Macroeconomic Theory II

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2022

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Distributional Macroeconomics

- Study of macroeconomic questions in terms of distributions rather than just aggregates.
  - Distributions of income and wealth across households.
  - Distribution of productivity, employment and capital across firms.
- More technically: macroeconomic theories in which relevant state variable is a distribution.
- More popular name: heterogeneous agent models.
- What is attractive about this approach?
  - conceptually: unified approach to macro and distribution
  - empirically: unified approach to micro and macro data

Important message

- Hard to coherently think about macro if ignore distribution.
- Rich interaction:

$$\text{distribution} \iff \text{macroeconomy}$$

- Or perhaps more precisely:

$$\text{macroeconomy is a distribution}$$

Distribution in Macro: A History of Thought

A pedagogical categorization of macroeconomic theories

- before modern macro: 1930 to 1970
- 1st generation modern macro: 1970 to 1990
- 2nd generation modern macro: 1990 to financial crisis
- 3rd generation modern macro: after the financial crisis

Main drivers of evolution in modern macro era

- better data
- better computers & algorithms
- current events (rising inequality, financial crisis)

(Warnings: Narrative won't be perfect. There is consensus on this.)

Before Modern Macro: 1930 to 1970

1. Keynesian IS/LM
   - about aggregates, no role for inequality/distribution by design
2. Distribution does play role in growth theory
   - mostly factor income distribution – capital vs labor
     Kaldor, Pasinetti, other Cambridge UK theorists
   - rarely personal income or wealth distribution
     exceptions: Tobin, Stiglitz, Blinder
3. Disconnected empirical work on inequality (Kuznets)

First Generation Macro Theories: 1970 to 1990

Representative agent models, e.g. RBC model

- again no role for inequality/distribution by design
- advertised as "microfounded" but rep agent assumption cuts 1st generation theories from much of micro research

What is wrong with that?

1. cannot speak to a number of important empirical facts, e.g.
    - unequally distributed growth
    - poorest hit hardest in recessions
2. cannot think coherently about welfare – "who gains, who loses?"

Second Generation Macro Theories: 1990 to 2008

- Incorporate micro heterogeneity, particularly in income and wealth – early "heterogeneous agent models"

  Aiyagari, Bewley, Huggett, Imrohoroğlu, Krusell-Smith, Den Haan,...

- Represent economy with a distribution that moves over time, responding to macroeconomic shocks, policies

- Can speak to facts on previous slide, useful for welfare analysis

Second Generation Theories: Inequality $\not\Rightarrow$ Macro

- Typical finding: heterogeneity does not matter much for macro aggregates
  Krusell-Smith (1998) "approximate aggregation"
- Reason: Linearity. Rich and poor differ in wealth but not consumption and saving behavior – rich = scaled version of poor.
- Hence, "inequality $\not\Rightarrow$ macro", but also a knife-edge result
- Problem: in data, rich $\neq$ scaled version of poor, e.g. rich have
  - lower MPCs out of transitory income changes
  - higher saving rates out of permanent income, wealth
- Note: some important contributions from 90s don't fit the narrative
  - Banerjee-Newman, Benabou, Galor-Zeira, Persson-Tabellini, ..

Third Generation Theories: after the Crisis

- 3rd generation theories take micro data more seriously
- Leads them to emphasize things like
  - household balance sheets
  - credit constraints
  - MPCs that are high on average but heterogeneous
  - non-homotheticicities, non-convexities

  $\Rightarrow$ move away from knife-edge case
- Typical finding: distribution matters for macro
- Example: HANK - Heterogeneous agent New Keynesian model.

Distribution in Macro: Summary

- Before modern macro: 1930 to 1970
  - it's complicated
- 1st generation: 1970 to 1990
  - representative agent models (RBC, New Keynesian etc)
  - no role for inequality by design
- 2nd generation: 1990 to financial crisis
  - early heterogeneous agent models
  - "macro $\Rightarrow$ inequality" but "macro $\Leftarrow$ inequality" (perception)
- 3rd generation: after the financial crisis
  - current heterogeneous agent models
  - rich interaction: "inequality $\iff$ macro"

Distribution in Macro: Summary

Janet Yellen speech "Macroeconomic Research After the Crisis"
http://www.federalreserve.gov/newsevents/speech/yellen20161014a.htm

- "Prior to the financial crisis, representative-agent models were the dominant paradigm for analyzing many macroeconomic questions [= 1st generation]."
- "However, a disaggregated approach seems needed to understand some key aspects of the Great Recession..."
- "While the economics profession has long been aware that these issues matter, their effects had been incorporated into macro models only to a very limited extent prior to the financial crisis [= 2nd generation]."
- " I am glad to now see a greater emphasis on the possible macroeconomic consequences of heterogeneity [= 3rd generation]."

