Escaping Uncertainty Traps*

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December 9, 2025

Abstract

Which policies help escape *uncertainty traps*, i.e., demand recessions plagued by self-fulfilling uncertainty? This problem is pervasive: uncertainty traps are endemic to standard New Keynesian models, and no conventional Taylor rule can eliminate them. When constrained by an effective lower bound, no interest rate rule, even beyond conventional Taylor rules, can eliminate uncertainty traps. By contrast, many fiscal policies escape uncertainty traps by sufficiently shifting focus away from debt sustainability and toward stimulus. One such policy is fiscal dominance. Another, *fiscal volatility targeting*, commits to not accommodate non-fundamental shocks and works without leading to the conventional fiscal equilibrium.

JEL Codes: E30, E40, G01.

Keywords: New Keynesian, Taylor rules, fiscal backstops, uncertainty trap, risk premia, sentiment, determinacy, escape clauses, fiscal dominance, volatility targeting.

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Policymakers wake up to an economy in demand recession with high levels of uncertainty. What can they do? Specifically, what policies help escape this type of recession? Should monetary policy adopt an aggressive rule promising to dramatically reduce interest rates, akin to the "Taylor principle?" What if monetary policy is ultimately constrained, say by a zero lower bound (ZLB)? Should fiscal policy ensure debt sustainability to prevent the recession from spiraling out of control, or should it engage in stimulus as an attempt to provide support?

To answer these questions, we start by describing *uncertainty trap* equilibria that can arise in New Keynesian (NK) models. As the prevailing model of demand recessions, the NK model is an appropriate setting to study. Given our focus on uncertainty, we deviate from the textbook analysis in one respect: we study the NK model in its nonlinear, stochastic form—i.e., without linearization. There exists an entire class of self-fulfilling equilibria characterized by volatility and risk premia that both rise in recessions. We refer to this as an uncertainty trap because it is precisely the uncertainty that self-justifies the recession. Furthermore, uncertainty trap equilibria can emerge under any conventional Taylor rule that responds to output and inflation, including nonlinear rules. In other words, once we take seriously the nonlinear, stochastic structure of the NK model, the Taylor principle simply fails. After explaining this, we explore policies that can help.

Uncertainty traps. The core mechanism in our paper is precautionary saving. A non-fundamental demand recession can arise today, roughly speaking, because a large-enough rise in uncertainty creates precautionary savings. In other words, if agents wake up and are sufficiently uncertain about the future, they will save today and consume less. The dynamics this creates, in addition to a recession today, is an expectation of recovery in the future, since precautionary savings shifts consumption into the future.

As an individual agent, why don't I just ignore all non-fundamental shocks and keep consuming as I would in the "fundamental equilibrium"? With nominal rigidities, output and wages are demand-determined. If I know that aggregate demand is low today but I expect it to recover in the future, I know my personal income will follow a similar path. Facing a reduction in my present-value of income, it is indeed optimal for me to consume less today. For this reason, namely that other agents' consumption influences my income hence my own consumption, demand volatility—an endogenous object—is not fully pinned down by equilibrium restrictions and can be shaped by coordination.

What, then, does discipline demand volatility? A volatility profile is consistent with equilibrium as long as households internalize that profile in their saving decisions and the implied output dynamics remain *globally stable*—that is, neither explosive nor implosive. The fact that uncertainty rises sufficiently in our story above, which then leads to a future demand recovery after the initial recession, is what ensures global stability. We must focus on global stability, rather than local stability, in our stochastic, nonlinear version of the model, which leads to a novel analysis and also some interesting insights. For example, when a non-fundamental demand recession occurs, uncertainty need not rise in the short run, but must rise sufficiently at some point, potentially far, in the future. In particular, it must rise once the recession becomes sufficiently deep. This logic suggests a large set of possible equilibrium mappings between output and volatility, which is what we find.

So far, the discussion has abstracted from monetary policy. Under a conventional Taylor rule, monetary policy tries to thwart non-fundamental recessions by lowering the interest rate. Lowering interest rates induces front-loaded consumption via the intertemporal substitution mechanism, in contrast to the back-loaded consumption induced by precautionary savings. Whether monetary policy succeeds or fails in eliminating an uncertainty trap depends on the balance of these two forces: intertemporal substitution versus precautionary savings. If uncertainty rises sufficiently, precautionary savings dominates in the sense that, on net, aggregate demand is back-loaded. In that case, a demand recession can emerge today, despite the Taylor rule, because agents rightly expect the economy to recover in the future. Taking this logic a step further suggests that *no Taylor rule can eliminate all uncertainty traps*: given a monetary rule, agents can always coordinate on a sufficiently countercyclical uncertainty such that precautionary savings dominates intertemporal substitution. This is what we prove (Theorem 1).

Now, we can understand why sufficient uncertainty is critical to our results. Consider a hypothetical demand recession that is accompanied by no rise in uncertainty at any horizon. Under the Taylor principle, the monetary authority lowers the path of interest rates to encourage front-loaded consumption. This implies that future aggregate demand, and hence the household's future income, must be even weaker. Facing a path of declining interest rates and vanishing income, the household's optimal choice is a collapsing consumption trajectory, which in turn induces an implosive path for output. Such implosive paths are ruled out as equilibria. (This essentially restates the well-known fact that the Taylor principle selects a unique equilibrium in the deterministic version of the NK model.) Uncertainty traps, by contrast, feature an income path that declines initially but then recovers over time rather than implodes.

How plausible are uncertainty trap dynamics? We argue that they possess several intuitive properties and are robust to some of the critiques levied against other types of multiplicity in NK models. First, all uncertainty trap equilibria are non-explosive, meaning they do not require any hyper-inflations or economic collapse. For this reason, uncertainty traps are very different from the hyper-inflationary indeterminacies raised by Cochrane (2011). Non-explosiveness also means that our equilibria are robust to certain popular types of "escape clauses" in the monetary economics literature, namely policies that eliminate explosive hyper-inflation paths (Obstfeld and Rogoff, 1983, 1986; Atkeson et al., 2010; Sims, 2013). Second, our equilibria feature countercyclical volatility, which is an intuitive property that emerges in real-world dynamics. Third, we provide numerical examples to show how our equilibria can be sustained by empirically-plausible levels of output-growth volatility. We also provide three extensions—one with idiosyncratic risk, one with alternative preferences, and one with the zero lower bound (ZLB)—to show that uncertainty traps can be sustained with even lower levels of aggregate volatility than our baseline model. Finally, uncertainty traps need not even stem from a "non-fundamental" shock; instead, volatility indeterminacy induced by sticky prices is a more general phenomenon that can operate for any source of aggregate uncertainty.

