Generalized Autoregressive Positive-valued Processes

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[1]The views expressed in this article are mine and do not necessarily reflect those of the Bank of Canada.

Plain Vanilla time-series models

- AR(1): $x_t = \omega + \rho x_{t-1} + \epsilon_t$, where ϵ_t is a white noise.
- AR(1) have been expended in several dimensions to better capture the observed autocorrelation function of a given timeseries : AFIRMA(p,q).
- This class of model is particularly useful for forecasting future outcome of x_t: $E_t[x_{t+\tau}] = \frac{\omega}{1-\rho} + \rho^\tau \left(x_t - \frac{\omega}{1-\rho} \right)$
- However in many applications in finance and macro-economy, we are interest in more than the point forecast of x_t. For instance, We would like to forecast the uncertainty around the point-forecast, that is $E_t[(x_{t+\tau} - E_t[x_{t+\tau}])^2]$.
- The valuation of financial derivatives required the forecast of all the cumulants of x, that is $E_t[(x_{t+\tau} - E_t[x_{t+\tau}])^n]$, $\forall n > 0$, or equivalently the whole density of $x_{t+\tau}$ at time t.

Dynamic model for higher moments

- Starting with the second moment, the most popular dynamic class of models are the ARCH-GARCH models: These are models for the squared-innovation: ϵ_t^2.
- A GARCH(1,1) model postulates that ϵ_t^2 is an ARMA(1,1), that is: $\epsilon_t^2 = \omega_2 + \rho_2 \epsilon_{t-1}^2 + \eta_t - \theta_2 \eta_{t-1}$, where η_t is a white noise.
- Hence, the 1-step ahead conditional variance is

$$\begin{aligned} h_t &\equiv E_t[(x_{t+1} - E_t[x_{t+1}])^2] = E_t[\epsilon_{t+1}^2] \\ &= \omega_2 + \rho_2 \epsilon_t^2 - \theta_2 \eta_t \\ &= \omega_2 + \rho_2 \epsilon_t^2 - \theta_2(\epsilon_t^2 - h_{t-1}) \\ h_t &= \omega_2 + \theta_2 h_{t-1} + (\rho_2 - \theta_2)\epsilon_t^2 \end{aligned}$$

- h_t is an AR(1), and $E_t[h_{t+\tau}] = \frac{\omega_2}{1-\rho_2} + \rho_2^\tau \left(h_t - \frac{\omega_2}{1-\rho_2} \right)$

Dynamic model for higher moments

- The τ-step ahead conditional variance is

$$
\begin{aligned}
h_t^{(\tau)} &\equiv Var_t[x_{t+\tau}] = E_t[h_{t+\tau-1}] + Var_t[E_{t+\tau-1}[x_{t+\tau}]] \\
&= \frac{\omega_2}{1-\rho_2} + \rho_2^{\tau-1}\left(h_t - \frac{\omega_2}{1-\rho_2}\right) + \rho^2 h_t^{(\tau-1)} \\
&= \ldots \\
h_t^{(\tau)} &= \frac{1-\rho^{2\tau}}{1-\rho^2}\frac{\omega_2}{1-\rho_2} + \frac{\rho_2^\tau - \rho^{2\tau}}{\rho_2-\rho^2}\left(h_t - \frac{\omega_2}{1-\rho_2}\right)
\end{aligned}
$$

- Similar GARCH-style dynamics for conditional skewness and conditional kurtosis have been studied. First define the standardized innovation, $z_t \equiv \frac{\epsilon_t}{\sqrt{h_{t-1}}}$, and postulate the following dynamic for the one-step ahead skewness (s_t) and kurtosis k_t

$$
\begin{aligned}
s_t &= \omega_3 + \theta_3 s_{t-1} + (\rho_3 - \theta_3) z_t^3 \\
k_t &= \omega_4 + \theta_4 s_{t-1} + (\rho_4 - \theta_4) z_t^4
\end{aligned}
$$