This course: Road-map (to be updated)

1. Income fluctuation problem
    - Discrete time
        - Long-term behavior of assets and consumption
        - Permanent Income Hypothesis
        - Precautionary Savings
        - Numerical methods
    - Continuous time
        - Optimization w/ uncertainty
        - Additional analytical results
        - Numerical tools
2. Incomplete markets: Closing the model (Bewley/Huggett/Aiyagari)
3. Incomplete markets with aggregate uncertainty
4. Examples: HANK model

Income fluctuation problem

Income fluctuation problem: Overview

1. Individuals are subject to exogenous income shocks. These shocks are not fully insurable because of the lack of a complete set of Arrow-Debreu contingent claims.
2. There is only a risk-free asset (i.e., and asset with non-state contingent rate of return) in which the individual can save/borrow, and that the individual faces a borrowing (liquidity) constraint.
3. A continuum of such agents subject to different shocks will give rise to a wealth distribution.
4. Integrating wealth holdings across all agents will give rise to an aggregate supply of capital.

Income fluctuation problem

$$\max_{\{a_{t+1},c_t\}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \qquad s.t.$$

$$c_t + a_{t+1} \leq y_t + Ra_t$$

$$\underline{a} \leq a_{t+1}$$

Assumptions

- Preferences
  - $u' > 0, u'' < 0$.
  - Inada conditions: $\lim_{c\to\infty} u'(c) = 0$ and $\lim_{c\to 0} u'(c) = \infty$.
- Income process
  - y_t drawn from a compact or finite set.
  - Expected present value bounded at any point in time.
  - Markov process (when stochastic).

Partial equilibrium exercise

- Exogenous interest rate: $R = 1 + r$.

Why are we studying the income fluctuations problem?

- Recall the complete markets outcome for a similar economy (see Yvan's slides on complete markets).
    - Individual consumption only depends on aggregate endowment.
    - CRRA case: $c_t^i = \alpha^i Y_t$.
    - Data rejects this prediction.
- Building block for general equilibrium model with heterogeneity on the household sector.

Income fluctuation problem: Road-map

1. In general, no analytic solution. However, we can make some progress.
   - Characterize the long-term behavior of consumption and assets.
   - Special case: Permanent Income Hypothesis.
   - Motives for precautionary savings.
2. Learn how to solve income fluctuation problem on a computer.
   - Bellman equation
   - Wealth distribution generated by optimal saving behavior
3. "Close the model": embed the income fluctuation problem in general equilibrium, thereby endogenizing r.
   - Different ways of doing this $\iff$ different assumptions on where capital demand comes from
   - Aiyagari: K demand of rep firm, Huggett: bonds in zero net supply, ...

Recursive formulation and Euler equation

- Recursive formulation

$$
\begin{aligned}
V(a,y) = \max_{c,a'} u(c) + \beta\mathbb{E}\left[V(a',y')|y\right] \qquad s.t.\\
c + a' \le y + Ra\\
\underline{a} \le a
\end{aligned}
$$

- Euler equation

$$u'(c) \ge \beta R\mathbb{E}\left[u'(c')|y\right]$$

equality if borrowing constraint is binding.

- Optimal allocations depend on
  1. Subjective discount rate vs interest rate: $\beta R \lesseqgtr 1$.
  2. Stochastic properties of income process y.
  3. Tightness of the borrowing constraint $\underline{a}$.
  4. Utility function.

Derivation of Euler

$$
\begin{aligned}
V(a,y) &= \max_{c,a'} u(c) + \beta\mathbb{E}\left[V(a',y')|y\right] \\
c + a' &\leq y + Ra & [\mu] \\
\underline{a} &\leq a & [\lambda]
\end{aligned}
$$

- FOC:

$$
\begin{aligned}
&[c] & u'(c) - \mu = 0 \\
&[a'] & \beta\mathbb{E}\left[V_a(a',y')|y\right] - \mu + \lambda = 0
\end{aligned}
$$

- Envelope

$$
[a] \qquad V_a(a,y) - R\mu = 0
$$

- Euler

$$
u'(c) = R\beta\mathbb{E}\left[u'(c')|y\right] + \lambda
$$

Recall that $\lambda \geq 0$.