Policy remedies. Are there policies that can help an economy escape an uncertainty trap? We already argued that conventional Taylor rules are insufficient in general. This susceptibility to uncertainty traps adds an additional critique of conventional Taylor rules, beyond the fact that they may permit hyperinflations (Cochrane, 2011) or may be time-inconsistent (Neumeyer and Nicolini, 2025). Next, we explore augmented monetary rules that directly target the risk premium. This can work: strong risk-premium targeting (i.e., an interest rate that responds more than one-for-one to the risk premium in recessions, and less than one-for-one in booms) eliminates the uncertainty trap. That being said, if interest rates are subject to an effective lower bound, as is the case in the real world, risk-premium targeting fails and uncertainty traps re-emerge. In this sense, the uncertainty trap phenomenon resembles the liquidity trap issue (Benhabib, Schmitt-Grohé, and Uribe, 2001) but with volatility as the source. Summarizing, monetary policy alone cannot escape or prevent uncertainty traps.

Given the limitations of monetary policy, we turn next to fiscal policy and ask how it can help. Until this point, we implicitly assumed that fiscal policy focuses on debt sustainability. This is known in the literature as "monetary dominance" paired with a fiscal regime that is "passive" (Leeper, 1991). Our first foray into fiscal policies examines the polar opposite, "active" fiscal policy, also known as "fiscal dominance." In this setup, the fiscal authority does not act to ensure debt sustainability under any and every possible shock. Some shocks can be "unbacked" and must be *absorbed* via inflation, debt valuations, or economic growth. Think of this as stimulus: unbacked government borrowing can create wealth effects and hence demand.

Fiscal dominance eliminates all uncertainty traps. It is worth understanding this point before considering other fiscal policies. The debt valuation equation says

$$\frac{\text{Nominal debt value}}{\text{Price level}} = \text{Current surplus} \times \text{Present value of real surplus growth} \tag{1}$$

Suppose a non-fundamental shock affects output and therefore current surpluses. Under passive fiscal policy, the fiscal authority would accommodate the surplus path to ensure government debt valuation equation holds. Under active fiscal policy, the non-fundamental shock must be absorbed by the price level, nominal debt valuation, or present-value of surplus growth (via changes to future surpluses or their discount rates).

If absorption cannot happen, then the uncertainty trap equilibrium is ruled out. Sticky prices say that prices cannot jump arbitrarily, and so the price level cannot absorb such a shock. Can the nominal debt value absorb the shock? In the baseline case with short-term debt, the debt price is fixed at 1, and so the quantity of debt is simply determined by the flow government budget constraint; thus, the nominal value of short-term debt is pinned down and cannot absorb the shock either. In the extension with long-term debt, the bond price is an additional forward-looking variable that could potentially respond to shocks. But with the additional variable comes an additional constraint, the bond-pricing equation, which is inconsistent with the originally conjectured output shock. A similar logic applies to the present-value of real surplus growth on the right-hand-side of (1). Given absorption fails, active fiscal policy effectively steers aggregate demand, disciplining its response to aggregate shocks and closing the door to the volatility required to sustain uncertainty traps (Theorem 2). The novelty and difficulty of this analysis stems from the nonlinearity of the environment.

Is fiscal dominance required? No. In the last part of the paper, we demonstrate that fiscal policy can eliminate uncertainty traps with a lighter touch that we call *fiscal volatility targeting* (Theorem 3). It can target zero non-fundamental volatility, and achieve that outcome, by eschewing debt sustainability and instead committing to provide stimulus in non-fundamental recessions. In other cases, where either times are normal or a recession is for fundamental reasons, fiscal policy continues to focus on debt sustainability. Meanwhile, monetary policy obeys the Taylor principle and steers the economy toward the fundamental solution. Thus, the resulting dynamics resemble the conventional "monetary equilibrium" as opposed to the one associated with fiscal dominance, even though fiscal policy is responsible for eliminating non-fundamental volatility. For example, whereas fiscal dominance causes demand to inherit fiscal shocks, this volatility targeting approach results in an equilibrium immune to fiscal shocks.

Our paper is primarily concerned with policy prescriptions for an uncertainty trap.

Accordingly, we do not take a stand on which regime or equilibrium is being played in the data. Given that uncertainty trap dynamics are reasonable, some economies may in fact be vulnerable to our equilibria. Conversely, even if a government takes appropriate steps to avoid uncertainty traps, we take no stand on whether its policy resembles fiscal dominance or something closer to fiscal volatility targeting.

Related literature. This paper relates to literatures on: (a) NK indeterminacies; (b) fiscal policy and monetary-fiscal regimes; (c) self-fulfilling uncertainty.

A well-developed literature exposits indeterminacies in monetary models. From Sargent and Wallace (1975), we know that exogenous interest rates do not pin down an equilibrium. Since then, several papers have established indeterminacies in NK models related to the ZLB—e.g., Benhabib et al. (2001) identify "deflationary traps" related to inflation expectations, and Benigno and Fornaro (2018) study "stagnation traps" related to growth expectations. There is additional scope for indeterminacy in heterogeneous-agent NK models if idiosyncratic income risk is countercyclical (Acharya and Dogra, 2020; Ravn and Sterk, 2021; Bilbiie, 2024; Acharya and Benhabib, 2024). Our uncertainty traps are novel in two ways: (i) unlike these deterministic results, our multiplicity vitally depends on aggregate risk and risk premia; (ii) our equilibria do not rely on the ZLB and hold under any conventional Taylor rule.

Uncertainty traps are nonlinear and stochastic by nature. In an important contribution, Caballero and Simsek (2020) study a nonlinear, stochastic NK model and illustrate how risk premia are critical to aggregate demand dynamics, but restricting attention to the "fundamental equilibrium." We directly connect to their setting: in our analysis, the uncertainty trap equilibrium keeps the economy stuck at the ZLB. Closely related to our study, contemporaneous work by Lee and Dordal i Carreras (2024) also studies a nonlinear IS curve with risk premia driving multiplicity. Like us, they also argue that standard "active" Taylor rules do not prune this type of volatility. Our results are more general in proving that self-fulfilling uncertainty can survive *any* Taylor rule, analyzing equilibria with inflation, and our analysis of the ZLB and other lower bounds. Most importantly, our key novelty is our exploration of fiscal policies. While the nonlinear NK literature is limited, other papers use perturbation methods to study risk premia in such models; to the best of our knowledge, these methods ignore the determinacy issues we raise.[1]

Angeletos and Lian (2023) has argued that the NK model, in the presence of a particular kind of small coordination friction, has as its locally unique equilibrium the "mini-

[1] Examples include Rudebusch and Swanson (2012), Kung (2015), and Basu and Bundick (2017), which all study a (particular) third-order perturbation around steady state.

mum state variable" (MSV) solution, even without the Taylor principle. Their results do not apply directly to our setting. Indeed, our uncertainty trap equilibria are intrinsically about nonlinearity and global (as opposed to local) indeterminacy. Still, future research should investigate the perturbations of Angeletos and Lian (2023) in a nonlinear setting.

