EEE 424 Analytical Assignment 2 Solutions Spring 2024-25

Q.1) For this problem, we use Parseval's theorem to express the least-squares error E_{LS} in the time domain:

$$E_{LS} = \sum_{n=-\infty}^{\infty} \big[h_d(n) - h(n)\big]^2.$$

If we assume that $h(n)$ is of order N, with $h(n) = 0$ for $n < 0$ and $n > N$, then

$$E_{LS} = \sum_{n=0}^{N} \big[h_d(n) - h(n)\big]^2 + \sum_{n=-\infty}^{-1} h_d(n)^2 + \sum_{n=N+1}^{\infty} h_d(n)^2.$$

Because the last two terms are constants that are not affected by the filter $h(n)$, the least-squares error E_{LS} is minimized by minimizing the first term. This is achieved by setting $h(n) = h_d(n)$ for $n = 0, 1, \ldots, N$ (i.e., using a rectangular window in the window design method).

Q.2) Designing a low-pass filter with the window design method generally produces a filter with ripples of the same amplitude in the passband and stopband. Therefore, because the passband and stopband ripples in the filter specifications are the same, we only need to be concerned about the stopband ripple requirement. A stopband ripple of $\delta_s = 0.01$ corresponds to a stopband attenuation of ≈ -40 dB. Therefore, from the given table it follows that we may use a Hanning window, which provides an attenuation of approximately 44 dB. The specification on the transition band is that $\Delta\omega = 0.05\pi$, or $\Delta f = 0.025$. Therefore, the required filter order is

$$N = \frac{3.1}{\Delta f} = \frac{3.1}{0.025} = 124$$

and we have

$$w(n) = 0.5 - 0.5\cos\left(\frac{2\pi n}{124}\right), \quad 0 \le n \le 124$$

With an ideal low-pass filter that has a cutoff frequency of $\omega_c = 0.325$ (the midpoint of the transition band), and a delay of $N/2 = 62$ so that $h_d(n)$ is placed symmetrically within the interval $[0, 124]$, we have

$$h_d(n) = \frac{\sin[0.325\pi(n-62)]}{\pi(n-62)}$$

Therefore, the filter is

$$h(n) = \left[0.5 - 0.5\cos\left(\frac{2\pi n}{124}\right)\right] \cdot \frac{\sin[0.325\pi(n-62)]}{\pi(n-62)}, \quad 0 \le n \le 124$$

Note that if we were to use a Hamming or a Blackman window instead of a Hanning window, the stopband and passband ripple requirements would have been exceeded, and the required filter order would have been larger. With a Blackman window, for example, the filter order required to meet the transition band requirement is

$$N = \frac{5.5}{\Delta f} = \frac{5.5}{0.025} = 220$$

Q.3) We start with the frequency response of the filter:

$$H(e^{j\omega}) = A + Be^{-j\omega} + Ce^{-j2\omega}.$$

The conditions given are:

(a) Unity Gain at $\omega = 0$:

$$H(e^{j0}) = A + B + C = 1.$$

(b) Zero Gain at $\omega = \pi$:

$$H(e^{j\pi}) = A + Be^{-j\pi} + Ce^{-j2\pi} = A - B + C = 0,$$

since $e^{-j\pi} = -1$ and $e^{-j2\pi} = 1$.

(c) Linear Phase Requirement: For a 3-tap FIR filter to have linear phase, its impulse response must be symmetric (or anti-symmetric). Here, "nonzero coefficients with linear phase" implies that the filter is symmetric since in the other case $B = 0$. Therefore, we require

$$A = C.$$

With $A = C$, the conditions simplify to:

Using unity gain condition:

$$A + B + A = 2A + B = 1. \tag{i}$$

From zero gain at $\omega = \pi$:

$$A - B + A = 2A - B = 0. \tag{ii}$$

Solve equation (ii) for B:

$$2A - B = 0 \quad \Longrightarrow \quad B = 2A.$$

Substitute $B = 2A$ into equation (i):

$$2A + 2A = 4A = 1 \quad \Longrightarrow \quad A = \frac{1}{4}.$$

Then,

$$B = 2A = \frac{2}{4} = \frac{1}{2},$$

and by symmetry,

$$C = A = \frac{1}{4}.$$

Q.4) Phase is zero which also makes it linear (or generalized linear) as a special case.