- Exercise: Derive Euler equation when $\underline{a}(a,y)$.

Long-term behavior: consumption and assets

- "Self-insurance": agent uses savings to insure himself against income fluctuations.
- Interested in the long-run properties of an optimal "self-insurance" scheme. Do $\{c_t\}$ and $\{a_t\}$ remain bounded as $t \to \infty$?

	Deterministic y	Stochastic y
$\beta R > 1$ (patient)	Diverging	Diverging
$\beta R = 1$	Stationary	Diverging
$\beta R < 1$ (impatient)	Stationary	Ambiguous

- Surprising result flip when $\beta R = 1$!

Next: sketch of some of the proofs.

Long-term behavior: consumption and assets

Case: deterministic y and patient $\beta R > 1$

- Euler: $u'(c_t) \geq \beta R u'(c_{t+1})$
- Given $\beta R > 1$, we have $u'(c_t) > u'(c_{t+1}) \iff c_t < c_{t+1}$.
- Define $M_t \equiv (\beta R)^t u'(c_t)$, then M_t is a decreasing sequence bounded below by zero ($M_t \geq 0$).
- Since M_t is bounded and $\lim_{t\to\infty} (\beta R)^t = \infty$, we must have $\lim_{t\to\infty} u'(c_t) = 0$.
- By Inada conditions, we have that $\lim_{t\to\infty} c_t \to \infty$.

- What about assets?
- From budget constraint $c_t = Ra_t + y_t - a_{t+1} \leq Ra_t + y_t - \underline{a}$.
- Since y_t is bounded, $\lim_{t\to\infty} a_t \to \infty$.

Long-term behavior: consumption and assets

Case: deterministic y and $\beta R = 1$

- Euler: $u'(c_t) \geq u'(c_{t+1})$.
- Consumption is non-decreasing
  - Constant if constraint does not bind
  - Increasing if constraint binds
- Optimal plan: constant consumption. Is it feasible?
  - Depends on the income path and the borrowing constraint.
- Assume there is no borrowing: $\underline{a} = 0$.
  - Assume the constraint binds finitely often and let $\tau - 1$ be the last period the constraint binds. Then,

$$c_s = x_\tau \equiv \frac{r}{1+r} \sum_{j=0}^{\infty} (1+r)^{-j} y_{\tau+j} \quad \forall s \geq \tau$$

    where x_t is the annuity return on the present value of the tail of the income process starting at period t.
  - Proposition: $\lim_{t\to\infty} c_t = \sup_t x_t$. Proof: see LS 17.3.1.
    Intuition: when x_t is decreasing, income today is high relative to average income going forward. The agent does not need to borrow to implement its optimal plan.

Long-term behavior: consumption and assets

Case: deterministic y and impatient $\beta R < 1$

- Euler: $u'(c_t) \geq \beta R u'(c_{t+1})$.
    - If the constraint does not bind, we have $u'(c_t) < u'(c_{t+1}) \iff c_t > c_{t+1}$.
    - Incentive to front load consumption.
    - If the constraint binds, consumption may increase.

- With $\underline{a} = 0$ and $y_t = \bar{y}$, we can show that
    - c_t and a_t decrease until some finite τ when $a_\tau = 0$.
    - After that, $c_t = \bar{y}$ and $a_t = 0$.
    - Sketch of proof: next slide

Long-term behavior: consumption and assets

- Constant income allow us to collapse state variables (a, t) into one "cash-in-hand" $x = Ra + y$.
- Recursive problem

$$V(x) = \max_{c,x'} u(c) + \beta V(x') \quad s.t.$$
$$x' = R(x - c) + y, \qquad x \geq c$$