A literature on the fiscal theory of the price level (FTPL) also explores determinacy in NK models under various policy profiles.[2] Unlike most of this literature, we analyze the fully nonlinear, stochastic, global dynamics of the model. Some papers have extended FTPL to stochastic nonlinear environments, but almost exclusively with *flexible prices*. For example, Bassetto and Cui (2018) and Brunnermeier, Merkel, and Sannikov (2023, 2024) study stochastic and nonlinear flexible-price models, focusing on the determinacy of a bubble and/or liquidity-service term in government debt valuation.[3] Two exceptions that do allow sticky prices, but sidestep our determinacy questions, are Mehrotra and Sergeyev (2021) and Li and Merkel (2025). Mehrotra and Sergeyev (2021) study fiscal sustainability with real debt, but in a setting with exogenous output. Li and Merkel (2025) study FTPL in a NK model with idiosyncratic risk, which can induce a government debt bubble; however, they avoid determinacy questions by assuming that endogenous objects like inflation and the output gap are Markovian in exogenous states and government bonds outstanding (i.e., they assume an MSV solution). Overall, we provide the first formal nonlinear FTPL-style analysis in textbook sticky-price models.

Importantly, our *fiscal volatility targeting* can eliminate uncertainty traps without resorting to fiscal dominance. Such targeting can sometimes work even if deployed only in deep recessions, analogous to "escape clauses" that prune liquidity traps (Benhabib, Schmitt-Grohé, and Uribe, 2002; Bassetto, 2004; Sims, 2013). Our key novelties are: (i) the policy itself has volatility in its design; and (ii) some specifications eliminate volatility regardless of how the monetary policy rule is set. Finally, by avoiding fiscal dominance, our volatility targeting policy avoids a key trade-off highlighted by Bianchi and Melosi (2017), whereby one bad outcome is eliminated (in our case, uncertainty traps; in their case, a deflationary trap) at the cost of introducing exposure to fiscal shocks.

Finally, our paper connects to a broader set of models that predict self-fulfilling uncertainty, with some examples in Bacchetta, Tille, and Van Wincoop (2012), Benhabib, Wang, and Wen (2015), Fajgelbaum, Schaal, and Taschereau-Dumouchel (2017), Khorrami and Mendo (2024, 2025), and the aforementioned contemporaneous NK paper by Lee and Dordal i Carreras (2024). Our policy analysis is novel to this literature.

[2]Seminal contributions to the FTPL include Leeper (1991), Sims (1994), Woodford (1994), Woodford (1995), Kocherlakota and Phelan (1999), and Cochrane (2001). Cochrane (2023) synthesizes many results.

[3]Others studying the FTPL in nonlinear, but deterministic, environments with liquidity premia include Berentsen and Waller (2018), Williamson (2018), Andolfatto and Martin (2018), and Miao and Su (2024).

1 Baseline Model

We present a canonical New Keynesian (NK) economy with complete markets and nominal rigidities (e.g., Galí, 2015). We study the model in continuous time for tractability.

Uncertainty. Our baseline model features no fundamental uncertainty in preferences or technologies. To transparently incorporate some uncertainty, we introduce a sunspot shock: a one-dimensional Brownian motion Z. All random processes will be adapted to Z. It is not critical that the shock be a pure sunspot. In Appendix B, we allow Z to be a fundamental shock in the form of monetary policy uncertainty, and the same results go through. The deep point is that uncertainty is not pinned down—the shock from which uncertainty stems is less important.

Preferences. The representative agent has rational expectations and time-separable utility with discount rate ρ, unitary EIS, and labor disutility parameter φ:

$$\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big(\log(C_t) - \frac{L_t^{1+\varphi}}{1+\varphi}\Big)dt\Big]. \tag{2}$$

Consumption C_t has the nominal price P_t and labor L_t earns the nominal wage W_t.

Technology. The consumption good is produced by a linear technology $Y_t = L_t$. We abstract from fundamental uncertainty (e.g., productivity shocks) for maximal clarity.

Behind the aggregate production function is a structure common to most of the NK literature. In particular, there are a continuum of firms who produce intermediate goods using labor in a linear technology. These intermediate goods are aggregated by a competitive final goods sector. The elasticity of substitution across intermediate goods is a constant ε. The intermediate-goods firms behave monopolistically competitively and set prices strategically, described next.

Price setting. Intermediate-goods firms set prices strategically, taking into consideration the impact prices have on their demand. Price setting is not frictionless: firms changing their prices are subject to quadratic adjustment costs, a la Rotemberg (1982). (For simplicity, we assume these adjustment costs are non-pecuniary, so that resource constraints are not directly affected by price adjustments.) In the interest of exposition, we relegate the statement of and solution to this standard problem to Appendix G.

Definition: inflation and output gap. Let P_t denote the aggregate price level and $\pi_t := \dot{P}_t/P_t$ its inflation rate. Note also that the flexible-price level of output is given by $Y^* =$

$(\frac{\varepsilon-1}{\varepsilon})^{\frac{1}{1+\varphi}}$. Following the literature, define the output gap $x_t := \log(Y_t/Y^*)$. Conjecture that x_t and π_t have dynamics of the form

$$dx_t = \mu_{x,t}dt + \sigma_{x,t}dZ_t \tag{3}$$

$$d\pi_t = \mu_{\pi,t}dt + \sigma_{\pi,t}dZ_t \tag{4}$$

for some $\mu_x, \sigma_x, \mu_\pi, \sigma_\pi$ to be determined in equilibrium.

Monetary policy. Let ι_t denote the nominal short-term interest rate, which is controlled by the central bank. Monetary policy follows a Taylor rule that targets the output gap and inflation with

$$\iota_t = \bar{\iota} + \Phi(x_t, \pi_t), \tag{MP}$$

for some long-run target rate $\bar{\iota}$ and some response function that satisfies $\Phi(0,0) = 0$. For the response function, a common linear example that we will use sometimes is

$$\Phi(x, \pi) = \phi_x x + \phi_\pi \pi. \tag{linear MP}$$

We address the zero lower bound (ZLB) in an extension below. For now, think of negative interest rates as a proxy for unconventional monetary policy that can work even when the short rate is zero.

