Received: 25 July 2020 | Revised: 9 February 2021 | Accepted: 27 March 2021

DOI: 10.1002/qre.2904

RESEARCH ARTICLE

WILEY

Efficient homogeneously weighted dispersion control charts with an application to distillation process

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Abstract

Monitoring disturbances in process dispersion using control chart is mostly based on the assumption that the quality characteristic follows normal distribution, which is not the case in many real-life situations. This paper proposes a set of new dispersion charts based on the homogeneously weighted moving average (HWMA) scheme, for efficient detection of shifts in process standard deviation (σ). These charts are based on a variety of σ estimators and are investigated for normal as well as heavy tailed symmetric and skewed distributions. The shift detection ability of the charts is evaluated using different run length characteristics, such as average run length (ARL), extra quadratic loss (EQL), and relative ARL measures. The performance of the proposed HWMA control charts is also compared with the existing EWMA dispersion charts, using different design parameters. Furthermore, an illustrative example is presented to monitor the vapor pressure in a distillation process.

KEYWORDS

control chart, process dispersion, HWMA, average run length, extra quadratic loss

1 | INTRODUCTION

Statistical Process Control (SPC) is a tool-kit to enhance process performance by monitoring changes in process parameters. Walter A. Shewhart originally proposed the SPC concept in the 1920s. One of the most popular tools of SPC is control chart. The main aim of using control chart is to improve the quality of the process by detecting shifts occurring in either the process mean or/and the process variance as soon as possible to prevent them from occurring again.

There are two types of control charts, which are memory-type control charts and memory-less-type control charts. In recent years, the focus toward memory-type control charts has increased massively because of their high ability to detect small-to-moderate shifts. Page[1] and Roberts[2] proposed the CUSUM and the EWMA charts, for quick detection of small shifts in process parameters, respectively. These are the two most commonly used memory-type control charts for process monitoring because of their simplicity and good performance. These charts make use of current and previous sample information by accumulating small continual shifts and are helpful for reducing large financial loss, by quick detection of small shifts. The well-known Shewhart chart structures are only based on current sample information and hence are only suitable for handling large process shifts. More details about basic control chart designs can be found in Montgomery.[3]

In process monitoring, the dispersion parameter is of more immediate concern. Indeed, there is an inverse relation between the improvement of the process and the process variance. Huwang et al[4] stated that monitoring shifts in process dispersion is more important compared to the monitoring of process mean. Many researchers, in recent years, devoted their studies on monitoring process dispersion using memory control charts. For instance, Abbas et al,[5–7] Abbasi and Miller,[8] Zhou et al,[9] Ahmad et al,[10] Castagliola et al,[11] Huang et al,[12] Haq,[13] Hossain et al,[14] Rajmanya and Ghute,[15] Osei-Aning and Abbasi,[16] Lee and Khoo,[17,18] Zaman et al,[19] and the references cited, therein.

The distributional assumption of most control chart structures for statistical process monitoring is that the quality characteristic follow the normal distribution; however, in practice, this assumption is mostly violated. Abbasi and Miller[20] indicated that control charts should be based on appropriate parent distribution of the process rather than just relying on normality assumption. Many authors have proposed EWMA control charts for monitoring process parameter when the assumption of normality is violated. Maravelakis et al[21] investigated robustness to nonnormality on EWMA dispersion charts based on individual measurements. Abbasi et al[22] investigated several EWMA dispersion charts for monitoring processes following normal and a variety of nonnormal distributions. Recently, Abbas[23] proposed a homogeneously weighted moving average (HWMA) control chart for monitoring small shifts in process mean. The HWMA chart assigns a specific weight to the current observation while the remaining weights are assigned equally to the past observations.

This study presents the HWMA dispersion control chart based on a set of σ estimators. These estimators are sample range (R), standard deviation (SD), inter quartile range (Q), Downton's estimator (D), mean absolute deviation from median (MD), median absolute deviation (MAD), and two estimators based on interpoint distances, namely, S_n and Q_n. We have investigated processes that follow normal and nonnormal parent distributions. The distributions considered are the normal, two heavy-tailed symmetric and five skewed distributions. The distributions and the dispersion estimates investigated in this study are the same as the ones investigated by Abbasi et al[22] for comparison purposes. Thus, for every distribution, we investigated the $\mathrm{R_H}$ chart based on R estimator, S_H chart based (SD), Q_H based on (Q), D_H chart based on (D), $\mathrm{MD_H}$ chart based on (MD), MAD_H chart based on (MAD), SN_H and QN_H charts based on S_n and Q_n estimators, respectively.

The remainder of the paper is organized as follows: The different dispersion estimates used in the study are given in Section 2. The design of the proposed HWMA dispersion chart is presented in Section 3 while Section 4 highlights the performance assessment and the steps taken in the simulation study. A comparison between the proposed HWMA and the EWMA dispersion control charts is given in Section 5 and an illustrative example is reported in Section 6. Finally Section 7 presents the conclusions of the study.

2 | DISPERSION ESTIMATORS

This section describes a variety of σ estimators as described in the previous section. Suppose X is the quality characteristic of interest and $x_1, x_2, \ldots, x_n$ be a random sample of size n. Moreover, suppose x_i be the ith order statistic (lowest to highest), $\bar{x}$ be the sample mean, $|X|$ be the absolute value of X and $\tilde{x}$ be the sample median. The different dispersion estimates are briefly described below:

(i) Sample range (R) based on order statistics: This is defined as

$$R = x_{(n)} - x_{(1)}. \tag{1}$$

It is the most widely used dispersion estimator. It is the difference between the highest and lowest observations in the sample. R is too sensitive for extreme values and the normality requirement is not satisfied.

(i) Standard deviation (S) based on squared deviations from the mean: it is defined as

$$S = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}. \tag{2}$$

Comparing S estimator with other estimators shows that S is the most efficient dispersion estimator for normal distribution but it is sensitive to outliers.

(i) Interquartile range (Q) based on order statistics: It is given as

$$Q = (Q_3 - Q_1)/1.34898, \tag{3}$$

where Q_3 and Q_1 are the upper and lower quantiles. For large sample size, Q loses its efficiency while it is used as a robust measure of scale with a breakdown point of 25%. Abbasi and Miller[20] showed that for some nonnormal distributions, the Q chart performed better than R and S charts.

(i) Downton's Estimator (D) based on order statistics: It is defined as:

$$D = \frac{2\sqrt{\pi}}{n(n-1)} \sum_{i=1}^{n} \left[i - \frac{1}{2}(n+1) \right] x_{(i)}. \tag{4}$$

For normally distributed processes, D is an unbiased estimator of σ. Control charts based on D showed superiority over R- and S-based charts considering nonnormal processes.[20,24]

(i) Mean absolute deviation (MD) based on absolute deviations from median: This is defined as

$$MD = \frac{1}{n} \sum_{i=1}^{n} |x_i - \tilde{x}|. \tag{5}$$

Abbasi and Miller[20] showed that MD is an efficient estimator for nonnormal distributions such as t and gamma distributions. Riaz and saghir[25] found that the MD chart is more sensitive than R and S charts.

(i) Median absolute deviation (MAD) based on absolute deviations from median: This is defined as

$$MAD = 1.4826\ med \frac{1}{n} |x_i - \tilde{x}|. \tag{6}$$

MAD is a very robust estimator of σ since it has the highest possible breakdown point (50%). However, its reliance on the distribution being symmetric and low Gaussian efficiency (36.74%) are the two main drawbacks.[26]

(i) σ estimators (S_n and Q_n)

S_n and Q_n estimators uses interpoint distances to estimate σ. S_n is given as

$$S_n = 1.1926\ med_i \left\{ med_i \left| x_i - x_j \right| ; i \neq j \right\}, \tag{7}$$

where the outer and inner medians represent [(n+1)/2]th and [(n/2)+1]th ranked statistic, respectively. These medians can be termed as the "high" and "low" medians.[26] Q_n is defined as:

$$Q_n = 2.2219\{\left| x_i - x_j \right| ; i < j\}_{(p)}, \tag{8}$$

where $p = \left(\frac{[\frac{n}{2}] + 1}{2} \right)$

Q_n is defined as the pth ranked statistic of the n-choose-2 interpoint distances. Q_n performs better than S_n and MAD estimates for symmetric and skewed distributions.[26]

The proposed HWMA charts will use all the above estimates to identify the best chart for normal and nonnormal process environments.

3 | PROPOSED HWMA DISPERSION CHART

Let $T_i(i = 1, 2, \ldots)$ represent a sequence of the observed values of an estimator T at time t, computed from a sample of size n. Here, T represents any of the dispersion estimators, mentioned in Section 2. The mean of the process is assumed to be stable as we are only interested in monitoring the process dispersion. Let the plotting statistic for HWMA dispersion charts be defined as:

$$H_i = \lambda T_i + (1-\lambda)\bar{T}_{i-1}, \tag{9}$$

where λ is the smoothing parameter and it is selected to be between zero and one, that is, $0 < \lambda \leq 1$. $\bar{T}_{i-1}$is the average of previous (i − 1) dispersion statistics and is given as:

$$\bar{T}_{i-1} = \frac{\sum_{j=1}^{t-1} T_j}{i-1}. \tag{10}$$

The initial value ($\bar{T}_0$) is taken to be equal to the target mean of T, that is, μ_0 (if target mean is unknown). Further, let Z defines relative dispersion statistic (ie, $Z = T/\sigma$). Let $E(Z) = t_2$ and $\sigma_z = t_3$. t_2, and t_3 are entirely dependent on sample size for a specific combination of T and parent distribution (see.......).

