ISA 444: Business Forecasting
17 - ARMA Models

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Spring 2021

Outline

Quick Refresher on What we Covered in Class so Far [1]

Main Learning Outcomes Discussed in Class so Far

☑ We have studied the basic components of time series, including trends, seasonal components, and cyclical components.

☑ We have considered why and how we forecast.

☑ We have learned how we evaluate forecast accuracy with measures such as Mean Absolute Error, Root Mean Square Error, Mean Absolute Percent Error.

☑ We have learned to compute forecast accuracy to evaluate both in-sample and out-of-sample performance of forecasts.

☑ We have considered Moving Average, Decomposition, and Smoothing Methods for exploring and forecasting time series.

Quick Refresher on What we Covered in Class so Far [2]

Main Learning Outcomes Discussed in Class so Far

☑ We have learned preliminaries regarding the autocorrelation structure of time series and how to plot the autocorrelation and partial autocorrelation over time.

☑ We have studied a random walk model and know how to recognize one using the ACF function.

☑ We know what it means for a time series to be nonstationary, and how to test for this formally.

Where We are Going

- Using all the information we have learned, we will learn to **Formally Model** a time series with statistical models.
- Some of these models will be **Extrapolative**, and some will be **Causal**.

Learning Objectives for Today's Class

Main Learning Outcomes

- **Describe the behavior of the ACF and PACF of an AR(p) process.**
- **Describe the behavior of the ACF and PACF of an MA(q) process.**
- **Describe the behavior of the ACF and PACF of an ARMA (p,q) process.**
- **Fit an ARMA model to a time series, evaluate the residuals of a fitted ARMA model to assess goodness of fit, use the Ljung-Box test for correlation among the residuals of an ARIMA model.**

Preface: ARMA Models

Models we consider here may have two components, an autoregressive component (AR) and a moving average component (MA).

Outline

The First Order Autoregressive Process [1]

The **First Order Autoregressive Process—AR(1)** is given by

$$y_t = \delta + \phi y_{t-1} + \epsilon_t,$$

where $|\phi| < 1$ is a weight, and ϵ_t is white noise. Essentially, this is similar (not exactly the same though) as regressing y_t on y_{t-1}.

$$E(y_t) = \mu = \frac{\delta}{1-\phi}$$

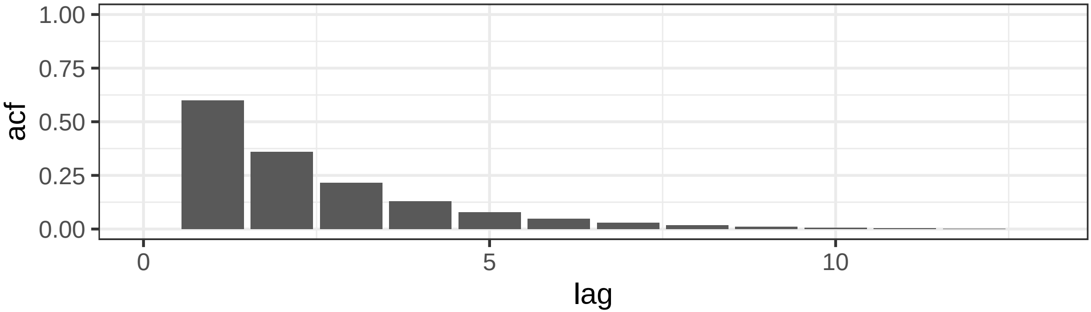
$$Var(y_t) = \sigma^2 \frac{1}{1-\phi^2}$$

The First Order Autoregressive Process [2]

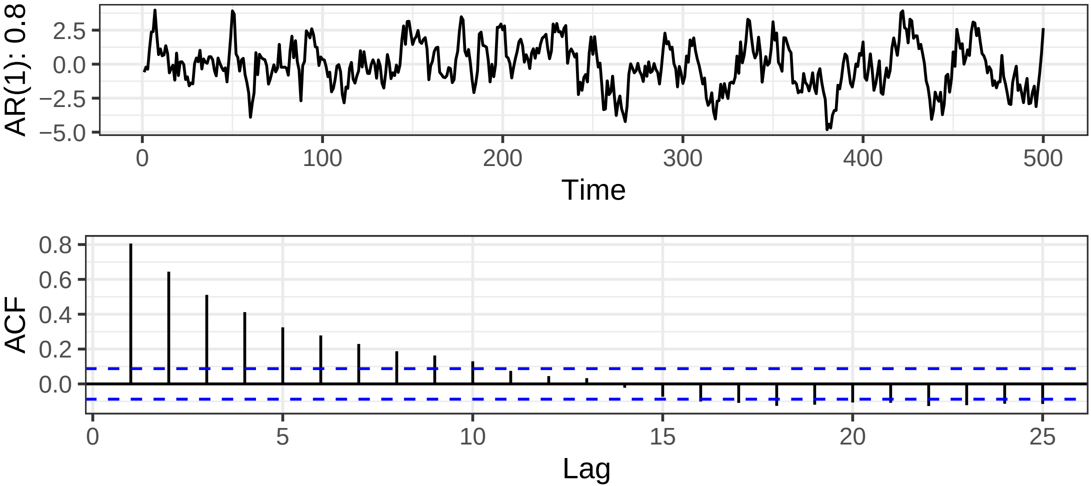
The *population* autocorrelation function of the AR(1) process at lag k is