- Sketch of proof.
  1. Consumption increasing w/ cash-in-hand.
     Envelope $V_x(x) = u'(c)$. Replace optimal policy $c(x)$ and differentiate w.r.t. x. Then, $c'(x) = V''(x)/u''(c) > 0$.
  2. When the borrowing constraint is not binding, x decreases over time.
     $\beta R < 1$, and envelope and Euler conditions imply
     $V'(x) = \beta R V'(x') < V'(x')$, which implies $x' < x$ by the concavity of V.
  3. If the constraint binds at t, then $c_s = y$ and $a_s = 0$ for all $s \geq t + 1$.
     Assume the constraint does not bind after $x = y$, i.e., $c < x$ or $a' > 0$.
     Then, $\beta R < 1$ and concavity of V imply a contradiction:
     $V(y) = \beta R V'((1 + r)a' + y) < V'((1 + r)a' + y) < V'(y)$.
  4. The constraint must bind in finite time.
     Assume the constraint never binds. Iterate Euler equation and recall $x > y$ when constraint did not bind last period and $c'(x) > 0$ to get a contradiction: $0 < u'(c_t) = \lim_{j\to\infty}(\beta R)^j u'(c_{t+j}) \leq \lim_{j\to\infty}(\beta R)^j u'(y) = 0$.

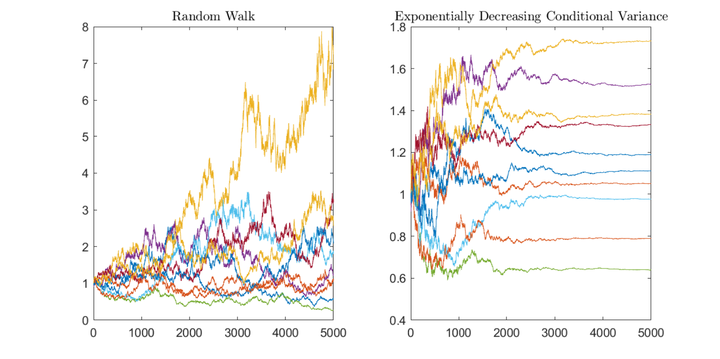
Supermartingale Convergence Theorem

- Tool for proofs in the stochastic case.
- Next: some "informal" statements to draw some intuition. More formal treatment: www.randomservices.org/random/martingales/index.html
- Let X_t be a sequence of random variables with $\mathbb{E}(|X_t|) < \infty$ for $t \in \mathbb{N}$.
    - If $X_t = \mathbb{E}_t(X_{t+1})$ $\forall t$, then $\{X_t\}_{t \geq 0}$ is a martingale.
    - If $X_t \geq \mathbb{E}_t(X_{t+1})$ $\forall t$, then $\{X_t\}_{t \geq 0}$ is a super-martingale.
    - If $X_t \leq \mathbb{E}_t(X_{t+1})$ $\forall t$, then $\{X_t\}_{t \geq 0}$ is a sub-martingale.
- **Theorem**: Let $\{X_t\}_{t \geq 0}$ be a non-negative super-martingale that satisfies $\sup_t \mathbb{E}(|X_t|) < \infty$, then there exist a random variable Z s.t.
    - $\lim_{t \to \infty} X_t \to_{a.s.} Z$
    - $\mathbb{E}(|Z|) < \infty$
- Intuition: "decreasing" sequence of random variables bounded below "converges" to a finite limit almost surely (i.e., for any shock history with positive probability).

Supermartingale Convergence Theorem (examples)

Consider $X_t = \sum_{s=0}^{t} f(s) Z_s$ with $Z_t \sim N(0,1)$. Then, X_t is a martingale and $\sup_t \mathbb{E}(|X_t|) = C_0 \sum_{s=0}^{t} f(s)^2$

- If $f(s) = 1$ (random walk), then $\sup_t \mathbb{E}(|X_t|) = \infty$
- If $f(s) = \beta^s$ with $\beta \in (0,1)$, then $\sup_t \mathbb{E}(|X_t|) < \infty$



Case: stochastic y and impatient $\beta R > 1$

- Euler: $u'(c_t) \geq \beta R \mathbb{E}_t[u'(c_{t+1})]$
- Define $M_t \equiv (\beta R)^t u'(c_t)$
- M_t is a non-negative super-martingale: $M_t \geq 0$ and $M_t \geq \mathbb{E}_t[M_t]$.
- By the Supermartingale Convergence Theorem, M_t converges to a finite random variable.
- Since $\lim_{t\rightarrow}(R\beta)^t \rightarrow \infty$, we must have that $\lim_{t\rightarrow} u'(c_t) = 0$ for any shock history with positive probability.
- By Inada conditions, we have that $\lim_{t\rightarrow\infty} c_t \rightarrow \infty$ for any shock history with positive probability.
- Also, $\lim_{t\rightarrow\infty} a_t \rightarrow \infty$ almost surely.