Financial markets. Financial markets are complete. Let M_t be the real stochastic discount factor induced by the real interest rate $r_t := \iota_t - \pi_t$ and the equilibrium price of risk h_t associated to the shock Z_t. The equity market is a claim on the profits of the intermediate-goods producers. Alternatively, we can think of these profits as being rebated to the consumers lump-sum. The risk-free bond market is in zero net supply—this will be generalized in Section 4 when we introduce "active" fiscal policies.[4]

Definition 1. An *equilibrium* is processes $(C_t, Y_t, L_t, W_t, P_t, M_t, B_t, \iota_t, r_t, \pi_t)_{t\geq 0}$, such that

(i) Taking (M_t, W_t, P_t) as given, consumers choose $(C_t, L_t)_{t\geq 0}$ to maximize (2) subject

[4]For our baseline results, it is not necessary to completely shut down fiscal policy and assume the absence of government debt. Instead, the reader can think of fiscal policy levying lump-sump taxes and transfers but acting "passively" to always ensure its own solvency. As we show in Section 4, this "passive fiscal" interpretation generates an identical set of equilibria to those we uncover in Section 2.

to lifetime budget and No-Ponzi constraints. The lifetime budget constraint is

$$\frac{B_0}{P_0} + \Pi_0 + \mathbb{E}\Big[\int_0^\infty M_t \frac{W_t L_t}{P_t} dt\Big] \geq \mathbb{E}\Big[\int_0^\infty M_t C_t dt\Big] \tag{5}$$

where Π represents the real present-value of producer profits and B represents the bond-holdings of the consumer. The No-Ponzi constraint is $\lim_{T\to\infty} M_T \frac{B_T}{P_T} \geq 0$.[5]

(ii) Firms set prices optimally, subject to their quadratic adjustment costs.

(iii) Markets clear, namely $C_t = Y_t = L_t$ and $B_t = 0$.

(iv) The central bank follows the interest rate rule (MP) for some given long-run target rate $\bar{\iota}$ and some given response function $\Phi(\cdot)$.

Equilibrium characterization. We first provide a summary characterization of all equilibria. Labor supply and consumption satisfy the following optimality conditions:

$$e^{-\rho t} L_t^{\varphi} = \lambda M_t \frac{W_t}{P_t} \tag{6}$$

$$e^{-\rho t} C_t^{-1} = \lambda M_t, \tag{7}$$

where λ is the Lagrange multiplier on the lifetime budget constraint (5).

On the firm side, Appendix G shows that optimal firm price setting gives rise to aggregate inflation dynamics that satisfy

$$\mu_{\pi,t} = \rho \pi_t - \eta \varepsilon \frac{W_t}{P_t} + \eta(\varepsilon - 1), \tag{8}$$

where η is each firm's degree of price flexibility. Notice that as $\eta \to 0$ (prices changes become infinitely costly), one possible equilibrium is to have $\pi_t \to 0$ for all times. We will assume this "rigid-price limit" is the equilibrium that obtains as $\eta \to 0$.

We use these conditions to obtain an "IS curve" and a "Phillips curve." Applying Itô's formula to (7), we obtain the consumption Euler equation, which may be rewritten in terms of the output gap as

$$\mu_{x,t} = \iota_t - \pi_t - \rho + \frac{1}{2}\sigma_{x,t}^2. \tag{IS}$$

[5]In addition, to prevent arbitrages like "doubling strategies" that can arise in continuous time, we must impose a uniform lower bound on borrowing, e.g., $M_t B_t / P_t \geq -\underline{b}$, although $\underline{b}$ can be arbitrarily large.

Equation (IS) is the IS curve. Next, divide the FOCs (6)-(7), and use goods and labor market clearing $C_t = Y_t = L_t$ to get $Y_t^{1+\varphi} = \frac{W_t}{P_t}$. Substitute this into (8) to obtain

$$\mu_{\pi,t} = \rho\pi_t - \kappa\Big(\frac{e^{(1+\varphi)x_t} - 1}{1+\varphi}\Big), \tag{PC}$$

where $\kappa := \eta(\varepsilon - 1)(1+\varphi)$. Equation (PC) is the Phillips curve.

The most important novelty in our paper is the presence of precautionary savings due to aggregate risk. This force is captured by the term $\frac{1}{2}\sigma_x^2$ in (IS). We will often refer to this term as a "risk premium" because σ_x^2 is exactly the equilibrium risk premium on the aggregate consumption claim.[6] When writing the IS curve in terms of log consumption, as is typically done, the Jensen correction of $\frac{1}{2}$ also shows up.

Transversality and non-explosiveness. Together with the monetary policy rule (MP), equations (IS) and (PC) form the nonlinear "three equation model" in standard NK models. An equilibrium is completely characterized by these three equations, along with conditions that ensure that any output gap or inflation explosions are consistent with optimization behavior. More specifically, agents' transversality conditions must hold, and they must not obtain minus infinite utility. These are usually regarded as technical conditions, but they are important to recognize in our setting.

For example, since $C_t = e^{x_t}Y^* = L_t$, the representative agent would obtain minus infinite utility if $x_t = \pm\infty$ in finite time, or even if $x_t \to \pm\infty$ too quickly. Clearly, this is not compatible with optimizing behavior if the agent has an alternative that delivers finite utility. Consumers would be individually better off choosing an alternative plan for their consumption and labor supply, unravelling such a proposed allocation. (In the case when $x_t \to \infty$, real wages are diverging towards plus infinity, and agents may obtain finite utility simply by working a finite amount forever. In the case when $x_t \to -\infty$ sufficiently fast, real wages are collapsing, so it is unclear that there is any plan delivering finite utility; however, since all plans are equally-bad, there is no reason for agents to coordinate on any particular plan.) Similarly, we show in Appendix G that firms' optimization rules out situations when $\pi_t \to \pm\infty$ too quickly, because that would induce an infinite amount of price adjustment costs.

In order to emphasize that the multiplicity unveiled later does not rely on explosive behavior (unlike the multiplicity exposited in Cochrane, 2011) and to streamline

[6] The aggregate risk premium is σ_x^2 for the following reason. Aggregate consumption growth volatility is σ_x. Due to log utility, agents optimally set their individual consumption growth volatility equal to the market price of risk (i.e., the sensitivity of $d\log M$ to dZ). Hence, the quantity of risk and price of risk are both equal to σ_x, implying the risk premium (as the product of risk quantity and risk price) is σ_x^2.

the analysis, we only consider equilibria satisfying a simple condition that conforms with most existing literature and rules out both finite-time and asymptotic explosions. As shown in Appendix A.1, these non-explosiveness criteria are sufficient to ensure all agents' transversality conditions hold (although they may be stronger than necessary). Obviously, relaxing any of these restrictions permits more equilibria than we unveil.