$$\rho(k) = \phi^k$$

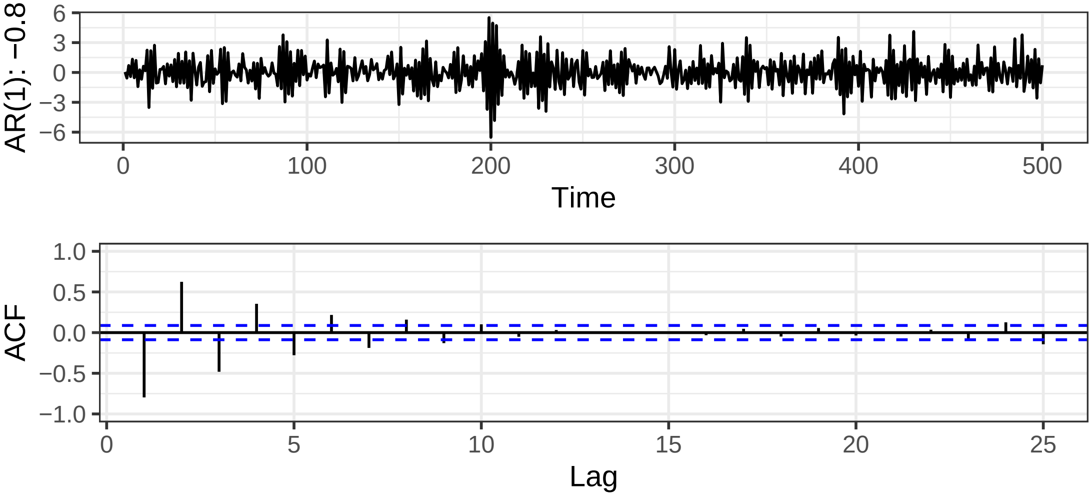
The theoretical/population ACF of an AR(1) process with $\phi = 0.6$ will look like this:



Example Plots of Simulated AR(1) Data [1]



Example Plots of Simulated AR(1) Data [2]



Example Plots of Simulated AR(1) Data [3]

Notice how the ACF "dies down" in each case.

The Second Order Autoregressive Process [1]

The **Second Order Autoregressive Process—AR(2)** is given by

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t,$$

where $|\phi_1| < 1$ and $|\phi_2| < 1$ are weights, and ϵ_t is white noise. Essentially, this is similar (not exactly the same though) as regressing y_t on y_{t-1} and y_{t-2}.

$$E(y_t) = \mu = \frac{\delta}{1 - \phi_1 - \phi_2}$$

$$Var(y_t) = \phi_1 \gamma(1) + \phi_2 \gamma(2) + \sigma^2,$$

where $\gamma(1)$ and $\gamma(2)$ are the autocovariance functions at lags 1 and 2, respectively.

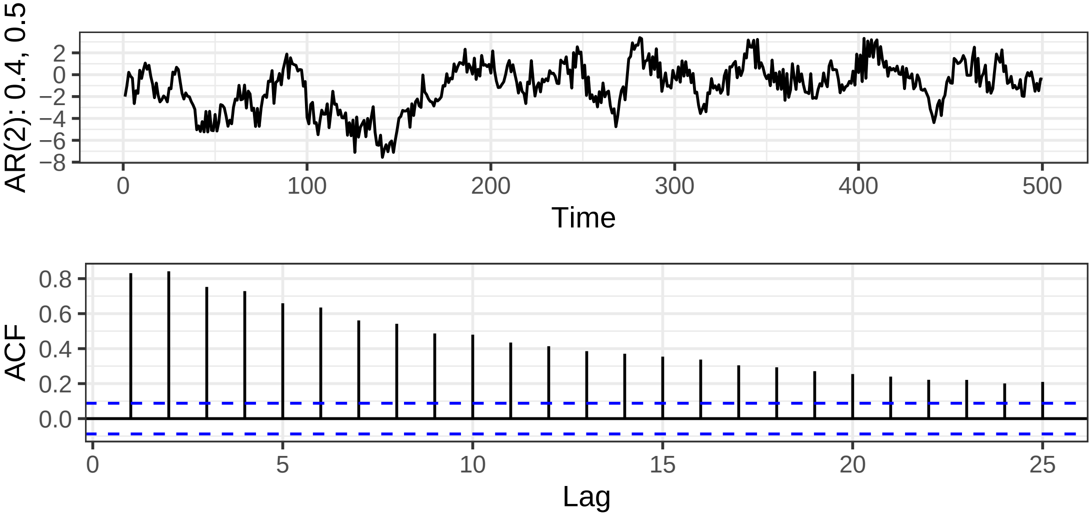
The Second Order Autoregressive Process [2]

The *population* autocorrelation function of the AR(1) process at lag k is

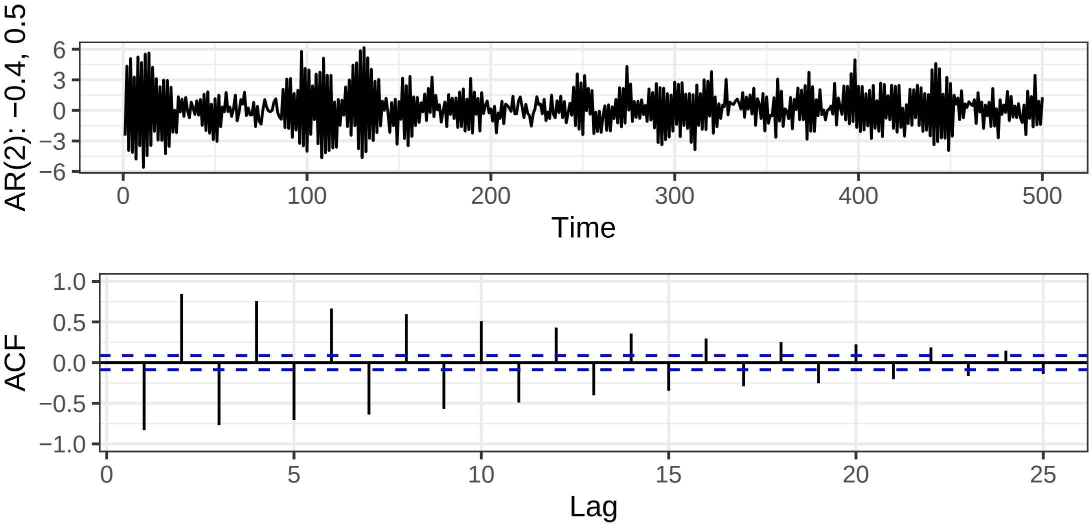
$$\rho(k) = \phi_1 \rho(k-1) + \phi_2 \rho(k-2)$$

The AR(2) model can be seen as an "adjusted" AR(1) model for which a single exponential decay expression as in the AR(1) model is not enough to describe the pattern in the ACF. Hence an additional term for the second lag is added.

