Technometrics

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/utch20

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TECHNOMETRICS
2023, VOL. 65, NO. 3, 418–431
https://doi.org/10.1080/00401706.2023.2177351

On Censoring Time in Statistical Monitoring of Lifetime Data

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ABSTRACT
Life tests for highly reliable products often take a long time even using accelerated life testing with censoring. When the production process is monitored by control charts with the product lifetime as the key quality characteristic, the time spent on life testing could incur significant delays for practitioners to make decisions after sampling. However, shortening the test duration, that results in excessive right-censored observations, inevitably degrades the test power for anomaly detection. This study pays close attention to the determination of censoring time in life tests when monitoring lifetime data with the likelihood-based control charts. To interpret the optimal censoring time, the performance metric—out-of-control average time to signal (OC ATS), is deconstructed into two parts: the original OC ATS and the delay caused by life testing. Finite-sample analytical and large-sample asymptotic expressions of ATS metrics are derived for Type-I censored exponential lifetimes. Similar analytical expressions are also derived for the Weibull case. For general distributions, a Monte Carlo simulation procedure is developed for obtaining approximate results. Our numerical investigation uncovers the 2-fold impact of censoring time on the actual performance of control charts under various scenarios and provides useful references for practitioners to set more sensible censoring times in life testing.

ARTICLE HISTORY
Received May 2022
Accepted February 2023

KEYWORDS
Average time to signal; Exponential distribution; Life testing; Optimal censoring time; Statistical process control; Weibull distribution

1. Introduction

The lifetime of a product, namely, the time-span during which a product is expected to function normally under appropriate operating conditions before a failure occurs, is a key characteristic of product quality (Dickinson et al. 2014). To ensure continuous production of qualified product units, it is necessary for practitioners to monitor the production process and identify if it still operates In Control (IC) or some unfavorable changes have occurred that might decrease the mean lifetime of units. As routine practice, practitioners perform suitable life tests on samples of the process output at regular intervals, that is, sampling points. Typically, a life test involves testing a small batch of n units until the units fail. Statistical Process Control (SPC) techniques can be used for monitoring these observed failure times or lifetimes so as to help detect underlying process changes as early as possible. Unlike classic industrial quality control problems, lifetime data cannot be immediately available when batches of product units are sampled from the production process.

Nowadays, owing to the rapid development of technology, the expected lifetime of modern products is prolonged. Some products, for example, hard disks, lithium batteries, liquid crystal displays, have very high reliability, meaning that the time to failure of such products could be fairly long, say, years. Even using accelerated testing, it still takes a considerable amount of time (and money) to conduct a complete life test by collecting the exact time to failure of each unit under testing. The time is much longer than the time interval of routine sampling. As a consequence, for each sample drawn from the production process, practitioners often need to wait at least a couple of sampling intervals in time until test results to be available. There is potentially a significant delay for the practitioner to take necessary remedial actions based on the test result since the sample was drawn. Such a delay of correct action signals would lead to huge economic penalties if the production process has gone Out-of-Control (OC). Because of this, terminating the life test at a suitable time point without waiting for all units to fail, is a common practice and is essential for rapid and regular feedbacks, especially earlier warning of abnormal operation. In this regard, right censoring arises and has been widely adopted in various industrial settings.

Right censoring can be classified into two main types (Meeker and Escobar 1998): Type-I censoring occurs if the life test is stopped at a predetermined time, c, and Type-II censoring occurs if the test is stopped right after a predetermined number, s, of the units fail. The testing of the remaining units that survive at that point is suspended and these suspended times are recorded as right-censored observations. In other words, an observation is right-censored if its time to failure is only known to be later than a specific time. This means that censoring inevitably leads to a certain degree of information loss. In consequence, traditional control charts designed for complete data become less effective when censoring occurs. A variety of new control charts have thus been proposed to monitor such right-censored data from different sources. Some early

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Supplementary materials for this article are available online. Please go to www.tandfonline.com/r/TECH.

attempts can be found in Steiner and MacKay (2000, 2001a, 2001b). Recent research on the monitoring of Type-I right-censored lifetime data includes, Dickinson et al. (2014), Raza, Riaz, and Ali (2016), Zhang, Tsung, and Xiang (2016), Xu and Jeske (2018), Bizuneh and Wang (2019), Panza and Vargas (2020), Lee (2021), Ali et al. (2021), and Ahmed, Ali, and Shah (2022), among others; in the case of Type-II censoring, readers are referred to Guo and Wang (2014), Chan, Han, and Pascual (2015), Haghighi, Pascual, and Castagliola (2015), Asadzadeh and Kiadaliry (2017), Huang, Yang, and Xie (2017), Wang, Bizuneh, and Cheng (2018), Khan et al. (2019), and Li et al. (2021), among others.

Censoring mechanisms shorten the test/waiting duration, but they also compromise the inference on the actual process state due to information loss (Asadzadeh 2022). In other words, it may not be an easy decision, which requires more in-depth analysis and validation. However, to our knowledge, in the existing literature, the observations related to "lifetime of products" are nearly always implicitly assumed to be immediately available as was the case in the classic SPC setup, whether censoring is considered or not. This is improper in many real applications and can cause problems, especially when monitoring the lifetime of highly reliable products. We can expect that the actual performance of those control charts would be greatly affected by delayed signals, if not properly treated.

An industrial example of lifetime monitoring was reported in Steiner and MacKay (2001a), where the manufacturer monitored the rust-resistant capabilities of the painting process in the manufacture of outdoor metal electrical boxes. This example has also been widely followed in the literature; see for example, Zhang and Chen (2004), Dickinson et al. (2014), Xu and Jeske (2018), and Bizuneh and Wang (2019). In this example, a life test consisted of placing three cut panels each from a test box, scratched in a prescribed manner, into a salt spray chamber maintained at 30°C. This is the typical application of accelerated salt fog endurance test, otherwise the boxes would be likely to take years at a minimum before rust is visible even under poor conditions. The test units in the chamber were checked daily for rust, and remained in the chamber until rust appeared or a maximum of 20 days, which can be seen as Type-I right censoring. This also implies that, to obtain the life test result for each batch of products sampled from the OC production process could take additional 20 days at most. It really goes on too long and would result in production of massive defectives along with economic losses to the manufacturer. Naturally, several questions arise: can one shorten the rust test duration, that is, < 20 days, and if yes, how short is it optimum? Unfortunately this fundamental issue is largely ignored by either Steiner and MacKay (2001a) or other relevant studies in their designs, and consequently, there is no concrete answer in the existing literature. More such examples can be found in such as Lee (2021) for monitoring the lifetime of LCD modules, Huang, Yang, and Xie (2017) for the failures of organic LEDs in accelerated life test, Li and Kong (2015) for the wheel failures on the train equipments, and Li et al. (2012) for the failure times of CT machines.

Although particular emphasis is placed on the statistical monitoring of censored lifetime data from industrial manufacturing processes, a similar situation also exists in other fields such as health care, service operation, and disaster management. For example, as a typical application in health care, the common 30 days rule (patients who died within the first 30 days after an operation) is widely adopted in the risk-adjusted monitoring of surgical outcomes. The 30 days limit serves as censoring time and the waiting times can result in delay in signaling a deterioration of a surgeon's performance. Several general reasons for using the 30 days limit were discussed in the literature, for example, Sego, Reynolds Jr, and Woodall (2009), Steiner and Jones (2010), Keefe et al. (2017), and Gan, Yuen, and Knoth (2020), but more in-depth analysis with well-developed mathematical models about the determination of the optimum limit is scarce. Admittedly, monitoring clinical outcomes tends to be more complicated than monitoring industrial manufacturing processes.

In this study, we attempt to shed some light on this fundamental issue. The discussion is confined to the case of Type-I right censoring. Under Type-I censoring, the duration of a life test is constrained by time, known as censoring time. The censoring time affects the observation of product lifetimes and subsequently, the estimation on the actual state of the process. This means that decreasing the predetermined censoring time leads to a shorter delay of correct signals, but also impairs the ability to recognize unfavorable changes in the expected lifetime of the product, and vice versa. Therefore, there could exist an optimum selection of censoring time (Lee 2021), when the criterion is to minimize the time to observe the correct action signal once the process has gone OC. This study aims at uncovering the overall impact of censoring time on the detection performance measured in terms of the Average Time to Signal (ATS) and exploring its optimal value under various scenarios.

The remainder of this article is organized as follows. Section 2 presents the general problem setting of monitoring type-I censored lifetimes and introduces the ATS metrics used for measuring the performance of the monitoring procedure, taking into account the delay incurred by the life testing. Section 3 discusses the monitoring of censored exponential lifetimes and derives the finite-sample analytical expressions and large-sample asymptotic expressions for our performance metrics. Section 4 further investigates the case of the Weibull distribution and also presents a procedure of approximating performance metrics via the Monte Carlo simulation for general lifetime distributions. In Section 5, numerical studies are carried out and some graphic results are reported. A synthetic example on lifetimes of Liquid Crystal Display (LCD) modules and the rust test example of Steiner and MacKay (2001a) are further analyzed and discussed in Section 6, with illustrations for practitioners and the answers to the aforementioned research questions. Finally, Section 7 closes this study with concluding remarks.