Limitations and introduction to affine dynamics

- We have been able to compute the multi-step ahead first and second moments, but it is not possible to characterize the multi-step ahead higher moments in-closed-form.
- In many applications in finance, in particular in option pricing, it is desirable to evaluate all the moments, at any given forecasting horizon in closed-form.
- One popular family of models which enables us to characterize the conditional distribution of x_t at any horizon in closed-form is the Affine family of dynamics.
- A discrete time process x_t is called affine when its conditional cumulant function, denoted $\psi_t(u)$, and defined as the logarithmic of the moment generating function,i.e.,

$$\psi_t(u) \equiv \log[E[\exp(ux_{t+1}) \mid x_\tau, \tau \leq t]],$$

is given by

$$\psi_t(u) = \omega(u) + \alpha(u)x_t. \tag{1}$$

Multi-Steps Ahead cumulant function

Given $\psi_t(u)$ in equation (1), $\psi_{t,\tau}(u) \equiv \log\left(E_t[\exp(ux_{t+\tau})]\right)$ is also an affine function of x_t, that is: $\psi_{t,\tau}(u) = \omega(u,\tau) + \alpha(u,\tau)x_t$.

- $$\begin{aligned}\psi_{t,\tau}(u) &= \log\left(E_t[E_{t+\tau-1}[\exp(ux_{t+\tau})]]\right)\\ &= \log\left(E_t[\exp(\psi_{t+\tau-1}(u))]\right)\\ &= \log\left(E_t[\exp(\omega(u) + \alpha(u)x_{t+\tau-1})]\right)\\ &= \omega(u) + \psi_{t,\tau-1}(\alpha(u))\end{aligned}$$

- Hence,

$$\begin{aligned}\omega(u,\tau) &= \omega(u) + \omega(\alpha(u), \tau-1)\\ \alpha(u,\tau) &= \alpha(\alpha(u), \tau-1)\end{aligned}$$

- with $\omega(u,1) = \omega(u)$ and $\alpha(u,1) = \alpha(u)$

Leading example of affine dynamics in finance: ARG

The Autoregressive gamma (ARG) processes is very popular in finance. Its corresponding functions $\omega(u)$ and $\alpha(u)$ for the scalar $u < 1/\varphi$ are given by:

$$\omega(u) = -\nu \log(1 - u\varphi), \text{ and } \alpha(u) = \frac{\phi u}{1 - u\varphi}, \tag{2}$$

with $\nu \geq 0$, $\varphi > 0$ and $\phi \geq 0$. It admits the following state space representation:

$$\begin{aligned} \frac{x_{t+1}}{\varphi} | U_{t+1}, I_t &\sim \gamma\left(\nu + U_{t+1}\right) \\ U_{t+1} | I_t &\sim P\left(\frac{\phi x_t}{\varphi}\right), \end{aligned}$$

where U_{t+1} is a latent process that follows a Poisson distribution denoted by $P(\cdot)$ and $\gamma(\cdot)$ is the standard gamma distribution.

Leading example of affine dynamic in option pricing: Heston-Nandi

The Heston-Nandi model is arguably the most popular discrete-time option pricing model. It is an affine-GARCH model where the dynamic of the conditional variance is given by

$$x_{t+1} = w + bx_t + a\left(\varepsilon_{t+1} - c\sqrt{x_t}\right)^2, \tag{3}$$

where

$$\varepsilon_{t+1} \sim i.i.dN\left(0,1\right).$$

The log-conditional moment generating function x_{t+1} is affine in x_t:

$$\psi_t(u) = \ln\left\{E_t\left[\exp\left(ux_{t+1}\right)\right]\right\} = \omega_{hn}\left(u\right) + \alpha_{hn}\left(u\right)x_t,$$

where

$$\begin{aligned}\omega_{hn}\left(u\right) &= uw - \frac{1}{2}\ln\left(1-2ua\right)\\ \alpha_{hn}\left(u\right) &= ub + \frac{uac^2}{1-2ua}.\end{aligned}$$

Limitations of the affine dynamic

- All the conditional cumulants (the derivatives of $\psi_t(u)$ at $u = 0$) are driven by the same factor x_t. Indeed, we have

$$\psi_t^{(n)}(0) = \omega^{(n)}(0) + \alpha^{(n)}(0)x_t, \tag{4}$$

  with $f^{(n)}(u)$, the n^{th} order derivative of function $f(\cdot)$ at u.
- This suggests that both the conditional expectation and the conditional variance of x_t are driven by x_t, and are, therefore, perfectly positively correlated, and that all moments are highly positively correlated.
- There is considerable empirical evidence contradicting the very tight restriction between the moments.
- This inability of affine dynamics to fit all the moments jointly implies that they cannot fit the conditional density and, hence, they generate large option pricing errors.

