Petri nets and weighted automata

Filip Mazowiecki

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What I could say that I work on

Formal verification

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Math puzzles

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What I will say?

A bit of both

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What are Petri nets and weighted automata?

- Mathematical models to verify programs, business models, etc

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What I will say?

A bit of both

What are Petri nets and weighted automata?

- Mathematical models to verify programs, business models, etc
- Automata with registers: Petri nets over $\mathbb{N}$ (with > 0 tests), weighted automata over $\mathbb{Q}$ (blind)

Petri nets geometric intuition

Petri net (d, T): d – dimension, $T \subseteq \mathbb{Z}^d$ – transitions

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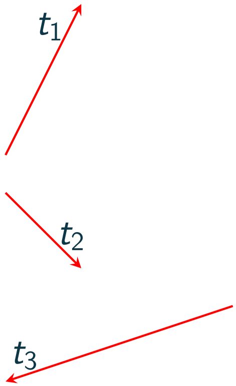

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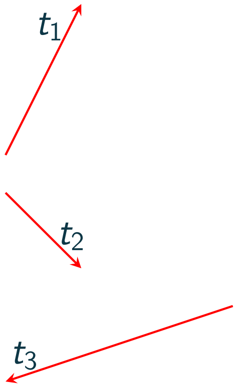

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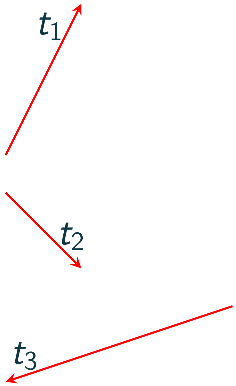

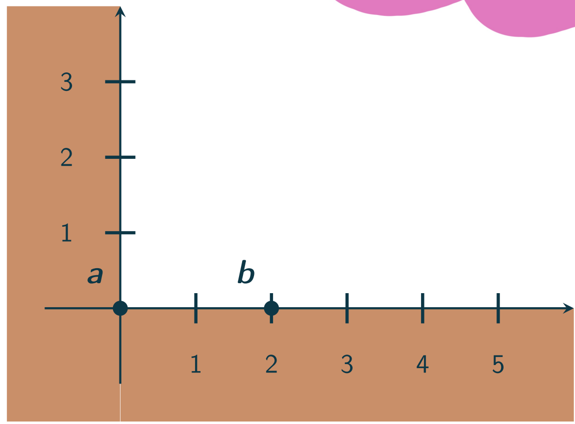

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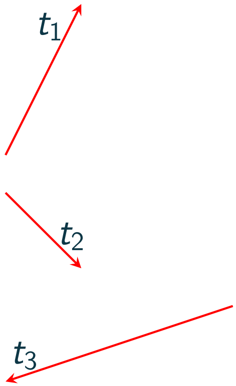

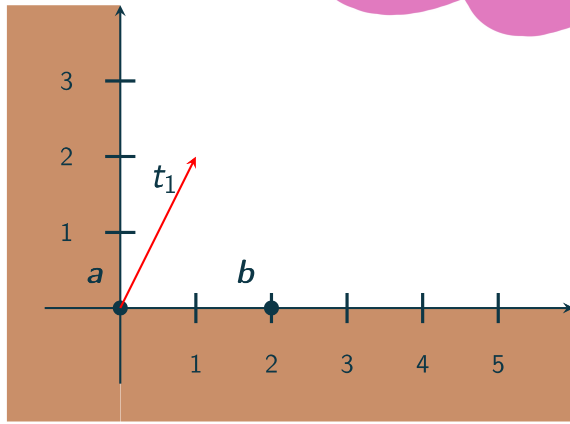

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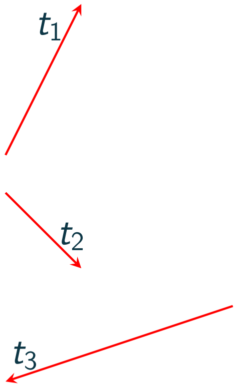

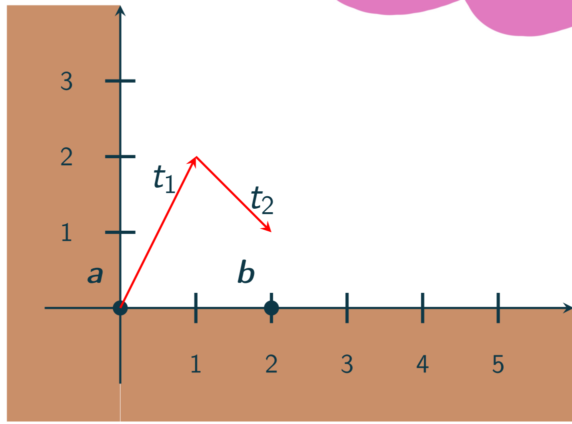

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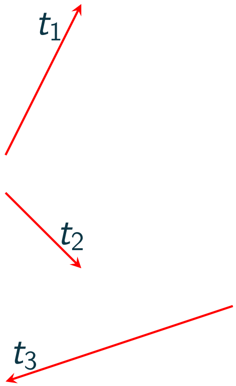

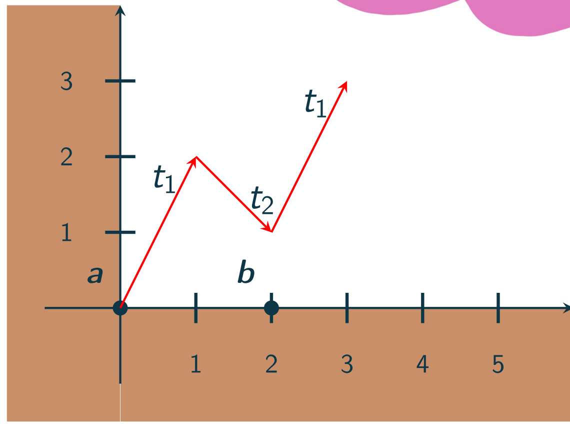

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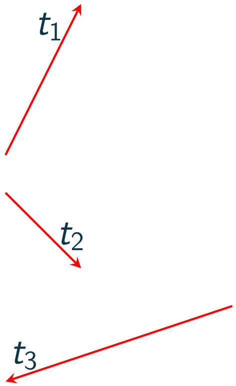

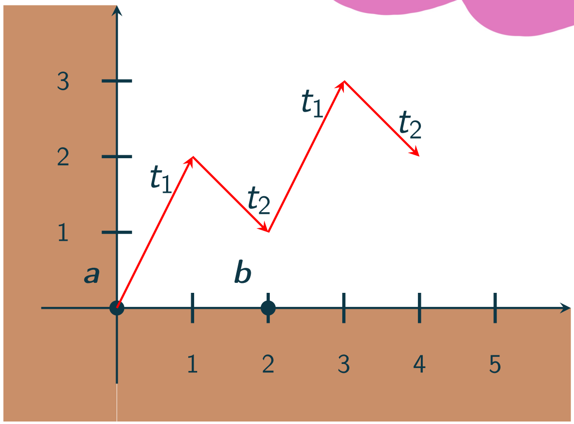

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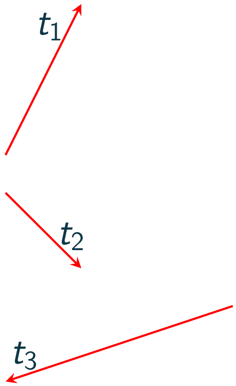

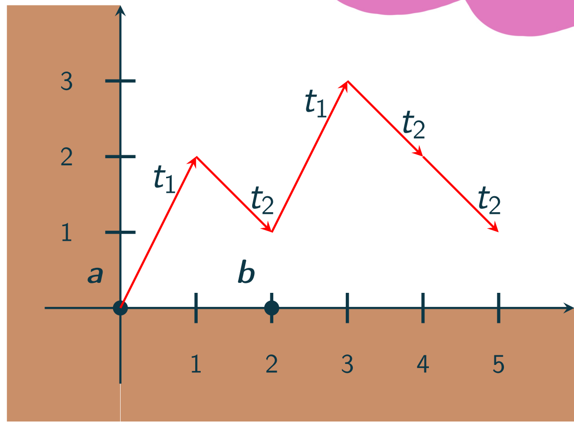

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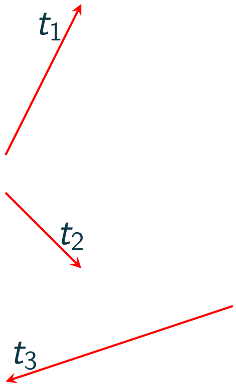

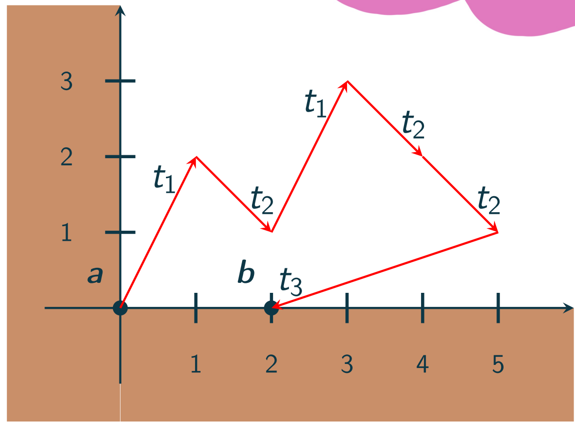

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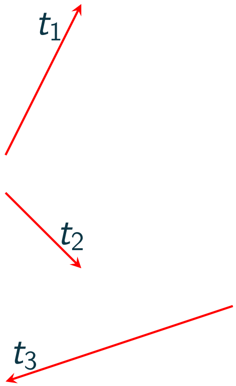

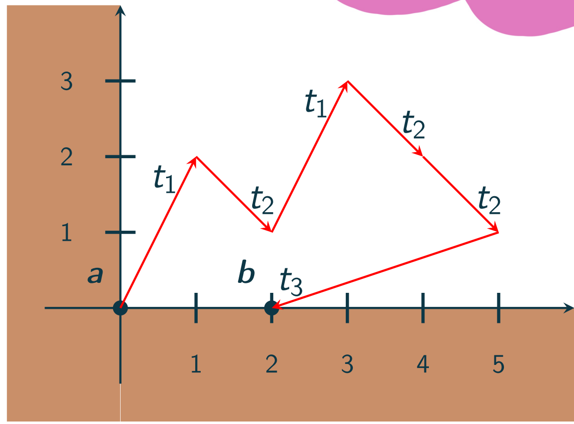

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Soundness problems: given (d, T) and starting in $(k, 0, 0, \ldots, 0)$

whatever I do can I always reach $(0, 0, \ldots, 0, k)$ staying within $\mathbb{N}^d$?

How can that be useful?

{x = True }
1: **goto** 2
2: **if** x **then goto** 3 **else goto** 1
3: x = !x
4: # critical section
5: x = !x, **exit**

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In the first line arbitrary many processes can appear

Problem: can two processes appear at once in the critical section?