2. Problem Formulation

2.1. Monitoring Type-I Censored Lifetimes

Consider monitoring a production process that is set up for certain products, say, hard disks. Reliability is often the primary quality characteristic of such products and the mean time to failure is a key factor when interpreting product reliability. It is customary to use a parametric lifetime distribution, for example, the exponential distribution, Weibull distribution, gamma

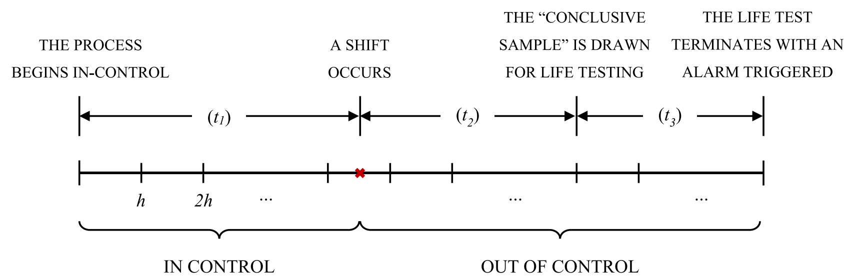

Figure 1. The IC period and OC period in a process monitoring cycle.

distribution, etc., to model the time to failure as a nonnegative continuous random variable (r.v.) T. The cdf and pdf of T are denoted by $F(t|\boldsymbol{\theta})$ and $f(t|\boldsymbol{\theta})$ where $\boldsymbol{\theta}$ is the parameter vector.

Define $\mu = \mathbb{E}[T]$ to be the mean time to failure of the product. When the process is operating IC, the distribution of T of units produced remains favorably stable with mean μ_0 where μ_0 is the IC value of μ. However, the process could be disturbed by the occurrence of some assignable causes at an unknown time point. The occurrence of an assignable cause is often not directly observable but incurs a shift in the parameter(s) of the underlying lifetime distribution and consequently, abnormal variation in T. In practice, the manufacturer is more interested in the occurrence of a decreasing shift in $\mathbb{E}[T]$ since this indicates deterioration in reliability. Thus, one wish to test the hypothesis

$$H_0 : \mu \geq \mu_0 \leftrightarrow H_1 : \mu < \mu_0.$$

Usually, μ_0 is designed by the manufacturer and is assumed known a priori.

To this end, an SPC procedure is implemented with samples of size n taken at intervals of h units of time, for example, hours or days, from the production process. Life testing is conducted on these samples under specified conditions in successive order over time. For each sample, all n units are put on test at once. Because of high reliability, the test is terminated at a predetermined censoring time c. Any surviving unit(s) are removed from the test and these are recorded as right-censored data. The observations of each sample, including failure times and censored lifetimes, are recorded and used for control charting. Since the purpose of this study is to detect the deterioration in reliability of products due to the occurrence of some assignable causes, the accelerated nature of the test may be ignored given that the testing is performed in the same manner and the relationship between accelerated and normal conditions does not change; see Steiner and MacKay (2001a). Therefore, we do not distinguish T under the accelerated or normal conditions.

In this study, a simple likelihood-based monitoring and charting procedure is applied to censored lifetime data. Specifically, we consider the plotting statistic based on the Maximum Likelihood Estimate (MLE) of μ, as follows

$$\hat{\mu}_i = u(T_{i1}^{(c)}, \ldots, T_{in}^{(c)}), i \geq 1$$

where $T_{ij}^{(c)} = \min(T_{ij}, c)$ is the observed value from the life test with right censoring for the jth unit of the ith sample, T_{ij}. $\hat{\mu}_i$ is plotted sequentially over time against a Lower Control Limit (LCL) H, on the chart. Determination of H is discussed later in subsequent sections for particular distributions. As long as the points plotted are above the LCL, the process is deemed to remain IC without having to do anything. However, if $\hat{\mu}_i$ falls below the LCL, as an evidence that the process is OC at the ith sample, an alarm is triggered. Then, an investigation for possible assignable cause(s) should be initiated. The remedial action will be implemented if the assignable cause is confirmed after the investigation. Otherwise, this signal will be regarded as a false alarm and, no more action is necessary.

2.2. Performance Metrics

The performance of the SPC procedure can be measured in terms of ATS, which is a widely-adopted performance metric (Montgomery 2013). The procedure is typically designed in a manner that its IC ATS (denoted as ATS^{ic}) is equal to a specified value while the OC ATS (denoted as ATS^{oc}) should be as small as possible. Specifically, here ATS^{ic} means the average time between two successive false alarms when the process is IC and, ATS^{oc} measures the average time from the change occurred in the process to the time of the first signal, that is, true alarm, by the chart. Therefore, this design aims at minimizing the expected length of the OC period while controlling the probability of false alarm at an acceptable level during the IC period; see Figure 1.

To elicit the ATS metrics, the following assumptions are considered: (a) Some time components, that is, the time to draw a sample for life testing and the time to analyze and interpret the results, are short enough to be negligible. (b) During the life test and the search for the assignable cause(s), the process continues in operation. (c) The sampling interval h is fixed based on operational conditions and let $h = 1$ without loss of generality.

By definition, the IC ATS remains to be of the simple form,

$$\text{ATS}^{\text{ic}} = \left(\frac{1}{\alpha}\right) \cdot h$$

where α is the acceptable probability of Type-I error. It is worth noting that, we did not use a more common definition of ATS^{ic}—the average time from the startup to the first false alarm. These two definitions are essentially equivalent in the classic SPC setup, but become different in the current context, as the time spent on life testing (until the false alarm triggered) can be significant. Nevertheless, the time spent on life testing for two successive false alarms can be offset on average when the process stays IC. Therefore, the metric—the average time between two successive false alarms, or alternatively the average number of false alarms per unit of time, is a better option to reflect the IC performance of a control chart. This can also be viewed as using the notion of controlling the False Alarm Rate (FAR) to design the chart.

The OC ATS in the current context is divided into two parts: t_2 and t_3 as in Figure 1. The first part of our OC ATS (denoted as ATS_1^{oc}) is the average time from the occurrence of the process shift to the time that the sample is drawn which will release the signal after the life test (called the "conclusive sample"). Precisely, ATS_1^{oc} here is equivalent in form to the OC ATS in traditional SPC research. This part is evaluated under the steady-state mode in which a process is assumed to have reached its stationary distribution when the shift occurs and the time of a process shift follows a uniform distribution within the sampling interval; see Reynolds, Amin, and Arnold (1990). Subsequently, the mean time that the shift occurs is at the center of a sampling interval. Hence, ATS_1^{oc} takes the following form,

$$\text{ATS}_1^{\text{oc}} = \left(\frac{1}{1-\beta} - 0.5\right) \cdot h, \tag{1}$$

where β is the probability of Type-II error and $1 - \beta$ is the probability of true alarm (power).

The second part of our OC ATS (denoted as ATS_2^{oc}) is the expected duration (i.e., the delay) of the life test that gives the correct action signal. This part is ignored in the relevant literature as previously discussed. In fact, this duration can be very long especially when monitoring the lifetime of highly reliable products, so that ATS_2^{oc} is comparable with ATS_1^{oc}. Hence, it is necessary to take ATS_2^{oc} into consideration when evaluating ATS^{oc} (from the shift to the signal). The complete expression of ATS_2^{oc} is a bit complex, but it has the following conceptual form as a conditional expectation,

$$\text{ATS}_2^{\text{oc}} = \mathbb{E}[M_i | \hat{\mu}_i < H \text{ when OC}] \tag{2}$$

where $M_i = \max(T_{i1}^{(c)}, \dots, T_{in}^{(c)}), i \geq 1$. By definition the actual duration of the life test $M_i \leq c$. One has $M_i < c$ only if all units fail during testing, in which case M_i is equal to the time to the last failure. The composite ATS^{oc} is thus the sum of ATS_1^{oc} and ATS_2^{oc} as

$$\text{ATS}^{\text{oc}} = \text{ATS}_1^{\text{oc}} + \text{ATS}_2^{\text{oc}}.$$

Actually, those earlier studies ignoring ATS_2^{oc}, can be regarded as implicitly relying on the simpler relationship $\text{ATS}^{\text{oc}} = \text{ATS}_1^{\text{oc}} + c$, using c instead of ATS_2^{oc}. When not focusing on the optimal determination of c that minimizes ATS^{oc}, explicit incorporation of c for setting up and comparing control charts is needless, as ATS^{oc} and ATS_1^{oc} are consistent for any prespecified and fixed c in this case. Although using c to approximate ATS_2^{oc} seems a natural starting point for the current study to search for the optimal value of c, later we will show that it is not a simple linear relationship between c and ATS_2^{oc}. It means that we need to be careful with this approximation.

It is not difficult to learn that c has the opposite impact on ATS_1^{oc} and ATS_2^{oc}. Intuitively, the power of the test $1 - \beta$ tends to increase if c becomes larger since more failure information would be observed which makes the identification of the process shift easier. Therefore, ATS_1^{oc} in (1) tends to decrease when a larger c is set. Nevertheless, with a larger c, the life test will last longer which increases ATS_2^{oc}. Since the composite ATS^{oc} consists of both ATS_1^{oc} and ATS_2^{oc}, the determination of the censoring time c and the corresponding (one-sided) LCL H is important for minimizing ATS^{oc} while controlling ATS^{ic}. By and large, the tradeoff between ATS_1^{oc} and ATS_2^{oc} plays a central role in this study.

Remark. According to the present setting, it may be the case that multiple samples, beginning the life test at intervals of h units of time, evolve in parallel. This could lead to an interesting phenomenon. It is not necessarily the first OC sample (say, i) that would find $\hat{\mu}_i < H$ triggering an alarm at the earliest. During the testing for this sample, if another sample j ($j > i$) that would also find $\hat{\mu}_j < H$, completes the test much more quickly, it is possible that sample j could trigger an alarm earlier than sample i (we call this phenomenon, competing alarms). This does make analysis more complicated. Our definition of ATS^{ic} and ATS^{oc} and related discussion above are nicely fit for a simple situation without competing alarms. It is easy to derive that, ATS_2^{oc} can serve as an upper bound of the second part of the composite ATS^{oc} and likewise ATS_1^{oc} as a lower bound of the first part of ATS^{oc}, when considering competing alarms. However, our preliminary numerical analysis reveals that the probability and impact of competing alarms are negligible or small in most cases, unless $1 - \beta$ is close to one and c as well as μ is large enough. Hence, for simplicity, we do not consider competing alarms in this study, but a more thorough investigation can be conducted as an interesting future study.