Intro to the Generalized Autoregressive Affine Dynamics

The key principle of our generalization of Affine processes to a GARP dynamic is simple: we want each cumulant $\psi_t^{(n)}(0)$ to be driven by its own specific factor (say, $m_t^{(n)}$), that is,

$$\psi_t^{(n)}(0) = \omega_n + \alpha_n m_t^{(n)}, \tag{5}$$

where $m_t^{(n)}$ is a moving average of x_t:

$$m_t^{(n)} = x_t + \theta_n m_{t-1}^{(n)}. \tag{6}$$

One way to achieve that with a minimal number of additional parameters is to set $\theta_n = \beta\theta^n$, which is equivalent to the following recursive formulation of the conditional cumulant generating function:

$$\psi_t(u) = \omega(u) + \alpha(u)x_t + \beta\psi_{t-1}(\theta u) \qquad \text{for } t \geq 1, \tag{7}$$

where functions $\alpha(\cdot)$ and $\omega(\cdot)$ are given in equation (2).

Will this works?

From equation (7), we have

$$\psi_t^{(n)}(0) = \omega^{(n)}(0) + \alpha^{(n)}(0)x_t + \beta\theta^n\psi_{t-1}^{(n)}(0)\,, \tag{8}$$

Equation (8) implies that:

$$\psi_t^{(n)}(0) = \frac{\omega^{(n)}(0)}{1-\beta\theta^n} + \alpha^{(n)}(0)\left(\sum_{j=0}^{\infty}(\beta\theta^n)^j x_{t-j}\right). \tag{9}$$

Consequently, each conditional cumulant (i.e., $\psi_t^{(n)}(0)$ for a given n) is driven by its own factor ($m_t^{(n)}$):

$$m_t^{(n)} = \sum_{j=0}^{\infty}(\beta\theta^n)^j x_{t-j}, \tag{10}$$

which is a moving average of the variable of interest x_t. Hence, with only two additional parameters (β and θ), we are able to generate a parsimonious generalization of ARG processes that disentangles the dynamics of all the conditional moments.

Can we characterize the multi-step distribution?

Like affine models, an important characteristic of GARP processes is the existence of a closed-form forecast of any nonlinear transformation of a GARP process at any horizon. The multi-horizon cumulant generating function defined as

$$\psi_t\left(u;\tau\right) \equiv \ln\left[E_t\left[\exp\left(ux_{t+\tau}\right)\right]\right]$$

is computed analytically in section 3.1 of the Appendix where we establish that:

$$\begin{aligned} \psi_t\left(u;\tau\right) &= \sum_{j=1}^{\tau}\beta^{j-1}\psi_t\left(\theta^{j-1}u_j\right) + \sum_{j=2}^{\tau}\sum_{i=0}^{j-2}\beta^{i}\omega\left(\theta^{i}u_j\right) \text{ for } \tau \geq 2 \quad (11) \\ u_\tau &= u, \quad u_l = \sum_{i=l+1}^{\tau}\beta^{i-(l+1)}\alpha\left(\theta^{i-(l+1)}u_i\right) \text{ for } 1 \leq l \leq \tau - 1. \end{aligned}$$

Main theoretical challenge

The main theoretical challenge of this paper is to build a family of processes whose the cumulant generating function has the recursive formulation in equation (7).

Generalized autoregressive gamma processes (GARG)

The GARG process through the following state space representation:

$$x_{t+1} = \bar{Z}_{t+1} + \mathbf{1}_{[t>0]} \left[\sum_{j=0}^{t-1} Z_{t+1}^{(j)} \right], \quad t \geq 0, \tag{12}$$

where $\mathbf{1}_{[\cdot]}$ is an indicator function, and for $t > 0$, $\bar{Z}_{t+1}$ and $Z_{t+1}^{(j)}$ with $j = 0, ..., t-1$ are $t+1$ conditionally (conditional on I_t) independent random variables with the following state-space representation:

$$\frac{Z_{t+1}^{(j)}}{\varphi_j} | U_{t+1}^{(j)}, I_t \sim \gamma\left(\nu_j + U_{t+1}^{(j)}\right) \tag{13}$$

$$U_{t+1}^{(j)} | I_t \sim P\left(\frac{\phi_j x_{t-j}}{\varphi_j}\right), \tag{14}$$

where

$$\nu_j = \nu\beta^j, \quad \varphi_j = \varphi\theta^j, \quad \phi_j = \phi\beta^j\theta^j. \tag{15}$$

