Demog 260 Mathematical Demography
Problem set 1 key *

2/19/2020

1. **True or False? "If every part grows exponentially at its own rate, then the whole will also grow exponentially." Explain your answer briefly.**

   False. From Keyfitz we know that the average growth rate of a heterogeneous population is:

$$\bar{r}(t) = \frac{\sum_i Q_i r_i e^{r_i t}}{\sum_i Q_i e^{r_i t}}$$

   .

   Here, $\bar{r}(t)$ is not constant for all t as there is a compositional effect. Need to read as "If every part grows at a constant exponential rate, then the whole will also grow exponentially.". We could argue that only in the long term it is true.

2. **List what you think are two of the best arguments in favor of doing a disaggregated projection? What are two of the best arguments in favor of doing an aggregated projection?**

   - Disaggregated projections:
     - Revealing the "true" path, patterns of individuals different from that of the aggregate.
     - Take into account size of each sub-population, in particular the smaller ones.
     - It may be more precise. Growth rates for sub-populations reflect their intrinsic characteristics, such that projections that do not use appropriate rates can lead to adverse social/policy implications.
   - Aggregated projections:
     - If population is homogeneous, we can estimate a simpler model than projecting many models for sub-populations.
     - We might care more about the overall rates rather than the group-specific rates.
     - More data available.
     - Easier.

3. $r_u = .0075$, $r_m = 0.035$, $K_u(1970) = 200$, $K_m(1970) = 50$

   (a) **Use the first few years of the projection to verify that rate of change in the aggregate growth rate equals the variance of the growth rate. Does it matter what time points and period you consider?**
   From the information given, we can obtain $\bar{r}(t)$ and $\sigma_r^2(t)$ by projecting each country's population

*This was compiled by Andrea Miranda-Gonzalez from the homework of all the class members. Special thanks to Maria Osborne for sharing her LaTex file and graphs.

using individual rates and then obtaining the total population ($\bar{K}(t)$). We use the formula for aggregate growth rate:

$$\bar{r}(t) = \frac{K_u(0)r_u(t)e^{r_u(t)t} + K_m(0)r_m(t)e^{r_m(t)t}}{K_u(0)e^{r_u(t)t} + K_m(0)e^{r_m(t)t}}$$

Then, we can calculate the variance of the growth rates as:

$$\begin{aligned}
\sigma_r^2(t) &= \frac{K_u(0)e^{r_u(t)t}(r_u(t) - \bar{r}(t))^2 + K_m(0)e^{r_m(t)t}(r_m(t) - \bar{r}(t))^2}{K_u(0)e^{r_u(t)t} + K_m(0)e^{r_m(t)t}} \\
&= \frac{K_u(t)r_u(t)^2 + K_m(t)r_u(t)^2}{K_u(t) + K_m(t)} - \bar{r}(t)^2
\end{aligned}$$

The table below show these results for projections at the beginning and ending of a 50 year period. Overall, the time points do not matter and we can verify that $\frac{d\bar{r}(t)}{d(t)} = \sigma_r^2(t)$

t	$\sigma_r^2(t)$	$K_U(t)$	$K_m(t)$	$\bar{K}(t)$	$\bar{K}(t) - \bar{K}(t-1)$	$\bar{r}(t)$	$\bar{r}'(t)$
0	0.00021	200	50	250	-	-	-
1	0.000122998	201.5056	51.78095	253.28662	3.2866	0.013060	-
2	0.000124998	203.0226	53.625	256.648	3.3614	0.01318364	0.000122993
3	0.0001269996	204.5510	55.5355	260.0865	3.4385	0.0133086	0.000124993
49	0.000188991	288.824	277.833	566.6577	-	-	-
50	0.00018905647	290.9983	287.730	578.7284	12.07068	0.021077	-
51	0.000189050	293.18897	297.978997	591.1679	12.4395	0.021266	0.0001890293