Example Plots of Simulated AR(2) Data [1]



Example Plots of Simulated AR(2) Data [2]



The General Order Autoregressive Process—AR(p) [1]

The **General Order Autoregressive Process—AR(p)** is given by

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

where $|\phi_i| < 1 \, \forall i = 1, 2, \ldots, p$ are weights, and ϵ_t is white noise. Essentially, this is similar (not exactly the same though) as regressing y_t on $y_{t-1}, \ldots, y_{t-p}$.

$$E(y_t) = \mu = \frac{\delta}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}$$

$$Var(y_t) = \sum_{i=1}^{p} \phi_i \gamma(i) + \sigma^2,$$

where $\gamma(i)$ is the autocovariance functions at lag i.

The General Order Autoregressive Process—AR(p) [2]

The *population* autocorrelation function of the AR(2) process at lag k is

$$\rho(k) = \sum_{i=1}^{p} \phi_i \rho(k-i) \text{ for } k > 0$$

The ACF of an AR(p) process, for $p > 1$ is a mixture of exponential decay and a damped sinusoid expression (damped sinusoid from the lag 2 and greater).

AR Model: Determining if the Data Can Be Modeled as an AR Process

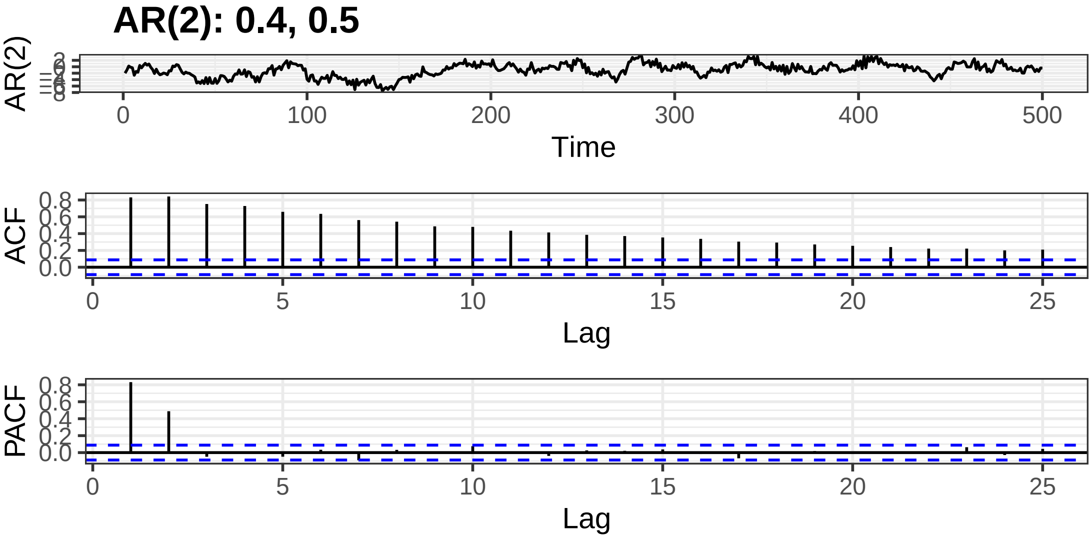
- We can usually tell from the ACF that there is an autoregressive (AR) component to the data because the ACF plot tends to geometrically decrease in magnitude (i.e., "die down").
- The **Order** of an AR Process refers to how many lags you include in the autoregressive model.
- Because the ACF of the AR model is a mixture, the **ACF is not useful for determining the order of the AR process**.
- Thus, the ACF helps us to know that we have an **AR model**, but not which AR model to fit!

AR Model: Determining the Order

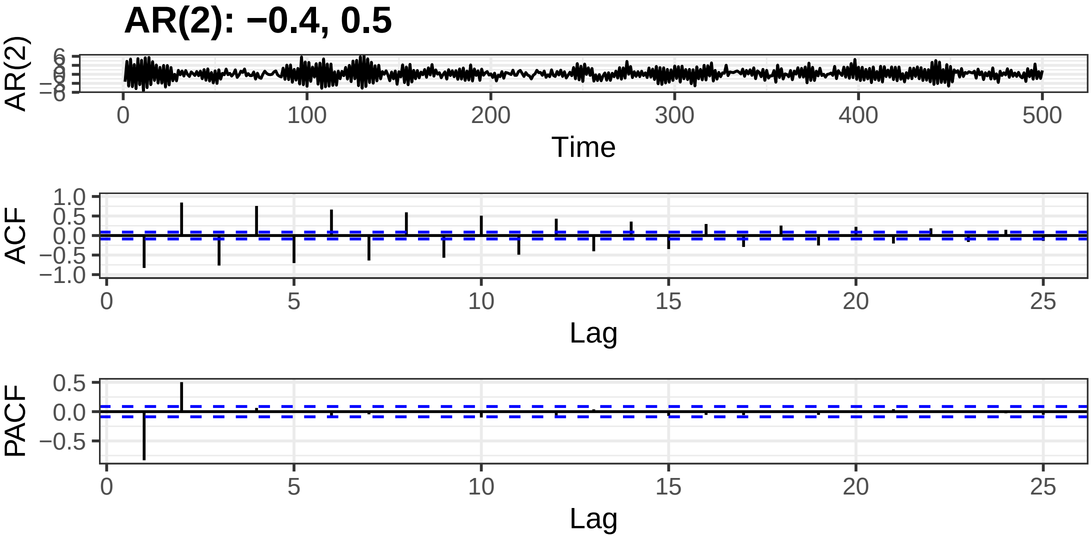
Recall the **Partial Autocorrelation:** The Partial Autocorrelation between y_t and y_{t+k} is the correlation between y_t and y_{t+k} removing the effects of $y_{t+1}, y_{t+2}, \ldots, y_{t+k-1}$.

- When plotted over multiple lags, we refer to the plot as the Partial Autocorrelation Function or PACF.
- For an AR(p) model, the PACF between y_t and y_{t+k} should be 0 $\forall k > p$.
- Thus, for an AR(p) process, the PACF should "cut off" after lag p.