Following Abbas,[23] we can show that:

$$E(H_i) = E(T_i)$$

$$V\ (H_i) = \begin{cases} \lambda^2\sigma_t^2,\ if\ t = 1 \\ \lambda^2\sigma_t^2 + \dfrac{(1-\lambda)^2\sigma_t^2}{(t-1)},\ if\ t > 1 \end{cases}, \tag{11}$$

where $E(T_i) = t_2\sigma$ and $\sigma_t = t_3\sigma$. For unknown process standard deviation, σ_t can be replaced by its estimator $\hat{\sigma}_t = \frac{\bar{T}}{t_2}$. Using these notations, the time varying upper control limit of the HWMA chart for the statistic T is defined as:

$$UCL_t = \begin{cases} \bar{T} + L_T \dfrac{t_3\bar{T}}{t_2}\sqrt{\lambda^2}, & if\ t = 1 \\ \bar{T} + L_T \dfrac{t_3\bar{T}}{t_2}\sqrt{\lambda^2 + \dfrac{(1-\lambda)^2}{t-1}}, & if\ t > 1 \end{cases}, \tag{12}$$

where L_T is the width of the control limit and it is adjusted to achieve the desired in-control average run length (ARL_0). Furthermore, only the UCL is considered in this study as we are interested in detecting an upward shift in the process dispersion. Abbasi et al[22] reported the t_2 and t_3 values for different parent normal and nonnormal distributions, considering a variety of sample sizes.

4 | PERFORMANCE EVALUATION OF HWMA CHART

The performance of control charts is usually evaluated using run length characteristics. The most common performance measures of control charts is the average run length (ARL). It is the average number of samples plotted before the detection of an out-of-control point. The in-control and the out-of-control average run lengths are denoted as ARL_0 and ARL_1, respectively. For a comparison between control charts, the ARL_0 is fixed at a desired level and ARL_1 is compared, the chart with the lowest ARL_1 value has the best performance in detecting shifts. The distribution of ARL is mostly highly skewed; hence few researchers recommended to use some other run length measures such as median run length and standard deviation of the run length distribution.[22] So, some other performance measures have been proposed to evaluate the overall effectiveness of HWMA charts over the entire range of shifts. In this study, some useful measures to assess the performance of the proposed HWMA dispersion control charts are the average run length (ARL), standard deviation of the run length (SDRL), median run length (MDRL), extra quadratic loss (EQL), and the relative average run length (RARL). Here, we used 10^4 Monte Carlo simulation runs developed in R language. Note that the results and report of study are based on subgroups size $n = 5$ and $n = 7$ using $\lambda = 0.05$ and $\lambda = 0.25$, respectively, for each HWMA dispersion chart with normal and nonnormal distributions. The main aim of this study is to investigate the performance of the HWMA charts and identify a chart (or group of charts) that is superior in the monitoring of dispersion for normal and nonnormal distribution processes. The nonnormal symmetric distributions are: logistic and Student's *t*. The skewed distributions are Weibull, chi-square, gamma and exponential, the extremely skewed distribution is log-normal. Furthermore, the density functions and the parameter values for different parent distributions used in this study are given below:

Normal (μ, σ^2), $-\infty < \mu \langle \infty, \sigma \rangle 0$

$$f\ \left(x\mu, \sigma^2\right) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-(x-\mu)^2/2\sigma},\ -\infty < x < \infty \tag{13}$$

Logistic (μ, k), $-\infty < \mu \langle \infty, k \rangle 0$

$$f\ (x\mu, k) = \frac{e^{-(x-\mu)^2/k}}{k\left(1 + e^{-(x-\mu)/k}\right)^2}\ ,\ -\infty < x < \infty, \tag{14}$$

Student's $t\,(k)$, $k > 0$

$$f\ (xk) = \frac{\Gamma\left[(k+1)/2\right]}{\sqrt{k\pi}\,\Gamma\,(k/2)}\left(1 + \frac{x^2}{k}\right)^{-(k+1)/2},\ -\infty < x < \infty, \tag{15}$$

Weibull(α, β), $\alpha > 0, \beta > 0$

$$f\ (x\alpha, \beta) = \frac{\alpha}{\beta}\, x^{\alpha-1} e^{-x^{\alpha}/\beta},\ x \geq 0, \tag{16}$$

Chi-square(k), $k > 0$

$$f\ (xk) = \frac{x^{(k/2)-1} e^{-x/2}}{2^{k/2}\Gamma\,(k/2)}\ ,\ x > 0, \tag{17}$$

Gamma(α, β), $\alpha > 0, \beta > 0$

$$f\ (x\alpha, \beta) = \frac{\beta^{\alpha}}{\Gamma\,(\alpha)}\, x^{\alpha-1} e^{-\beta x},\ x > 0, \tag{18}$$

Exponential(λ), $\lambda > 0$

$$f\ (x\lambda) = \lambda e^{-\lambda x},\ x \geq 0, \tag{19}$$

Log-normal (μ, σ^2), $-\infty < \mu \langle \infty, \sigma \rangle 0$

$$f\ \left(x\mu, \sigma^2\right) = \frac{1}{x\sqrt{2\pi\sigma^2}}\, e^{-(ln(x)-\mu)^2/2\sigma^2},\ x \geq 0. \tag{20}$$

Without loss of generality, we considered $\mu = 0$ and $\sigma = 1$ for normal distribution, $\mu = 0$ and $k = 1$ for logistic distribution, $k = 5$ for Student's *t*-distribution, $\alpha = 1.5$ and $\beta = 1$ for Weibull distribution, $\alpha = 2$ and $\beta = 1$ for gamma distribution, $\lambda = 1$ for exponential distribution and $\mu = 0$ and $\sigma = 1$ for the log-normal distribution.

The extra quadratic loss (EQL) and relative average run length (RARL) measures are used to evaluate the overall shift detection ability of the proposed HWMA charts. These are briefly defined below:

4.1 | Extra quadratic loss (EQL) and relative ARL (RARL)

Reynolds Jr. and Stoumbos[27] and Wu et al[28] recommended the use of extra quadratic loss (EQL) and relative ARL (RARL) measures to evaluate the overall detection ability of control charts over the entire shift domain. EQL uses a weight δ^2 to compute the weighted average of ARL over the whole shift range. It is defined as:

$$\mathrm{EQL} = \frac{1}{\delta_{\max} - \delta_{\min}} \int_{\delta_{\min}}^{\delta_{\max}} \delta^2 \mathrm{ARL}(\delta)\, d\delta, \tag{21}$$

where ARL(δ) is the ARL value of a specific chart at shift δ. A chart with the smallest EQL is considered better than competing charts.

RARL is defined as the ratio of the ARL value of a specific chart at a given shift δ and the ARL of benchmark chart. Mathematically, it is given by

$$\mathrm{RARL} = \frac{1}{\delta_{\max} - \delta_{\min}} \int_{\delta_{\min}}^{\delta_{\max}} \frac{\mathrm{ARL}(\delta)}{\mathrm{ARL}_{\mathrm{benchmark}}} d\delta, \tag{22}$$

where ARL(δ) is the ARL value of the chart for a specific shift δ and $\mathrm{ARL}_{\mathrm{benchmark}}$ is the ARL value of the benchmark chart, is mostly taken as a chart with the lowest EQL value.

5 | PERFORMANCE COMPARISON

In this section, the ARL, SDRL, MDRL as well as the EQL and RARL performance measures of the proposed HWMA charts are compared for processes following normal and nonnormal distributions. Also, the performance of the proposed HWMA charts is compared with their respective EWMA dispersion charts. Table 1 reports the run length characteristics (ARL, MDRL, and SDRL) of the HWMA dispersion charts under the idea assumption of normality. To save space, only the ARL results are presented in Tables 2–9, for processes following other parent nonnormal distributions. In these tables, results are only reported for $n = 7$ and $\lambda = 0.25$. The control charts multipliers for each chart are given to achieve a desired ARL_0 set at 200 in the tables. The overall performance measures EQL and RARL for the different HWMA dispersion charts are reported in Table 9 (using $n = 7$ and $\lambda = 0.25$) and Table 10 (using $n = 5$ and $\lambda = 0.05$). To save space, we did not provide detailed results for other combinations of n and λ. The RARL comparison of the proposed charts is provided in Figure 1, considering using $n = 5$ and $\lambda = 0.05$. The control chart is said to be superior if its ARL_1 values are lower and have the minimum EQL compared to other competing charts. Furthermore, the sensitivity of the proposed HWMA chart for different smoothing constant λ and varying sample sizes n are evaluated for some special cases.

