Mortality Heterogeneity: Multiplicative Fixed Frailty
Dem260 Math Demog
Spring 2020
Lecture 2

Joshua R. Goldstein

February 6, 2020

Agenda for today

1. Conclusions from last time
2. Review of Mortality Mathematics
3. Multiplicative-fixed-frailty and alternatives to it.
4. Population Survival and Hazards under fixed frailty

* Break

5. Gamma frailty

Conclusions from last time

- Heterogeneity as variation in risk (not just outcome)

Conclusions from last time

- Heterogeneity as variation in risk (not just outcome)
- Constantly growing parts $\neq$ constantly growth whole

Conclusions from last time

- Heterogeneity as variation in risk (not just outcome)
- Constantly growing parts $\neq$ constantly growth whole
- Keyfitz result: Change in growth rate $=$ variance of growth rates

Conclusions from last time

- Heterogeneity as variation in risk (not just outcome)
- Constantly growing parts $\neq$ constantly growth whole
- Keyfitz result: Change in growth rate $=$ variance of growth rates
- Poisson growth gives us a closed-form solution.

Review of mortality mathematics

$\ell(x)$ or $S(x)$	probability of survival to age x
$\mu(x)$ or $h(x)$	hazard rate at age x ("minus the exponential rate of change in survival")

Let's treat μ as a definition.

$$\mu(x) \equiv -\frac{d}{dx} \log \ell(x)$$

Can anti-differentiate (integrate) to solve for survival:

$$\ell(x) = s(x) = e^{-\int_0^x \mu(a)\, da}$$

Application: what is $\ell'(x)$?

- in words?
- taking derivative of $\ell(x)$
- interpretation

Two special cases

- Constant hazards $\mu(x) = \mu$. What's $\ell(x)$?
- Gompertz hazards $\mu(x) = ae^{bx}$. What's $\ell(x)$?

Extending Keyfitz to mortality

$$\frac{d}{dx}\bar{\mu}(x) = \text{average rate of change} - \sigma_\mu^2$$

What is $\bar{\mu}$? It's a weighted average:

$$\bar{\mu}(x) = \frac{\int \mu(x|z)\ell(x|z)p(z)\,dz}{\int \ell(x|z)p(z)\,dz}$$

To derive Keyfitz extension, differentiate with respect to age x. (See V&M eq (13)). A good exercise.

Multiplicative Fixed frailty

For individual i,

$$\mu_i(x) = z_i \mu_0(x).$$

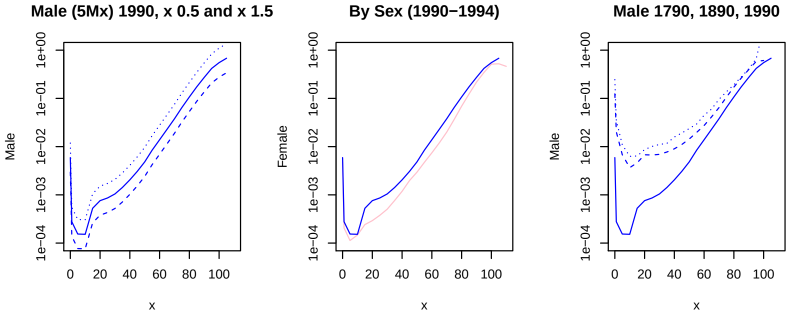
z_i "frailty" of the ith individual. (Usually thought of as a random variable with mean 1.)

$\mu_0(x)$ "Baseline hazard" schedule. (Also, the schedule of a person with $z = 1$).

What are some alternatives?

Let's think of at least three.
(β, i, Δ)

Which look like multiplicative fixed frailty?



Part II. Results from Fixed Frailty

2. A simulation

Our questions

- How do we do a micro-simulation, with individuals?
- How does fixed frailty fit in?
- How do we compute pop survival, hazards, etc.
- How does life table of heterogeneous pop differ from baseline?

Homework

- Look at frailty, distribution and mean frailty of survivors.
- How does life expectancy $e(x)$ differ in pop and baseline?

Let's derive pop survival (Note: $\bar{s} = \bar{\ell}$)

Pop survival will be a weighted average of group survival curves

$$\bar{s}(x) = \frac{p(z_1)s_1(x) + p(z_2)s_2(x) + \dots}{p(z_1) + p(z_2) + \dots}$$

With continuous z (what are limits of integration?)

$$\bar{s}(x) = \int s(x|z)p(z)\,dz$$

Under Multiplicative Fixed Frailty use

$$\mu(x|z) = z\mu_0(x)$$

to derive

$$\bar{s}(x) = \int s_0(x)^z p(z)\,dz.$$

Now population hazards (stepping stones)

Definition of hazards:

$$\bar{\mu}(x) = -\frac{d}{dx}\log \bar{s}(x)$$

$$\bar{\mu}(x) = \mu_0(x)\frac{\int z s_0(x)^z p(z)\, dz}{\int s_0(x)^z p(z)\, dz}$$

$$\bar{\mu}(x) = \mu_0(x)\bar{z}(x)$$

Let's fill in steps.

