Mortality Heterogeneity: Gamma Frailty with Applications
Dem260 Math Demog
Spring 2020
Lecture 3

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Agenda for today

1. Conclusions from last time
2. Hazards and Average Frailty for Gamma
3. CenSoc Application
* Cookie Break
4. Cross-overs and (Inversion Formula)
5. Plateaus (we'll skip)
6. Distorted progress?

Conclusions from class 1

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- Poisson growth gives us a closed-form solution.

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- Gave us analytical expressions for population survival and hazards including $\bar{\mu}(x) = \mu_0(x)\bar{z}(x)$
- Extended Keyfitz result to age-changing hazards
- Survival curve for Gamma

From pop survival to pop hazards

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So,

$$\bar{\mu}(x) = \mu_0(x) \frac{1}{1 + \sigma^2 H_0(x)}$$

Sketch $\bar{z}(x)$. Hint: what form does $H_0(x)$ have?

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Example: Gamma-Gompertz

- If $H_0(x)$ be Gompertz, we have closed-form expression. What is it?
- Does $\bar{z}$ have the form

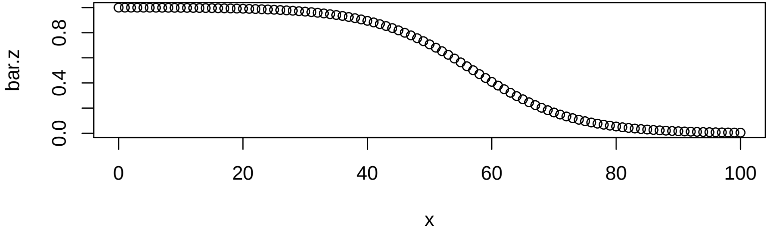
$$\frac{1}{1 + v * e^{wx}}$$

- This is a backwards S, going down.

A picture

```
sigma.sq = .2
x = 0:100
a = 5 * 10^-4
b = 1/8
H0.x = (a/b) * (exp(b*x) - 1)
bar.z = 1 / (1 + sigma.sq * H0.x)
plot(x, bar.z)
```

A picture



Look at the apparent exponential decline in tail
(Homework: what is proportional rate of change in $\bar{z}$ as x gets big? Is it close to Gompertz b?)

Average frailty in terms of survival

$$\bar{z}(x) = [\bar{S}(x)]^{\sigma^2}!$$

Let's derive.

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In real life, we observe $\bar{S}(x)$. So this allows us to say something about implied $\bar{z}$ from hazards – assuming we have a value for σ^2.

Reversing the logic: if we see a characteristic changing with age, then maybe we can estimate "σ^2" (I put in quotes because its the variance of the proportional effect of the observed characteristic.)

Shift to CenSoc_mortality_selection.Rmd

Cookie Break

Convergence and cross-overs

What happens to mortality disparities at older ages?

- *Cumulative disadvantage*
- *Age as a leveler*
  Individual adaptation/plasticity, gov support, separation from unequal structures like labor market
- *Bad data / measurement*
  Unreliable ages, institutionalization changes sample, etc.
- Nothing.
  It's all selection ("frailty"), pop hazards but individual hazards would have remained "parallel".

Our goal is to examine this last "null hypothesis." What can frailty explain, and what do we need other explanations for?

A possible null-model

- 2 groups, each with internal gamma-frailty
- proportional baseline hazards

$$\mu_2(x) = R\mu_1(x)$$

(see V&M (38))

$$\mu_1(x|z_1) = \mu_1 z_1$$
$$\mu_2(x|z_2) = \mu_2 z_2$$

And, frailty terms are each gamma, with mean 1 and own variances.

A result: V&M (5E)

$$\bar{R}(x) \equiv \frac{\bar{\mu}_2(x)}{\bar{\mu}_1(x)} = \frac{R + R\sigma_1^2 H_1(x)}{1 + R\sigma_2^2 H_1(x)}$$

Is there a typo? Look at subscripts.