5.1 | Proposed HWMA chart

Normal distribution: The ARL performance of a variety of HWMA dispersion charts is compared for normally distributed quality characteristics in Table 1 whereas EQL and RARL comparison is provided in Figure 1 and Tables 9–10. We can clearly observe that the S_H chart has the lowest ARL values among the other competing charts, indicating that it has the best performance. Moreover, the S_H chart has the lowest EQL and RARL values, which shows that the overall shift detection ability of S_H chart is also better, compared to other charts. The D_H chart is considered the second best control chart, as the results were close to S_H chart, followed by the R_H and the MD_H charts. Furthermore, the MAD_H, SN_H, and QN_H have the worst performance as their ARLs, EQLs, and RARLs are high and thus their abilities of detecting shifts are low. The SDRL and MDRL criteria gave the same conclusions as the ARL. The MDRL values for all charts are the same when the sift size is $3 \le \delta \le 4$. We conclude that for the normal distribution, S_H has the best performance while D_H is close competitor with S_H. MAD_H, SN_H, and QN_H have the worst performance in detecting shifts whereas R_H and MD_H charts lie between the best and worst chart performances. Also, we have computed the ARL values for all

TABLE 1 Run length profiles of the HWMA dispersion chart for normal distribution when $\lambda = 0.25$, $n = 7$, and ARL0 = 200

δ		R_H	S_H	Q_H	D_H	MD_H	MAD_H	SN_H	QN_H
1.0	ARL	201.70	200.67	201.48	200.22	201.54	199.39	200.11	199.36
	SDRL	191.50	187.65	185.92	188.04	186.66	183.39	184.74	186.80
	MDRL	145.00	146.00	147.00	145.00	146.00	146.00	144.00	145.00
1.1	ARL	34.38	29.78	47.33	30.75	33.02	55.31	46.44	47.06
	SDRL	28.42	24.39	39.45	25.56	26.59	47.54	39.50	39.13
	MDRL	27.00	23.00	36.00	24.00	27.00	42.00	36.00	37.00
1.2	ARL	13.64	11.87	20.87	12.20	13.21	25.25	20.53	20.59
	SDRL	10.28	8.88	16.15	9.06	9.86	20.00	16.38	16.45
	MDRL	11.00	10.00	17.00	10.00	11.00	20.00	16.00	16.00
1.3	ARL	7.80	6.80	12.13	7.11	7.55	14.98	11.94	12.03
	SDRL	5.66	4.81	9.02	5.05	5.33	11.57	9.10	9.01
	MDRL	7.00	6.00	10.00	6.00	6.00	12.00	10.00	10.00
1.4	ARL	5.26	4.62	8.21	4.74	5.17	10.13	8.10	8.05
	SDRL	3.60	3.17	6.01	3.32	3.58	7.64	5.99	5.93
	MDRL	5.00	4.00	7.00	4.00	4.00	8.00	7.00	7.00
1.5	ARL	3.93	3.52	6.04	3.55	3.77	7.57	5.97	6.04
	SDRL	2.66	2.33	4.35	2.39	2.53	5.59	4.34	4.38
	MDRL	3.00	3.00	5.00	3.00	3.00	6.00	5.00	4.00
1.6	ARL	3.12	2.79	4.80	2.87	3.02	5.94	4.77	4.77
	SDRL	2.03	1.85	3.37	1.85	2.01	4.34	3.42	3.40
	MDRL	3.00	3.00	4.00	3.00	3.00	5.00	4.00	3.00
1.8	ARL	2.22	2.04	3.35	2.08	2.18	4.15	3.35	3.30
	SDRL	1.39	1.27	2.23	1.30	1.36	2.94	2.37	2.26
	MDRL	2.00	2.00	3.00	2.00	2.00	4.00	3.00	2.00
2.0	ARL	1.78	1.62	2.63	1.68	1.76	3.27	2.63	2.60
	SDRL	1.07	0.95	1.73	0.99	1.05	2.27	1.80	1.74
	MDRL	1.00	1.00	2.00	1.00	1.00	3.00	2.00	1.00
2.5	ARL	1.28	1.23	1.77	1.23	1.28	2.15	1.80	1.76
	SDRL	0.62	0.55	1.09	0.55	0.61	1.40	1.13	1.10
	MDRL	1.00	1.00	1.00	1.00	1.00	2.00	1.00	1.00
3.0	ARL	1.11	1.09	1.42	1.10	1.11	1.71	1.46	1.42
	SDRL	0.38	0.34	0.78	0.34	0.39	1.04	0.84	0.78
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
3.5	ARL	1.05	1.04	1.24	1.05	1.05	1.45	1.30	1.24
	SDRL	0.25	0.22	0.57	0.23	0.24	0.82	0.65	0.58
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
4.0	ARL	1.02	1.01	1.15	1.02	1.02	1.30	1.19	1.16
	SDRL	0.15	0.14	0.45	0.15	0.17	0.65	0.51	0.46
	MDRL	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
L		**2.780**	**2.660**	**3.000**	**2.680**	**2.722**	**2.950**	**2.780**	**2.840**

charts when $n = 5$ and $\lambda = 0.05$; however, due to limitation of space, the result is not provided but available upon request from the authors. We found that S_H, D_H, R_H, and MD_H charts have the best performance compared to other charts as their ARL values are the lowest for all shift sizes, following them is the SN_H and QN_H charts. The MAD_H and Q_H charts have the worst performance, as their ARL_1 values are the highest. The EQL and RARL results for this case are shown in Table 10, S_H has the lowest values and thus it is the best chart and Q_H has the highest values and thus it is has the worst performance.

TABLE 2 Average run length profiles of the HWMA dispersion chart for logistic distribution when $\lambda = 0.25$, $n = 7$, and ARL0 = 200

δ	R_H	S_H	Q_H	D_H	MD_H	MAD_H	SN_H	QN_H
1.0	200.53	201.74	199.63	199.14	200.19	201.41	200.88	200.56
1.1	52.42	46.00	53.73	42.94	42.13	63.04	56.01	56.79
1.2	22.14	19.35	23.96	17.59	17.34	29.66	25.33	25.45
1.3	12.63	10.96	14.08	10.01	11.69	17.45	14.51	14.74
1.4	8.40	7.29	9.33	6.65	6.61	11.89	9.83	9.97
1.5	6.10	5.33	6.86	4.89	4.93	8.87	7.25	7.35
1.6	4.84	4.09	5.41	3.88	3.85	6.84	5.66	5.69
1.8	3.29	2.85	3.79	2.72	2.71	4.81	3.94	4.00
2.0	2.53	2.25	2.90	2.10	2.12	3.58	3.07	3.04
2.5	1.68	1.51	1.92	1.46	1.45	2.35	2.04	2.01
3.0	1.32	1.22	1.53	1.21	1.21	1.81	1.59	1.55
3.5	1.17	1.11	1.31	1.09	1.10	1.55	1.39	1.34
4.0	1.09	1.06	1.20	1.05	1.05	1.36	1.26	1.23
L	**3.190**	**3.066**	**3.030**	**2.950**	**2.920**	**3.050**	**2.900**	**3.010**

TABLE 3 Average run length profiles of the HWMA dispersion chart for Student's *t* distribution when $\lambda = 0.25$, $n = 7$, and ARL0 = 200

δ	R_H	S_H	Q_H	D_H	MD_H	MAD_H	SN_H	QN_H
1.0	200.87	200.98	199.92	200.91	199.66	199.18	201.22	201.01
1.1	81.97	77.61	58.10	64.48	56.75	64.93	61.99	62.78
1.2	39.09	36.48	26.24	28.23	24.86	31.36	28.45	28.36
1.3	22.39	20.30	15.46	15.63	13.90	18.78	16.75	16.72
1.4	14.71	12.98	10.16	10.28	8.98	12.98	11.23	11.12
1.5	10.43	9.08	7.53	7.43	6.48	9.40	8.07	8.20
1.6	7.75	6.92	5.94	5.68	5.04	7.35	6.35	6.29
1.8	5.17	4.58	4.06	3.82	3.46	5.11	4.35	4.34
2.0	3.77	3.36	3.11	2.85	2.63	3.87	3.40	3.37
2.5	2.31	2.08	2.04	1.83	1.70	2.49	2.19	2.16
3.0	1.73	1.56	1.58	1.41	1.36	1.92	1.69	1.66
3.5	1.42	1.31	1.37	1.21	1.19	1.62	1.44	1.41
4.0	1.24	1.17	1.23	1.11	1.09	1.42	1.31	1.27
L	**3.710**	**3.630**	**3.148**	**3.400**	**3.249**	**3.220**	**3.138**	**3.166**

Nonnormal symmetric distributions (logistic, and Student's t): From Tables 2 and 3, it can be observed that the MD_H and D_H charts for both parent distributions have the best performance in terms of their ARL values, which indicate that less samples are required to detect shifts and thus their abilities of detecting shifts are high, in addition, they have the least EQL and RARL values, as it is seen from Table 9. The R_H and S_H charts are less efficient for the Student's *t* distribution, compared with the logistic distribution. The MAD_H, SN_H, and QN_H are the worst performing charts. For $n = 5$ and $\lambda = 0.05$, considering logistic distribution, both D_H and MD_H charts have the best performance in detecting small shifts ($1.1 \leq \delta \leq 1.6$), while for Student's *t* distribution, they are sensitive in detecting large shifts only and the Q_H chart is sensitive in detecting small shifts. The R_H, S_H, SN_H, and QN_H lie between the superior and inferior charts, and the MAD_H charts for both distributions have the worst performance in terms of all the run length measures. Furthermore, for both distributions, MD_H chart is the best in terms of their EQL and RARL values as can be seen from Table 10, while MAD_H chart is the worst.