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$\{x = \text{True}\}$	{bathroom free}
1: **goto** 2	go to the door
2: **if** x **then goto** 3 **else goto** 1	if free enter, otherwise go back
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- dimension d: every line and possible values of x,
- transitions: how processes/people move and change the values of x

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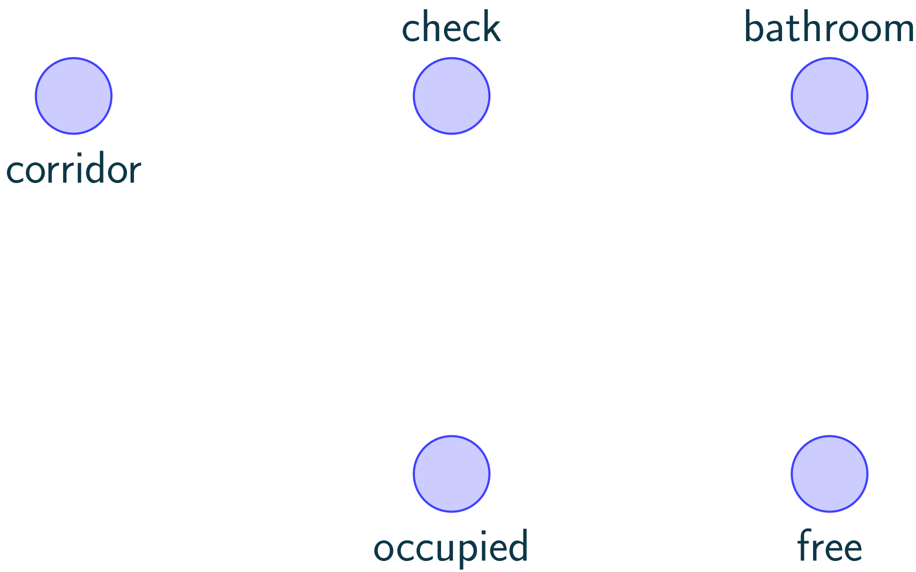

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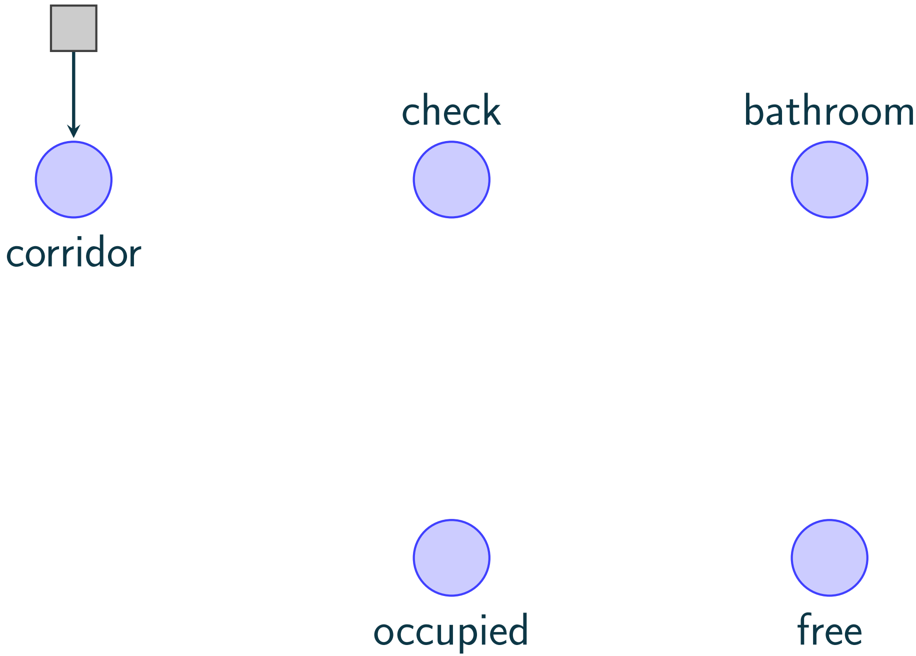

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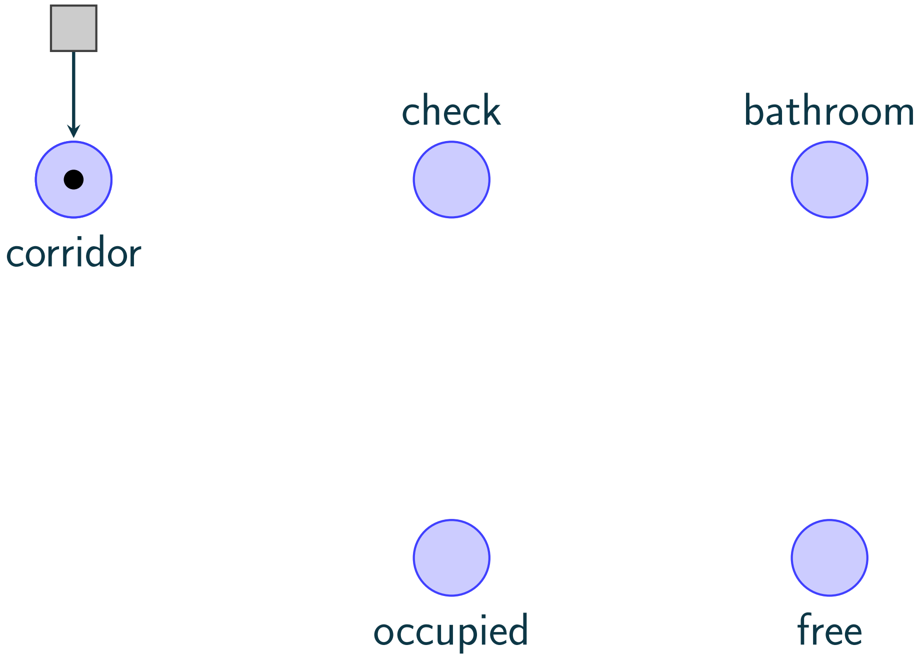

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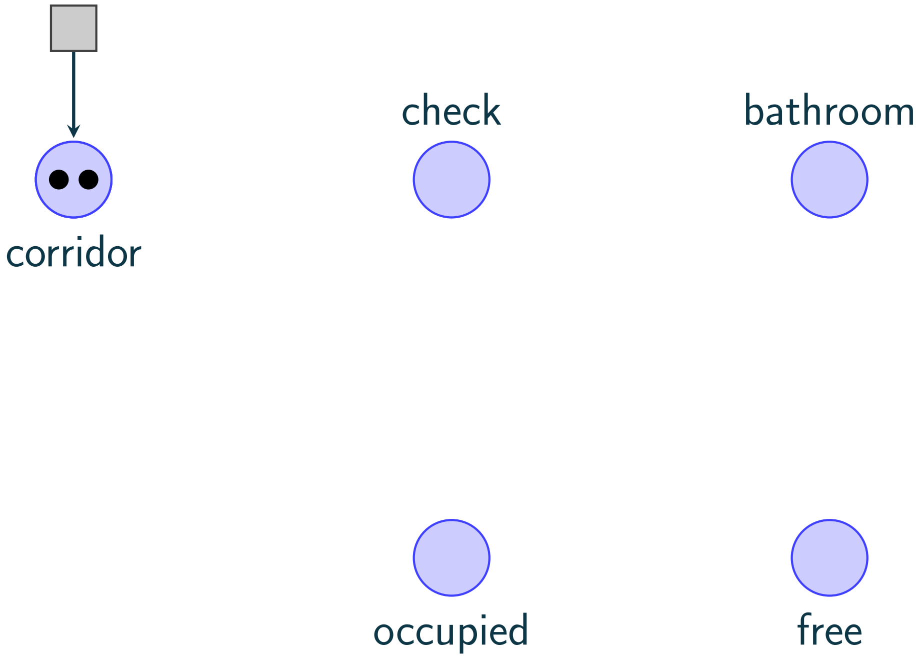

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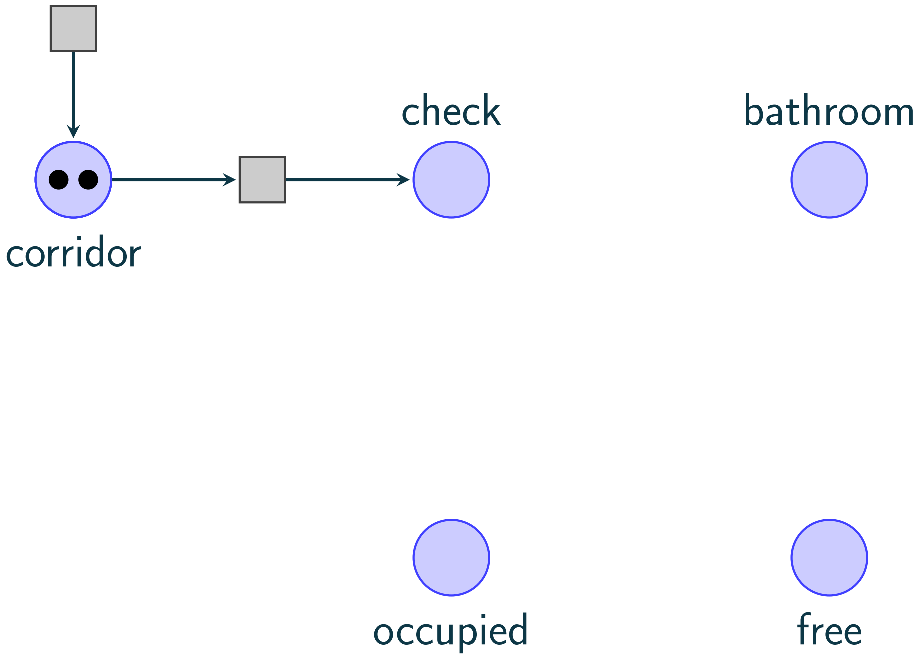

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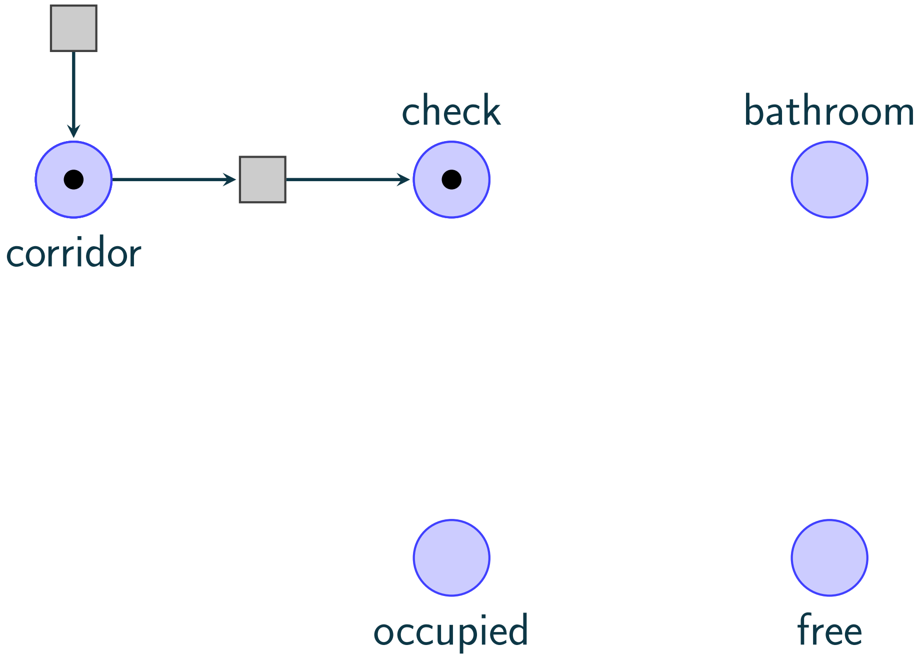