3. Monitoring Censored Exponential Lifetime Data

In this section, our primary concerns is with the exponential lifetime data. The exponential distribution is the most representative lifetime distribution model and also a special case of the Weibull distribution and gamma distribution. Later, the discussion will also be extended to other distribution (i.e., the Weibull distribution) for more general settings.

3.1. Censored Exponential MLE Control Chart

Let $T \sim \text{Exp}(\lambda)$ with rate parameter $\lambda > 0$ and cdf $F_e(t) = 1 - e^{-\lambda t}$. For monitoring censored exponential lifetimes, we use a control chart (hereafter, censored exponential MLE) based on the well-known MLE of μ as the plotting statistic, as follows

$$\hat{\mu}_i = \frac{1}{r_i} \sum_{j=1}^{n} T_{ij}^{(c)}, i \geq 1$$

where r_i is the number of units that have failed before the censoring time, of the ith sample. In particular, when there is no failure occurring in the ith sample, that is, $r_i = 0$, one can define $\hat{\mu}_i = +\infty > H$. In this case, it is more convincing that the process is IC. To avoid division by zero, one can use $1/\hat{\mu}_i$ as the plotting statistic against an Upper Control Limit (UCL). In what follows, the analytical expressions of ATS metrics for the censored exponential MLE control chart is derived and their asymptotic properties are investigated.

3.2. Derivation of Closed-Form ATS Metrics

This section presents the first theoretical result, that is, the analytical expressions for ATS^{ic}, ATS_1^{oc}, and ATS_2^{oc}. Let $T_1, \dots, T_n \sim \text{Exp}(\lambda)$ and $T_j^{(c)} = \min(T_j, c)$ where c is the censoring time. $T_j^{(c)}$ has a point mass at c as $\Pr(T_j^{(c)} = c) = \Pr(T_j \geq c) = e^{-\lambda c}$. Here, the subscript i in $T_{i1}, \dots, T_{in}$ is dropped for simplicity. Let $S = \sum_{j=1}^{n} T_j^{(c)}$ and $\hat{\mu} = \sum_{j=1}^{n} T_j^{(c)}/r$ where r is the observed number of failure in the sample of size n. We further introduce the truncated exponential distribution $\tilde{T}_j^{(\tau)}$ on the interval $[0, \tau]$ with the cdf as $F(t) = \frac{1-e^{-\lambda t}}{1-e^{-\lambda \tau}}$.

Lemma 1 (Bartholomew 1963). The cdf of $\hat{\mu} = \sum_{j=1}^{n} T_i^{(c)}/r$ conditional upon r being nonzero is given by

$$F_{\hat{\mu}|r>0}(t) = \Pr(\hat{\mu} \le t | r > 0) = 1 - \frac{1}{1-e^{-n\lambda c}} \sum_{j=1}^{n} \binom{n}{j} \sum_{k=0}^{j} (-1)^k \binom{j}{k} e^{-(n-j+k)\lambda c} [1 - Q_{2j}(x)]$$

where $x = 2\lambda \max[tj - c(n-j+k), 0]$ and, $Q_v(u)$ denotes $\Pr(\chi_v^2 < u)$ where χ_v^2 is a Chi-squared distributed r.v. with v degrees of freedom.

Lemma 1 is required to derive the analytical forms of ATS^{ic} and ATS_1^{oc}. It concerns the cdf of the plotting statistic given that the failure number is nonzero, that is, $\hat{\mu}|r > 0$. The Chi-squared distributions with even degrees of freedom in Lemma 1 are essentially Erlang distributions which characterize the sum of iid exponential r.v.s. Simply following Lemma 1, the signal probability regardless of the nonzero restriction on r is

$$\begin{aligned}\Pr(\hat{\mu} < H) &= \Pr(\hat{\mu} < H | r > 0) \Pr(r > 0) \\ &\quad + \Pr(\hat{\mu} < H | r = 0) \Pr(r = 0) \\ &= F_{\hat{\mu}|r>0}(H) \cdot (1 - e^{-n\lambda c}).\end{aligned} \tag{3}$$

The second term in (3) diminishes as one always has $\hat{\mu} = +\infty > H$ if $r = 0$ where no signal is triggered. The first term can be calculated easily as a product of two probabilities, that is, $\Pr(\hat{\mu} < H | r > 0)$ given by Lemma 1 and $\Pr(r > 0) = 1 - \Pr(r = 0) = 1 - \Pr(T_1 > c, \ldots, T_n > c) = 1 - e^{-n\lambda c}$ the probability of observing at least one failure.

Hence, it leads to the analytical expressions of ATS^{ic} and ATS_1^{oc} as

$$\text{ATS}^{\text{ic}} = \left(\frac{1}{\alpha}\right) \cdot h = \left(\frac{1}{F_{\hat{\mu}|r>0;\lambda=\lambda_0}(H) \cdot (1 - e^{-n\lambda_0 c})}\right) \cdot h, \tag{4}$$

$$\begin{aligned}\text{ATS}_1^{\text{oc}} &= \left(\frac{1}{1-\beta} - 0.5\right) \cdot h \\ &= \left(\frac{1}{F_{\hat{\mu}|r>0;\lambda=\lambda_1}(H) \cdot (1 - e^{-n\lambda_1 c})} - 0.5\right) \cdot h,\end{aligned} \tag{5}$$

where λ_0 and λ_1 are the IC and OC rate parameters, respectively. Note that, one can also replace λ_0 and λ_1 with the reciprocals of μ_0 and μ_1, where μ_0 and μ_1 are the corresponding expected IC and OC lifetimes. Then, by setting ATS^{ic} equal to some desired value in (4) depending on α and simplifying the expression of ATS^{ic} for a given c, the LCL $H_n(c|\lambda_0)$ can be determined by solving a quantile of $\hat{\mu}|r > 0; \lambda = \lambda_0$ as

$$H_n(c|\lambda_0) = F^{-1}_{\hat{\mu}|r>0;\lambda=\lambda_0}\left(\frac{\alpha}{1 - e^{-n\lambda_0 c}}\right). \tag{6}$$

Since $F_{\hat{\mu}|r>0;\lambda=\lambda_0}$ is a cdf, $H_n(c|\lambda_0)$ is uniquely determined by (6) if $\alpha < 1 - e^{-n\lambda_0 c}$. This can always be achieved by selecting a censoring time $c > -\log(1-\alpha)/(n\lambda_0)$ given any sample size n. Therefore, it is natural to make the following Assumption 1.

Assumption 1. There exists an $\epsilon > 0$, such that, $c \ge \epsilon - \log(1-\alpha)/(n\lambda_0)$, that is, the censoring time c must be bounded away from $-\log(1-\alpha)/(n\lambda_0)$.

The analytical expression of ATS_2^{oc} is more complicated as follows. Details of derivations are given in the Supplementary Materials. The analytical forms of ATS can help us determine the optimal c without seeking the aid of Monte Carlo simulation which can be inefficient when dealing with the conditional expectation. However, one should be cautious that these analytical forms can be numerically unstable if n goes large since the formulas involve quite a few combinations and permutations. From our numerical studies in R and Matlab, it is found that the analytical forms work well when $n < 30$ which includes reasonable sample sizes for SPC applications.

$$\text{ATS}_2^{\text{oc}} = \begin{cases} c - \dfrac{c + \sum_{j=1}^{n} (-1)^j \binom{n}{j} (1 - e^{-j\lambda_1 c})/(\lambda_1 j)}{(1 - e^{-n\lambda_1 c}) \cdot F_{\hat{\mu}|r>0;\lambda=\lambda_1}(H)}, & H > c, \\[2ex] c - \dfrac{H + \sum_{j=1}^{n} (-1)^j \binom{n}{j} (1 - e^{-j\lambda_1 H})/(\lambda_1 j) + \int_H^c (1 - e^{-\lambda_1 t})^n \cdot F_{\tilde{S}}^{(t)}(nH) \mathrm{d}t}{(1 - e^{-n\lambda_1 c}) \cdot F_{\hat{\mu}|r>0;\lambda=\lambda_1}(H)}, & H \le c. \end{cases} \tag{7}$$

3.3. Asymptotic Analysis

Computing exact values of OC ATS metrics based on the analytical formulas in (5) and (7) are formidable for large sample sizes. This limitation directs us to an asymptotic analysis on ATS. If $n \to \infty$, involved combinatorics in (5) and (7) can be approximated by smooth functions with simple forms. Eventually, the asymptotic ATS metrics lead to meaningful theoretical investigation on optimal censoring time.

