ISA 444: Business Forecasting

18: ARMA Model Identification and Fitting

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Quick Refresher from Last Class

- ✅ Utilize time-series plots (line charts and ACF) to identify whether a ts is stationary.
- ✅ Apply transformations to a nonstationary time series to bring it into stationarity (**review**).
- ✅ Conduct formal tests for stationarity using the ADF and KPSS tests.

Assignment #12 - Review

We will go over any questions you may have had pertaining to assignment 12.

Learning Objectives for Today's Class

- Describe the behavior of the ACF and PACF of an AR(p) process.
- Describe the behavior of the ACF and PACF of an MA(q) process.
- Describe the behavior of the ACF and PACF of an ARMA (p,q) process.
- Fit an ARMA model to a time series, evaluate the residuals of a fitted ARMA model to assess goodness of fit, use the Ljung-Box test for correlation among the residuals of an ARIMA model.

Preface: ARMA Models

Models we consider here may have two components, an **autoregressive component** (AR) and a **moving average** component (MA).

Autoregressive Processes

The First Order Autoregressive Process

The **First Order Autoregressive Process—AR(1)** is given by

$$y_t = \delta + \phi y_{t-1} + \epsilon_t,$$

where $|\phi| < 1$ is a weight, and ϵ_t is white noise. Essentially, this is similar (not exactly the same though) as regressing y_t on y_{t-1}.

The mean and variance of an AR(1) process are as follows:

$$E(y_t) = \mu = \frac{\delta}{1 - \phi}$$

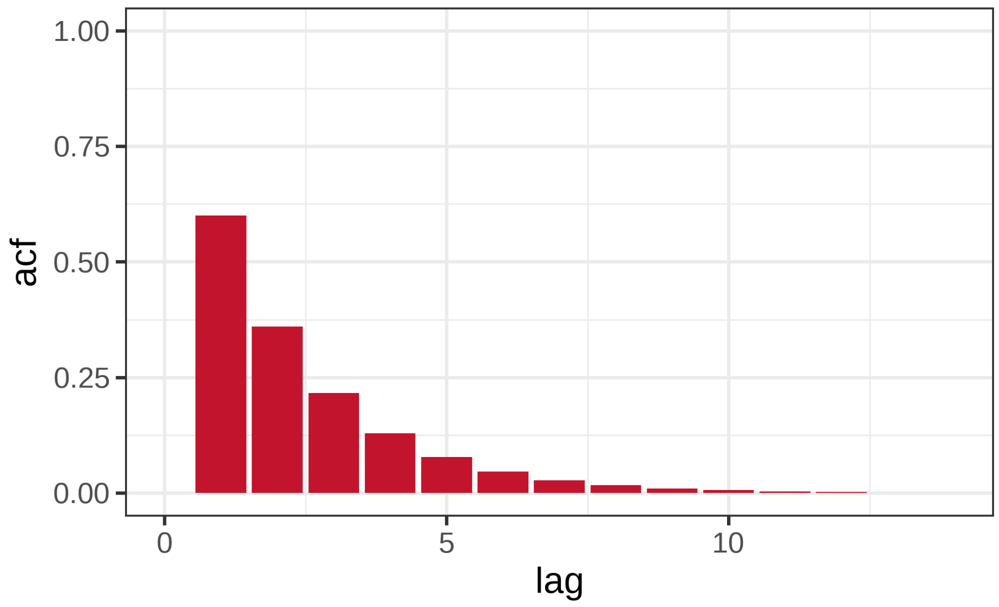
$$Var(y_t) = \sigma^2 \frac{1}{1 - \phi^2}$$

The First Order Autoregressive Process

The *population* autocorrelation function of the AR(1) process at lag k is

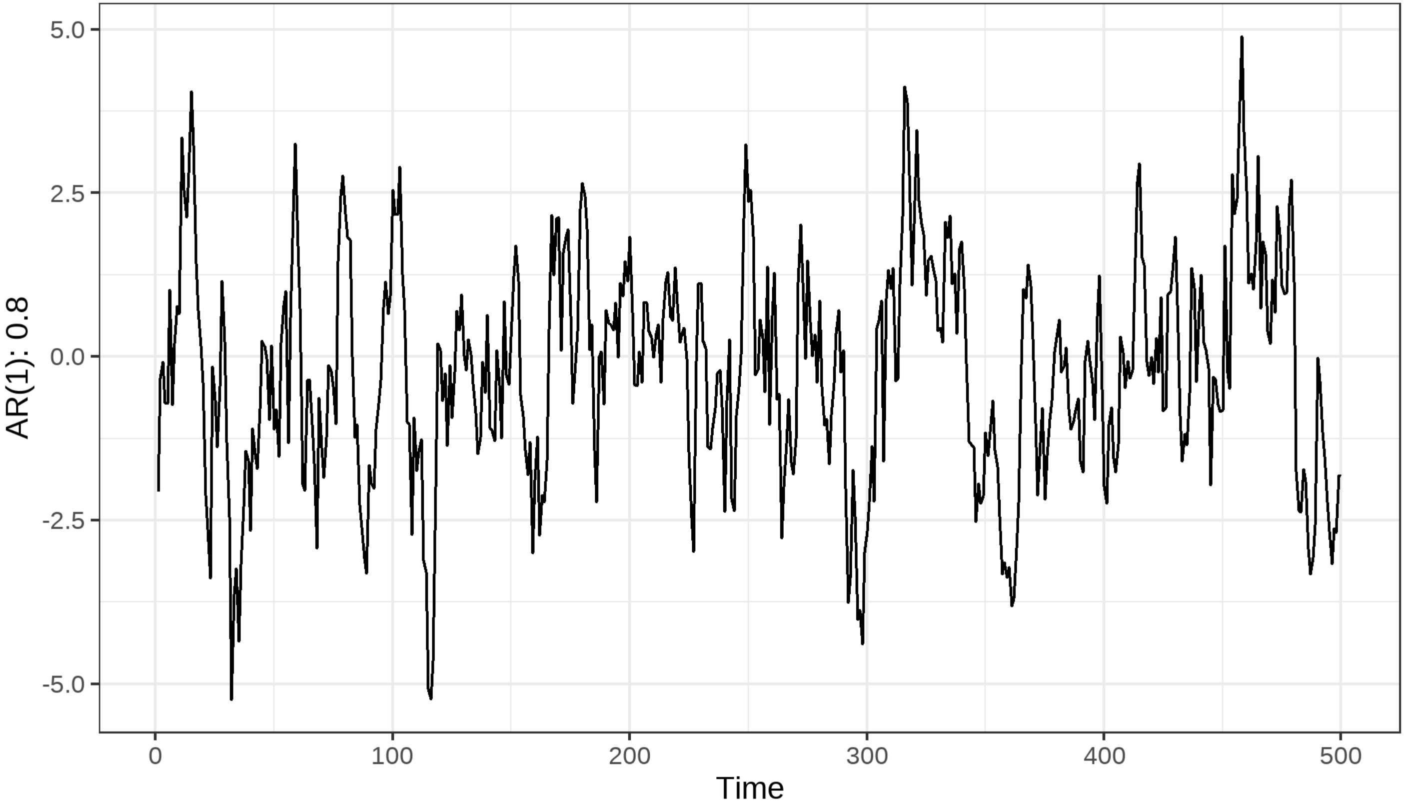
$$\rho(k) = \phi^k$$

The theoretical/population ACF of an AR(1) process with $\phi = 0.6$ will look like this:



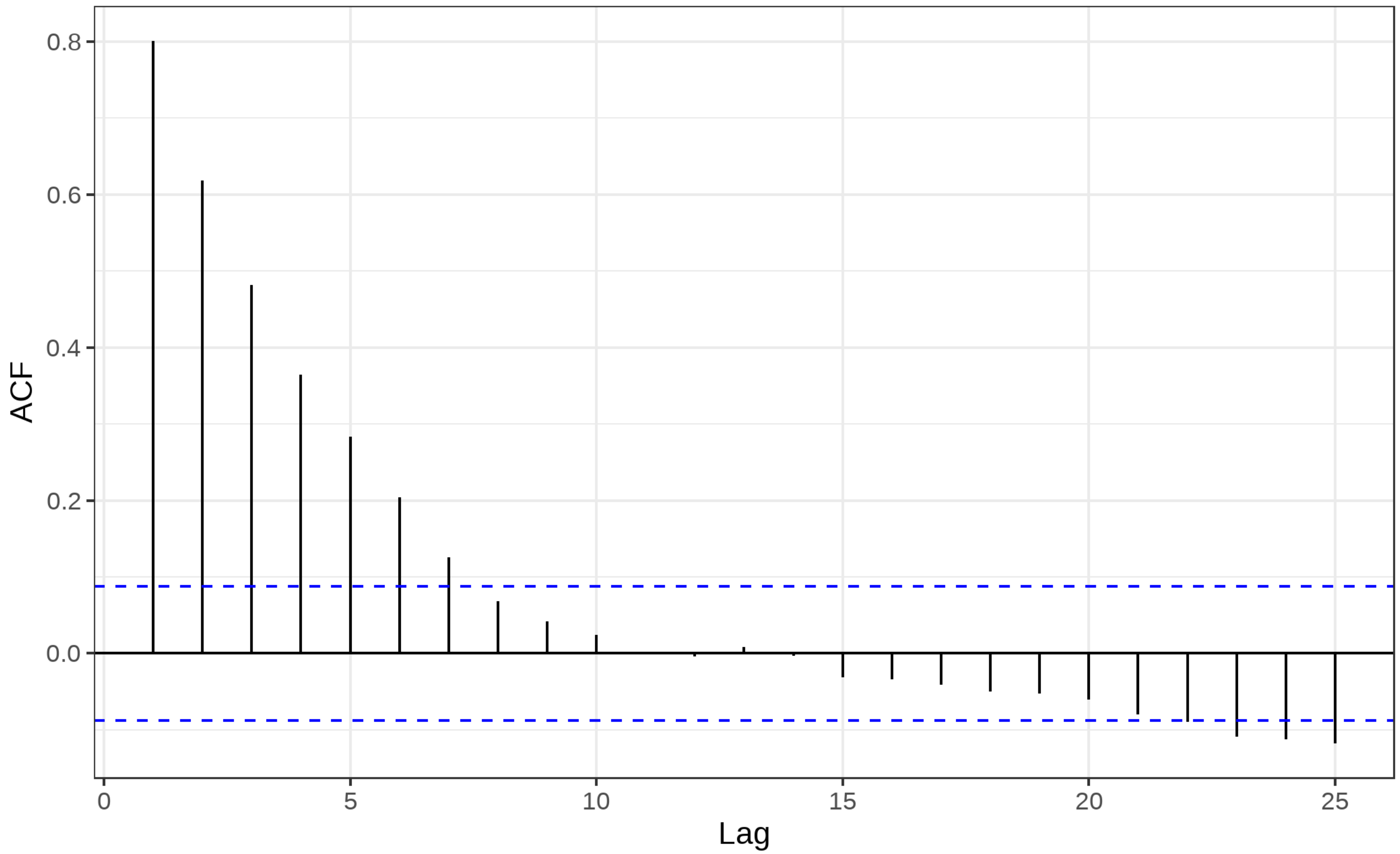
TS that can be Modeled as an AR(1) Process

TS that can be approximated using an AR(1) model will be **stationary**.



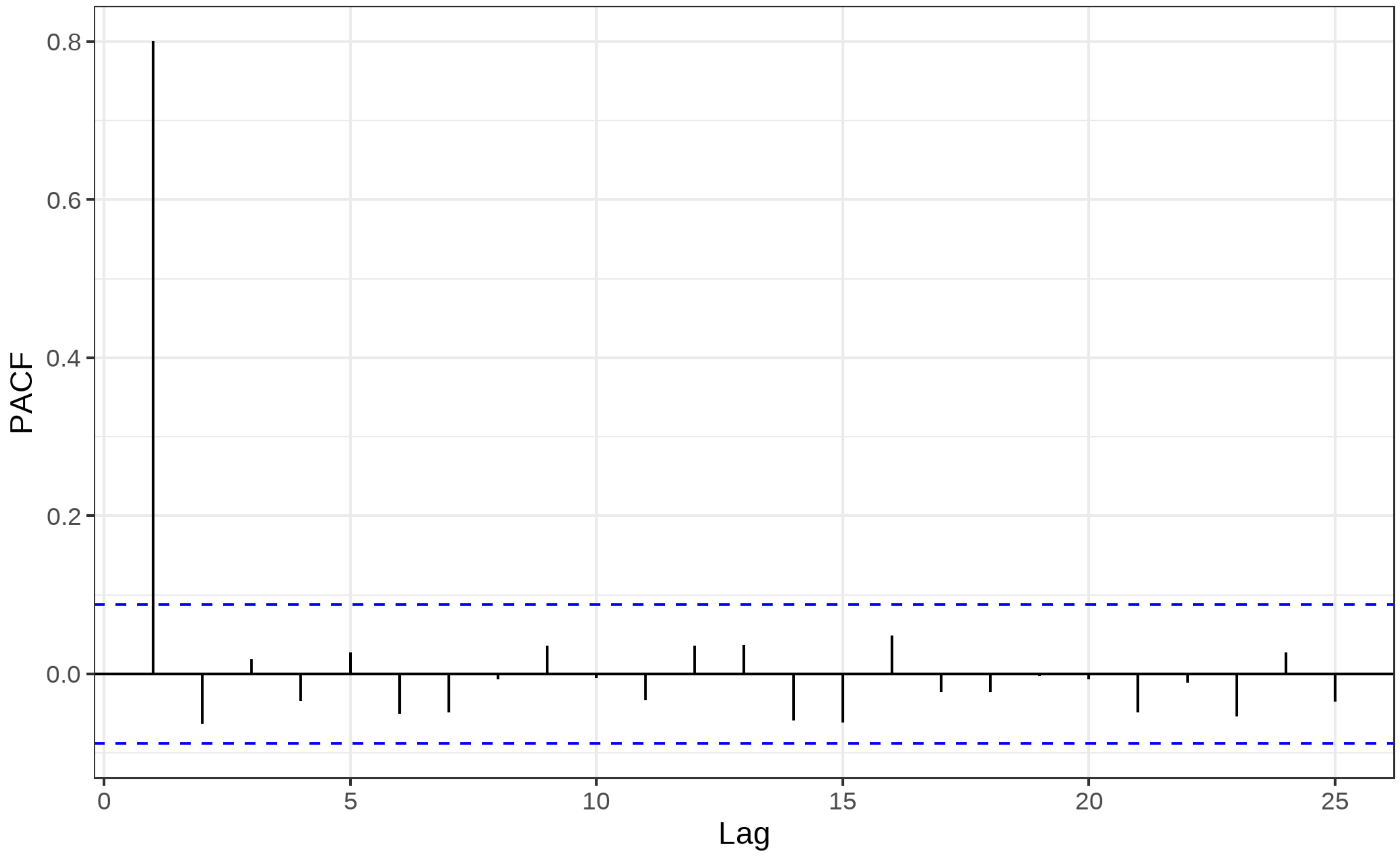
TS that can be Modeled as an AR(1) Process

TS that can be approximated using an AR(1) model will have **an ACF that dies down**.



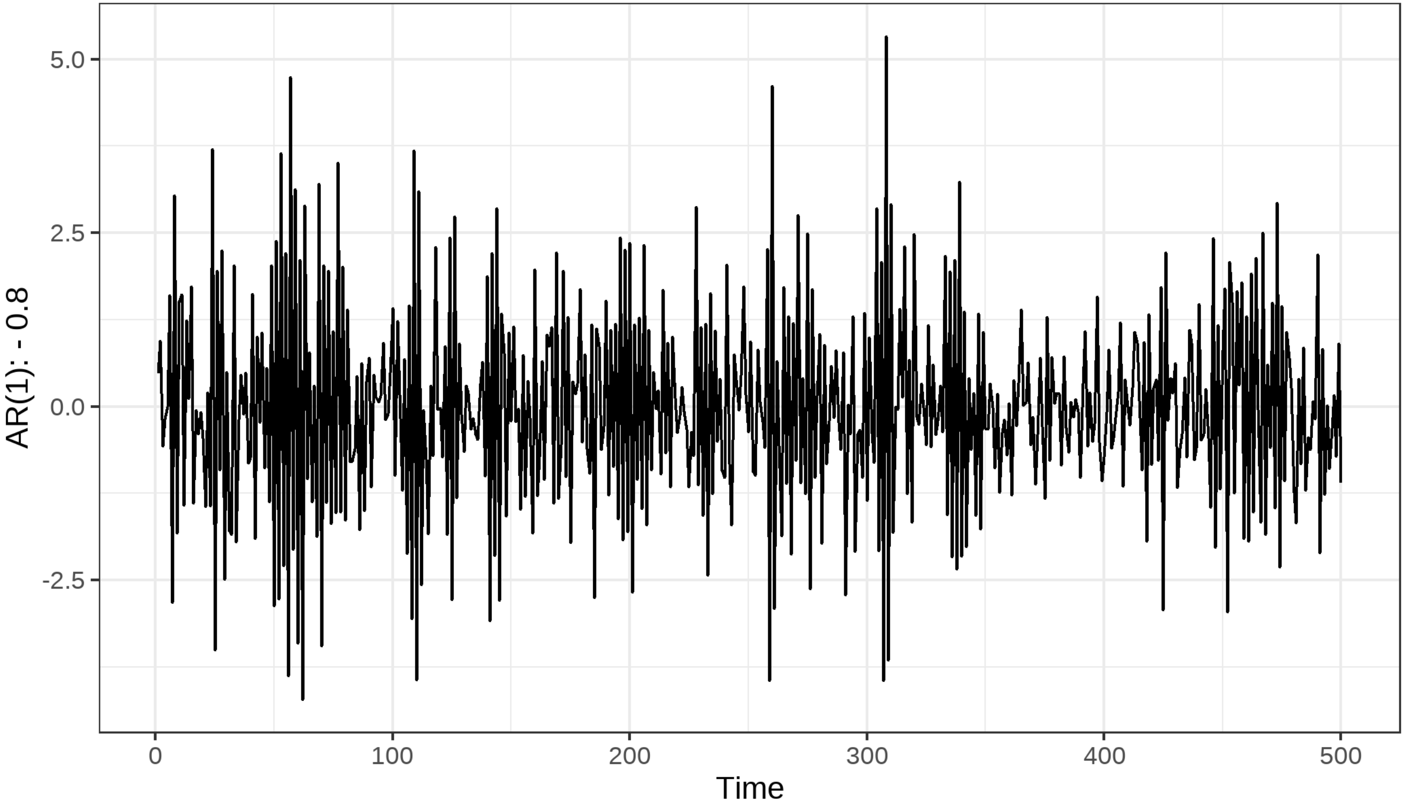
TS that can be Modeled as an AR(1) Process