(b) **The growth rate changes over the course of the 50 years, but there is a constant growth rate that will produce the exact same population after 50 years. A reasonable choice of the constant growth rate to apply is the value of the changing growth rate at year 25 (half-way through the period). We can estimate this using a Taylor series approximation:**

$$\bar{r}(25) \approx \bar{r}(0) + 25\bar{r}'(0) + (25)^2\bar{r}''(0)$$

i. **Show that**

$$\bar{r}''(t) = \bar{r}_3(t) - \bar{r}_2(t)\bar{r}(t) - 2\bar{r}(t)\sigma_r^2(t)$$

Proof. From before, we know that

$$\begin{aligned}
\sigma_r^2(t) &= \bar{r}'(t) \\
&= \frac{\sum K_i(0)r_i^2e^{r_it}}{\sum K_i(0)e^{r_it}} - \left(\frac{\sum K_i(0)r_ie^{r_it}}{\sum K_i(0)e^{r_it}}\right)^2 \\
&= \frac{\sum K_i(0)r_i^2e^{r_it}}{\sum K_i(0)e^{r_it}} - \bar{r}^2(t)
\end{aligned}$$

Using the quotient rule $\left(\frac{u}{v}\right)' = \frac{u'v - vu'}{v^2}$ we can take the second derivative of the average growth rate

$$\begin{aligned}
\bar{r}''(t) &= \frac{(\sum K_i(0)e^{r_it})(\sum K_i(0)r_i^3e^{r_it}) - (\sum K_i(0)r_i^2e^{r_it})(\sum K_i(0)r_ie^{r_it})}{(\sum K_i(0)e^{r_it})^2} - 2\bar{r}(t) \times \bar{r}'(t) \\
&= \frac{\sum K_i(0)r_i^3e^{r_it}}{\sum K_i(0)e^{r_it}} - \frac{\sum K_i(0)r_i^2e^{r_it}}{\sum K_i(0)e^{r_it}} \times \frac{\sum K_i(0)r_ie^{r_it}}{\sum K_i(0)e^{r_it}} - 2\bar{r}(t)\sigma_r^2(t) \\
&= \bar{r}_3(t) - \bar{r}_2(t)\bar{r}(t) - 2\bar{r}(t)\sigma_r^2(t)
\end{aligned}$$

where $\bar{r}_n$ is the nth moment of $\bar{r}(t)$, and it is known that $\bar{r}'(t) = \sigma_r^2(t)$.

□

ii. **Calculate the combined US-Mexico population after 50 years according to the following five (5) methods, plot the total population after 50 years according to these 5 methods on a graph**

A. *Dissaggregated ("true") forecast, with each country growing at its own rate*

$$\begin{aligned}\bar{K}(2020) &= K_u(2020) + K_m(2020)\\ &= K_u(1970)e^{50r_u} + K_m(1970)e^{50r_m}\\ &= 200e^{.0075\times 50} + 50e^{.035\times 50}\\ &= 578.728\end{aligned}$$

B. *Aggregated forecast, pretending it's one country, growing at $\bar{r}(0)$) for 50 years*

$$\begin{aligned}\bar{r}(0) &= \frac{\sum K_i(0)r_ie^{r_it}}{\sum K_i(0)e^{r_it}}\\ &= \frac{K_u(1970)r_u + K_m(1970)r_m}{K_u(1970)+K_m(1970)}\\ &= \frac{200\times .0075 + 50\times .035}{200+50} = 0.013\end{aligned}$$

$$\begin{aligned}\bar{K}(2020) &= \bar{K}(1970)e^{\bar{r}(0)50}\\ &= 250e^{.013\times 50} \approx 478.8852\end{aligned}$$

C. *Aggregated forecast, growing at the "true" value of $\bar{r}(25)$ for the whole period. (Use the value of $\bar{r}(25)$) that you calculate from the disaggregated forecast):*

$$\begin{aligned}\bar{r}(25) &= \frac{\sum K_i(0)r_ie^{r_it}}{\sum K_i(0)e^{r_it}}\\ &= \frac{K_u(1970)r_ue^{25r_u} + K_m(1970)r_me^{25r_m}}{K_u(1970)e^{25r_u}+K_m(1970)e^{25r_m}}\\ &= \frac{200\times .0075e^{.0075(25)} + 50\times .035e^{.035(25)}}{200e^{.0075(25)}+50e^{.035(25)}} \approx 0.0176632\\ \bar{K}(2020) &= 250e^{0.016632(50)} \approx 574.25316\end{aligned}$$