Following Tableman and Kim (2003), the asymptotic distribution of $\hat{\mu}$ is $\sqrt{n}\left(\hat{\mu} - 1/\lambda\right) \to^{d} N(0, 1/[\lambda^2 q(c|\lambda)])$ with $q(c|\lambda) = 1 - e^{-\lambda c}$ where $\to^{d}$ denotes convergence in distribution. If $c \to 0$, $q(c|\lambda) \to 0$ and the asymptotic distribution become ill-posed while Assumption 1 prevents us from this. However, directly using the asymptotic distribution of $\hat{\mu}$ to approximate the sampling distributions of finite sample sizes is rather crude since the exact distributions of $\hat{\mu}$ could be heavily skewed for even moderate n with a small c. Tableman and Kim (2003) suggested a better approximation based on a log transform on $\hat{\mu}$. With the delta method, the asymptotic distribution of $\log(\hat{\mu})$ is given as,

$$\sqrt{n}\left(\log(\hat{\mu}) + \log(\lambda)\right) \to^{d} N\left(0, \frac{1}{q(c|\lambda)}\right). \tag{8}$$

The rest analysis is then based on the asymptotic distribution in (8). Therefore, for a sufficiently large n, the signal probability can be approximated as

$$\begin{aligned}\Pr(\hat{\mu} < H|\lambda) &= \Pr\left(\sqrt{nq(c|\lambda)}\left[\log(\hat{\mu}) + \log(\lambda)\right]\right. \\ &\quad \left. < \sqrt{nq(c|\lambda)}\left[\log(H) + \log(\lambda)\right]\right) \\ &\approx \Phi\left(\sqrt{nq(c|\lambda)}\log(\lambda H)\right), \qquad (9)\end{aligned}$$

where $\Phi(\cdot)$ is the standard normal cdf. Denote the approximate signal probability defined in (9) for the IC process $\lambda = \lambda_0$ and OC process $\lambda = \lambda_1$ by $\xi_n(c|\lambda_i) = \Phi\left(\sqrt{nq(c|\lambda_i)}\log(\lambda_i H)\right)$, which is a function of c indexed by the sample size n with $i = 0, 1$. For the IC process $\lambda = \lambda_0$, by solving $\xi_n(c|\lambda_0) = \alpha$, an approximate LCL $\tilde{H}_n(c|\lambda_0)$ is

$$\tilde{H}_n(c|\lambda_0) = \frac{1}{\lambda_0}\exp\left(\frac{\Phi^{-1}(\alpha)}{\sqrt{nq(c|\lambda_0)}}\right). \qquad (10)$$

Some proprieties of this approximate LCL can be found in the Supplementary Materials.

If the process is OC with $\lambda = \lambda_1 > \lambda_0$, the power can then be approximated by

$$\begin{aligned}1 - \beta &\approx \xi_n(c|\lambda_1) \\ &= \Phi\left(\sqrt{nq(c|\lambda_1)}\log(\lambda_1\tilde{H}_n(c|\lambda_0))\right) \\ &= \Phi\left(\sqrt{nq(c|\lambda_1)}\log\left(\frac{\lambda_1}{\lambda_0}\right) + \Phi^{-1}(\alpha)\sqrt{\frac{q(c|\lambda_1)}{q(c|\lambda_0)}}\right). \qquad (11)\end{aligned}$$

Intuitively, this approximate power shall be monotonically increasing under certain conditions. The desired monotonicity can be interpreted as that, with a larger c, more information on lifetime will be obtained in the test and the power tends to be higher. Therefore, Theorem 2 establishes the monotonicity of the approximate power $\xi_n(c|\lambda_1)$ under the Type-I censored exponential lifetime testing set-up. Details on the proof can be found in the supplementary materials. Theorem 2 also ensures that the approximate power $\xi_n(c|\lambda_1)$ is upper-bounded by $\Phi(\sqrt{n}\log(\lambda_1/\lambda_0) + \Phi^{-1}(\alpha))$ as $c \to \infty$ in (11).

Lemma 2. The approximate power $\xi_n(c|\lambda_1)$ is monotonically increasing in c if $\alpha \in (0, 1/2)$.

From (1), by setting $h = 1$, one has $\text{ATS}_1^{\text{oc}} \approx [1/\xi_n(c|\lambda_1) - 0.5]$. Furthermore, with a sufficiently large n, it is easy to figure out $\text{ATS}_2^{\text{oc}} \approx c$ since the probability of observing all units failed within the censoring time c goes to zero. Therefore, an approximate ATS^{oc} is constructed as a function of c $\tilde{\text{ATS}}^{\text{oc}}(c) = c + \frac{1}{\xi_n(c|\lambda_1)} - 0.5$. By taking the derivative of $\tilde{\text{ATS}}^{\text{oc}}(c)$, the potential optimal c could be obtained by solving the following equation

$$[\tilde{\text{ATS}}^{\text{oc}}(c)]' = 1 - \frac{\xi_n'(c|\lambda_1)}{[\xi_n(c|\lambda_1)]^2} = 0. \qquad (12)$$

Finally, Theorem 1 on the existence of a potential optimal c is presented as follows. Details on the proof can be found in the supplementary materials. With the guarantee of Theorem 1, the bisection algorithm could be used to search the solution to (12). However, one must be aware of that any feasible solution shall satisfy Assumption 1. Since the finite-sample analytical expressions could be intractable for most common lifetime distributions, the asymptotic results establish a more general way to optimize the censoring time in SPC applications on lifetimes with large sample sizes. Numerical comparisons between asymptotic analysis and finite-sample analytical formulas are summarized in the Supplementary Materials for censored exponential lifetimes.

Theorem 1. There exists at least one $c^* \in (0, \infty)$ satisfying the condition in (12).

4. Monitoring Censored Lifetime Data for Other Distributions

In Section 3, the study focuses on the exponential distribution and derive the finite-sample and asymptotic expressions of ATS metrics, and based on that, the impact of censoring time can be evaluated. This section extends discussions to statistical monitoring of lifetime data for other distributions. First, the focus moves onto the Weibull distribution, as a straightforward extension of the exponential case, and correspondingly derive relevant analytical expressions of ATS metrics. Then, a simulation-based procedure is introduced for approximate ATS metrics which can apply to any lifetime distribution.

4.1. Extension to the Censored Weibull Case

Let $T \sim \text{Weibull}(\gamma, \eta)$, where $\gamma > 0$ is the shape parameter and $\eta > 0$ is the scale parameter of the Weibull distribution. The cdf for the Weibull distribution is $F_{\text{w}}(t) = 1 - \mathrm{e}^{-(t/\eta)^\gamma}$. It is well-known that the Weibull distribution reduces to an exponential distribution when $\gamma = 1$. The mean time to failure μ for the Weibull case is $\mathbb{E}[T] = \eta\Gamma(1 + 1/\gamma)$, where $\Gamma(\cdot)$ is the gamma function. In line with Zhang and Chen (2004), Dickinson et al. (2014), Xu and Jeske (2018), among others, this study is more interested in detecting an unwanted drop in $\mathbb{E}[T]$ caused by a decrease in η assuming γ is fixed. It is easy to see that, monitoring η is equivalent to monitoring μ in this case.

Hence, for censored Weibull lifetimes, one can use a control chart similar to the censored exponential MLE control chart but with the plotting statistic changed to the MLE of η (see Balakrishnan and Kateri 2008), as follows

$$\hat{\eta}_i = \sqrt[\gamma]{\frac{1}{r_i}\sum_{j=1}^{n}\left(T_{ij}^{(c)}\right)^\gamma}, i \geq 1$$

where γ remains constant during the production process and r_i is the number of units that have failed before the censoring time, of the ith sample. Analogously, it is called the censored Weibull MLE control chart. In particular, when there is no failure occurring in the life testing for the ith sample, that is, $r_i = 0$, one can again define $\hat{\eta}_i = +\infty > H$ or simply use $1/\hat{\eta}_i$ as the plotting statistic with an UCL to avoid possible division by zero.

Interestingly, it is tractable to derive similar analytical expressions for ATS^{ic}, ATS_1^{oc} and ATS_2^{oc} after performing a power transform on $\hat{\eta}_i$. Following the invariant property of MLE and the monotonicity of the power function, the transformed plotting statistics $\hat{\eta}_i^{\gamma}$ can be used instead of $\hat{\eta}_i$ without loss of generality if the shape parameter γ is known a priori. It appears that T^{γ} has an exponential distribution of rate parameter $\eta^{-\gamma}$ (Cox and Oakes 1984). Therefore, the transformed censored Weibull lifetime $\left(T^{(c)}\right)^{\gamma}$ follows an exponential distribution of rate parameter $\eta^{-\gamma}$ truncated at c^{γ} with a point mass at $t = c^{\gamma}$.

The analytical expressions of $\mathrm{ATS}^{\mathrm{ic}}$ and $\mathrm{ATS}_1^{\mathrm{oc}}$ derived in (4) and (5) can be adapted to the censored Weibull case after the power transform with the transformed rate $\lambda_0' = \eta_0^{-\gamma}$ and $\lambda_1' = \eta_1^{-\gamma}$ and the transformed censoring threshold $c' = c^{\gamma}$ as follows

$$\mathrm{ATS}^{\mathrm{ic}} = \left(\frac{1}{F_{\hat{\mu}|r>0;\lambda=\eta_0^{-\gamma}}(H') \cdot (1 - \mathrm{e}^{-n(c/\eta_0)^{\gamma}})} \right) \cdot h, \qquad (13)$$

$$\mathrm{ATS}_1^{\mathrm{oc}} = \left(\frac{1}{F_{\hat{\mu}|r>0;\lambda=\eta_1^{-\gamma}}(H') \cdot (1 - \mathrm{e}^{-n(c/\eta_1)^{\gamma}})} - 0.5 \right) \cdot h, \qquad (14)$$

where the LCL H' is determined by solving a quantile of $\hat{\mu}|r > 0; \lambda = \eta_0^{-\gamma}$ as

$$H_n'(c|\eta_0, \gamma) = F^{-1}_{\hat{\mu}|r>0;\lambda=\eta_0^{-\gamma}} \left(\frac{\alpha}{1 - \mathrm{e}^{-n(c/\eta_0)^{\gamma}}} \right). \qquad (15)$$

$\mathrm{ATS}_2^{\mathrm{oc}}$ for the censored Weibull MLE control chart is slightly more complicated than that in (7) as follows.

$$\mathrm{ATS}_2^{\mathrm{oc}} = \begin{cases} c - \dfrac{c + \sum_{j=1}^{n} (-1)^j \binom{n}{j} \zeta(1/\gamma, j(c/\eta_1)^{\gamma}) \eta_1 / (\gamma j^{1/\gamma})}{(1 - \mathrm{e}^{-n(c/\eta_1)^{\gamma}}) \cdot F_{\hat{\mu}|r>0;\lambda=\eta_1^{-\gamma}}(H')}, & H' > c^{\gamma}, \\ c - \dfrac{(H')^{1/\gamma} + \sum_{j=1}^{n} (-1)^j \binom{n}{j} \zeta(1/\gamma, jH'/\eta_1^{\gamma}) \eta_1 / (\gamma j^{1/\gamma}) + \int_{H'}^{c^{\gamma}} \left(1 - \mathrm{e}^{-t/\eta_1^{\gamma}}\right)^n \cdot F_{\tilde{S}}^{(t)}(nH') \mathrm{d}(t^{1/\gamma})}{(1 - \mathrm{e}^{-n(c/\eta_1)^{\gamma}}) \cdot F_{\hat{\mu}|r>0;\lambda=\eta_1^{-\gamma}}(H')}, & H' \le c^{\gamma}, \end{cases}$$

where $\zeta(a, x) = \int_0^x t^{a-1} e^{-t} \mathrm{d}t$ is the lower incomplete gamma function. The derivation is similar to that of (7) and details are omitted. With analytical expressions, the impact of censoring time can be accurately evaluated when monitoring censored Weibull lifetimes.