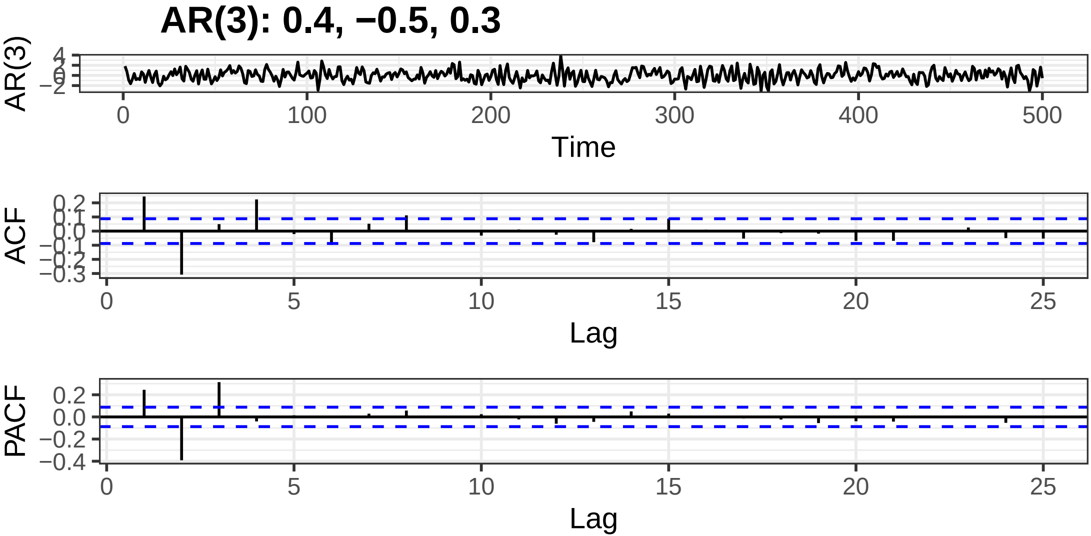
Example Plots of AR Processes [1]



Example Plots of AR Processes [2]



Example Plots of AR Processes [3]



Example Plots of AR Processes [4]

Note how the PACF is not significant after lag (p) in each case.

Outline

The Moving Average Process [1]

The moving average process of order $q, MA(q)$, process is given as

$$y_t = \mu + \epsilon_t - \theta_1 \epsilon_{t-1} - \cdots - \theta_q \epsilon_{t-q}$$

where θ_i is a weight, and ϵ_i is white noise. **An MA(q) process is always stationary regardless of the weights.**

$$\begin{aligned} E(y_t) &= E(\mu + \epsilon_t - \theta_1 \epsilon_{t-1} - \cdots - \theta_q \epsilon_{t-q}) \\ &= \mu \end{aligned}$$

$$\begin{aligned} Var(y_t) &= Var(\mu + \epsilon_t - \theta_1 \epsilon_{t-1} - \cdots - \theta_q \epsilon_{t-q}) \\ &= \sigma^2 (1 + \theta_1^2 + \cdots + \theta_q^2) \end{aligned}$$

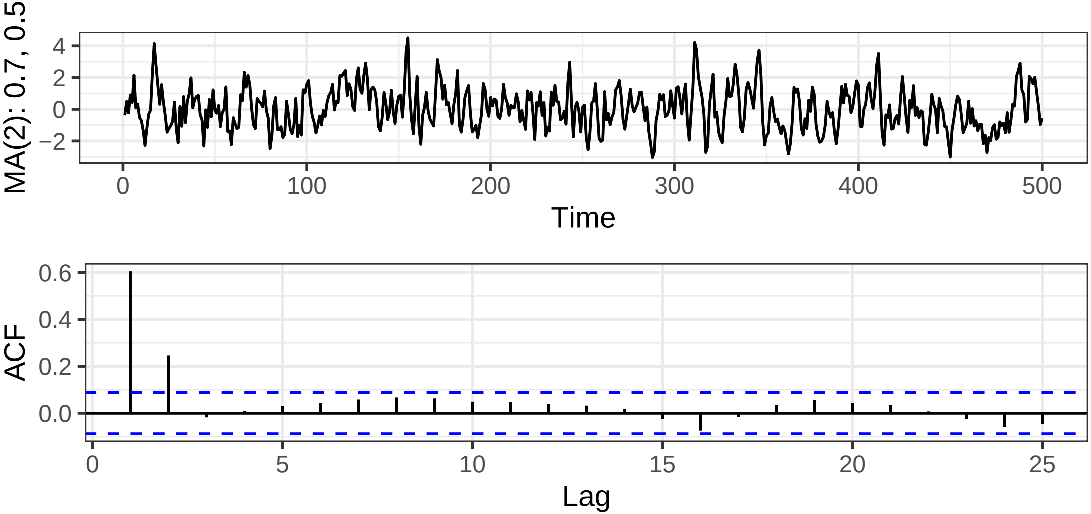
The Moving Average Process [2]

The POPULATION autocorrelation function of the MA(q) process at lag k is

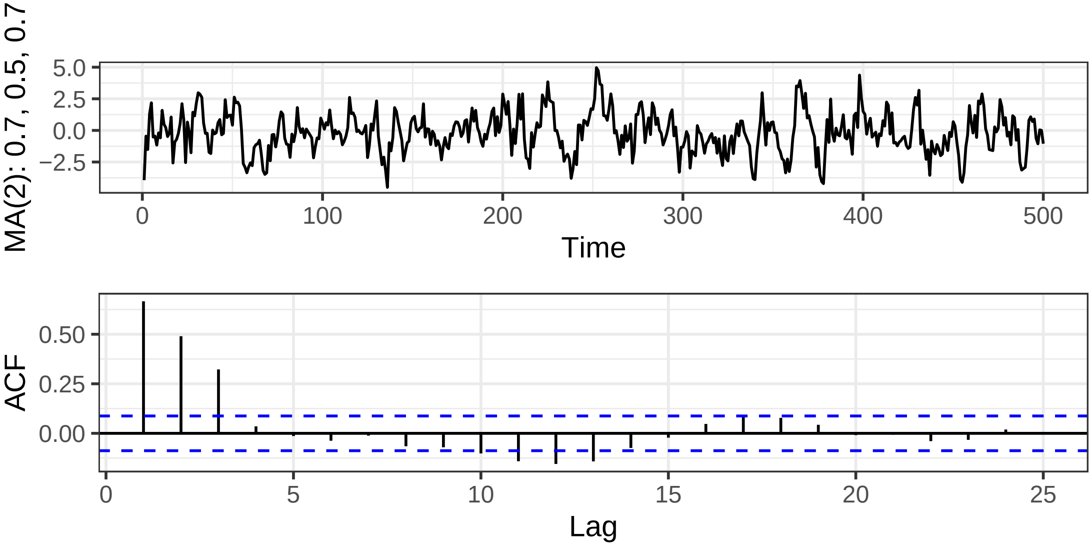
$$\rho(k) = \begin{cases} \frac{(-\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q)}{1+\theta_1^2+\cdots+\theta_q^2}, & k = 1, 2, \ldots, q \\ 0, & k > q \end{cases}$$

