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Robust inference for nondestructive one-shot device testing under step-stress model with exponential lifetimes

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Abstract

One-shot devices analysis involves an extreme case of interval censoring, wherein one can only know whether the failure time is either before or after the test time. Some kind of one-shot devices do not get destroyed when tested, and so can continue within the experiment, providing extra information for inference, if they did not fail before an inspection time. In addition, their reliability can be rapidly estimated via accelerated life tests (ALTs) by running the tests at varying and higher stress levels than working conditions. In particular, step-stress tests allow the experimenter to increase the stress levels at prefixed times gradually during the life-testing experiment. The cumulative exposure model is commonly assumed for step-stress models, relating the lifetime distribution of units at one stress level to the lifetime distributions at preceding stress levels. In this paper, we develop robust estimators and Z-type test statistics based on the density power divergence (DPD) for testing linear null hypothesis for nondestructive one-shot devices under the step-stress ALTs with exponential lifetime distribution. We study asymptotic and robustness properties of the estimators and test statistics, yielding point estimation and confidence intervals for different lifetime characteristic such as reliability, distribution quantiles, and mean lifetime of the devices. A simulation study is carried out to assess the performance of the methods of inference developed here and some real-life data sets are analyzed finally for illustrative purpose.

KEYWORDS
multiple step-stress accelerated life test, nondestructive one-shot devices, robustness

1 | INTRODUCTION

One-shot device testing is an increasingly important problem in the area of reliability. This involves an extreme case of interval censoring, wherein one only knows if the device works when it is tested. Most of the existing literature considers

the case of "destructive" one-shot devices. This is the case when, once the device is used, it is either destroyed or must be rebuilt. Some typical examples are automobile air bags, fuel injectors, disposable napkins, missiles Olwell[34] and fire extinguishers and munition Newby.[33] Mainly motivated by the work of Fan et al.,[18] Balakrishnan and Ling[9–11] developed efficient EM algorithms for the estimation of model parameters under the assumption of exponential, Weibull and gamma lifetime distributions, respectively. One may refer to the recent book by Balakrishnan et al.[8] for a detailed review of all these works. Balakrishnan and Castilla[3] recently developed results for the lognormal lifetime distribution. Some other related works about one-shot devices can be found in Mun et al.,[31] Sharma and Upadhyay,[36] and Zhu et al.[42] However, the destructiveness assumption is not always necessary as in many experiments, the tested devices can be reused if have not failed during the test. We will refer to this type of devices as "nondestructive" one-shot devices. Their major advantage is that operating devices can continue in the experiment, providing extra information about their lifetime characteristics. Some typical examples are metal fatigue, thermal ageing of electrical insulation, spare wheels, safety valves, hot spare disks, electronics components, light bulbs, electric motors, and stability of pharmaceuticals.

A common practice in reliability is to employ accelerated life tests (ALTs) to shorten the lifetime of a product by increasing some stress factors associated with it, such as temperature, pressure, or humidity. This way, the experimental time and cost can be reduced. After suitable inference is developed, we can then extrapolate the results to normal operating conditions; see Meeter and Mecker[29] and Meeker et al.[30] There are different types of ALTs resulting in different statistical models. For example, constant-stress ALTs assume that each device is subject to only prespecified stress levels, while step-stress ALTs apply stress to devices in such a way they will get changed at prespecified times, and progressive-stress ALT continuously increases the stress level. The constant-stress and step-stress ALTs have been widely studied for destructive one-shot devices; see Ling,[26] Lee and Bae,[25] Wu et al.,[40] and Ling and Hu,[27] among others. We focus here on the step-stress model for nondestructive one-shot devices. In particular, we adopt a parametric approach in which the lifetimes of the devices are assumed to follow exponential distribution.

Nondestructive one-shot devices set-up arises in many applications where experimental units cannot be monitored continuously and so interval censored data are the only available information for inference. Interval censored inferential methods are widely applied in engineering and in clinical biological studies, where experimental units are human beings or animal subjects who are examined at discrete intervals (for example, quarterly check-ups), and events of interest are only known to occur at some time between examinations.[1] On the other hand, Nelson[32] pointed out that the step-stress accelerated life test (SSALT) design is especially useful in experimentation with electronics components, considering the nature of such products. In this vein, Guono[21] inferred the reliability of electronics components in storage under the nondestructive one-shot devices SSALT set-up, Han and Bai illustrated the method in solar lighting devices subjected to a multilevel step-stress ALT and Kohl and Kateri applied the model to evaluate the reliability of light-emitting diodes (LEDs). Two real nondestructive one-shot device datasets on electronic components and light bulbs are analyzed in Section 8 to illustrate the applicability of the method in industrial studies. Applications of the SSALT are mostly in the field of engineering but it could also be adapted to the medical context. For example, Lu and Storer[28] used a simple SSALT test plan, which has only two stress levels, for the evaluation of drug toxicity, where cumulative exposure to the drug may lead to toxicity at a faster rate.

While classical estimation methods are based on the maximum likelihood estimators (MLE), recent works have shown the advantage of using divergence-based methods in terms of robustness, with an unavoidable loss of efficiency in the case of uncontaminated data. Balakrishnan et al.[4–7,12] developed robust estimation methods based on density power divergence (DPD) for the constant-stress model and then constructed robust test statistics for testing linear hypothesis. In this paper, we develop robust estimators and test statistics for nondestructive one-shot devices under the multiple step-stress model and exponential lifetimes, and illustrate their robustness features both theoretically and empirically.

The rest of the paper is organized as follows: Section 2 describes the multiple SSALT under exponential lifetimes and introduces the classical MLE for the SSALT model, and Section 3 presents the minimum DPD and the minimum restricted DPD estimators under linear constraints, with the corresponding asymptotic results. In Section 4, the robustness of the proposed estimators is examined through their influence function analysis. Section 5 describes point estimation and confidence intervals for the reliability, distributional quantile, and mean lifetime of the device based on minimum DPD estimators. In Section 6, robust test statistics are developed, including Z-type and Rao-type tests, and their asymptotic properties are examined, providing approximate power functions of the tests. Sections 7 and 8 empirically illustrate the performance of the proposed methods thorough an extensive simulation study and real data analysis, respectively. Finally, Section 9 presents some concluding remarks.

2 | MODEL FORMULATION AND THE MAXIMUM LIKELIHOOD ESTIMATOR

Let us consider a SSALT with k ordered stress levels, $x_1 < x_2 < \cdots < x_k$, and N one-shot devices under test. At prefixed times τ_i, called times of stress change, we increase the stress level from x_i to x_{i+1}, for $i = 1, \ldots, k-1$, and we denote τ_k the time at which the experiment terminates. Let us also consider a sequence of length L of inspection times during the experiment, including the times of stress change τ_i, $i = 1, \ldots, k$,

$$0 < t_1 < \cdots < t_{l_1} = \tau_1 < t_{l_1+1} < \cdots < t_{l_k} = \tau_k,$$

where l_i denotes the number of inspection times before the ith stress change and $L = l_k$. Under this set-up, the simple step-stress model corresponds to the case when $k = 2$, $l_1 = 1$, and $l_2 = 2$. This model has been widely studied in the literature; for example, Nelson[32] discussed general cumulative exposure model, including the simple stress model, while Balakrishnan[2] reviewed exact inferential procedures for exponential step-stress models.

We further assume that the lifetime, T, of a device follows an exponential distribution, under stress level x_i, with failure rate λ_i depending on the stress. The distribution of the lifetime of a device during the test is then formed under applying the cumulative exposure model, which relates the lifetime distribution of a device at one stress level to the distributions at preceding stress levels by assuming the residual life of that device depends only on the cumulative exposure it had experienced, with no memory of how this exposure was accumulated. Then, if $G_i(\cdot)$ denotes the exponential lifetime distribution function at the ith stress level, the distribution function of T, $G_T(t)$, is given by

$$G_T(t) = \begin{cases} G_1(t) = 1 - e^{-\lambda_1 t}, & 0 < t < \tau_1 \\ G_2(t + a_1 - \tau_1) = 1 - e^{-\lambda_2(t+a_1-\tau_1)}, & \tau_1 \le t < \tau_2 \\ \vdots & \vdots \\ G_k(t + a_{k-1} - \tau_{k-1}) = 1 - e^{-\lambda_k(t+a_{k-1}-\tau_{k-1})}, & \tau_{k-1} \le t < \infty, \end{cases} \quad (1)$$

with

$$a_{i-1} = \frac{\sum_{l=1}^{i-1}(\tau_l - \tau_{l-1})\lambda_l}{\lambda_i}, \quad (2)$$

for $i = 1, \ldots, k-1$. For notational convenience, we set $a_{-1} = \tau_{-1} = 0$. The corresponding density function of T is given by

$$g_T(t) = \begin{cases} g_1(t) = \lambda_1 e^{-\lambda_1 t}, & 0 < t < \tau_1 \\ g_2(t + a_1 - \tau_1) = \lambda_2 e^{-\lambda_2(t+a_1-\tau_1)}, & \tau_1 \le t < \tau_2 \\ \vdots & \vdots \\ g_k(t + a_{k-1} - \tau_{k-1}) = \lambda_k e^{-\lambda_k(t+a_{k-1},-\tau_{k-1})} & \tau_{k-1} \le t < \infty. \end{cases} \quad (3)$$

Although the distribution function is continuous in $(0, \infty)$, the density function has k points of discontinuity at times of stress change. We further assume that at stress level x_i, the rate parameter λ_i of a device has a log-linear relationship with stress level given by

$$\lambda_i(\boldsymbol{\theta}) = \theta_0 \exp(\theta_1 x_i), \quad i = 1, .., k, \quad (4)$$

where $\boldsymbol{\theta} = (\theta_0, \theta_1) \in \mathbb{R}^+ \times \mathbb{R} = \Theta$ is an unknown parameter vector of the model. Note that the mean lifetime of a device is inverse of the exponential parameter, and so it would decrease with an increase in the level. The log-linear relation in Equation (4) is frequently assumed in ALT models, as it can be shown to be equivalent to the well-known inverse power law model or the Arrhenius reaction rate model.

Suppose n_j failures of test devices are observed in the interval $(t_{j-1}, t_j]$, $j = 1, .., k$, and for notational ease, let us denote n_{L+1} for the number of surviving devices at the end of the experiment. Then, the probability of failure of a device in the jth interval is

$$\pi_j(\boldsymbol{\theta}) = G_T(t_j) - G_T(t_{j-1}), \quad j = 1, .., L, \quad (5)$$

and the probability of survival at the end of the experiment is $\pi_{L+1}(\theta) = 1 - G_T(t_L)$. Accordingly, a multinomial model with probability vector $\pi(\theta) = (\pi_1(\theta)), \ldots, \pi_{L+1}(\theta))^T$ and N trials can be used to present the likelihood function of the model as

$$\mathcal{L}(\theta; n_1, \ldots, n_{L+1}) = \frac{N!}{n_1! \cdots n_{L+1}!} \prod_{j=1}^{L+1} \pi_j(\theta)^{n_j}.$$

From the above likelihood function, the MLE of θ would simply be

$$\widehat{\theta}^{MLE} = \left(\widehat{\theta}_0^{MLE}, \widehat{\theta}_1^{MLE}\right) = \arg\max_{\theta \in \Theta} \mathcal{L}(\theta; n_1, \ldots, n_{L+1}).$$

Remark 1. We could have alternatively derived the likelihood function of the model using binomial distribution for each interval, by using conditional probabilities of failure, given that the device did not fail in earlier time intervals. However, both approaches yield the same likelihood function.

Now, let $\widehat{p} = (n_1/N, \ldots, n_{L+1}/N)$ be the empirical probability vector obtained from the observed data. Then, the Kullback–Leibler divergence between the empirical and theoretical probability vectors, $\widehat{p}$ and $\pi(\theta)$, is given by

$$d_{KL}(\widehat{p}, \pi(\theta)) = \sum_{j=1}^{L+1} \widehat{p}_j \log\left(\frac{\widehat{p}_j}{\pi_j(\theta)}\right).$$

It is straightforward to see that the Kullback–Leibler divergence is related to the log likelihood function in the form

$$d_{KL}(\widehat{p}, \pi(\theta)) = c - \frac{1}{N} \log \mathcal{L}(\theta; n_1, \ldots, n_{L+1}) \tag{6}$$

where the constant $c = \sum_{i=1}^{L+1} \widehat{p}_i \log(\widehat{p}_i)$ does not depend on θ. Hence, the MLE can equivalently be defined as

$$\widehat{\theta}^{MLE} = \arg\min_{\theta \in \Theta} d_{KL}(\widehat{p}, \pi(\theta)). \tag{7}$$

From an asymptotic point of view, it is well-known that the MLE is a Best Asymptotically Normal (BAN) estimator, and it has, therefore, been widely used for the SSALT model. However, despite its high efficiency, the MLE lacks robustness as contaminated data could influence the parameter estimation considerably. In the next section, we present a robust family of estimators for the SSALT model based on the DPD.

3 | MINIMUM DENSITY POWER DIVERGENCE ESTIMATOR

The DPD family, introduced by Basu et al.,[14] a rich class of density-based divergences, produces robust estimators with relative small loss in efficiency. Given two density or mass functions, f_θ and g, the DPD between them is defined as

$$d_\beta(g, f_\theta) = \int \left\{ f_\theta^{1+\beta}(y) - \frac{\beta+1}{\beta} f_\theta^\beta(y) g(y) + \frac{1}{\beta} g^{1+\beta}(y) \right\} dy \quad \text{for } \beta > 0.$$

The parameter β, indexing the DPD divergence, controls the trade-off between efficiency and robustness. In fact, the DPD can be defined at $\beta = 0$ by taking continuous limits leading to the Kullback–Leibler divergence.

Following the discussion in the last section, we consider the DPD between the empirical and theoretical probability vectors, $\widehat{p}$ and $\pi(\theta)$,

$$d_\beta(\widehat{p}, \pi(\theta)) = \sum_{j=1}^{L+1} \left(\pi_j(\theta)^{1+\beta} - \left(1 + \frac{1}{\beta}\right) \widehat{p}_j \pi_j(\theta)^\beta + \frac{1}{\beta} \widehat{p}_j^{\beta+1} \right), \tag{8}$$

and correspondingly define the minimum DPD estimator (MDPPE) as

$$\widehat{\theta}^\beta = \left(\widehat{\theta}_0^\beta, \widehat{\theta}_1^\beta\right) = \arg\min_{\theta\in\Theta} d_\beta(\widehat{\boldsymbol{p}}, \boldsymbol{\pi}(\theta)). \tag{9}$$

Note that the value $\beta = 0$ corresponds to the MLE of θ. Hence, the proposed family could be considered as a generalization of the MLE with a tuning parameter β accounting for the compromise between efficiency and robustness. Moreover, the last term of each addend in Equation (8) does not depend on the model parameter, and so it can be ignored in the minimization process.

The next result presents the estimating equations for the MDPDE.