4.2. Simulation Algorithm for ATS Metrics for General Distributions

When turning to more general lifetime distributions, analytical expressions of ATS metrics may become intractable. Nevertheless, our main idea, that is, compromising the efficacy and time owing to conducting the life test, can be generalized to any other distributions. Although one may not always be able to derive analytical results, an alternative is to use the Monte Carlo simulation with adequate replicates (say, 100,000 times) to obtain their approximate estimates. For convenience, the censored Weibull case discussed above is used to illustrate the simulation-based procedure, which is outlined in Algorithm 1.

For other lifetime distributions, the procedure is similar and only those steps related to generating random observations and computing the plotting statistics need to be adapted accordingly.

Algorithm 1: Monte Carlo simulation algorithm for approximating ATS metrics

Input: γ_0, IC Weibull shape; η_0, IC Weibull scale; n, sample size; c, censoring time; $h = 1$, sampling interval; $\alpha \in (0, 1)$, acceptable probability of Type-I error; $\rho \in (0, 1)$, OC variation factor; L_0 and L_1, Monte Carlo sample sizes.

Output: $\hat{H}$, $\hat{\mathrm{ATS}}_1^{\mathrm{oc}}$, $\hat{\mathrm{ATS}}_2^{\mathrm{oc}}$, $\hat{\mathrm{ATS}}^{\mathrm{oc}}$, Monte Carlo estimates of the lower control limit and ATS metrics.

for *i in* $1 : L_0$ **do**
  $T_{i1}, \ldots, T_{in} \sim \mathrm{Weibull}(\gamma_0, \eta_0)$ iid ; `// Generate an IC random sample`
  $T_{ij}^{(c)} \leftarrow \min(T_{ij}, c), j = 1, \ldots, n$; `// Generate the IC censored sample`
  $r_i \leftarrow \#\{T_{ij} \le c\}$; `// Count the number of observations failed`
  $\hat{\eta}_i \leftarrow \sqrt[\gamma]{\frac{1}{r_i} \sum_{j=1}^{n} (T_{ij}^{(c)})^{\gamma}}$; `// Compute the IC plotting statistic`
end
$\hat{H} \leftarrow$ the α lower empirical quantile of $\{\hat{\eta}_i\}$;
$\eta_1 \leftarrow (1 - \rho)\eta_0$; `// Get OC Weibull scale`
for *k in* $1 : L_1$ **do**
  $\ell \leftarrow 0$; `// Initialize the counter of repetitions to the first OC signal`
  $\hat{\eta}_{\ell} \leftarrow +\infty$; `// Initialize the OC plotting statistic`
  while $\hat{\eta}_{\ell} \ge \hat{H}$ **do**
    $\ell \leftarrow \ell + 1$; `// Update the counter of repetitions to the first OC signal`
    $T_{\ell 1}, \ldots, T_{\ell n} \sim \mathrm{Weibull}(\gamma_0, \eta_1)$ iid ; `// Generate an OC random sample`
    $T_{\ell j}^{(c)} \leftarrow \min(T_{\ell j}, c)$; `// Generate the OC censored sample`
    $r_{\ell} \leftarrow \#\{T_{\ell j} \le c\}$; `// Count the number of observations failed`
    $\hat{\eta}_{\ell} \leftarrow \sqrt[\gamma]{\frac{1}{r_{\ell}} \sum_{j=1}^{n} (T_{\ell j}^{(c)})^{\gamma}}$; `// Compute the OC plotting statistic`
  end
  $q_k \leftarrow \ell$; `// Record the number of repetitions to the first OC signal`
  $t_k \leftarrow \max_j\{T_{\ell j}^{(c)}\}$; `// Record the maximum of the last censored OC sample`
end
$\hat{\mathrm{ATS}}_1^{\mathrm{oc}} \leftarrow (\frac{1}{L_1} \sum_{k=1}^{L_1} q_k - 0.5) \cdot h$;
$\hat{\mathrm{ATS}}_2^{\mathrm{oc}} \leftarrow \frac{1}{L_1} \sum_{k=1}^{L_1} t_k$;
$\hat{\mathrm{ATS}}^{\mathrm{oc}} \leftarrow \hat{\mathrm{ATS}}_1^{\mathrm{oc}} + \hat{\mathrm{ATS}}_2^{\mathrm{oc}}$.

Additionally, we have compared the $\mathrm{ATS}^{\mathrm{oc}}$ results obtained by the Monte Carlo simulation with the analytical results for the censored exponential and Weibull cases. We can find a high degree of consistency between them, so that it helps confirm that

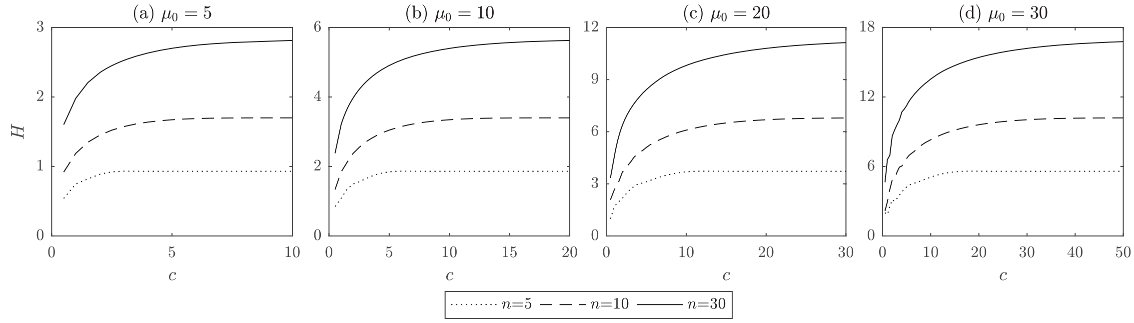

Figure 2. Plots of *H* against *c* for the censored exponential cases.

our analytical expressions of ATS metrics as well as the simulation Algorithm 1 are appropriate. These results are available from the source codes provided.

5. Numerical Investigation based on Censored Exponential

In this section, the performance of the likelihood-based control charts, designed to detect decreasing shifts in $\mathbb{E}[T]$ with various censoring time c, is evaluated in terms of the composite $\text{ATS}^{\text{oc}} = \text{ATS}_1^{\text{oc}} + \text{ATS}_2^{\text{oc}}$. Particularly, this section focuses on the censored exponential cases so that the analytical expressions of ATS metrics can be directly used to obtain accurate results. More numerical results on censored Weibull cases are reported in the supplementary materials. Throughout this section, the LCL H, given each value of c, is specified to achieve the same IC condition of $\alpha = 0.0027$; that is $\text{ATS}^{\text{ic}} = 370$.

5.1. Results on H against c

First of all, it is essential to understand the relationship between H and c under various scenarios (for the exponential case). For illustration, in Figure 2 four representative cases of the expected IC lifetime $\mu_0 = 5$, 10, 20, and 30, and three cases of the sample size $n = 5$, 10, and 30, are taken into account in order to give an overall picture of their relationship. From Figure 2, one can clearly observe that, when n is fixed, H rapidly increases with c and then approaches toward the limit—the LCL for complete samples, that is, no censoring. For a larger n, H is larger and approaches more slowly to its limit. These observations are exactly consistent in the four cases of μ_0. Moreover, when μ_0 is larger, H is also larger given that other conditions are the same.

5.2. Results on OC ATS against c

Then, we look deep into the corresponding ATS^{oc} results. Figures 3–5 show the ATS^{oc} results for $n = 5$, 10, and 30, respectively, with $\mu_0 = 5$, 10, 20, and 30. The expected OC lifetime is defined as $\mu_1 = (1 - \rho)\mu_0$, where $0 < \rho < 1$ represents the variation factor and $\rho = 0.10$, 0.25, 0.50, and 0.75, are studied for different magnitudes of the change. The three ATS^{oc} curves (ATS_1^{oc}, ATS_2^{oc} and ATS^{oc}) are plotted against c in each panel.