Condition 1. *A non-explosive allocation has* $\mathbb{P}\{|x_t| < \infty, |\pi_t| < \infty; \forall t \geq 0\} = 1$,

$$\limsup_{T\to\infty} \mathbb{E}|x_T| < \infty, \quad \limsup_{T\to\infty} \mathbb{E}|\pi_T| < \infty, \quad \textit{and} \quad \limsup_{T\to\infty} \mathbb{E}[e^{(1+\varphi)x_T - \rho T}] = 0. \tag{9}$$

We summarize our characterization in the following lemma. The proof is standard except for a careful treatment of potentially explosive behavior (see Appendix A.1).

Lemma 1. *Suppose processes* $(x_t, \pi_t, \iota_t)_{t\geq 0}$ *satisfy (IS), (PC), (MP), and Condition 1. Then,* $(x_t, \pi_t, \iota_t)_{t\geq 0}$ *corresponds to an equilibrium of Definition 1.*

Going forward, we will want to make reference only to equilibria which satisfy Condition 1. For that reason, we include the following definition.

Definition 2. A *non-explosive equilibrium* is an equilibrium in which Condition 1 holds.

Linearized Phillips curve approximation. As mentioned above, our critically novel aspect is the nonlinearity in (IS), due to precautionary savings. Less important for us is the nonlinearity in (PC). To make this particularly clear, the remainder of the paper uses a linearized Phillips curve instead, as in the overwhelming majority of the literature. An additional benefit is that the resulting linearized Phillips curve here (under Rotemberg pricing) is identical to the linearized Phillips curve obtained with alternative pricing frictions (e.g., under Calvo pricing), implying an amount of robustness to our results.[7] Appendix F.2 presents complementary analysis of deterministic dynamics based on the fully nonlinear Phillips curve (PC).

By the first-order approximation $\frac{e^{(1+\varphi)x}-1}{1+\varphi} \approx x$, the linearized Phillips curve is

$$\mu_{\pi,t} = \rho\pi_t - \kappa x_t. \tag{linear PC}$$

[7] That being said, the nonlinear Phillips curves in the Rotemberg and Calvo models are different. As one example, Ascari and Rossi (2012) show that Rotemberg and Calvo models can be quite different in settings with trend inflation. The key difference: the nonlinear Calvo model features a state variable capturing the cross-sectional dispersion in prices, which drops out in the linearization. We cannot guarantee that all of our sunspot equilibria would survive the inclusion of such a state variable, although we see no obvious reason why this would matter besides complicating the analysis.

In this approximation, going forward we refer to non-explosive equilibrium as $(x_t, \pi_t, \iota_t)_{t\geq 0}$ that satisfy (IS), (linear PC), (MP), and Condition 1.

2 Uncertainty Trap Equilibria

We obtain several new results pertaining to volatility in NK models. For theoretical clarity, this section assumes a sufficiently aggressive monetary policy (i.e., the "Taylor principle"). As is well known, the Taylor principle renders the deterministic equilibrium unique among non-explosive equilibria—we verify and extend such results in Appendix F. Nevertheless, a large class of non-explosive stochastic equilibria exists for *any* degree of monetary responsiveness. We begin with some intuition and a series of examples to build understanding, followed by more general theoretical propositions. After that, we present three extensions that all reduce the required aggregate volatility in an uncertainty trap, thus helping to make our equilibria more plausible quantitatively. Appendix A.3 contains the proofs of this section's results, relying on some key stochastic stability results in Appendix A.2.

2.1 Understanding uncertainty traps

Before turning to the formal arguments, we outline the core intuition. Why can a non-fundamental recession today, $x_0 < 0$, be an equilibrium outcome once we incorporate *precautionary savings* into the analysis? For simplicity, we illustrate the core intuition in the rigid-price limit, $\kappa \to 0$. This makes clear that the indeterminacy we study is real rather than nominal and reduces the analysis to a one-dimensional stochastic system for the output gap.

Using $\bar{\iota} = \rho$ as the policy target and imposing $\pi_t = 0$ in the rigid-price limit, combining (IS) with (MP) collapses the standard NK system to

$$\mu_{x,t} = \Phi(x_t) + \frac{1}{2}\sigma_{x,t}^2. \tag{10}$$

We assume monetary policy satisfies the Taylor principle, i.e., $\Phi'(x) > 0$.

The novelty in our analysis is the σ_x^2 term, which captures agents' optimal response to risk—that is, precautionary saving. An equilibrium is a stochastic process for x_t that satisfies (10) and the non-explosiveness Condition 1. Non-explosiveness disciplines the dynamics, but seemingly only weakly: we conjecture some initial condition x_0 and a process for $\sigma^2_{x,t}$, recover $\mu_{x,t}$ from (10), and the resulting process for x_t constitutes an equilibrium if and only if it is non-explosive—that is, globally stable.

When the σ^2_x term is ignored, as in standard linearized analysis, global and local stability coincide, and the Taylor principle tightly disciplines the dynamics of x_t. Why? Suppose hypothetically a recession is observed today ($x_0 < 0$). The household understands that the downturn is occurring *despite* the monetary authority's adherence to the Taylor principle, which calls for lowering interest rates in recessions to stimulate current consumption through intertemporal substitution. Hence, the household correctly infers that aggregate demand—already depressed today—is in fact front-loaded. Since the household's income equals aggregate demand, it understands the observed recession implies a vanishing future income path combined with an ever-decreasing path of interest rates. Confronted with this outlook, the household's optimal choice is a collapsing consumption trajectory. This entire story is conveniently summarized by equation (10) as $\mu_{x,t} = \Phi(x_t) < 0$ when $x_t < 0$. The resulting output implosion is in violation of Condition 1, and so the initial hypothetical recession cannot be part of a non-explosive equilibrium.

Now incorporate precautionary saving and assume that output volatility $\sigma^2_{x,t}$ rises during recessions. Again suppose hypothetically that $x_0 < 0$. If $\sigma^2_{x,t}$ rises sufficiently, then precautionary savings can dominate monetary policy, in the sense that $\mu_{x,t} = \Phi(x_t) + \frac{1}{2}\sigma^2_{x,t} > 0$ is possible. If so, demand is expected to recover over time, rather than collapse, and consequently monetary policy is no longer expected to choose an ever-decreasing path of interest rates. Faced with a recession that is only temporary, with an expected future recovery in income and interest rates, a household's optimal consumption plan is to reduce consumption today in favor of consumption in the future. The initially conjectured recession thus becomes part of a non-explosive equilibrium.