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Questions:

- If variances are equal. What happens at age 0? What happens at very old ages.
- If the higher mortality group has bigger frailty variance, what happens at older ages?
- Same if higher mortality group has smaller frailty variance?

(Homework: prove this, simulate this. see if cross of is when cumulative hazards satisfy the condition at the end of 5E. (harder problem: can you solve for x0 in terms of variances 1 and 2 and R with gamma gompertz?)

Inversion

Given observed hazards, how do we get baseline?
(Impossible without assumptions; but what if we have gamma-frailty with σ^2?)

Our challenge

Our pop hazards formula is not so easy to invert

$$\bar{\mu}(x) = \frac{\mu_0(x)}{1 + \sigma^2 H_0(x)}$$

because we have both hazards and cumulative hazards on right.

A solution

Hazards are slope of log survival
Recall for Gamma,

$$\bar{S}(x) = \frac{1}{(1 + \sigma^2 H_0(x))^{1/\sigma^2}}$$

We write down the hazard as the derivative of log survival

$$\bar{\mu}(x) = \frac{1}{\sigma^2}\frac{d}{dx}\log(1 + \sigma^2 H_0(x)).$$

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The anti-derivative of both sides, gives

$$\bar{H}(x) = \frac{1}{\sigma^2} \log(1 + \sigma^2 H_0(x)).$$

And now we have only 1 expression involving the baseline hazards on the right.

Our inversion formula

Solving

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for the cumulative baseline hazard gives

$$H_0(x) = \frac{1}{\sigma^2} \left(e^{\sigma^2 \bar{H}(x)} - 1 \right).$$

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$$H_0(x) = \frac{1}{\sigma^2} \left(e^{\sigma^2 \bar{H}(x)} - 1 \right).$$

And differencing, gives us a remarkably simple expression for the baseline hazard in terms of the observed population hazard

$$\mu_0(x) = \bar{\mu}(x) e^{\sigma_2 \bar{H}(x)}$$

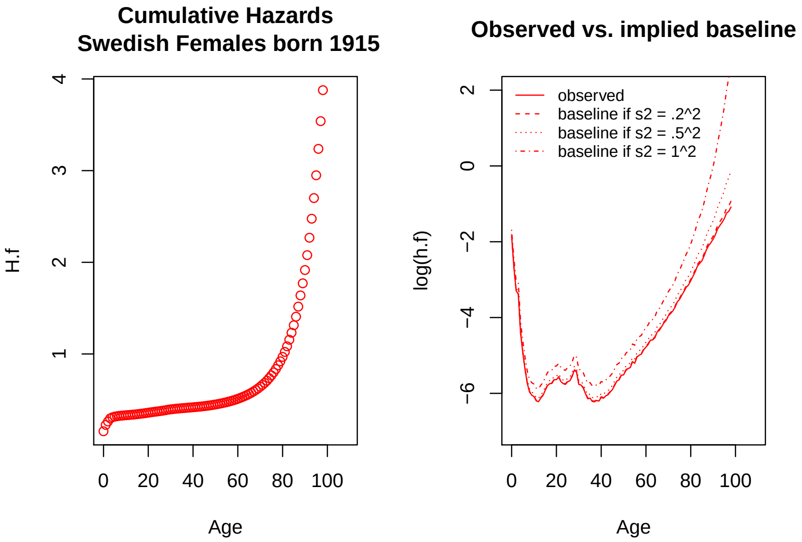
Something for nothing?

$$\mu_0(x) = \bar{\mu}(x) e^{\sigma_2 \bar{H}(x)}$$

- We don't observe underlying baseline hazard μ_0 on left
- What is observed (and unobserved) on right?