Long-term behavior: consumption and assets

Case: stochastic y and $\beta R = 1$

- Assume y_t is i.i.d. We can reduce the state space to a single state variable: $x_t \equiv Ra_t + y_t$.
- Envelope condition $V'(x) = u'(c) > 0$. Euler: $u'(c_t) \geq \mathbb{E}_t[u'(c_{t+1})]$
- Then, $V'(x_t)$ is a non-negative super-martingale. Hence, $V'(x_t) \to_{a.s.} Z$, where Z is a random variable with $\mathbb{E}[Z] < \infty$.
- Claim: $Z \equiv 0$. Proof: next slide.
- Since $u'(c_t) \to_{a.s.} 0$, we must have $c_t \to_{a.s.} \infty$ and $x_t \to_{a.s.} \infty$
- Chamberlain and Wilson (2000) generalize this argument for an arbitrary stochastic endowment.

Long-term behavior: consumption and assets

Claim: $Z \equiv 0$. Proof:

- Fix a history of shocks y^∞, then $Z(y^\infty)$ is a number. Recall $0 \leq Z < \infty$.
- Assume $Z(y^\infty) > 0$, then $\lim_{t\to\infty} V'(x_t)(y^\infty) = Z(y^\infty)$
- Since $V' > 0$ and $V'' < 0$ (Benveniste-Scheinkman), we can invert $V'(x)$.
- Hence, $\lim_{t\to\infty} x_t(y^\infty) = \tilde{z}(y^\infty)$.
- A symmetric argument delivers that c_t converges to some constant. Recall $u'(c_t) = V'(x_t)$.
- Let t be sufficiently large, so that x_t and c_t must be within an ϵ of their limits and must remain within those neighborhood.
- However, the budget constraint dictates $x_{t+1} - R(x_t - c_t) = y_{t+1}$ and y_t does not converge (we picked an arbitrary shock history). This is a contradiction.

- Q: Why doesn't the argument work if we consider a deterministic income path, e.g., y^∞?

Long-term behavior: consumption and assets

Case: stochastic y and patient $\beta R < 1$

Under additional restriction on utility and on the income process we can prove that the assets are bounded but one cannot prove it in general.

Consider the iid y_t case.

- If absolute risk aversion that goes to zero as consumption explodes, assets are bounded.
  - CRRA utility satisfies this. Obviously, CARA utility does not.
  - Intuition: vanishing absolute risk aversion means that the consumer is less worried about income fluctuations as he gets rich, so he will consume more and accumulate less.
- It is a sufficient condition, i.e. if βR is sufficiently low it might be the case that even with constant absolute risk aversion the asset space is bounded.

Permanent Income Hypothesis (PIH)

Consider a particular income fluctuation problem

1. $\beta R = 1$.
2. y_t stochastic.
3. Replace borrowing constraint w/ No Ponzi: $\mathbb{E}[\lim_{\tau\to\infty} R^{\tau} a_{t+\tau}] \geq 0 \;\; \forall t$.
4. Quadratic utility $u(c) = \alpha c - \frac{\gamma}{2}c^2$, $u'(c) = \alpha - \gamma c$.

Result 1: Consumption is a martingale

- Euler: $u'(c_t) = \mathbb{E}_t[u'(c_{t+1})]$.
- Replacing utility function: $c_t = \mathbb{E}_t[c_{t+1}]$
- Law of Iterated Expectations: $c_t = \mathbb{E}_t[c_{t+j}] \;\; \forall j \geq 1$ (Martingale)

Permanent Income Hypothesis (PIH)

Result 2: Consumption equals permanent income & Certainty equivalence

- Budget constraint + No Ponzi condition

$$Ra_t + \sum_{j=0}^{\infty} R^{-j} y_{t+j} = \sum_{j=0}^{\infty} R^{-j} c_{t+j}$$

- Taking expectation + c_t martingale

$$c_t = \frac{r}{1+r} \left[\underbrace{Ra_t + \sum_{j=0}^{\infty} R^{-j} \mathbb{E}_t [y_{t+j}]}_{\equiv H_t} \right]$$

  Consumption is the annuity value of the expected present value of total wealth (agent consumes its "permanent income").

- Certainty equivalence
  - c_t is a function of the income process only through H_t.
  - Only first moments matter even though the agent is risk-averse!
  - More variance of income $\Rightarrow$ same c_t but lower value function.