Skewed distributions (Weibull, chi-square, gamma, and exponential): From Tables 4–7 and 9, we can see that the performance of MD_H and D_H charts are the best, as compared to all other competing charts. The S_H, R_H, and Q_H charts have almost the same performance among the four distributions but are more efficient for parent Weibull distribution, compared to the other distributions, in terms of their ARL. Again, the MAD_H, SN_H, and QN_H were the least efficient charts as their ARLs, EQLs, and RARLs are high. The ARL values for $n = 5$ and $\lambda = 0.05$ were also computed and they gave similar

TABLE 4 Average run length profiles of the HWMA dispersion chart for Weibull distribution when $\lambda = 0.25$, $n = 7$, and ARL0 = 200

δ	R_H	S_H	Q_H	D_H	MD_H	MAD_H	SN_H	QN_H
1.0	199.99	199.70	201.97	199.90	200.80	199.62	201.35	199.41
1.1	48.81	47.47	56.07	40.52	42.38	62.07	55.13	54.67
1.2	20.86	19.76	25.22	16.93	17.62	29.19	25.27	24.55
1.3	11.90	11.20	14.96	9.86	10.03	17.43	14.87	14.39
1.4	7.77	7.45	10.03	6.50	6.79	11.91	9.89	9.54
1.5	5.73	5.46	7.45	4.82	4.98	8.79	7.37	7.08
1.6	4.50	4.24	5.76	3.77	3.94	6.88	5.85	5.55
1.8	3.12	3.00	4.02	2.67	2.78	4.74	4.10	3.83
2.0	2.41	2.30	3.07	2.13	2.19	3.64	3.14	2.96
2.5	1.60	1.57	2.05	1.46	1.49	2.37	2.06	1.95
3.0	1.30	1.26	1.59	1.20	1.23	1.84	1.63	1.54
3.5	1.14	1.13	1.36	1.10	1.11	1.55	1.41	1.33
4.0	1.07	1.07	1.23	1.05	1.06	1.37	1.28	1.22
L	**3.100**	**3.062**	**3.190**	**2.922**	**2.944**	**3.120**	**3.030**	**3.040**

TABLE 5 Average run length profiles of the HWMA dispersion chart for chi-square distribution when $\lambda = 0.25$, $n = 7$, and ARL0 = 200

δ	R_H	S_H	Q_H	D_H	MD_H	MAD_H	SN_H	QN_H
1.0	200.86	200.96	200.78	199.21	201.72	201.11	201.07	201.13
1.1	59.21	56.46	60.50	47.66	47.78	64.85	60.00	59.05
1.2	26.48	24.66	27.79	21.01	20.15	30.98	27.15	26.39
1.3	15.19	14.21	16.40	11.67	11.40	18.42	16.04	15.24
1.4	10.00	9.32	10.84	7.77	7.63	12.58	10.83	10.29
1.5	7.25	6.91	8.04	5.70	5.63	9.24	8.01	7.62
1.6	5.68	5.29	6.28	4.49	4.44	7.26	6.29	5.84
1.8	3.84	3.63	4.28	3.12	3.09	5.03	4.32	4.04
2.0	2.90	2.76	3.30	2.42	2.42	3.82	3.33	3.12
2.5	1.86	1.80	2.12	1.63	1.59	2.48	2.16	2.04
3.0	1.46	1.39	1.66	1.30	1.30	1.90	1.68	1.59
3.5	1.24	1.21	1.40	1.16	1.16	1.60	1.43	1.37
4.0	1.13	1.11	1.27	1.08	1.08	1.42	1.29	1.23
L	**3.310**	**3.225**	**3.161**	**3.050**	**3.001**	**3.160**	**3.088**	**3.100**

results to that of $n = 7$ and $\lambda = 0.25$ and their EQL and RARL values are given in Table 10, where it shows that D_H chart is the best among all the dispersion charts.

Extremely skewed distribution (log-normal): From Table 8, we can observe that the MAD_H, SN_H, and QN_H charts have the least ARLs, thus they are the best charts in detecting shifts in the process dispersion. The Q_H, MD_H, and D_H charts by order come right after the best charts. Both S_H and R_H charts have the worst performance as all their run length measures are higher than other charts. For $n = 5$ and $\lambda = 0.05$, the SN_H, QN_H, and MAD_H performances are considered superior in terms of their ARLs, EQLs, and RARLs. Both S_H and R_H charts have the worst performances as their ARL and other performance measures.

5.2 | Effect of sample size (n)

We evaluated the performance of S_H and D_H charts for different levels of sample size $n = 5, 7, 10, 12$, and 15, when $\lambda = 0.05$. Table 11 reports the ARL values of the proposed S_H chart for normally distributed quality characteristic whereas for D_H chart, the ARL comparison is presented graphically in Figure 2. It can be observed from Table 11 and Figure 2 that the

TABLE 6 Average run length profiles of the HWMA dispersion chart for gamma distribution when $\lambda = 0.25$, $n = 7$, and ARL0 = 200

δ	R_H	S_H	Q_H	D_H	MD_H	MAD_H	SN_H	QN_H
1.0	201.64	199.74	200.45	199.37	200.33	201.21	201.14	199.90
1.1	62.36	59.77	63.36	50.97	51.71	68.21	63.04	61.42
1.2	28.62	27.05	29.56	22.09	21.94	33.02	29.23	27.88
1.3	16.56	15.49	17.32	12.74	12.57	19.57	17.10	16.27
1.4	10.96	10.30	11.60	8.33	8.35	13.33	11.70	11.18
1.5	7.87	7.46	8.57	6.12	6.05	9.90	8.54	8.11
1.6	6.10	5.79	6.61	4.84	4.73	7.65	6.69	6.31
1.8	4.11	3.96	4.59	3.26	3.33	5.27	4.56	4.37
2.0	3.16	2.98	3.50	2.54	2.56	3.99	3.53	3.30
2.5	2.00	1.93	2.22	1.72	1.71	2.55	2.30	2.13
3.0	1.53	1.49	1.72	1.35	1.34	1.97	1.76	1.66
3.5	1.28	1.26	1.45	1.19	1.18	1.63	1.50	1.41
4.0	1.17	1.156	1.31	1.10	1.09	1.45	1.36	1.28
L	**3.288**	**3.230**	**3.210**	**3.040**	**3.030**	**3.210**	**3.130**	**3.150**

TABLE 7 Average run length profiles of the HWMA dispersion chart for exponential distribution when $\lambda = 0.25$, $n = 7$, and ARL0 = 200

δ	R_H	S_H	Q_H	D_H	MD_H	MAD_H	SN_H	QN_H
1.0	200.69	200.41	201.63	200.65	199.48	200.17	199.75	201.38
1.1	76.40	75.43	72.66	64.14	62.04	80.31	75.31	74.93
1.2	38.26	36.94	36.11	30.06	28.28	41.75	38.71	37.46
1.3	22.66	22.12	21.91	17.80	16.54	25.67	23.34	23.01
1.4	15.18	14.68	15.22	11.79	11.22	17.84	16.16	15.59
1.5	11.20	10.71	11.04	8.65	8.09	13.32	11.93	11.46
1.6	8.56	8.32	8.53	6.71	6.39	10.42	9.24	8.87
1.8	6.78	5.64	5.89	4.61	4.42	7.09	6.29	5.99
2.0	4.31	4.19	4.45	3.50	3.32	5.24	4.75	4.49
2.5	2.66	2.62	2.77	2.22	2.16	3.28	3.03	2.85
3.0	1.99	1.96	2.10	1.69	1.64	2.47	2.27	2.14
3.5	1.65	1.60	1.73	1.44	1.41	2.02	1.85	1.76
4.0	1.44	1.41	1.53	1.27	1.25	1.75	1.63	1.55
L	**3.470**	**3.450**	**3.370**	**3.240**	**3.180**	**3.450**	**3.340**	**3.440**

ARL_1 of the S_H and the D_H charts decrease with an increase in the sample size n. We observed this phenomenon for all dispersion HWMA charts that the detection ability of the charts improves as n increases.

5.3 | Effect of smoothing parameter (λ)

The performance of S_H and MD_H charts for varying levels of sample size $\lambda = 0.03$, 0.05, 0.1, and 0.25, when $n = 7$ is evaluated, considering normal, logistic, and Gamma processes. Tables 13 and 14 present the ARL and EQL values of HWMA charts (together with EWMA charts). A graphical comparison of the ARL is also presented for MD_H chart for Gamma distributed process in Figure 3. From Tables 12 and 13 and Figure 3, it is observed that the ARL_1 of the HWMA dispersion charts reduces with the decrease in λ. It indicates that the proposed HWMA dispersion charts are more sensitive in the detection of shifts for small values of λ. It is expected that for $\lambda > 0.5$, the HWMA dispersion chart will perform better for the detection of large shifts as compared to using small values of λ. As the purpose of the study is the efficient detection of small process shifts, hence we recommend the use of $\lambda < 0.25$ for the proposed HWMA dispersion charts.