Our stories above suggest that uncertainty traps persist for any degree of monetary-policy aggressiveness. Even if $\Phi'(x) > 0$ is very large, there is always a level of σ^2_x that overwhelms it and makes $\mu_x > 0$. Because equilibrium disciplines volatility only through global stability, one can always find a sufficiently countercyclical volatility profile that preserves global stability—and thus sustains an uncertainty trap.

Two technicalities require some care. First, what is the precise speed of recovery, measured by $\mu_{x,t}$, that is required to justify a demand recession? To obtain an exact condition, our global stability analysis adapts methodological tools from the literature on stochastic stability. Second, since global stability is the appropriate equilibrium condition, we only require that $\mu_{x,t} = \Phi(x_t) + \frac{1}{2}\sigma^2_{x,t}$ be sufficiently positive when x_t is sufficiently low. That is, demand recessions can be justified by sufficiently high long-horizon uncertainty, even without an increase in short-horizon uncertainty.

2.2 Constructing example uncertainty traps

The intuition outlined in the previous section now needs to be formalized. For concreteness, consider a monetary-policy rule with target rate $\bar{\imath} = \rho$ and nonlinear response function

$$\Phi(x) = \phi_x(e^x - e^{-x}), \qquad \phi_x > 0. \tag{11}$$

This rule satisfies the Taylor principle; indeed, it is a super-aggressive policy rule, responding more strongly to output gap fluctuations than its linear approximation $2\phi_x x$. Such aggressiveness gives monetary policy a better chance of preventing uncertainty traps. Nevertheless, we now demonstrate that the model admits an entire family of uncertainty-trap equilibria.

Before turning to the analysis of uncertainty traps, we emphasize that under the interest-rate rule (11), the standard analysis that ignores precautionary savings yields a unique non-explosive equilibrium in which the output gap is identically zero. In fact, the policy rule considered here drives the deterministic economy (with $\sigma_{x,t}^2 = 0$) to a finite-time explosion unless $x_t = 0$ for all t (see Appendix F.3).

It is important to stress, however, that the source of multiplicity is not output volatility per se—that is, the mere fact that $\sigma_{x,t}^2 > 0$—but rather the effect of risk on agents' behavior, namely precautionary savings. Put differently, a stochastic economy in which precautionary savings are artificially shut down has $\mu_{x,t} = \phi_x(e^{x_t} - e^{-x_t})$ and still features a unique non-explosive equilibrium.

We now turn our attention to the stochastic economy in which agents engage in precautionary savings. For mathematical convenience, it will be easier to study the level output gap $y_t := e^{x_t}$. To confirm a non-explosive equilibrium, we need to verify that y_t remains within $(0, \infty)$ forever and does not approach either boundary too rapidly in expectation. With monetary policy (11), the drift and diffusion of y_t are, by Itô's formula,

$$\mu_y = \phi_x(y^2 - 1) + y\sigma_x^2 \quad \text{and} \quad \sigma_y = y\sigma_x \tag{12}$$

Given a candidate volatility function $\sigma_x(x)$, we can completely characterize whether or not these dynamics are explosive or not. We start with two illustrative examples. In Example 1, the non-explosion condition holds, and so we have an uncertainty trap. In Example 2, volatility is "insufficiently countercyclical," and so it is not an equilibrium.

Following their descriptions, Figure 1 compares the dynamics from these examples.

Example 1. *Consider, for some constant $\nu > 0$, the volatility function*

$$\sigma_x^2 = \begin{cases} \left(\frac{\nu}{y}\right)^2 + \phi_x \frac{1-y^2}{y}, & \text{if } y < 1; \\ 0, & \text{if } y \geq 1. \end{cases} \tag{13}$$

Uncertainty rises as demand falls, with the inverse relationship particularly strong when demand is very depressed. Plugging this into equation (12), the dynamics for y_t would be

$$dy_t = \begin{cases} \frac{\nu^2}{y_t} dt + \sqrt{\nu^2 + \phi_x y_t (1 - y_t^2)} dZ_t, & \text{if } y_t < 1; \\ \phi_x (y_t^2 - 1) dt & \text{if } y_t \geq 1. \end{cases} \tag{14}$$

It is relatively intuitive to see that $y_t > 0$ forever in this example: if y ever approached 0, the drift $\frac{\nu^2}{y}$ would explode fast enough to push y back up. Formalizing this mathematically, the process in (14) behaves asymptotically as $y \to 0$ like a Bessel(3) process, which never hits 0 (more precisely, 0 is an "entrance boundary" for this process).[8] *And consequently, $x_t = \log(y_t)$ satisfies the non-explosiveness condition provided $y_0 \leq 1$. Starting from any $y_0 \leq 1$, this economy will display sunspot volatility until it eventually converges to and remains stuck at the efficient level $y = 1$. This entire construction works for any $\nu > 0$. In summary, despite the super-aggressive response function (11), many equilibria exist with different σ_x.*

Example 2. *Consider, instead, the volatility function*

$$\sigma_x^2 = \begin{cases} \tilde{\nu}^2 + \phi_x \frac{1-y^2}{y}, & \text{if } y < 1; \\ 0, & \text{if } y \geq 1. \end{cases} \tag{15}$$

This uncertainty is countercylical but less so than Example 1. Plugging this into (12), we have

$$dy_t = \begin{cases} y_t \tilde{\nu}^2 dt + \sqrt{y_t^2 \tilde{\nu}^2 + \phi_x y_t (1 - y_t^2)} dZ_t, & \text{if } y_t < 1; \\ \phi_x (y_t^2 - 1) dt & \text{if } y_t \geq 1. \end{cases} \tag{16}$$

This alternative ensures positive expected output growth, but it fails the non-explosiveness condition because the drift is not increasing fast enough as y falls. Formalizing this mathematically, the process in (16) behaves asymptotically as $y \to 0$ like a Feller square root process (also known

[8] A Bessel(n) process corresponds to the solution of $dX_t = \frac{n-1}{2} X_t^{-1} dt + dZ_t$ where dZ_t is a one-dimensional Brownian motion. A Bessel(n) process is also equivalent to the Euclidean norm of a n-dimensional Brownian motion and therefore it satisfies $X_t > 0$ for all $t > 0$, provided $n \geq 2$. Taking the limit of the dy_t evolution equation as $y \to 0$, we can see that $\nu^{-1} y$ behaves exactly as a Bessel(3) process.

in economics as a "Cox-Ingersoll-Ross" process) with a long-run mean of zero, which then clearly suggests that y_t will eventually hit 0.[9] *This process is explosive and is not an equilibrium.*