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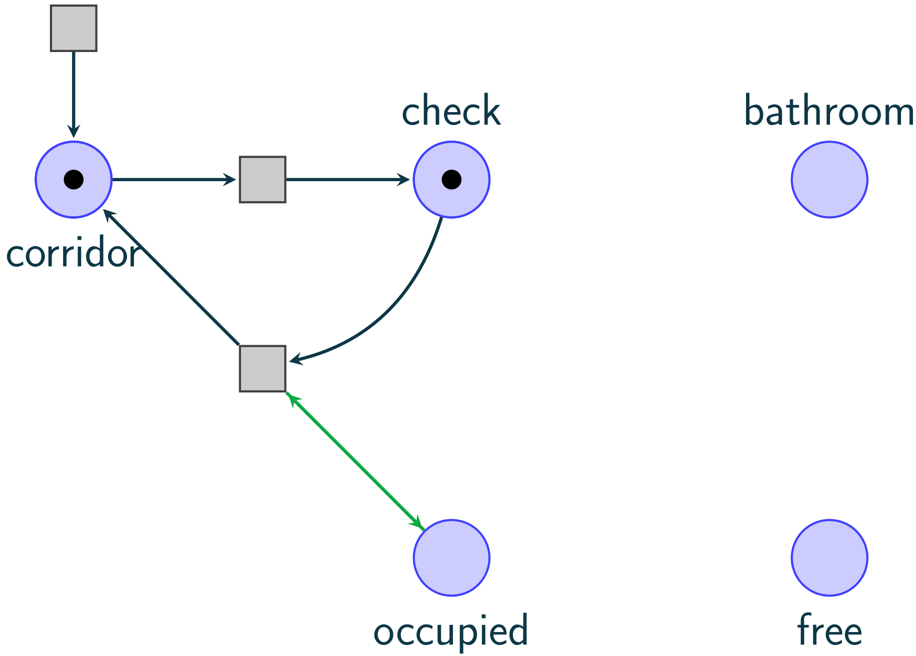

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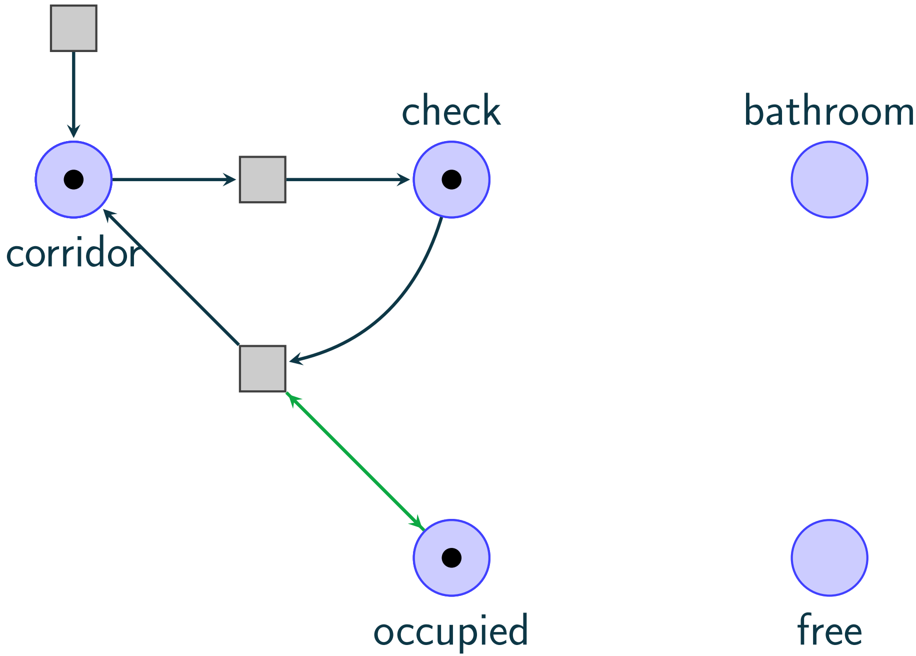

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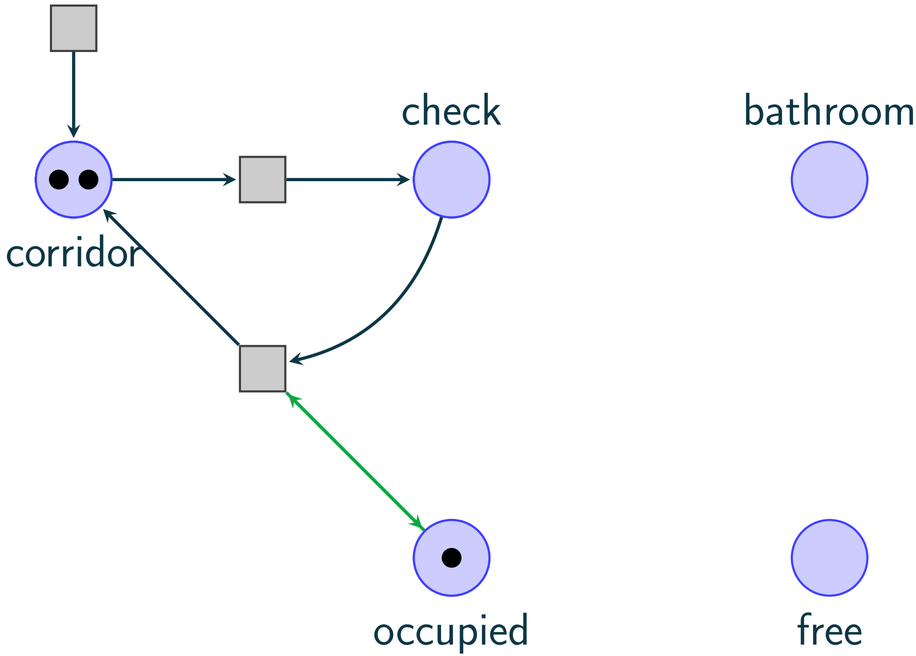

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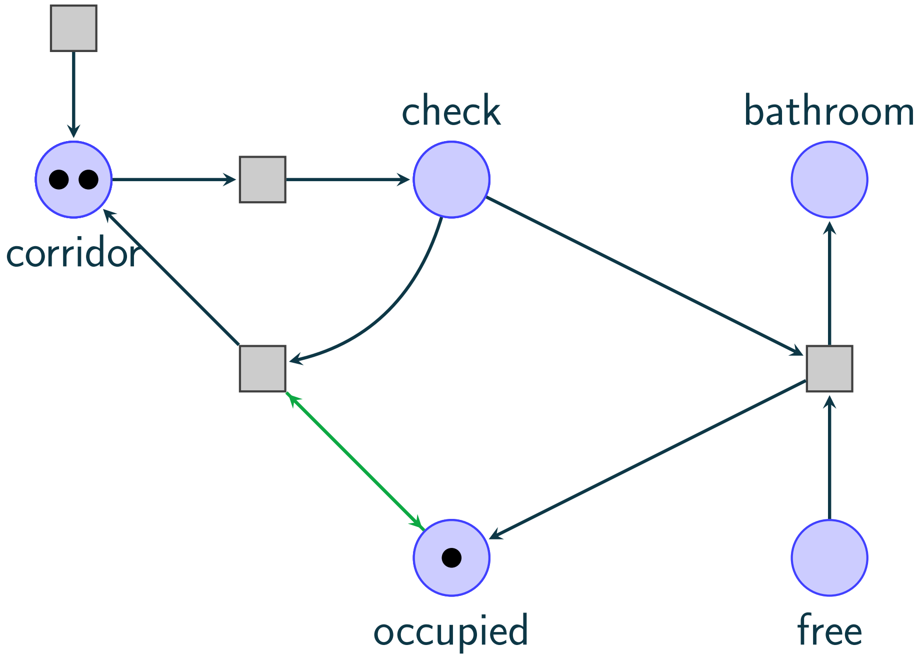

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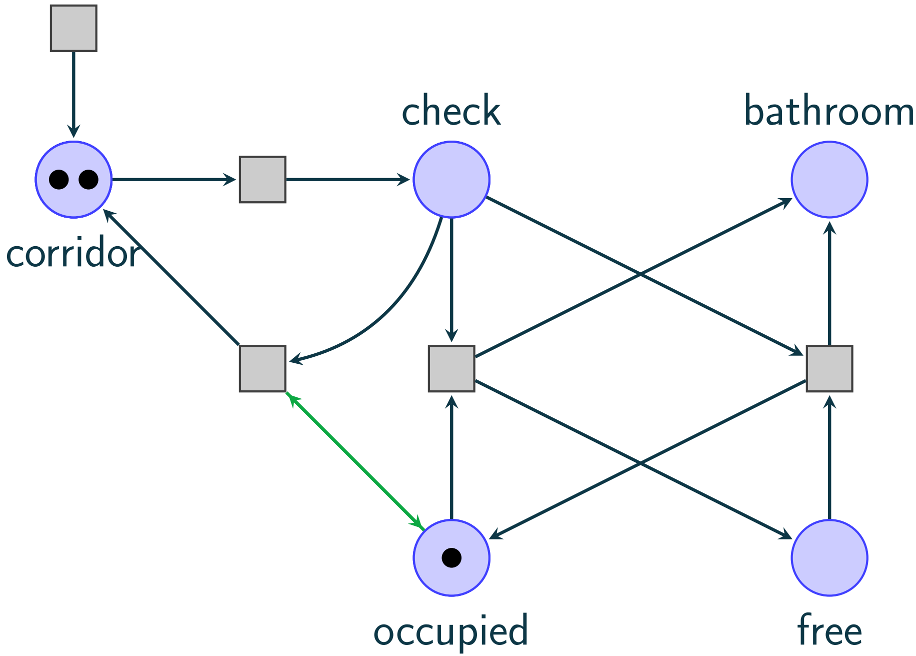

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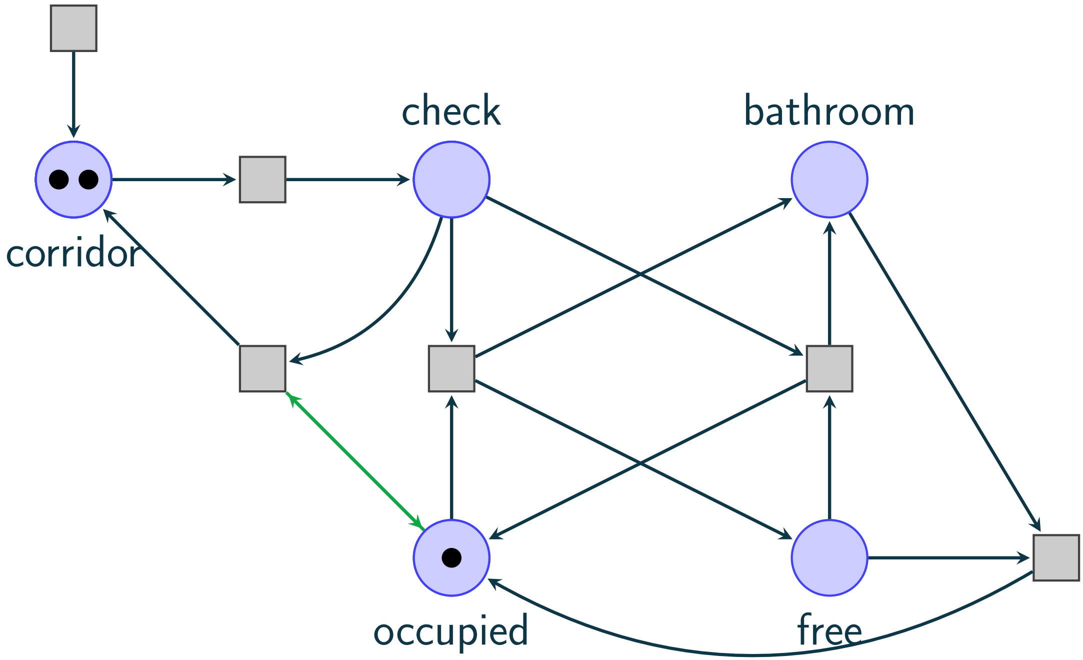