TABLE 8 Average run length profiles of the HWMA dispersion chart for log-normal distribution when $\lambda = 0.25$, $n = 7$, and ARL0 = 200

δ	R_H	S_H	Q_H	D_H	MD_H	MAD_H	SN_H	QN_H
1.0	200.66	199.27	199.68	200.48	201.45	200.10	201.82	199.02
1.1	131.71	131.37	103.66	122.06	116.14	100.98	99.19	99.55
1.2	91.67	90.53	61.33	80.08	72.95	58.19	55.64	54.37
1.3	65.21	65.21	40.20	55.21	48.21	37.50	36.36	34.21
1.4	49.29	49.29	28.15	39.83	34.00	26.51	24.87	23.81
1.5	38.22	38.35	20.85	29.63	25.47	19.92	18.40	17.54
1.6	30.47	30.62	16.33	23.24	19.82	15.47	14.06	13.61
1.8	20.72	21.12	11.03	15.52	13.21	10.16	9.46	9.27
2.0	15.22	15.53	8.01	11.25	9.44	7.63	7.08	6.53
2.5	8.60	8.63	4.74	6.44	5.45	4.49	4.19	3.97
3.0	5.83	5.80	3.35	4.36	3.80	3.19	3.03	2.87
3.5	4.44	4.41	2.65	3.35	2.92	2.59	2.40	2.30
4.0	3.57	3.52	2.22	2.75	2.41	2.19	2.04	1.94
L	**4.740**	**4.740**	**3.880**	**4.511**	**4.350**	**3.930**	**3.890**	**3.900**

TABLE 9 EQL and RARL for the HWMA dispersion charts for $\lambda = 0.25$, $n = 7$, and $ARL_0 = 200$

Distribution		R_H	S_H	Q_H	D_H	MD_H	MAD_H	SN_H	QN_H
Normal	EQL	14.69	14	18.54	14.17	14.54	21.51	18.7	18.44
	RARL	1.05	1.00	1.32	1.01	1.04	1.54	1.34	1.32
Logistic	EQL	18.31	16.9	19.99	16.28	16.36	23.47	20.86	20.69
	RARL	1.12	1.04	1.23	1.00	1.00	1.44	1.28	1.27
Student's *t*	EQL	24.95	23.08	21.05	20.26	18.97	24.57	22.34	22.15
	RARL	1.32	1.22	1.11	1.07	1.00	1.30	1.18	1.17
Weibull	EQL	17.65	17.24	20.86	16.13	16.48	23.47	21.1	20.27
	RARL	1.09	1.07	1.29	1.00	1.02	1.46	1.31	1.26
Gamma	EQL	21.18	20.48	22.76	18.38	18.35	25.25	23.06	21.98
	RARL	1.15	1.12	1.24	1.00	1.00	1.38	1.26	1.20
Chi-square	EQL	20.13	19.39	21.87	17.67	17.59	24.39	21.98	21.12
	RARL	1.14	1.10	1.24	1.00	1.00	1.39	1.25	1.20
Exponential	EQL	26.98	26.21	27.12	22.76	22.03	31.07	28.75	27.71
	RARL	1.22	1.19	1.23	1.03	1.00	1.41	1.31	1.26
Log-normal	EQL	69.82	69.85	42.66	55.37	48.73	40.94	38.76	37.25
	RARL	1.87	1.88	1.15	1.49	1.31	1.10	1.04	1.00

5.4 | Proposed HWMA charts versus EWMA charts

In this section, the performance of the proposed HWMA control charts is compared with the EWMA charts for monitoring shifts in process variability. Tables 12 and 13 present the ARL values of the HWMA and EWMA charts using the *S* and *MD* estimators, considering normal, logistic, and Gamma distributions at varying levels of $\lambda = 0.03, 0.05, 0.1, 0.25$. When the smoothing parameter λ is small (ie, $\lambda = 0.03$ or 0.05), the proposed HWMA charts are more sensitive than the EWMA dispersion charts. For example when $\lambda = 0.03$, considering *MD* estimator for Gamma distributed process, the proposed MD_H chart requires on average 7 samples to detect a shift of magnitude 1.2σ while the corresponding MD_E chart requires 15 samples to detect a shift of same magnitude. Similar comparisons can be drawn for other charts, considering specific choices of λ and n. This shows that the proposed charts are significantly better than their respective EWMA counterparts. As the smoothing parameter λ increases, the ARL of both HWMA and EWMA charts gets closer and closer to each other, with a slight advantage to HWMA dispersion charts. Also, it can be seen that the EQL values for HWMA charts in all cases

TABLE 10 EQL and RARL for the HWMA dispersion charts for $\lambda = 0.05$, $n = 5$, and $\text{ARL}_0 = 200$

Distribution		R_H	S_H	Q_H	D_H	MD_H	MAD_H	SN_H	QN_H
Normal	EQL	13.97	13.76	20.71	13.85	14.09	20.43	18.85	18.6
	RARL	1.02	1.00	1.51	1.01	1.02	1.48	1.37	1.35
Logistic	EQL	16.26	15.7	21.72	15.4	15.37	22.18	21.03	20.92
	RARL	1.06	1.02	1.41	1.00	1.00	1.44	1.37	1.36
Student's t	EQL	19.5	19.28	19.67	18.12	17.86	23.58	22.55	22.13
	RARL	1.09	1.08	1.10	1.01	1.00	1.32	1.26	1.24
Weibull	EQL	16.08	16.1	23.67	15.69	16.09	23.84	21.95	22.05
	RARL	1.02	1.03	1.51	1.00	1.03	1.52	1.40	1.41
Gamma	EQL	18.2	18.19	23.94	17.26	17.6	23.96	23.12	22.29
	RARL	1.05	1.05	1.39	1.00	1.02	1.39	1.34	1.29
Chi-square	EQL	17.03	16.76	22.1	15.97	15.98	22.59	21.77	23.37
	RARL	1.07	1.05	1.38	1.00	1.00	1.41	1.36	1.46
Exponential	EQL	22.7	22.94	29.89	21.11	21.2	28.36	27.13	27.21
	RARL	1.08	1.09	1.42	1.00	1.00	1.34	1.29	1.29
Log-normal	EQL	46.24	48.37	41.84	41.46	37.99	35.77	34.15	34.31
	RARL	1.35	1.42	1.23	1.21	1.11	1.05	1.00	1.00

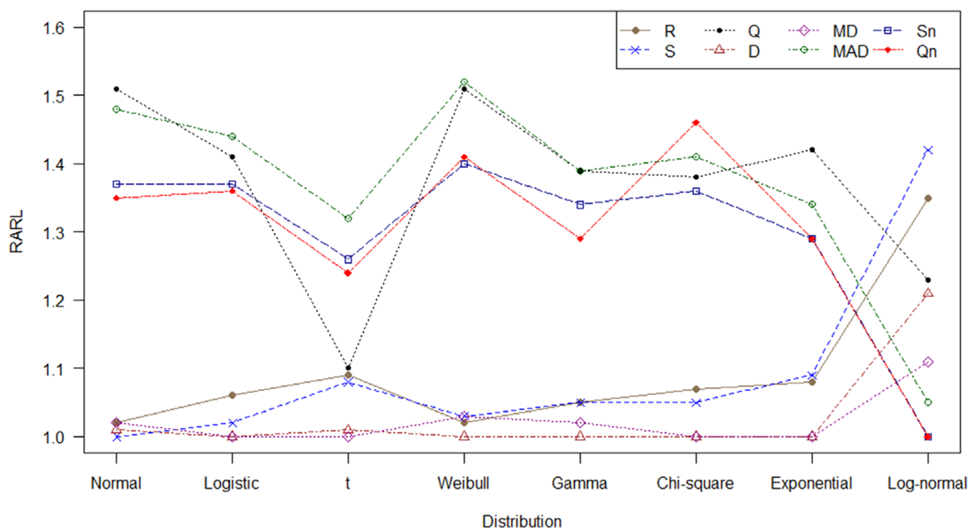

FIGURE 1 RARL comparison for dispersion HWMA control charts when $n = 5$ and $\lambda = 0.05$

is less than the EQL values of EWMA charts. It indicates that the overall performances of the HWMA charts are always better than their respective EWMA counterparts.

Moreover, Table 14 reports results of HWMA and EWMA charts based on R, S, D, MD, and QN dispersion estimators, considering all the parent distributions when $n = 5$ and when $\lambda = 0.05$. HWMA dispersion charts are better than the corresponding EWMA dispersion charts, among all distributions except for the extremely skewed distribution (log-normal) where the sensitivity of the EWMA chart is higher than that of the HWMA charts. For example, when $\lambda = 0.05$, the ARL values of the HWMA chart for logistic distribution when shift size $\delta = 1.5$ for S_H and D_H estimators are 4.12 and 3.97, respectively, while that of EWMA chart are 6.69 and 6.57, respectively.

5.5 | Proposed HWMA chart

Normal distribution:

TABLE 11 Average run length profiles of HWMA standard deviation SD chart for normal distribution when $\lambda = 0.05$ and ARL0 = 200 with varying sample size $n = 5, 7, 10, 12$, and 15

		δ												
n	**L**	**1.0**	**1.1**	**1.2**	**1.3**	**1.4**	**1.5**	**1.6**	**1.8**	**2.0**	**2.5**	**3.0**	**3.5**	**4.0**
5	**1.570**	201.34	23.12	9.2	5.51	3.89	3.12	2.58	2.02	1.69	1.3	1.15	1.08	1.05
7	**1.530**	200.58	17.35	6.82	4.08	3.01	2.4	2.01	1.6	1.34	1.31	1.04	1.01	1.00
10	**1.364**	200.74	11.87	4.76	2.93	2.19	1.8	1.54	1.25	1.14	1.02	1.01	1.00	1.00
12	**1.638**	200.55	11.57	4.56	2.98	2.19	1.77	1.52	1.23	1.09	1.01	1.00	1.00	1.00
15	**1.600**	199.12	9.75	4.01	2.52	1.89	1.53	1.32	1.12	1.04	1.01	1.00	1.00	1.00

