Bilkent University

Electrical and Electronics Department

EE321-02 Lab 4 Report:

14/11/2024

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Introduction:

In this lab, we learned the Fourier series expansion and some related approximations.

“lab4q1.m” is the MATLAB file where all the code for part 1 lies within. **(Appendix 1)**

“lab4q2.m” is the MATLAB file where all the code for part 2 lies within. **(Appendix 2)**

“lab4q3.m” is the MATLAB file where all the code for part 3 lies within. **(Appendix 3)**

Lab Work:

Question 1:

a)

Here is the signal $y_a(t)$ and the plot of the discretized signal $y_a(t)$ with sampling period 1/10 seconds **(Figures 1.a.1 & 1.a.2)**:

$$y_a(t) = \begin{cases} 0 & t \in [0,7)\text{s.} \\ 8 & t \in [7,10)\text{s.} \\ 0 & t \in [10,18)\text{s.} \end{cases}$$

Figure 1.a.1: $y_a(t)$

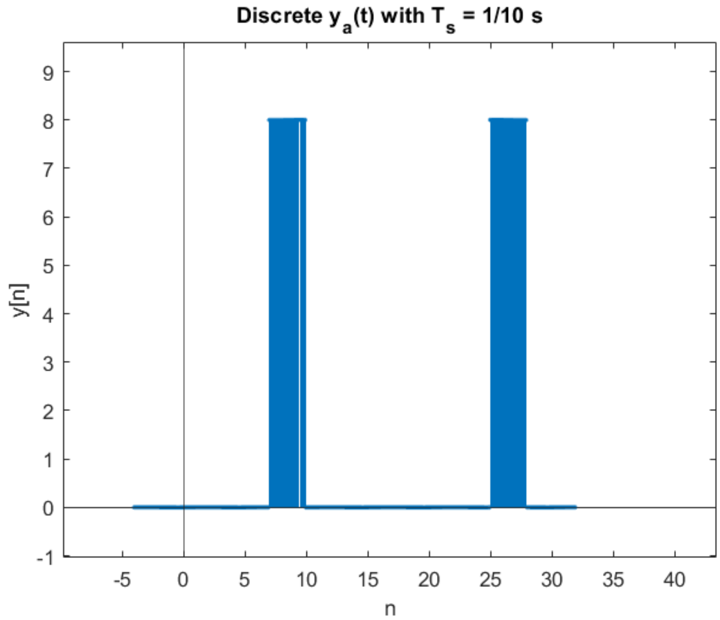

Figure 1.a.2: Plot of the Discrete $y_a(t)$ with T_s = 1/10 seconds

b)

Here you can see the Fourier series expansion of $y_a(t)$ **(Figure 1.b)**:

1-) $y_a(t) = \begin{cases} 0, & t \in [0,7)\ s \\ 8, & t \in [7,10)\ s \\ 0, & t \in [10,18)\ s \end{cases}$

$y_a(t) = y_a(t+18n);\ n \in Z$

$$a_k = \frac{1}{T}\int_{o.p} y_a(t)\cdot e^{-j\frac{2\pi}{T}kt}\,dt = \frac{1}{18}\int_{7}^{10} 8\cdot e^{-j\frac{2\pi}{18}\cdot k\cdot t}\,dt =$$

$$= \frac{-4}{9}\cdot\frac{18}{j\cdot 2\pi k}\cdot\left(e^{-j\frac{2\pi}{18}\cdot 10k} - e^{-j\frac{2\pi}{18}\cdot 7k}\right) = \frac{4}{j\pi k}\left(e^{j\frac{22\pi}{18}k} - e^{j\frac{16\pi}{18}k}\right) =$$

$$\underset{k\neq 0}{a_k} = \frac{4}{j\pi k}\cdot\left[\cos\left(\frac{11\pi}{9}\cdot k\right) - \cos\left(\frac{8\pi}{9}k\right) + j\cdot\sin\left(\frac{11\pi}{9}k\right) - j\cdot\sin\left(\frac{8\pi}{9}\cdot k\right)\right]$$

$$a_k = \frac{4}{\pi k}\cdot\underbrace{\left[\sin\left(\frac{11\pi}{9}k\right) - \sin\left(\frac{8\pi}{9}k\right) + j\cdot\cos\left(\frac{8\pi}{9}k\right) - j\cdot\cos\left(\frac{11\pi}{9}k\right)\right]}_{t_k}$$

$$a_0 = \frac{1}{T}\int_{o.p} y_a(t)\,dt = \frac{1}{18}\int_{7}^{10} 8\,dt = \frac{24}{18} = 4/3$$

$$F.S.E\,(y_a(t)) = \boxed{\frac{4}{3} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{4\cdot t_k}{\pi k}\cdot e^{j\frac{\pi}{9}k\cdot t}}$$

Figure 1.b: F.S.E. of $y_a(t)$

c)

(1) shows the relationship between the coefficients k and their respective a_k. Also, here is the spectrum of coefficients for both the real and imaginary parts of a_k **(Figures 1.c.1 & 1.c.2)**:

$$a_k = \begin{Bmatrix} \frac{4}{3}\,;\, k=0 \\ \frac{4}{j\pi k}\cdot\left(e^{j\frac{22}{18}\pi k} - e^{j\frac{16}{18}\pi k}\right);\, k\neq 0 \end{Bmatrix} \qquad \textbf{(1)}$$

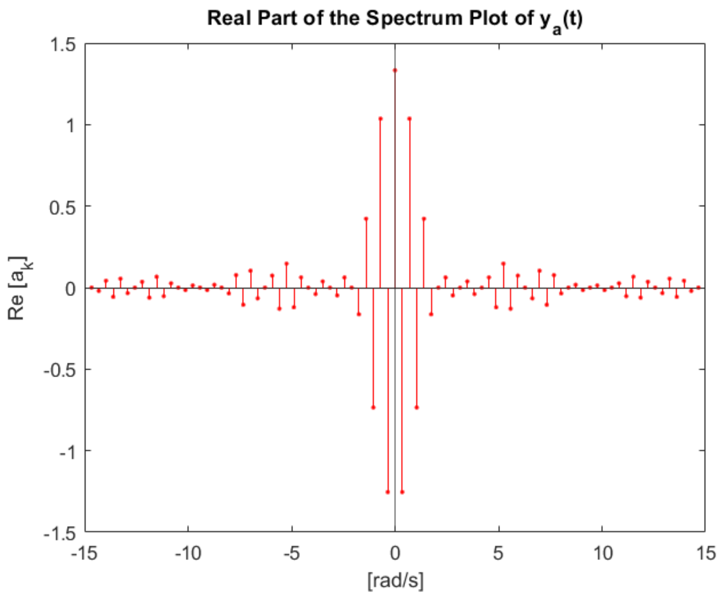

Figure 1.c.1: Spectrum Plot of the Real Part of a_k

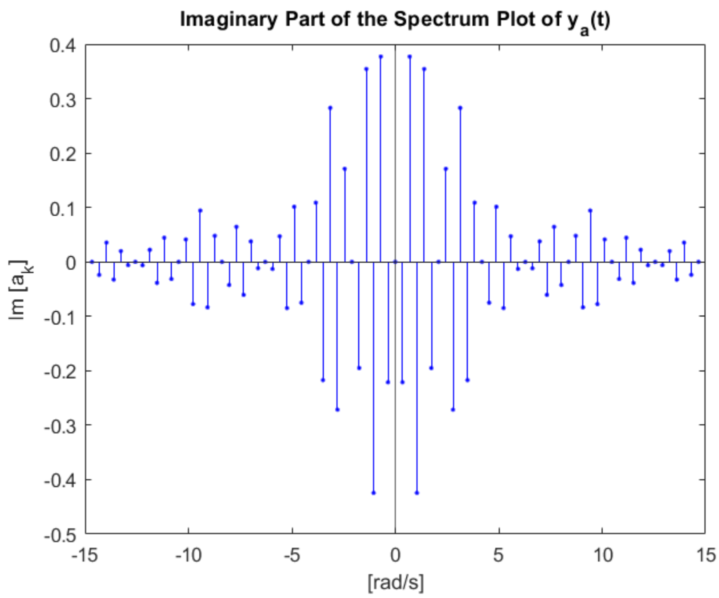

Figure 1.c.2: Spectrum Plot of the Imaginary Part of a_k

d)

By using the F.S.E from part b, we can express zN[n] as in **(2)**.

$$z_N[n] \; = \; \frac{4}{3} + \; \sum_{k=-N\,;\,k\neq 0}^{N} a_k \cdot e^{j\frac{\pi}{9}\cdot k\cdot\frac{n}{9}}\; for\; n\; \in [-40\,,\,319] \qquad \textbf{(2)}$$

Here is the plot of $z_N[n]$ when N = 150 **(Figure 1.d)**:

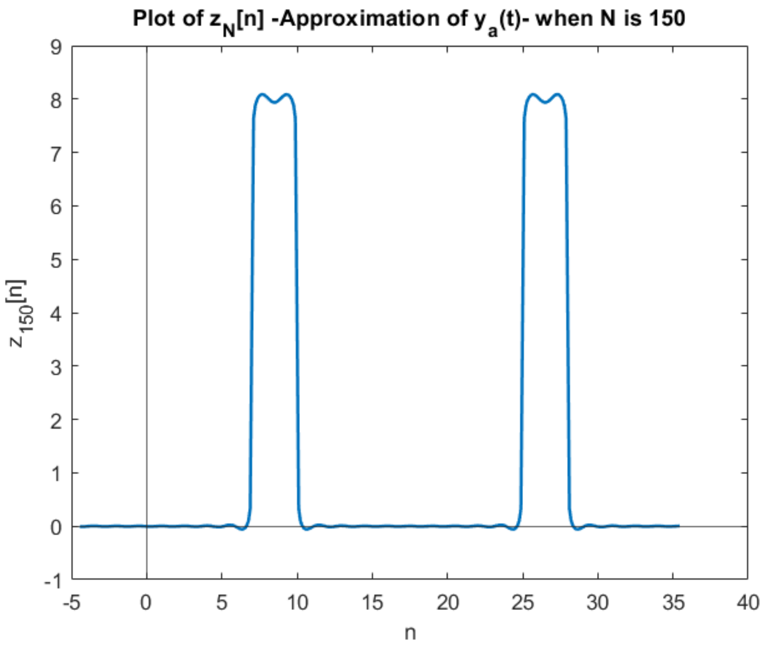

Figure 1.d: Plot of $z_N[n]$ when N = 150

The plot does look like the original function $y_a(t)$ in some sense, but it is not completely the same. The reason for this is because N should go all the way to infinity for our signal to look more and more like $y_a(t)$. As N gets closer to zero, the approximation should start to look more different from the original function.

e)

Here is the plot of $z_N[n]$ when N = 75 **(Figure 1.e)**:

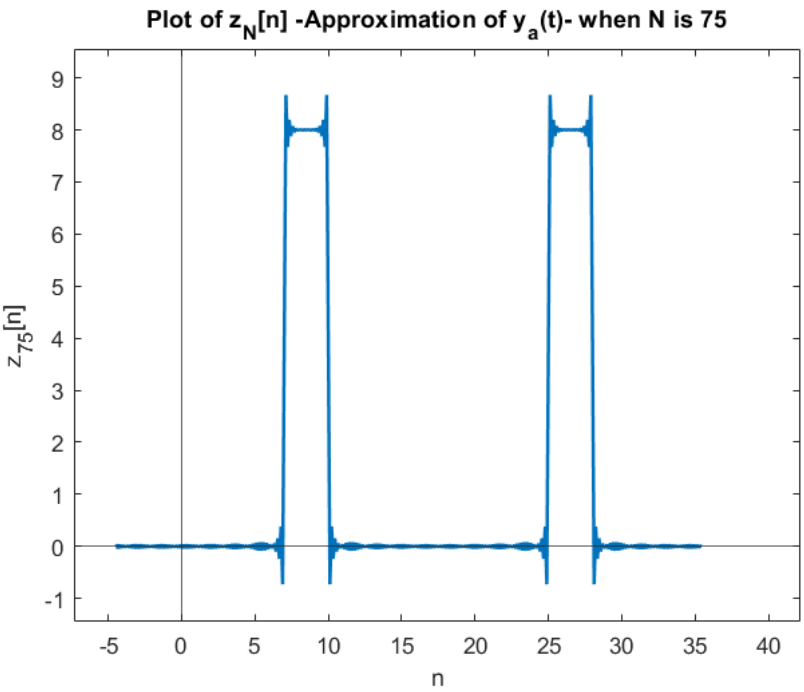

Figure 1.e: Plot of z_N[n] when N = 75

f)

Here is the plot of z_N[n] when N = 30 **(Figure 1.f)**:

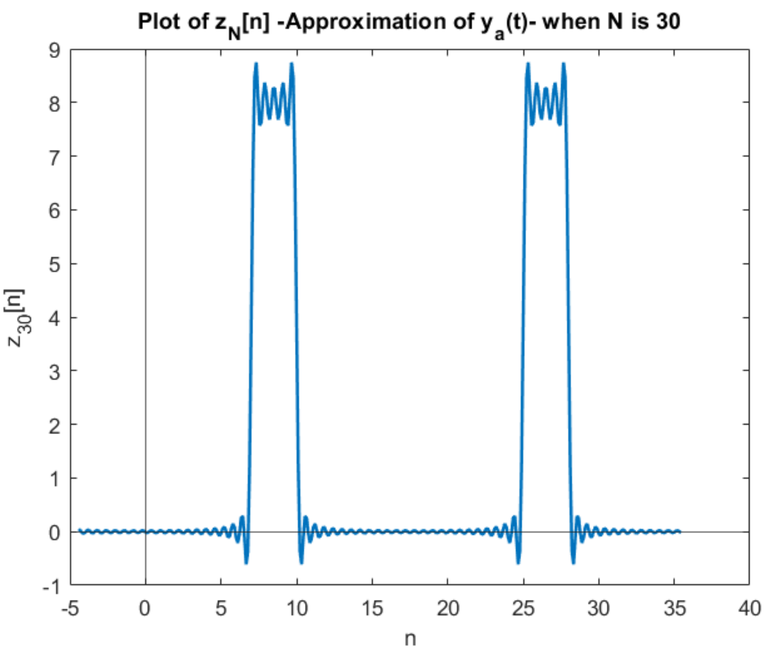

Figure 1.f: Plot of z_N[n] when N = 30

g)

Here is the plot of $z_N[n]$ when N = 5 **(Figure 1.g)**:

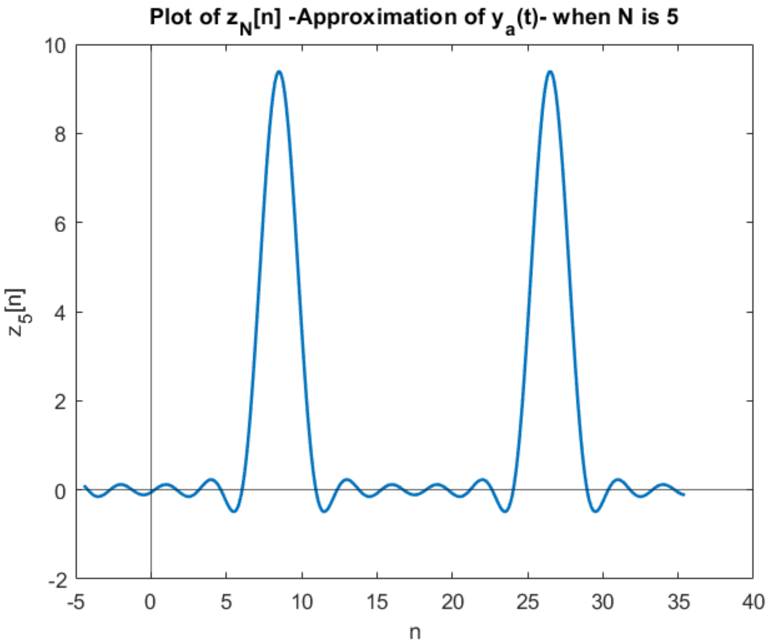

Figure 1.g: Plot of $z_N[n]$ when N = 5

h)

Here is the plot of $z_N[n]$ when N = 3 **(Figure 1.h)**:

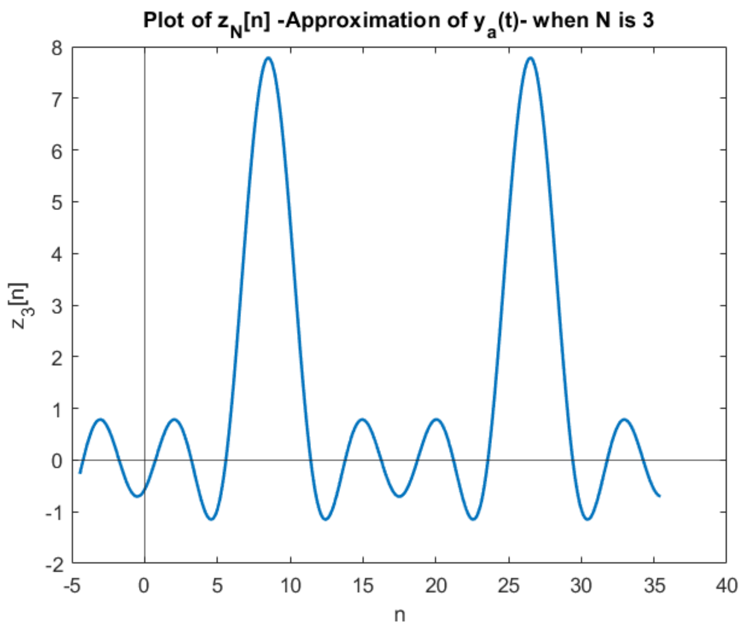

Figure 1.h: Plot of $z_N[n]$ when N = 3

i)

Here is the plot of $z_N[n]$ when N = 3 **(Figure 1.i)**:

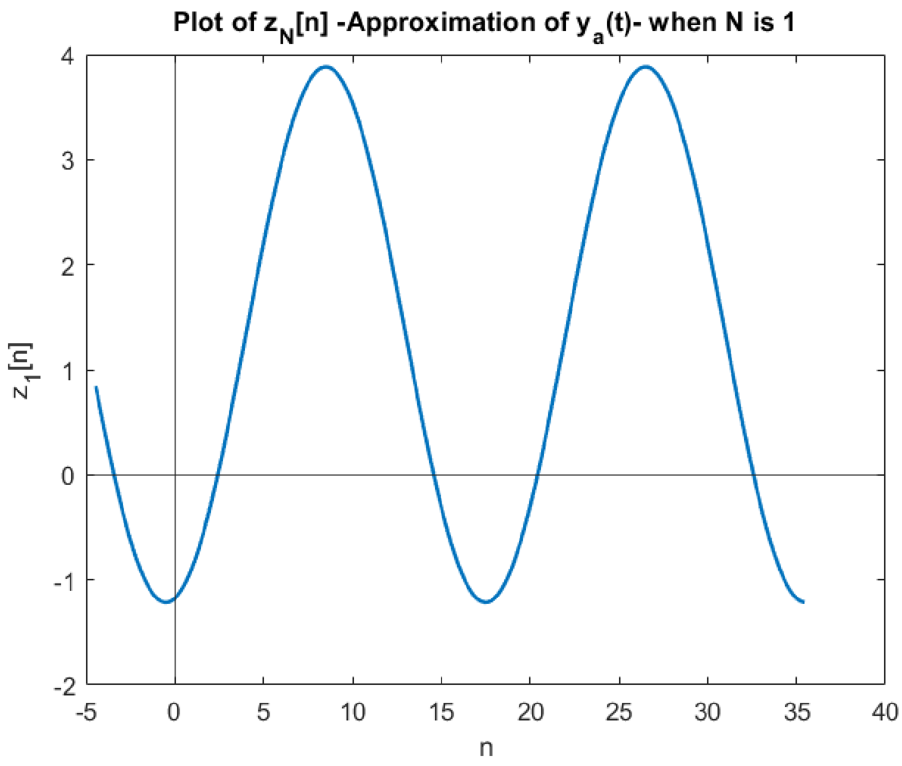

Figure 1.i: Plot of $z_N[n]$ when N =1

As we predicted in part d, indeed as N went closer to zero, the quality of the approximation went down. This is because we are missing more and more frequency components as N goes to zero. The missing components makes the approximation quality worse.

Also, we can also say that the components with relatively higher a_k's affect the approximation more than the ones with lower coefficients.

