Lecture 2

Linear recursive sequences

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- **Fibonacci sequence** F_n

$$f_0 = 0, \qquad f_1 = 1, \qquad f_n = f_{n-1} + f_{n-2} \quad \text{for} \quad n \geqslant 2$$

f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}
0	1	1	2	3	5	8	13	21	34	55	89

Linear recursive sequences (LRS)

Fix a semiring $\mathbb{S}(\oplus, \odot, \mathbb{0}, \mathbb{1})$ (I often omit $\odot$)

$$u_{n+k} = a_k u_{n+k-1} \oplus a_{k-1} u_{n+k-1} \oplus \ldots \oplus a_1 u_n$$

fixing $a_1, \ldots, a_k \in \mathbb{S}$ and $u_0, \ldots, u_{k-1} \in \mathbb{S}$

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Definition

u_n is (homogeneous) linear recursive if there is

$L(x_1, x_2, \ldots, x_k) = a_1 x_1 \oplus \ldots \oplus a_k x_k, \quad a_i \in \mathbb{S}$

s.t. $u_{n+k} = L(u_n, \ldots, u_{n+k-1})$ for all n

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Fibonacci: $L(x_1, x_2) = x_1 + x_2$

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  This gives us a homogeneous definition for $k = 2$

$$u_{n+2} = 2u_{n+1} - u_n$$

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it's not a coincidence

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Number of previous elements referred to (k)

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- Idea: system of linear sequences

  Example: Fibonacci F_n with an extra sequence G_n ($= F_{n+1}$)

$$\begin{cases} F_0 = 0 \\ G_0 = 1 \end{cases} \qquad \begin{cases} F_{n+1} = G_n \\ G_{n+1} = F_n + G_n \end{cases}$$

System of linear sequences

Definition

A sequence is defined by a system of linear sequences if $u_n = u_n^1$ and

$$\begin{cases} u_0^1 = c_1 \\ u_0^2 = c_2 \\ \quad\vdots \\ u_0^k = c_k \end{cases} \qquad \begin{cases} u_{n+1}^1 = L_1(u_n^1, u_n^2, \ldots, u_n^k) \\ u_{n+1}^2 = L_2(u_n^1, u_n^2, \ldots, u_n^k) \\ \quad\vdots \\ u_{n+1}^k = L_k(u_n^1, u_n^2, \ldots, u_n^k) \end{cases}$$

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- This is equivalent to

  We have k dimensional vector of sequences: $\vec{u_n}^\top = (u_n^1, \ldots, u_n^k)$

  With a k dimensional vector of initial values $I^\top = (c_1, \ldots, c_k)$

  And a matrix M of dimension $k \times k$ s.t. $\vec{u_n}^\top = I^\top M^n$

  $(M[i, \bullet] = L_i)$

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where L_i are linear, $c_i \in \mathbb{S}$

$$\begin{cases} F_0 = 0 \\ G_0 = 1 \end{cases} \qquad \begin{cases} F_{n+1} = G_n \\ G_{n+1} = F_n + G_n \end{cases}$$

$$(F_n, G_n) = (0, 1)\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$$

Example definable by a system of linear equations

Over $(\mathbb{Q}, +, \cdot, 0, 1)$

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$$I = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

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$$I = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$(a_n, b_n, c_n) = I^{\mathsf{T}} M^n$

Can you define it as a linear sequence?

Equivalence of the definitions

Theorem

If $\mathbb{S}$ is a commutative ring then

(1) a_n is homogeneous linear recursive iff

(2) a_n is nonhomogeneous linear recursive iff

(3) a_n is definable as a system of linear equations

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  (3) $\implies$ (1) the hard part

From nonhomogeneous to a system

Let $L(x_1, x_2, \ldots, x_k) = a_1 x_1 + \ldots + a_k x_k + a_0$ (semiring $+$)

Such that $u_{n+k} = L(u_n, \ldots, u_{n+k-1})$

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- We use k extra sequences $u_n^0, u_n^1, \ldots, u_n^k$, where $u_n = u_n^1$

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- Finally $u_{n+1}^k = a_1 u_n^1 + \ldots + a_k u_n^k + u_n^0$

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- Example

$$u_{n+3} = 3u_{n+2} - 2u_{n+1} + 4u_n - 5 \rightsquigarrow \begin{cases} x_0 = -5 \\ u_0 = u_0 \\ b_0 = u_1 \\ c_0 = u_2 \\ d_0 = u_3 \end{cases} \begin{cases} x_{n+1} = x_n \\ u_{n+1} = b_n \\ b_{n+1} = c_n \\ c_{n+1} = d_n \\ d_{n+1} = 3u_n - 2b_n + 4c_n + x_n \end{cases}$$