From Figures 3–5, one can observe that, ATS_1^{oc} decreases while ATS_2^{oc} increases with c, as expected. The shape of ATS^{oc} depends on the particular scenario of interest (relevant to μ_0, n, and ρ, especially the latter two), combining the contradictory impacts of ATS_1^{oc} and ATS_2^{oc}. One shall notice that there are three types of ATS^{oc} curves, namely, monotone decreasing, U-shape, and monotone increasing. These findings are reported as follows: (i) The monotonically decreasing ATS^{oc} occurs with a small ρ, where ATS_1^{oc} is large relative to ATS_2^{oc} and is thus dominant in ATS^{oc}; see scenario (a) of Figures 3–5 which is a typical example. It is worth noting that, due to the existence of ATS_2^{oc}, the ATS^{oc} curve is more gentle than the ATS_1^{oc} curve, so as to moderate the difference with various c. (ii) On the contrary, the monotonically increasing ATS^{oc} appears when μ_0 is small but n and ρ are large that leads to a small ATS_1^{oc}. Among all the scenarios considered, the monotonically increasing ATS^{oc} only occurs in scenario (d) of Figure 5, where ATS^{oc} is almost only affected by ATS_2^{oc}, as ATS_1^{oc} is close to 0.5 and nearly invariable. (iii) In other cases, there exists an (apparent or unapparent) U-shape relationship between ATS^{oc} and c. In particular, some scenarios show that ATS_1^{oc} and ATS_2^{oc} dominate successively and generate an intersection. When n is larger (say, $n = 30$), such a U-shape becomes more apparent even if ρ is relatively small. (iv) In any case, when c is large enough, ATS^{oc} (as well as ATS_1^{oc} and ATS_2^{oc}) approaches to the constant for no censoring (more slowly with a large μ_0).

Figures 3–5 and related discussion above, contribute to a deeper understanding of the relationship between ATS^{oc} and c. One can see that c plays an important role in monitoring censored exponential lifetime data and determining ATS^{oc}. Table 1 further summarizes the results on the minimum ATS^{oc} for each scenario. Obviously, given a larger n, or a larger ρ, the minimum ATS^{oc} significantly drops off. But, a larger μ_0 slightly increases the minimum ATS^{oc} instead. In addition to the minimum ATS^{oc}, the results on the range of choices of c that can obtain a ATS^{oc} no more than 101% the corresponding minimum, are also included in Table 1, which can be regarded as favorable choices. Note that, if the ATS^{oc} curve follows the monotone decreasing type, or has an unapparent U-shape, the range of favorable choices of c can be wide; but if the ATS^{oc} curve has an apparent U-shape, usually one has only a few choices of c (the attained ATS^{oc} is much larger with other choices of c).

To sum up, we have learned that the optimal determination of c in terms of ATS^{oc}, depends on the concerned scenario. Moreover, we gain rules of thumb for a reasonable choice of c. At least, it is not necessary to adopt a large c, even with the monotonically decreasing ATS^{oc}, as ATS^{oc} tends to be stable earlier than ATS_1^{oc}. In many cases, setting a large c will result

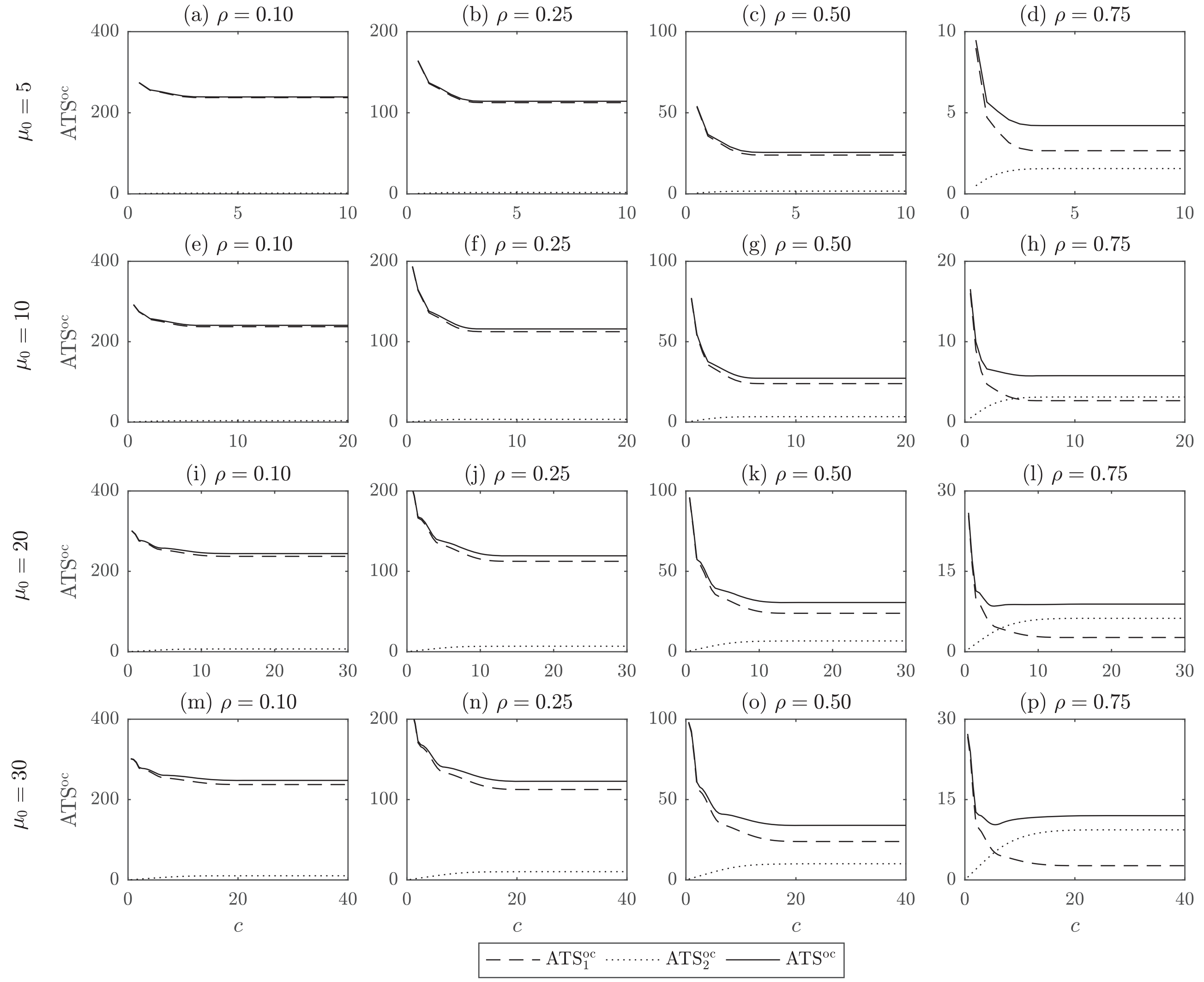

Figure 3. Plots of ATS^{OC} against c when $n = 5$ for the censored exponential cases.

in a significant increase in ATS^{oc}. Because of the existence of ATS$_2^{oc}$, the traditional view that focuses on improving the ability to recognize unfavorable changes by lengthening the life testing for more failure data, cannot ensure more rapid feedback. On the contrary, in some cases, this would make things worse.

6. Case Studies

6.1. A Synthetic Example on Lifetimes of LCD Modules

Through above numerical results, we gain insight into the impact of censoring time on monitoring lifetime data. In this section, an example of monitoring the lifetime of LCD modules is presented to demonstrate its practical application. This illustrative example is a synthetic replicate of the real-world problem described in Lee (2021). According to the plan, a fixed number of units of the LCD module (that comprises an LCD, a driver integrated circuit, a backlight board, and a printed circuit board) are sampled from the production process at intervals of $h = 1$ hour for a reliability life test. The life test is conducted at a temperature of 70°C and humidity of 80%. The lifetime distribution of the LCD module under the test condition follows an exponential distribution with $\mu_0 = 10.01$. The censored exponential chart is an option for monitoring the production process.

To apply the proposed method for early detection of underlying process changes, we start with the design part. As $\mu_0 \approx 10$, we can quickly (but roughly) identify the LCL for any given c based on Figure 2(b) when using $n = 5, 10$, or 30 for $\alpha = 0.0027$. If an exact LCL is warranted or a different n is considered, we need to use (6). For example, we roughly find H is about 3.1 from Figure 2(b) for $n = 10$ and $c = 5$, where the exact value is 3.05. Moreover, if the targeted shift is confirmed as $\rho = 0.50$ (other value can also be considered), we can derive that the optimal c is $5 \sim 6$ with the minimum ATS^{oc} ≈ 14 considering the U-shape relationship between c and ATS^{oc}; see Figure 4(g). Hence, the design part is completed, and we can prescribe $c = 5$ and $H = 3.05$ given $n = 10$ and $\alpha = 0.0027$.