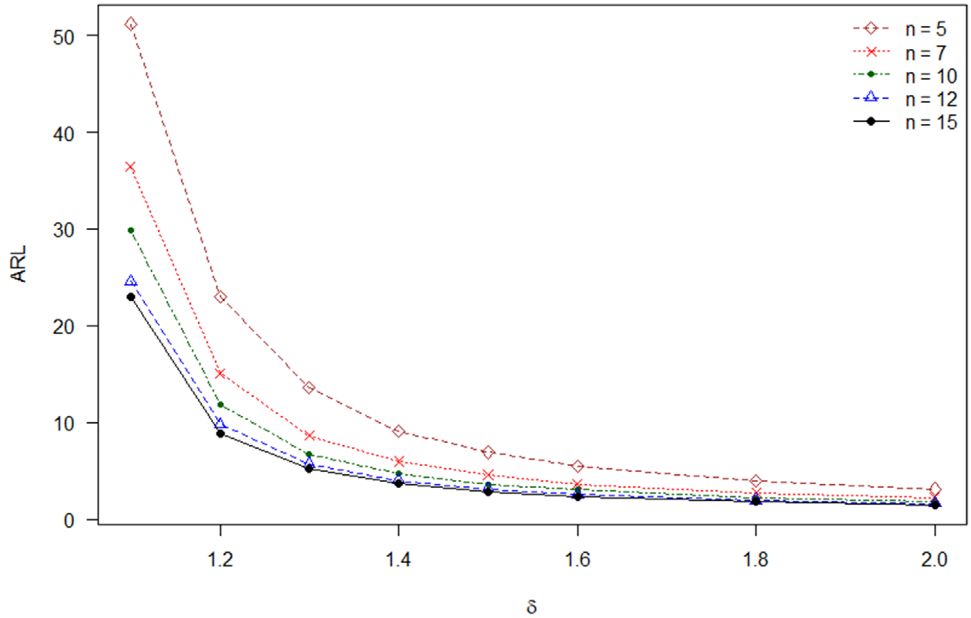

FIGURE 2 Effect of sample size on the ARL of D_H control chart for exponentially distributed process when $\lambda = 0.05$

For $n = 7$ and $\lambda = 0.25$ (Table 1)

- S_H has the best performance as it has the lowest ARL, EQL, and RARL values among the other competing charts.
- D_H is close competitor with S_H; they are the second best charts.
- R_H and MD_H charts lie between the best and worst chart performances.
- MAD_H, Q_H, SN_H, and QN_H have the worst performances in detecting shifts.

For $n = 5$ and $\lambda = 0.05$ (Table 10)

- S_H, D_H, R_H, and MD_H charts have the best performance compared to other charts as their ARL values are the lowest for all shift sizes.
- SN_H and QN_H charts lie between the best and worst chart performances.
- Q_H has the highest ARL values and thus it is has the worst performance

Nonnormal symmetric distributions (logistic, and Student's t):
For $n = 7$ and $\lambda = 0.25$ (Tables 2 and 3)

- MD_H and D_H charts for both parent distributions have the best performance in terms of their ARL values.
- The R_H and S_H charts are less efficient for the Student's t distribution, compared with the logistic distribution.
- The MAD_H, SN_H, and QN_H are the worst performing charts in both distributions.

TABLE 12 Average run length profiles of HWMA and EWMA standard deviation SD chart for normal, logistic, and gamma distributions with varying smoothing constant $\lambda = 0.03$, 0.05, 0.1, and 0.25 when $n = 7$ and ARL0 = 200

				δ									
λ	Distribution	Chart	L	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.8	2.0	EQL
0.03	**Normal**	**HWMA**	**1.120**	199.90	11.85	4.88	3.20	2.45	1.99	1.74	1.41	1.25	**15.68**
		EWMA	**0.190**	199.37	25.17	11.79	7.84	5.84	4.72	3.98	3.09	2.57	**22.80**
	Logistic	**HWMA**	**1.280**	200.08	18.16	7.17	4.38	3.22	2.58	2.18	1.73	1.46	**17.72**
		EWMA	**0.199**	200.31	31.20	14.83	9.63	7.14	5.77	4.82	3.74	3.06	**25.74**
	Gamma	**HWMA**	**1.150**	201.55	21.56	8.67	5.16	3.77	3.01	2.54	1.98	1.65	**19.14**
		EWMA	**0.188**	201.92	36.51	17.09	11.13	8.35	6.60	5.51	4.22	3.45	**28.20**
0.05	**Normal**	**HWMA**	**1.530**	200.58	17.35	6.82	4.08	3.01	2.40	2.01	1.60	1.34	**17.27**
		EWMA	**0.295**	201.87	25.94	11.92	7.62	5.68	4.55	3.81	2.96	2.46	**22.73**
	Logistic	**HWMA**	**1.740**	199.27	26.35	10.22	6.02	4.13	3.23	2.72	2.01	1.68	**20.19**
		EWMA	**0.303**	200.51	32.04	14.74	9.26	6.85	5.47	4.60	3.50	2.89	**25.35**
	Gamma	**HWMA**	**1.680**	199.61	32.83	13.11	7.54	5.21	3.96	3.28	2.40	1.97	**22.62**
		EWMA	**0.294**	199.22	37.82	17.83	11.08	8.20	6.44	5.36	4.13	3.33	**28.09**
0.10	**Normal**	**HWMA**	**2.130**	200.39	24.55	10.05	5.71	3.94	3.07	2.53	1.88	1.55	**19.67**
		EWMA	**0.500**	198.22	27.26	11.69	7.14	5.25	4.14	3.48	2.66	2.17	**21.98**
	Logistic	**HWMA**	**2.460**	201.07	37.57	15.37	8.87	5.91	4.39	3.51	2.57	2.05	**24.29**
		EWMA	**0.521**	201.20	34.86	14.93	9.16	6.59	5.12	4.28	3.24	2.62	**25.21**
	Gamma	**HWMA**	**2.500**	199.92	47.28	20.91	11.93	7.88	5.83	4.61	3.25	2.54	**28.49**
		EWMA	**0.520**	200.79	44.26	19.31	11.58	8.18	6.28	5.22	3.88	3.12	**28.91**
0.25	**Normal**	**HWMA**	**2.660**	200.67	29.78	11.87	6.80	4.62	3.52	2.79	2.04	1.62	**21.23**
		EWMA	**0.950**	201.73	32.34	12.24	7.03	4.84	3.64	3.02	2.25	1.89	**22.09**
	Logistic	**HWMA**	**3.066**	201.74	46.00	19.35	10.96	7.29	5.33	4.09	2.85	2.25	**27.23**
		EWMA	**1.023**	198.27	45.85	17.99	9.95	6.73	4.98	3.99	2.89	2.31	**26.50**
	Gamma	**HWMA**	**3.230**	199.74	59.77	27.05	15.49	10.30	7.46	5.79	3.96	2.98	**33.41**
		EWMA	**1.068**	199.96	55.95	24.56	13.68	8.97	6.60	5.24	3.73	2.93	**31.46**

For $n = 5$ and $\lambda = 0.05$ (Table 10)

- MD_H chart is the best in terms of their EQL and RARL values.
- The R_H, S_H, SN_H, and QN_H lie between the superior and inferior charts.
- The MAD_H charts for both distributions have the worst performance in terms of all the run length measures.

Skewed distributions (Weibull, chi-square, gamma, and exponential):
For $n = 7$ and $\lambda = 0.25$ (Tables 4–7)

- The performances of MD_H and D_H charts are the best, as compared to all other competing charts.
- The S_H, R_H, and the Q_H charts have almost the same performance among the four distributions but are more efficient for parent Weibull distribution, compared to the other distributions.
- The MAD_H, SN_H, and QN_H were the least efficient charts as their ARLs, EQLs, and RARLs are high.

For $n = 5$ and $\lambda = 0.05$ (Table 9)

- The performances of MD_H and D_H charts are the best, as compared to all other competing charts.
- The S_H, R_H, and the Q_H charts have almost the same performance among the four distributions but are more efficient for parent Weibull distribution, compared to the other distributions.
- The MAD_H, SN_H, and QN_H were the least efficient charts as their ARLs, EQLs, and RARLs are high.

TABLE 13 Average run length profiles of HWMA and EWMA mean deviation from sample median MD chart for normal, logistic, and gamma distribution with varying smoothing constant $\lambda = 0.03$, 0.05, 0.1, and 0.25 when $n = 7$ and ARL0 = 200