Also, according to the Gibbs phenomena, the jump discontinuity can't disappear with increasing number of components added to our approximated function. The problem that the graph for N= 150 has a smoother curve compared with the N = 75 is because MATLAB itself fits the points to the lines smoothly when they become really close to each other. Therefore, the first approximation graph should not look like as it does in MATLAB. The discontinuities must be sharper in the plot where N is 150.

j)

Here are 4 different harmonics of the original function $y_a(t)$ **(Figures 1.j.1 to 1.j.4)**:

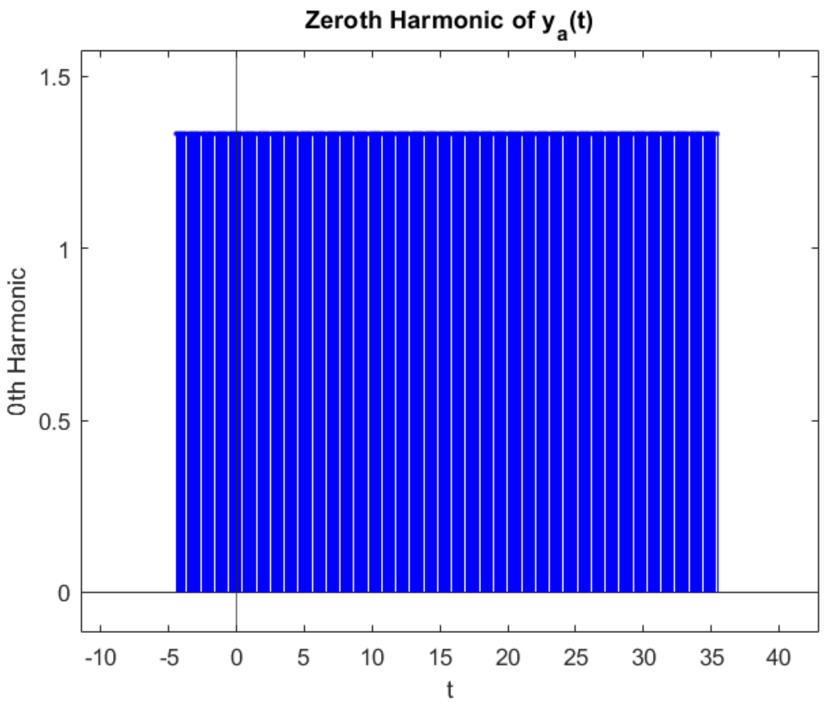

Figure 1.j.1: Zeroth Harmonic

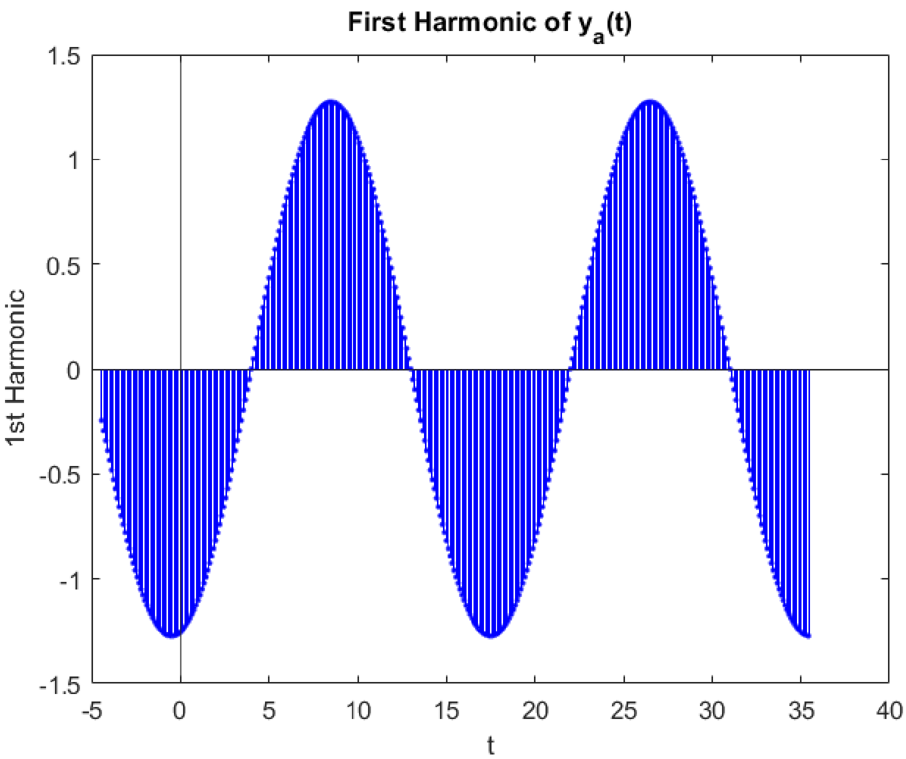

Figure 1.j.2: First Harmonic

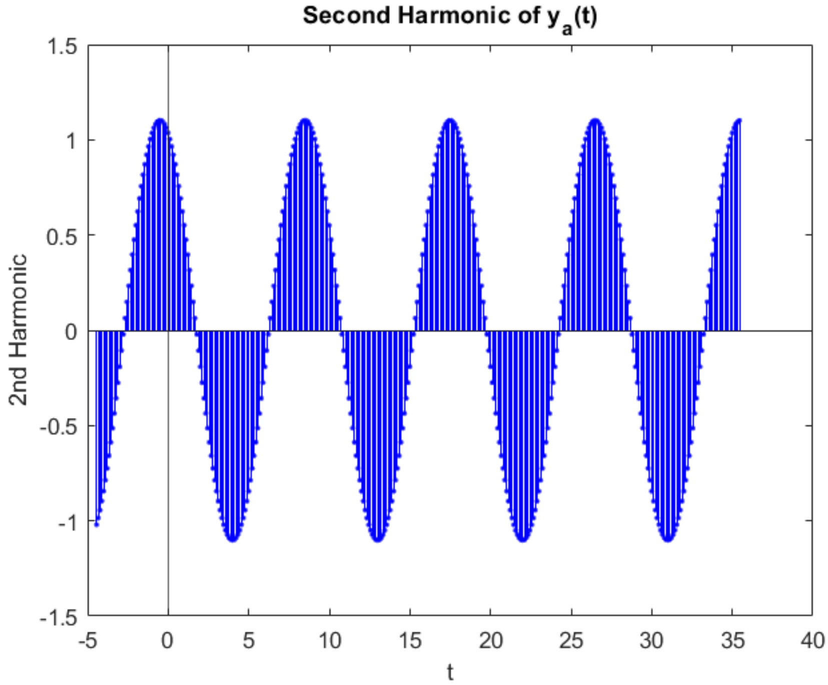

Figure 1.j.3: Second Harmonic

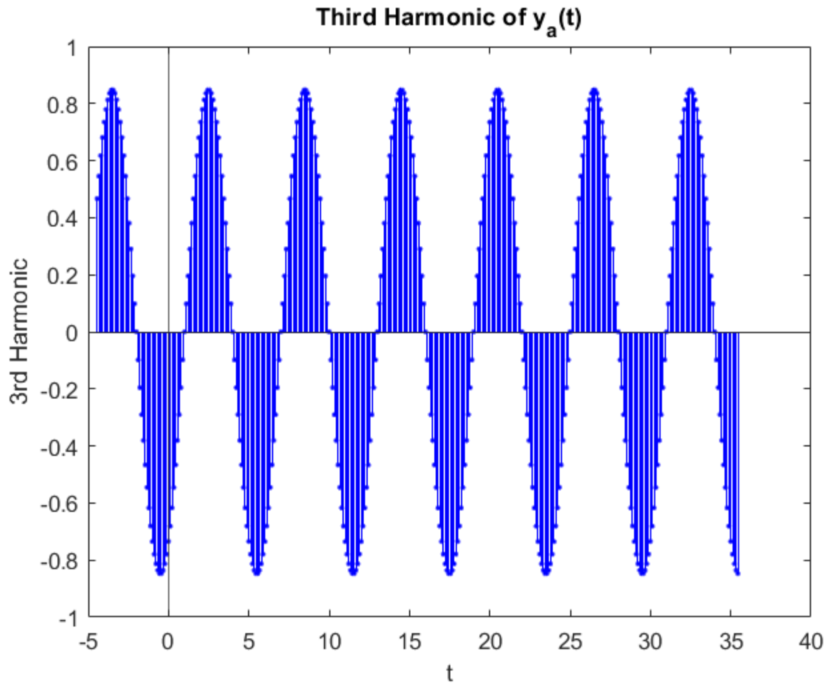

Figure 1.j.4: Third Harmonic

Question 2:

a)

Here is the signal $y_a(t)$ and the plot of the discretized signal $y_a(t)$ with sampling period 1/10 seconds **(Figures 2.a.1 & 2.a.2)**: The fundamental period for this signal is 18-unit time.

$$y_a(t) = \left|5\cos\left(\frac{\pi}{9}t\right)\right|$$

Figure 2.a.1: $y_a(t)$

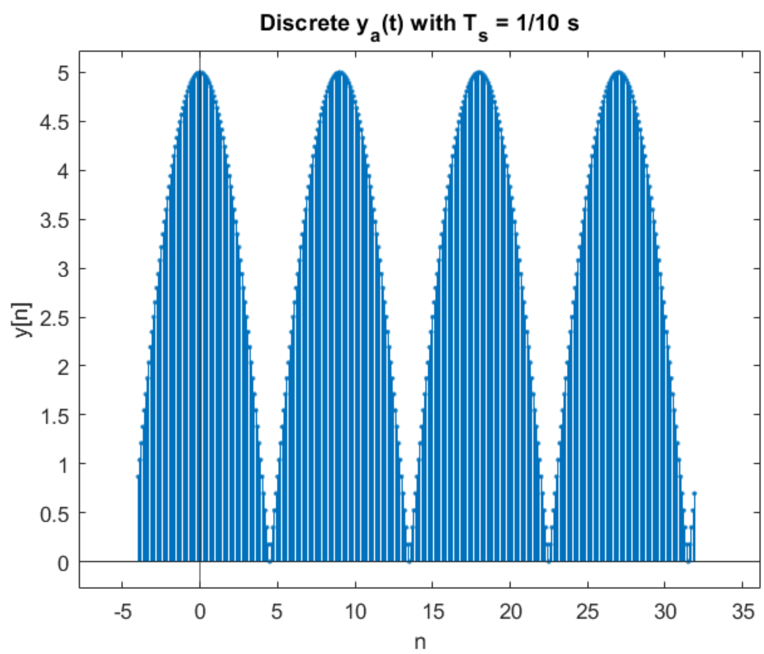

Figure 2.a.2: Plot of the Discrete $y_a(t)$ with T_s = 1/10 seconds

b)

Here you can see the Fourier series expansion of $y_a(t)$ **(Figure 2.b)**:

2-) $y_{a_2}(t) = \left|5\cos\left(\frac{\pi}{9}t\right)\right|$ Fund. Period $= 9 = T$

$$\underset{k\neq 0}{a_k} = \frac{1}{T}\int_{oop} y_a(t)\cdot e^{-j\frac{2\pi}{T}kt}\,dt = \frac{1}{9}\int_{-4.5}^{4.5} 5\cdot\cos\left(\frac{\pi}{9}t\right)\cdot e^{-j\frac{2\pi}{9}kt}\,dt =$$

$$= \frac{5}{18}\int_{-4.5}^{4.5}\left(e^{-j\frac{2\pi}{9}kt}\cdot e^{j\frac{\pi}{9}t} + e^{-j\frac{2\pi}{9}kt}\cdot e^{-j\frac{\pi}{9}t}\right)dt = \frac{5}{18}\cdot\int_{-4.5}^{4.5} e^{j\frac{\pi}{9}(1-2k)t} + e^{-j\frac{\pi}{9}(1+2k)t}\,dt$$

$$= \frac{5}{18}\left[\frac{+9}{\pi(1-2k)j}\cdot e^{j\frac{\pi}{9}(1-2k)t} + \frac{-(+9)}{\pi(1+2k)j}\cdot e^{-j\frac{\pi}{9}(1+2k)t}\right]_{t=-4.5}^{t=4.5} =$$

$$= \frac{5}{18}\cdot\left[\frac{+9}{\pi(1-2k)}\cdot\underbrace{\left(e^{j\frac{\pi}{2}(1-2k)} - e^{-j\frac{\pi}{2}(1-2k)}\right)}_{2\cdot\sin\left(\frac{\pi}{2}(1-2k)\right)} + \frac{9}{\pi(2k+1)}\cdot\underbrace{\left(e^{-j\frac{\pi}{2}(1+2k)} - e^{j\frac{\pi}{2}(1+2k)}\right)}_{+2\cdot\sin\left(\frac{\pi}{2}(1+2k)\right)}\right] =$$

$$= \boxed{5\cdot\left[\frac{\sin\left(\frac{\pi}{2}\cdot(1-2k)\right)}{\pi(1-2k)} + \frac{\sin\left(\frac{\pi}{2}(1+2k)\right)}{\pi(2k+1)}\right]} \underset{k\neq 0}{=} a_k$$

$$a_0 = \frac{1}{T}\int_{oop} y_a(t)\,dt = \frac{1}{9}\int_{-4.5}^{4.5} 5\cdot\cos\left(\frac{\pi}{9}t\right)dt = \frac{1}{\cancel{9}}\cdot\frac{\cancel{9}}{\pi}\cdot 5\cdot\left(\frac{\sin\left(\frac{\pi}{2}\right) - \sin\left(\frac{-\pi}{2}\right)}{2}\right) =$$

$$a_0 = \frac{10}{\pi}$$

$$F.S.E\left[y_a(t)\right] = \frac{10}{\pi} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} a_k\cdot e^{j\frac{2\pi}{9}kt}$$

$$F.S.E\left[y_a(t)\right] = \frac{10}{\pi} + \sum_{k=1}^{\infty} 2\cdot a_k\cdot\cos\left(\frac{2\pi}{9}k\cdot t\right)$$

: since a_k is odd

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Figure 2.b: F.S.E. of $y_a(t)$

c)

(3) shows the relationship between the coefficients k and their respective a_k. Also, here is the spectrum of coefficients **(Figure 2.c)**:

$$a_k = \left\{ \begin{array}{ll} \frac{10}{\pi} & ;\, k = 0 \\ 5 \cdot \left[\frac{\sin\left(\frac{\pi}{2} \cdot (1-2k)\right)}{\pi \cdot (1-2k)} + \frac{\sin\left(\frac{\pi}{2} \cdot (1+2k)\right)}{\pi \cdot (1+2k)} \right] & ;\, k \neq 0 \end{array} \right\} \quad \textbf{(3)}$$

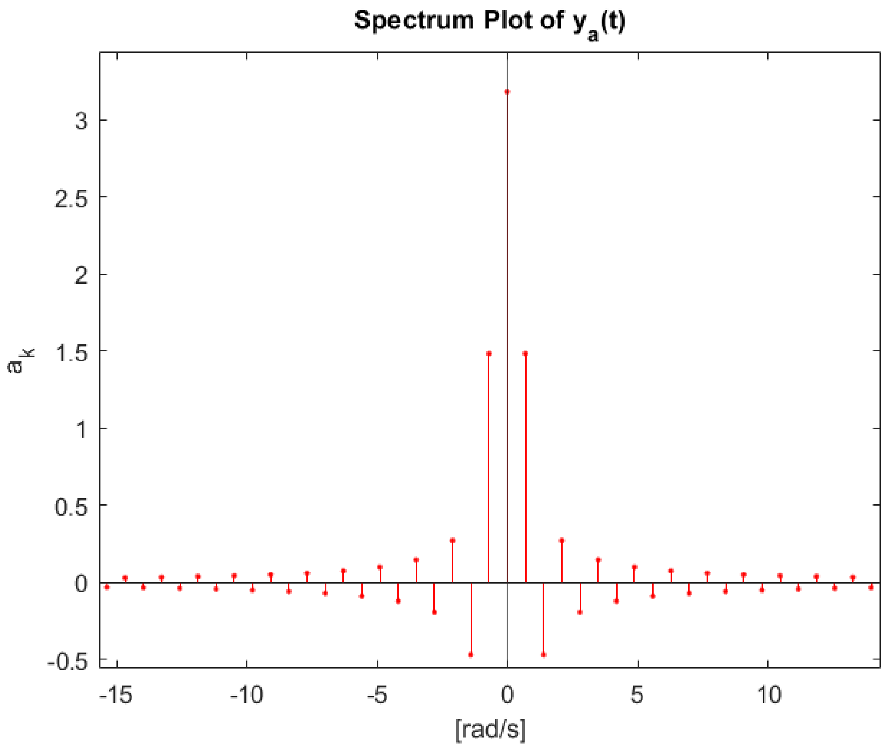

Figure 2.c: Spectrum Plot of y_a(t)

d)

By using the F.S.E from part b, we can express $z_N[n]$ as in **(4)**.

$$z_N[n] = \frac{10}{\pi} + \sum_{k=1\,;\,k\neq 0}^{N} 2 \cdot a_k \cdot \cos\left(\frac{2\pi}{9} \cdot k \cdot \frac{n}{9}\right) \;\; for\; n \in [-40\,,\,319] \quad \textbf{(4)}$$

Here is the plot of $z_N[n]$ when N = 150 **(Figure 2.d)**:

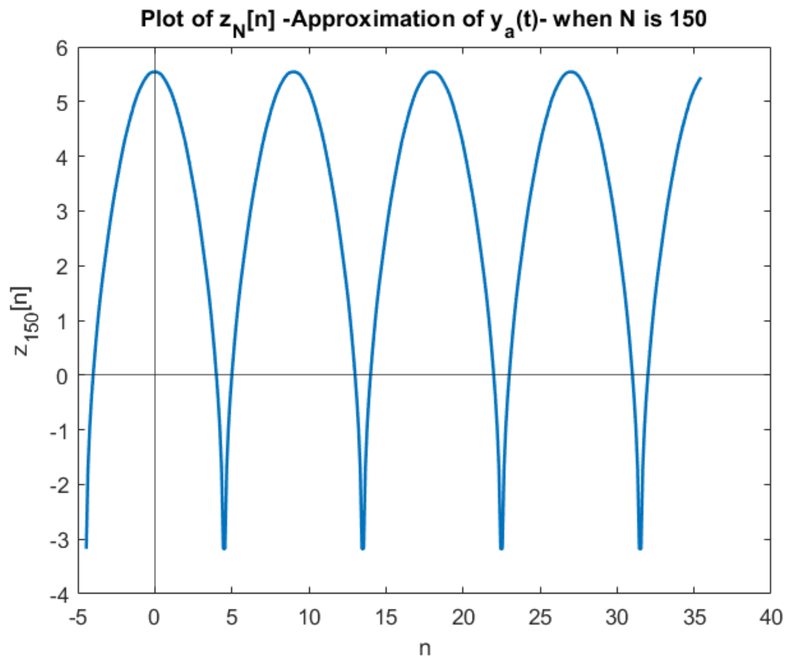

Figure 2.d: Plot of z_N[n] when N = 150

e)

Here is the plot of z_N[n] when N = 75 **(Figure 2.e)**:

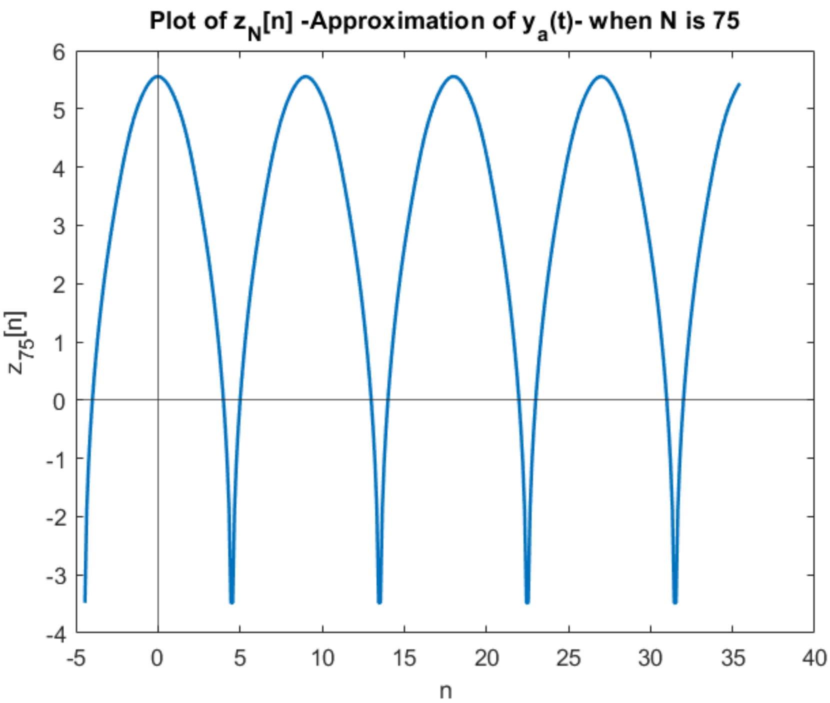

Figure 2.e: Plot of z_N[n] when N = 75

f)

Here is the plot of $z_N[n]$ when N = 30 **(Figure 2.f)**:

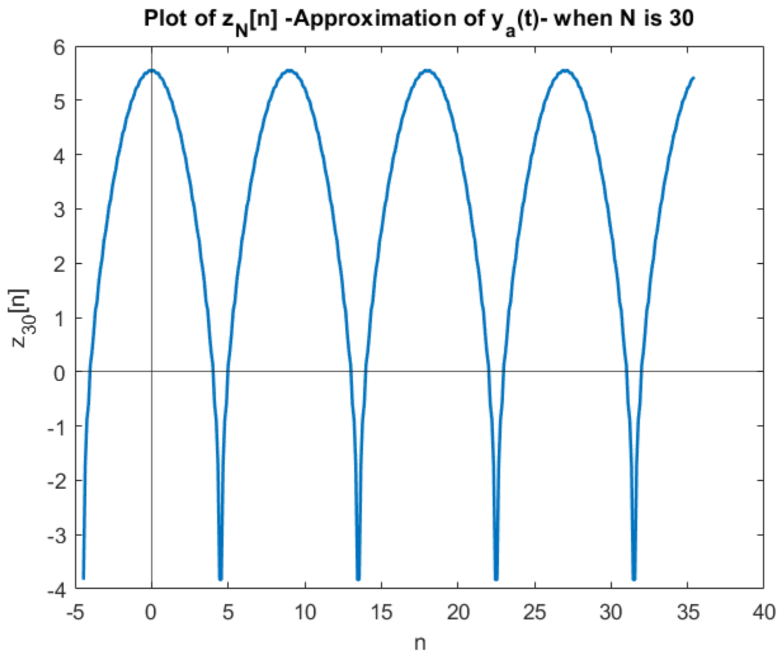

Figure 2.f: Plot of $z_N[n]$ when N = 30

g)