Q.5) For $n \in \mathbb{Z}_+$ and $k \in \{0, 1, 2, \dots, 2^n - 1\}$, we define

$$\Phi_{n,k}(x) = \begin{cases} 2^{n/2} & \text{if } k2^{-n} \le x < (k+1/2)2^{-n} \\ -2^{n/2} & \text{if } (k+1/2)2^{-n} \le x \le (k+1)2^{-n} \\ 0 & \text{else} \end{cases}$$

Consider any two pairs (n, k) and (n', k') where $(n, k) \neq (n', k')$.

If $n = n'$, then for $k \neq k'$, the supports of the functions $\Phi_{n,k}$ and $\Phi_{n',k'}$, i.e., the set of points on which the function is non-zero, are disjoint. Therefore, their inner product

$$\langle \Phi_{n,k}, \Phi_{n',k'} \rangle = \int_0^1 \Phi_{n,k}(x)\Phi_{n',k'}(x)dx = 0,$$

for all $n = n'$ and $k \neq k'$.

If $n \neq n'$, let $n > n'$ without loss of generality. We will check when the supports of $\Phi_{n,k}$ and $\Phi_{n',k'}$ intersect. Note that the support set of $\Phi_{n,k}$ is

$$[k2^{-n}, (k+1)2^{-n}].$$

Accordingly, for some constant c, we have

$$\int_0^1 c \cdot \Phi_{n,k}(x)dx = \int_{k2^{-n}}^{(k+1)2^{-n}} c \cdot \Phi_{n,k}(x)dx = \int_{k2^{-n}}^{(k+1/2)2^{-n}} c \cdot 2^{n/2}dx + \int_{(k+1/2)2^{-n}}^{(k+1)2^{-n}} c \cdot (-2^{n/2})dx$$

$$= c \cdot 2^{n/2}((k+1/2)2^{-n} - k2^{-n}) + c \cdot (-2^{n/2})((k+1)2^{-n} - (k+1/2)2^{-n}) = 0$$

Therefore, the inner product of $\Phi_{n,k}$ and any function that is constant on the support of $\Phi_{n,k}$ is just 0. In other words, $\Phi_{n,k}$ is orthogonal to any function that is constant in its support. Accordingly, checking the following conditions over the support sets of $\Phi_{n,k}$ and $\Phi_{n',k'}$:

- $[k2^{-n}, (k+1)2^{-n}] \cap [k'2^{-n'}, (k'+1)2^{-n'}] = \emptyset \Rightarrow \langle \Phi_{n,k}, \Phi_{n',k'} \rangle = 0$
- $[k2^{-n}, (k+1)2^{-n}] \subset [k'2^{-n'}, (k'+1/2)2^{-n'}] \Rightarrow \Phi_{n',k'}(x) = 2^{n'/2} \Rightarrow \langle \Phi_{n,k}, \Phi_{n',k'} \rangle = 0$

- $[k2^{-n},(k+1)2^{-n}] \subset [(k'+1/2)2^{-n'},(k'+1)2^{-n'}] \Rightarrow \Phi_{n',k'}(x) = -2^{n'/2} \Rightarrow \langle \Phi_{n,k}, \Phi_{n',k'} \rangle = 0$

Furthermore, we have $\langle \Psi_{0,0}, \Phi_{n,k} \rangle = 0$ for all n, k since $\Psi_{0,0}$ is constant on $[0,1]$.
For any allowable pair n, k, we also have

$$\|\Phi_{n,k}\| = \int_0^1 \Phi_{n,k}(x)\Phi_{n,k}(x)dx = 2^n 2^{-n} = 1$$

Therefore, we can conclude that the set is **orthonormal**.

Q.6) (a) Let $x(t) = e^{-\alpha t}u(t)$
Fourier Transform:

$$\mathcal{X}(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_0^{\infty} e^{-\alpha t}e^{-j\omega t}dt = \int_0^{\infty} e^{-(\alpha+j\omega)t}dt$$

$$= \left[\frac{e^{-(\alpha+j\omega)t}}{-(\alpha+j\omega)}\right]_0^{\infty} = \frac{1}{\alpha+j\omega}$$

So,

$$\mathcal{X}(j\omega) = \frac{1}{\alpha+j\omega}$$

(b) Short-Time Fourier Transform (STFT):

$$\mathcal{X}_w(\tau,\omega) = \int_{-\infty}^{\infty} x(t)w(t-\tau)e^{-j\omega t}dt$$