Generalized autoregressive gamma processes (GARG)

The cumulant generating function of $\bar{Z}_{t+1}$ is $\beta^t \psi_0 \left(\theta^t u\right)$, with

$$\psi_0 \left(u\right) = \frac{\phi}{1-\beta\theta} \frac{\theta u}{1-\theta\varphi u} \mu - \frac{\nu}{1-\beta\theta} \ln \left(1-\theta\varphi u\right). \tag{16}$$

The GARG has five parameters, ν, φ, ϕ, β and θ, with the following restrictions:

$$\nu, \varphi, \phi, \beta, \theta \geq 0, \quad \beta\theta < 1. \tag{17}$$

Main Result

Let us assume that the positive-valued univariate process of interest x_t follows the dynamics described in equations (12), (13), and (14). Then, for a scalar u such that $1 - u\varphi_j > 0$ for all $j > 0$, the conditional cumulant generating function of x_{t+1} ($\psi_t(u)$) exists and evolves according to the following recursive dynamic:

$$\psi_t(u) = \omega(u) + \alpha(u)x_t + \beta\psi_{t-1}(\theta u) \qquad \text{for } t \geq 1, \tag{18}$$

where functions $\alpha(\cdot)$ and $\omega(\cdot)$ are given in equation (2).

Proof of the Main Result

Using equation (12) and the fact that all the Zs on the right-hand side are conditionally independent, we have:

$$\psi_t(u) = \ln E_t\left[\exp\left(u\bar{Z}_{t+1}\right)\right] + \sum_{j=0}^{t-1} \ln E_t\left[\exp\left(uZ_{t+1}^{(j)}\right)\right].$$

By assumption,

$$\ln E_t\left[\exp\left(u\bar{Z}_{t+1}\right)\right] = \beta^t \psi_0\left(\theta^t u\right),$$

and the state-space representation given by equations (13) and (14) implies that

$$\ln E_t\left[\exp\left(uZ_{t+1}^{(j)}\right)\right] = \beta^j \omega(\theta^j u) + \beta^j \alpha(\theta^j u) x_{t-j}. \qquad (19)$$

Proof of the Main Result

Hence,

$$
\begin{aligned}
\psi_t(u) &= \beta^t \psi_0\left(\theta^t u\right) + \sum_{j=0}^{t-1} \beta^j \left[\omega(\theta^j u) + \alpha(\theta^j u) x_{t-j}\right] \\
&= \omega(u) + \alpha(u) x_t + \beta^t \psi_0\left(\theta^t u\right) + \sum_{j=1}^{t-1} \beta^j \left[\omega(\theta^j u) + \alpha(\theta^j u) x_{t-j}\right] \\
&= \omega(u) + \alpha(u) x_t + \\
&\quad \beta \left\{ \beta^{t-1} \psi_0\left(\theta^{t-1} \theta u\right) + \sum_{k=0}^{t-2} \beta^k \left[\omega(\theta^k \theta u) + \alpha(\theta^k \theta u) x_{t-1-k}\right] \right\} \\
&= \omega(u) + \alpha(u) x_t + \beta \psi_{t-1}(\theta u),
\end{aligned}
$$

which establishes the main result.

Ergodicity and unconditional distribution

- Weak ergodicity conditions, are conditions under which the distribution at horizon τ tends to a limiting distribution.
- Weak ergodicity is equivalent to the convergence of the multi-horizon cumulant generating function, which is also equivalent to the convergence of the τ-step ahead n-th conditional cumulant $\psi_t^{(n)}(0,\tau)$ as τ increases and for all n.
- We show the following result: $\psi_t^{(n)}(0,\tau)$ converges as τ increases if and only if $\rho < 1$ and $\beta\theta^j < 1$ for $j = 1, ..., n$, with $\rho \equiv \alpha'(0) + \beta\theta = \phi + \beta\theta$.
- Thus the following corollary: If $\phi < 1$, $\beta\theta < 1 - \phi$, and $\theta \leq 1$, the h-step ahead conditional distribution of a GARP process converges as τ increases.