Result 2. *The estimating equations associated with the MDPDE for the SSALT model, under exponential lifetimes, satisfying the log-linear relation in Equation (4), are given by*

$$\boldsymbol{W}^T \boldsymbol{D}_{\pi(\theta)}^{\beta-1}(\widehat{\boldsymbol{p}} - \boldsymbol{\pi}(\theta)) = \boldsymbol{0}_2,$$

where $\boldsymbol{0}_2$ is the two-dimensional null vector, $\boldsymbol{D}_{\pi(\theta)}$ denotes a $(L+1)\times(L+1)$ diagonal matrix with diagonal entries $\pi_j(\theta)$, $j = 1, \ldots, L+1$, and $\boldsymbol{W}$ is a $(L+1)\times 2$ matrix with rows $\boldsymbol{w}_j = \boldsymbol{z}_j - \boldsymbol{z}_{j-1}$, where

$$\boldsymbol{z}_j = g_T(t_j)\begin{pmatrix} \dfrac{t_j + a_{i-1} - \tau_{i-1}}{\theta_0} \\ (t_j + a_{i-1} - \tau_{i-1})x_i + a_{i-1}^* \end{pmatrix}, \quad j = 1, \ldots, L, \tag{10}$$

$$a_{i-1}^* = \frac{1}{\lambda_i}\sum_{l=1}^{i-1}\lambda_l(\tau_l - \tau_{l-1})(-x_i + x_l), \quad i = 2, \ldots, k, \tag{11}$$

$\boldsymbol{z}_{-1} = \boldsymbol{z}_{L+1} = \boldsymbol{0}$ and i is the stress level at which the units are tested after the jth inspection time.

For the MLE, the estimating equations are obtained by deriving the Kullback–Leibler divergence given in Equation (6), yielding

$$\boldsymbol{W}^T \boldsymbol{D}_{\pi(\theta)}^{-1}(\widehat{\boldsymbol{p}} - \boldsymbol{\pi}(\theta)) = \boldsymbol{0}_2.$$

Next we present the asymptotic distribution of the proposed estimator, for any positive value of β.

Result 3. *Let θ_0 be the true value of the parameter θ. Then, the asymptotic distribution of the MDPDE, $\widehat{\theta}^\beta$, for the SSALT model, under exponential lifetime, is given by*

$$\sqrt{N}\left(\widehat{\theta}^\beta - \theta_0\right) \to \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{J}_\beta^{-1}(\theta_0)\boldsymbol{K}_\beta(\theta_0)\boldsymbol{J}_\beta^{-1}(\theta_0)\right),$$

where

$$\boldsymbol{J}_\beta(\theta_0) = \boldsymbol{W}^T \boldsymbol{D}_{\pi(\theta_0)}^{\beta-1}\boldsymbol{W}, \qquad \boldsymbol{K}_\beta(\theta_0) = \boldsymbol{W}^T\left(\boldsymbol{D}_{\pi(\theta_0)}^{2\beta-1} - \boldsymbol{\pi}(\theta_0)^\beta\boldsymbol{\pi}(\theta_0)^{\beta T}\right)\boldsymbol{W}, \tag{12}$$

$\boldsymbol{D}_{\pi(\theta_0)}$ denotes the diagonal matrix with entries $\pi_j(\theta_0)$, $j = 1, \ldots, L+1$, and $\boldsymbol{\pi}(\theta_0)^\beta$ denotes the vector with components $\pi_j(\theta_0)^\beta$.

For $\beta = 0$, the Fisher information matrix associated with the SSALT model under exponential lifetimes coincides with the matrices $\boldsymbol{J}_\beta(\theta_0)$ and $\boldsymbol{K}_\beta(\theta_0)$, and so we obtain the asymptotic distribution of the MLE as a particular case, that is,

$$\sqrt{N}\left(\widehat{\theta}^0 - \theta_0\right) \to \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{I}_F^{-1}(\theta_0)\right),$$

where $\boldsymbol{I}_F(\theta_0) = \boldsymbol{W}^T \boldsymbol{D}_{\pi(\theta_0)}^{-1}\boldsymbol{W}$.

Remark 4. As $\widehat{\theta}^{\beta}$ is a consistent estimator of θ_0, the asymptotic variances of $\widehat{\theta}_0^{\beta}$ and $\widehat{\theta}_1^{\beta}$ for $\beta > 0$, denoted by $\sigma^2(\theta_0^{\beta})$ and $\sigma^2(\theta_1^{\beta})$, respectively, can be estimated by the diagonal entries of $\boldsymbol{J}_{\beta}^{-1}(\widehat{\theta}^{\beta})\boldsymbol{K}_{\beta}(\widehat{\theta}^{\beta})\boldsymbol{J}_{\beta}^{-1}(\widehat{\theta}^{\beta})$. Therefore, asymptotic confidence intervals for θ_0 and θ_1, at confidence level $(1 - \alpha)$, are given by

$$\widehat{\theta}_i^{\beta} \pm \frac{\widehat{\sigma}(\theta_i^{\beta})}{\sqrt{N}} z_{\alpha/2}, \quad i = 0, 1, \tag{13}$$

with $z_{\alpha/2}$ being the lower $\alpha/2$-quantile of a standard normal distribution. Moreover, we have

$$N(\widehat{\theta}^{\beta} - \theta)^T \left(\boldsymbol{J}_{\beta}^{-1}(\theta_0)\boldsymbol{K}_{\beta}(\theta_0)\boldsymbol{J}_{\beta}^{-1}(\theta_0)\right)^{-1}(\widehat{\theta}^{\beta} - \theta) \to \chi_2^2$$

and consequently an associated ellipsoidal confidence region for $\theta = (\theta_0, \theta_1)$ is given by

$$C_{N,\beta}^{\alpha} = \left\{\theta | N(\widehat{\theta}^{\beta} - \theta)^T \left(\boldsymbol{J}_{\beta}^{-1}(\theta_0)\boldsymbol{K}_{\beta}(\theta_0)\boldsymbol{J}_{\beta}^{-1}(\theta_0)\right)^{-1}(\widehat{\theta}^{\beta} - \theta) \leq c\right\}.$$

Choosing $c = \chi_{2,\alpha}^2$, the $100(1 - \alpha)$-percentile of the chi-square distribution with two degrees of freedom, we have $\mathbb{P}_{\theta}(C_{N,\beta}^{\alpha})$ tending to $1 - \alpha$ as $n \to \infty$. Hence, $C_{n,\beta}^{\alpha}$ represents an ellipsoidal confidence region for θ having limiting confidence coefficient $1 - \alpha$ as $n \to \infty$ (see Serfling[35] for more details). The volume of the ellipsoidal region $C_{N,\beta}^{\alpha}$, based on Cramer,[16] is given by

$$\chi_{2,\alpha}^2 \pi \det\left(\left(\boldsymbol{J}_{\beta}^{-1}(\widehat{\theta}^{\beta})\boldsymbol{K}_{\beta}(\widehat{\theta}^{\beta})\boldsymbol{J}_{\beta}^{-1}(\widehat{\theta}^{\beta})\right)^{-1}\right)^{1/2}.$$

Thus, a measure of the asymptotic relative efficiency of $\widehat{\theta}^{\beta}$, for $\beta > 0$, with respect to the MLE, $\widehat{\theta}^0$, is given by

$$\left(\frac{\det\left(\boldsymbol{I}_F(\widehat{\theta})\right)}{|\boldsymbol{J}_{\beta}^{-1}(\widehat{\theta}^{\beta})\boldsymbol{K}_{\beta}(\widehat{\theta}^{\beta})\boldsymbol{J}_{\beta}^{-1}(\widehat{\theta}^{\beta})|}\right)^{1/2}.$$

4 | INFLUENCE FUNCTION OF THE MDPDE

The influence function (IF), first introduced by Hampel et al.,[22] plays a central role in the study of robustness properties of an estimator. Intuitively, it quantifies the impact of an infinitesimal perturbation in the true distribution underlying the data on the asymptotic value of the resulting parameter estimate. An estimator is said to be robust if its influence function is bounded.

Mathematically, the IF of an estimator is computed in terms of its corresponding statistical functional. Let F_{θ} and $\boldsymbol{G}$ be the assumed distribution of the model and the true density underlying the data, respectively. We use $\boldsymbol{T}(\boldsymbol{G})$ to denote the statistical functional associated with the estimator $\widehat{\theta}$. Then, the IF of the estimator $\widehat{\theta}$ at a point $\boldsymbol{t}$ is computed as

$$\text{IF}(\boldsymbol{t}, \boldsymbol{T}, \boldsymbol{G}) = \lim_{\varepsilon \to 0} \frac{\boldsymbol{T}(\boldsymbol{G}_{\varepsilon}) - \boldsymbol{T}(\boldsymbol{G})}{\varepsilon} = \left.\frac{\partial \boldsymbol{T}(\boldsymbol{G}_{\varepsilon})}{\partial \varepsilon}\right|_{\varepsilon=0}, \tag{14}$$

where $\boldsymbol{G}_{\varepsilon} = (1 - \varepsilon)\boldsymbol{G} + \varepsilon\Delta_{\boldsymbol{t}}$ is the contaminated version of $\boldsymbol{G}$, with ε being the contamination proportion, and $\Delta_{\boldsymbol{t}}$ being the degenerate distribution at the contamination point $\boldsymbol{t}$. An estimator is said to be robust if its IF at the model distribution F_{θ} is bounded. For the SSALT model, we could consider only one cell contamination, and so the contamination point $\boldsymbol{t}$ would have all elements equal to zero except for only one component.

Let us denote F_{θ} for the assumed distribution of the multinomial model with mass function $\pi(\theta)$ given by the SSALT model with exponential lifetimes and $\boldsymbol{G}$ denote the true distribution underlying the data, with mass function $\boldsymbol{g}$. We define the statistical functional $\boldsymbol{T}_{\beta}(\boldsymbol{G})$ as the minimizer of the DPD between the two mass functions, π_{θ} and $\boldsymbol{g}$, given in Equation (3). Then, an expression of the IF can be computed from Equation (14) as stated in the following result.

Result 5. *The IF of the MDPDE of the SSALT model, $\widehat{\theta}^{\beta}$, at a point contamination $\boldsymbol{n}$ and the assumed model distribution F_{θ_0} is given by*

$$IF(\boldsymbol{n}, \boldsymbol{T}_{\beta}, F_{\theta_0}) = \boldsymbol{J}_{\beta}^{-1}(\theta_0)\boldsymbol{W}^T \boldsymbol{D}_{\pi(\theta_0)}^{\beta-1}(-\pi(\theta_0) + \Delta_{\boldsymbol{n}}). \tag{15}$$

Remark 6. The matrix $\boldsymbol{J}_{\beta}(\theta_0)$ is assumed to be bounded, and so the robustness of the estimators depends on the boundedness of the second factor of the IF, given by

$$\boldsymbol{W}^T \boldsymbol{D}_{\pi(\theta_0)}^{\beta-1}(-\pi(\theta_0) + \Delta_{\boldsymbol{n}}) = \sum_{j=1}^{L+1}(\boldsymbol{z}_j - \boldsymbol{z}_{j-1})\pi_j(\theta_0)^{\beta-1}(-\pi_j(\theta_0) + \Delta_{\boldsymbol{n}j}), \tag{16}$$

where $\boldsymbol{z}_j$ is as defined in Equation (10). All the terms in Equation (16) are bounded for fixed stress levels and inspection times at any contamination point $\boldsymbol{n}$, as the maximum value of its components is the number of trials N. Then, any of the proposed MDPDE for $\beta \geq 0$ is robust against vertical outliers, including the MLE. Conversely, the IF boundedness is affected by high-leverage points, that is, outliers on the independent variables involved in the model. In particular, we will discuss two classes of leverage points, namely, extreme values of inspection times and extreme stress levels.

Let us first consider the situation wherein an inspection time, t_j, tends to infinity, for fixed j. We denote i the fixed stress level corresponding to the jth inspection time. As the inspection times are ordered, there will be no more terms in the summation after the jth term. Then, we can write

$$(\boldsymbol{z}_j - \boldsymbol{z}_{j-1})\pi_j(\theta_0)^{\beta-1} = \left(g_i(T_j)\begin{pmatrix} \frac{T_j}{\theta_0} \\ T_j x_i + a_{i-1}^* \end{pmatrix} - g_i(T_{j-1})\begin{pmatrix} \frac{T_{j-1}}{\theta_0} \\ T_{j-1} x_i + a_{i-1}^* \end{pmatrix}\right)(G_i(T_j) - G_i(T_{j-1}))^{\beta-1}$$

$$= \left[\lambda_i(\theta_0)\exp(-\lambda_i(\theta_0)T_j)\begin{pmatrix} \frac{T_j}{\theta_0} \\ T_j x_i + a_{i-1}^* \end{pmatrix} - \lambda_i(\theta_0)\exp(-\lambda_i(\theta_0)T_{j-1})\begin{pmatrix} \frac{T_{j-1}}{\theta_0} \\ T_{j-1} x_i + a_{i-1}^* \end{pmatrix}\right]$$

$$\times \left(-\exp(-\lambda_i(\theta_0)T_j) + \exp(-\lambda_i(\theta_0)T_{j-1})\right)^{\beta-1}$$

with $T_j = t_j + a_{i-1} - \tau_{i-1}$. All the terms depending on times before t_j are bounded, and the values $\lambda_i(\theta_0), a_{i-1}, a_{i-1}^*$ and τ_{i-1} are positive constants. Then, taking limits as $T_j \to \infty$, we get

$$\lim_{t_j \to \infty} (\boldsymbol{z}_j - \boldsymbol{z}_{j-1})\pi_j(\theta_0)^{\beta-1} = \begin{cases} +\infty & \text{if } \beta = 0, \\ < \infty & \text{if } \beta > 0. \end{cases}$$

Hence, the IF of the MDPDEs, for positive values of β, is bounded when any inspection time gets increased, whereas the IF of the MLE is unbounded for this class of leverage points.

Similarly, let us consider a stress level x_i and let $x_i \to \infty$. We take t_j such that $t_j = \tau_i$, the time of stress change for the ith stress level. Again, as the stress levels are ordered, we can consider that the devices at subsequent steps are subjected to the same stress x_i. Then, we need to establish the boundedness of all terms from j onwards. The lifetime rates are not constant since they depend on the stress level. Therefore, taking limits on Equation (4), we have

$$\lim_{x_i \to \infty} \lambda_i(\theta) = \begin{cases} 0 & \text{if } \theta_1 \leq 0, \\ \infty & \text{if } \theta_1 > 0. \end{cases}$$

The limiting behavior of the IF of θ_0 and θ_1 may be different, since the first only depends on the stress level at $g_T(T_j)$, whereas the IF of θ_1 includes a term in $g_T(T_j)T_j x_i$. Therefore, for $\theta_1 > 0$,

$$\lim_{x_i \to \infty}\left(g_i(T_j)\begin{pmatrix} \frac{T_j}{\theta_0} \\ T_j x_i + a_{i-1}^* \end{pmatrix} - g_i(T_{j-1})\begin{pmatrix} \frac{T_{j-1}}{\theta_0} \\ T_{j-1} x_i + a_{i-1}^* \end{pmatrix}\right)(G_i(T_j) - G_i(T_{j-1}))^{\beta-1}$$

$$= \lim_{x_i \to \infty}\left(\lambda_i(\theta_0)\exp(-\lambda_i(\theta_0)T_j)\begin{pmatrix} \frac{T_j}{\theta_0} \\ T_j x_i + a_{i-1}^* \end{pmatrix} - \lambda_i(\theta_0)\exp(-\lambda_i(\theta_0)T_{j-1})\begin{pmatrix} \frac{T_{j-1}}{\theta_0} \\ T_{j-1} x_i + a_{i-1}^* \end{pmatrix}\right)$$

$$\times \left(-\exp(-\lambda_i(\theta_0)T_j) + \exp(-\lambda_i(\theta_0)T_{j-1})\right)^{\beta-1}$$

$$= \begin{cases} -\infty & \text{if } \beta = 0, \\ < \infty & \text{if } \beta > 0. \end{cases}$$

For $\theta_1 < 0$, we must deal with the IF of θ_0 and the IF of θ_1 separately. For the IF of the first parameter θ_0, we have

$$\lim_{x_i \to \infty} \left(g_i(T_j)\frac{T_j}{\theta_0} - g_i(T_{j-1})\frac{T_{j-1}}{\theta_0}\right)(G_i(T_j) - G_i(T_{j-1}))^{\beta-1} < \infty \quad \forall \beta \geq 0,$$

whereas taking limits in the IF of the second parameter θ_1, we obtain

$$\lim_{x_i \to \infty} \left(g_i(T_j)(T_j x_i + a_{i-1}^*) - g_i(T_{j-1})(T_{j-1}x_i + a_{i-1}^*)\right)(G_i(T_j) - G_i(T_{j-1}))^{\beta-1} = \begin{cases} +\infty & \text{if } \beta = 0, \\ < \infty & \text{if } \beta > 0. \end{cases}$$

Thus, the IF of the proposed MDPDE is bounded for all $\beta > 0$ regardless of the sign of the true parameter value θ_1. In contrast, the IF of the MLE of the parameter θ_0 is unbounded for positive true parameter values of θ_1 and so is the IF of the MLE of the parameter θ_1. This means that the proposed estimators are also robust for all type of outliers, whereas the MLE lacks robustness against this "bad" leverage points.