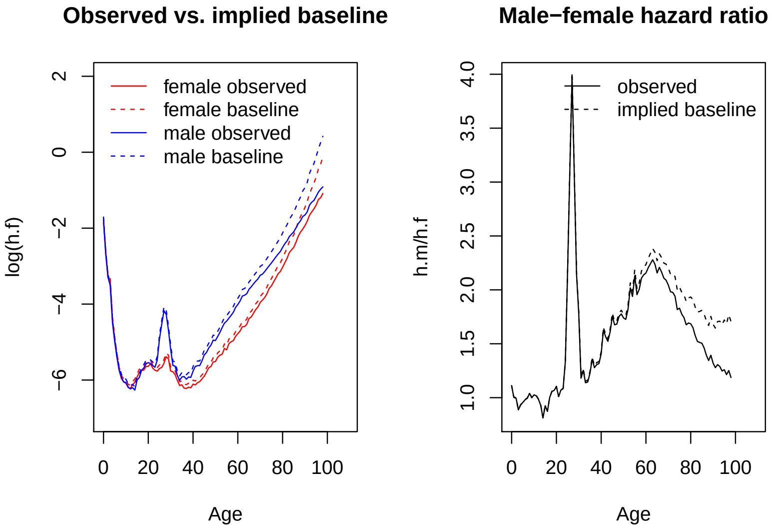
An example

Italian Females, born 1915 ($\sigma^2 = .2^2, .5^2, 1^2$)



Less convergence in baseline?

Italian Females vs Males born 1915 ($\sigma^2 = .5^2$)



Much bigger convergence in "observed" than in baseline

German's App for Black-White Crossover

Old-age mortality plateaus

We're going to skip for now, but ...

- The student group presentation will introduce
- Ken will spend a whole class on it.
- I note that the main interest is not in the plateau itself, but rather what clues it gives to evolutionary theories of senescence.

Heterogeneity slows mortality improvement (1)

Define $\bar{\rho}(x,t)$ be the rate of mortality improvement

$$\bar{\rho}(x,t) = -\frac{d}{dt}\log\bar{\mu}(x,t)$$

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Extending our gamma result for 1 cohort to the surface x, t,

$$\bar{\mu}(x, t) = \mu_0(x, t) \bar{S}_c(x, t)^{\sigma^2}$$

Heterogeneity slows mortality improvement (2)

We take the log and the time-derivative of hazards give V&M (39*)

$$\bar{\rho}(x,t) = \rho_0(x,t) - \sigma^2 \frac{d}{dt} \log \bar{S}_c(x,t)$$

(Note: this formula should be correct. V&M contains a mistake: an extra minus sign in the definition of $\acute{\bar{s}}$.)
Reversing perspective,

$$\rho_0(x,t) = \bar{\rho}(x,t) + \sigma^2 \frac{d}{dt} \log \bar{S}_c(x,t),$$

which tells us that individual-level hazards are improving faster across cohorts than the observed, aggregate hazard.

An example: 80-year old Italians, born 1880-1900

```
sigma.sq = .2 ## assume we know sigma^2
dt <- fread("~/Documents/hmd/hmd_statistics/c_lt_both/bltcoh_1x1/ITA.bltcoh_1x1
mx.80.c1880 <- dt[Year == 1880 & Age == "80"]$mx
mx.80.c1900 <- dt[Year == 1900 & Age == "80"]$mx
(rho.bar.80 <- -log(mx.80.c1900/mx.80.c1880)/20) ## about 0.9%

## [1] 0.008729961

lx.80.c1880 <- dt[Year == 1880 & Age == "80"]$lx / 100000
lx.80.c1900 <- dt[Year == 1900 & Age == "80"]$lx / 100000

(d.log.Sx <- log(lx.80.c1900/lx.80.c1880)/20) ## about 2%

## [1] 0.02177952

(rho.0.80 = rho.bar.80 + sigma.sq * d.log.Sx) ## from last slide

## [1] 0.01308586

##  0.9% + (.2)(2%) =  1.3%
```

So individual-level mortality progress is nearly 50% faster (1.3% vs. 0.9%) than it appears!

Discussion of group mini-projects/presentations

Conclusions

- Gamma frailty gives simple expressions for population survival, hazard, and average frailty.
- Gamma frailty gives a plateau
- Gamma frailty gives us a predicted rate of convergence and cross-over with age
- All of this means it is a useful null model.
- Takes us away from "it could be selection" to "what if it were selection"

The usual caveat: don't confuse model with reality. (Ken will come back to this.)