The GARP is an ARMA(1,1) dynamic

x_t is an ARMA(1,1) with the autoregressive parameter $\rho = \alpha'(0) + \beta\theta$ and the moving average parameter $\beta\theta$. Indeed, we have

$$\begin{aligned} x_{t+1} &= \psi_t'(0) + \underbrace{x_{t+1} - \psi_t'(0)}_{u_{t+1}} \\ &= \underbrace{\omega'(0) + \alpha'(0)\,x_t + \beta\theta\psi_{t-1}'(0)}_{=\psi_t'(0)} + u_{t+1} \\ &= \omega'(0) + \alpha'(0)\,x_t + \beta\theta\underbrace{(x_t - u_t)}_{\psi_{t-1}'(0)} + u_{t+1} \\ &= \omega'(0) + \left(\alpha'(0) + \beta\theta\right)x_t + u_{t+1} - \beta\theta u_t. \end{aligned}$$

Hence

$$Corr\left(x_t, x_{t+h}\right) = \rho^{h-1} Corr(x_t, x_{t+1}) \text{ if } h \geq 1,$$

$$Corr\left(x_t, x_{t+1}\right) = \alpha'(0)\left[\frac{1 - (\beta\theta)^2 - \alpha'(0)\beta\theta}{1 - (\beta\theta)^2 - 2\alpha'(0)\beta\theta}\right].$$

Application: Option Pricing

The day t stock price and return by S_t and R_t, with $R_t \equiv \ln(S_t/S_{t-1})$. We design an option pricing model where returns and realized variances (RV_t) are modeled jointly:

$$
\begin{aligned}
R_{t+1} &= \ln(S_{t+1}/S_t) = r + \left(\lambda - \frac{1}{2}\right) RV_{t+1} + \sqrt{RV_{t+1}}\varepsilon_{t+1} \\
\varepsilon_{t+1} &\sim i.i.dN(0,1) \\
RV_{t+1} &\sim GARG(\phi, \varphi, \nu, \beta, \theta),
\end{aligned}
$$

Benchmark models are variants of the ARG(p,q)

$$\psi_t(u) \equiv \ln[E[\exp(uRV_{t+1})|I_t]] = \omega(u) + \alpha(u)m_t,$$

where

1. ARG0: $m_t = RV_t$
2. ARG1: $m_t = RV_t + \theta_1 m_{t-1}$
3. ARG2: $m_t = RV_t + \theta_1 m_{t-1} + \theta_2 m_{t-2}$.
4. MARG: $RV_t = x_{1,t} + x_{2,t}$, where $x_{j,t} \sim ARG(\nu_j, \varphi, \phi_j)$

MLE

		ARG Models					
		ARG0		ARG1		ARG2	
Parameters		Est	SE	Est	SE	Est	SE
ϕ		0.711	5.14E-03	0.362	6.53E-03	0.371	7.54E-03
φ		8.19E-03	5.93E-05	6.71E-03	4.79E-05	6.69E-03	4.83E-05
ν		1.017	0.034	0.978	0.042	0.975	0.042
θ_1		0		0.531	8.29E-03	0.440	0.027
θ_2		0		0		0.081	0.021
Model Properties	Obs						
Avg	16.98	16.98		17.00		17.00	
Vol	18.27	16.91		15.44		15.37	
Skew	2.74	1.96		1.49		1.47	
Kurt	12.07	8.69		6.20		6.13	
AC(1)	0.67	0.71		0.57		0.57	
Log Likelihoods		13320		13577		13580	
BIC		-5.94		-6.05		-6.05	
LR P-Value, H0: ARG0				0.00		0.00	