5 | POINT ESTIMATION AND CONFIDENCE INTERVALS OF RELIABILITY AND MEAN LIFETIME

One may be interested in studying the reliability of nondestructive one-shot devices or in estimating its expected lifetime. Technically, the reliability of a device is the probability that it will perform its intended function, under operating condition, for a specified period of time. Then, the reliability can be measured as the probability of survival until a prespecified time under normal operating conditions. Following the notation in Section 2, the reliability or survival function of the lifetime T of a device is given by

$$R_T(t) = 1 - G_T(t) = \begin{cases} R_1(t) = e^{-\lambda_1 t}, & 0 < t < \tau_1, \\ R_2(t + a_1 - \tau_1) = e^{-\lambda_2(t+a_1-\tau_1)}, & \tau_1 \leq t < \tau_2, \\ \vdots \\ R_K(t + a_{k-1} - \tau_{k-1}) = e^{-\lambda_k(t+a_{k-1}-\tau_{k-1})}, & \tau_{k-1} \leq t < \infty, \end{cases}$$

where a_{i-1} is as defined in Equation (2) and $R_i(t), i = 1, \dots k$, is the reliability function at the ith stress level, which depends in turn on the model parameters $\theta = (\theta_0, \theta_1)$. Therefore, an estimated reliability at a certain time can be obtained from the above formula. For cumulative exposure model with exponential lifetime distributions, the reliability of the device at the inspection interval $[\tau_i, \tau_{i+1}]$ (assuming that the stress level will not be increased) corresponds to the reliability function of a translated exponential distribution with parameter $\lambda_i, i = 1, \dots, k$. If the device is subjected to a constant stress level x_i, then its reliability at time t can be computed as the reliability function $R_i(t)$. Let us denote x_0 for the stress level at normal operating conditions. Then, the reliability of the device is given by, for a fixed time t,

$$R_0(t) = \exp(-\lambda_0 t) = \exp(-\theta_0 \exp(\theta_1 x_0)t). \tag{17}$$

Also, for planning purposes, one may need to estimate the time at which more than a certain percentage of devices are expected to fail under normal operating conditions. Mathematically, those times are the distribution quantiles, computed as the inverse distribution (or reliability) function,

$$Q_{1-\alpha} = R_0^{-1}(1 - \alpha) = G_0^{-1}(\alpha) = -\frac{\log(1 - \alpha)}{\lambda_0}, \tag{18}$$

with $1 - \alpha$ being the proportion of surviving units. Further, the mean lifetime of a device under an exponential lifetime distribution with parameter λ is $\mathbb{E}[T] = 1/\lambda$. From Equation (4), the mean lifetime of the device depends on the stress

level through a log-linear relationship as

$$\frac{1}{\lambda_i} = \frac{1}{\theta_0} \exp(-\theta_1 x_i), i = 1, \dots, k.$$

Hence, under normal operating conditions, the expected lifetime of the device is simply

$$\mathrm{E}_T = \mathbb{E}[T] = \frac{1}{\lambda_0} = \frac{1}{\theta_0} \exp(-\theta_1 x_0). \tag{19}$$

It is straightforward to obtain point estimation of the reliability, distribution quantiles, and mean lifetime of devices under normal operating conditions by plugging in the MDPDE, $\widehat{\theta}^\beta$, in Equations (17)–(19). We denote the estimators of the reliability at a mission time t, $1 - \alpha$ upper quantile and mean lifetime by $\widehat{R}_0^\beta(t)$, $\widehat{Q}_{1-\alpha}^\beta$, and $\widehat{E}_T^\beta$, respectively. Additional interest may be on confidence intervals (CI) for such quantities. We first present the asymptotic distribution of the reliability, quantiles, and mean lifetime estimators based on the MDPDEs, $\widehat{\theta}^\beta$, under normal operating conditions. These results can be obtained readily from the asymptotic distribution of the MDPDE by employing the Delta method.

Result 7. *Let θ_0 be the true value of the parameter θ. Let $\widehat{\theta}^\beta$ be the MDPDE, with tuning parameter β. Then, the asymptotic distribution of the estimated reliability at a mission time t, under normal operating conditions, based on the MDPDE $\widehat{\theta}^\beta$, $\widehat{R}_0^\beta(t)$, is given by*

$$\sqrt{N}\big(\widehat{R}_0^\beta(t) - R_0^\beta(t)\big) \xrightarrow[N\to\infty]{L} \mathcal{N}\left(\mathbf{0}, \sigma\big(R_0^\beta(t)\big)^2\right),$$

with

$$\sigma\big(R_0^\beta(t)\big)^2 = \nabla h(\theta_0)^T \mathbf{J}_\beta^{-1}(\theta_0) \mathbf{K}_\beta(\theta_0) \mathbf{J}_\beta^{-1}(\theta_0) \nabla h(\theta_0),$$

where the matrices $\mathbf{J}_\beta(\theta_0)$ and $\mathbf{K}_\beta(\theta_0)$ are as defined in Equation (12) and $\nabla h(\theta)^T = (-R_0(t)\frac{\lambda_0 t}{\theta_0}, -R_0(t)\lambda_0 t x_0)$ is the gradient of the function $h(\theta) = \exp(-\theta_0 \exp(\theta_1 x_0)t)$.

Result 8. *Under the same assumptions as in Result 7, the asymptotic distribution of the estimated $(1 - \alpha)$-quantile, under normal operating conditions, based on the MDPDE $\widehat{\theta}^\beta$, $\widehat{Q}_{1-\alpha}^\beta$, is given by*

$$\sqrt{N}\big(\widehat{Q}_{1-\alpha}^\beta - Q_{1-\alpha}\big) \xrightarrow[N\to\infty]{L} \mathcal{N}\left(\mathbf{0}, \sigma(Q_{1-\alpha})^2\right),$$

with

$$\sigma(Q_{1-\alpha})^2 = \nabla h_1(\theta_0)^T \mathbf{J}_\beta^{-1}(\theta_0) \mathbf{K}_\beta(\theta_0) \mathbf{J}_\beta^{-1}(\theta_0) \nabla h_1(\theta_0)$$

where the matrices $\mathbf{J}_\beta(\theta_0)$ and $\mathbf{K}_\beta(\theta_0)$ are as defined in Equation (12) and

$$\nabla h_1(\theta)^T = \left(\frac{\log(1-\alpha)}{\theta_0^2} \exp(-\theta_1 x_0), \frac{\log(1-\alpha)x_0}{\theta_0} \exp(-\theta_1 x_0)\right)$$

is the gradient of the function $h_1(\theta) = \frac{1}{\theta_0} \exp(-\theta_1 x_0)$.

Result 9. *Under the same assumptions as in Result 7, the asymptotic distribution of the estimated mean lifetime, under normal operating conditions, based on the MDPDE $\widehat{\theta}^\beta$, $\widehat{E}_T^\beta$, is given by*

$$\sqrt{N}\big(\widehat{E}_T^\beta - E_T\big) \xrightarrow[N\to\infty]{L} \mathcal{N}\left(\mathbf{0}, \sigma(E_T)^2\right),$$

with

$$\sigma(E_T)^2 = \nabla h_2(\boldsymbol{\theta}_0)^T \boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}_0) \boldsymbol{K}_\beta(\boldsymbol{\theta}_0) \boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}_0) \nabla h_2(\boldsymbol{\theta}_0)$$

where the matrices $\boldsymbol{J}_\beta(\boldsymbol{\theta}_0)$ and $\boldsymbol{K}_\beta(\boldsymbol{\theta}_0)$ are as defined in Equation (12) and $\nabla h_2(\boldsymbol{\theta})^T = (\frac{-1}{\theta_0^2} \exp(-\theta_1 x_0), \frac{-x_0}{\theta_0} \exp(-\theta_1 x_0))$ is the gradient of the function $h_2(\boldsymbol{\theta}) = \frac{1}{\theta_0} \exp(-\theta_1 x_0)$.

As $\widehat{\boldsymbol{\theta}}^\beta$ are consistent estimators, from the above results, we can easily obtain approximate two-sided $100(1-\alpha)\%$ CI for the reliability, $(1-\alpha)$-quantile and mean lifetime, under normal operating conditions, to be

$$\widehat{R}_0^\beta(t) \pm z_{\alpha/2} \frac{\sigma(\widehat{R}_0^\beta(t))}{\sqrt{N}}, \quad \widehat{Q}_{1-\alpha}^\beta \pm z_{\alpha/2} \frac{\sigma(\widehat{Q}_{1-\alpha}^\beta)}{\sqrt{N}} \quad \text{and} \quad \widehat{E}_T^\beta \pm z_{\alpha/2} \frac{\sigma(\widehat{E}_T^\beta)}{\sqrt{N}}$$

where $\sigma(\widehat{R}_0^\beta(t))$, $\sigma(\widehat{Q}_{1-\alpha}^\beta)$, and $\sigma(\widehat{E}_T^\beta)$ are as defined in Results 7–9, respectively.

The above asymptotic confidence intervals are based on the asymptotic properties of the estimators and so they may be satisfactory only for large sample sizes. In small samples, we may have to truncate the confidence intervals as the mean lifetime and quantiles must be positive and the reliability should be between 0 and 1. In this regard, Viveros and Balakrishnan[37] employed a logit transformation of the estimated reliability to obtain more accurate CIs based on the MLE. The transformed reliability is defined as

$$\phi = \phi(R_0(t)) = \text{logit}(R_0(t)) = \log\left(\frac{R_0(t)}{1-R_0(t)}\right), \tag{20}$$

where $\phi \in \mathbb{R}$. Thus, the range for this transformed reliability would not require truncation. The logit transformation is a natural choice when dealing with parameters that represent probabilities, since it results in $\mathbb{R}$. Estimated values of the transformed reliabilities based on the MDPDEs, $\widehat{\phi}^\beta$, can be easily obtained by substituting the corresponding estimated reliabilities $\widehat{R}_0^\beta(t)$ in Equation (20), and their asymptotic distribution and CIs can be derived by using Delta method. Inverting such a CI, after some algebra, we obtain the asymptotic CI for the reliability as

$$\left[\frac{\widehat{R}_0^\beta(t)}{\widehat{R}_0^\beta(t) + (1-\widehat{R}_0^\beta(t))S}, \frac{\widehat{R}_0^\beta(t)}{\widehat{R}_0^\beta(t) + (1-\widehat{R}_0^\beta(t))/S}\right],$$

with $S = \exp(\frac{z_{\alpha/2}}{\sqrt{N}} \frac{\sigma(\widehat{R}_0^\beta(t))}{\widehat{R}_0^\beta(t)(1-\widehat{R}_0^\beta(t))})$ and $\sigma(R_0(t))$ as defined in Result 7.

A similar idea can be applied for the quantiles and mean lifetimes in Equations (18) and (19). As both quantities must be positive, the logarithm is a natural choice for transforming them to $\mathbb{R}$. Transformed quantiles and mean lifetimes are then

$$\phi_1 = \log(Q_{1-\alpha}) \text{ and } \phi_2 = \log(E_T). \tag{21}$$

Again, using Delta method for deriving the asymptotic distributions, and then inverting the logarithmic transformations, we obtain CIs for $Q_{1-\alpha}$ and E_T as

$$\left[\widehat{Q}_{1-\alpha}^\beta \exp\left(-\frac{z_{\alpha/2}}{\sqrt{N}} \frac{\sigma(\widehat{Q}_{1-\alpha}^\beta)}{\widehat{Q}_{1-\alpha}^\beta}\right), \widehat{Q}_{1-\alpha}^\beta \exp\left(\frac{z_{\alpha/2}}{\sqrt{N}} \frac{\sigma(\widehat{Q}_{1-\alpha}^\beta)}{\widehat{Q}_{1-\alpha}^\beta}\right)\right]$$

and

$$\left[\widehat{E}_T^\beta \exp\left(-\frac{z_{\alpha/2}}{\sqrt{N}} \frac{\sigma(\widehat{E}_T^\beta)}{\widehat{E}_T^\beta}\right), \widehat{E}_T^\beta \exp\left(\frac{z_{\alpha/2}}{\sqrt{N}} \frac{\sigma(\widehat{E}_T^\beta)}{\widehat{E}_T^\beta}\right)\right],$$

respectively, with $\sigma(E_T)$ and $\sigma(Q_{1-\alpha})$ as defined in Results 8 and 9.

6 | ROBUST TESTS OF HYPOTHESES

In this section, we consider linear hypothesis tests on the model parameter θ, of the form

$$H_0 : \boldsymbol{m}^T \theta = d, \tag{22}$$

where $\boldsymbol{m} = (m_0, m_1)^T \in \mathbb{R}^2$. In particular, the linear hypothesis with $\boldsymbol{m}^T = (0, 1)$ and $d = 0$ would test if the stress level affects the lifetime of the one-shot devices or not. We present here the a testing procedure based on the MDPDE and then we study it robustness and asymptotic behavior.

Specifically, we define the Z-type statistics based on the MDPDE and then study theoretically its asymptotic distribution under the null and contiguous hypotheses and robustness properties

Definition 10. The Z-type statistic based on the MDPDE $\widehat{\theta}^\beta$, for testing null hypothesis (22), is given by

$$Z_N(\widehat{\theta}^\beta) = \sqrt{N} \left(\boldsymbol{m}^T \boldsymbol{J}_\beta^{-1}(\widehat{\theta}^\beta) \boldsymbol{K}_\beta(\widehat{\theta}^\beta) \boldsymbol{J}_\beta^{-1}(\widehat{\theta}^\beta) \boldsymbol{m} \right)^{-1/2} \left(\boldsymbol{m}^T \widehat{\theta}^\beta - d \right). \tag{23}$$

The asymptotic distribution of this statistic is given in the following result

Result 11. *The asymptotic distribution of the Z-type statistic (23), under the null hypothesis (22), is a standard normal distribution.*

Based on Result 11, for any $\beta \geq 0$ and $\boldsymbol{m} \in \mathbb{R}^2$, the critical region with significance level α for the hypothesis test with linear null hypothesis in Equation (22), with level α is given by

$$\mathcal{R}_\alpha = \{(n_1, \dots, n_{L+1}) \text{ s.t. } |Z_N(\widehat{\theta}^\beta)| > z_{\alpha/2}\}, \tag{24}$$

where $z_{\alpha/2}$ denotes the upper $\alpha/2$-quantile of the standard normal distribution.