D. *Aggregated forecast, growing at the first-order Taylor series estimate of $\bar{r}(25)$ for the whole period:*

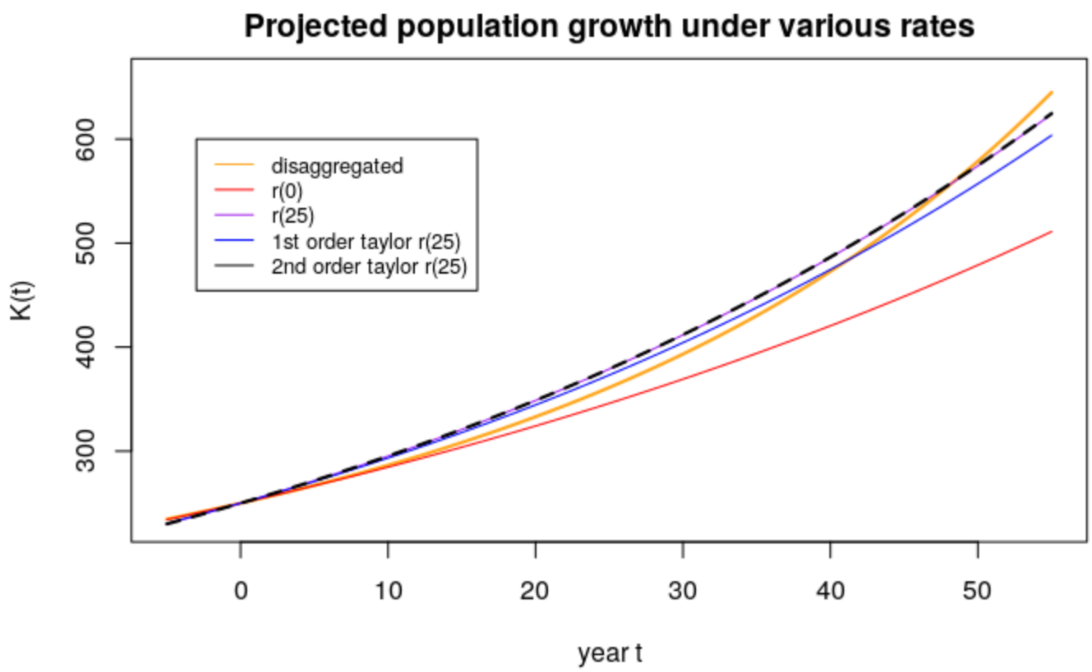
$$\begin{aligned}\sigma_r^2(0) &= \frac{K_u(1970)r_u^2 + K_m(1970)r_m^2}{K_u(1970)+K_m(1970)} - \bar{r}^2(0)\\ &= \frac{(200)(0.0075)^2 + (50)(0.035)^2}{250} - (0.013)^2\\ &= 0.000121\\ \hat{\bar{r}}(25) &\approx \bar{r}(0) + 25\sigma_r^2(0) = 0.016025\\ \bar{K}(2020) &= 250e^{0.016025(50)} \approx 557.0811485\end{aligned}$$

