EEE 424 Analytical Assignment 1 Spring 2024-25

Due-date: 13 March 2025, 23:59

Q.1) [6 pts] Consider the following transfer function:

$$H(\omega) = \frac{1 - a^4 \cdot e^{-j\omega 4}}{1 - a^4 \cdot e^{-j\omega}}$$

Find the corresponding impulse response. Also clearly state for what values of a is this filter stable?

Q.2) [6 pts] We have the Hilbert space of vectors in $\mathbb{R}^4$ with one of its basis $\{e_0, e_1, e_2, e_3\}$. Given the first three basis vectors

$$e_0 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} e_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix} e_2 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

How many possibilities are there for e_3, such that the basis set above is an orthogonal basis of $\mathbb{R}^4$? Also give one example of e_3.

Q.3) [10 pts] A stable discrete-time LTI system is defined by the difference equation:

$$y[n+1] - \frac{5}{2}y[n] + y[n-1] = x[n+1] + x[n-1]$$

(a) Find its system function $H(z)$ and indicate its ROC. Is it causal?

(b) Find its impulse response and frequency response $H(e^{j\omega})$. Show that $H(e^{j\omega})$ is real-valued.

(c) Find the output $y[n]$ if $x[n] = 10\cos\left[\frac{\pi}{3}n - \frac{1}{2}\right]$.

Q.4) [15 pts] A gear has 8 identical teeth spaced evenly around its circumference. An optical sensor is fixed beside the gear such that each time a tooth passes in front of it, the sensor outputs a brief signal "pulse." However, your measurement system does not record continuously. Instead, you instruct a microcontroller to sample the sensor's state once every T seconds.

a) Initially, the gear rotates at a constant speed of 30 revolutions per minute (rpm). As such, the gear makes one full revolution every $\frac{60}{30} = 2$ seconds. Since there are 8 identical teeth, the sensor's "pattern" effectively repeats every $\frac{1}{8}$ of a revolution. Describe in your own words why this "8x repetition" can cause aliasing if our sampling interval T is too large.

b) If $T = 0.4$ seconds, determine how many teeth pass in front of the sensor between samples. In other words, how much rotation occurs from one sample to the next?

c) Suppose $T = 1.0$ seconds. Now how many teeth pass between samples? Does it appear (based on discrete samples) that the gear might be rotating *forward*, *backward*, or *not at all*?

d) For each sampling interval above (0.4s and 1.0s), find the apparent rotational speed in "teeth per second" or "revolutions per minute" according to the sampled data. (Hint: A forward jump of +7 teeth looks the same as a -1 tooth jump, etc. You must account for the possibility that the gear might appear to move less than 4 teeth per sample if it can be aliased to a smaller or negative step.)

e) If you want to be absolutely certain the sampled data always shows correct forward motion (never backward or stationary), what is the maximum allowable T? Think in terms of Nyquist-style reasoning: the gear's effective "tooth-pass frequency" is 8 times the rotation frequency.

f) Now suppose the gear speed varies between 30rpm and 60rpm. How must you adjust your sampling interval T so that—even at the *highest* speed—you never misinterpret forward rotation as backward?

Q.5) [8 pts] Consider a signal $\underline{x} = [6 \;\; 2]$. Find its DFT of size 2. Clearly state what the basis vectors are in the frequency domain and the time domain (suggested to use the natural basis for the time domain). Plot the signal itself, its frequency components corresponding to each basis vector, and its time and frequency domain basis vectors all on the same 2D plot. Explain in detail the process of evaluating the DFT in such a way.

Q.6) [10 pts] Suppose we have a finite-length sequence defined by

$$x[n] = n, \quad n = 0, 1, 2, \ldots, N-1,$$

and its N–point DFT

$$X[k] = \sum_{n=0}^{N-1} n\, e^{-j\frac{2\pi}{N}kn}, \quad k = 0, 1, 2, \ldots, N-1.$$

Find $X[k]$. **Hint:** You may find the Geometric Series and its derivative(s) useful.

Q.7) [9 pts] Let $S = L_2[0,1]$ and $V = \text{span}\{1, t, t^2\}$, the set of quadratic polynomials over the unit interval. Define $p_1(t) = 1$, $p_2(t) = t$, and $p_3(t) = t^2$. Clearly, V is a subspace of S. What is the projection of $x(t) = \cos\left(\frac{\pi t}{2}\right)$ onto V? In other words, what values of $\{c_1, c_2, c_3\}$ best approximates $x(t)$ on $[0,1]$?

Q.8) [15 pts] Consider two sequences as follows. The first sequence is given by $x(0) = 1$, $x(1) = 2$, $x(2) = 3$ (with $x(n) = 0$ for $n \geq 3$), and the second sequence is given by $h(0) = 4$, $h(1) = 5$, $h(2) = 6$, $h(3) = 7$ (with $h(n) = 0$ for $n \geq 4$).

a) Perform a 4-point circular convolution of $x(n)$ and $h(n)$ using any time-domain method of your choice or using a Toeplitz Circulant matrix.

Compute it again by following these steps:

- Compute the 4-point DFT of $x(n)$.
- Compute the 4-point DFT of $h(n)$, which is already defined over $0 \leq n \leq 3$.
- Multiply the two DFTs pointwise to obtain the spectrum of the circular convolution.
- Compute the 4-point inverse DFT to obtain the time-domain result, $y(n)$ for $n = 0, 1, 2, 3$.

Verify your results using your two methods.

b) Compute the linear convolution of $x(n)$ and $h(n)$ by convolving the sequences in the usual way. Since $x(n)$ has length 3 and $h(n)$ has length 4, the linear convolution will have length 3 + 4 - 1 = 6. Carefully compute the six output values.

c) Analyze the relationship between the two results. Determine the minimum DFT length that would be necessary in order to use the DFT method to recover the full linear convolution of $x(n)$ and $h(n)$.

Q.9) [5 pts] An audio transmitter can transmit only two signals: it either transmits a sine wave at 23 kHz or it transmits a sine wave at 25 kHz. The receiver has the following sampling frequencies available: 16 kHz, 32 kHz and 48 kHz.

Which is the best sampling frequency for the receiver to know when is 23 kHz being transmitted, and when 25 kHz is. Explain your choice in detail.

Q.10) [16 pts] This question is related to the Fast Fourier Transform (FFT) algorithm.

(a) Consider the sequence

$$x[n] = \{1, 1, -1, -1, -1, 1, 1, -1\}. \tag{1}$$

Determine the DFT $X[k]$ of $x[n]$ using the decimation-in-time FFT algorithm.

(b) Now, compute the DFT $Y[k]$ of the following sequence $y[n]$ using the decimation-in-time FFT algorithm:

$$y[n] = \{1, -2, 1, -1, 1, -1, 2, -1\}. \tag{2}$$

(c) Compare the efficiencies of computing the 8-point FFT of $x[n]$ and $y[n]$. Is it possible to compute one of them more efficiently?

(d) Apart from computing the FFT using the decimation-in-time algorithm, you have also been introduced to the computation of FFT using decimation-in-frequency. Plot the flow graph for an 8-point decimation-in-frequency FFT algorithm.