Window function:

$$w(t) = \text{rect}\left(\frac{t}{T_0}\right) = \begin{cases} 1, & |t| \le \frac{T_0}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$w(t-\tau) = \text{rect}\left(\frac{t-\tau}{T_0}\right) = \begin{cases} 1, & \tau - \frac{T_0}{2} \le t \le \tau + \frac{T_0}{2} \\ 0, & \text{otherwise} \end{cases}$$

So,

$$\mathcal{X}_w(\tau,\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} e^{-\alpha t}u(t)w(t-\tau)e^{-j\omega t}dt = \int_{\tau-T_0/2}^{\tau+T_0/2} e^{-\alpha t}u(t)w(t-\tau)e^{-j\omega t}dt$$

Case 1: $\tau \ge \frac{T_0}{2}$

$$\mathcal{X}_w(\tau,\omega) = \left[\frac{e^{-(\alpha+j\omega)t}}{-(\alpha+j\omega)}\right]_{\tau-T_0/2}^{\tau+T_0/2} = \frac{1}{\alpha+j\omega}\left(e^{-(\alpha+j\omega)(\tau-T_0/2)} - e^{-(\alpha+j\omega)(\tau+T_0/2)}\right)$$

Case 2: $\tau < \frac{T_0}{2}$

$$\mathcal{X}_w(\tau,\omega) = \int_0^{\tau+T_0/2} e^{-(\alpha+j\omega)t}dt = \left[\frac{e^{-(\alpha+j\omega)t}}{-(\alpha+j\omega)}\right]_0^{\tau+T_0/2} = \frac{1}{\alpha+j\omega}\left(1 - e^{-(\alpha+j\omega)(\tau+T_0/2)}\right)$$

Thus:

$$\mathcal{X}_w(\tau,\omega) = \begin{cases} \frac{1}{\alpha+j\omega}\left(e^{-(\alpha+j\omega)(\tau-T_0/2)} - e^{-(\alpha+j\omega)(\tau+T_0/2)}\right), & \tau \ge \frac{T_0}{2} \\ \frac{1}{\alpha+j\omega}\left(1 - e^{-(\alpha+j\omega)(\tau+T_0/2)}\right), & \tau < \frac{T_0}{2} \end{cases}$$

Evaluation:

$$\mathcal{X}_w(0,\omega) = \frac{1}{\alpha+j\omega}\left(1 - e^{-\alpha T_0/2}\right)$$

$$\mathcal{X}_w(\tau,0) = \begin{cases} \frac{1}{\alpha}\left(e^{-\alpha(\tau-T_0/2)} - e^{-\alpha(\tau+T_0/2)}\right), & \tau \ge \frac{T_0}{2} \\ \frac{1}{\alpha}\left(1 - e^{-\alpha(\tau+T_0/2)}\right), & \tau < \frac{T_0}{2} \end{cases}$$

Q.7) (a)

$$d_0[n] = x_0[2n]$$

$$D_0(z) = \sum_{n=-\infty}^{\infty} x_0[2n]z^{-n} = \sum_{n=-\infty}^{\infty} x_0[n]w_n z^{-n/2},$$

where

$$w_n = \begin{cases} 1, & (n)_2 = 0 \\ 0, & (n)_2 \neq 0 \end{cases}$$

Note that

$$\frac{1}{M}\sum_{k=0}^{M-1} e^{j\frac{2\pi}{M}kn} = \frac{1}{M}\sum_{k=0}^{M-1}\left(e^{j\frac{2\pi}{M}n}\right)^k = \frac{1}{M}\frac{1-e^{j\frac{2\pi}{M}Mn}}{1-e^{j\frac{2\pi}{M}n}}$$

$$= \frac{1}{M}\frac{1-e^{j2\pi n}}{1-e^{j\frac{2\pi}{M}n}} = \begin{cases} 1, & (n)_M = 0 \\ 0, & (n)_M \neq 0 \end{cases}$$

Now use this for $M = 2$:

$$w_n = \frac{1}{2}\sum_{k=0}^{1} e^{j\frac{2\pi}{2}kn}$$

$$D_0(z) = \sum_{n=-\infty}^{\infty} x_0[n]\left(\frac{1}{2}\sum_{k=0}^{1} e^{j\frac{2\pi}{2}kn}\right) z^{-n/2}$$

$$= \frac{1}{2}\sum_{k=0}^{1}\left(\sum_{n=-\infty}^{\infty} x_0[n]\left(z^{1/2}e^{-j\frac{2\pi}{2}k}\right)^{-n}\right)$$