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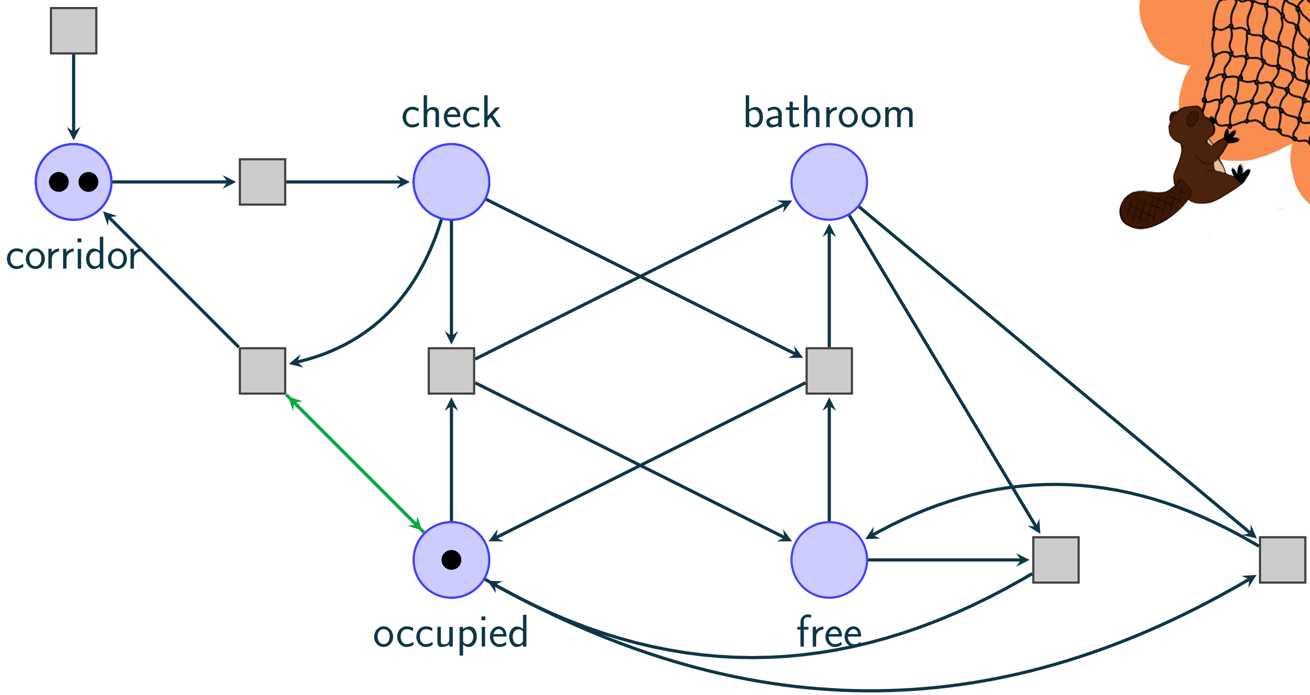

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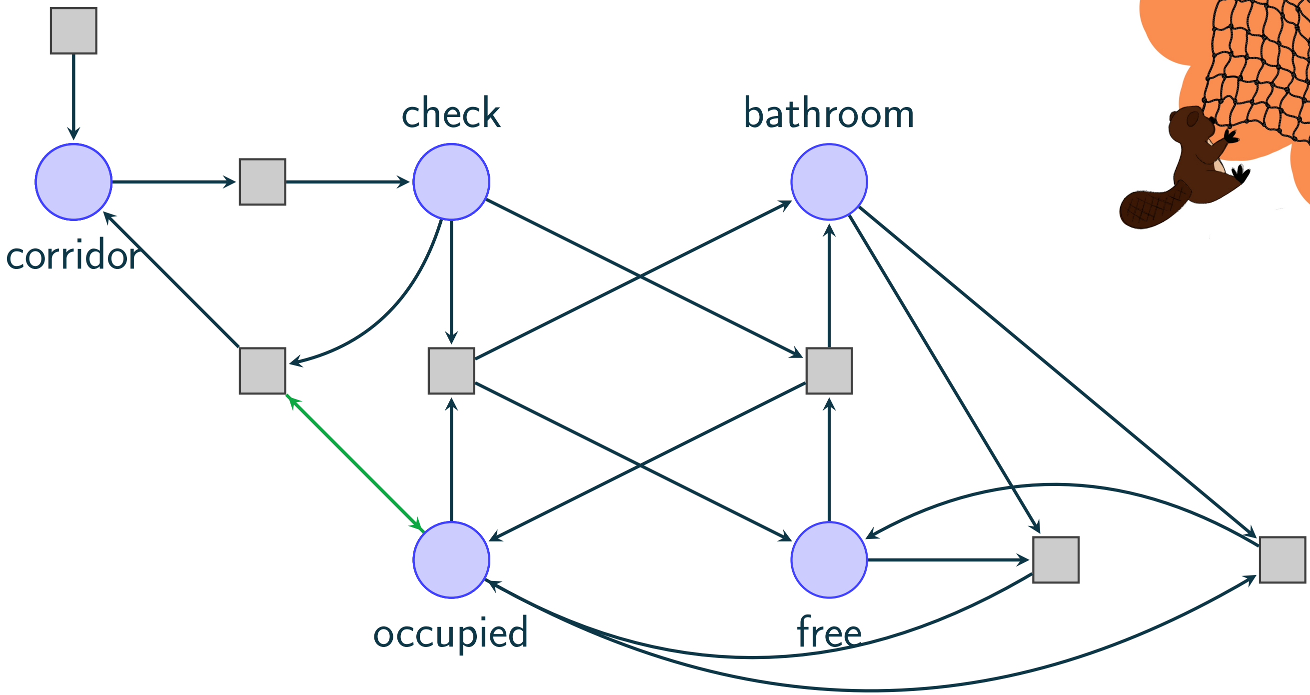

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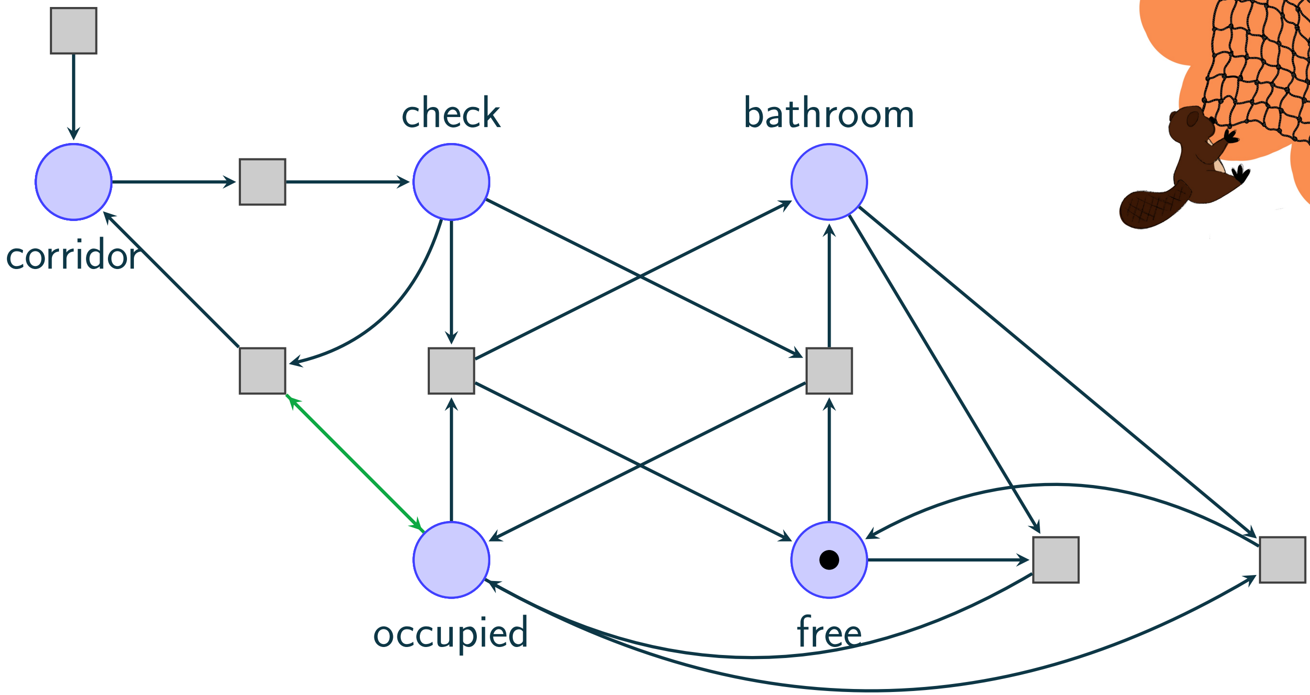

bathroom is free

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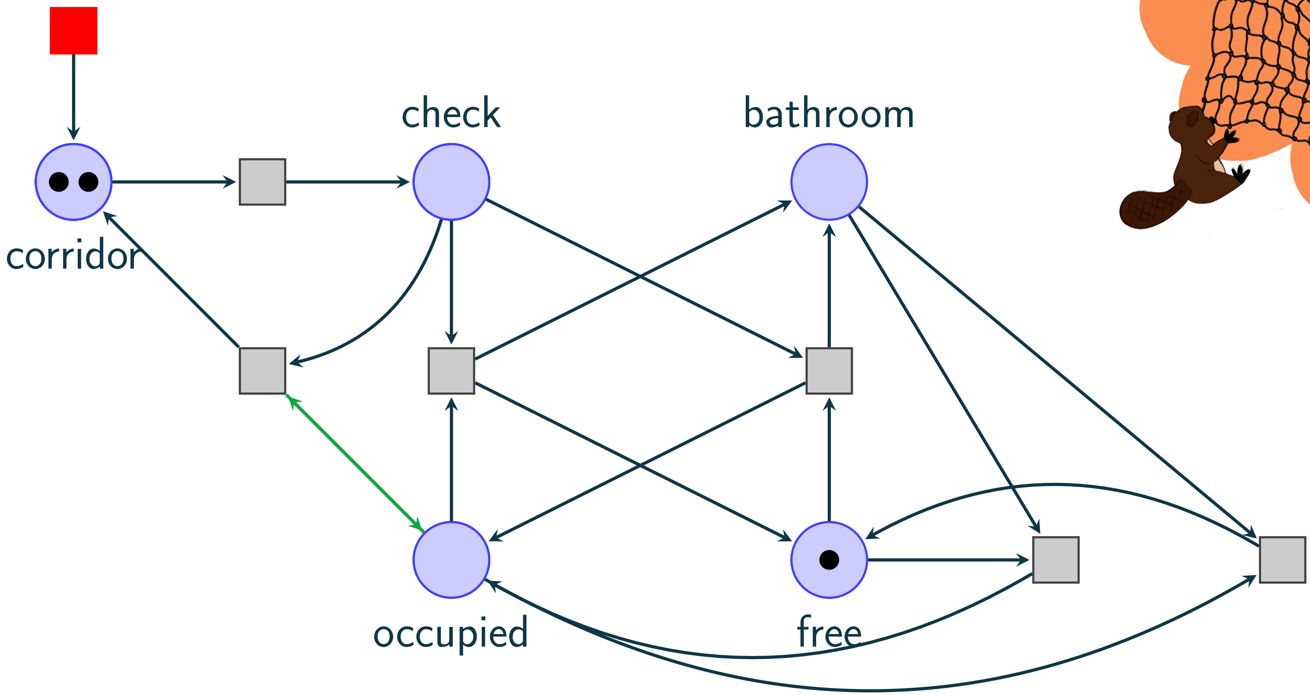