				δ									
λ	Distribution	Chart	L	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.8	2.0	EQL
0.03	**Normal**	**HWMA**	**1.114**	200.12	12.76	5.30	3.37	2.52	2.09	1.80	1.47	1.28	**16.00**
		EWMA	**0.190**	199.66	15.39	12.57	8.13	6.11	4.94	4.17	3.21	2.65	**22.07**
	Logistic	**HWMA**	**1.200**	201.22	16.12	6.72	4.06	3.04	2.44	2.08	1.66	1.43	**17.25**
		EWMA	**0.195**	199.85	30.01	14.23	9.34	6.95	5.60	4.71	3.63	2.97	**25.22**
	Gamma	**HWMA**	**1.100**	201.28	17.74	7.36	4.52	3.29	2.64	2.27	1.78	1.54	**17.91**
		EWMA	**0.191**	199.08	33.63	15.97	10.47	7.74	6.26	5.23	4.03	3.28	**26.94**
0.05	**Normal**	**HWMA**	**1.500**	199.35	17.96	7.13	4.28	3.09	2.50	2.09	1.63	1.40	**17.47**
		EWMA	**0.294**	201.18	27.13	12.46	8.06	5.92	4.74	4.01	3.09	2.53	**23.27**
	Logistic	**HWMA**	**1.650**	200.84	24.01	9.46	5.49	3.84	3.04	2.56	1.93	1.60	**19.54**
		EWMA	**0.300**	199.70	31.46	14.33	9.14	6.71	5.41	4.53	3.44	2.83	**25.03**
	Gamma	**HWMA**	**1.600**	201.16	27.64	10.95	6.46	4.42	3.44	2.87	2.18	1.80	**20.94**
		EWMA	**0.299**	201.21	35.84	16.37	10.45	7.70	6.11	5.07	3.86	3.16	**27.11**
0.10	**Normal**	**HWMA**	**2.180**	201.56	27.69	11.11	6.29	4.36	3.32	2.71	2.01	1.64	**20.69**
		EWMA	**0.500**	201.32	28.66	12.27	7.56	5.47	4.34	3.64	2.77	2.27	**22.72**
	Logistic	**HWMA**	**2.360**	201.19	34.79	14.27	8.17	5.47	4.15	3.38	2.42	1.93	**23.34**
		EWMA	**0.515**	201.23	34.21	14.63	8.94	6.39	5.06	4.17	3.15	2.58	**24.88**
	Gamma	**HWMA**	**2.380**	201.05	40.85	17.69	9.93	6.65	4.98	3.96	2.85	2.27	**25.92**
		EWMA	**0.516**	201.07	39.75	16.98	10.34	7.45	5.83	4.72	3.55	2.90	**27.10**
0.25	**Normal**	**HWMA**	**2.722**	201.54	33.02	13.21	7.55	5.17	3.77	3.02	2.18	1.76	**22.38**
		EWMA	**0.965**	200.59	35.07	13.50	7.57	5.18	3.96	3.22	2.41	1.94	**22.97**
	Logistic	**HWMA**	**2.920**	200.19	42.13	17.34	11.69	6.61	4.93	3.85	2.71	2.12	**26.06**
		EWMA	**0.992**	199.23	42.25	16.43	9.28	6.29	4.72	3.80	2.75	2.23	**25.43**
	Gamma	**HWMA**	**3.030**	200.33	51.71	21.94	12.57	8.35	6.05	4.73	3.33	2.56	**29.55**
		EWMA	**1.040**	199.58	65.22	29.60	17.62	11.80	8.59	6.75	4.79	3.71	**36.53**

Extremely skewed distribution (log-normal):
For $n = 7$ and $\lambda = 0.25$ (Table 8)

- The MAD_H, SN_H, and QN_H charts have the least ARLs, thus they are the best charts in detecting shifts in the process dispersion.
- The Q_H, MD_H, and D_H charts by order come right after the best charts.
- Both S_H and R_H charts have the worst performance as all their run length measures are higher than other charts.

For $n = 5$ and $\lambda = 0.05$

- The SN_H, $QN_{H,}$ and MAD_H performances are considered superior in terms of their ARLs, EQLs, and RARLs.
- Both S_H and R_H charts have the worst performances as their ARL and other performance measures.

5.6 | Effect of sample size (n)

- It can be observed from Table 11 and Figure 2 that the ARL_1 of the S_H and the D_H charts decrease with an increase in the sample size n.
- For all dispersion HWMA charts that the detection ability of the charts improves as n increases.

TABLE 14 ARL for HWMA and EWMA for some dispersion charts when $\lambda = 0.05$, $n = 5$, and $ARL_0 = 200$

Distribution	δ	HWMA R_H	S_H	D_H	MD_H	QN_H	EWMA R_E	S_E	D_E	MD_E	QN_E
Normal	1.1	24.22	23.12	23.68	24.65	39.68	33.53	32.65	33.48	34.33	49.93
	1.2	9.83	9.20	9.36	10.06	16.97	15.76	15.19	15.24	15.56	24.08
	1.5	3.19	3.12	3.12	3.22	5.17	5.91	5.72	5.78	5.92	8.87
	2.0	1.73	1.69	1.71	1.78	2.59	3.06	3.00	3.01	3.08	4.50
Logistic	1.1	36.22	33.46	32.07	31.08	48.44	40.09	38.85	37.91	38.74	53.66
	1.2	14.65	13.66	12.86	12.71	21.73	19.35	18.17	17.80	17.85	26.06
	1.5	4.39	4.12	3.97	3.89	6.42	7.07	6.69	6.57	6.63	9.61
	2.0	2.20	2.08	2.03	2.02	3.06	3.58	3.46	3.40	3.43	4.79
Student's t	1.1	51.43	51.30	44.95	43.88	52.84	59.04	56.51	54.07	53.03	63.16
	1.2	22.10	22.02	19.13	18.01	23.58	29.61	28.86	27.40	27.04	33.51
	1.5	6.20	6.18	5.48	5.32	7.10	11.71	11.44	11.10	10.97	13.37
	2.0	2.79	2.77	2.51	2.52	3.29	5.99	5.92	5.81	5.69	6.90
Weibull	1.1	34.08	34.49	32.45	33.79	52.18	39.73	40.35	38.16	38.18	52.45
	1.2	14.29	14.35	13.31	14.01	23.69	18.59	18.65	17.74	17.45	25.01
	1.5	4.39	4.36	4.16	4.30	6.99	6.65	6.74	6.44	6.54	9.24
	2.0	2.12	2.18	2.08	2.13	3.30	3.46	3.46	3.30	3.32	4.60
Gamma	1.1	42.67	43.64	38.72	40.37	52.25	62.36	59.77	50.97	51.71	61.42
	1.2	18.27	18.19	16.56	17.12	24.16	28.62	27.05	22.09	21.94	27.88
	1.5	5.48	5.48	4.93	5.05	7.18	7.87	7.46	6.12	6.05	8.11
	2.0	2.57	2.61	2.38	2.44	3.29	3.16	2.98	2.54	2.56	3.30
Chi-square	1.1	38.63	37.82	33.64	33.61	49.61	45.71	45.29	42.07	42.55	54.37
	1.2	16.34	15.92	13.88	13.95	22.38	21.35	21.41	20.11	20.12	26.58
	1.5	4.78	4.71	4.19	4.27	6.83	7.95	7.90	7.41	7.41	9.77
	2.0	2.35	2.32	2.14	2.13	3.15	4.01	4.01	3.83	3.81	4.90
Exponential	1.1	56.32	56.48	51.19	51.41	65.42	76.40	75.43	64.14	62.04	74.93
	1.2	26.35	26.42	22.96	22.86	31.82	38.26	36.94	30.06	28.28	37.46
	1.5	7.77	7.85	6.91	6.87	9.89	11.20	10.71	8.65	8.09	11.46
	2.0	3.44	3.46	3.14	3.18	4.32	4.31	4.19	3.50	3.32	4.49
Log-normal	1.1	105.55	108.79	98.46	92.87	84.33	131.71	131.37	122.06	116.14	99.55
	1.2	63.12	66.00	57.46	50.15	45.05	91.67	90.53	80.08	72.95	54.37
	1.5	21.79	23.47	18.76	16.87	14.27	38.22	38.35	29.63	25.47	17.54
	2.0	8.64	8.99	7.30	6.66	5.67	15.22	15.53	11.25	9.44	6.53

5.7 | Effect of smoothing parameter (λ)

- From Tables 12 and 13 and Figure 3, it is observed that the ARL_1 of the HWMA dispersion charts reduces with the decrease in λ.
- The proposed HWMA dispersion charts are more sensitive in the detection of shifts for small values of λ.

5.8 | Proposed HWMA charts versus EWMA charts

Tables 12 and 13 present the ARL values of the HWMA and EWMA charts using the S and MD estimators, considering normal, logistic, and Gamma distributions at varying levels of $\lambda = 0.03, 0.05, 0.1, 0.25$.

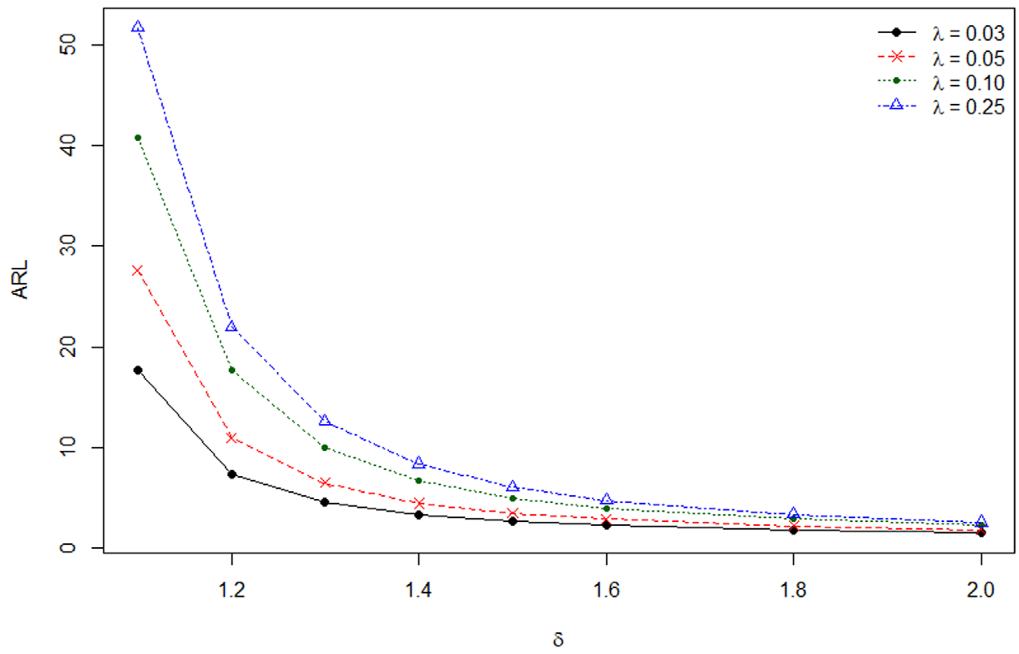

FIGURE 3 Effect of λ on the ARL of MD_H control chart for gamma distributed process when $n = 7$

- When the smoothing parameter λ is small (ie, $\lambda = 0.03$ or 0.05), the proposed HWMA charts are more sensitive than the EWMA dispersion charts.
- As the smoothing parameter λ increase, the ARL of both HWMA and EWMA charts gets closer and closer to each other, with a slight advantage to HWMA dispersion charts.
- It can be seen that the EQL values for HWMA charts in all cases are less than the EQL values of EWMA charts. It indicates that the overall performances of the HWMA charts are always better than their respective EWMA counterparts.