$$D_0(z) = \frac{1}{2}\sum_{k=0}^{1} X_0(z^{1/2}e^{-j\pi k})$$

$$= \frac{1}{2}\left(X_0(z^{1/2}) + X_0(-z^{1/2})\right)$$

Hence,

$$D_0(z) = \frac{1}{2}X_0(z^{1/2}) + \frac{1}{2}X_0(-z^{1/2})$$

Similarly,

$$D_1(z) = \frac{1}{2}X_1(z^{1/2}) + \frac{1}{2}X_1(-z^{1/2})$$

(b)

$$u_0[n] = \begin{cases} d_0\left[\frac{n}{2}\right], & \text{if } (n)_2 = 0 \\ 0, & \text{otherwise} \end{cases}$$

$$U_0(z) = \sum_{n=-\infty}^{\infty} u_0[n]z^{-n} = \sum_{n=-\infty}^{\infty} d_0[n](z^2)^{-n} = D_0(z^2)$$

$$\Rightarrow U_0(z) = D_0(z^2), \text{ and similarly, } U_1(z) = D_1(z^2).$$

(c) Using LTI filters' z-transforms,

$$X_0(z) = X(z)H_0(z), \text{ and similarly, } X_1(z) = X(z)H_1(z)$$

(d) Overall system: $f(x[n]) = y[n]$

LTI:

$$f(ax_a[n-n_1] + bx_b[n-n_2]) = ay_a[n-n_1] + by_b[n-n_2]$$

where f is the filter representing the overall system and

$$f(x_a[n-n_1]) = y_a[n-n_1]$$
$$f(x_b[n-n_2]) = y_b[n-n_2]$$

Define the frequency response of f as $F(z)$. Then, for the system to be LTI,

$$\left(aX_a(z)z^{-n_1} + bX_b(z)z^{-n_2}\right) F(z)$$

$$aY_a(z)z^{-n_1} + bY_b(z)z^{-n_2}.$$

From the overall system:

$$y[n] = y_0[n] + y_1[n]$$
$$Y(z) = Y_0(z) + Y_1(z)$$

$$\begin{aligned} Y_0(z) &= U_0(z)G_0(z) \\ Y_0(z) &= D_0(z^2)G_0(z) = \frac{1}{2}G_0(z)\left(X_0(z) + X_0(-z)\right) \\ &= \frac{1}{2}G_0(z)\left(X(z)H_0(z) + X(-z)H_0(-z)\right) \end{aligned}$$

Similarly,

$$Y_1(z) = \frac{1}{2}\left(G_1(z)H_1(z)X(z) + G_1(z)H_1(-z)X(-z)\right)$$

Therefore,

$$Y(z) = X(z) \cdot \frac{1}{2}(G_0(z)H_0(z) + G_1(z)H_1(z)) + X(-z) \cdot \frac{1}{2}(G_0(z)H_0(-z) + G_1(z)H_1(-z))$$

Consider *only* time invariance,

$$f(x[n-n_1]) = y[n-n_1] \quad \rightarrow \quad X(z)F(z)z^{-n_1} = Y(z)z^{-n_1}$$

$$\Rightarrow X(z)F(z) = Y(z) \text{ for time invariant system.}$$

The system is $Y(z)$ is not of the form $X(z)F(z)$ **in general**, so the system is **not necessarily LTI.**

(e) If $H_0(-z)G_0(z) + H_1(-z)G_1(z) = 0$, then:

$$Y(z) = \frac{1}{2}X(z)(G_0(z)H_0(z) + G_1(z)H_1(z)) = X(z)F(z)$$

This fits the LTI system form with $F(z) = G_0(z)H_0(z) + G_1(z)H_1(z)$.
Linearity can be easily shown by putting $aX_0(z)$ instead of $X_0(z)$ for some $a \in \mathbb{R}$.
Hence, when the condition is satisfied, the system can be claimed to be LTI.

(f) If $G_0(z) = H_1(-z)$ and $G_1(z) = -H_0(-z)$, then:

$$\begin{aligned} Y(z) &= \frac{1}{2}X(z)(G_0(z)H_0(z) + G_1(z)H_1(z)) \\ &\quad + \frac{1}{2}X(-z)(H_1(-z)H_0(-z) + (-H_0(-z))H_1(-z)) \\ &= \frac{1}{2}X(z)(G_0(z)H_0(z) + G_1(z)H_1(z)) = X(z)F(z) \end{aligned}$$

So again, the system is **LTI**.

