EEE 424 Analytical Assignment 2 Spring 2024-25

Due-date: 16th May, 2025

Q.1) [5 pts] Given a desired frequency response $H_d(e^{j\omega})$, show that the rectangular window (assume that $h[n]$ is of order N) design minimizes the least-squares error

$$E_{LS} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| H_d(e^{j\omega}) - H(e^{j\omega}) \right|^2 d\omega.$$

Q.2) [10 pts] Consider the following specifications for a low-pass filter:

$$0.99 \leq \left| H\left(e^{j\omega}\right) \right| \leq 1.01 \quad \text{for} \quad 0 \leq |\omega| \leq 0.3\pi,$$

$$\left| H\left(e^{j\omega}\right) \right| \leq 0.01 \quad \text{for} \quad 0.35\pi \leq |\omega| \leq \pi.$$

Design a linear-phase FIR filter to meet these specifications using the window design method. Provide the $h[n]$ corresponding to your design. Plot the ideal lowpass filter and your design on the same plot. Choose an appropriate window for your design from the following table.

Window	Side-Lobe Amplitude (dB)	Transition Width (Δf)	Stopband Attenuation (dB)
Rectangular	-13	$0.9/N$	-21
Hanning	-31	$3.1/N$	-44
Hamming	-41	$3.3/N$	-53
Blackman	-57	$5.5/N$	-74

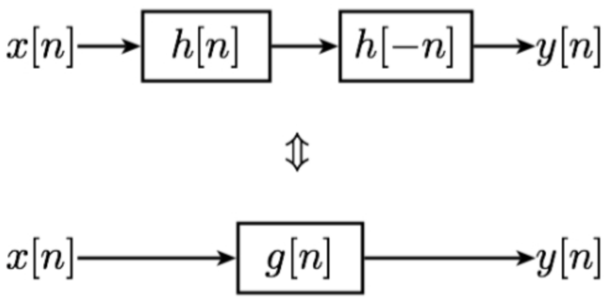
Figure 1: The Peak Side-Lobe Amplitude of Some Common Windows and the Approximate Transition Width and Stopband Attenuation of an Nth-Order Low-Pass Filter Designed Using the Given Window.

Q.3) [5 pts] Consider the LTI system defined by $y[n] = A\,x[n] + B\,x[n-1] + C\,x[n-2]$. This system has the following characteristics:

- It has unity gain at $\omega = 0$.
- It has zero gain at $\omega = \pi$.
- All coefficients A, B, and C are nonzero, and the system exhibits linear phase.

Determine the values of A, B, and C. The term gain can be used interchangeably with amplitude in this case.

Q.4) [5 pts] Consider a filter with a real-valued impulse response $h[n]$. The filter is cascaded with another filter whose impulse response is $h'[n] = h[-n]$, i.e., the time-reversed version of $h[n]$. The cascade of these two filters can be viewed as a single filter with impulse response $g[n]$. In block diagram form:



What is the phase of $G(e^{j\omega})$? Is the phase linear?

Q.5) [15 pts] Consider the definition of the Haar wavelet. The scaling function of the Haar wavelet is defined as

$$\phi(t) = \begin{cases} 1 & \text{if } t \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

The mother wavelet is defined as

$$\psi_{0,0}(t) = \begin{cases} 1 & \text{if } t \in [0, \frac{1}{2}) \\ -1 & \text{if } t \in [\frac{1}{2}, 1] \\ 0 & \text{otherwise} \end{cases}$$

The corresponding daughter wavelets are defined as $\psi_{j,k}(t) = 2^{j/2}\psi(2^j t - k)$ (you can consider $\psi(t) = \psi_{0,0}(t)$.

Is the family of functions defined by the Haar wavelets an orthonormal family? If your answer is yes, prove that the functions define an orthonormal family. If your answer is no, prove that at least a pair of the given signals is not orthonormal.

Q.6) [10 pts] Consider the function $x(t) = e^{-\alpha t}u(t)$, where $u(t)$ is the unit step function defined as:

$$u(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{otherwise} \end{cases} \qquad (\alpha > 0)$$

(a) Derive the Fourier Transform $X(\omega)$ of the signal $x(t)$.

(b) Derive the Short Time Fourier Transform (STFT) of $x(t)$ in terms of $X_w(\tau, \omega)$.
Use the window function $w(t) = \text{rect}\left(\frac{t}{T_0}\right)$. Evaluate with τ as a parameter.
Based on your answer, evaluate

$$X(\tau,\omega)\big|_{\tau=0} = X(0,\omega), \quad \text{and} \quad X(\tau,\omega)\big|_{\omega=0} = X(\tau, 0).$$

You can assume $T_0 > 0$ for simplicity.

Q.7) [25 pts] Consider the discrete time system in Fig. 2. All four filters $H_0(z)$, $H_1(z)$, $G_0(z)$, and $G_1(z)$ are linear and time invariant filters.