Remark 12. We can generalize the null hypothesis in Equation (22) to

$$H_0 : \boldsymbol{M}^T \theta = \boldsymbol{d}$$

with $\boldsymbol{M}$ being a $r \times 2$ ($r \leq 2$) matrix and $\boldsymbol{d}$ being a r-dimensional vector. Then, we can define the corresponding test statistic as

$$Z_N^*(\widehat{\theta}^\beta) = N \left(\boldsymbol{M}^T \widehat{\theta}^\beta - \boldsymbol{d} \right)^T \left(\boldsymbol{M}^T \boldsymbol{J}_\beta^{-1}(\widehat{\theta}^\beta) \boldsymbol{K}_\beta(\widehat{\theta}^\beta) \boldsymbol{J}_\beta^{-1} \boldsymbol{M} \right)^{-1} \left(\boldsymbol{M}^T \widehat{\theta}^\beta - \boldsymbol{d} \right). \tag{25}$$

Note that the matrix $\boldsymbol{M}^T \boldsymbol{J}_\beta^{-1}(\widehat{\theta}^\beta) \boldsymbol{K}_\beta(\widehat{\theta}^\beta) \boldsymbol{J}_\beta^{-1}(\widehat{\theta}^\beta) \boldsymbol{M}$ is symmetric, and so the statistic is well-defined. It is not difficult to stablish that, under the generalized null hypothesis above,

$$\sqrt{N} \left(\boldsymbol{M}^T \boldsymbol{J}_\beta^{-1}(\widehat{\theta}^\beta) \boldsymbol{K}_\beta(\widehat{\theta}^\beta) \boldsymbol{J}_\beta^{-1}(\widehat{\theta}^\beta) \boldsymbol{m} \right)^{-1/2} \left(\boldsymbol{M}^T \widehat{\theta}^\beta - \boldsymbol{d} \right) \xrightarrow[L \to \infty]{L} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}),$$

and therefore,

$$Z_N^*(\widehat{\theta}^\beta) \xrightarrow[L \to \infty]{L} \chi_r^2.$$

Consequently, a critical region corresponding to the generalized null hypothesis is

$$\mathcal{R}_\alpha = \{(n_1, \dots, n_{L+1}) \text{ s.t. } Z_N^*(\widehat{\theta}^\beta) > \chi_{r,\alpha}^2\},$$

where $\chi_{r,\alpha}^2$ denotes the upper α-quantile of a chi-square distribution with r degrees of freedom.

The robustness of the proposed Z-type test statistic can be established by its IF. The IF of a testing procedure at a contamination point $\boldsymbol{n}$ is defined as the Gateaux derivative of the functional, defining the test statistic at the contamination

direction given by Δ_n. In the present context, the functional associated with the proposed Z-type test statistic, $Z_N(\widehat{\theta}^\beta)$, under the null hypothesis is given by

$$Z_N(T_\beta(G)) = \sqrt{\frac{N}{m^T J_\beta^{-1}(\theta_0) K_\beta(\theta_0) J_\beta^{-1}(\theta_0) m}} \left(m^T T_\beta(G) - d\right).$$

Therefore, the IF of the proposed Z-type test statistic can be easily derived from the IF of the MDPDE, as

$$\begin{aligned}
\text{IF}(n, Z_N, G) &= \left.\frac{\partial Z_N(T_\beta(G_\varepsilon))}{\partial \varepsilon}\right|_{\varepsilon=0} \\
&= \sqrt{\frac{N}{m^T J_\beta^{-1}(\theta_0) K_\beta(\theta_0) J_\beta^{-1}(\theta_0) m}} \, m^T \left.\frac{\partial T_\beta(G_\varepsilon)}{\partial \varepsilon}\right|_{\varepsilon=0} \\
&= \sqrt{\frac{N}{m^T J_\beta^{-1}(\theta_0) K_\beta(\theta_0) J_\beta^{-1}(\theta_0) m}} \, m^T \text{IF}(n, T_\beta, G).
\end{aligned}$$

The boundedness of the IF of the Z-type test statistic at a contamination point n and the true distribution F_{θ_0} can be discussed by the boundedness of the IF of the corresponding MDPDE, and thus, robust estimators results in robust test statistics.

On the other hand, we can obtain the asymptotic distribution of the Z-type test in Equation (23) at a contiguous alternative hypothesis. Let $\theta_L \in \Theta \setminus \Theta_0$ be an alternative and take θ_0 as the closest element to the boundary of Θ_0 in the sense of Euclidean distance. We consider contiguous alternative hypothesis of the form

$$H_{1,L} : \theta = \theta_L, \tag{26}$$

with $\theta_L = \theta_0 + \frac{1}{\sqrt{N}}\ell$, for a fixed vector $\ell \in \mathbb{R}^2$. Note that, defining $\ell^* = m^T \ell$, we have

$$m^T \theta_L - d = m^T(\theta_L - \theta_0) = m^T \frac{\ell}{\sqrt{N}}$$

so that the contiguous hypothesis in Equation (26) can be equivalently stated by the condition $g(\theta_L) = \frac{1}{\sqrt{N}}\ell^*$.

Result 13. *The asymptotic distribution of the Z-type statistic in Equation (23), under the contiguous hypothesis (26), is a normal distribution, with mean $\left(m^T J_\beta^{-1}(\widehat{\theta}^\beta) K_\beta(\widehat{\theta}^\beta) J_\beta^{-1}(\widehat{\theta}^\beta) m\right)^{-1/2} m^T \ell$ and unit variance.*

From the above result, we can obtain an approximation for the power function of the test statistic in Equation (22) at the contiguous hypothesis in Equation (26), as

$$\begin{aligned}
\beta_N(\theta_L) &= \mathbb{P}\left(|Z_N(\widehat{\theta}^\beta)| > z_{\alpha/2} | \theta = \theta_L\right) \\
&\approx 2\left[1 - \Phi\left(z_{\alpha/2} - \sqrt{\frac{N}{m^T J_\beta^{-1}(\widehat{\theta}^\beta) K_\beta(\widehat{\theta}^\beta) J_\beta^{-1}(\widehat{\theta}^\beta) m}} \, m^T \ell\right)\right]
\end{aligned}$$

It is clear that $\lim_{N\to\infty} \beta_N(\theta_L) = 1$ and so the Z-type statistic is consistent in the sense of Fraser.[19] More generally, the following result provides an asymptotic approximation to the power function.

Result 14. *Let $\theta^* \in \Theta$ be the true value of the parameter θ with $m^T \theta^* \neq d$. Then, the approximate power function of the test statistic in Equation (22) is given by*

$$\beta_N(\theta^*) \approx 2\left[1 - \Phi\left(1 - \sqrt{N}\left(m^T J_\beta^{-1}(\widehat{\theta}^\beta) K_\beta(\widehat{\theta}^\beta) J_\beta^{-1}(\widehat{\theta}^\beta) m\right)^{-1/2} (m^T \theta^* - d)\right)\right],$$

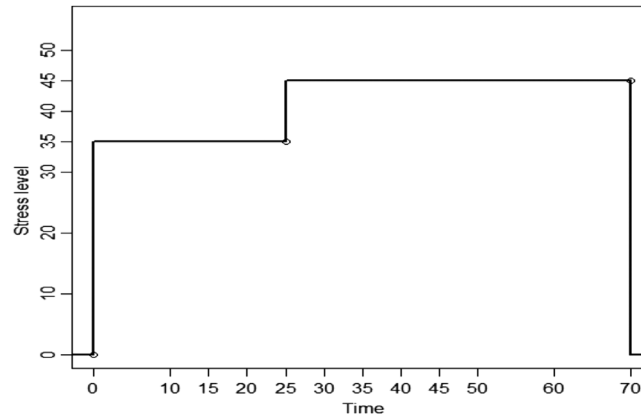*where $\Phi(\cdot)$ denotes the standard normal distribution function.*

FIGURE 1 Step-stress design during the experimental time.

7 | SIMULATION STUDY

In this section, we examine the behavior of the proposed robust MDPDEs, Z-type tests, and Rao-type tests for the SSALT model with exponential lifetime distribution under different contamination scenarios.

An outlying observation is an abnormal value that is distinctly different from the rest of the data. For nondestructive one-shot device sampling, the only available information is the number of failures within intervals, with no knowledge on exact lifetimes. Therefore, an outlying lifetime would bet inappropriate for one-shot device set-up. For multinomial sampling, data contamination will be produced by abnormal counts of failures within an interval, that is, outlying cells of the multinomial sample. Then, to introduce contamination in our context, we should increase (or decrease) the probability of failure in Equation (5) for (at least) one interval. So, the probability of failure is switched in such contaminated cells as

$$\tilde{\pi}_j(\boldsymbol{\theta}) = G_\theta(IT_j) - G_{\tilde{\theta}}(IT_{j-1}) \tag{27}$$

for some $j = 2, \dots, L$, where $\tilde{\theta} = (\widetilde{\theta}_0, \widetilde{\theta}_1)$ is a contaminated parameter with $\tilde{\theta}_0 \leq \theta_0$ and $\tilde{\theta}_1 \leq \theta_1$. It is important to point out that, after the contamination of the probability of failure in a cell, the probability vector of the multinomial model must be normalized to add up to 1.

The restrictions $\widetilde{\theta}_0 \leq \theta_0$ and $\widetilde{\theta}_1 \leq \theta_1$ are chosen to increase the probability of failure within an interval. Indeed, decreasing the exponential parameter $\lambda(\theta) = \theta_0 \exp(\theta_1 x)$ for a fixed inspection time results in a lower value of the distribution function and consequently, the probability of failure within the interval gets increased. Increasing (or decreasing) the probability of failure within an interval would result in an unexpectedly greater (or lower) failure count. This phenomenon often appears in real-life experiments wherein an external factor possibly causes the failure of a group of devices. For example, when the devices under test are produced by different manufacturers, a group of devices from a particular producer may be less reliable. Then, these devices tend to cluster together and fail (earlier than expected) as a group. The greater (or smaller) is the unusual count, the more influenced will be the estimated model. Then, the proposed contamination scheme represents contamination by early failures.

We consider a two-step stress ALT experiment with $L = 11$ inspection times and a total of $N = 180$ devices under test. At the beginning of the experiment, all the devices are subjected to a stress level $x_1 = 35$ until the first time of stress change $\tau_1 = 25$. Then, the surviving units are subjected to an increased stress level, $x_2 = 45$, until the end of the experiment at $\tau_2 = 70$. During the experiment, inspection is performed at a grid of inspection times containing the times of stress change, IT $= (10, 15, 20, 25, 30, 35, 40, 45, 50, 60, 70)$.

Figure 1 plots the SSALT design during the experimental time.

We set the value of the true parameter $\theta_0 = (0.003, 0.03)$, and then generate data from the corresponding multinomial model described in Section 2, under exponential lifetimes. The simulation set-up is summarized in Table 1.

Moreover, we contaminate the data by increasing the probability of failure in the third interval as explained in Equation (5), with the contaminated parameter being either $\widetilde{\theta} = ((1 - \varepsilon)\theta_0, \theta_1)$ (θ_0-contaminated cell) or $\widetilde{\theta} = (\theta_0, (1 - \varepsilon)\theta_1)$ (θ_1-contaminated cell), $\varepsilon \in (0, 1)$. Here, ε represents the contamination rate. We repeat the same simulation scheme over $R = 1000$ repetitions for each scenario. The contamination scheme used is summarized in Table 2.

TABLE 1 Model design parameters.

Stress levels	$x_1 = 35, x_2 = 45$
Times of stress change	$\tau_1 = 25, \tau_2 = 70$
Inspection times	$t = 10, 15, 20, 25, 30, 35, 40, 45, 50, 60, 70$
Sample size	$N = 180$
Model parameters	$\theta_0 = (0.003, 0.03)$

TABLE 2 Contaminated parameters for both θ_0 and θ_1-contamination for a contamination rate ε.

	Contaminated parameter
θ_0-contamination	$((1-\varepsilon)\theta_0, \theta_1)$
θ_1-contamination	$(\theta_0, (1-\varepsilon)\theta_1)$

(A) θ_0-contaminated cell **(B)** θ_1-contaminated cell
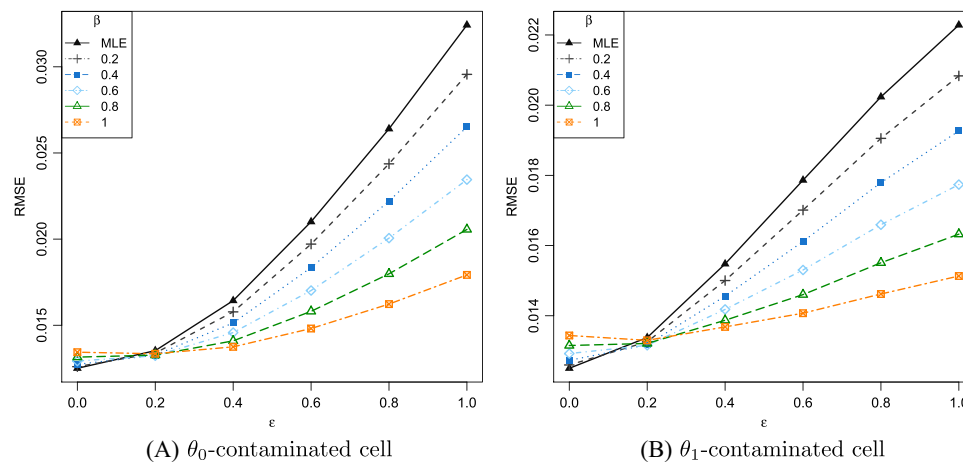
FIGURE 2 RMSE of different estimators against data contamination in $R = 1000$ replications. RMSE, root mean square error.

Note that the contamination is introduced in both scenarios by increasing the probability of failure within an interval. However, as the two parameters, θ_0 and θ_1, play a different role in the model, it is naturally of interest to evaluate the performance of the estimators when switching the values of them.

7.1 | Minimum density power divergence estimators

In order to evaluate the performance of the proposed estimators, we calculate the root mean square error (RMSE) of the MDPDE for different values of $\beta \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$, including the MLE for $\beta = 0$. Further, to assess the efficiency loss of an estimator with respect to the MLE, we define a measure $\rho(\widehat{\theta}^\beta)$, quantifying the relative RMSE of an estimator with respect to the RMSE of the MLE, as

$$\rho(\widehat{\theta}^\beta) = \frac{||\widehat{\theta}^\beta - \theta_0||_2}{||\widehat{\theta}^0 - \theta_0||_2} - 1.$$

Then, $\rho(\widehat{\theta}^\beta)$ measures the efficiency loss of an estimator with respect to the MLE. Clearly, when $\rho(\widehat{\theta}^\beta) < 0$, the MDPDE is more accurate than the MLE, and evidently $\rho(\widehat{\theta}^0) = 1$.

As previously discussed, we test different scenarios of contamination. In the first scenario, we generate an outlying cell in the third interval by decreasing the value of the first parameter, θ_0, and in the second scenario, we perform similarly, but decreasing the second parameter θ_1. In both cases, the lifetime rate $\lambda(\widetilde{\theta})$, is decreased; the smaller is the contamination parameter, the greater is the contamination.