Here is the plot of $z_N[n]$ when N = 5 **(Figure 2.g)**:

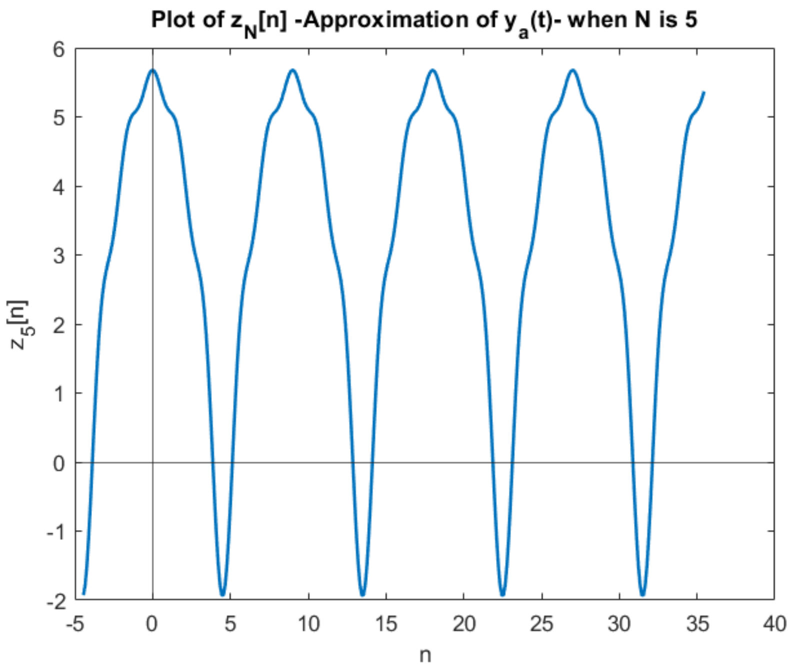

Figure 2.g: Plot of $z_N[n]$ when N = 5

h)

Here is the plot of $z_N[n]$ when N = 3 **(Figure 2.h)**:

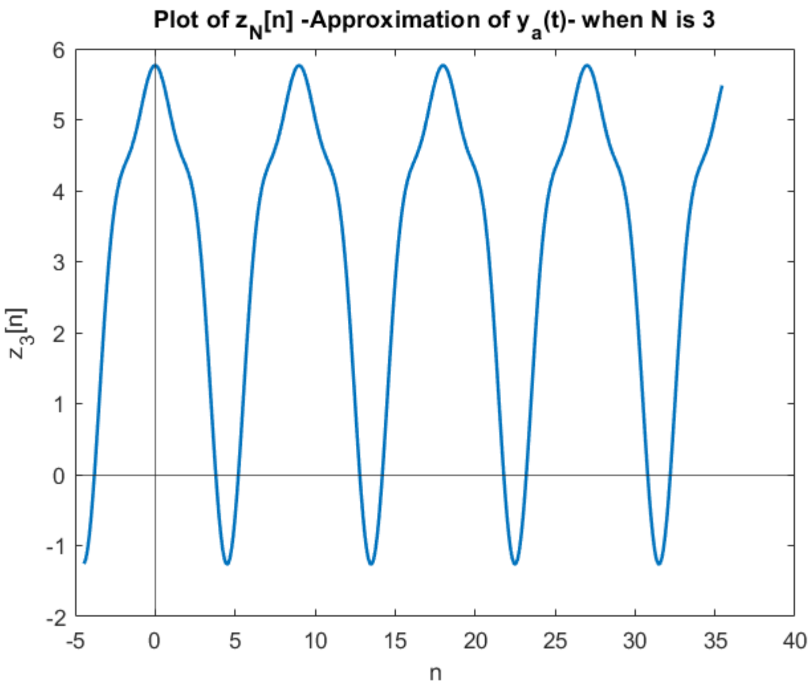

Figure 2.h: Plot of $z_N[n]$ when N = 3

i)

Here is the plot of $z_N[n]$ when N = 1 **(Figure 2.i)**:

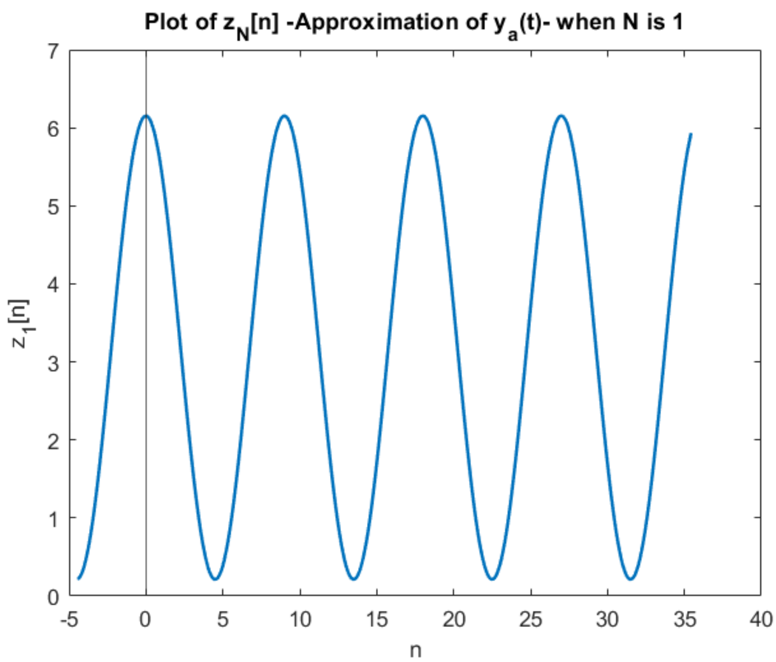

Figure 2.i: Plot of $z_N[n]$ when N =1

Similar to the first question, indeed as N went closer to zero, the quality of the approximation went down. This is because we are missing more and more frequency components as N goes to zero. The missing components makes the approximation quality worse. Also, we can also say that the components with relatively higher a_k's affect the approximation more than the ones with lower coefficients.

However, different from the first question, since there was no sudden jumps or discontinuities in the original function, we could not observe the Gibbs effect in this part of the lab.

j)

Here are 4 different harmonics of the original function $y_a(t)$ **(Figures 2.j.1 to 2.j.4)**:

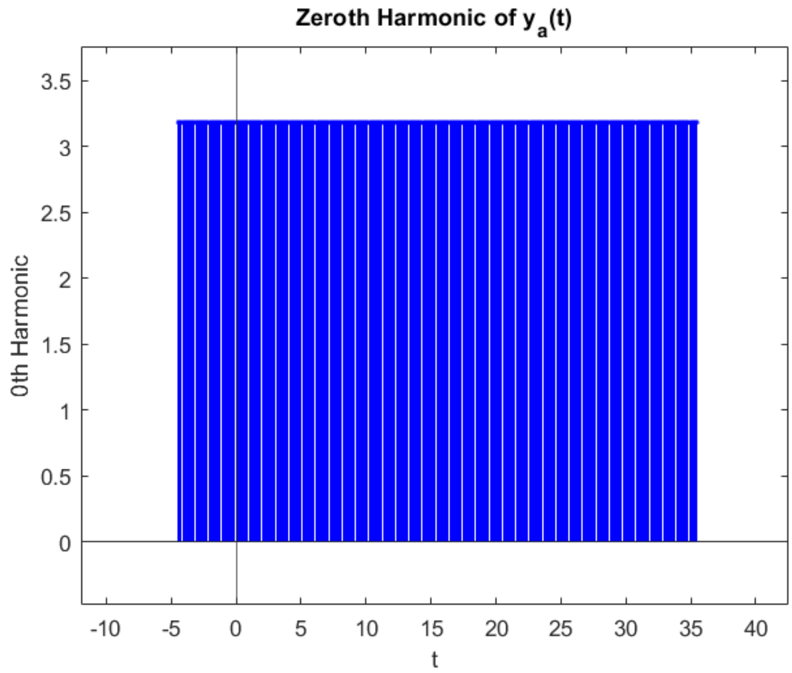

Figure 2.j.1: Zeroth Harmonic

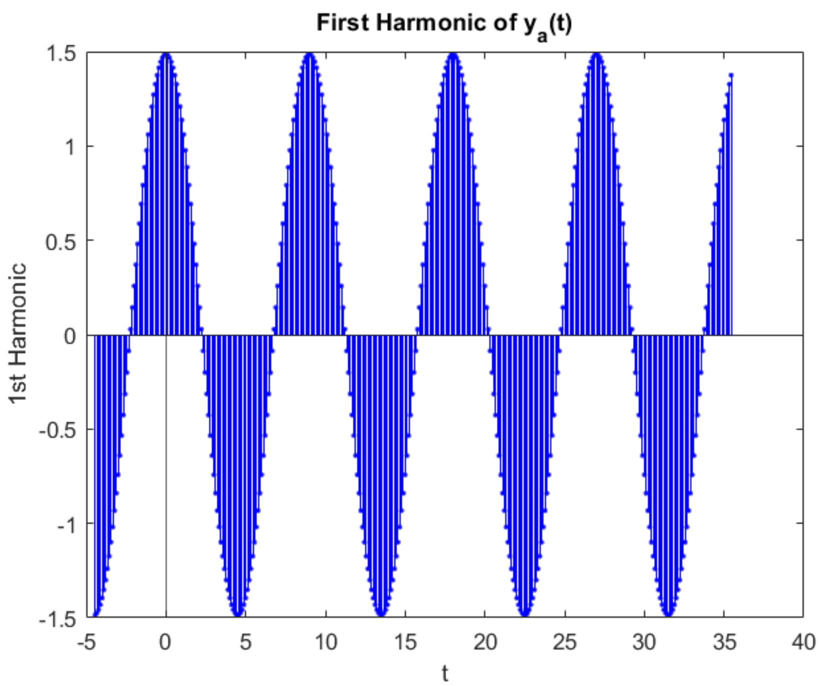

Figure 2.j.2: First Harmonic

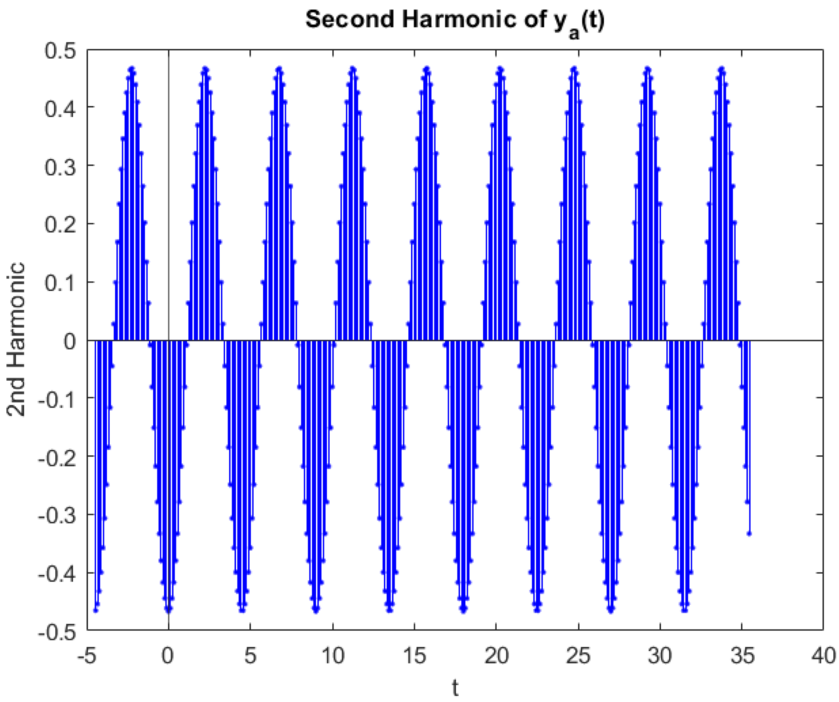

Figure 2.j.3: Second Harmonic

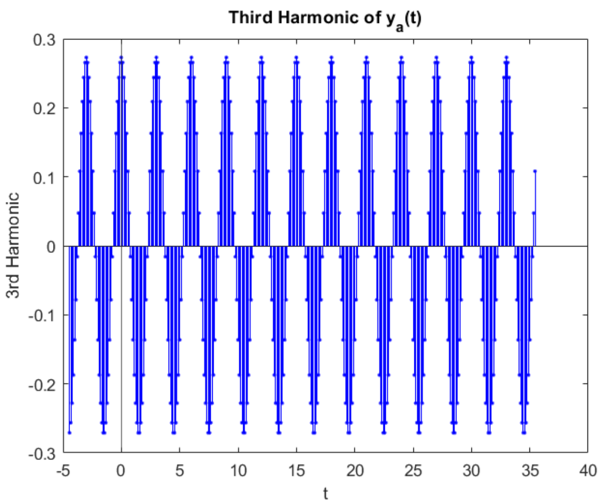

Figure 2.j.4: Third Harmonic

Question 3:

a)

Here is the signal $y_a(t)$ and the plot of the discretized signal $y_a(t)$ with sampling period 1/10 seconds **(Figures 3.a.1 & 3.a.2)**: The fundamental period for this signal is 18-unit time.

$$y_a(t) = \begin{cases} \left|5\cos\left(\frac{\pi}{9}t\right)\right| & t \in [-4.5, 4.5)\,\mathrm{s}. \\ 0 \quad t \in [4.5, 13.5)\mathrm{s}. \end{cases}$$

Figure 3.a.1: $y_a(t)$

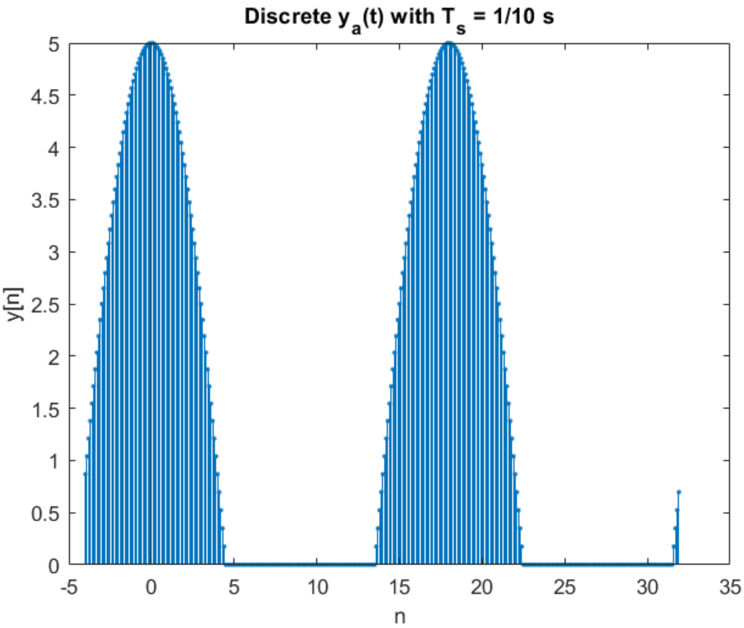

Figure 3.a.2: Plot of the Discrete $y_a(t)$ with $T_s = 1/10$ seconds

b)

Here you can see the Fourier series expansion of $y_a(t)$ **(Figure 3.b)**:

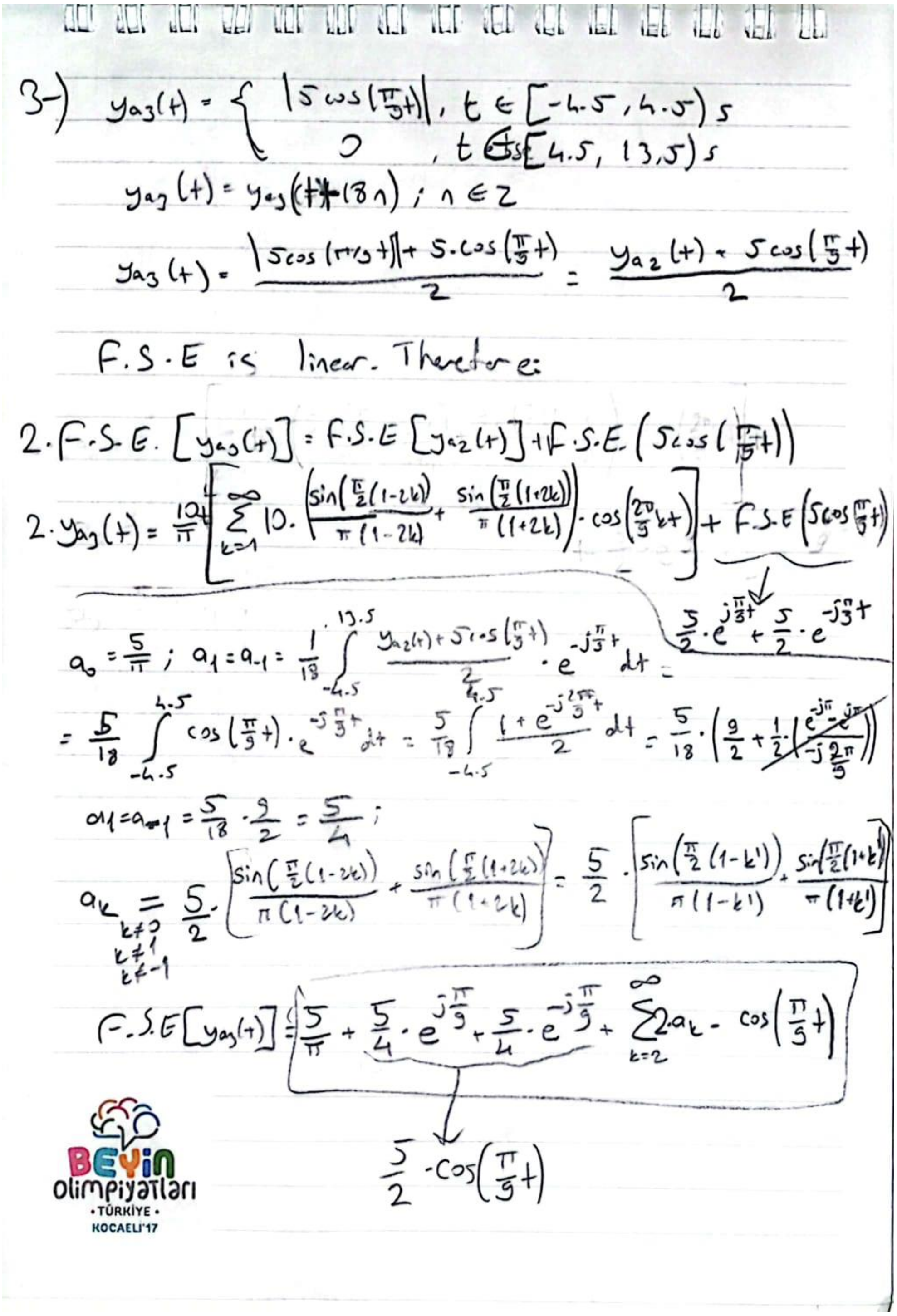

Figure 2.b: F.S.E. of $y_a(t)$

c)

(5) shows the relationship between the coefficients k and their respective a_k. Also, here is the spectrum of coefficients **(Figure 3.c)**:

$$a_k = \begin{cases} \frac{5}{2}\cdot\left(\frac{\sin\left(\frac{\pi}{2}\cdot(1-k)\right)}{\pi\cdot(1-k)} + \frac{\sin\left(\frac{\pi}{2}\cdot(1+k)\right)}{\pi\cdot(1+k)}\right) & ; k\neq 0,1,-1 \\ \frac{5}{\pi} & ; k = 0 \\ \frac{5}{4} & ; k = 1,-1 \end{cases} \quad \textbf{(5)}$$

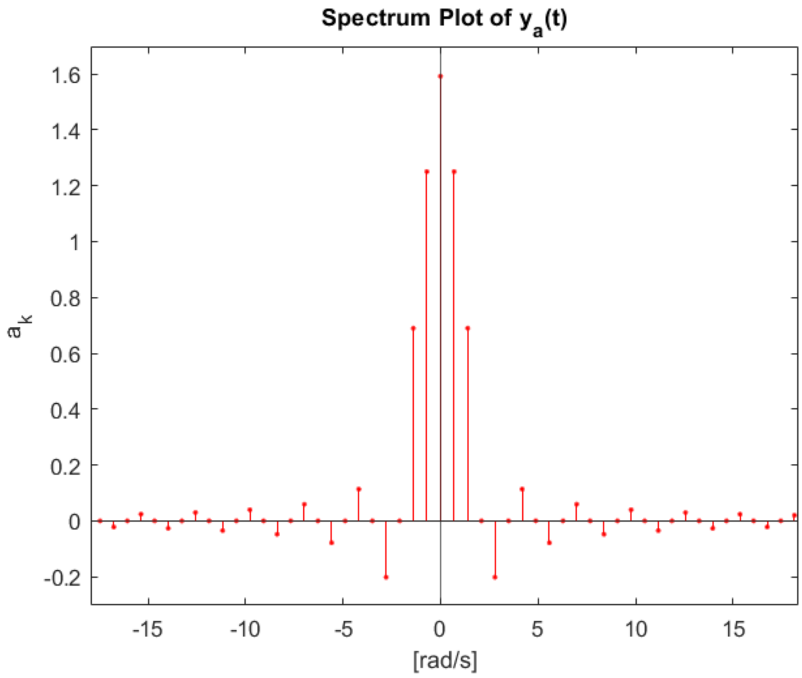

Figure 3.c: Spectrum Plot of $y_a(t)$

d)