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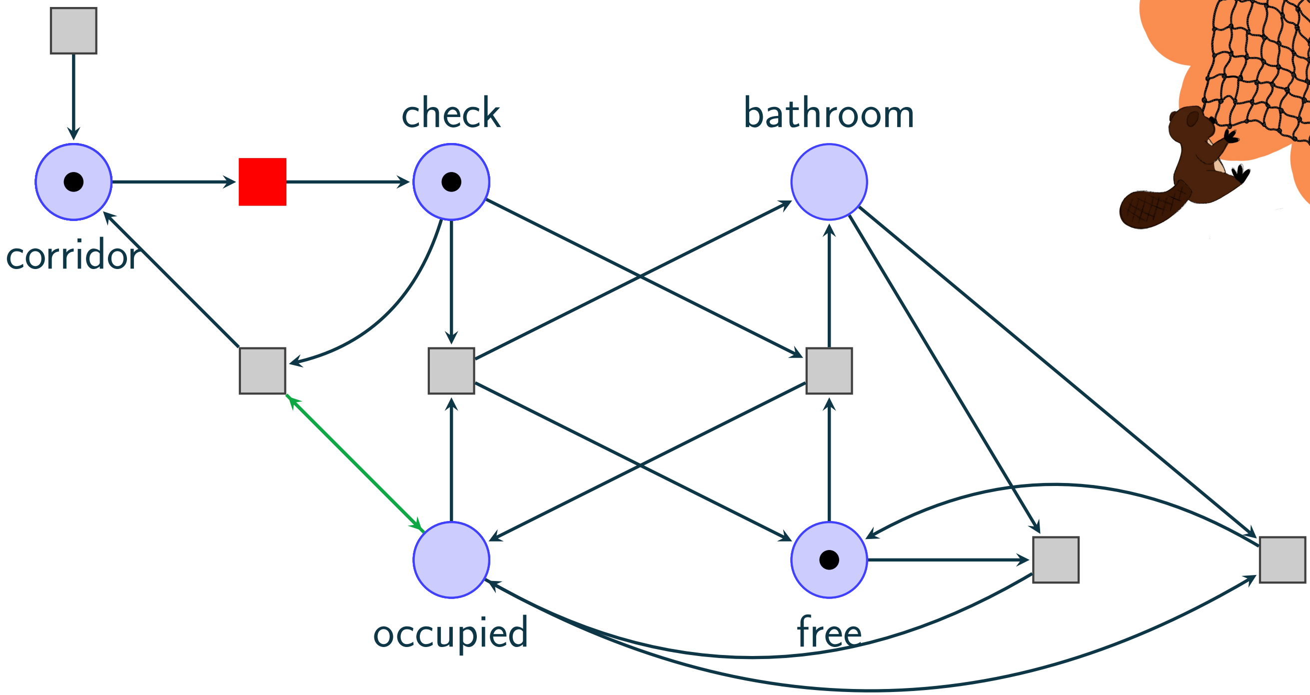

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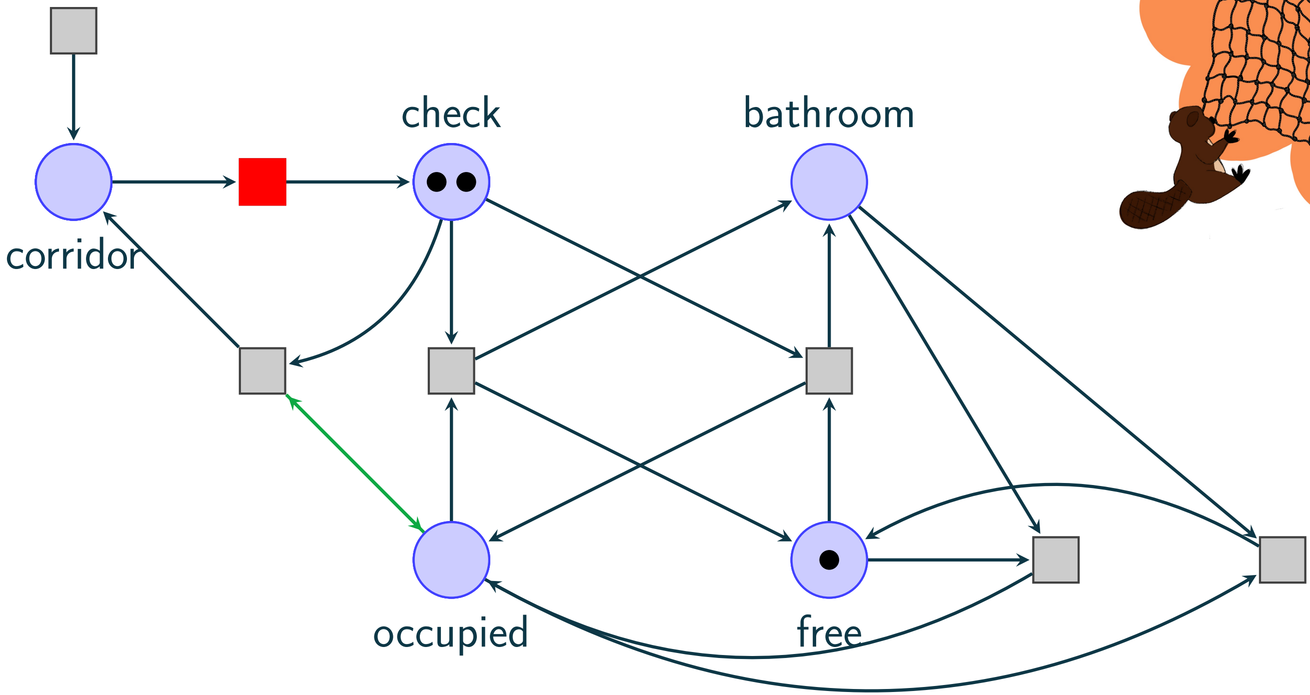

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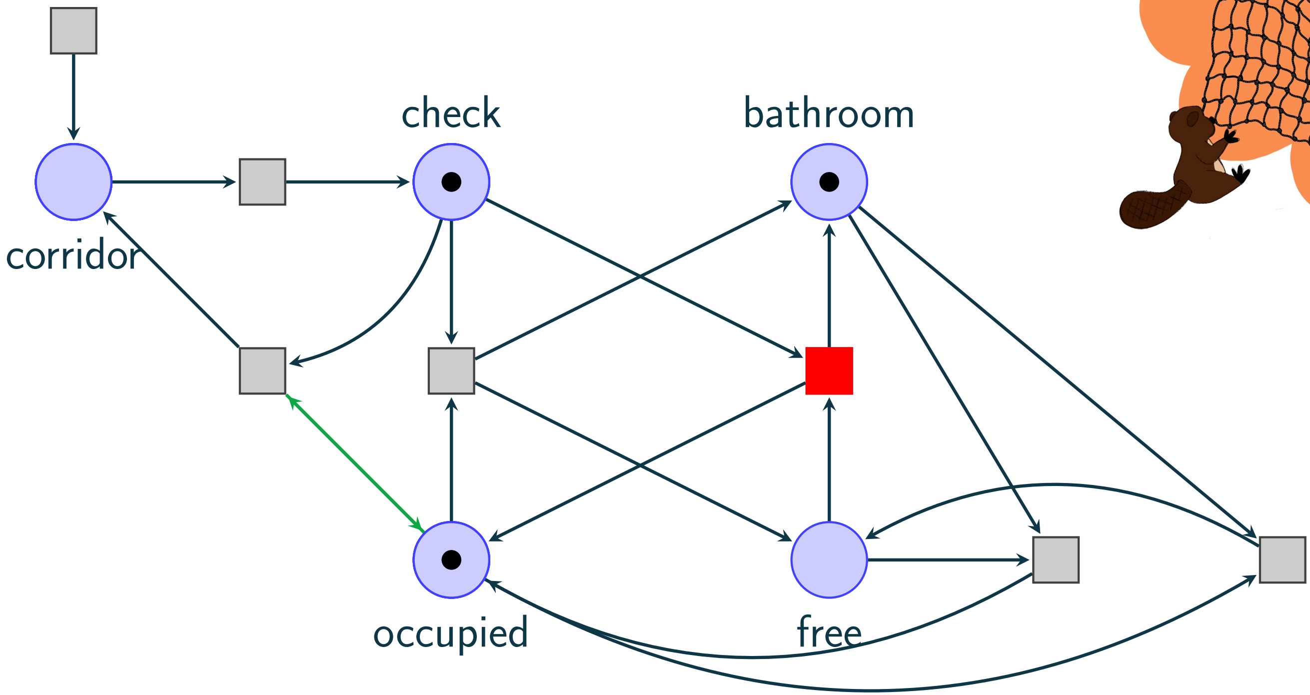