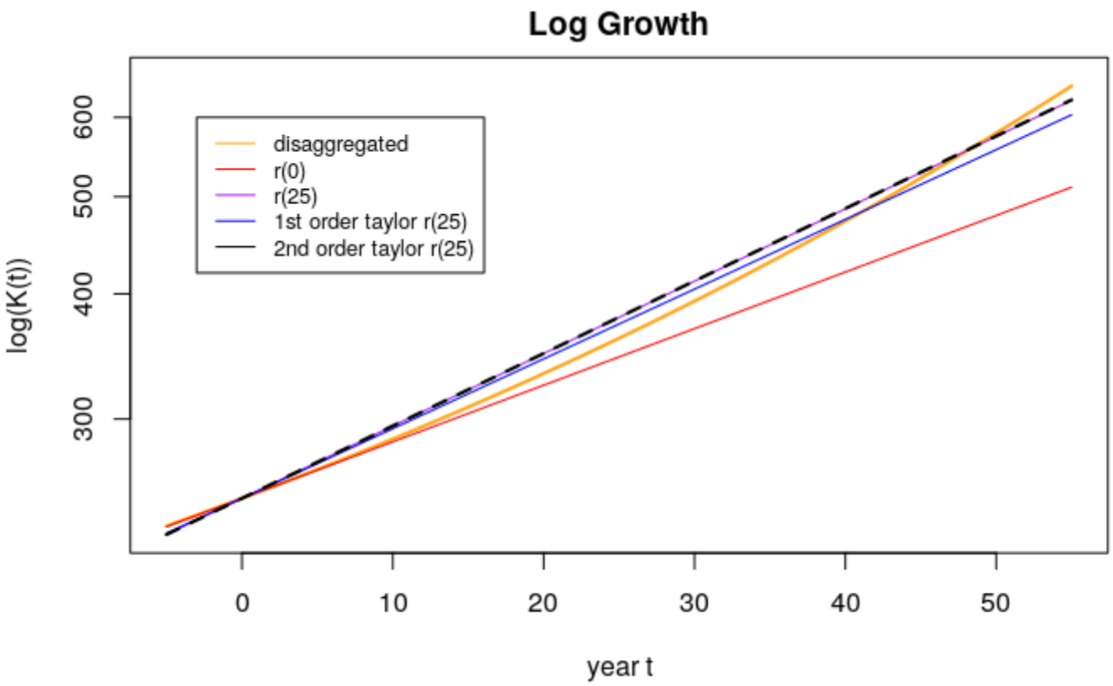
E. *Aggregated forecast, growing at the second-order Taylor series estimate of $\bar{r}(25)$ for the whole period:* For this exercise we need to calculate the extra second-order term from the

Taylor series, $25^2\bar{r}''(0)$:

$$\begin{aligned}
\bar{r}''(0) &= \frac{200(0.0075)^3 + 50(0.035)^3}{250} - \frac{200(0.0075)^2 + 50(0.035)^2}{250} \times (0.013) - 2(0.013)(0.000121) \\
\hat{\bar{r}}(25) &\approx \bar{r}(0) + 25\sigma_r^2(0) + (25)^2\bar{r}''(0) \\
&= 0.01664891 \\
\bar{K}(2020) &= 250e^{0.01664891(50)} \\
&\approx 574.73337
\end{aligned}$$

iii. *Graphs*
There are many methods to project populations.





4. *For Ken's Poisson-Exponential model,*

   (a) *What is the closed-form expression for $\hat{r}(t)$* (derived in class)

$$\hat{r}(t) = \frac{d}{dt} log(\hat{k}) \tag{1}$$
$$= r_0 - \alpha\lambda e^{-\alpha t} \tag{2}$$

   (b) *What is the variance of the growth rate?*
   The variance of the growth rate is $\sigma_r^2(t)$:

$$\bar{r}'(t) = \sigma_r^2(t) = \alpha^2 \lambda e^{-\alpha t}$$

   (c) *Write down an expression for the distortion index. What variables and parameters in the model does it depend on? Are there any variables or parameters that it doesn't depend on?* (difference of $\bar{r}(t)$ and $\bar{r}(0)$)

$$\bar{r}(t) - \bar{r}(0) = r_0 - \lambda\alpha e^{-\alpha t} - (r_0 - \lambda\alpha e^0) = \lambda\alpha - \lambda\alpha e^{-\alpha t}$$
$$\text{or,} \qquad \lambda\alpha(1 - e^{-\alpha t})$$

   Depends on λ, α, t, but not r_0 or s. That is, it depends on the the gap between growth rates, the relative population sizes, fastest growth rate, the poisson distribution parameter, but not the poisson distributed integer s.