Q.8) (a) Minimize $\|s[n] - h[n] * x[n]\|^2$

$$\begin{aligned} x[n] &= s[n] + w[n] \\ \mu_x &= \mu_s + \mu_w \quad (E[x] = E[s] + E[w]) \\ R_{xx}[m] &= \mathbb{E}[x[n+m]x^*[n]] \\ &= \mathbb{E}[(s[n+m] + w[n+m])(s[n] + w[n])^*] \\ &= \mathbb{E}[s[n+m]s^*[n]] + \mathbb{E}[s[n+m]w^*[n]] + \mathbb{E}[w[n+m]s^*[n]] + \mathbb{E}[w[n+m]w^*[n]] \\ &= R_{ss}[m] + R_{sw}[m] + R_{ws}[m] + R_{ww}[m] \end{aligned}$$

Assume $R_{sw}[m] = R_{ws}[m] = 0$:

$$R_{xx}[m] = R_{ss}[m] + R_{ww}[m]$$

Consequently,

$$S_{xx}[m] = S_{ss}[m] + S_{ww}[m]$$

Define $y[n] := x[n] * h[n]$. Minimize $\|s[n] - y[n]\|^2$:

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} h[m]x[n-m]$$
$$y[n] \in \text{span}\{x[n-m]\} =: S$$

Minimize $\|s[n] - y[n]\|^2 \Rightarrow s[n] - y[n] \perp S \Rightarrow \langle s[n] - y[n], x[n-m] \rangle = 0$

$$\begin{aligned}
\langle s[n], x[n-m] \rangle &= \langle y[n], x[n-m] \rangle \\
\mathbb{E}[s[n]x^*[n-m]] &= \mathbb{E}[y[n]x^*[n-m]] \\
R_{sx}[m] = R_{yx}[m] &\Rightarrow S_{sx}(\omega) = S_{yx}(\omega)
\end{aligned}$$

Since $y[n] = h[n] * x[n]$, then $Y(\omega) = H(\omega)X(\omega) \Rightarrow S_{yx}(\omega) = H(\omega)S_{xx}(\omega)$

$$H(\omega) = \frac{S_{sx}(\omega)}{S_{xx}(\omega)}$$

To find $S_{yx}(\omega)$:

$$\begin{aligned}
R_{yx}[m] &= \mathbb{E}\left[y[n+m]x^*[n]\right] \\
&= \mathbb{E}\left[y[n+m]\left(s[n] + w[n]\right)^*\right] \\
&= \mathbb{E}\left[y[n+m]s^*[n]\right] + \mathbb{E}\left[y[n+m]w^*[n]\right] \\
&= R_{ys}[m] + R_{yw}[m] \\
&= R_{ys}[m] \quad \text{(Assume } R_{yw}[m] = 0) \\
R_{yx}[m] &= R_{ys}[m]
\end{aligned}$$

Using the condition $R_{sx}[m] = R_{yx}[m]$ to minimize MSE:

$$\begin{aligned}
R_{yx}[m] = R_{sx}[m] &= \mathbb{E}\left[s[m+n]x^*[n]\right] \\
&= \mathbb{E}\left[s[m+n]\left(s[n] + w[n]\right)^*\right] \\
&= \mathbb{E}\left[s[m+n]s^*[n]\right] + \mathbb{E}\left[s[m+n]w^*[n]\right]
\end{aligned}$$

$$\begin{aligned}
R_{sx}[m] &= \mathbb{E}\left[s[m+n]s^*[n]\right] + \mathbb{E}\left[s[m+n]w^*[n]\right] \\
&= R_{ss}[m] + R_{sw}[m] = R_{ss}[m] \quad \text{(assuming } R_{sw}[m] = 0) \\
R_{sx}[m] &= R_{ss}[m] \\
S_{sx}(\omega) &= S_{ss}(\omega) \\
R_{sx}[m] &= R_{yx}[m] \Rightarrow S_{sx}(\omega) = S_{yx}(\omega) \\
\Rightarrow S_{yx}(\omega) &= S_{ss}(\omega)
\end{aligned}$$

$$\begin{aligned}
\therefore \quad H(\omega) &= \frac{S_{yx}(\omega)}{S_{xx}(\omega)} \\
&= \frac{S_{ss}(\omega)}{S_{xx}(\omega)} \\
&= \frac{S_{ss}(\omega)}{S_{ss}(\omega) + S_{ww}(\omega)}
\end{aligned}$$