This feature of the ACF is very helpful in identifying the MA model and its appropriate order because the ACF function of a MA model is not significant (i.e., "cuts off") after lag q.

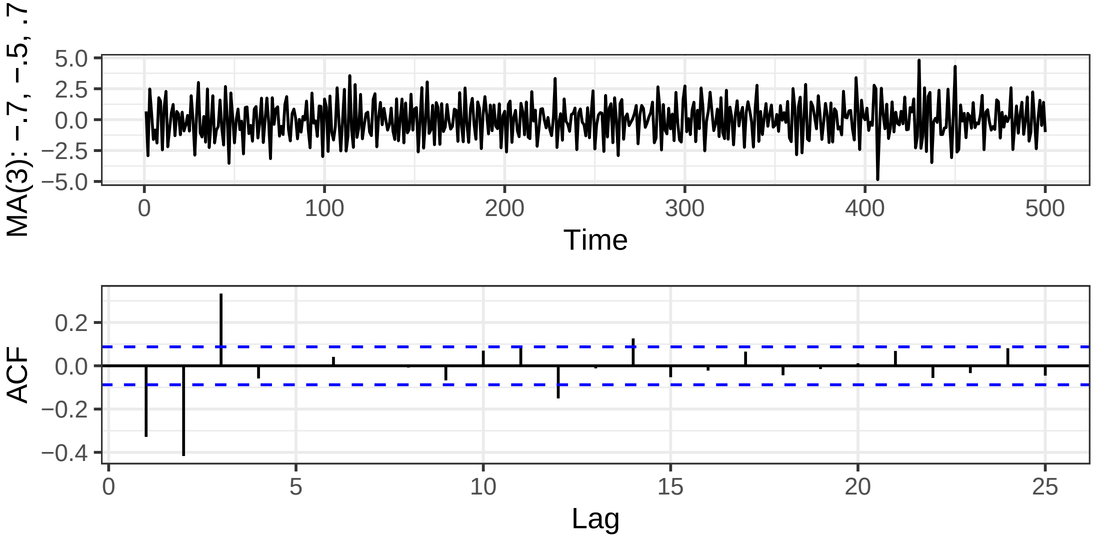
Example Plots of MA Processes [1]



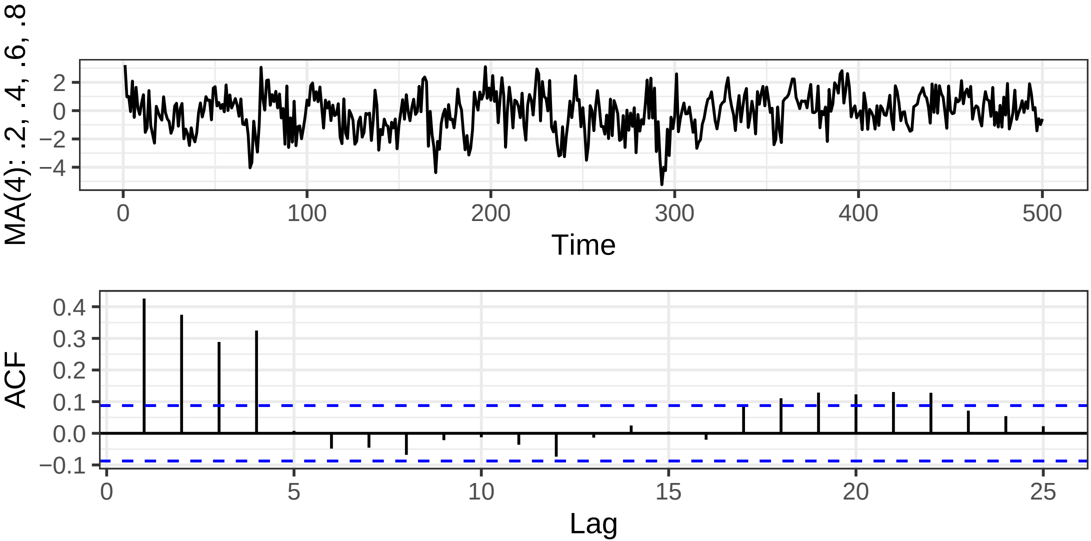
Example Plots of MA Processes [2]



Example Plots of MA Processes [3]



Example Plots of MA Processes [4]



Outline

Mixed Autoregressive Moving Average Processes [1]

Sometimes, if a really high order seems needed for an AR process, it may be better, instead, to add one or more MA term. This results in a mixed autoregressive moving average or an ARMA model.