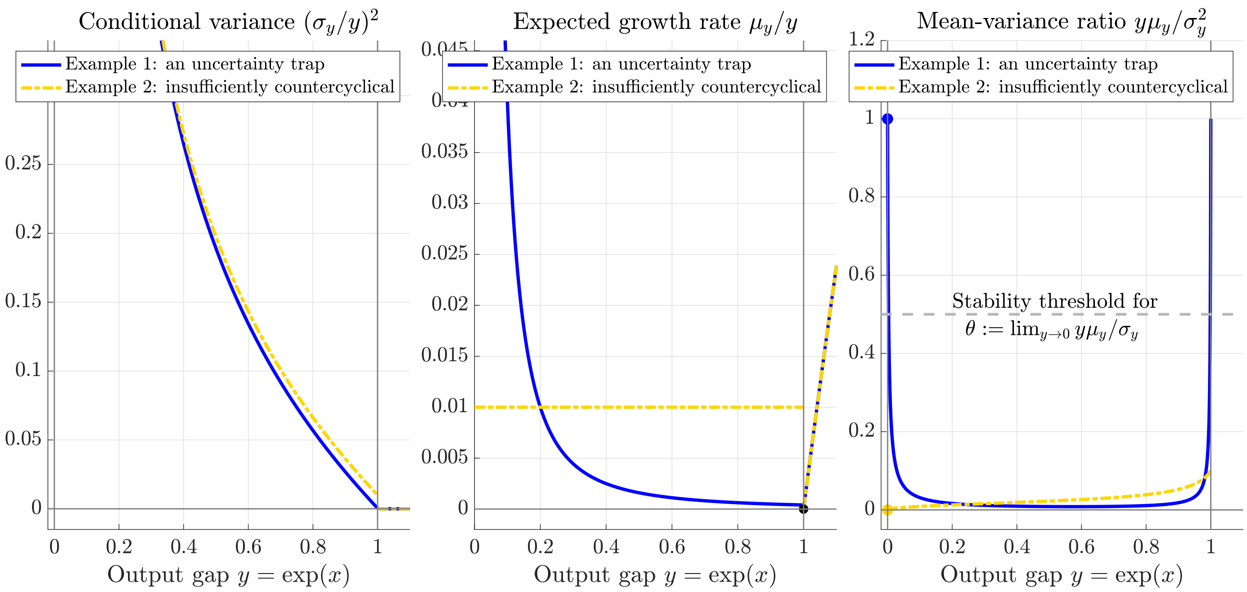

Figure 1: Output gap dynamics in Examples 1-2. Parameters: $\rho = 0.04$, $\nu = 0.02$, $\phi_x = 0.125$, $\tilde{\nu} = 0.10$.

Figure 1 visualizes the difference between the dynamics in the two examples. The left panel shows that uncertainty is recessionary and countercyclical in both examples, but slightly moreso in Example 1. The discrepancy is modest for most values of y and becomes significant only asymptotically as $y \to 0$, emphasizing the importance of the tail behavior in deep recessions. In fact, volatility in Example 2 is slightly higher for most values of y, but as we will see, this is immaterial for whether or not an uncertainty trap can take hold. The middle panel shows the consequences of uncertainty on the expected growth rate μ_y/y. In Example 1, this growth rate rises sufficiently fast as y falls, which is enough to prevent $y \to 0$. This is a case of a strongly countercyclical risk premium. By contrast, the growth rate in Example 2 remains constant as $y \to 0$, because the risk premium is not as countercyclical.

How countercyclical must the risk premium be to create an uncertainty trap? We provide a precise answer. Specifically, as shown in Lemma A.1 in the appendix, a sufficient condition for non-explosiveness towards $x = -\infty$, as required by Condition 1, is

$$\theta := \lim_{y\to 0} \frac{y\mu_y}{\sigma_y^2} > \frac{1}{2}. \tag{17}$$

[9] A Feller square-root process corresponds to the solution of $dX_t = a(b - X_t)dt + c\sqrt{X_t}dZ_t$ where dZ_t is a one-dimensional Brownian motion. The parameter b corresponds to the long-run mean. The necessary and sufficient condition for X_t to never hit 0 is $2ab \geq c^2$.

In other words, the expected growth rate μ_y/y must rise sufficiently as $y \to 0$ to offset one-half of growth rate variance $(\sigma_y/y)^2$. The idea is that the forces impacting dynamics in expectation must outweigh the shocks hitting the system. Importantly, this is only a relevant consideration in the tail, in deep recessions.

Using (17) as a tool, we can see immediately why Example 1 is an equilibrium and Example 2 is not. The right panel of Figure 1 plots the ratio $y\mu_y/\sigma_y^2$ for the two examples, showing a dramatically different behavior as $y \to 0$: Example 1 has $\theta = 1$, satisfying condition (17), while Example 2 has $\theta = 0$, violating this condition.

We can go beyond these examples to develop a generic understanding of the forces at play. Returning to the general Taylor rule (MP) and remaining in the rigid-price limit, the relevant ratio takes the form

$$\lim_{y\to 0} \frac{y\mu_y}{\sigma_y^2} = 1 + \lim_{x\to -\infty} \frac{\Phi(x)}{\sigma_x^2}. \tag{18}$$

Assuming that monetary policy satisfies the Taylor principle, we have $\Phi(x) < 0$ for $x < 0$. Hence, there is a clear race between monetary policy and risk. Monetary policy lowers expected future growth by front-loading aggregate demand through the intertemporal substitution channel, thereby reducing the key growth-to-variance ratio. By contrast, risk σ_x^2 raises expected output growth, because precautionary savings back-load aggregate demand. An uncertainty trap can arise as an equilibrium only if the risk effect in sufficiently deep recessions dominates the effect of monetary policy.

The equilibrium in Example 1 features only transitory volatility. This was chosen only for expositional clarity and is easily generalized. If volatility fades before the economy has fully recovered, precautionary saving motives disappear. Consequently, the economy cannot complete its recovery. Instead, it remains trapped in a volatile recession. We exploit this idea in the next example to create a permanent uncertainty trap.