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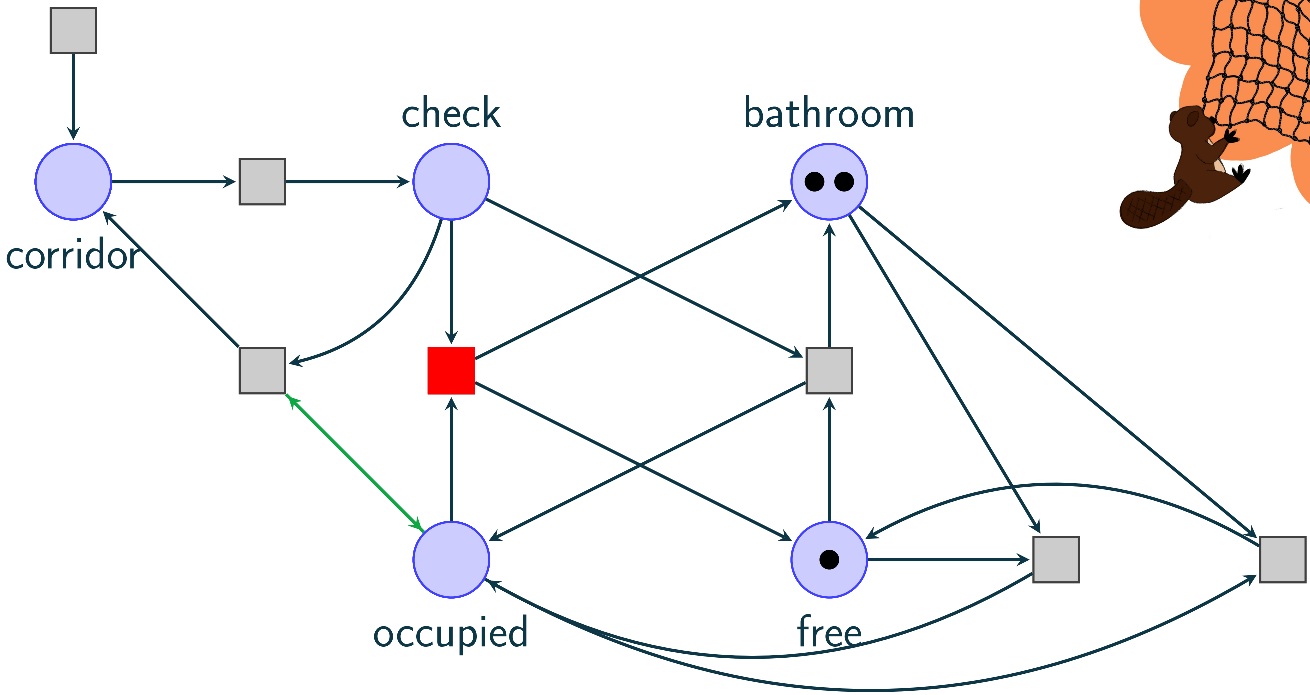

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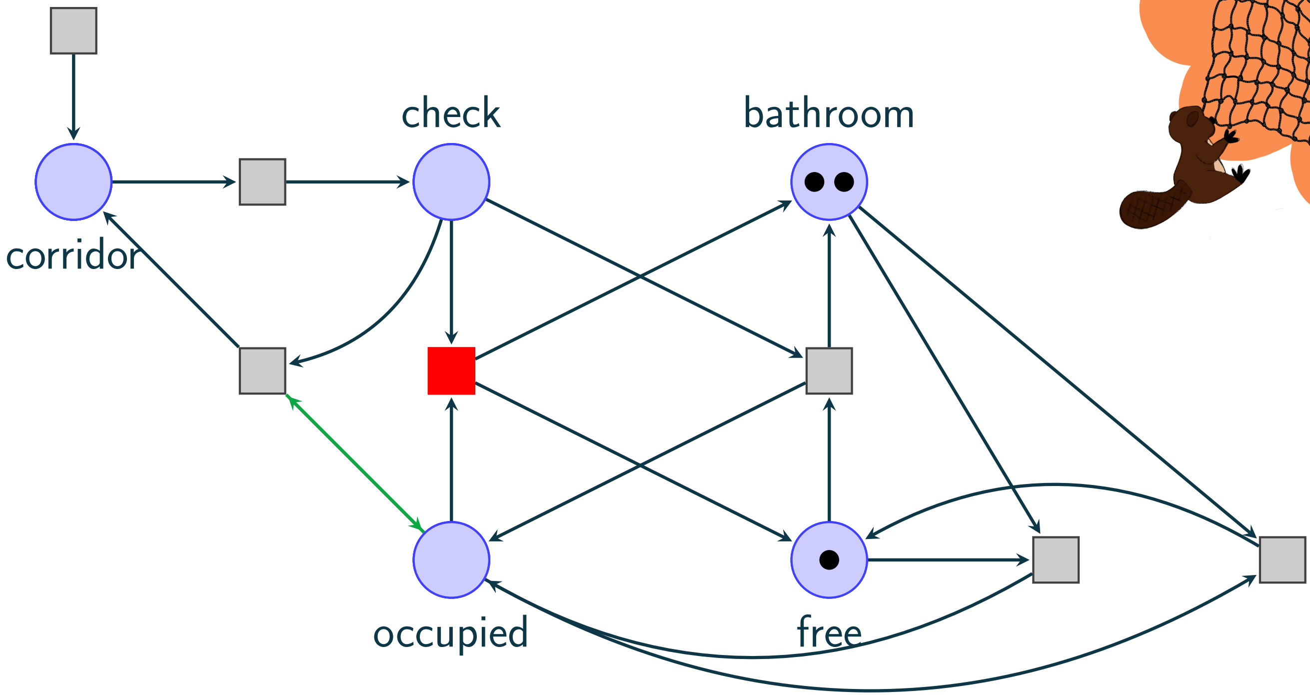

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It can be fixed, but this way we only detect errors

Modelling 2 (soundness problem)

Suppose the administration wants to distribute tasks

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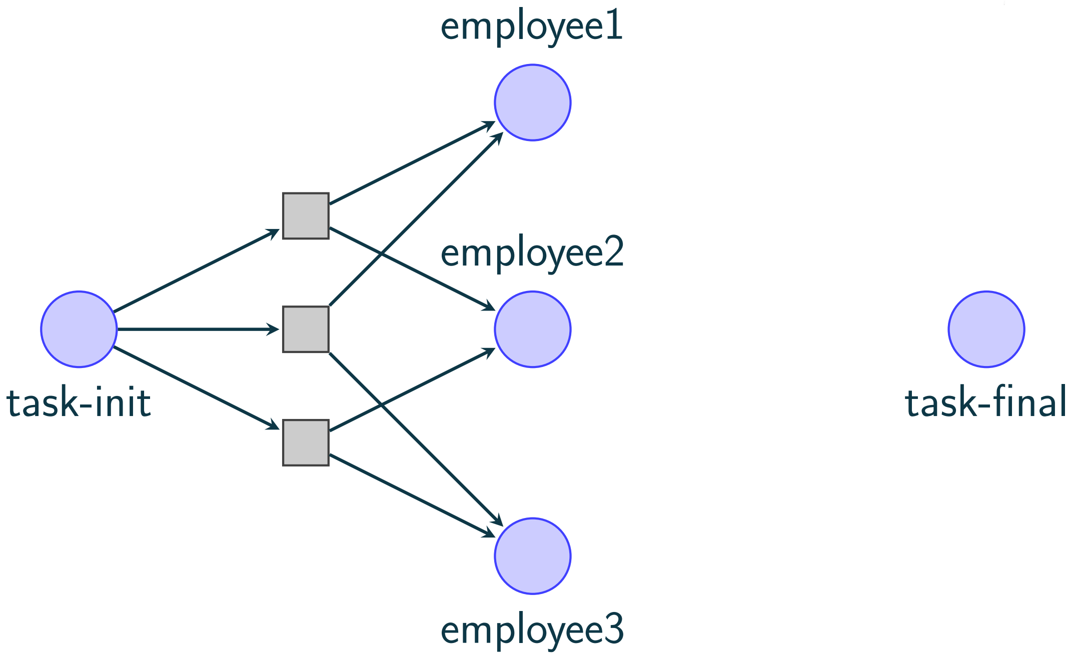

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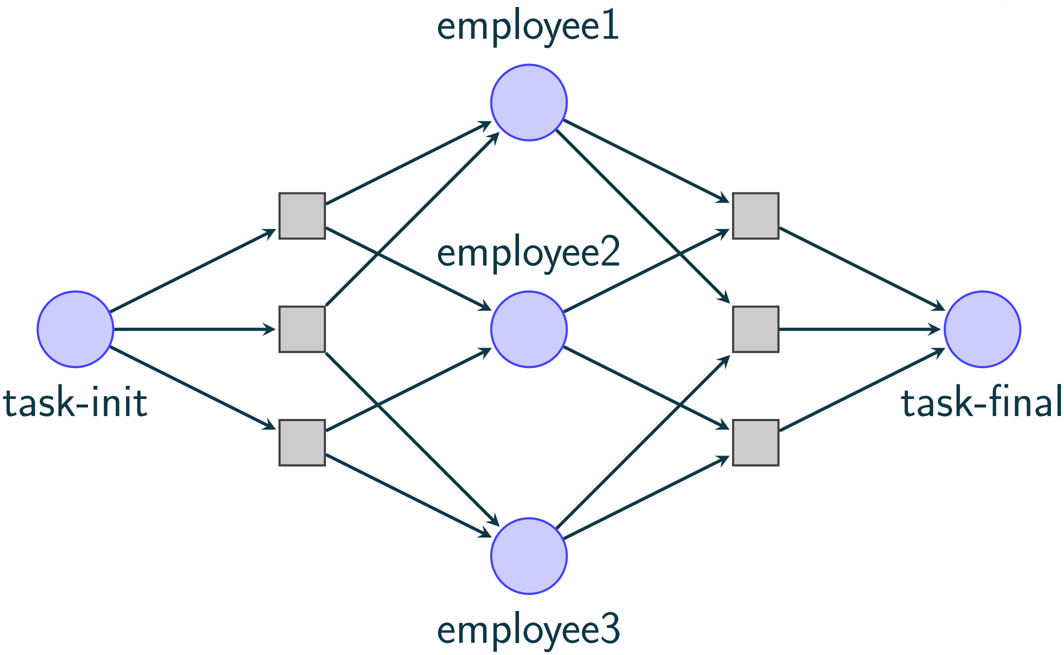

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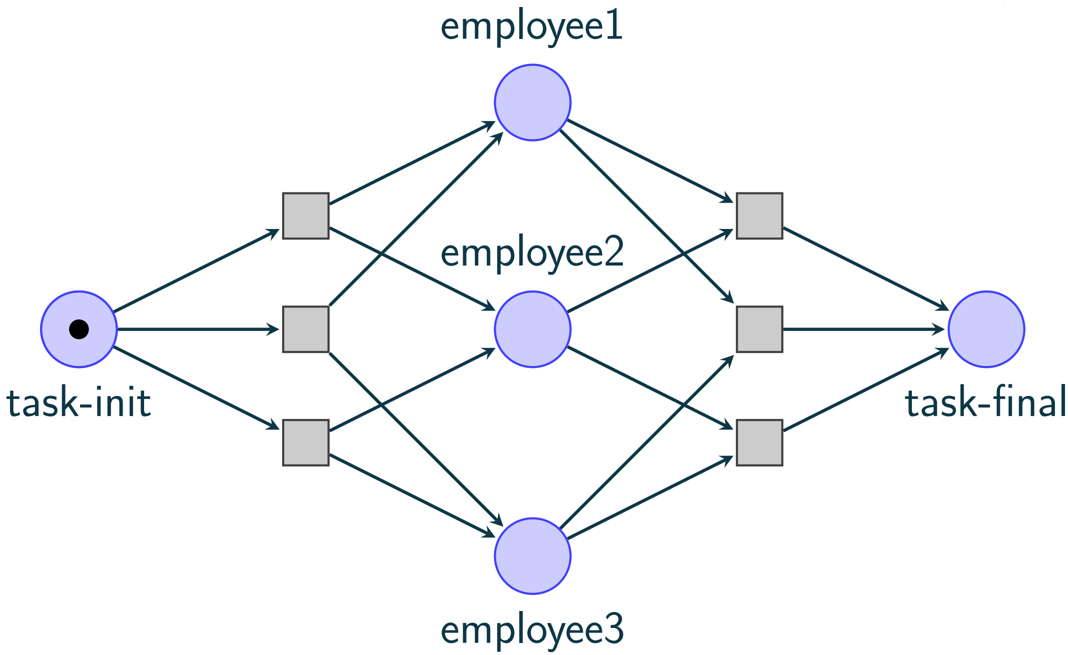