Table 14 reports results of HWMA and EWMA charts based on *R, S, D, MD*, and *QN* dispersion estimators, considering all the parent distributions when $n = 5$ and when $\lambda = 0.05$.

- HWMA dispersion charts are better than the corresponding EWMA dispersion charts, among all distributions except for the extremely skewed distribution (log-normal) where the sensitivity of the EWMA chart is higher than that of the HWMA charts.

6 | REAL DATA APPLICATION

This section provides the real-life application of the proposed charts for monitoring the vapor pressure of the distillation process. The dataset is obtained from https://openmv.net/info/distillation-tower. A brief explanation of the distillation process and the application of the proposed charts are provided below:

6.1 | Distillation process

Distillation is an important process in chemical and industrial engineering. It separates the liquid components from a mixture based on their selective boiling and condensation, that is, vapor- and liquid-phase distribution. During this process, a distillation tower allows the liquids to distribute in multiple liquids by passing through different stages. In the first stage, time is a most important factor that allows the chemical mixtures to separate. However, distribution in later stages relies on the properties of each chemical, for example, vapor pressure, evaporation point, the properties of mixtures being separated, and the desired purity of the end distillate. In principle, these properties decide the fate of molecule to fall or rise depending on their heavy and light components followed by separation of different liquid components. A brief

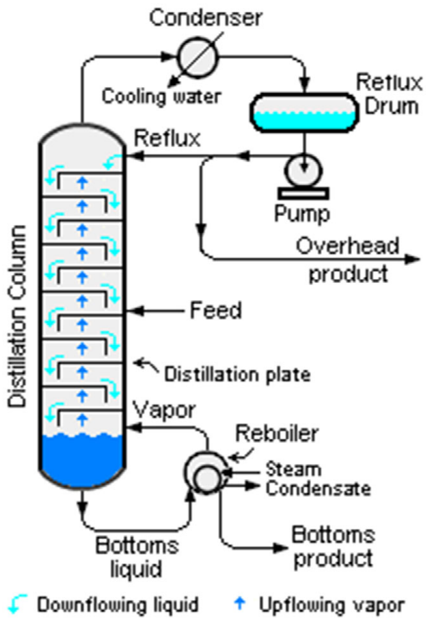

FIGURE 4 A representation of distillation column (https://en.wikipedia.org/wiki/Distillation)

explanation of the distillation process is shown in Figure 4 (https://en.wikipedia.org/wiki/Distillation). The schematic diagram of distillation unit represents a single feed that separates liquid mixtures into two produces: overhead product and bottom product. Usually, the bottom products are liquids (heavy components) while the overhead products can be either liquid or vapor or both (light components).[29,30] Furthermore, for necessary vaporization, the reboiler is used where it works on providing heat from the bottom and a vapor is produced where it flows upward through the distillation column. Thereon, the condenser is used for cooling and condensing the vapor from the top. A liquid is produced in reflux drum and it flows downward in the distillation column to be recycled back.

6.2 | Quality control application

This section uses the vapor pressure variable, which is a measure of the tendency of a material to change into the vapor state. It is one of the chemical properties that decide the separation of liquids into different components in the distillation process and thus it is an important variable to be monitored. The following steps were taken to apply the proposed charts on this dataset:

First, we find the distribution of the data to be used in constructing the upper control limits of the proposed HWMA charts. We used the *fitdist* function in *fitdistrplus* package in R and tested the data for four potential distributions which are normal, gamma, Weibull and lognormal. We identified the best-fitted distribution for this data as gamma distribution with scale and rate parameters 31.08 and 0.848, respectively. Moreover, we also used the *descdist* function to plot the Cullen and Frey Graph, given below in Figure 5; this graph plots the square of skewness against the kurtosis. It also indicated that gamma is the best-fitted distribution.

After distribution fitting, 50 samples size of $n = 5$ are drawn from the data and control limits are computed for a fixed $ARL_0 = 200$ using $\lambda = 0.05$. These samples were standardized to have mean and standard deviation equals 0 and 1, respectively. For illustration purpose, we used S_H, $D_{H,}$ and MAD_H control charts. The control chart coefficients for three charts are found to be $L_S = 1.350$, $L_D = 1.361$, and $L_{MAD} = 1.607$.

To check the detection ability of the proposed HWMA charts, shift of $\delta = 3$ was introduced in the standard deviation of last 10 samples. Figure 6 depicts the resulting control chart plots. As it can be seen in Figure 6, the S_H chart detected 6 out-of-control points, 7 out-of-control points were detected by D_H chart and 5 out-of-control points were detected by MAD_H chart with one false alarm.

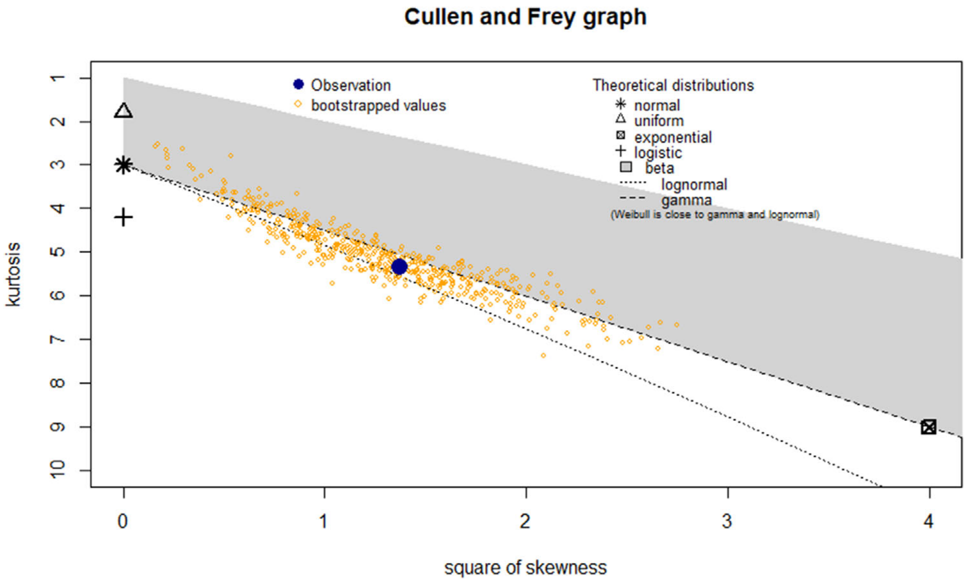

FIGURE 5 Cullen and Frey graph for the real-life data

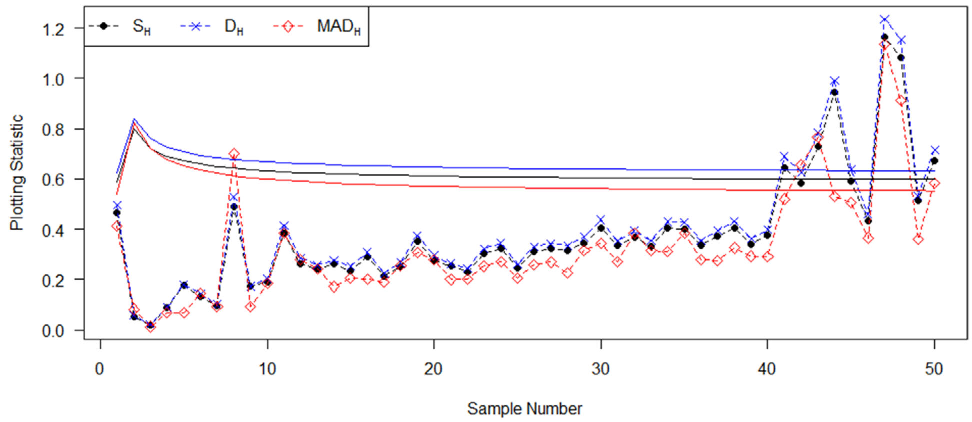

FIGURE 6 Control chart displays for the real-life data

7 | CONCLUSION

This paper proposed and investigated a set of HWMA charts for detecting shifts in process dispersion for normal and nonnormal parent distributions, considering a wide range of dispersion estimators. The study revealed that the S_H chart followed by the R_H and the MD_H charts perform better for normally distributed process while for most of the skewed and heavy symmetric parent distributions, MD_H and D_H charts are the best performing charts. For extremely skewed case of lognormal distribution, SN_H, QN_H, and MAD_H charts had superior performances compared to other charts. Comparison with the EWMA dispersion charts indicated superiority of the HWMA dispersion charts, particularly when for small choices of smoothing parameter. Hence, we recommend the use of HWMA dispersion charts for efficient monitoring of process dispersion.

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How to cite this article: Abbasi SA, Nassar SH, Aldosari MM, Adeoti OA. Efficient homogeneously weighted dispersion control charts with an application to distillation process. *Qual Reliab Eng Int*. 2021;37:3221–3241. https://doi.org/10.1002/qre.2904