TS that can be approximated using an AR(1) model will have **a PACF that cuts-off at lag 1**.



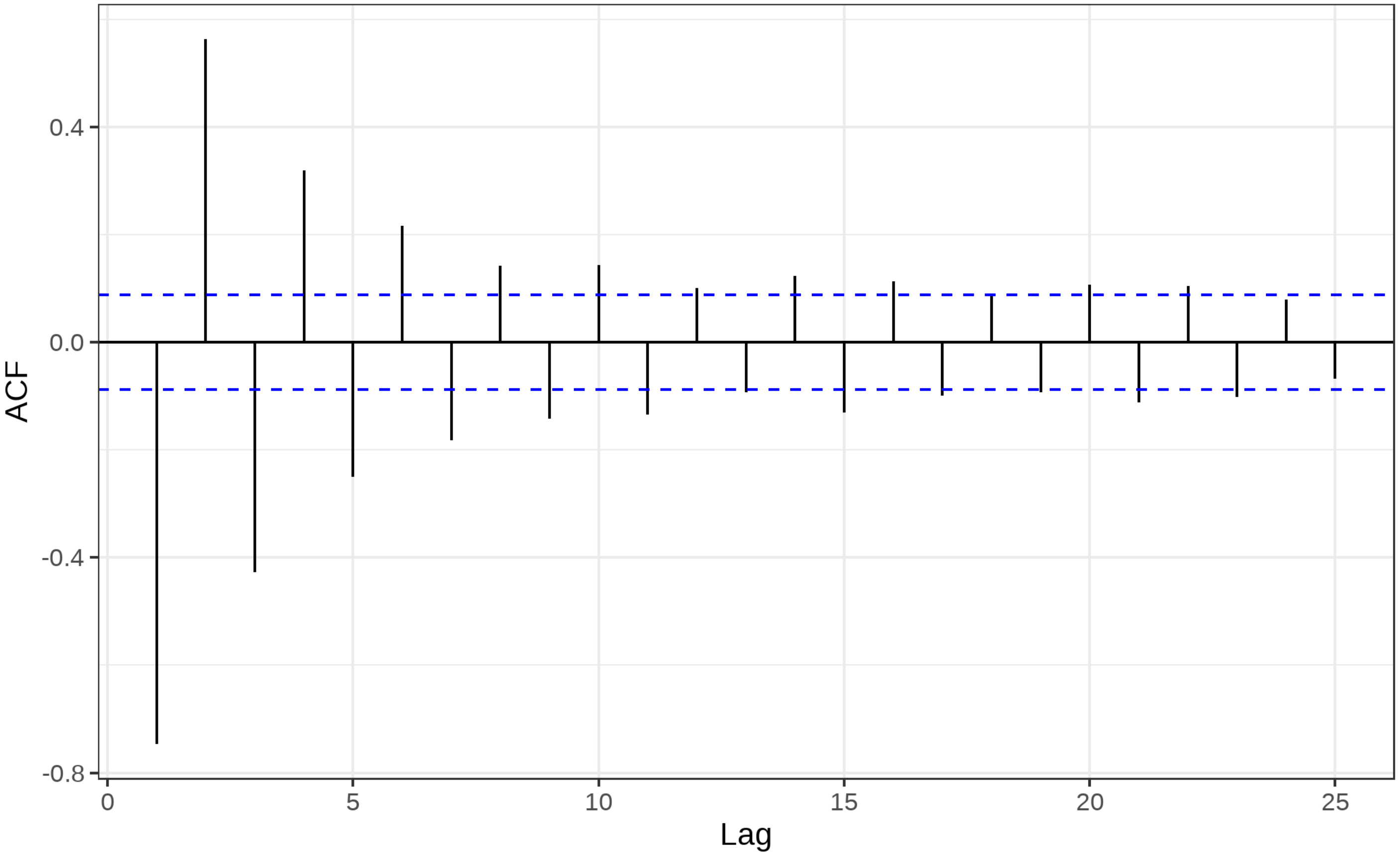
Another Visual Example for an AR(1) Process

TS that can be approximated using an AR(1) model will **be stationary**.



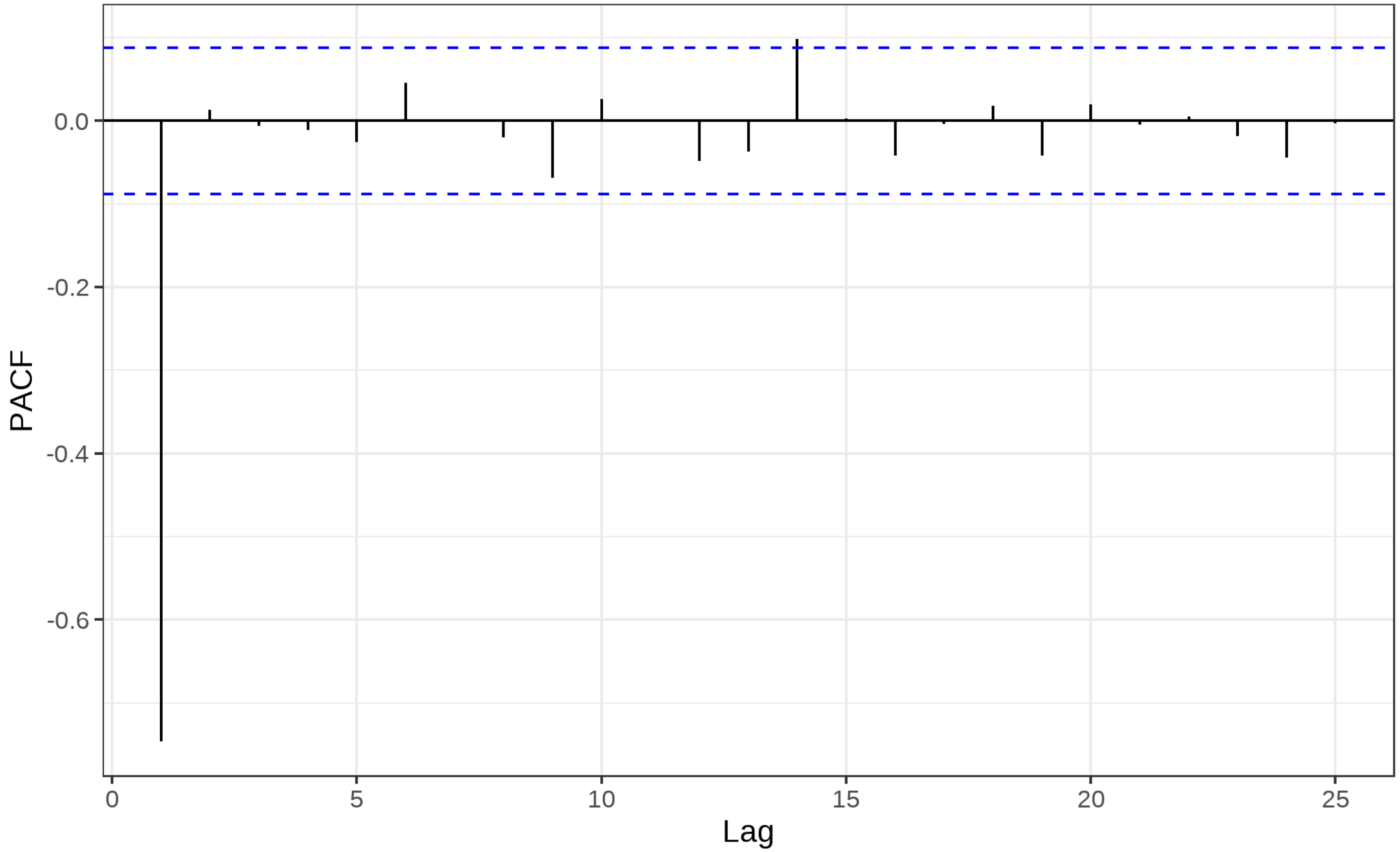
Another Visual Example for an AR(1) Process

TS that can be approximated using an AR(1) model will have **an ACF that dies down (damped sinusoidal)**.