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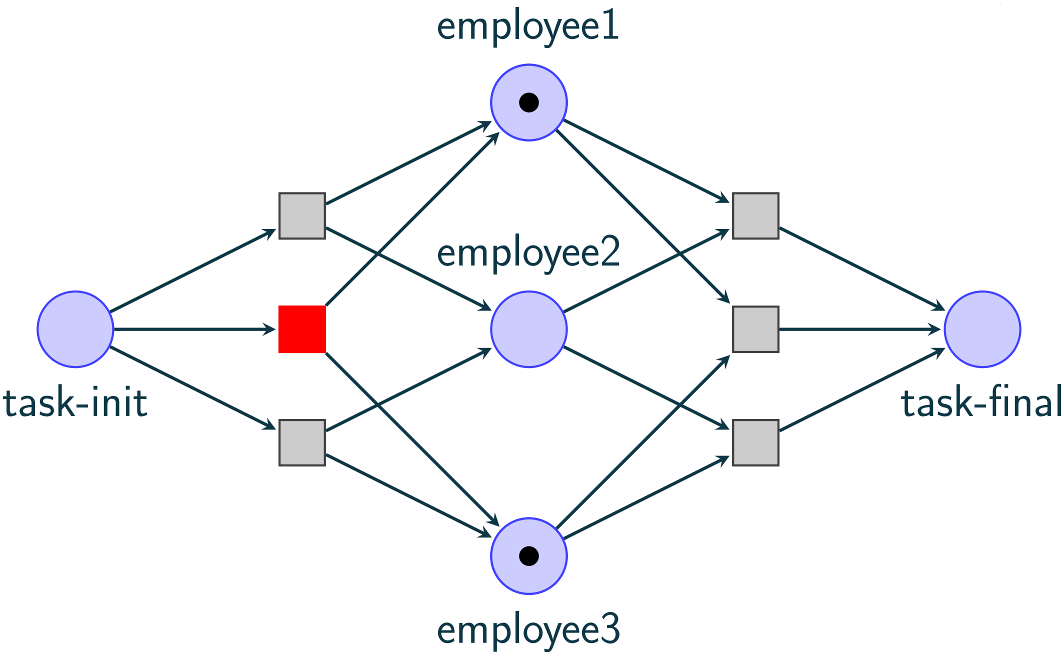

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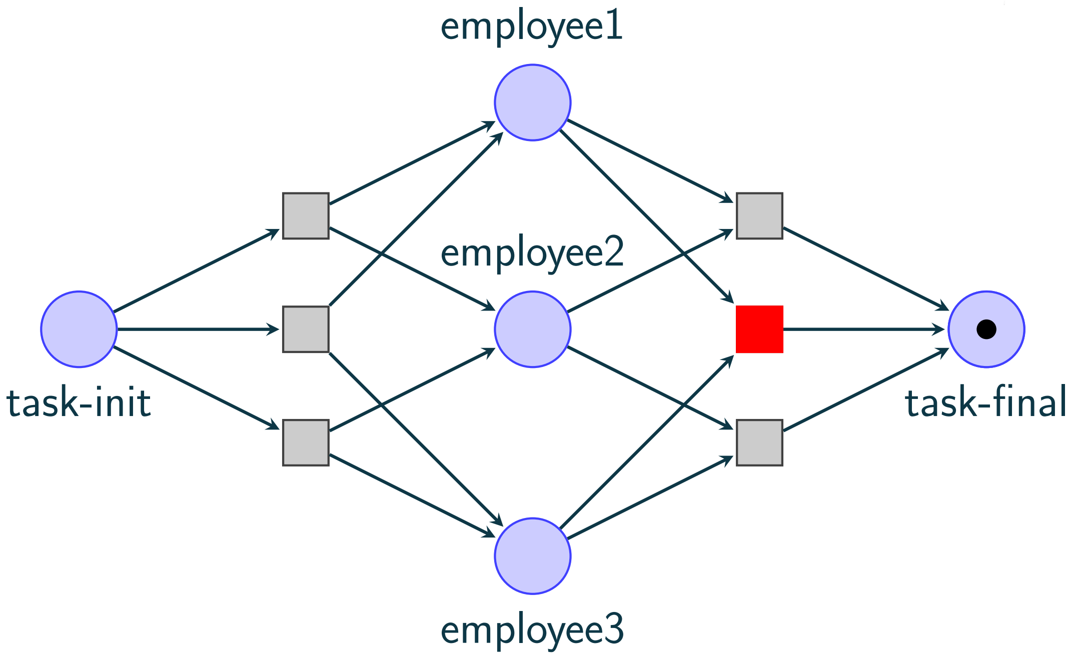

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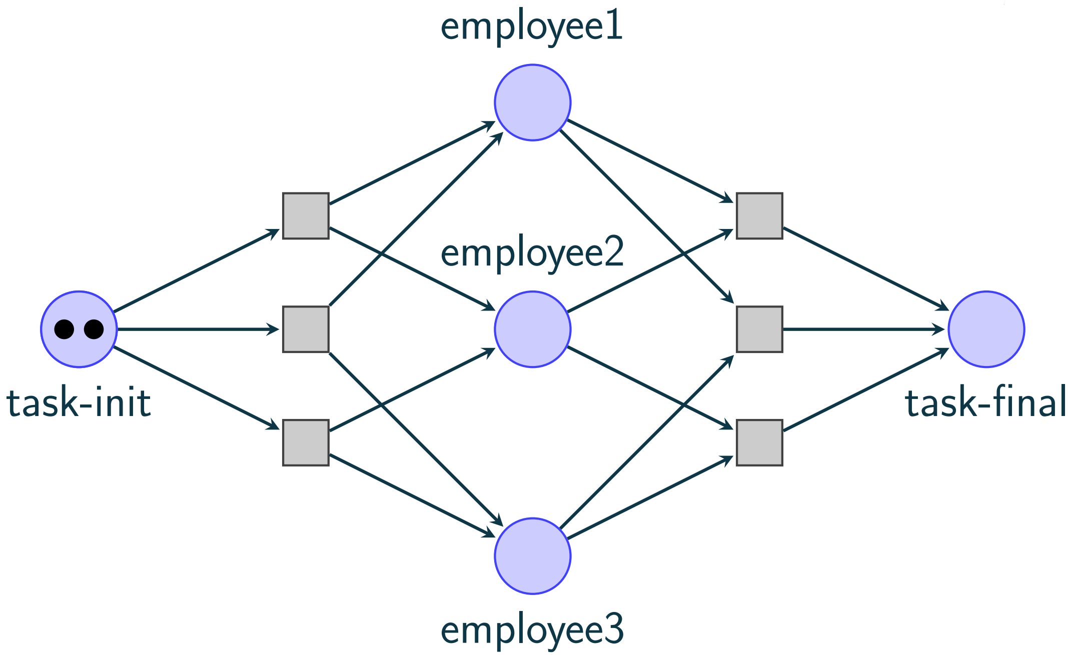

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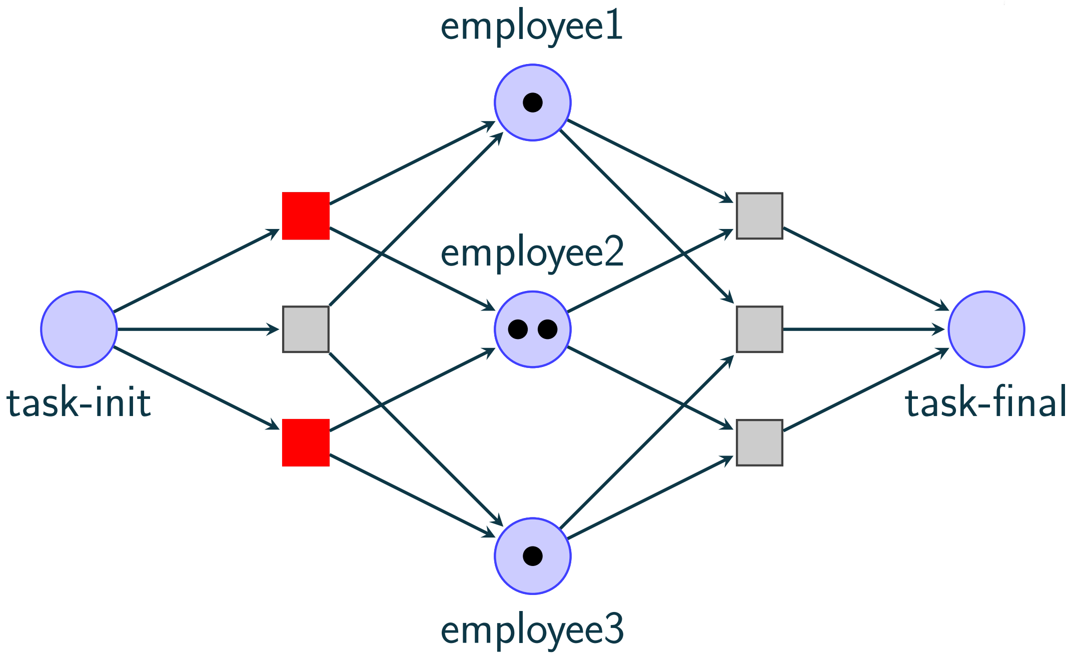

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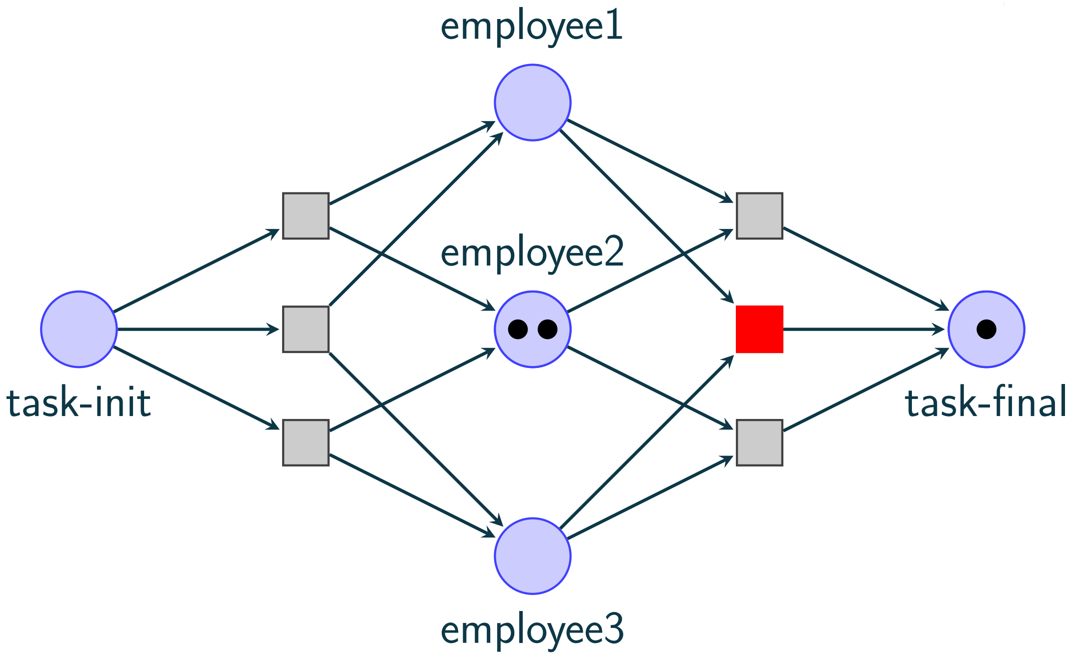

