i)) < f(w(S, i+1))$

Example: $f(w) = \sum_{i=0}^{k} \min(\#_a(w_i), \#_b(w_i))$

$w = (\underline{a^N}\ \underline{b^N}\#)^m$

$\{1 \dots n\} = \{(1,1), (2,1) \dots (1,m), (2,m)\}$

$S_j = \{(1,j), (2,j)\}, \quad f(w(S_j, i)) < f(w(S_j, i+1))$

**Theorem** (Pumping lemma 3)

(...) there exists $N$ s.t. for all $(n, N)$-representations there exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$ s.t. for every partition $S_1, \ldots, S_m$ of $\{1, \ldots, n\}$ either:

- there exists $j$ s.t. $f(w(S_j, i)) = f(w(S_j, i+1))$
- there exists a selector $S$ s.t. $f(w(S, i)) < f(w(S, i+1))$

Example: $f(w) = \sum_{i=0}^{k} \min(\#_a(w_i), \#_b(w_i))$

$w = (\underline{a^N}\,\underline{b^N}\#)^m$

$\{1 \ldots n\} = \{(1,1), (2,1) \ldots (1,m), (2,m)\}$

$S_j = \{(1,j), (2,j)\}, \quad f(w(S_j, i)) < f(w(S_j, i+1))$

for every selector $f(w(S, i)) = f(w(S, i+1))$

**Theorem** (Pumping lemma 3)

(...) there exists $N$ s.t. for all $(n, N)$-representations

there exists refinement $w = u'_0 \cdot \underline{y_1} \cdot u'_1 \cdot \underline{y_2} \cdot \ldots u'_{n-1} \cdot \underline{y_n} \cdot u'_n$

s.t. for every partition $S_1, \ldots, S_m$ of $\{1, \ldots, n\}$ either:

- there exists $j$ s.t. $f(w(S_j, i)) = f(w(S_j, i+1))$
- there exists a selector $S$ s.t. $f(w(S, i)) < f(w(S, i+1))$

Example: $f(w) = \sum_{i=0}^{k} \min(\#_a(w_i), \#_b(w_i))$

$w = (\underline{a^N}\,\underline{b^N}\#)^m$

$\{1 \ldots n\} = \{(1,1), (2,1) \ldots (1,m), (2,m)\}$

$S_j = \{(1,j), (2,j)\}, \quad f(w(S_j, i)) < f(w(S_j, i+1))$

for every selector $f(w(S, i)) = f(w(S, i+1))$

Corollary: PA-WA $\subsetneq$ WA over $(\min, +)$

# Conclusions

## Conclusions

- Corollary: U-WA $\subsetneq$ FA-WA $\subsetneq$ PA-WA $\subsetneq$ WA over $(\min, +)$

(only the last one is new)

## Conclusions

- Corollary: U-WA $\subsetneq$ FA-WA $\subsetneq$ PA-WA $\subsetneq$ WA over $(\min, +)$
(only the last one is new)

- Pumping Lemmas 1 and 2 hold for more (WA over $\mathbb{N}(+, \cdot, 0, 1)$)

## Conclusions

- Corollary: U-WA $\subsetneq$ FA-WA $\subsetneq$ PA-WA $\subsetneq$ WA over $(\min, +)$

(only the last one is new)

- Pumping Lemmas 1 and 2 hold for more (WA over $\mathbb{N}(+, \cdot, 0, 1)$)

- Can we hope for characterizations?

## Conclusions

- Corollary: U-WA $\subsetneq$ FA-WA $\subsetneq$ PA-WA $\subsetneq$ WA over $(\min, +)$

(only the last one is new)

- Pumping Lemmas 1 and 2 hold for more (WA over $\mathbb{N}(+, \cdot, 0, 1)$)

- Can we hope for characterizations?

Not for free. For every $f : \Sigma^* \to \mathbb{N}$ consider $g(w) = f(w) + |w|$

## Conclusions

- Corollary: U-WA $\subsetneq$ FA-WA $\subsetneq$ PA-WA $\subsetneq$ WA over $(\min, +)$

(only the last one is new)

- Pumping Lemmas 1 and 2 hold for more (WA over $\mathbb{N}(+, \cdot, 0, 1)$)

- Can we hope for characterizations?

Not for free. For every $f : \Sigma^* \to \mathbb{N}$ consider $g(w) = f(w) + |w|$

$f$ and $g$ should be in the same class

## Conclusions

- Corollary: U-WA $\subsetneq$ FA-WA $\subsetneq$ PA-WA $\subsetneq$ WA over $(\min, +)$

(only the last one is new)

- Pumping Lemmas 1 and 2 hold for more (WA over $\mathbb{N}(+, \cdot, 0, 1)$)

- Can we hope for characterizations?

Not for free. For every $f : \Sigma^* \to \mathbb{N}$ consider $g(w) = f(w) + |w|$

$f$ and $g$ should be in the same class

but $g$ satisfies all lemmas.

## Conclusions

- Corollary: U-WA $\subsetneq$ FA-WA $\subsetneq$ PA-WA $\subsetneq$ WA over $(\min, +)$

(only the last one is new)

- Pumping Lemmas 1 and 2 hold for more (WA over $\mathbb{N}(+, \cdot, 0, 1)$)

- Can we hope for characterizations?

Not for free. For every $f : \Sigma^* \to \mathbb{N}$ consider $g(w) = f(w) + |w|$

$f$ and $g$ should be in the same class

but $g$ satisfies all lemmas.

- Beyond weighted automata

Pumping lemmas for weighted logic and cost-register automata?