MLE

		MARG		GARG	
Parameters		Est	SE	Est	SE
ϕ		0.954	1.53E-02	0.252	4.18E-04
φ		0.011	3.36E-05	0.017	6.97E-05
ν		0.013	2.06E-02	0.039	1.62E-04
β				1.171	5.91E-03
θ				0.619	3.12E-03
ϕ_2		0.914	6.00E-03		
ν_2		0.198	1.88E-02		
Model Properties	Obs				
Avg	16.98	16.98		16.98	
Vol	18.27	18.27		18.27	
Skew	2.74	2.05		2.19	
Kurt	12.07	10.04		9.80	
AC(1)	0.67	0.93		0.67	
Log Likelihoods		13597		14025	
BIC		-6.07		-6.25	
LR P-Value, H0: ARG0		0.00		0.00	

Estimation using Options

Parameters	ARG Models					
	ARG0		ARG1		ARG2	
	Est	SE	Est	SE	Est	SE
ϕ	0.938	7.65E-06	0.016	8.17E-05	0.016	8.28E-05
φ	2.90E-05	1.74E-07	9.50E-04	3.72E-05	9.49E-04	1.07E-05
ν	0.219	1.51E-05	0.032	1.20E-03	0.032	6.20E-05
θ_1	0		0.974	1.41E-04	0.963	2.12E-04
θ_2			0		0.011	7.18E-05
Model Properties						
Log Likelihoods	31554		37271		37273	
BIC	-2.96		-3.50		-3.50	
LR P-Value, H0: ARG0			0.00		0.00	
Avg. Model IV	20.54		20.74		20.74	
Variance Persistence						
	0.938		0.9905		0.9904	
Option Errors						
IVRMSE	5.493		4.199		4.199	
Ratio to ARG0	1.000		0.764		0.764	
DM test P-Value			0.00		0.156	

Estimation using Options

	MARG		GARG	
Parameters	Est	SE	Est	SE
ϕ	0.962	1.43E-04	0.020	5.15E-05
φ	1.88E-05	8.82E-07	1.75E-04	1.23E-05
ν	0.145	9.43E-03	4.84E-03	3.07E-06
β			1.079	1.13E-06
θ			0.897	4.66E-07
ϕ_2	0.547	1.17E-02		
ν_2	0.697	4.77E-02		
Model Properties				
Log Likelihoods	37845		38236	
BIC	-3.55		-3.59	
LR P-Value, H0: ARG0	0.00		0.00	
Avg. Model IV	20.78		20.80	
Variance Persistence				
	0.998		0.9801	
Option Errors				
IVRMSE	3.945		3.862	
Ratio to ARG0	0.720		0.703	
DM test P-Value	0.00		0.00	

IVRMSE Option Error by Moneyness, Maturity

	OTM Call		
	Delta < 0.3	$0.3 \leq$ Delta < 0.4	$0.4 \leq$ Delta < 0.5
Panel A: IVRMSE by Moneyness			
ARG0	6.284	4.721	4.739
ARG1	5.183	2.985	3.046
ARG2	5.184	2.985	3.045
MARG	4.032	3.078	2.924
GARG	3.985	2.757	2.910
	DTM < 30	$30 \leq$ DTM < 60	$60 \leq$ DTM < 90
Panel B: IVRMSE by Maturity			
ARG0	5.845	5.205	5.525
ARG1	4.344	4.270	4.076
ARG2	4.344	4.270	4.075
MARG	4.208	4.094	3.943
GARG	3.736	3.953	3.710

IVRMSE Option Error by Moneyness, Maturity

	OTM Put		
	$0.5 \leq \text{Delta} < 0.6$	$0.6 \leq \text{Delta} < 0.7$	$\text{Delta} \geq 0.7$
Panel A: IVRMSE by Moneyness			
ARG0	5.465	5.737	5.318
ARG1	3.335	4.018	4.414
ARG2	3.335	4.017	4.413
MARG	3.676	4.202	5.727
GARG	3.034	3.375	4.490
	$90 \leq \text{DTM} < 120$	$120 \leq \text{DTM} < 150$	$\text{DTM} \geq 150$
Panel B: IVRMSE by Maturity			
ARG0	5.673	5.532	5.525
ARG1	3.937	4.265	4.356
ARG2	3.937	4.264	4.355
MARG	3.528	3.886	3.724
GARG	3.677	4.293	3.947

Next Steps?

- β is constant, can we build a generalized autoregressive cumulant generating function model where the autoregressive function $\beta(\cdot)$ varies?
- Answer, yes, but I am still working on the properties of this creature.
- Stay-turn and thanks a lot for hosting me and/or attending the talk.