Another Visual Example for an AR(1) Process

TS that can be approximated using an AR(1) model will have **a PACF that cuts-off at lag 1**.



General Order Autoregressive Process: AR(p)

The **General Order Autoregressive Process—AR(p)** is given by

$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t,$$

where $|\phi_i| < 1 \, \forall i = 1, 2, \ldots, p$ are weights, and ϵ_t is white noise. Essentially, this is similar (not exactly the same though) as regressing y_t on $y_{t-1}, \ldots, y_{t-p}$. The mean and variance of an AR(p) process are as follows:

$$E(y_t) = \mu = \frac{\delta}{1 - \phi_1 - \phi_2 - \cdots - \phi_p}$$

$$Var(y_t) = \sum_{i=1}^{p} \phi_i \gamma(i) + \sigma^2,$$

where $\gamma(i)$ is the autocovariance functions at lag i.

General Order Autoregressive Process: AR(p)

The *population* autocorrelation function of the AR(2) process at lag k is

$$\rho(k) = \sum_{i=1}^{p} \phi_i \rho(k-i) \text{ for } k > 0$$

The ACF of an AR(p) process, for $p > 1$ is a mixture of exponential decay and a damped sinusoidal expression (damped sinusoidal from the lag 2 and greater).

AR Model: Determining if the Data Can Be Modeled as an AR Process

- We can usually tell from the ACF that there is an autoregressive (AR) component to the data because the ACF plot tends to geometrically decrease in magnitude (i.e., "die down").
- The **Order** of an AR Process refers to how many lags you include in the autoregressive model.
- Because the ACF of the AR model is a mixture, the **ACF is not useful for determining the order of the AR process**.
- Thus, the ACF helps us to know that we have an **AR model**, but not which AR model to fit!

AR Model: Determining the Order

Recall the **Partial Autocorrelation:** The Partial Autocorrelation between y_t and y_{t+k} is the correlation between y_t and y_{t+k} removing the effects of $y_{t+1}, y_{t+2}, \ldots, y_{t+k-1}$.

- When plotted over multiple lags, we refer to the plot as the Partial Autocorrelation Function or PACF.
- For an AR(p) model, the PACF between y_t and y_{t+k} should be 0 $\forall k > p$.
- Thus, for an AR(p) process, the PACF should "cut off" after lag p.

An In-Class Exercise

02:00

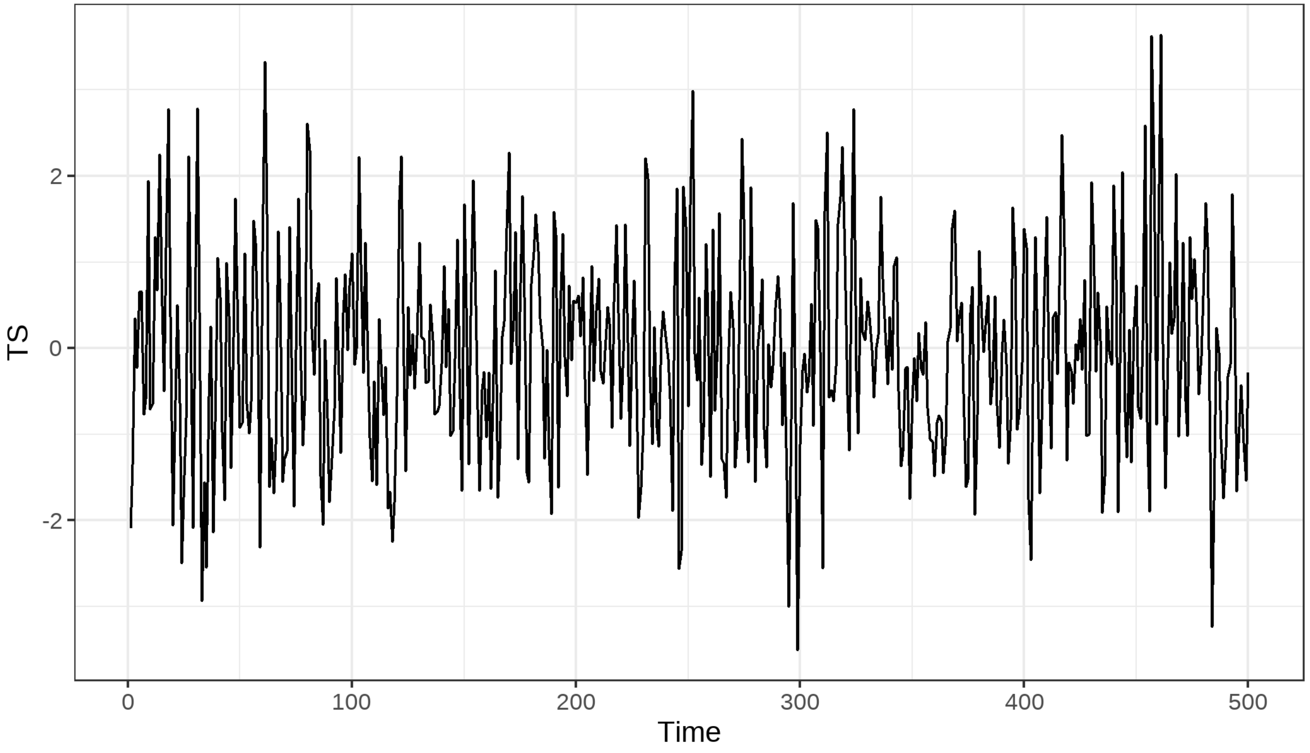
Activity | TS plot | ACF plot | PACF plot | Solution

> Over the next 2 minutes, please identify whether this time series can be modeled using an AR process and if yes, what is the order for this model.

An In-Class Exercise

02:00

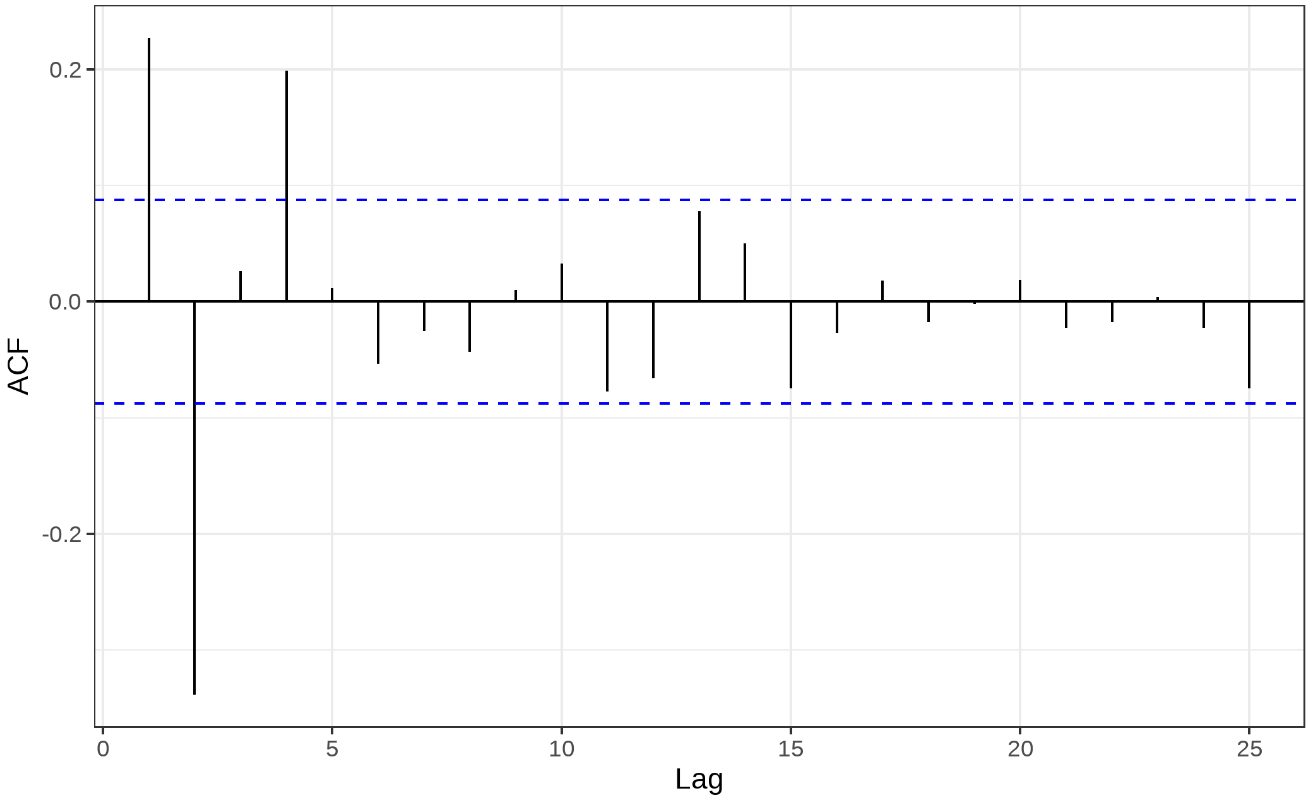
Activity | TS plot | ACF plot | PACF plot | Solution