Figures 2 and 3 show the RMSE and the RMSE ratio, ρ, produced with different values of β and the two contamination scenarios determined from $R = 1000$ replications. In the left side plots, the contamination is introduced by decreasing
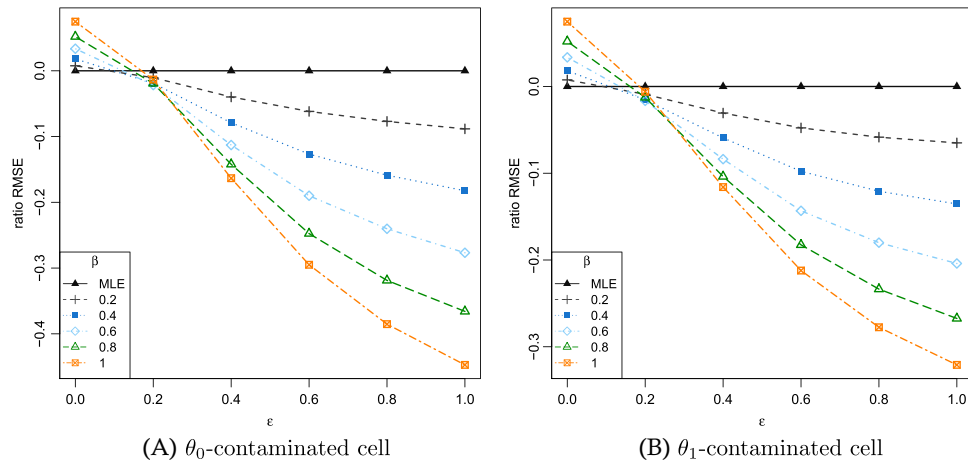
FIGURE 3 Efficiency loss with respect to the MLE against data contamination in $R = 1000$ replications. MLE, maximum likelihood estimators.
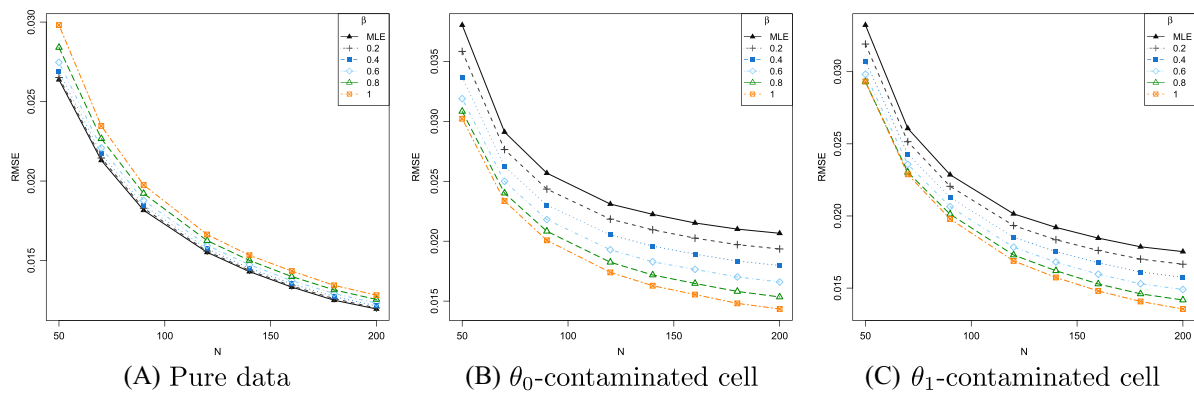

FIGURE 4 RMSE of different estimators against simple in $R = 1000$ replications for pure and contaminated data with $\varepsilon = 0.6$. RMSE, root mean square error.

θ_0, yielding a contamination rate of $\varepsilon = 1 - \frac{\widetilde{\theta}_0}{\theta_0}$ while on the right side plots, they are computed by decreasing the second parameter θ_1, and so the corresponding contamination rate is computed as $\varepsilon = 1 - \frac{\widetilde{\theta}_1}{\theta_1}$.

These results show the advantage of the proposed MDPDE in terms of robustness. The larger the parameter β, the more robust is the corresponding estimator. Indeed, MDPDEs with moderate and large values of β maintain their low RMSE even under heavy contamination. Furthermore, similar behaviors are seen under both scenarios of contamination, but θ_0-contamination seems to affect more strongly any MDPDE. As expected, in the absence of contamination, the MLE is the most efficient estimator, even though all proposed MDPDEs perform competitively in this uncontaminated scenario. In contrast, when a "great" outlier cell is generated, that is, the count of failures within the contaminated interval is unexpectedly high, the efficiency loss of MLE with respect to the proposed MDPDE is seen to be quite pronounced. More specifically, when the contamination rates are greater than 20% in either of the two parameters, the MDPDEs considerably outperform the MLE, and that difference is seen to be pronounced in θ_0-contaminated scenario. As previously discussed, such great outliers would be produced in SSALT experiments when a group of devices have low reliability compared to the rest of devices under test, and so they fail (as a group) earlier than expected.

Based on the asymptotic distribution of the MDPDE, the performance of the estimators should get increased with large sample sizes. Conversely, in many practical applications, the sample size is usually quite small, presenting a challenging situation for inference. Figure 4 presents the RMSE of the MDPDE for different values of the tuning parameter β against sample size under pure data (left), θ_0-contamination (middle) and θ_0-contamination (right). In the absence of contamination all estimators performs quite similar, although fewer values of β result in more efficient MDPDEs. When increasing

TABLE 3 Empirical coverage probability of the direct (left) and transformed (left) confidence intervals for the reliability at $t = 60$, mean lifetime, and 5% quantile under normal operating conditions $x_0 = 25$ and θ_1-contamination.

β	0	0.2	0.4	0.6	0.8	1	0	0.2	0.4	0.6	0.8	1
	Direct						Transformed					
ε	Reliability at $t = 60$											
0	0.93	0.93	0.94	0.94	0.94	0.93	0.96	0.96	0.96	0.96	0.96	0.96
0.2	0.94	0.94	0.94	0.94	0.94	0.95	0.93	0.94	0.94	0.95	0.95	0.95
0.4	0.89	0.90	0.91	0.92	0.93	0.94	0.88	0.89	0.90	0.92	0.93	0.94
0.6	0.81	0.84	0.87	0.89	0.91	0.93	0.80	0.82	0.85	0.89	0.91	0.93
0.8	0.74	0.77	0.82	0.86	0.89	0.92	0.72	0.74	0.79	0.84	0.88	0.91
1	0.68	0.71	0.76	0.82	0.87	0.90	0.65	0.68	0.73	0.79	0.85	0.90
	Mean lifetime											
0	0.93	0.94	0.94	0.93	0.93	0.92	0.96	0.96	0.96	0.96	0.95	0.95
0.2	0.82	0.83	0.84	0.85	0.86	0.87	0.92	0.92	0.92	0.93	0.94	0.94
0.4	0.70	0.71	0.73	0.75	0.78	0.82	0.83	0.85	0.88	0.89	0.90	0.91
0.6	0.60	0.62	0.65	0.68	0.71	0.76	0.74	0.76	0.79	0.83	0.87	0.89
0.8	0.51	0.54	0.58	0.62	0.65	0.70	0.66	0.69	0.72	0.77	0.82	0.87
1	0.43	0.48	0.52	0.56	0.62	0.67	0.59	0.63	0.67	0.72	0.77	0.84
	5% Quantile											
0	0.93	0.94	0.94	0.93	0.93	0.92	0.96	0.96	0.96	0.96	0.95	0.95
0.2	0.82	0.83	0.84	0.85	0.86	0.87	0.92	0.92	0.92	0.93	0.94	0.94
0.4	0.70	0.71	0.73	0.75	0.78	0.82	0.83	0.85	0.88	0.89	0.90	0.91
0.6	0.60	0.62	0.65	0.68	0.71	0.76	0.74	0.76	0.79	0.83	0.87	0.89
0.8	0.51	0.54	0.58	0.62	0.65	0.70	0.66	0.69	0.72	0.77	0.82	0.87
1	0.43	0.48	0.52	0.56	0.62	0.67	0.59	0.63	0.67	0.72	0.77	0.84

the sample size, the efficiency loss with respect to the MLE is reduced, as all estimators converges to the true value of the model parameter. In contrast, in the presence of contamination, the robustness improvement of the MDPDE with positive values of β with respect to the MLE is remarkable under all sample sizes, and it is preserved for large sample size. That is, the previous discussion about the trade-off between efficiency and robustness is well-founded for poorer sample sizes too.

We finally analyze the performance of the direct and transformed confidence intervals derived in Section 5 for the reliability at $t = 60$, mean lifetime and 5%-quantile under normal operating conditions, $x_0 = 25$. We contaminate the model as before, but for brevity, we only report the results for the θ_1-contamination direction, as similar conclusions are obtained for θ_0-direction. Table 3 presents the empirical coverage probabilities of the direct and transformed CIs for three lifetime characteristics based on the MDPDE with different values of the tuning parameter, obtained over $R = 1000$ repetitions. The results illustrate, on the one hand, the certain advantage of the direct and transformed CIs based on MDPDE with moderate and large values of β in terms of robustness, with a low loss of coverage in the absence of contamination. On the other hand, transformed CIs outperform the direct method in all scenarios, and their benefit is emphasized under contaminated scenarios, where the gain in coverage is remarkable.

7.2 | Z-type tests

We empirically examine the performance of the Z-type test statistic based on the MDPDE. We adopt again the two-step stress ALT experiment with $L = 11$ inspection times and $N = 180$ devices described in Section 7.1, and we consider testing the hypothesis

$$H_0 : \theta_1 = 0.03 \quad \text{versus} \quad H_1 : \theta_1 \neq 0.03. \tag{28}$$

The true value of the parameter is set to be $\boldsymbol{\theta}_0 = (0.003, 0.03)^T$ so as to fit the null hypothesis. Then, the Z-type test statistic is defined using Equation (23) with $\boldsymbol{m} = (0, 1)^T$ and $d = 0.03$, and the critical region of the test is given by Equation (24).
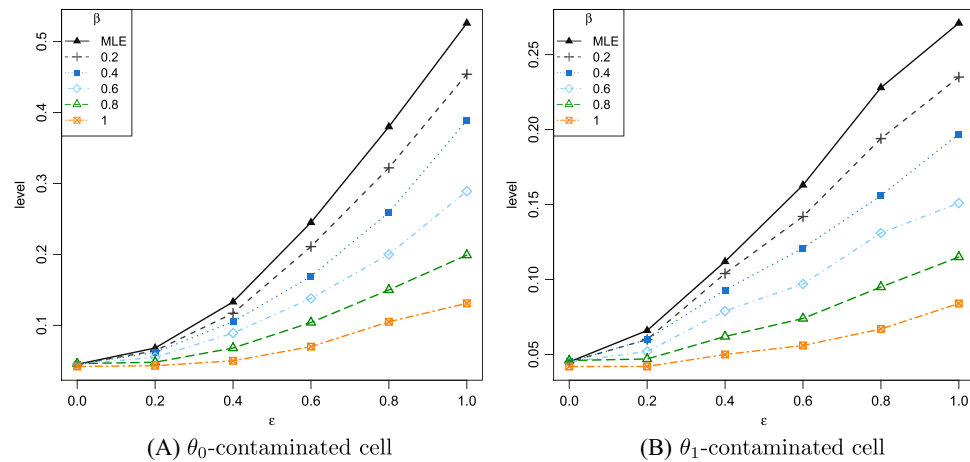
FIGURE 5 Empirical significance level against contamination cell proportion in $R = 1000$ replications.
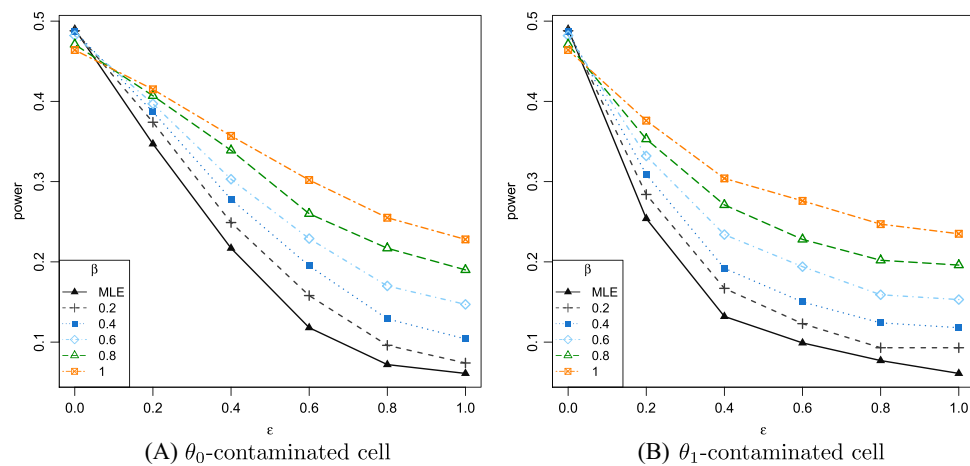

FIGURE 6 Empirical power against contamination cell proportion in $R = 1000$ replications.

Figure 5 shows the empirical level of the test against cell contamination, ε, with the two different contaminated scenarios considered in the last section, θ_0-contaminated third cell (left), and θ_1-contaminated third cell (right). The empirical level is computed as the proportion of rejected Z-type test statistic over $R = 1000$ replications of the model under the null hypothesis for a significance level of $\alpha = 0.05$. The empirical level of the Z-type tests based on the MLE promptly grows when the contamination rate gets increased, whereas the empirical level of the Z-type test based on the MDPDE, with large values of β, remains low in heavily contaminated scenarios, highlighting its robustness property. Again, θ_0-contamination strongly affects the performance of Z-type test statistics, and so empirical levels are higher under the first scenario of contamination for similar contamination rates. Indeed, the MLE doubles its empirical level under moderately contaminated scenarios in θ_0 with respect to θ_1-contamination.

Next, we examine the empirical power of the linear hypothesis test by considering a different true parameter value, $\theta = (0.003, 0.6)^T$, not satisfying the null hypothesis in Equation (28). Note that the true parameter is close to the null hypothesis, and thus the power of the test will be quite low even under pure data.

Figure 6 displays the empirical power of the Z-type test statistics over $R = 1000$ replications under the two different contamination scenarios. The advantage of the robust Z-type test statistics based on MDPDEs with moderate and large values of the tuning parameter β stands out even in moderately contaminated scenarios, with contamination rates being over 20%. When increasing the contamination to greater levels, the empirical power of the classical Z-type test statistic based on MLE rapidly falls down to critical values below 5%–10%. In contrast, the Z-type tests based on robust MDPDEs remain satisfactory even under most heavily contaminated scenarios. Here, contamination of the first parameter, θ_1, results in a faster drop in power, and so θ_1-contamination strongly affects the performance of the Z-type test statistics in terms of power, contrary to the effect on the level.
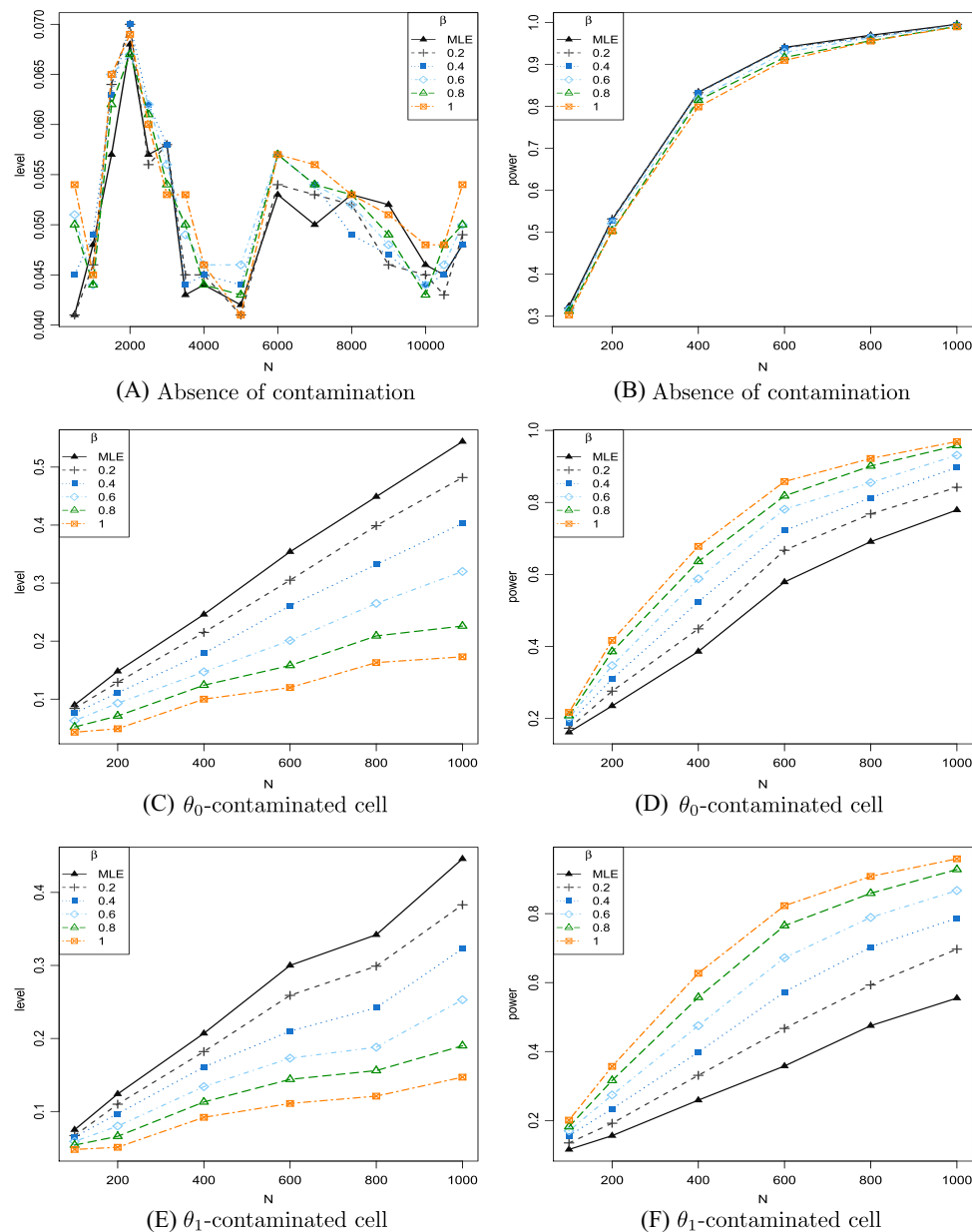
FIGURE 7 Empirical level and power against sample size with different contaminated scenarios in $R = 1000$ replications.