Permanent Income Hypothesis (PIH)

Consumption and wealth dynamics

- Δc_{t+1} is proportional to the revision in expected earnings due to the new information accruing in that same time interval.

$$\Delta c_{t+1} = \frac{r}{1+r}(\mathbb{E}_{t+1} - \mathbb{E}_t)H_{t+1} = \frac{r}{1+r}\sum_{j=0}^{\infty} R^{-j}(\mathbb{E}_{t+1} - \mathbb{E}_t)y_{t+1+j}$$

  Q: why does financial wealth does not appear in the formula above?

- Δa_{t+1} corresponds to minus the expected present value of earning changes.

$$\Delta a_{t+1} = -\sum_{j=1}^{\infty} R^{-j}\mathbb{E}_t(\Delta y_{t+j})$$

  - Save ($\uparrow a$) if you expect a decreasing income path.
  - Borrow ($\downarrow a$) if you expect an increasing income path.

Exercise: derive Δa_{t+1}. Tip: First, find a_{t+1} using c_t, then re-write the expression.

Permanent Income Hypothesis (PIH)

Specific Income Process

$$y_{t+1} = \rho y_t + \nu_{t+1}, \qquad \nu_t \text{ is a white noise}$$

- After some algebra ...

$$\Delta c_t = \left(\frac{r}{1+r-\rho}\right)\nu_t \qquad \Delta a_{t+1} = \left(\frac{1-\rho}{1+r-\rho}\right)(\rho y_{t-1} + \nu_t)$$

- Random walk y_t ($\rho = 1$): all income is permanent so all is consumed.

$$\Delta c_t = \nu_t \qquad \Delta a_{t+1} = 0$$

  The borrowing constraint will never be binding (provided $a_0 \geq \underline{a}$).

- I.I.D. y_t ($\rho = 0$): all income is transitory so only the annuity value is consumed.

$$\Delta c_t = \left(\frac{r}{1+r}\right)\nu_t \qquad \Delta a_{t+1} = \left(\frac{1}{1+r}\right)\nu_t$$

  Assets are a random walk, so any constraint tighter than No Ponzi condition will bind w/ probability 1.

Cross-section distribution

- Assume y_t is i.i.d. across households and there is a continuum of them.
- Result 1: Cross- sectional mean is constant in time
    - Let $C_t = \mathbb{E}[c_{i,t}]$ be the cross-section consumption mean at t.
    - Since consumption is a martingale, $c_{i,t+1} = c_{i,t} + \varepsilon_{i,t+1}$ with $\mathbb{E}[\varepsilon_{i,t+1}|c_{i,t}] = 0$. Then,

$$C_{t+1} = \mathbb{E}[c_{i,t+1}] = \mathbb{E}[c_{i,t}]$$

- Result 2: Cross sectional variance explodes
    - $\lim_{t\to\infty} \mathbb{V}(c_{i,t}) = \infty$ as long as $\mathbb{V}(\varepsilon_t) > 0\ \forall t$. Random earnings $\mathbb{V}(y_t) > 0$ ensure this.

Precautionary Savings

We can break the certainty equivalence result if we either:

- Relax quadratic utility and allow $u''' > 0$ (Prudence)
- Incorporate an occasionally binding borrowing constraint, i.e., $\underline{a}$ more stringent than no Ponzi.

Saving motives

- Intertemporal motive βR vs 1
- Smoothing motive: $u'' > 0$
- Precautionary motive: saving due to income uncertainty over and above certainty equivalence.
- Life cycle and bequests motives.

Precautionary Savings: Prudence

Two-period example

$$\max_{c0,a_1} u(c_0) + \beta\mathbb{E}(u(c_1)) \qquad s.t.$$

$$c_0 + a_1 = y_0, \qquad c_1 = Ra_1 + y_1$$

- y_1 is stochastic, $u' > 0$, $u'' < 0$ and $u''' > 0$. Also, $\beta R = 1$ for simplicity.
- Euler $u'(y_0 - a_1) = \mathbb{E}[u'(Ra_1 + y_1)]$, which delivers a unique solution for a_1 because $u'' < 0$. Let a_1^* be this solution.
- Consider a mean preserving spread: $\tilde{y}_1 = y_1 + \varepsilon$ where ε has zero mean and positive variance.
- u' is convex because $u''' > 0$. So,

$$\mathbb{E}[u'(Ra_1 + \tilde{y}_1)] > \mathbb{E}[u'(Ra_1 + y_1)]$$

  (Draw marginal utility)
- Then, Euler equation implies that more uncertainty implies more savings, i.e., $\tilde{a}_1^* > a_1^*$.