From systems to homogeneous linear recursive sequences

- Suppose $\vec{u_n}^{\top} = I^{\top} M^n$
  where $\vec{u_n}^{\top} = (u_n^1, \ldots, u_n^k)$ and M is $k \times k$ dimensional
  $I^{\top} = (u_0^1, \ldots, u_0^k)$
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  $p(A) = 0$ (Cayley–Hamilton theorem)

  $a_k M^k + \ldots + a_1 M^1 + a_0 I = 0$

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  $a_k M^k + \ldots + a_1 M^1 + a_0 I = 0$

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From systems to homogeneous linear recursive sequences

- $u_n = I^{\mathsf{T}} M^n F$

  $a_k M^k + \ldots + a_1 M^1 + a_0 I = 0$

  $a_k = 1$

- $M^{n+k} = \sum_{i=0}^{k-1} -a_i M^{n+i}$

From systems to homogeneous linear recursive sequences

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This defines u_n as a homogeneous linear recursive sequence

$\blacksquare$

Example

Consider $a_n = n^2$, recall $(n+1)^2 = n^2 + 2n + 1$

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Verify that $(n+3)^2 = 3(n+2)^2 - 3(n+1)^2 + n^2$

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  (1) $\implies$ (2)

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- On tutorials: for some semirings that are not rings the inclusions:

  (1) $\subseteq$ (2) $\subseteq$ (3) are strict

Connections with weighted automata

Recall: $\llbracket \mathcal{A} \rrbracket : \Sigma^* \to \mathbb{S}$

$$\llbracket \mathcal{A} \rrbracket (a_1 a_2 \ldots a_n) = I^\top M_{a_1} M_{a_2} \ldots M_{a_n} F$$

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- For $F = (1, 0, \ldots, 0)$ these are sequences definable by systems

  On tutorials you'll see that for any F we are still in the same class

From sequences to weighted automata

- Suppose $\vec{u_n}^\top = I^\top M^n$

  where $\vec{u_n}^\top = (u_n^1, \ldots, u_n^k)$ and M is $k \times k$ dimensional

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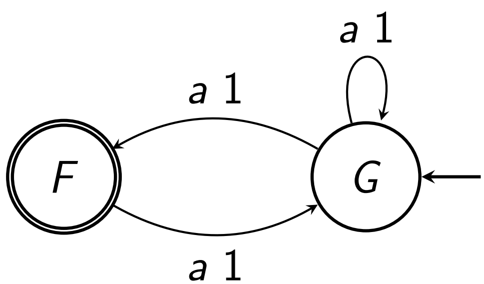
  Fibonacci over $(\mathbb{Q}, +, \cdot, 0, 1)$

  $$(F_n, G_n) = (0, 1) \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n$$



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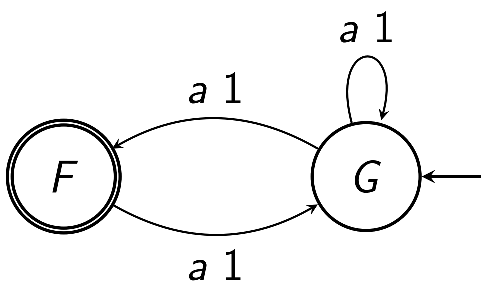

- When weights are 1 then it is equivalent to counting the accepting runs

Closed form

Fix $(\mathbb{Q}, +, \cdot, 0, 1)$

- $u_n = I^\top M^n F$ characteristic polynomial $p(x) = (x - \lambda_1)^{m_1} \ldots (x - \lambda_s)^{m_s}$

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Note $\lambda_i \in \mathbb{C}$

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- $u_n = I^\top M^n F = \sum_{i=1}^{s} p_i(n) \lambda_i^n$

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  Characteristic polynomial is $det\left(x\mathrm{Id} - \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right) = x^2 - x + 1$

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- One can verify that $F_n = \frac{1}{\sqrt{5}}\frac{1+\sqrt{5}}{2} - \frac{1}{\sqrt{5}}\frac{1-\sqrt{5}}{2}$

Asymptotics

Lemma

If u_n is a linear recursive sequence over $(\mathbb{Q}, +, \cdot, 0, 1)$

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$\left(\frac{n}{e}\right)^n = 2^{n(\log(n) - \log(e))}$

■