We finally examine the performance of the Z-type test statistic against sample size under different contamination scenarios: in the absence of contamination (top), θ_0-contaminated third cell with a 40% reduction of the parameter (middle), and θ_1-contaminated third cell with a 40% reduction of the parameter (bottom). Figure 7 shows the empirical level (left) and power (right) obtained with different sample sizes over $R = 1000$ replications. SSALT for one-shot device data usually deals with small sample sizes. However, to better illustrate the consistency of the proposed Z-type test statistics based on the MDPDE in the absence of contamination, the empirical level is plotted for large sample sizes under pure data. In the absence of contamination (top), all Z-type tests based on MDPDEs with different values of β perform similarly in terms of level and power, but the power of the Z-type test based on the MLE is slightly better. Furthermore, all test statistics are consistent, as their empirical level are quite close to the significance level $\alpha = 0.05$ and their empirical power are close to 1. On the other hand, when an outlying cell is introduced by decreasing any of the model parameters (middle and bottom graphs), the Z-type test based on MDPDE with low values of the tuning parameter (including the MLE) has its performance to worsen rapidly in both level and power measures. In contrast, MDPDEs based on large values of β outperform the test based on the MLE in both contamination scenarios, especially in terms of empirical level, and they remain stable and competitive under large sample sizes and all contamination scenarios.

7.3 | Choice of the tuning parameter

The tuning parameter β of the DPD loss function controls the trade-off between efficiency and robustness of the resulting MDPPE. Following the discussions in the preceding sections, larger values of β produce more robust but less efficient estimators. Therefore, the optimal value of β will be good to determine. From our empirical results, a moderately large value of β (over $\beta = 0.4$) is expected to provide robust estimators without a high loss of efficiency with respect to the MLE in the absence of contamination. Determining this optimal value for the best compromise is, therefore, of great practical interest. Optimal values of β will produce robust estimators without coming at the cost of a high efficiency loss. Then, a criterion measuring the efficiency loss in favor of robustness gain should be adopted.

Warwick and Jones[39] introduced an useful data-based procedure for the choice of the tuning parameter for the MDPDE. However, this method depends on the choice of a pilot estimator and Basak et al.[13] improved the method by removing the dependency on an initial estimator. The approach of Warwick and Jones[39] minimizes the asymptotic MSE of the MDPDE given by

$$\widehat{MSE}(\beta) = \left(\widehat{\theta}^\beta - \theta_P\right)^T \left(\widehat{\theta}^\beta - \theta_P\right) + \frac{1}{N} \text{Tr} \left\{ \boldsymbol{J}_\beta(\widehat{\theta}^\beta)^{-1} \boldsymbol{K}_\beta(\widehat{\theta}^\beta) \boldsymbol{J}_\beta(\widehat{\theta}^\beta)^{-1} \right\}, \tag{29}$$

where θ_P is a pilot estimator and Tr denote the trace of the matrix. Several proposals of this pilot estimator have been studied in the literature. However, the choice of the pilot significantly impact on the optimal tuning parameter, as it invariably draws the final estimator towards itself. To overcome this drawback, Basak et al.[13] proposed an iterative algorithm that replaces in each step, the value of the pilot estimator by the estimator obtained with the optimal value of β until the optimal choice of the tuning parameter (or equivalently, the pilot estimator) gets stabilized. The process should be initialized with a suitable robust pilot estimator, but the final choice of β gets more pilot-independent. Basak et al.[13] empirically showed that when the pilot estimators are within the MDPDE class, all robust pilots lead to the same iterated optimal choice, and moreover the performance of the algorithm improves even with pure data. These are all summarized in the following algorithm:

Algorithm [Choice of the tuning parameter]

1. Fix the convergence rate ε and choose an initial pilot estimator θ_P from the MDPDE family;
2. Update the optimal value of the tuning parameter, β^*, using the minimum asymptotic MSE in Equation (29);
3. If the optimal estimate $\widehat{\theta}^{\beta^*}$ differs from the pilot estimator by less than the convergence rate: Stop;
   Else, replace the pilot estimator by the optimal $\widehat{\theta}^{\beta^*}$ and return to Step 2.

Let us consider the previous simulation set up with true parameter value $\theta = (0.003, 0.03)^T$, but now let us choose the optimal value of β according to the presented data-based procedure detailed in the above algorithm. We initialize the method with the MDPDE with different tuning parameters $\beta_p = 0, 0.5$, and 1, yielding the pilot estimators $\widehat{\theta}^0, \widehat{\theta}^{0.5}$, and $\widehat{\theta}^1$, respectively, and we fix the convergence rate to be $\varepsilon = 0.001$. The minimization of Equation (29) is carried out over a grid search in [0,1] of size 100.

Figure 8 shows the optimal values of the tuning parameter β against data contamination over $R = 1000$ repetitions. As expected, optimal values of β are greater with high contamination rates. Further, the choice of optimal β is almost entirely independent of the pilot estimator, and so the presented algorithm does not seem to get affected by this initial choice. However, it tends to be slightly conservative and selects moderately high values of β when a high contamination rate is introduced. Optimal values of the tuning parameter are generally larger when the contamination is introduced on the first parameter. Then, the model is more sensitive to contamination in such direction. Additionally, Figure 9 compares the RMSE of the resulting (optimal) estimator to the RMSE of the estimators with fixed values of $\beta \in \{0, 0.2, 0.4, 0.6, 0.8, 1\}$ based on $R = 1000$ repetitions. As expected, the data-based method for choosing optimal β outperforms any of the methods with a prefixed value of β, since it adapts the tuning parameter value to the amount of contamination present in the data. The optimal choice achieves a suitable trade-off between efficiency and robustness, approaching to the MLE under uncontaminated scenarios and to the most robust estimators under heavily contamination.
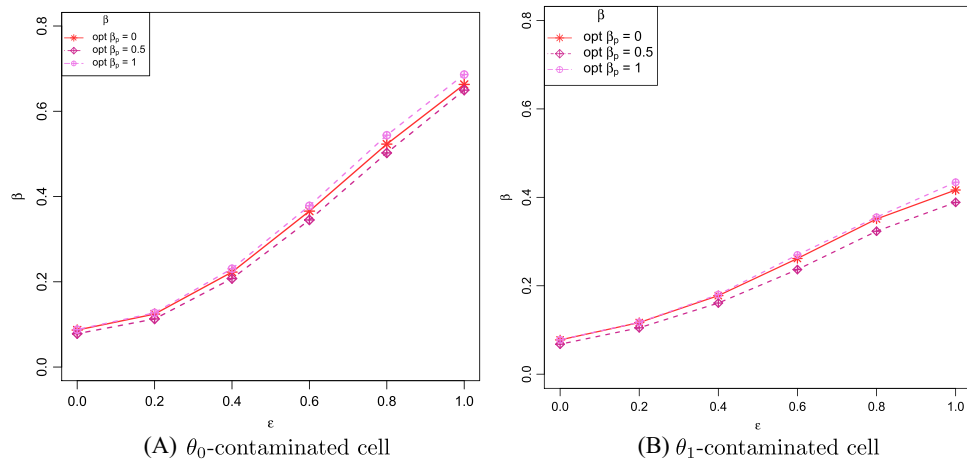
FIGURE 8 Optimal values of β against contamination cell proportion for different pilot estimators based on $R = 1000$ repetitions.
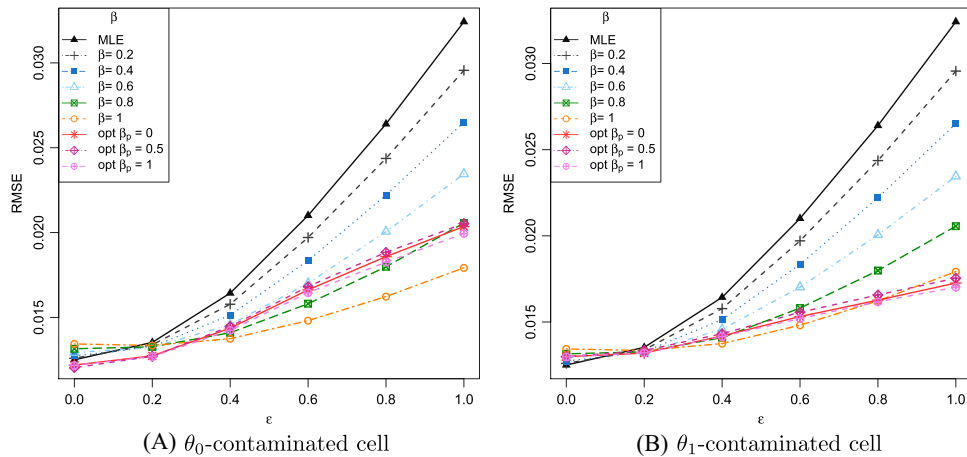

FIGURE 9 RMSE against contamination cell proportion for different values of the tuning parameter β and optimum values obtained with three different pilot estimators based on $R = 1000$ repetitions. RMSE, root mean square error.

8 | DATA ANALYSIS

In this section, we discuss two real-life applications of the MDPDE for the SSALT model, developed in the preceding sections.

8.1 | Electronic components data

The first example was studied by Wang and Fei[38] to get the reliability indices of a kind of electronic components at the normal temperature of $x_0 = 25°C$. $N = 100$ items from a batch of products were randomly selected for a simple SSALT ($k = 2$), with two stress levels $x_1 = 100°C$ and $x_2 = 150°C$. In the original experiment, the stress level rises when 30 products had failed and the test continues until 20 more products had failed, obtaining a total of 50 failures. Their failure times are as follows:

- Failure times at the first stress level x_1: 32, 54, 59, 86, 117, 123, 213, 267, 268, 273, 299, 311, 321, 333, 339, 386, 408, 422, 435, 437, 476, 518, 570, 632, 666, 697, 796, 854, 858, 910.
- Failure times at the second stress level, x_2: 16, 19, 21, 36, 37, 63, 70, 75, 83, 95, 100, 106, 110, 113, 116, 135, 136, 149, 172, 186.

TABLE 4 Estimated parameters for the electronic components data for different values of β.

β	$\log(\widehat{\theta}_0)$	IC ($\log(\theta_0)$)	$\widehat{\theta}_1$ ($\times 10^2$)	IC (θ_1) ($\times 10^2$)
MLE	−10.857	[−12.243, −9.470]	3.021	[1.887, 4.155]
0.2	−10.842	[−12.236, −9.448]	3.003	[1.862, 4.143]
0.4	−10.833	[−12.236, −9.429]	2.992	[1.843, 4.141]
0.6	−10.827	[−12.243, −9.411]	2.986	[1.827, 4.146]
0.8	−10.830	[−12.260, −9.399]	2.989	[1.819, 4.160]
1	−10.837	[−12.284, −9.389]	2.996	[1.813, 4.180]
0.027*	**−10.856**	**[−12.243, −9.468]**	**3.019**	**[1.884, 4.154]**

Bold values indicate the estimated values with optimal tuning parameter

TABLE 5 Estimated mean lifetime and asymptotic (direct and transformed) confidence intervals (in hours) of the electronic components under three constant temperatures.

	Mean lifetime	Direct CI	Transformed CI
$x_0 = 25$			
MLE	6.772	[0, 14.290]	[2.231, 20.553]
0.2	6.704	[0, 14.182]	[2.197, 20.455]
0.4	6.660	[0, 14.142]	[2.166, 20.480]
0.6	6.630	[0, 14.145]	[2.135, 20.594]
0.8	6.643	[0, 14.250]	[2.114, 20.878]
1	6.679	[0, 14.417]	[2.096, 21.278]
0.027*	**6.768**	**[0, 14.589]**	[2.0982, 21.287]
$x_1 = 100$			
MLE	0.702	[0.452, 0.953]	[0.492, 1.004]
0.2	0.705	[0.453, 0.957]	[0.493, 1.008]
0.4	0.706	[0.452, 0.960]	[0.493, 1.011]
0.6	0.706	[0.451, 0.961]	[0.492, 1.013]
0.8	0.706	[0.449, 0.962]	[0.491, 1.015]
1	0.706	[0.447, 0.965]	[0.489, 1.018]
0.027*	**0.703**	**[0.446, 0.961]**	**[0.488, 1.014]**
$x_2 = 150$			
MLE	0.155	[0.087, 0.223]	[0.100, 0.241]
0.2	0.157	[0.087, 0.227]	[0.101, 0.245]
0.4.	0.158	[0.088, 0.229]	[0.101, 0.247]
0.6	0.159	[0.088, 0.230]	[0.101, 0.248]
0.8	0.158	[0.087, 0.230]	[0.101, 0.248]
1	0.158	[0.087, 0.229]	[0.100, 0.248]
0.027*	**0.155**	**[0.086, 0.225]**	**[0.099, 0.243]**

Bold values indicate the estimated values with optimal tuning parameter

For illustrating the performance of the MPDPE for the SSALT model with one-shot devices, we will assume that we only know how many devices had failed before certain prespecified inspection times, $t = 270, 430, 600, 910, 975, 1015, 1040, 1096$. Additionally, the time of stress change, τ_1, is prefixed at $t = 910$.

Table 4 shows the estimated model parameters with different values of the tuning parameter β, along with an approximate CI constructed from Equation (13). The last row of the table contains the estimates with the optimal value of β obtained with the algorithm presented in Section 7.3. The algorithm is initialized with the pilot estimate $\widehat{\theta}^{0.5}$ and the optimum value of β is reached at $\beta = 0.027$, implying moderately low contamination in the data. Further, to fairly analyze the role of each of the parameters in the model, we present the logarithm of the first parameter, θ_0, so both model parameters are reported on the same scale. The mean lifetime of the electronic component at low temperatures is appreciably high,

TABLE 6 Estimated reliability at $t = 600$ s and asymptotic (direct and transformed) confidence intervals of the electronic components under three constant temperatures.