(b)

$$\begin{aligned}
\text{MSE} &= \mathbb{E}[|s[n] - h[n] * x[n]|^2] \\
&= \mathbb{E}[(s[n] - y[n])(s[n] - y[n])^*] \\
&= \mathbb{E}[s[n]s^*[n]] - \mathbb{E}[s[n]y^*[n]] - \mathbb{E}[y[n]s^*[n]] + \mathbb{E}[y[n]y^*[n]] \\
&= R_{ss}[0] - \sum_{m=-\infty}^{\infty} h[m]R_{sx}[m] - \sum_{m=-\infty}^{\infty} h^*[m]R_{xs}[-m] + \sum_{l=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} h[l]h^*[m]R_{xx}[l-m]
\end{aligned}$$

$$
\begin{aligned}
R_{xx}[l-m] &= R_{ss}[l-m] + R_{ww}[l-m] \\
\Rightarrow R_{sx}[m] &= R_{ss}[m] \text{ assuming } R_{sw}[m] = 0 \\
R_{xs}[m] &= R_{ss}[m] \text{ likewise.}
\end{aligned}
$$

$$
\text{MSE} = R_{ss}[0] - \sum_m h[m] R_{ss}[m] - \sum_m h^*[m] R_{ss}[m] + \sum_l \sum_m h[l] h^*[m] (R_{ss}[l-m] + R_{ww}[l-m])
$$

(c)

$$
\begin{aligned}
\mathbb{E}\left[|S(\omega) - Y(\omega)|^2\right] &= \mathbb{E}\left[(S(\omega) - Y(\omega))(S(\omega) - Y(\omega))^*\right] \\
&= \mathbb{E}\left[S(\omega)S^*(\omega)\right] - \mathbb{E}\left[S(\omega)Y^*(\omega)\right] \\
&\quad - \mathbb{E}\left[Y(\omega)S^*(\omega)\right] + \mathbb{E}\left[Y(\omega)Y^*(\omega)\right] \\
&= S_{ss}(\omega) - S_{sy}(\omega) - S_{ys}(\omega) + S_{yy}(\omega)
\end{aligned}
$$

$$
\begin{aligned}
Y(\omega) &= X(\omega)H(\omega) = \left(S(\omega) + W(\omega)\right) H(\omega) \\
\mathbb{E}\left[S(\omega)S^*(\omega)\right] &= S_{ss}(\omega) \\
\mathbb{E}\left[S(\omega)(S(\omega) + W(\omega))H(\omega)\right]^* &= H^*(\omega)\mathbb{E}\left[S(\omega)S^*(\omega)\right] \\
&\quad + H^*(\omega)\mathbb{E}\left[S(\omega)W^*(\omega)\right] \\
&= H^*(\omega)S_{ss}(\omega) + H^*(\omega)S_{sw}(\omega) \\
&= H^*(\omega)S_{ss}(\omega) \quad \text{(since } R_{sw}(m) = 0 \text{ assumed)} \\
\mathbb{E}\left[(S(\omega) + W(\omega))H(\omega)S^*(\omega)\right] &= H(\omega)S_{ss}(\omega) \\
\mathbb{E}\left[Y(\omega)Y^*(\omega)\right] &= \mathbb{E}\left[(S(\omega) + W(\omega))H(\omega)H^*(\omega)(S(\omega) + W(\omega))^*\right] \\
&= |H(\omega)|^2 \left(S_{ss}(\omega) + S_{ww}(\omega)\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[|S(\omega) - Y(\omega)|^2\right] &= S_{ss}(\omega) + H^*(\omega)S_{ss}(\omega) + H(\omega)S_{ss}(\omega) \\
&\quad + |H(\omega)|^2 \left(S_{ss}(\omega) + S_{ww}(\omega)\right) \\
&= S_{ss}(\omega)\left(1 - H(\omega) + H^*(\omega) + H(\omega)H^*(\omega)\right) \\
&\quad + |H(\omega)|^2 S_{ww}(\omega)
\end{aligned}
$$

In terms of $e^{j\omega}$ instead of ω:

$$
\begin{aligned}
\mathbb{E}\left[|S(e^{j\omega}) - Y(e^{j\omega})|^2\right] &= S_{ss}(e^{j\omega}) - H^*(e^{j\omega})S_{ss}(e^{j\omega}) - H(e^{j\omega})S_{ss}(e^{j\omega}) \\
&\quad + |H(e^{j\omega})|^2 S_{ss}(e^{j\omega}) + |H(e^{j\omega})|^2 S_{ww}(e^{j\omega})
\end{aligned}
$$