By using the F.S.E from part b, we can express $z_N[n]$ as in **(6)**.

$$z_N[n] = \frac{5}{\pi} + \frac{5}{2}\cdot\cos\left(\frac{\pi}{9}\cdot\frac{n}{9}\right) + \sum_{k=1\,;\,k\neq 0}^{N} 2\cdot a_k\cdot\cos\left(\frac{\pi}{9}\cdot k\cdot\frac{n}{9}\right) \quad for\ n \in [-40\,,\,319]$$

(6)

Here is the plot of $z_N[n]$ when N = 150 **(Figure 3.d)**:

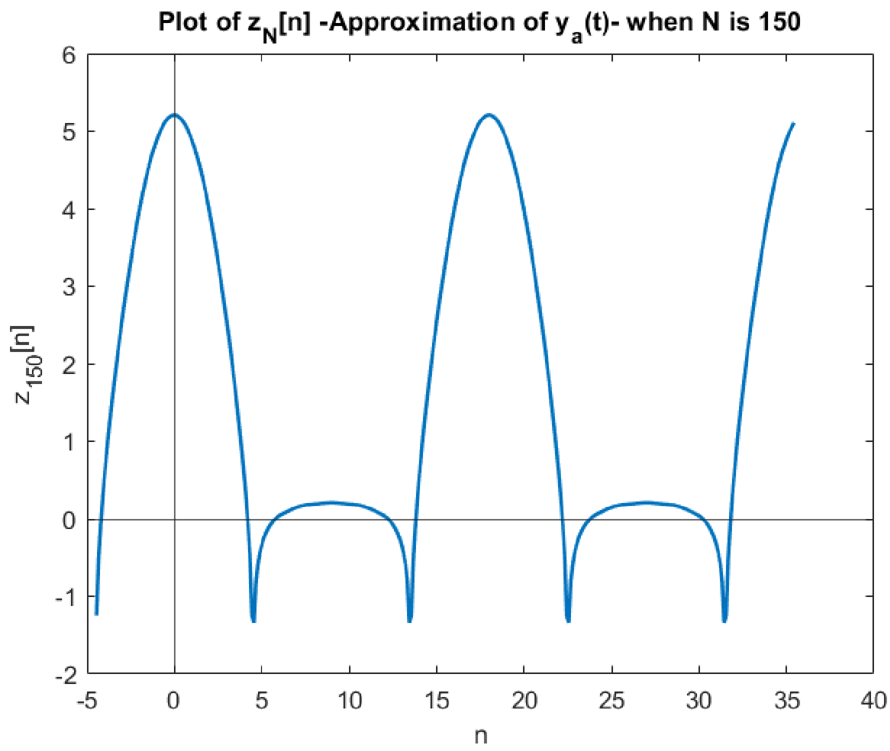

Figure 3.d: Plot of $z_N[n]$ when N = 150

e)

Here is the plot of $z_N[n]$ when N = 75 **(Figure 3.e)**:

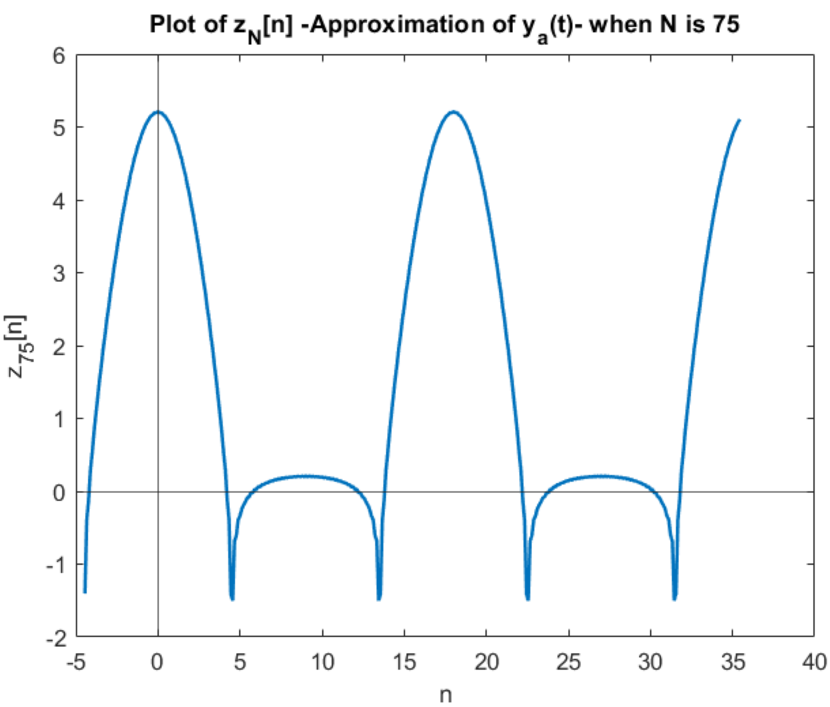

Figure 3.e: Plot of $z_N[n]$ when N = 75

f)

Here is the plot of $z_N[n]$ when N = 30 **(Figure 3.f)**:

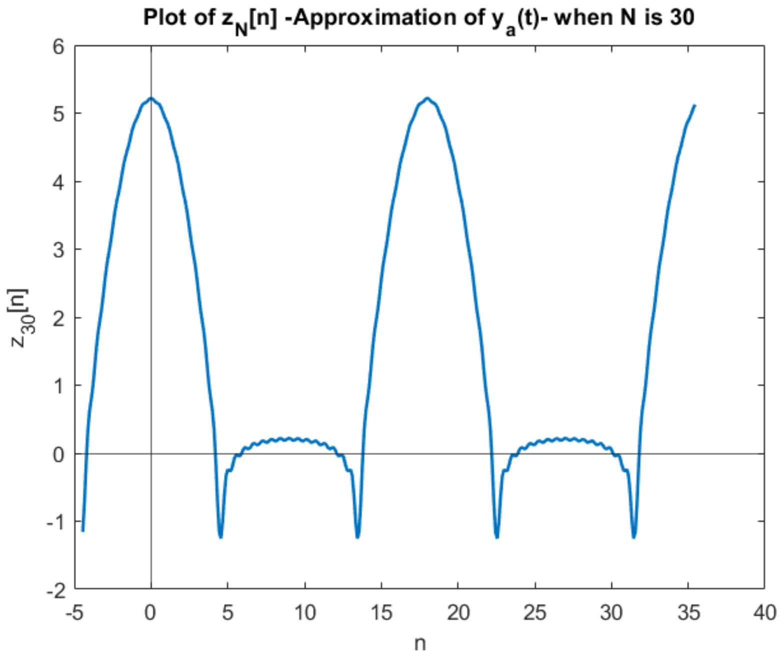

Figure 3.f: Plot of $z_N[n]$ when N = 30

g)

Here is the plot of $z_N[n]$ when N = 5 **(Figure 3.g)**:

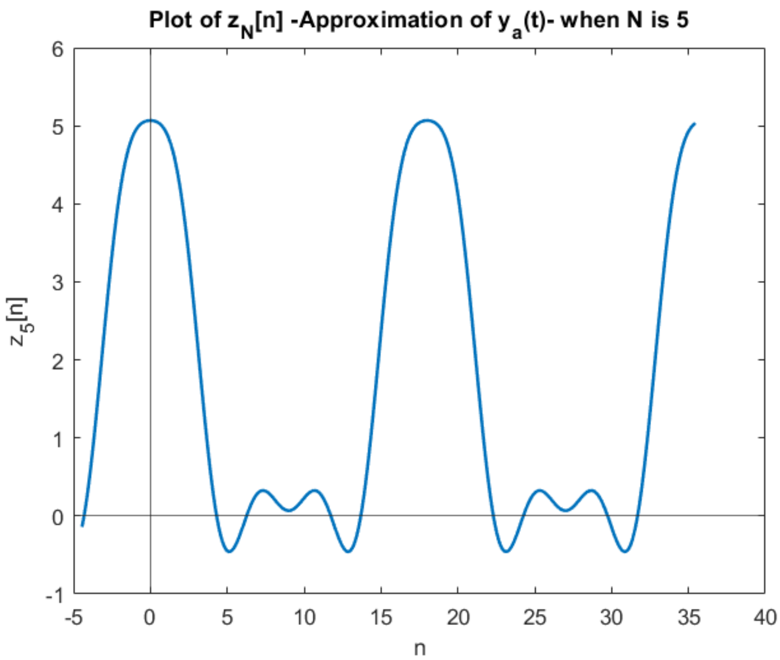

Figure 3.g: Plot of $z_N[n]$ when N = 5

h)

Here is the plot of $z_N[n]$ when N = 3 **(Figure 3.h)**:

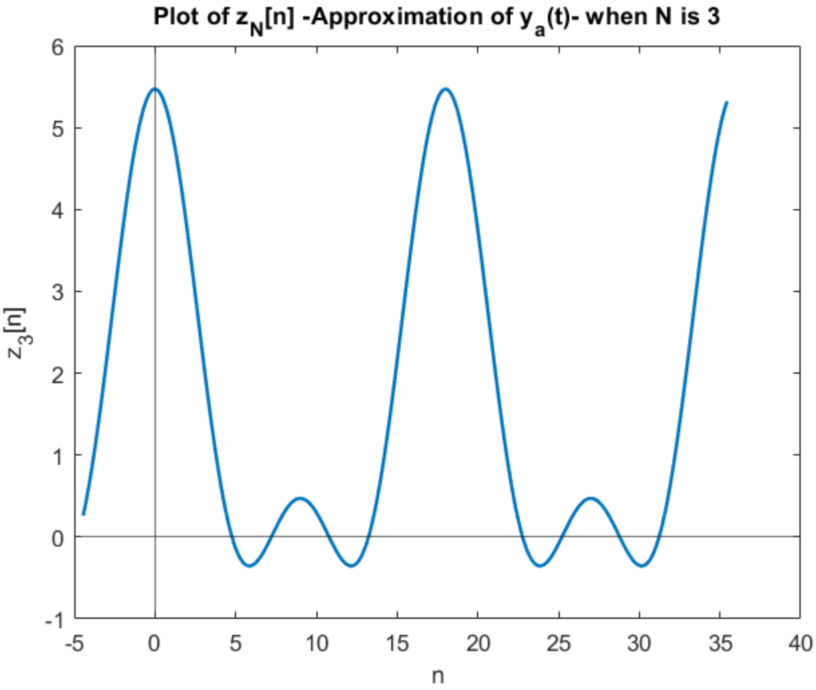

Figure 3.h: Plot of z_N[n] when N = 3

i)