In general, an ARMA(p,q) model is given as

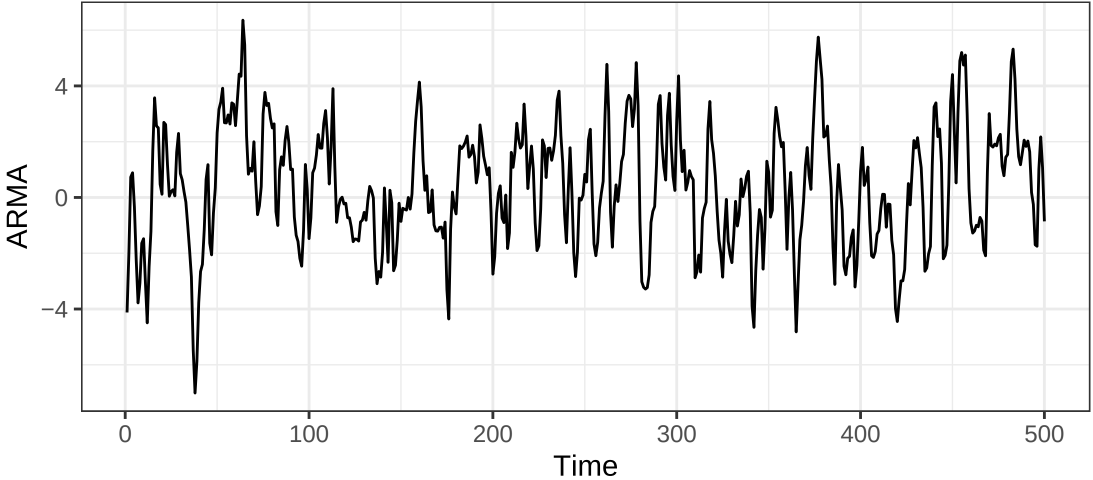
$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t - \theta_1 \epsilon_{t-1} - \cdots - \theta_q \epsilon_{t-q}$$

The ACF and PACF of the ARMA(p,q) process exhibit exponential decay exponential decay and/or damped sinusoid patterns. This makes identification of the order of the ARMA(p,q) process difficult.

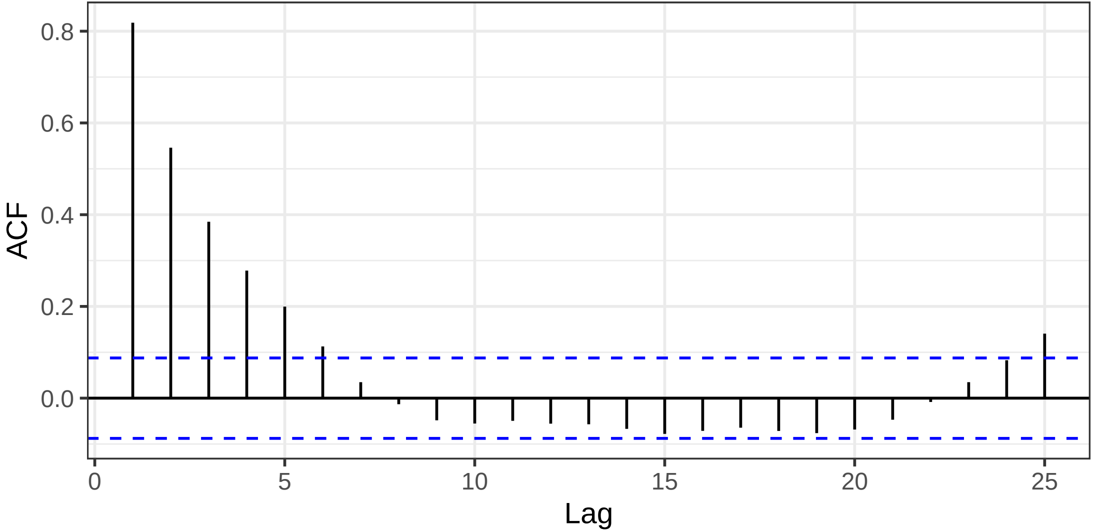
Mixed Autoregressive Moving Average Processes [2]

Model	**ACF**	**PACF**
AR(p)	Exponentially decays or damped sinusoidal pattern	Cuts off after lag p
MA(q)	Cuts off after lag q	Exponentially decays or damped sinusoidal pattern
ARMA(p,q)	Exponentially decays or damped sinusoidal pattern	Exponentially decays or damped sinusoidal pattern

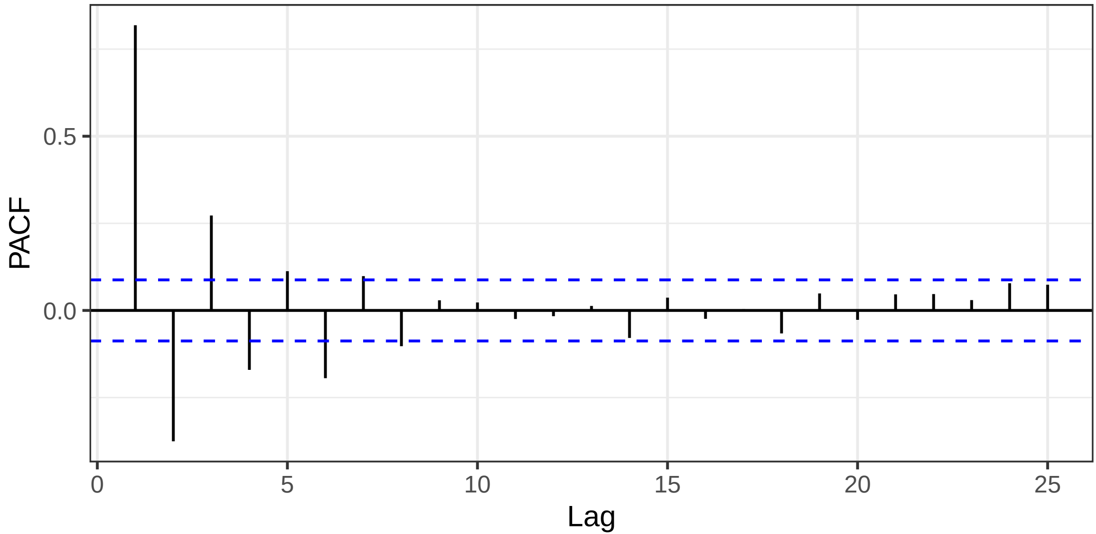
Example Plot of an ARMA(1,1): AR=.6, MA=.8 [1]