An In-Class Exercise

02:00

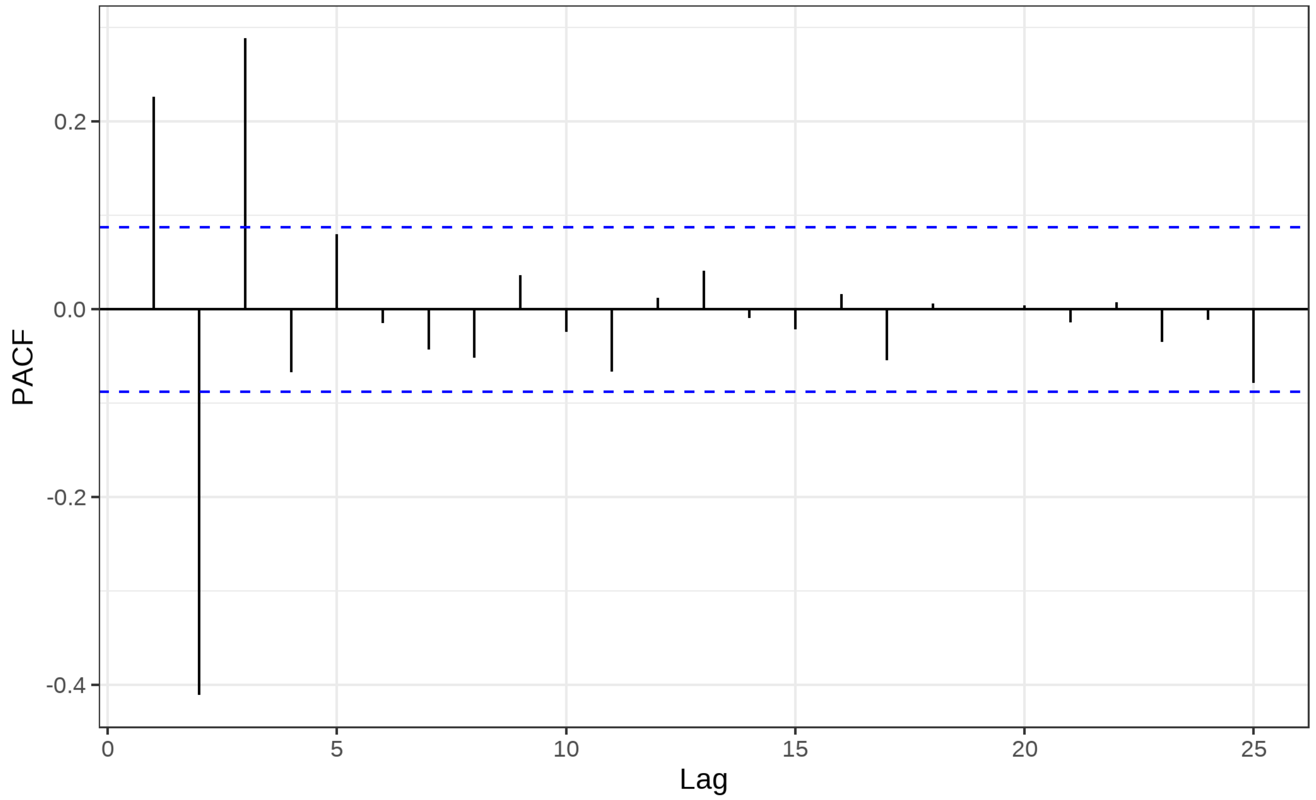
Activity | TS plot | **ACF plot** | PACF plot | Solution



An In-Class Exercise

02:00

Activity | TS plot | ACF plot | PACF plot | Solution



An In-Class Exercise

02:00

Activity | TS plot | ACF plot | PACF plot | **Solution**

- **Can the TS be modeled using an AR process?:** ....
- **If so, what is the order?:** p = ...

The Moving Average (MA) Process

The Moving Average Process

The moving average process of order q, $MA(q)$, process is given as

$$y_t = \mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}$$

where θ_i is a weight, and ϵ_i is white noise. **An MA(q) process is always stationary regardless of the weights.**

$$\begin{aligned} E(y_t) &= E(\mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots - \theta_q \epsilon_{t-q}) \\ &= \mu \end{aligned}$$

$$\begin{aligned} Var(y_t) &= Var(\mu + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}) \\ &= \sigma^2 (1 + \theta_1^2 + \cdots + \theta_q^2) \end{aligned}$$

The Moving Average Process

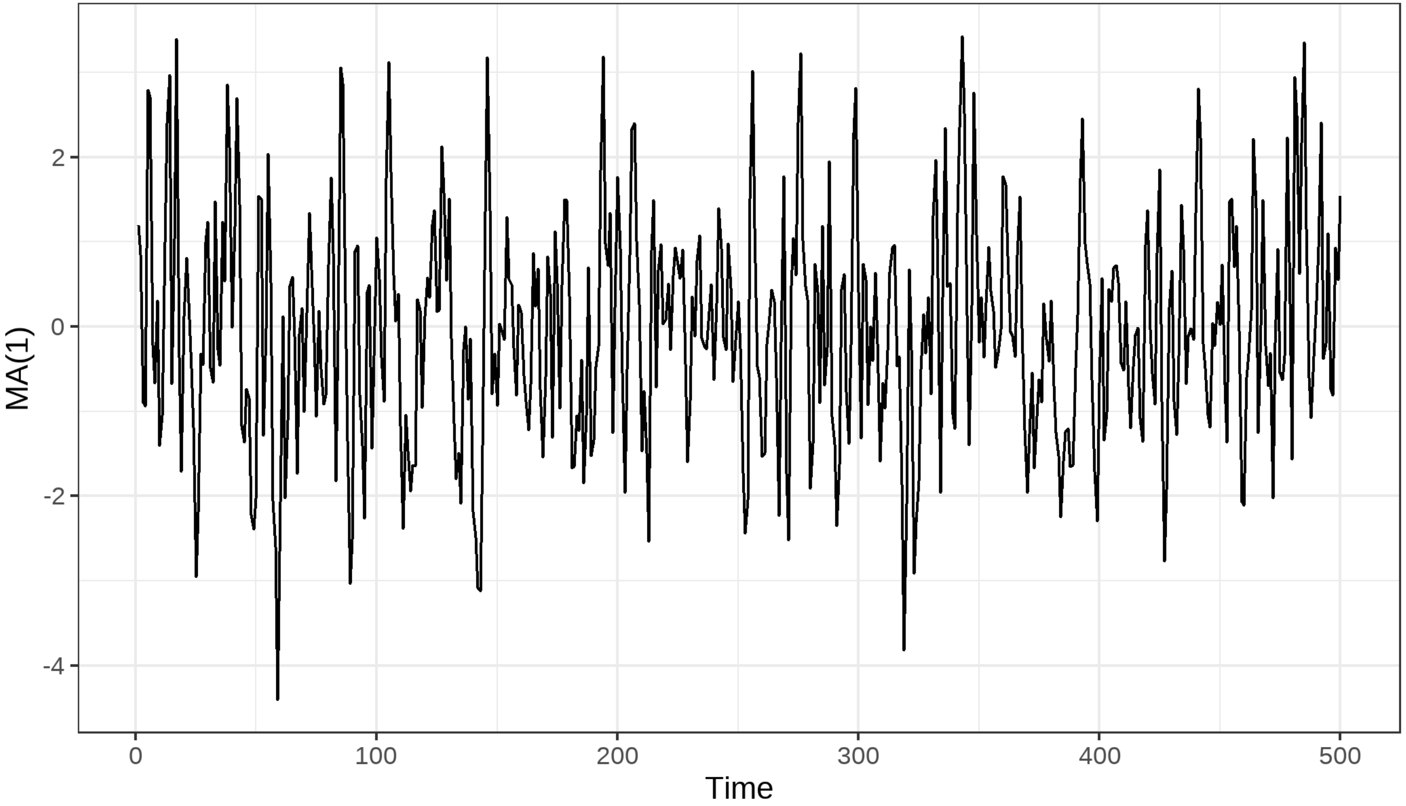
The **population** autocorrelation function of the MA(q) process at lag k is

$$\rho(k) = \begin{cases} \frac{(\theta_k + \theta_1\theta_{k+1} + \cdots + \theta_{q-k}\theta_q)}{1+\theta_1^2+\cdots+\theta_q^2}, & k = 1, 2, \ldots, q \\ 0, & k > q \end{cases}$$

This feature of the ACF is very helpful in identifying the MA model and its appropriate order because the ACF function of a MA model is not significant (i.e., “cuts off”) after lag q.