To illustrate the implementation of the censored exponential MLE control chart, we use synthetic lifetime data. 50 lifetime observations for the LCD modules are simulated—30 IC data randomly drawn from the exponential distribution with $\mu_0 = 10.01$ and 20 OC data with $\mu_1 = 5.00$, both censored at $c = 5$. The data can be found in Part (G) of the supplementary mate-

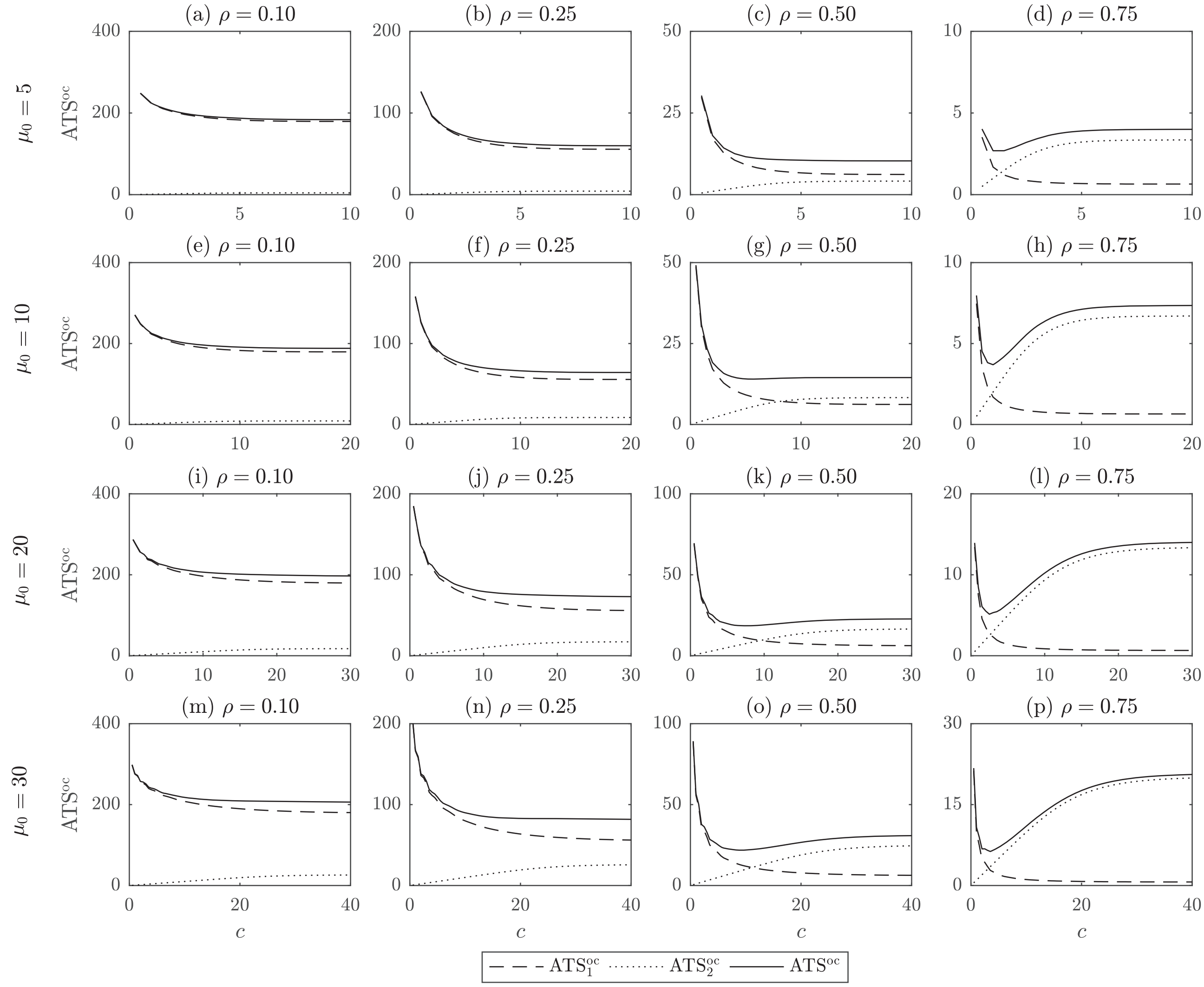

Figure 4. Plots of ATS^{OC} against c when $n = 10$ for the censored exponential cases.

rials. The sample statistics are computed based on the observations in each row of Table 4 in Part (G) of the supplementary materials and plotted sequentially on the control chart. From Figure 6, we see that there are no points plotted below the LCL for the first 30 (IC) samples, which verifies the process is IC. The first OC signal occurs at the 36th sample. Then, the practitioner should initiate an investigation for possible assignable cause(s) and take remedial actions (if necessary). Note that, this alarm is issued until the test result to be available, which is 5 hr after that sample was taken.

6.2. Revisit of the Rust Test Example

In this section, we revisit the rust test example (Steiner and MacKay 2001a) and aim to answer the questions arose in the Introduction about whether the censoring time being used for the rust test necessitates a reduction. An original dataset of 100 samples of size 3 was taken from historical records, where c was set at 20 days. Steiner and MacKay (2001a) reported that a Weibull distribution with $\gamma_0 = 1.51, \eta_0 = 48.04$ was a reasonable fit to the data. The censored Weibull chart can thus be used for monitoring and analyzing the data.

First, we examine the impact of c on ATS^{OC} in this example. Figure 7 shows the resulting ATS^{OC} curves for $\rho = 0.10$, 0.25, 0.50, and 0.75, respectively. ATS^{IC} is set to 370 as well. As for $n = 3$ and $\gamma_0 > 1$, the ATS^{OC} curves resemble those in the last row of Fig. 2 in Part (F) of the supplementary materials. From Figure 7, we observe that, though there exist some fluctuations in the ATS^{OC} curve, it almost achieves stability in all the four cases when c increases to 20. The (first) optimal point, where ATS^{OC} minimizes, is marked by the asterisk '$*$' in Figure 7. For $\rho = 0.10$, 0.25, and 0.50, there is no difference between ATS^{OC} at $c = 20$ and the minimum ATS^{OC} ($c = 16$). For $\rho = 0.75$, the optimal $c = 5$ and there appears an unapparent U-shape between $c = 1$ and $c = 7$. The corresponding ATS^{OC} at $c = 20$ is 12.93, slightly higher than 12.36, the minimum ATS^{OC} ($c = 5$). According to these observations, we may get the answer to the questions: the censoring time $c = 20$ is close to the optimum and shortening the rust test duration would not significantly reduce ATS^{OC} (and if shortening it too much, ATS^{OC} would increase a lot, instead).

With $c = 20$, the LCL for the censored Weibull MLE control chart can be derived and is 9.77. The resulting control chart for the original data (of 100 samples of size 3) is given by the first 100 samples in Figure 8. Figure 8 shows that there are no OC

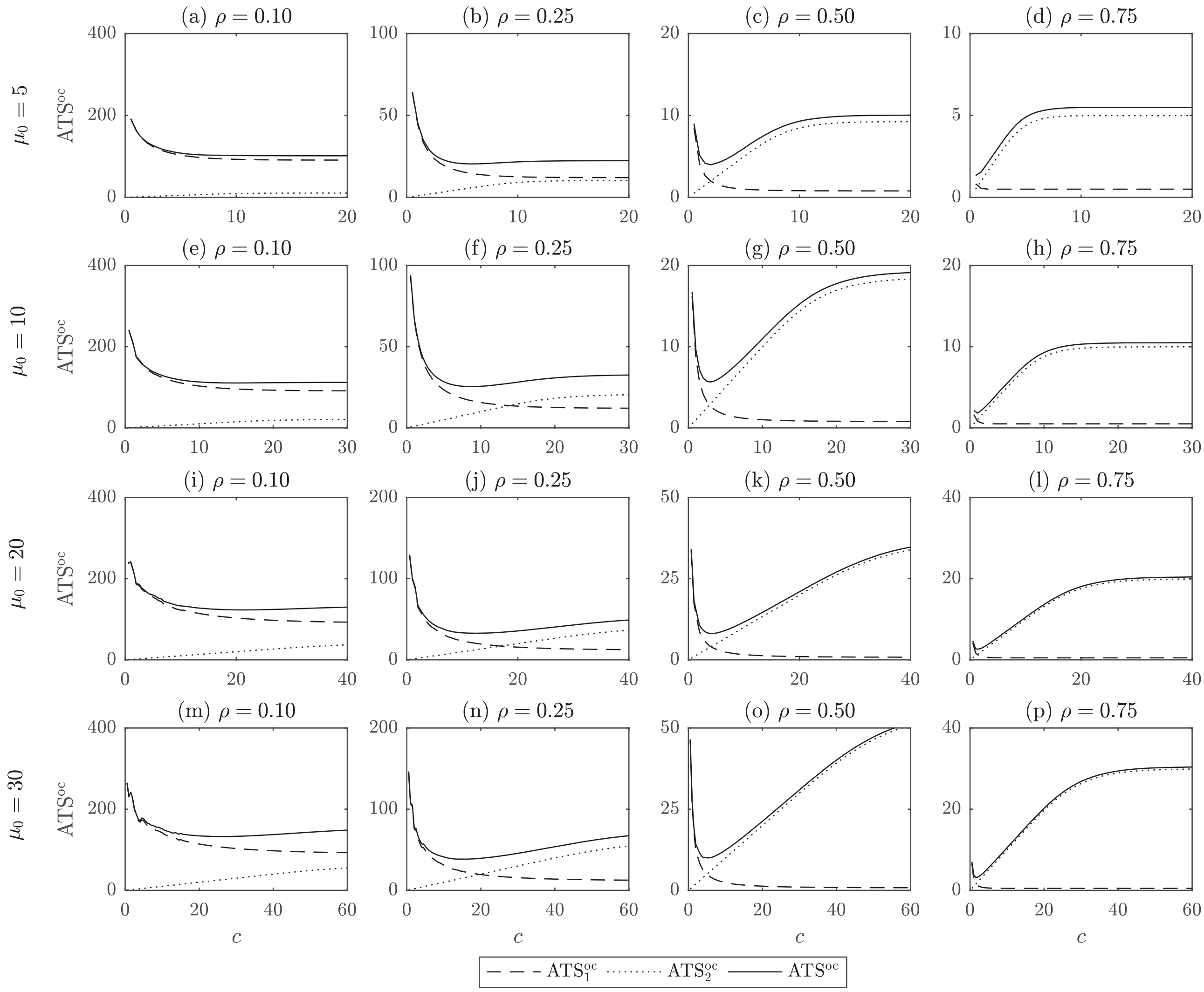

Figure 5. Plots of $\mathrm{ATS^{OC}}$ against c when $n = 30$ for the censored exponential cases.

points within this range. Therefore, the original data appears to come from an IC process. We should have obtained reasonably accurate parameter estimates. As a result, we may continue to monitor the process for deterioration using the control chart with the LCL. The final 20 samples are simulated assuming the process shifted to $\eta_1 = 12$ as in Steiner and MacKay (2001a). From Figure 8, we notice that the point for the 103th sample first falls below the LCL (and later, more points fall below the LCL), signaling a decrease in the mean. We locate this earliest OC sample with three simulated observations, that is, 11.99, 7.27, and 4.77. In other words, the actual rust test duration for this sample is about 12 days. In this case, the total time required to produce a signal needs 15 days since the shift—3 days for the conclusive sample to be drawn and another 12 days for the test to be finished.