	$\widehat{S}(600)$	Direct CI	Transformed CI
$x_0 = 25$			
MLE	0.976	[0.949, 1.00]	[0.929, 0.992]
0.2	0.975	[0.948, 1.00]	[0.928, 0.992]
0.4	0.975	[0.948, 1.00]	[0.927, 0.992]
0.6	0.975	[0.947, 1.00]	[0.926, 0.992]
0.8	0.975	[0.947, 1.00]	[0.925, 0.992]
1	0.975	[0.947, 1.00]	[0.924, 0.992]
0.027*	**0.976**	**[0.948, 1.00]**	**[0.926, 0.992]**
$x_1 = 100$			
MLE	0.789	[0.722, 0.856]	0.714, 0.848]
0.2	0.790	[0.723, 0.856]	0.715, 0.849]
0.4	0.790	[0.723, 0.857]	0.715, 0.849]
0.	0.790	[0.722, 0.857]	0.715, 0.849]
0.8	0.790	[0.722, 0.857]	0.714, 0.850]
1	0.790	[0.721, 0.858]	0.713, 0.850]
0.027*	**0.789**	**[0.721, 0.857]**	**[0.713, 0.849]**
$x_2 = 150$			
MLE	0.341	[0.180, 0.503]	[0.202, 0.516]
0.2	0.346	[0.183, 0.509]	[0.205, 0.521]
0.4	0.349	[0.185, 0.513]	[0.206, 0.524]
0.6	0.350	[0.185, 0.514]	[0.207, 0.526]
0.8	0.349	[0.184, 0.514]	[0.206, 0.526]
1	0.348	[0.182, 0.514]	[0.204, 0.526]
0.027*	**0.342**	**[0.178, 0.507]**	**[0.200, 0.520]**

Bold values indicate the estimated values with optimal tuning parameter

and increasing a degree on the temperature multiplies the lifetime of the device, according to all estimators, in approximately 0.97 times. Consequently, the increase of the temperature from $x_1 = 100°$ to $x_2 = 150°$ shortens the lifetime by more than 0.22 times. Robust methods tend to estimate with a higher value the first parameter, and consequently decrease the estimate of the second one. Conversely, the variance of the estimator increases with β, producing wider intervals.

Table 5 shows the mean lifetime (in hours) of the electronic components under different (constant) stress level, and their associated direct and transformed CI. Under normal operating conditions ($x_0 = 25$), the devices are expected to last for more than 6 h, but their lifetime gets severely decreased when exposed to very high temperatures, which allows to infer about the reliability in a short period of time. It is interesting to note that direct CI of the mean lifetime under normal operating conditions gets truncated due to the positivity constraint. On the other hand, transformed CI is wider, and the right end point is quite far from than the the one obtained with direct CIs. This difference gets reduced at increased temperatures.

On the other hand, one may be interested in estimating the reliability of the devices when it is exposed to different constant temperatures. We fix a "mission time" at $t = 600$ s and we report the estimated reliabilities and CIs under different stress levels in Table 6. Again, direct CIs had to be truncated under normal operating conditions so as to remain within the interval (0,1). As expected, the reliability of the devices decreases when increasing the stress level, and here all estimates remain close for all values of the tuning parameter.

Finally, one may be interested in determining the time at which 10% of the devices are expected to fail, under different (constant) temperatures. Table 7 presents the estimated 0.9-quantiles of the lifetime distribution (or equivalently the 0.1-quantiles of the reliability). Here, the direct CI of the estimated quantiles under normal operating conditions are again truncated, demonstrating again the drawback of the direct method, while transformed CI provides a good alternative for such intervals without the problem of constraints.

TABLE 7 Estimated 90% quantile and asymptotic (direct and transformed) confidence intervals (in seconds) of the electronic components under three constant temperatures.

	$\widehat{Q}_{0.9}$	Direct CI	Transformed CI
$x_0 = 25$			
MLE	2568.46	[0, 5420.08]	[846.25, 7795.56]
0.2	2542.69	[0, 5379.29]	[833.30, 7758.67]
0.4	2526.14	[0, 5363.87]	[821.48, 7768.17]
0.6	2514.90	[0, 5365.10]	[809.70, 7811.19]
0.8	2519.62	[0, 5405.00]	[801.67, 7919.06]
1	2533.16	[0, 5468.50]	[795.09, 8070.66]
0.027*	**2567.25**	**[0, 5533.47]**	**[808.50, 8151.84]**
$x_1 = 100$			
MLE	266.45	[171.42, 361.49]	[186.52, 380.65]
0.2	267.47	[171.79, 363.14]	[187.03, 382.49]
0.4	267.82	[171.62, 364.03]	[187.00, 383.57]
0.6	267.77	[171.07, 364.46]	[186.61, 384.22]
0.8	267.71	[170.39, 365.04]	[186.12, 385.08]
1	267.73	[169.63, 365.84]	[185.60, 386.22]
0.027*	**266.75**	**[169.13, 364.37]**	**[185.00, 384.62]**
$x_2 = 150$			
MLE	58.83	[32.89, 84.77]	[37.85, 91.43]
0.2	59.60	[33.16, 86.04]	[38.24, 92.88]
0.4	59.99	[33.23, 86.76]	[38.41, 93.72]
0.6	60.15	[33.19, 87.11]	[38.42, 94.17]
0.8	60.06	[33.04, 87.07]	[38.30, 94.17]
1	59.85	[32.83, 86.87]	[38.11, 94.00]
0.027*	**58.96**	**[32.50, 85.42]**	**[37.64, 92.35]**

Bold values indicate the estimated values with optimal tuning parameter

8.2 | Light bulbs data

Zhu[41] conducted an accelerated life testing experiment in the Quality and Reliability Engineering Laboratory of the Industrial and Systems Engineering Department of Rutgers University so as to examine the reliability of light bulbs. Two sets of 32 miniature light bulbs were placed in a temperature and humidity chamber where humidity was held constant, and the long-term failure due to bulb filament fatigue was then studied. When the switch was turned on, full current suddenly flowed to the filament at the speed of light. This sudden massive vibration caused the filament to wildly bounce causing fatigue behavior of the filament, which resulted in breakage of the filament. Long-term failure occurred when the filament eventually become so fatigued that its electrical resistance increased to the point that current would not flow. Each light bulb was connected with a resistor, across which Voltage was measured to monitor the status of the light bulbs. Normal operating condition of the light bulbs is 2V. To carry out the SSALT, they applied 2.25 V for 96 h and then increased the voltage to 2.44 V. The step-voltage test got stopped at 140 h. Failure times during the experiment are as follows:

12.07, 19.5, 22.1, 23.11, 24, 25.1, 26.9, 36.64, 44.1, 46.3, 54, 58.09, 64.17, 72.25, 86.9, 90.09, 91.22, 102.1, 105.1, 109.2, 114.4, 117.9, 121.9, 122.5, 123.6, 126.5, 130.1, 14 17.95, 24, 26.46, 26.58, 28.06, 34, 36.13, 40.85, 41.11 42.63, 52.51, 62.68, 73.13, 83.63, 91.56, 94.38, 97.71, 101.53, 105.11 112.11, 119.58 ,120.2, 126.95, 129.25, 136.31.

The remaining 11 light bulbs continued to provide light when the experiment was terminated. To illustrate the performance of the MDPDE of the SSALT model, we transformed the collected data into one-shot devices data with inspection times $t = 25, 50, 96, 110, 120, 140$. Table 8 shows the estimated values of the model parameters with different values of the tuning parameter β. Applying the data-based choice of β described in Section 7.3, the optimum value is approximately $\beta = 0.12$, thus showing slightly higher contamination to be present in this data than in the last electronic component

TABLE 8 Estimated parameters for the light bulb data for different values of β.

β	$\log(\widehat{\theta}_0)$	IC ($\log(\theta_0)$)	$\widehat{\theta}_1$	IC (θ_1)
MLE	−10.727	[−11.718, −9.736]	5.285	[2.282, 8.287]
0.2	−10.734	[−11.725, −9.743]	5.308	[2.305, 8.310]
0.4	−10.739	[−11.731, −9.746]	5.326	[2.320, 8.332]
0.6	−10.747	[−11.742, −9.752]	5.354	[2.343, 8.364]
0.8	−10.755	[−11.753, −9.757]	5.381	[2.364, 8.398]
1	−10.764	[−11.766, −9.763]	5.411	[2.387, 8.434]
0.12*	**−10.729**	**[−11.720, −9.738]**	**5.293**	**[2.290, 8.295]**

Bold values indicate the estimated values with optimal tuning parameter

TABLE 9 Estimated mean lifetime (in minutes) and asymptotic confidence intervals for the light bulbs under different voltages.

	Mean lifetime	**Direct CI**	**Transformed CI**
$x_0 = 2$			
MLE	483.84	[4.48, 963.19]	[179.65, 1303.08]
0.2	486.96	[4.33, 969.59]	[180.74, 1311.99]
0.4	489.43	[3.56, 975.31]	[181.37, 1320.78]
0.6	493.47	[2.42, 984.51]	[182.43, 1334.82]
0.8	497.58	[0.96, 994.20]	[183.40, 1349.95]
1	502.22	[0, 1005.14]	[184.50, 1367.09]
0.12*	**484.82**	**[0, 970.90]**	**[177.89, 1321.32]**
$x_0 = 2.25$			
MLE	129.10	[85.30, 172.90]	[91.96, 181.25]
0.2	129.19	[85.33,173.06]	[92.00, 181.42]
0.4	129.25	[85.29, 173.21]	[91.99, 181.61]
0.6	129.41	[85.28, 173.54]	[92.02, 182.00]
0.8	129.61	[85.27, 173.95]	[92.06, 182.48]
1	129.85	[85.25, 174.44]	[92.10, 183.06]
0.12*	**129.11**	**[84.86, 173.35]**	**[91.64, 181.88]**
$x_0 = 2.44$			
MLE	47.30	[25.60, 69.00]	[29.90, 74.83]
0.2	47.13	[25.53, 68.72]	[29.80, 74.52]
0.4	46.99	[25.46, 68.52]	[29.72, 74.30]
0.6	46.80	[25.36, 68.23]	[29.60, 73.98]
0.8	46.63	[25.27, 67.98]	[29.49, 73.70]
1	46.45	[25.19, 67.71]	[29.39, 73.41]
0.12*	**47.23**	**[25.46, 69.01]**	**[29.79, 74.90]**

Bold values indicate the estimated values with optimal tuning parameter

example. Results for this optimum value are presented in the last row of Table 8. As in the previous example, we report the values of $\log(\widehat{\theta}_0)$ so both model parameters are in same scale. Unlike in the previous example, now robust estimators give a higher value to the second parameter and decrease the value of the first one.

Table 9 shows the estimated mean lifetime and their corresponding direct and transformed CIs, with different values of β, under three stress levels: $x_0 = 2$V corresponding to working condition, and two stress levels at which devices were subjected during the experiment, $x_1 = 2.25$V and $x_2 = 2.44$V. It is striking that robust methods provide larger mean lifetime under normal operating condition ($x_0 = 2$V) than the MLE, but shorter lifetimes under a higher stress level ($x_2 = 2.44$V). Furthermore, direct and transformed CIs of the mean lifetime markedly differ under working condition, but that difference gets reduced when the voltage gets increased.

TABLE 10 Estimated reliability at $t = 50$ s and asymptotic (direct and transformed) confidence intervals for the light bulbs under three constant voltages.

	$\widehat{S}(50)$	Direct CI	Transformed CI
$x_0 = 2$			
MLE	0.902	[0.809, 0.994]	[0.764, 0.963]
0.2	0.902	[0.811, 0.994]	[0.765, 0.963]
0.4	0.903	[0.811, 0.994]	[0.766, 0.964]
0.6	0.904	[0.813, 0.995]	[0.767, 0.964]
0.8	0.904	[0.814, 0.995]	[0.768, 0.964]
1	0.905	[0.815, 0.995]	[0.769, 0.965]
0.12*	**0.902**	**[0.809, 0.995]**	**[0.762, 0.964]**
$x_0 = 2.25$			
MLE	0.679	[0.590, 0.768]	[0.584, 0.761]
0.2	0.679	[0.590, 0.768]	[0.584, 0.761]
0.4	0.679	[0.590, 0.769]	[0.584, 0.761]
0.6	0.680	[0.590, 0.769]	[0.584, 0.762]
0.8	0.680	[0.590, 0.770]	[0.584, 0.762]
1	0.680	[0.590, 0.770]	[0.585, 0.763]
0.12*	**0.679**	**[0.589, 0.769]**	**[0.583, 0.762]**
$x_0 = 2.44$			
MLE	0.347	[0.179, 0.516]	[0.202, 0.528]
0.2	0.346	[0.178, 0.514]	[0.201, 0.527]
0.4	0.345	[0.177, 0.513]	[0.200, 0.526]
0.6	0.344	[0.175, 0.512]	[0.199, 0.524]
0.8	0.342	[0.174, 0.510]	[0.198, 0.523]
1	0.341	[0.173, 0.509]	[0.197, 0.522]
0.12*	**0.347**	**[0.178, 0.516]**	**[0.201, 0.529]**

Bold values indicate the estimated values with optimal tuning parameter

Next, Table 10 shows the estimated reliability at mission time $t = 50$ s under constant voltage, with different values of β. The reliability of the light bulbs at working condition is sufficiently high, exceeding a 90% reliability, but gets radically decreased at the highest voltage $x_2 = 2.44$.

Finally, Table 11 presents the estimated 0.9-quantile of the lifetime distribution. At that times, 10% of light bulbs are expected to fail if they are subjected to constant voltage. Under normal operating condition $x_0 = 2$, 10% of the light bulbs are expected to fail by (approximately) 52 s. Higher quantiles are predicted with low values of β, including the MLE, but when increasing voltage to $x_2 = 2.44$, the situation turns around, and quantiles based on MDPDE with higher values of β predict lower times at which 10% of the light bulbs are expected to fail.

9 | CONCLUDING REMARKS

In this paper, we have developed robust estimation methods and test procedures for nondestructive one-shot devices under the SSALT model with exponential lifetimes. The proposed MDPDEs, indexed by a tuning parameter β controlling the trade-off between efficiency and robustness, generalize the classical likelihood approach to a wider family, including the MLE for $\beta = 0$. MDPDEs are consistent, asymptotically normal and also enjoy robustness properties for positive values of the tuning parameter; they offer a competitive and robust alternative to the classical estimators based on MLEs. Additionally, we have presented a data-based criterion for choosing an optimal value of the tuning parameter, β, that does not depend on any initial (pilot) estimator. Through MDPDEs, point estimation and direct and transformed CIs of some lifetime characteristics of interest, such as the reliability at certain mission times, distribution quantiles, and mean lifetimes, have been proposed.

TABLE 11 Estimated 90% quantile and asymptotic (direct and transformed) confidence intervals (in seconds) of the light bulbs under three constant stress levels.