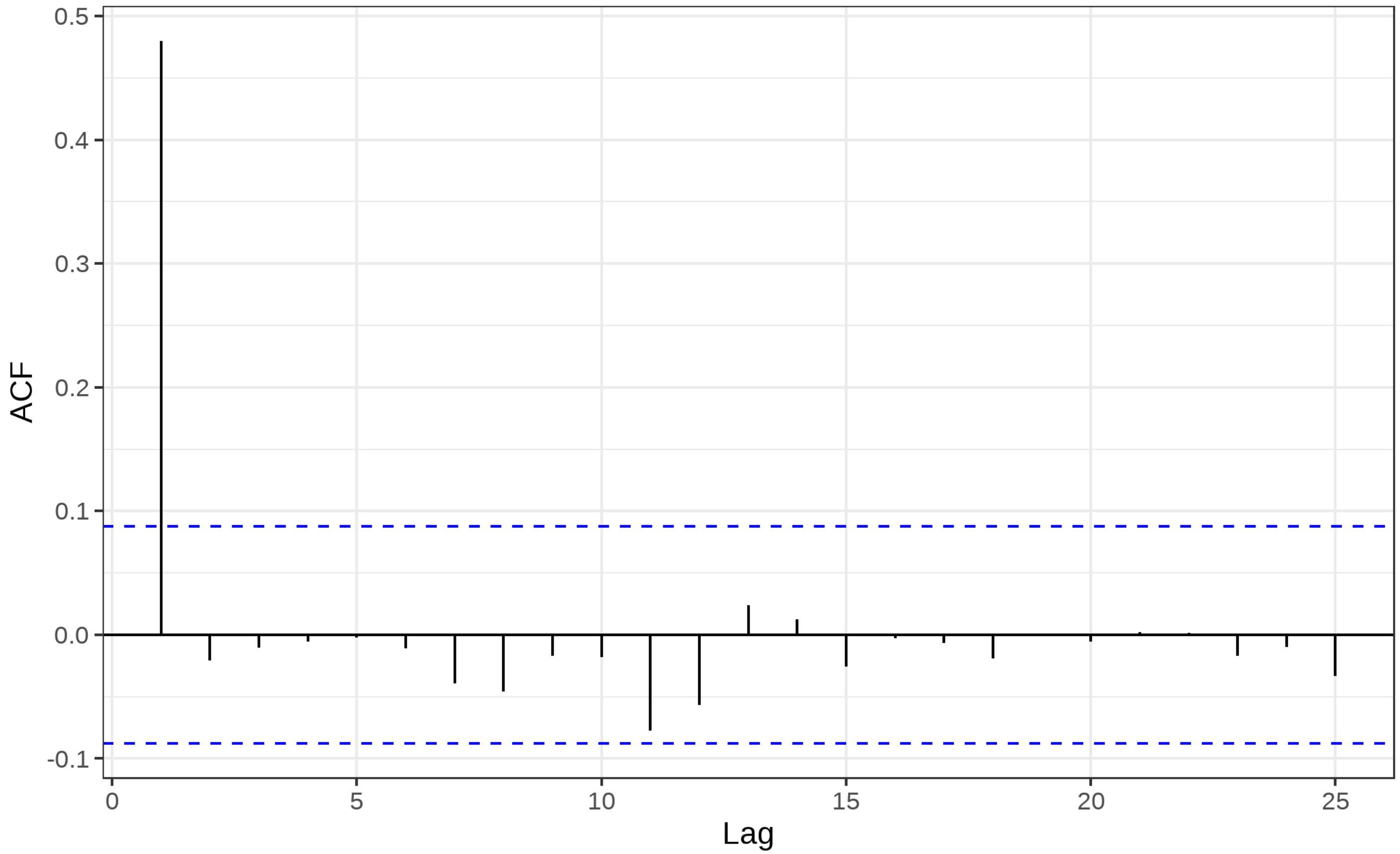
TS that can be Modeled as an MA(1) Process

TS that can be approximated using an MA(1) model will be **stationary**.



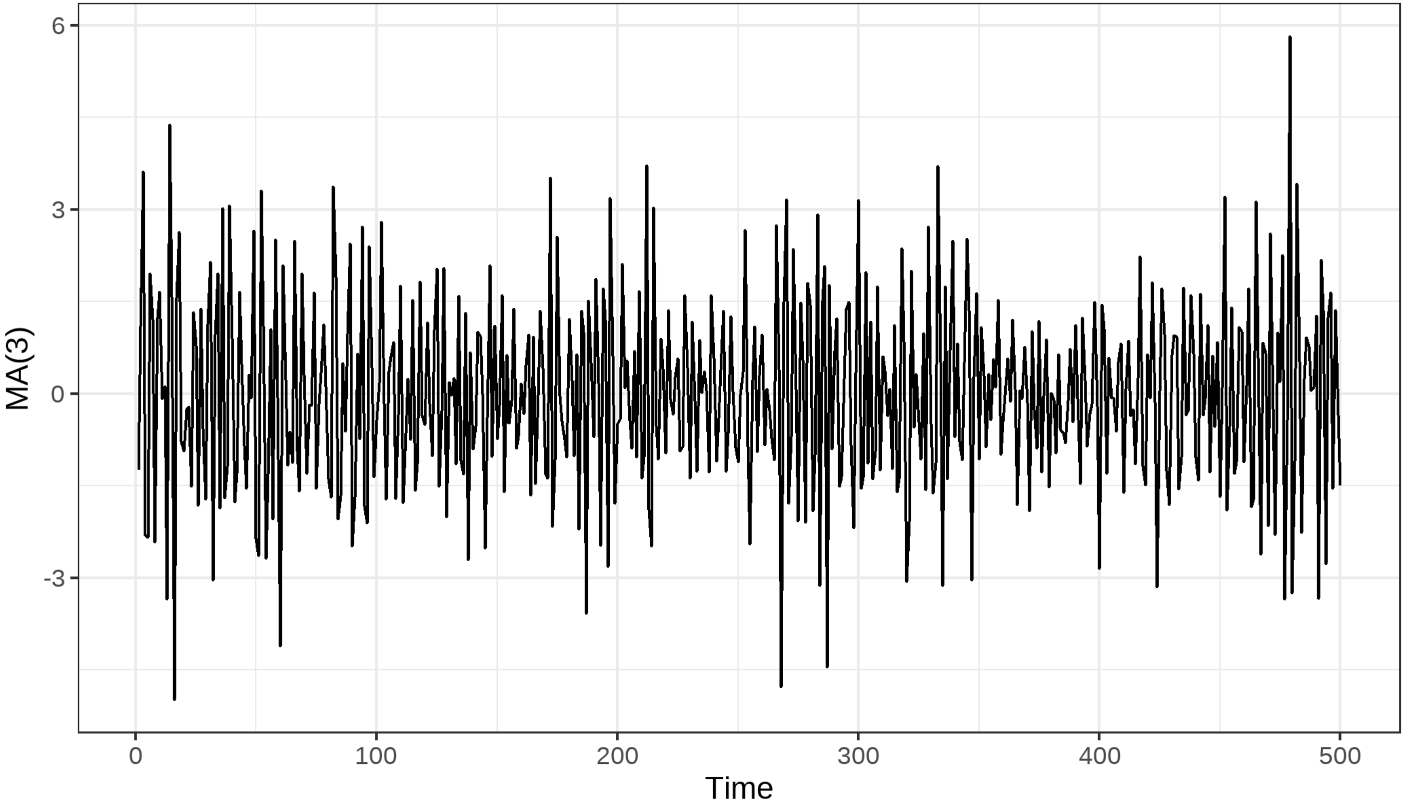
TS that can be Modeled as an MA(1) Process

TS that can be approximated using an MA(1) model will have **an ACF that cuts off at lag 1**.