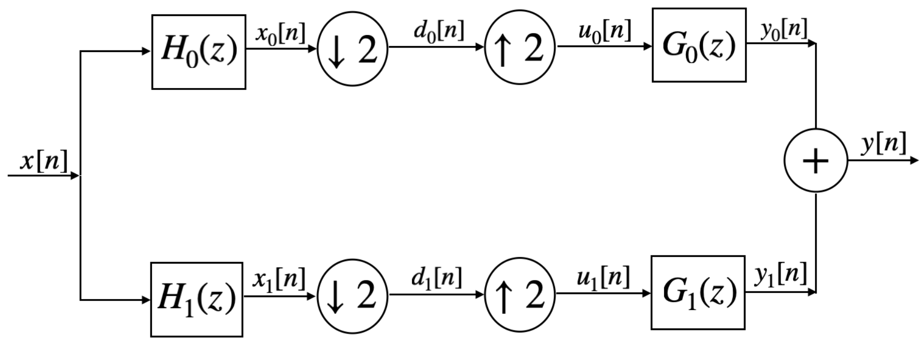


Figure 2: The discrete-time system for Q.7)

$$X(z) = \mathbb{Z}\{x[n]\} \text{ for } i = 0, 1.$$
$$Y(z) = \mathbb{Z}\{y[n]\} \text{ for } i = 0, 1.$$
$$D_i(z) = \mathbb{Z}\{d_i[n]\} \text{ for } i = 0, 1.$$
$$X_i(z) = \mathbb{Z}\{x_i[n]\} \text{ for } i = 0, 1.$$
$$U_i(z) = \mathbb{Z}\{u_i[n]\} \text{ for } i = 0, 1.$$
$$Y_i(z) = \mathbb{Z}\{y_i[n]\} \text{ for } i = 0, 1.$$

(a) Find $D_0(z)$ in terms of $X_0(z)$. Also find $D_1(z)$ in terms of $X_1(z)$.
(b) Find $U_0(z)$ in terms of $D_0(z)$. Also find $U_1(z)$ in terms of $D_1(z)$.
(c) Find $X_0(z)$ in terms of $X(z)$. Also find $X_1(z)$ in terms of $X(z)$.
(d) Show that the overall system is not necessarily LTI for arbitrary filters $H_0(z)$, $H_1(z)$, $G_0(z)$, and $G_1(z)$.
(e) Show that the overall system is LTI if

$$H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0.$$

(f) Show that if

$$G_0(z) = H_1(-z) \quad \text{and} \quad G_1(z) = -H_0(-z),$$

the overall system is LTI.

Q.8) [25 pts] Consider the setting in Fig. 3, where $s[n]$ is a WSS process, $x[n]$ is the observation of the signal $s[n]$ under a noise process $w[n]$. The goal is to design a filter $h[n]$ that minimizes the mean square error between the output process $y[n]$ and the input process $s[n]$, $E[(s[n] - h[n] * x[n])^2]$, where $*$ represents convolution. The processes being evaluated are discrete-time processes.

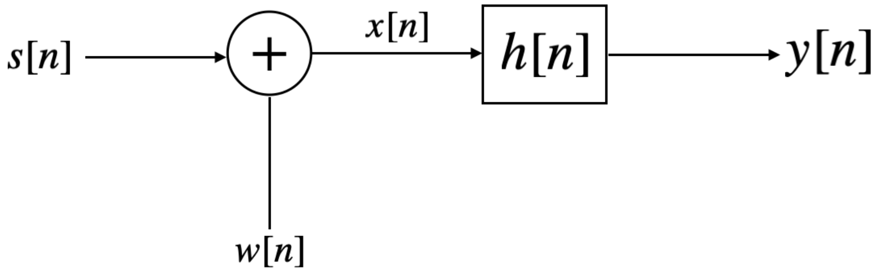


Figure 3: WSS process filtering under additive noise for Q.7)

(a) Derive the optimal Wiener filter for this setting that minimizes the mean squared error

$$E\left[(s[n] - h[n] * x[n])^2\right].$$

Explicitly derive the optimal filter step-by-step similar to the way done in the lecture notes. Derive the filter expression in terms of $S_{s,s}(e^{j\omega})$ and $S_{w,w}(e^{j\omega})$.
Hint: You can use the assumptions (e.g. regarding the correlation between noise process and the input process) used in the lecture notes if necessary, however, you should indicate their usage when you incorporate them in your solution.

(b) Derive an expression for the mean squared error of the optimal filter

$$E\left[|s[n] - h[n] * x[n]|^2\right]$$

in terms of $R_{s,s}[n]$, $R_{w,w}[n]$, and $h[n]$.

(c) Derive an expression for the mean squared error of the optimal filter at each frequency, that is, derive an expression for

$$E\left[|S(e^{j\omega}) - H(e^{j\omega})X(e^{j\omega})|^2\right]$$

in terms of $S_{s,s}(e^{j\omega})$, $S_{w,w}(e^{j\omega})$, and $H(e^{j\omega})$.
Note: Do not confuse $S(e^{j\omega})$, the Fourier Transform of $s[n]$, with the Power Spectral Density.
Hint: If $y[n]$ is a stationary random process,

$$S_{y,y}(e^{j\omega}) = E[Y(e^{j\omega})Y^*(e^{j\omega})].$$

$$S(e^{j\omega}) = \mathcal{DTFT}\{s[n]\}, \quad X(e^{j\omega}) = \mathcal{DTFT}\{x[n]\}, \quad H(e^{j\omega}) = \mathcal{DTFT}\{h[n]\}$$

$S_{s,s}(e^{j\omega})$ is the Power Spectral Density of the signal $s[n]$.

$S_{w,w}(e^{j\omega})$ is the Power Spectral Density of the noise $w[n]$.