	$\widehat{Q}_{0.9}$	Direct CI	Transformed CI
$x_0 = 2$			
MLE	50.98	[0.47, 101.48]	[18.93, 137.29]
0.2	51.31	[0.46, 102.16]	[19.04, 138.23]
0.4	51.57	[0.37, 102.76]	[19.11, 139.16]
0.6	51.99	[0.25, 103.73]	[19.22, 140.64]
0.8	52.42	[0.10, 104.75]	[19.32, 142.23]
1	52.91	[0, 105.90]	[19.44, 144.04]
0.12*	**51.08**	**[0, 102.29]**	**[18.74, 139.22]**
$x_1 = 2.25$			
MLE	13.60	[8.99, 18.22]	[9.69, 19.10]
0.2.	13.61	[8.99, 18.23]	[9.69, 19.11]
0.4.	13.62	[8.99, 18.25]	[9.69, 19.13]
0.6	13.64	[8.99, 18.28]	[9.70, 19.18]
0.8	13.66	[8.98, 18.33]	[9.70, 19.23]
1	13.68	[8.98, 18.38]	[9.70, 19.29]
0.12*	**13.60**	**[8.94, 18.26]**	**[9.66, 19.16]**
$x_2 = 2.44$			
MLE	4.98	[2.70, 7.27]	[3.15, 7.88]
0.2	4.97	[2.69, 7.24]	[3.14, 7.85]
0.4	4.95	[2.68, 7.22]	[3.13, 7.83]
0.6	4.93	[2.67, 7.19]	[3.12, 7.79]
0.8	4.91	[2.66, 7.16]	[3.11, 7.77]
1	4.89	[2.65, 7.13]	[3.10, 7.73]
0.12*	**4.98**	**[2.68, 7.27]**	**[3.14, 7.89]**

Bold values indicate the estimated values with optimal tuning parameter

Further, robust Z-type test statistics based on the MDPDEs have been developed for linear null hypothesis and its level and power functions have been empirically and theoretically studied. Robust estimators and the associated Z-type test statistics perform slightly worse than MLE in low contaminated scenarios, but they exhibit worthwhile gain in terms of robustness when contamination is present in a cell through a contamination in any of the model parameters.

Finally, two real data examples have been analyzed to illustrate all inferential methods developed here. The data-based choice of the optimal β assists us in understanding the level of contamination present in the sample, and then choosing an estimator suitably.

DATA AVAILABILITY STATEMENT

Data sharing is not applicable to this article as no new data were created or analyzed in this study. The real datasets are publicly available on the corresponding referenced papers.

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APPENDIX A: PROOFS OF THE MAIN RESULTS

Proof of Result 2

Proof. The MDPDE is defined as the minimizer of the DPD-based loss in Equation (8), and so must satisfy:

$$\frac{\partial d_\beta(\widehat{\boldsymbol{p}}, \boldsymbol{\pi}(\boldsymbol{\theta}))}{\partial \boldsymbol{\theta}} = (\beta + 1) \sum_{j=1}^{L+1} \left(\pi_j(\boldsymbol{\theta})^{\beta-1} (\pi_j(\boldsymbol{\theta}) - \widehat{p}_j) \frac{\partial \pi_j(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} \right) = \boldsymbol{0}_2.$$

Next, the derivative of the probability of success $\pi_j(\boldsymbol{\theta})$ depends on the stress level at which the device is being tested. We denote x_i for the stress level at which the units are tested after the τ_ith inspection time. Taking derivatives in Equation

(5), we get

$$\frac{\partial \pi_j(\theta)}{\partial \theta} = \frac{\partial G_T(t_j)}{\partial \theta} - \frac{G_T(t_{j-1})}{\partial \theta}.$$

Upon, using

$$\frac{\partial \lambda_i(\theta)}{\partial \theta} = (\exp(\theta_1 x_i), \theta_0 \exp(\theta_1 x_i) x_i)^T \quad \text{and} \quad \frac{\partial a_{i-1}(\theta)}{\partial \theta} = (0, a_{i-1}^*)^T$$

with a_{i-1}^* defined in Equation (11), we have

$$\mathbf{z}_j = \frac{\partial G_T(t_j)}{\partial \theta} = e^{-\lambda_i(t_j + a_{i-1} - \tau_{i-1})} \left(\frac{\partial \lambda_i(\theta)}{\partial \theta}(t_j + a_{i-1} - \tau_{i-1}) + \lambda_i(\theta)\frac{\partial a_{i-1}(\theta)}{\partial \theta} \right)$$

$$= g_T(t_j) \begin{pmatrix} \frac{t_j + a_{i-1} - \tau_{i-1}}{\theta_0} \\ (t_j + a_{i-1} - \tau_{i-1})x_i + a_{i-1}^* \end{pmatrix}.$$

Defining the matrix $\mathbf{W}$ with rows $\mathbf{w}_j = \mathbf{z}_j - \mathbf{z}_{j-1}$, we obtain the desired expression. □

Proof of Result 3

Proof. Following Basu et al. [14], the matrices $\mathbf{J}_\beta(\theta_0)$ and $\mathbf{K}_\beta(\theta_0)$ are given by

$$\mathbf{J}_\beta(\theta_0) = \sum_{j=1}^{L+1} \mathbf{u}_j \mathbf{u}_j^T \pi_j(\theta)^{\beta+1}$$

$$\mathbf{K}_\beta(\theta_0) = \sum_{j=1}^{L+1} \mathbf{u}_j \mathbf{u}_j^T \pi_j(\theta_0)^{2\beta+1} - \left(\sum_{j=1}^{L+1} \mathbf{u}_j \pi_j(\theta_0)^{\beta+1} \right)\left(\sum_{j=1}^{L+1} \mathbf{u}_j \pi_j(\theta_0)^{\beta+1} \right)^T,$$

where

$$\mathbf{u}_j(\theta) = \frac{\partial \log(\pi_j(\theta))}{\partial \theta} = \frac{1}{\pi_j(\theta)} \frac{\partial \pi_j(\theta)}{\partial \theta} = \frac{\mathbf{w}_j}{\pi_j(\theta)}.$$

Hence, we can write

$$\mathbf{J}_\beta(\theta_0) = \sum_{j=1}^{L+1} \mathbf{w}_j \mathbf{w}_j^T \pi_j(\theta_0)^{\beta-1} = \mathbf{W}^T D_{\pi(\theta_0)}^{\beta-1} \mathbf{W}$$

$$\mathbf{K}_\beta(\theta_0) = \sum_{j=1}^{L+1} \mathbf{w}_j \mathbf{w}_j^T \pi_j(\theta_0)^{2\beta-1} - \left(\sum_{j=1}^{L+1} \mathbf{w}_j \pi_j(\theta_0)^{\beta} \right)\left(\sum_{j=1}^{L+1} \mathbf{w}_j \pi_j(\theta_0)^{\beta} \right)^T$$

$$= \mathbf{W}^T \left(D_{\pi(\theta_0)}^{2\beta-1} - \pi(\theta_0)^{\beta} \pi(\theta_0)^{\beta T} \right) \mathbf{W}. \qquad \square$$

Proof of Result 5

Proof. For notational convenience, let us define $\varepsilon, \theta_\varepsilon = \mathbf{T}_\beta(\mathbf{G}_\varepsilon)$. Here, $\mathbf{G}_\varepsilon = (1-\varepsilon)F_{\theta_0} + \varepsilon\Delta_n$. The MDPDE is the minimum of the DPD between $\boldsymbol{\pi}(\theta)$ and $\mathbf{g}_\varepsilon$, and so must satisfy

$$\sum_{j=1}^{L+1} \pi_j(\theta_\varepsilon)^{\beta-1} \left[(\pi_j(\theta_\varepsilon) - \mathbf{g}_{\varepsilon,j})\frac{\partial \pi_j(\theta_\varepsilon)}{\partial \theta} \right] = \mathbf{0}. \tag{A.1}$$

Implicitly differentiating in the estimating equation (A.1), we obtain

$$\sum_{j=1}^{L+1} (\beta - 1)\pi_j(\theta_\varepsilon)^{\beta-2} \frac{\partial \pi_j(\theta_\varepsilon)}{\partial \theta} \frac{\partial \theta_\varepsilon}{\partial \varepsilon} \left[(\pi_j(\theta_\varepsilon) - \boldsymbol{g}_{\varepsilon,j}) \frac{\partial \pi_j(\theta_\varepsilon)}{\partial \theta} \right]$$

$$+ \pi_j(\theta_\varepsilon)^{\beta-1} \left[\left(\frac{\partial \pi_j(\theta_\varepsilon)}{\partial \theta} \frac{\partial \theta_\varepsilon}{\partial \varepsilon} - \frac{\boldsymbol{g}_{\varepsilon,j}}{\partial \varepsilon} \right) \frac{\partial \pi_j(\theta_\varepsilon)}{\partial \theta} + (\pi_j(\theta_\varepsilon) - \boldsymbol{g}_{\varepsilon,j}) \frac{\partial^2 \pi_j(\theta_\varepsilon)}{\partial \theta^2} \frac{\partial \theta_\varepsilon}{\partial \varepsilon} \right] = \boldsymbol{0}.$$

Upon using $\boldsymbol{g}_0 = \boldsymbol{\pi}(\theta_0)$ and evaluating at $\varepsilon = 0$, we get

$$\sum_{j=1}^{L+1} \pi_j(\theta_0)^{\beta-1} \left[\left(\frac{\partial \pi_j(\theta_0)}{\partial \theta} \right)^2 \mathrm{IF}(\boldsymbol{n}, \boldsymbol{T}_\beta, \boldsymbol{G}) - \left(\frac{\partial \pi_j(\theta_0)}{\partial \theta} \right)(-\pi_j(\theta_0) + \Delta_{\boldsymbol{n}}) \right] = \boldsymbol{0}.$$

Writing the obtained equations in matrix form

$$\boldsymbol{W}^T D_{\pi(\theta_0)}^{\beta-1} \boldsymbol{W} \cdot \mathrm{IF}(\boldsymbol{n}, \boldsymbol{T}_\beta, \boldsymbol{G}) - \boldsymbol{W}^T D_{\pi(\theta_0)}^{\beta-1}(-\boldsymbol{\pi}(\theta_0) + \Delta_{\boldsymbol{n}})$$

and solving for $\mathrm{IF}(\boldsymbol{n}, \boldsymbol{T}_\beta, \boldsymbol{G})$, we obtain the desired expression. □

Proof of the Result 11
Proof. Under the null hypothesis, we have

$$\boldsymbol{m}^T \widehat{\theta}^\beta - d = \boldsymbol{m}^T(\widehat{\theta}^\beta - \theta_0).$$

Then, from Result 3, we know that

$$\sqrt{N}\left(\widehat{\theta}^\beta - \theta_0 \right) \to \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{J}_\beta^{-1}(\theta_0)\boldsymbol{K}_\beta(\theta_0)\boldsymbol{J}_\beta^{-1}(\theta_0) \right)$$

from which it follows that

$$\sqrt{N}\left(\boldsymbol{m}^T \widehat{\theta}^\beta - d \right) \to \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{m}^T \boldsymbol{J}_\beta^{-1}(\theta_0)\boldsymbol{K}_\beta(\theta_0)\boldsymbol{J}_\beta^{-1}(\theta_0)\boldsymbol{m} \right),$$

and then transforming it, we obtain

$$\sqrt{N}\left(\boldsymbol{m}^T \boldsymbol{J}_\beta^{-1}(\widehat{\theta}^\beta)\boldsymbol{K}_\beta(\widehat{\theta}^\beta)\boldsymbol{J}_\beta^{-1}(\widehat{\theta}^\beta)\boldsymbol{m} \right)^{-1/2} \left(\boldsymbol{m}^T \widehat{\theta}^\beta - d \right) \to \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_{r \times r}).$$

Now, as $\widehat{\theta}^\beta$ is a consistent estimator of θ_0, the stated result follows from Slutsky's theorem. □

Proof of the Result 13
Proof. We can rewrite

$$\boldsymbol{m}^T \widehat{\theta}^\beta - d = \boldsymbol{m}^T \theta_L - d + \boldsymbol{m}^T(\widehat{\theta}^\beta - \theta_L) = \frac{1}{\sqrt{N}} \boldsymbol{m}^T \ell + \boldsymbol{m}^T(\widehat{\theta}^\beta - \theta_L)$$

and then

$$\sqrt{N}(\boldsymbol{m}^T \widehat{\theta}^\beta - d) = \boldsymbol{m}^T \ell + \boldsymbol{m}^T \sqrt{N}(\widehat{\theta}^\beta - \theta_L).$$

But, from Result 3, we know that

$$\sqrt{N}\left(\widehat{\theta}^\beta - \theta_L \right) \xrightarrow[L \to \infty]{L} \mathcal{N}\left(\boldsymbol{0}, \boldsymbol{J}_\beta^{-1}(\theta_L)\boldsymbol{K}_\beta(\theta_L)\boldsymbol{J}_\beta^{-1}(\theta_L) \right).$$

Therefore,

$$\sqrt{N}(\boldsymbol{m}^T\widehat{\boldsymbol{\theta}}^\beta - d) \xrightarrow[L\to\infty]{L} \mathcal{N}\left(\boldsymbol{m}^T\boldsymbol{\ell}, \boldsymbol{m}^T\boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}_L)\boldsymbol{K}_\beta(\boldsymbol{\theta}_L)\boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}_L)\boldsymbol{m}\right)$$

and so

$$\frac{\sqrt{N}(\boldsymbol{m}^T\widehat{\boldsymbol{\theta}}^\beta - d) - \boldsymbol{m}^T\boldsymbol{\ell}}{\sqrt{\boldsymbol{m}^T\boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}_L)\boldsymbol{K}_\beta(\boldsymbol{\theta}_L)\boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}_L)\boldsymbol{m}}} \xrightarrow[L\to\infty]{L} \mathcal{N}(0,1).$$

As $\widehat{\boldsymbol{\theta}}^\beta \xrightarrow{P} \boldsymbol{\theta}_L$, the stated result follows from Slutsky's theorem. $\qquad\square$

Proof of the Result 14

Proof. The power function is the probability of rejection, given the critical region in Equation (24). Thus,

$$\beta_N(\boldsymbol{\theta}^*) = \mathbb{P}\left(|Z_N(\widehat{\boldsymbol{\theta}}^\beta)| > z_{\alpha/2}|\boldsymbol{\theta} = \boldsymbol{\theta}^*\right)$$

$$= 2\mathbb{P}\left(Z_N(\widehat{\boldsymbol{\theta}}^\beta) > z_{\alpha/2}|\boldsymbol{\theta} = \boldsymbol{\theta}^*\right)$$

$$= 2\mathbb{P}\left(\sqrt{\frac{N}{\boldsymbol{m}^T\boldsymbol{J}_\beta^{-1}(\widehat{\boldsymbol{\theta}}^\beta)\boldsymbol{K}_\beta(\widehat{\boldsymbol{\theta}}^\beta)\boldsymbol{J}_\beta^{-1}(\widehat{\boldsymbol{\theta}}^\beta)\boldsymbol{m}}}\left(\boldsymbol{m}^T\widehat{\boldsymbol{\theta}}^\beta - \boldsymbol{\theta}^*\right)\right.$$

$$\left. > z_{\alpha/2} - \sqrt{\frac{N}{\boldsymbol{m}^T\boldsymbol{J}_\beta^{-1}(\widehat{\boldsymbol{\theta}}^\beta)\boldsymbol{K}_\beta(\widehat{\boldsymbol{\theta}}^\beta)\boldsymbol{J}_\beta^{-1}(\widehat{\boldsymbol{\theta}}^\beta)\boldsymbol{m}}}(\boldsymbol{\theta}^* - d)\right).$$

As $\widehat{\boldsymbol{\theta}}^\beta \xrightarrow{P} \boldsymbol{\theta}^*$ and $\sqrt{N}(\widehat{\boldsymbol{\theta}}^\beta - \boldsymbol{\theta}^*) \xrightarrow[L\to\infty]{L} \mathcal{N}(\boldsymbol{0}, \boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}_L)\boldsymbol{K}_\beta(\boldsymbol{\theta}_L)\boldsymbol{J}_\beta^{-1}(\boldsymbol{\theta}_L))$, the result follows from Slutsky's theorem. $\qquad\square$