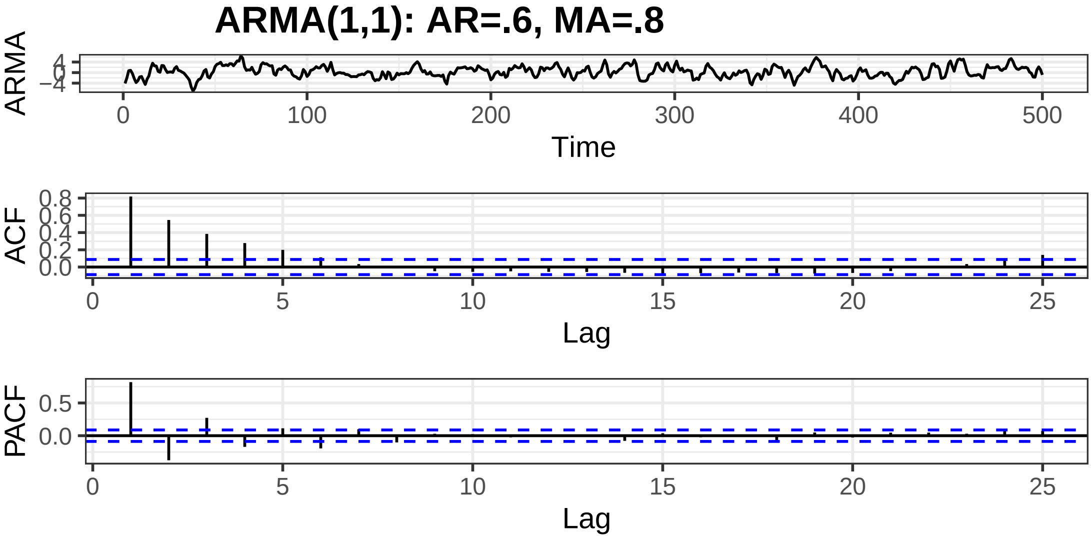
Example Plot of an ARMA(1,1): AR=.6, MA=.8 [2]



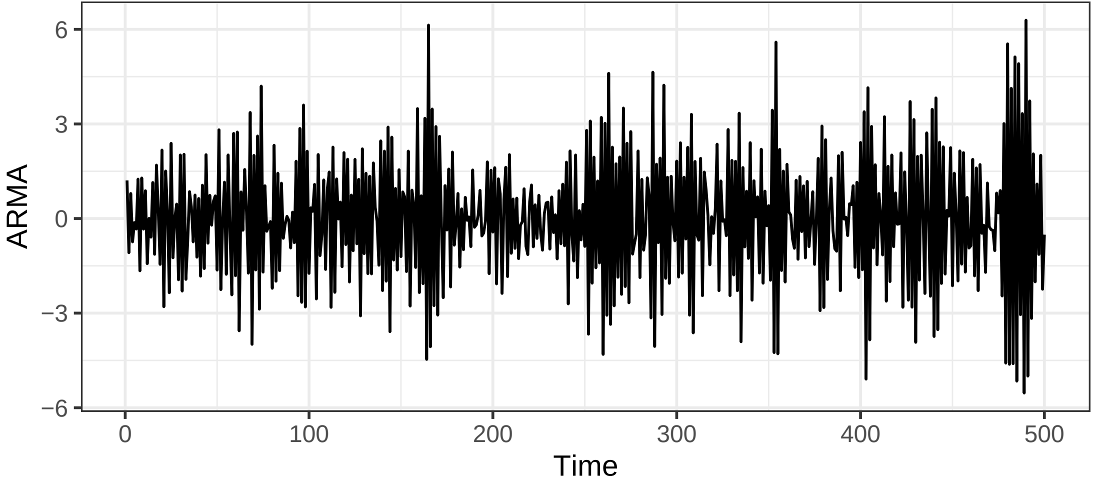
Example Plot of an ARMA(1,1): AR=.6, MA=.8 [3]



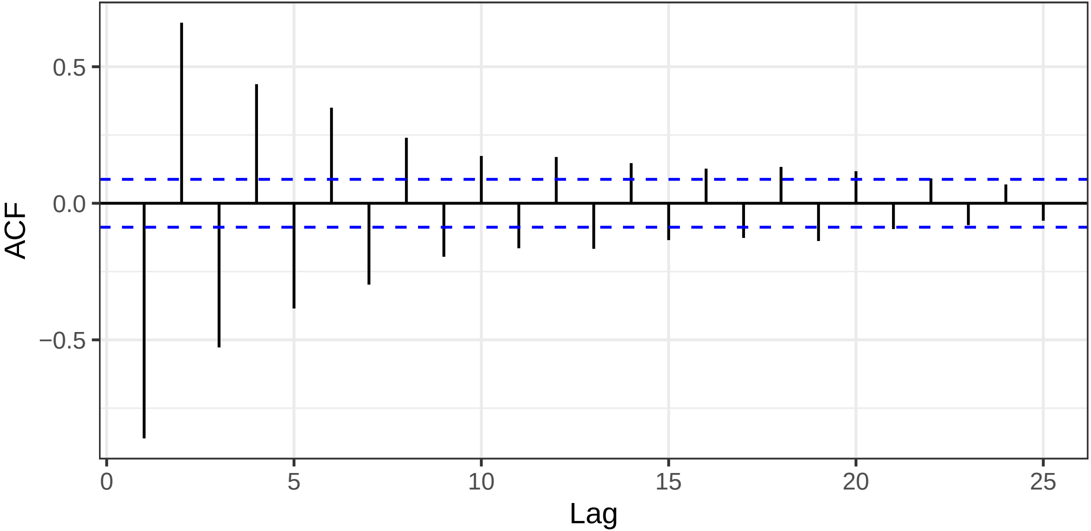
Example Plot of an ARMA(1,1): AR=.6, MA=.8 [4]