Rodriguez question

Why isn't population hazard a (simple) average of individual hazards?

Rodriguez question

Why isn't population hazard a (simple) average of individual hazards?
Answer: selected survival means that the distribution of frailty at age x differs from the starting frailty distribution at age 0.

The rate of increase in hazards

(AKA "LAR: Lifetable Aging Rate")

$$\beta(x) = \frac{d}{dx} \log \mu(x)$$

Example:
What is $\beta(x)$ for Gompertz: $\mu(x) = ae^{bx}$?

Vaupel's result

$$\bar{\beta}(x) = \beta_0(x) - \bar{\mu}(x) CV_z^2(x)$$

The same kind of tedious derivation ... but conclusion is more interesting.

- Hazards rise less slowly in pop than in baseline
- If pop hazards plateau, then $\bar{\beta}(x) = 0$
- Two special cases
  - Homogeneous pop and plateau in baseline
  - Gompertz and constant CV_z (e.g., from Gamma)

Part II. Introduction to Gamma Frailty

Gamma agenda (today)

- What do we want in a frailty distribution?
- What's the Gamma?
- Last math: closed form pop survival

What do we want in a frailty distribution?

What do we want in a frailty distribution?

- positive?
- a single dimension summarizing multiple factors? (Normal?)
- flexible?
- tractable?

The Gamma distribution

$$p(z|k,\lambda) = \frac{\lambda^k}{\Gamma(k)} z^{k-1} e^{-\lambda z}$$

.

- z the random variable representing frailty
- k, λ parameters
- $\Gamma(k)$ A normalizing constant.

Gamma in R

Our parameterization:

$$\text{Mean: } k/\lambda$$
$$\text{Variance: } k/\lambda^2$$

```
## with k and lambda
k = 3; lambda = 6
x <- rgamma(10000, shape = k, rate= lambda)
mean(x)

## [1] 0.4985961

sd(x)

## [1] 0.2892092
```

Alternate parameterization

```r
## with mean 1, sigma.sq
sigma.sq <- .25
z <- rgamma(10000, shape = 1/sigma.sq, rate = 1/sigma.sq)
mean(z)

## [1] 1.007105

var(z)

## [1] 0.2508519
```

Population Survival of Gamma Frailty

Big picture

$$\bar{s}(x) = \int s_0(x)^z p(z)\, dz$$

Or, using our definition of survival,

$$\bar{s}(x) = \int e^{-zH_0(x)} p(z)\, dz$$

“completing the gamma”

$$\bar{s}(x) = \int e^{-zH_0(x)} \frac{\lambda^k}{\Gamma(k)} z^{k-1} e^{-\lambda z}\, dz$$

Rearranging,

$$\bar{s}(x) = \lambda^k \int \frac{1}{\Gamma(k)} z^{k-1} e^{-z(H_0(x)+\lambda)}\, dz$$

Integral is like a $Gamma(z|k, H_0(x) + \lambda)$, but missing something. What?

Our Result

$$\bar{S}(x) = \frac{\lambda^k}{[H_0(x) + \lambda]^k}$$

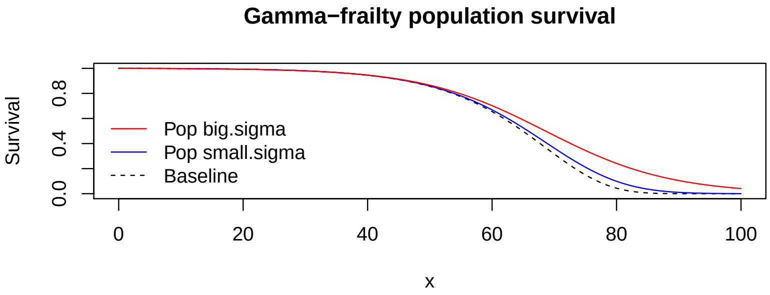
If mean $= 1.0$, then we can let $\lambda = k = 1/\sigma^2$,

$$\bar{S}(x) = \frac{1/\sigma^2}{(H_0(x) + 1/\sigma^2)^{1/\sigma^2}} = \frac{1}{(1 + \sigma^2 H_0(x))^{1/\sigma^2}}$$

Interpreting Gamma-frailty survival

$$\bar{S}(x) = \frac{1}{(1 + \sigma^2 H_0(x))^{1/\sigma^2}}$$

- Older ages, smaller survival.
- Variance not so clear



Conclusions

- What is multiplicative fixed frailty?
- Analytical expressions for aggregate/pop/unconditional survival and hazards
- Extension of Keyfitz result
- Gamma intro

Next time

We'll see the power of the Gamma for

- Explicit expression for hazards
- Inversion formula (baseline from aggregate)
- Plateaus
- Cross-overs
- Inequality and change over time