7. Conclusions

In this study, we investigate an overlooked issue in traditional SPC research related to monitoring lifetime data of highly reliable products, that is, the 2-fold impact of censoring time on the timeliness of anomaly detection. We derive the finite-sample and asymptotic analytical expressions of ATS metrics for the likelihood-based control charts for the censored exponential and Weibull cases, and moreover, provide a Monte Carlo simulation procedure that can be used to obtain approximate results for more general lifetime distributions. Our investigation shows that the duration of the life test cannot be simply ignored when evaluating $\mathrm{ATS^{oc}}$, as we can see that $\mathrm{ATS_2^{oc}}$ is influential even dominant in many cases, unless the variation factor ρ as well as the sample size n is small. The overall impact of censoring time c and related recommendations under various scenarios are thoroughly discussed. The findings can offer practitioners a useful reference basis regarding the favorable choice of c.

Although this study deals with online monitoring of lifetime data under Type-I censoring using the likelihood-based control charts, the idea generally applies to other censoring mechanisms and charts. These are readily achieved based on the proposed Monte Carlo simulation procedure. At the end of Section 2.2, we mentioned an interesting phenomenon called competing alarms. A more thorough investigation on the impact of competing alarms is left for future study. In the current context, we are focusing on determining the optimal c. A potential direction is to develop an integrated optimization scheme, taking into account

Table 1. Minimum $\mathrm{ATS}^{\mathrm{oc}}$ and choice(s) of c obtaining a $\mathrm{ATS}^{\mathrm{oc}}$ no more than 101% the minimum for the censored exponential cases.

$\mu_0 = 5$	$n = 5$		$n = 10$		$n = 30$	
	choice(s) of c	minimum $\mathrm{ATS}^{\mathrm{oc}}$	choice(s) of c	minimum $\mathrm{ATS}^{\mathrm{oc}}$	choice(s) of c	minimum $\mathrm{ATS}^{\mathrm{oc}}$
$\rho = 0.10$	$[3,+\infty)$	239.06	$[6,+\infty)$	183.71	$[8.5,+\infty)$	101.57
$\rho = 0.25$	$[3,+\infty)$	114.19	$[7,+\infty)$	59.90	[5,7]	20.43
$\rho = 0.50$	$[3,+\infty)$	25.59	$[6,+\infty)$	10.34	[2]	3.99
$\rho = 0.75$	$[3,+\infty)$	4.21	[1,1.5]	2.69	[0.5]	1.33
$\mu_0 = 10$	$n = 5$		$n = 10$		$n = 30$	
	choice(s) of c	minimum $\mathrm{ATS}^{\mathrm{oc}}$	choice(s) of c	minimum $\mathrm{ATS}^{\mathrm{oc}}$	choice(s) of c	minimum $\mathrm{ATS}^{\mathrm{oc}}$
$\rho = 0.10$	$[5.5,+\infty)$	240.77	$[11.5,+\infty)$	188.08	[11.5,23)	110.65
$\rho = 0.25$	$[6,+\infty)$	115.89	$[13,+\infty)$	64.21	[7.5,10)	25.46
$\rho = 0.50$	$[6,+\infty)$	27.26	[5,7]	14.02	[3]	5.66
$\rho = 0.75$	$[5,+\infty)$	5.76	[2]	3.69	[1]	1.83
$\mu_0 = 20$	$n = 5$		$n = 10$		$n = 30$	
	choice(s) of c	minimum $\mathrm{ATS}^{\mathrm{oc}}$	choice(s) of c	minimum $\mathrm{ATS}^{\mathrm{oc}}$	choice(s) of c	minimum $\mathrm{ATS}^{\mathrm{oc}}$
$\rho = 0.10$	$[10,+\infty)$	244.23	$[22,+\infty)$	196.81	[17,27]	123.01
$\rho = 0.25$	$[11,+\infty)$	119.32	$[24.5,+\infty)$	72.84	[11,14.5]	32.78
$\rho = 0.50$	$[11.5,+\infty)$	30.60	[6.5,9]	18.52	[4,4.5]	8.10
$\rho = 0.75$	[4]	8.49	[2.5]	5.13	[1.5]	2.65
$\mu_0 = 30$	$n = 5$		$n = 10$		$n = 30$	
	choice(s) of c	minimum $\mathrm{ATS}^{\mathrm{oc}}$	choice(s) of c	minimum $\mathrm{ATS}^{\mathrm{oc}}$	choice(s) of c	minimum $\mathrm{ATS}^{\mathrm{oc}}$
$\rho = 0.10$	$[15,+\infty)$	247.66	$[30,+\infty)$	205.54	[20.5,32.5]	132.61
$\rho = 0.25$	$[16.5,+\infty)$	122.72	$[33,+\infty)$	81.47	[13,17]	38.21
$\rho = 0.50$	$[16.5,+\infty)$	33.94	[8.5,10]	21.81	[5,6]	10.01
$\rho = 0.75$	[5,6]	10.31	[3.5]	6.31	[1.5,2]	3.20

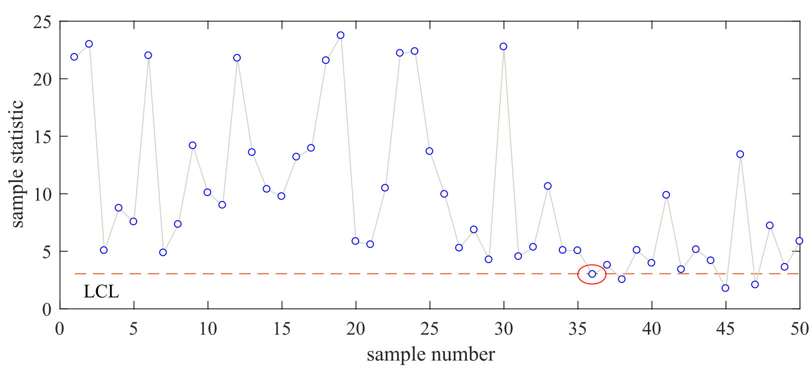

Figure 6. The censored exponential MLE control chart for the LCD modules.

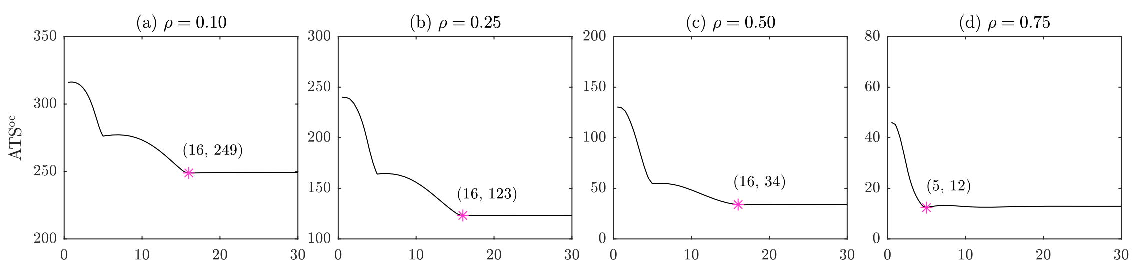

Figure 7. Plots of $\mathrm{ATS}^{\mathrm{OC}}$ against c for the rust test example.

not only the choice of c but also n as well as other controllable parameters. In this manner, the performance of anomaly detection can be further enhanced. Another critical issue is that, the underlying lifetime distribution is often unknown or cannot be exactly estimated based on historical data. Therefore, robust design and nonparametric monitoring of this type of control charts are worth further investigation.

Supplementary Materials

Technical Materials: (A) Derivation of $\mathrm{ATS}_2^{\mathrm{oc}}$; (B) Proof of Figure 2; (C) Proof of Theorem 1; (D) Lower bound of approximate LCL; (E) Numerical results on asymptotics of censored exponential; (F) Numerical results on censored Weibull; (G) Simulated lifetime observations used in Section 6.1.
MATLAB Codes: For the computation of the control limit H and ATS metrics of the likelihood-based control charts for the censored exponential and Weibull cases.

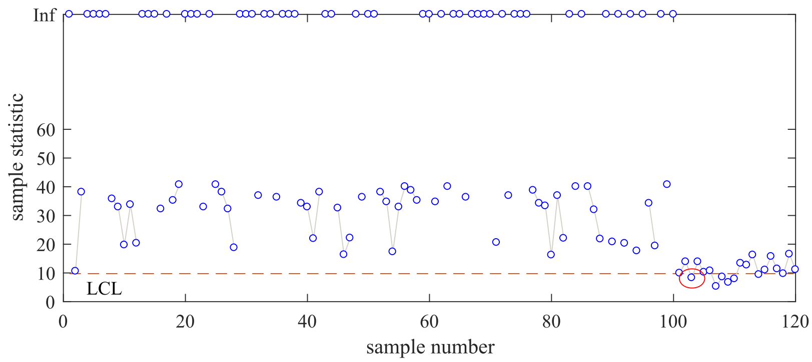

Figure 8. The censored Weibull MLE control chart for the rust test example.

Acknowledgments

We thank the two anonymous reviewers, the Associate Editor and the Editor for their valuable suggestions which improved considerably the earlier versions of this article.

Disclosure Statement

No potential competing interest was reported by the authors.

Funding

This work has been supported by National Natural Science Foundation of China (no. 71801179).

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