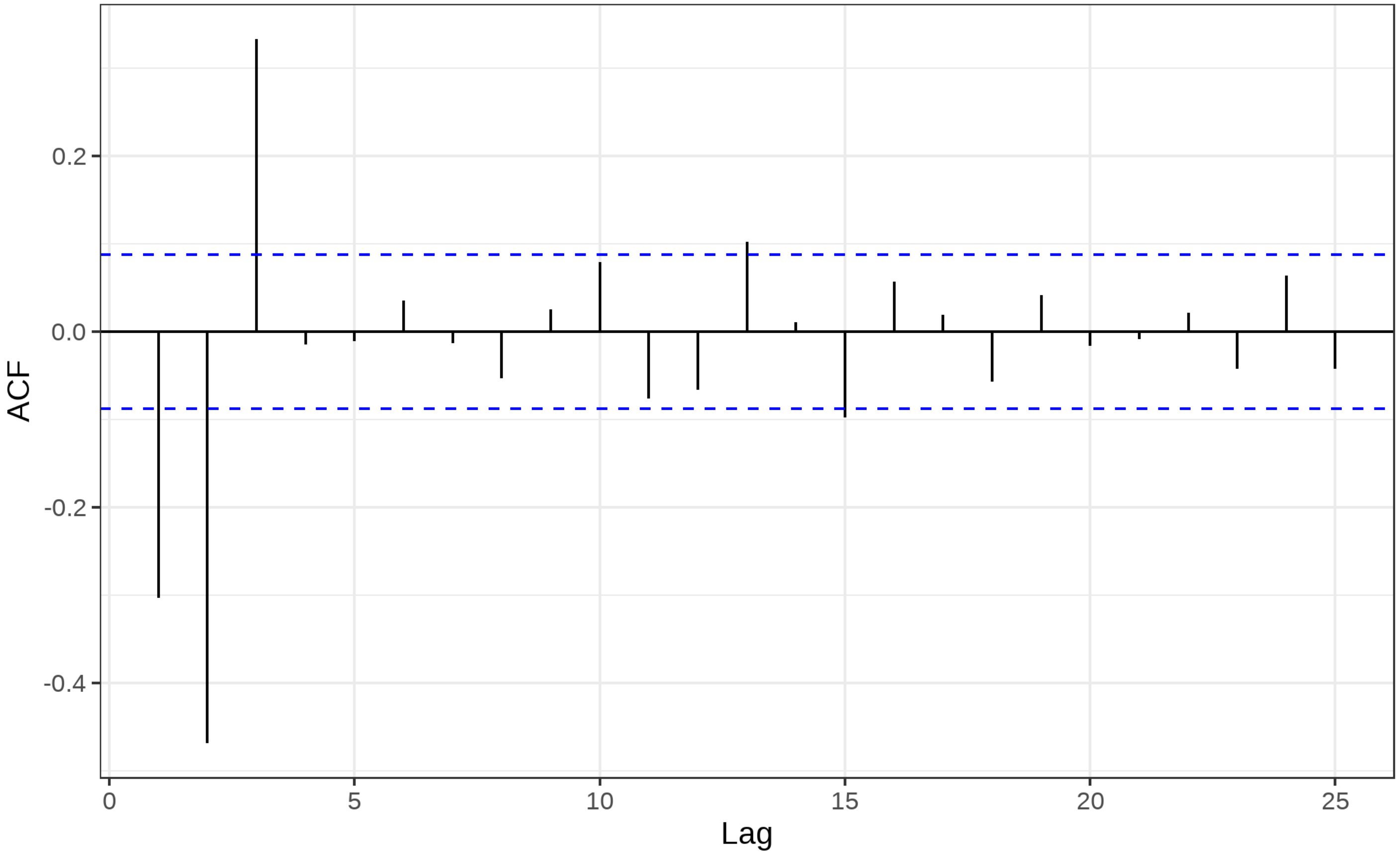
TS that can be Modeled as an MA(3) Process

TS that can be approximated using an MA(3) model will be **stationary**.



TS that can be Modeled as an MA(3) Process

TS that can be approximated using an MA(3) model will have **an ACF that cuts off at lag 3**.



ARMA Models

ARMA Processes

Sometimes, if a really high order seems needed for an AR process, it may be better to add one or more MA term. This results in a mixed autoregressive moving average (ARMA) model.

In general, an ARMA(p,q) model is given as

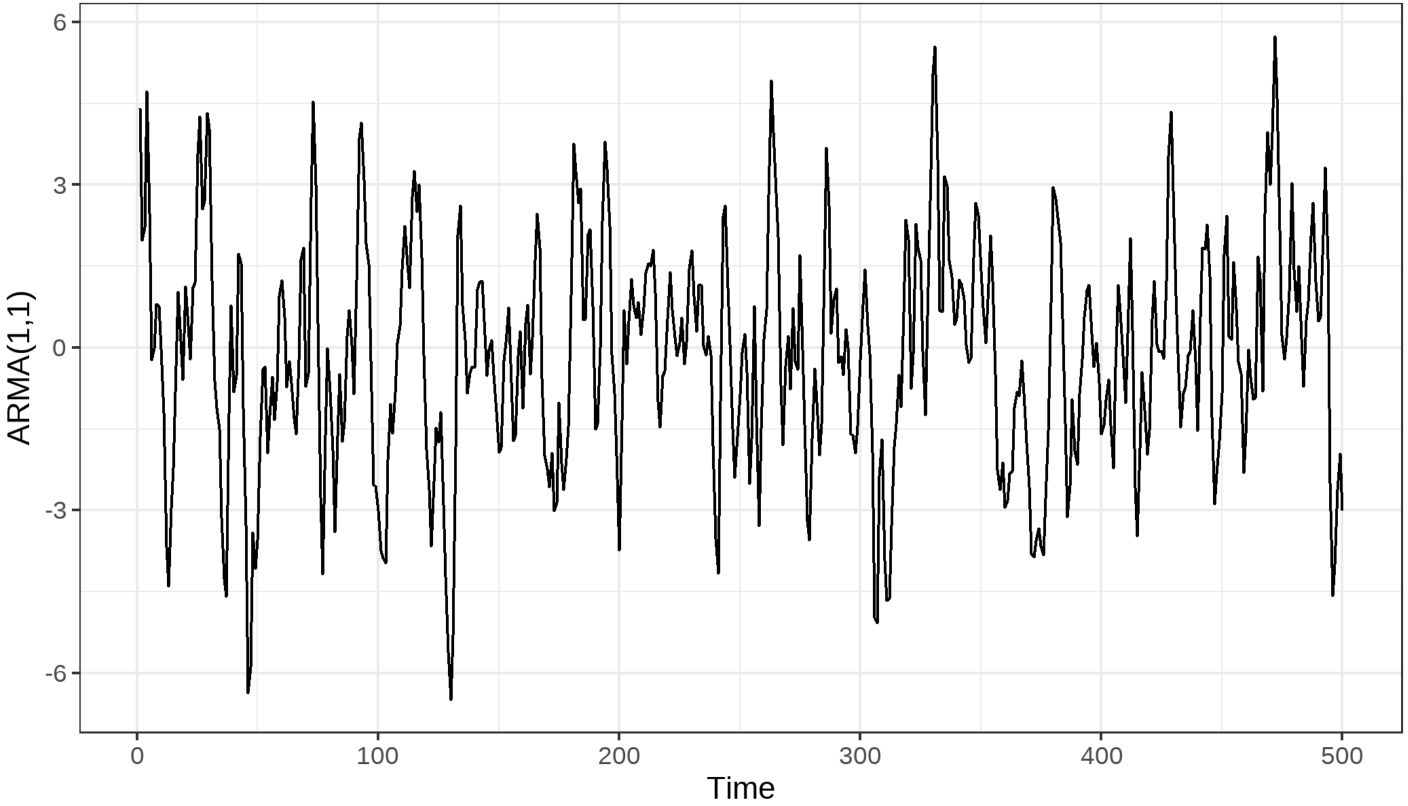
$$y_t = \delta + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q}$$

The ACF and PACF of the ARMA(p,q) process exhibit exponential decay exponential decay/ damped sinusoidal patterns. This makes identifying the order of the ARMA(p,q) difficult.