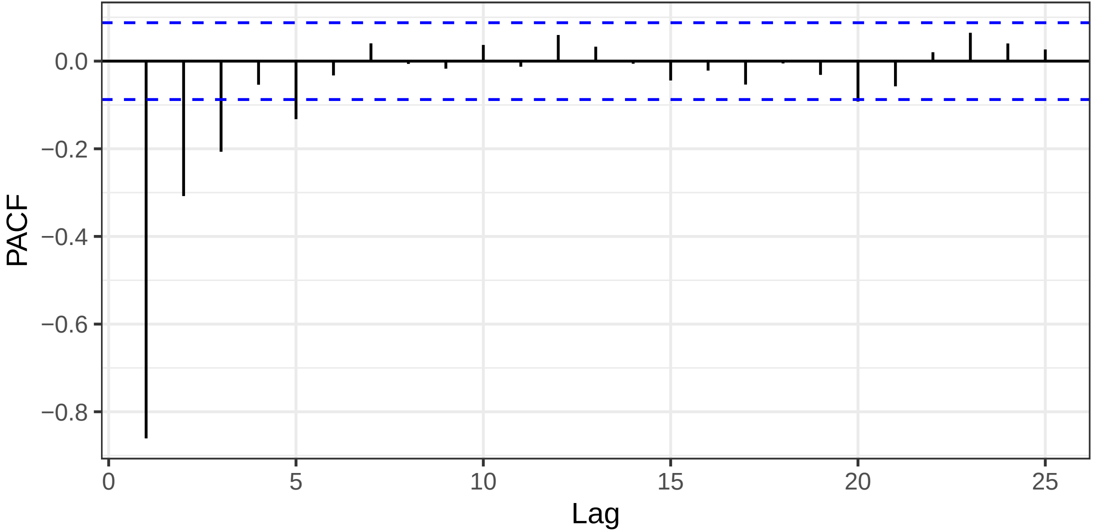
Example Plots of an ARMA(1,1): AR=-.7, MA=-.6 [1]



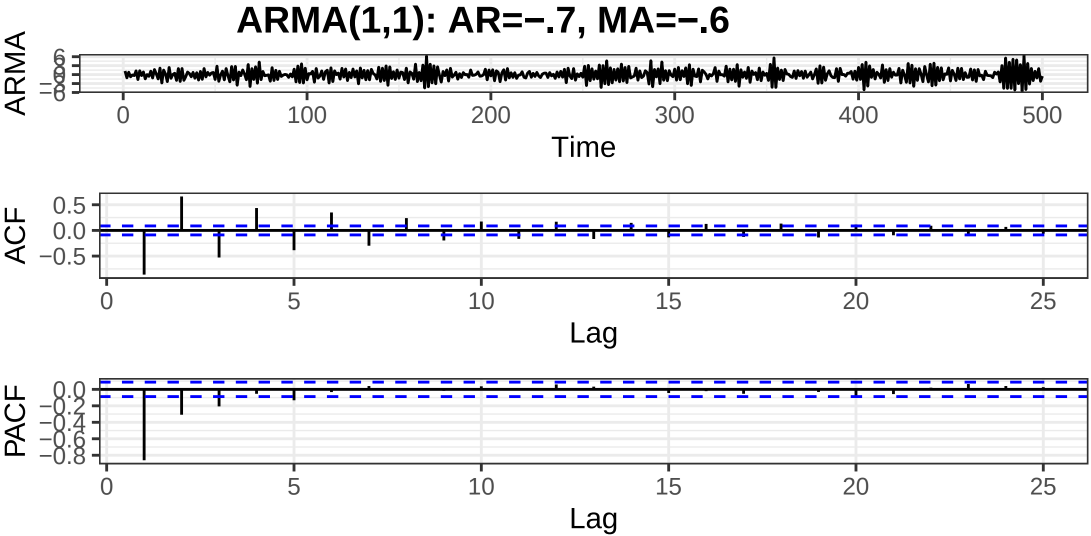
Example Plots of an ARMA(1,1): AR=-.7, MA=-.6 [2]



Example Plots of an ARMA(1,1): AR=-.7, MA=-.6 [3]



Example Plots of an ARMA(1,1): AR=-.7, MA=-.6 [4]



Outline

The 5-Step Procedure [1]

1. Plot the data over time.
2. Do the data seem stationary? If necessary, conduct a test for stationarity.
3. Once you can assume stationarity, find the ACF plot.
   a. If the ACF plot cuts off, fit an MA(q), where $q =$ the cutoff point.
   b. If the ACF plot dies down, find the PACF plot.
      i. If the PACF plot cuts off, fit an AR(p) model, where $p =$ the cutoff point.
      ii. If the PACF plot dies down, fit an ARMA (p, q) model.
      You must iterate through p and q using a guess and check method starting with ARMA(1,1) models – increment each by 1.
4. Evaluate the model residuals and consider the ACF and PACF of the residuals.
5. If model fit is good, forecast future values.

The 5-Step Procedure [2]

Note:

- Often you will fit multiple models in Step 3 and compare models in Step 4 to select the best fit.

Live Demo

Viscosity of a fluid is a measure that corresponds to "thickness." For example, honey has a higher viscosity than water. A chemical company needs precise forecasts of the viscosity of a product in order to control product quality. Using the 17 - viscosity.csv, we have 95 daily readings to use to develop a forecast.

In order to develop a forecast, let us first figure out what type of ARMA(p, q) model to fit and then develop the forecast.

Outline

Summary of Main Points

Main Learning Outcomes

- Describe the behavior of the ACF and PACF of an AR(p) process.
- Describe the behavior of the ACF and PACF of an MA(q) process.
- Describe the behavior of the ACF and PACF of an ARMA (p,q) process.
- Fit an ARMA model to a time series, evaluate the residuals of a fitted ARMA model to assess goodness of fit, use the Ljung-Box test for correlation among the residuals of an ARIMA model.

Summary Table

Model	**ACF**	**PACF**
AR(p)	Exponentially decays or damped sinusoidal pattern	Cuts off after lag p
MA(q)	Cuts off after lag q	Exponentially decays or damped sinusoidal pattern
ARMA(p,q)	Exponentially decays or damped sinusoidal pattern	Exponentially decays or damped sinusoidal pattern

Things to Do to Prepare for Next Class

- Thoroughly read Chapters 6.2.1 - 6.2.4 of our textbook.
- Go through the slides, examples and make sure you have a good understanding of what we have covered.
- Complete the graded assignment. Will be posted by 10 AM on March 25, 2021.

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