If the marginal utility is convex $u''' > 0$, then the individual is "prudent" and a rise in future income uncertainty leads to a rise in current savings and a decline in current consumption.

Precautionary Savings: Prudence

Multiperiod model (i.i.d. income process)

- Recursive problem

$$V(x) = \max_{c,x'} u(c) + \beta \mathbb{E} V(x') \quad s.t.$$

$$x' = R(x-c) + y, \qquad x \geq c$$

- Euler + Envelope

$$u'(c) = \beta R \mathbb{E}_t[V'(R(x-c)+y')]$$

  If $V''' > 0$, the marginal value of wealth V' is a convex function and the RHS increases with a mean-preserving spread for y'.
  Then, the optimal response is to decrease consumption and increase savings (i.e., precautionary savings).

- Sibley (1975) shows that $u''' > 0$ implies $V''' > 0$ for the income fluctuation problem with i.i.d. earnings and finite horizon.

Precautionary Savings: Borrowing constraint

Consider the PIH framework but replace the No Ponzi condition with a no borrowing constraint $a_t \geq 0$.

- Euler equation $c_t \leq \mathbb{E}_t[c_{t+1}]$ implies

$$\begin{aligned} c_t &= \min\{Ra_t + y_t, \mathbb{E}_t[c_{t+1}]\} \\ &= \min\{Ra_t + y_t, \mathbb{E}_t[\min\{Ra_{t+1} + y_{t+1}, \mathbb{E}_{t+1}[c_{t+2}]]\} \end{aligned}$$

- Assume the constraint is not binding at t. Then, a mean preserving spread of y_{t+1} increases savings (and decreases consumption) at t.
  - Low realizations of y_{t+1} become more likely.
  - More likely that borrowing constraint binds at $t+1$.
  - The value of $\mathbb{E}_t[c_{t+1}] = \mathbb{E}_t[\min\{Ra_{t+1} + y_{t+1}, \mathbb{E}_{t+1}[c_{t+2}]]$ decreases.
  - Since the constraint is not binding today, we have that c_t declines and a_{t+1} increases.
- Even in absence of prudence (e.g. with quadratic preferences), in presence of borrowing constraints a rise in future income uncertainty can lead to a rise in current savings and a decline in current consumption, so certainty equivalent does not hold.

Natural Borrowing Limit

- Find the maximum amount the agent can repay with probability 1 using his future income. To do so, set $c_t = 0$ and future income to its lowest possible realization. We assume $c_t \geq 0$.
- Deterministic case

$$
\begin{aligned}
c_t &= Ra_t + y_t - a_{t+1} \geq 0 \\
a_t &\geq R^{-1}(a_{t+1} - y_t) \\
a_t &\geq \underbrace{-R^{-1} \sum_{j=0}^{\infty} R^{-j} y_{t+j}}_{NBL} + \underbrace{\lim_{j \to \infty} R^{-j} a_{t+j}}_{=0 \text{ by No Ponzi}}
\end{aligned}
$$

- Stochastic case
  - Same calculation but with future income path fixed at its lowest possible value y_{min}.

$$
a_t \geq -R^{-1} \sum_{j=0}^{\infty} R^{-j} y_{min} + \lim_{j \to \infty} R^{-j} a_{t+j} = -\frac{y_{min}}{R-1}
$$

  - Example: If $\log y_t \sim N(\mu, \sigma)$, then the NBL is zero.
- NBL never binds if $\lim_{c \to 0} u'(c) = \infty$ or $\lim_{c \to 0} u(c) = -\infty$

Numerical solution

Deterministic and constant income $y_t = y$

- Recursive formulation of household problem: Bellman equation

$$V(a) = \max_{c,a'} u(c) + \beta \mathbb{E}\left[V(a')\right] \qquad s.t.$$

$$c + a' \leq y + Ra$$

$$\underline{a} \leq a$$

- Functional equation: solve for unknown function.
- Arguments of value function are called state variables.
- Solution is
  - Value function: $V(a)$
  - Policy functions: $c(a)$, $a'(a)$.

Value Function Iteration (refresher)

- Easiest method to numerically solve Bellman equation for $V(a)$.
- Guess value function on RHS of Bellman equation then maximize to get value function on LHS.
- Update guess and iterate to convergence right until convergence.
- Contraction Mapping Theorem: guaranteed to converge if $\beta < 1$.
- We will learn other methods later, but this is simplest (and slowest).