Here is the plot of $z_N[n]$ when N = 1 **(Figure 3.i)**:

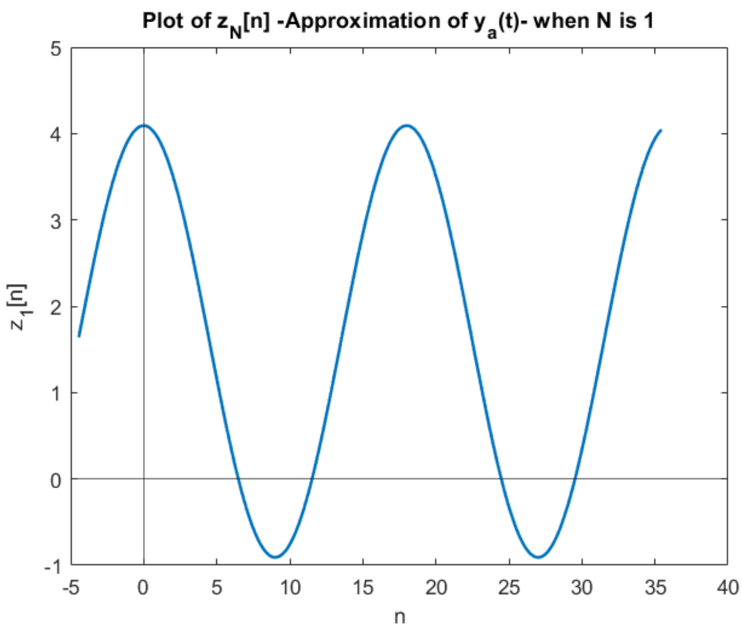

Figure 3.i: Plot of z_N[n] when N =1

Similar analysis to the first two questions can be made in this one as well. The quality of the approximation went significantly down when we decreased the number of harmonics we take into account while approximating. The harmonics with higher coefficients contributed more to the approximated function than the ones with smaller coefficients.

j)

Here are 4 different harmonics of the original function $y_a(t)$ **(Figures 3.j.1 to 3.j.4)**:

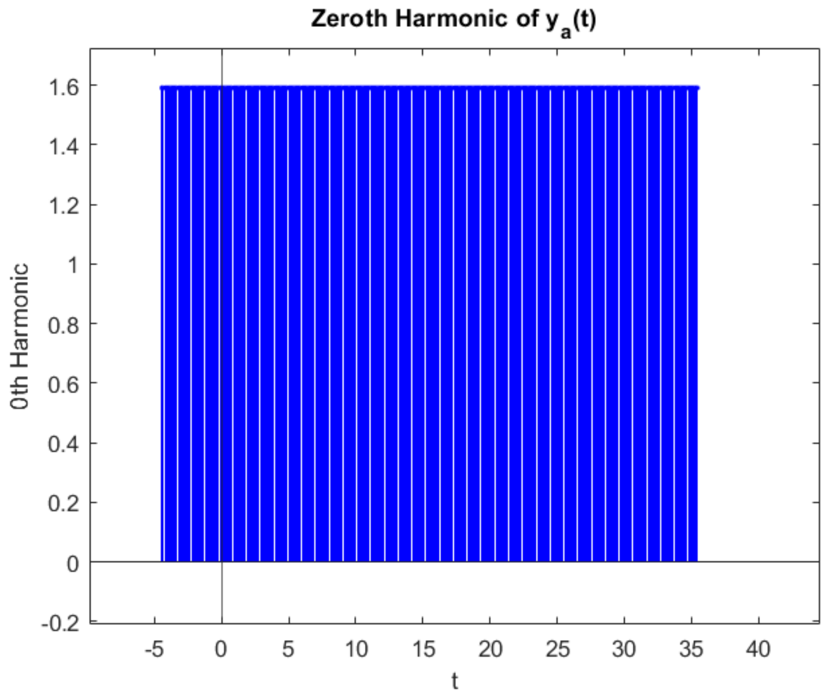

Figure 3.j.1: Zeroth Harmonic

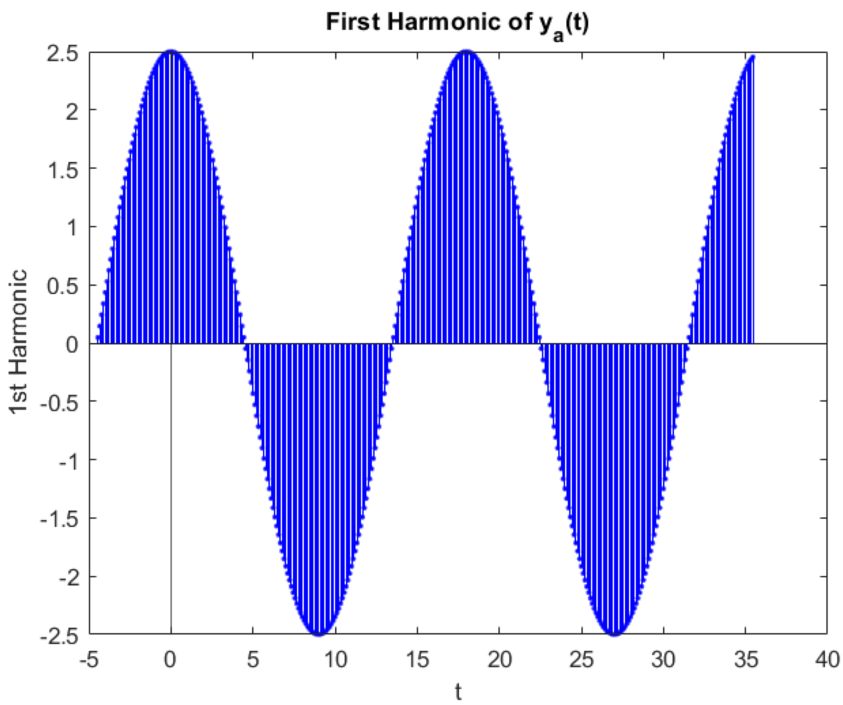

Figure 3.j.2: First Harmonic

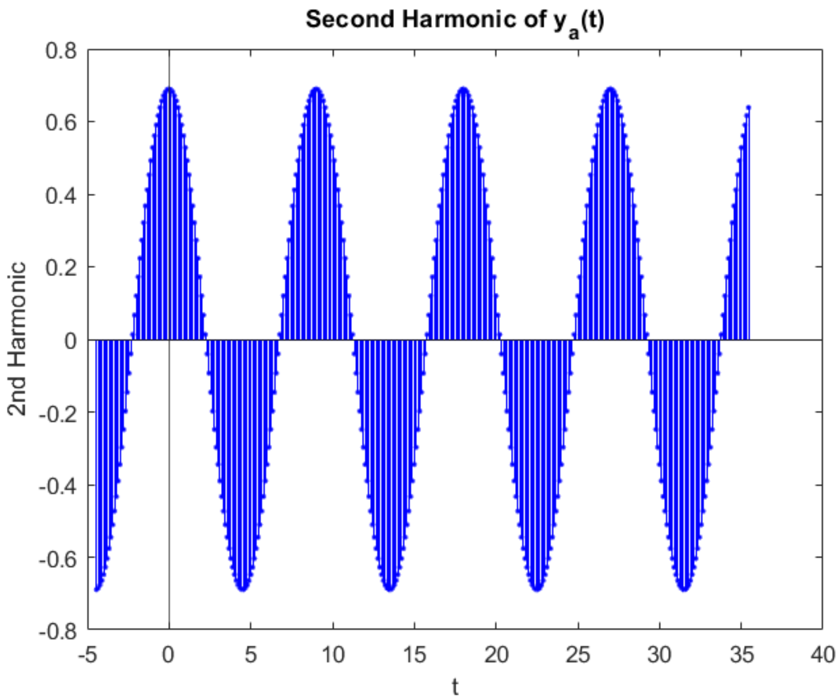

Figure 3.j.3: Second Harmonic

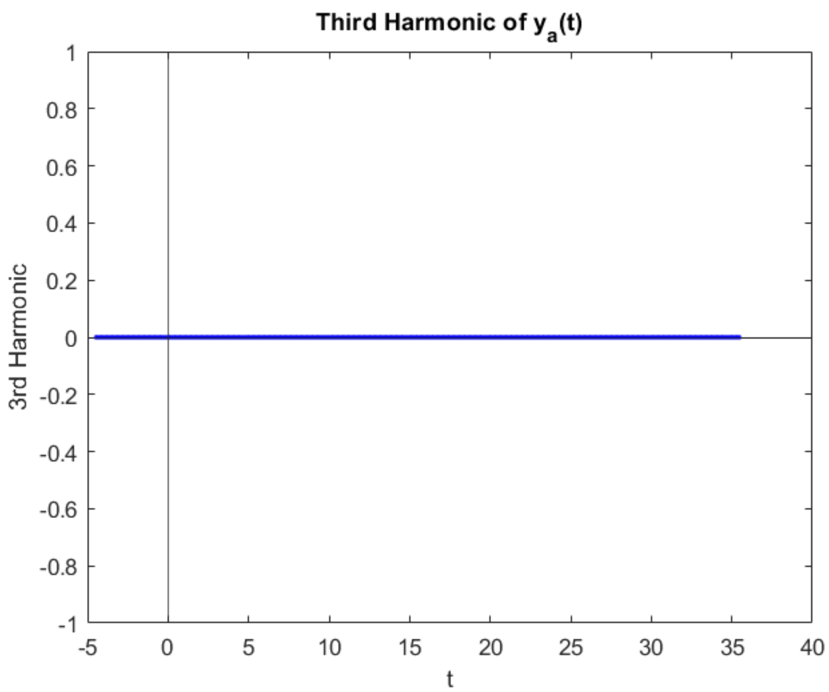

Figure 3.j.4: Third Harmonic

Conclusion & Comments:

In this lab, we learned the Fourier series expansion and some related approximations. The lab was a total success, and every single requirement was met. I believe the lab was very helpful to me in understanding more about the fourier transform's insights. I also find this lab extremely helpful in visualizing the harmonics of a function.

Appendices:

1. https://github.com/fmcetin7/Bilkent-EEE-321/blob/main/lab4/lab4q1.m
2. https://github.com/fmcetin7/Bilkent-EEE-321/blob/main/lab4/lab4q2.m
3. https://github.com/fmcetin7/Bilkent-EEE-321/blob/main/lab4/lab4q3.m