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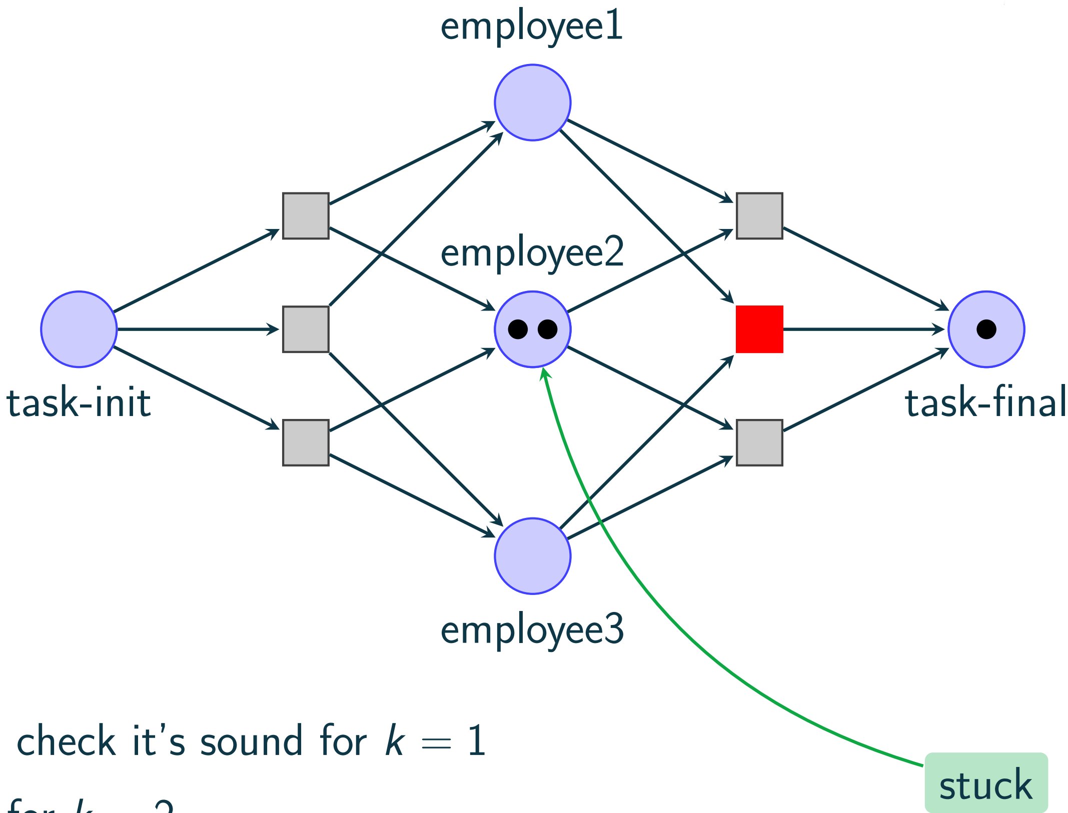

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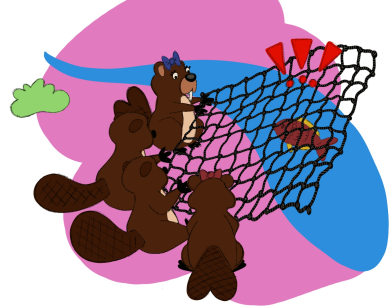
Results



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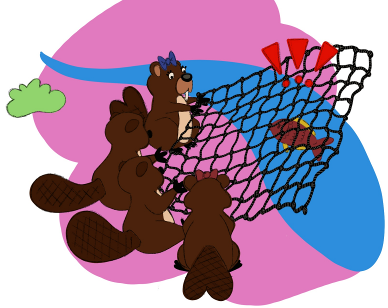

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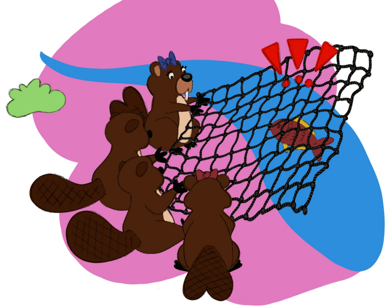

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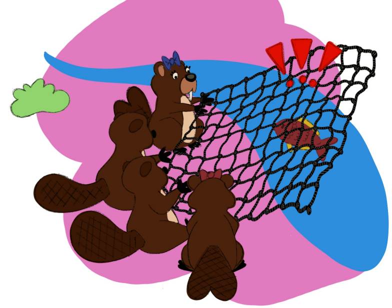

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The second results even has a follow-up implementation

Weighted automata

For simplicity just linear recursive sequences

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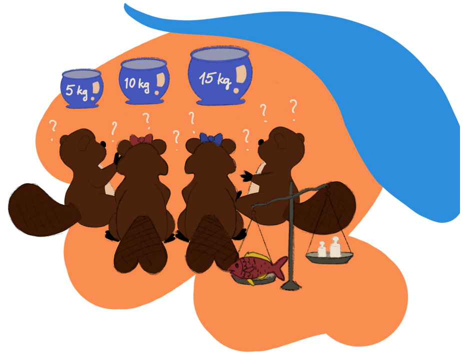
$$\text{e.g. } F_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(1 - \frac{1-\sqrt{5}}{2}\right)^n$$

Polynomial recursive sequences



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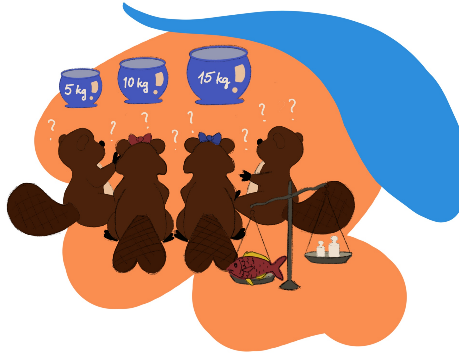

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For example $a_n = n!$

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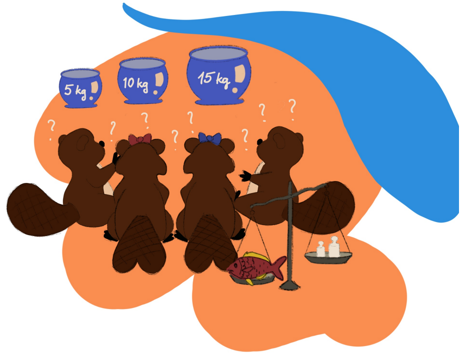

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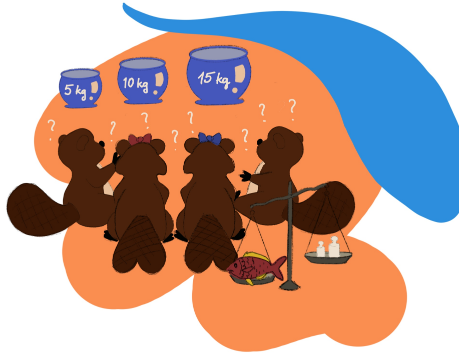

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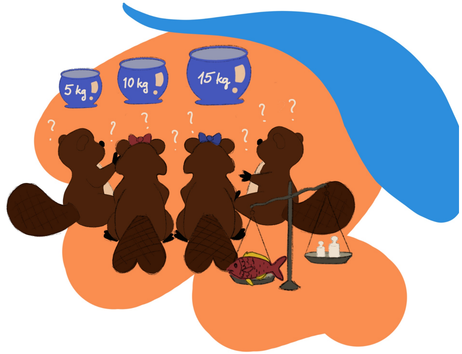

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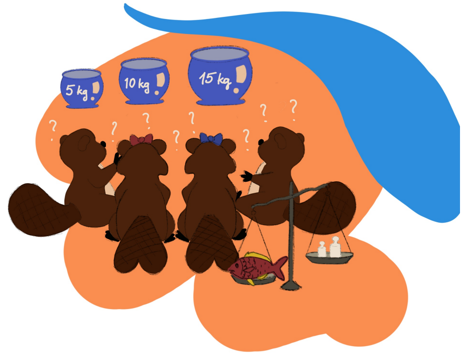

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e.g. for $n!$ we have $\quad a_{n+2} = \frac{(a_{n+1})^2}{a_n} + a_{n+1}$

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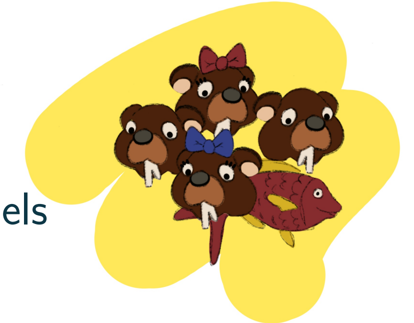

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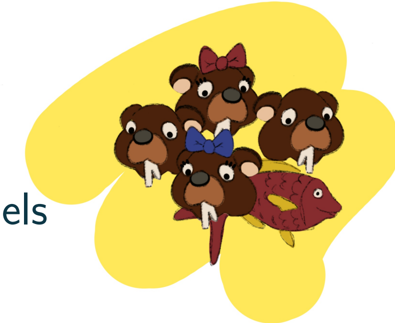

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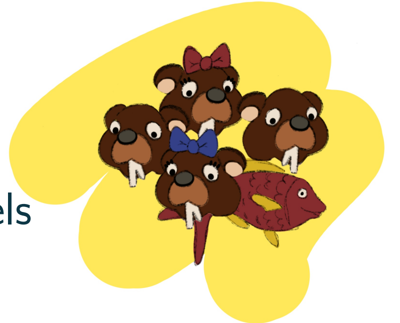

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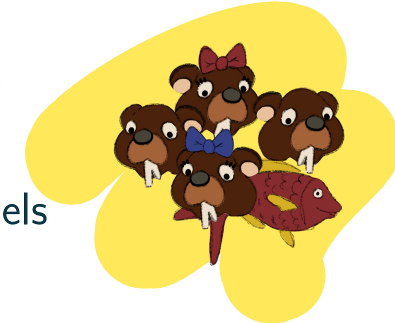

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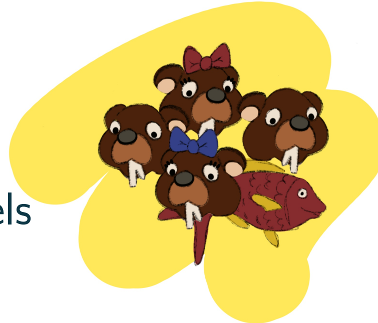

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- Michał Skrzypczak (for letting me butcher his slide package)