Value Function Iteration (refresher)

- Step 1: Discretized asset space $\mathcal{A} = a_1, a_2, ..., a_N$. Set $a_1 = \underline{a}$.
- Step 2: Guess initial $V_0(a)$. Good guess is the value of consuming the permanent income.

$$V_0(a) = \sum_{t=0}^{\infty} \beta^t u(ra + y) = \frac{u(ra + y)}{1 - \beta}.$$

- Step 3: Set $\ell = 1$. Loop over all $\mathcal{A}$ and solve

$$\begin{aligned} a_{\ell+1}(a_i) &= \arg\max_{a' \in \mathcal{A}} u(y + (1+r)a_i - a') + \beta V_\ell(a') \\ V_{\ell+1}(a_i) &= \max_{a' \in \mathcal{A}} u(y + (1+r)a_i - a') + \beta V_\ell(a') \\ &= \max_{a' \in \mathcal{A}} u(y + (1+r)a_i - a_{\ell+1}(a_i)) + \beta V_\ell(a_{\ell+1}(a_i)) \end{aligned}$$

Value Function Iteration (refresher)

- Step 4: Check for convergence $\epsilon_\ell < \bar{\epsilon}$

$$\epsilon_\ell = \max_i |V_{\ell+1}(a_i) - V_\ell(a_i)|$$

  - if $\epsilon_\ell \geq \bar{\epsilon}$, go to step 2 with $\ell := \ell + 1$
  - if $\epsilon_\ell < \bar{\epsilon}$, then next step.

- Step 5: Extract optimal policy functions:
  - $a'(a) = a_{\ell+1}$
  - $V(a) = V_{\ell+1}$
  - $c(a) = y + Ra - a'(a)$

- Code: vfi_deterministic.m

Consumption function restricted to implied grid so not very accurate.

Finite Horizon Dynamic Programming (for completeness)

- Value function depends on time t

$$V_t(a) = \max_{c,a'} u(c) + \beta \mathbb{E}\left[V_t(a')\right] \qquad s.t.$$

$$c + a' \leq y_t + Ra$$

$$\underline{a} \leq a$$

- Solution consists of sequence of value functions $\{V_t(a)\}_{t=0}^T$ and sequence of policy functions $\{c_t(a), a_t'(a)\}_{t=0}^T$
- Solve by backward induction. Last period:

$$a_T' = 0$$

$$c_T(a) = y_T + Ra$$

$$V_T(a) = u(y_T + Ra)$$

- Code: vfi_deterministic_finite.m

Stochastic Markov Income Process y_t

- Sequence Formulation

$$\max_{\{a_{t+1},c_t\}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u(c_t) \right] \qquad s.t.$$

$$c_t + a_{t+1} \leq y_t + Ra_t$$

$$\underline{a} \leq a_{t+1}$$

- Assume y_t is a Markov Process: CDF F satisfies

$$F(y_{t+1}|y_t) = F(y_{t+1}|y_t)$$

where $y^t := \{y_0, y_1, ...y_t\}$ denotes the history of income realizations.

Stochastic Markov Income Process y_t

- Recursive formulation

$$\begin{aligned} V(a,y) &= \max_{c,a'} u(c) + \beta \mathbb{E}\left[V(a',y')|y\right] \qquad s.t. \\ c + a' &\leq y + Ra \\ \underline{a} &\leq a \end{aligned}$$

- Solution is
  - Value function: $V(a,y)$
  - Policy functions: $c(a,y)$, $a'(a,y)$.

Discrete-State Markov Process for Income

- Finite number of income realizations: $y \in \{y_1, ..., y_J\}$
- $P_{J \times J}$ is a Markov transition matrix where
  - the (i, j) element of P is $\Pr(y_{t+1} = y_j | y_t = y_i) = p_{ij}$
  - $\forall (i, j) \quad p_{ij} \in [0, 1]$
  - $\forall i \quad \sum_j p_{ij} = 1$
- Stationary distribution is vector π with elements π_j
  - Solves
    $$\pi = P^T \pi, \qquad P^T = \text{transpose of } P$$
    π is the eigenvector of P^T associated with an eigenvalue of 1.
    $Av = \lambda v$
  - Easy method for finding π in practice: take some π_0 and N large
    $$\pi \approx (P^T)^N \pi_0$$
  - Logic: $\pi_{t+1} = P^T \pi_t$.