Model	ACF	PACF
AR(p)	Exponentially decays or damped sinusoidal pattern	Cuts off after lag p
MA(q)	Cuts off after lag q	Exponentially decays or damped sinusoidal pattern
ARMA(p,q)	Exponentially decays or damped sinusoidal pattern	Exponentially decays or damped sinusoidal pattern

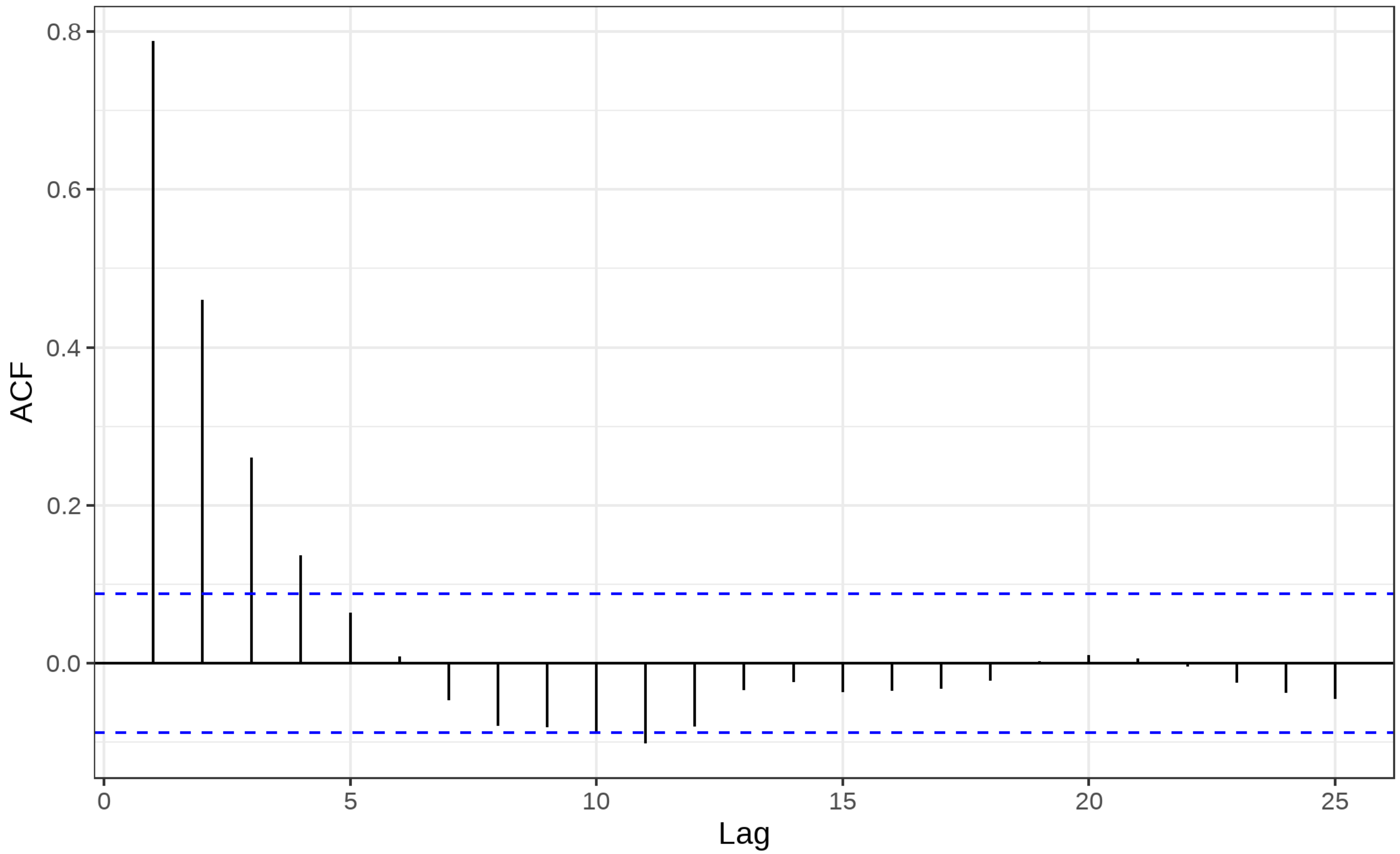
TS that can be Modeled as an ARMA Process

TS that can be approximated using an ARMA(1,1) model will be **stationary**.



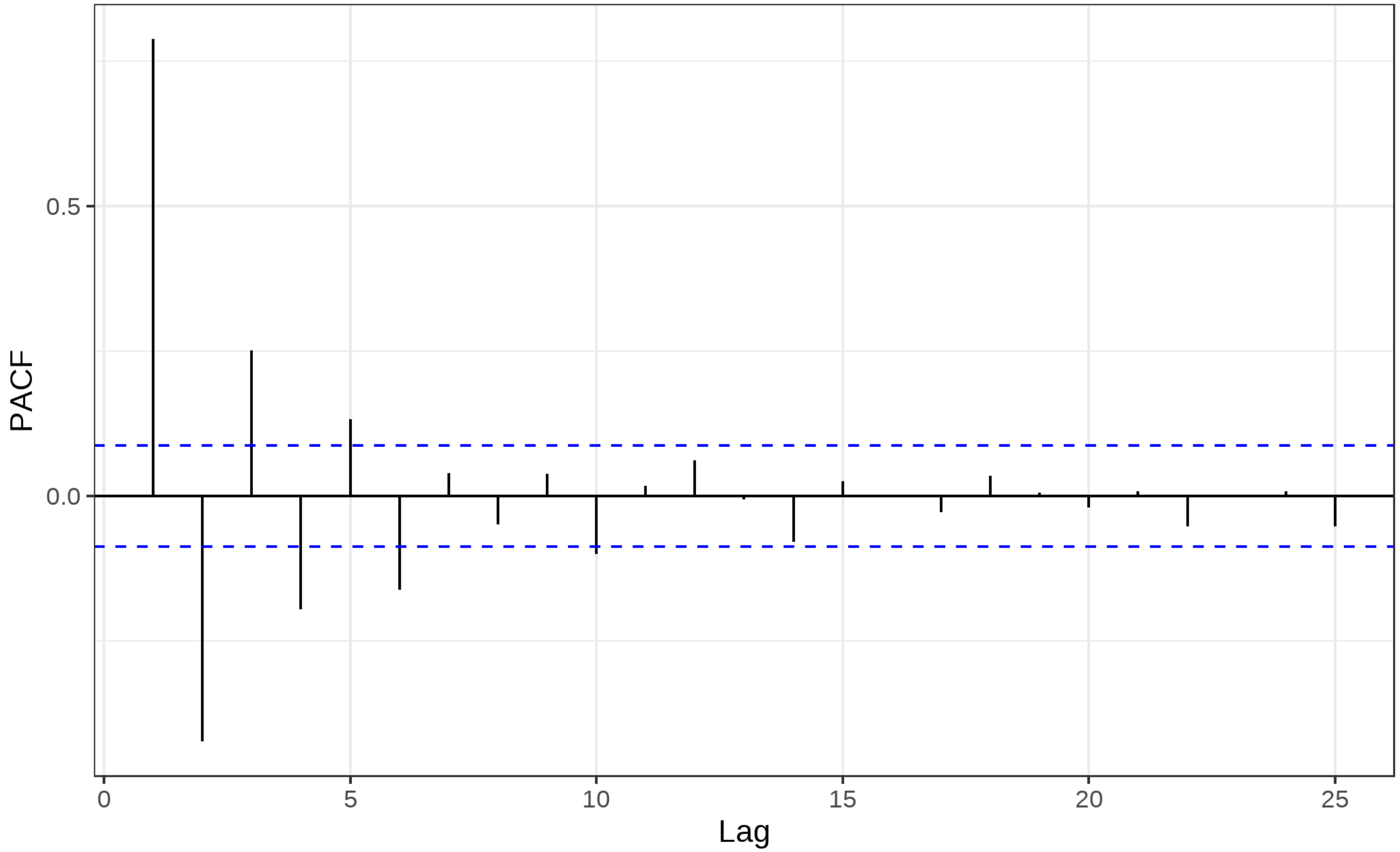
TS that can be Modeled as an ARMA Process

TS that can be approximated using an ARMA(1,1) model will have **an ACF that dies down (damped sinusoidal)**.



TS that can be Modeled as an ARMA Process

TS that can be approximated using an ARMA(1,1) model will have **a PCF that dies down (damped sinusoidal)**.



Fitting AR, MA, or ARMA Models

Fitting an ARMA Model

- Plot the data over time.
- Do the data seem stationary? If necessary, conduct a test for stationarity.
- Once you can assume stationarity, find the ACF plot.
  - If the ACF plot cuts off, fit an MA(q), where $q =$ the cutoff point.
  - If the ACF plot dies down, find the PACF plot.
  - If the PACF plot cuts off, fit an AR(p) model, where $p =$ the cutoff point.
  - If the PACF plot dies down, fit an ARMA (p,q) model.
    - You must iterate through p and q using a guess and check method starting with ARMA(1,1) models -- increment each by 1.
- Evaluate the model residuals and consider the ACF and PACF of the residuals.
- If model fit is good, forecast future values.

Note: Often you will fit multiple models in Step 3 and compare models in Step 4 to select the best fit.

A Live Demo

Viscosity of a fluid is a measure that corresponds to “thickness”. For example, honey has a higher viscosity than water. A chemical company needs precise forecasts of the viscosity of a product in order to control product quality. Using the viscosity.csv, we have 95 daily readings to use to develop a forecast.

In order to develop a forecast, let us first figure out what type of ARMA(p, q) model to fit and then develop the forecast.

Recap

Summary of Main Points

By now, you should be able to do the following:

- Describe the behavior of the ACF and PACF of an AR(p) process.
- Describe the behavior of the ACF and PACF of an MA(q) process.
- Describe the behavior of the ACF and PACF of an ARMA (p,q) process.
- Fit an ARMA model to a time series, evaluate the residuals of a fitted ARMA model to assess goodness of fit, use the Ljung-Box test for correlation among the residuals of an ARIMA model.