Example 3. *Consider the volatility function*

$$\sigma_x^2 = \begin{cases} \left(\frac{v}{y}\right)^2 + \phi_x \frac{1-y^2}{y}, & \text{if } y \leq 1-\delta_H; \\ \tilde{\sigma}_x^2(x), & \text{if } y \in (1-\delta_H, 1-\delta_L) \\ 0, & \text{if } y \geq 1-\delta_L. \end{cases} \tag{19}$$

for any $0 < \delta_L < \delta_H < 1$, any function $\tilde{\sigma}_x^2(x) \geq 0$, and any $y_0 < 1$. This volatility coincides with Example 1 in a deep recession, is arbitrary when the economy is in an intermediate recession, and is zero in a small recession. It follows from our previous analysis that the volatility dynamics *near $y = 0$ and $y = 1$ ensure Condition 1 is satisfied. Most interestingly, there exists a non-degenerate stationary distribution for y with full support in the region $y \leq 1 - \delta_L$.*[10]

We use Example 3 in a numerical exercise to illustrate that our uncertainty traps can have empirically-reasonable properties. Figure 2 displays the configuration and the stationary distribution. We set $\delta_L = 0.005$ and $\delta_H = 0.15$, meaning the economy is always depressed by 0.5% below potential but only encounters extreme volatility when output is 15% below potential. The choice of $\tilde{\sigma}_x^2(x)$ is illustrated by the black dotted-dashed line in the left panel of Figure 2. This function is specified to remain around 2% during ordinary recessions ($x_t \geq -0.03$) and to rise sharply only during the deepest downturns ($x_t < -0.04$). The stationary distribution indicates that the economy spends approximately 20% of the time below $x = \log(1 - \delta_L)$, whereas excursions into deep recessions are rare. Specifically, the probability of observing x below -0.05 is only 0.90%. These tail excursions are infrequent because even modest uncertainty is enough to create enough precautionary savings that induces a recovery on average. For example, the right panel shows that the drift is positive for all $x \leq -0.04$.

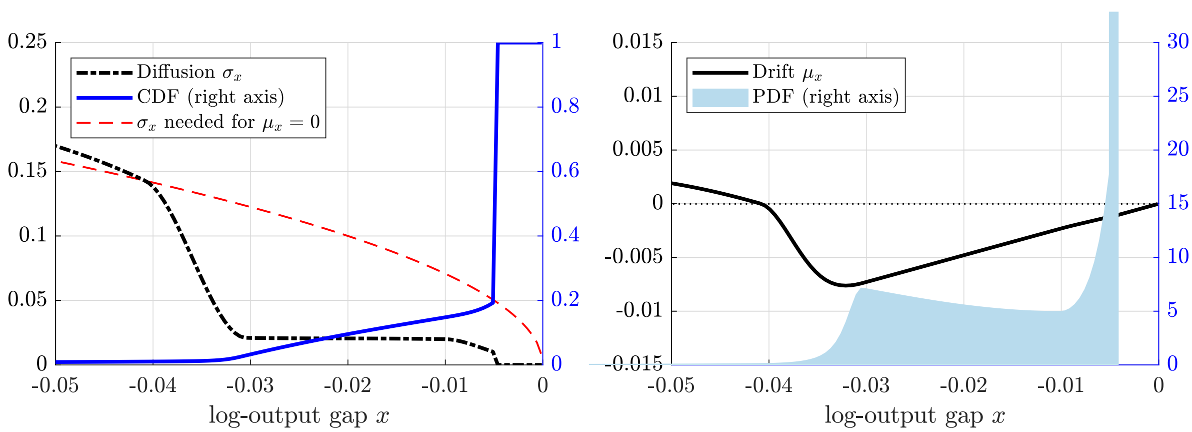

Figure 2: Uncertainty trap equilibrium from Example 3. The stationary distribution is obtained from a discretized Kolmogorov Forward Equation and features a mass point at $x = \log(1 - \delta_L)$. Parameters: $\rho = 0.04$, $\nu = 0.50$, $\delta_L = 0.005$, $\delta_H = 0.15$, $\phi_x = 0.125$. The function $\tilde{\sigma}_x^2(x)$ is constructed as a shape-preserving piecewise cubic interpolation of the following points. We set $\tilde{\sigma}_x(\log(0.97)) = 0.021$, $\tilde{\sigma}_x(\log(0.99)) = 0.02$, $\tilde{\sigma}_x(\log(0.9949)) = 0.01$, and choose $\tilde{\sigma}_x(\log(0.96))$ such that $\mu_x(\log(0.96)) = 0$. Continuity of $\sigma_x^2(x)$ is imposed at $x = \log(1 - \delta_H)$.

Because of infrequent tail excursions, this example features quantitatively reasonable output dynamics. The unconditional standard deviation of annual output growth is

[10]To see this point, note that the process has zero volatility and negative drift when $y \in (1 - \delta_L, 1)$; therefore, the process exits the region $(1 - \delta_L, 1)$ and enters $(0, 1 - \delta_L)$ in finite time almost-surely. Upon entering the volatile region $(0, 1 - \delta_L)$, the process can move around but will never reach $y = 0$, by the same Bessel(3) argument established in Example 1. Finally, note that the stationary distribution will additionally have a mass point at $y = 1 - \delta_L$, because the dynamics induce y_t to visit the point $1 - \delta_L$ infinitely often (i.e., the drift of y_t is positive just below $1 - \delta_L$).

2.56%, close to the data. This standard deviation increases to 5.87% conditional on output growth being at least half a standard deviation below its unconditional mean, so volatility is also countercyclical here.[11]

2.3 General results

The analysis so far is confined to some particular examples with a specific monetary policy. But perhaps monetary policy could act even more aggressively and eliminate uncertainty traps. Is there some Taylor rule that can kill these equilibria? No. There always exists a level of volatility that keeps the dynamics "stable" for *any* level of aggression in the Taylor rule satisfying the following mild regularity assumption (which all rules considered in this paper and in the literature satisfy):

Assumption 1. *There exists $\beta > 0$ such that $\Phi(x)$ satisfies $\lim_{x \to -\infty} e^{\beta x} \Phi(x) > -\infty$.*

Theorem 1. *Suppose prices are rigid ($\kappa \to 0$). Consider any Taylor rule (MP) with $\bar{\iota} = \rho$, increasing in x, and satisfying Assumption 1. Then,*

(i) There exist a continuum of non-explosive equilibria indexed by $x_0 < 0$ and the volatility function $\sigma_x(x)$. The volatility can be any mapping $\sigma_x : \mathbb{R} \mapsto \mathbb{R}$ that is finite for all $x \in (-\infty, 0)$ and satisfies suitable boundary conditions as $x \to -\infty$ and $x \to 0$.

(ii) If $\inf_x \Phi'(x) > 0$, then all non-explosive equilibria have $x_t \leq 0$ forever, and hence any stochastic equilibrium is recessionary.

The idea behind statement (i) of Theorem 1 is contained in the example constructions above. For any Taylor rule, there exists a large-enough volatility that "undoes" the effect of interest rates on output gap dynamics. We emphasize the fact that $\sigma_x(x)$ can essentially be any function satisfying suitable "boundary conditions": when one cares about global stability as we do, all that matters are boundary conditions on the equilibrium dynamics, rather than a local analysis around the fundamental equilibrium $(x, \pi) = (0, 0)$.[12]

Statement (ii) of Theorem 1 says that self-fulfilling volatility is *necessarily recessionary*. Risk premia σ_x^2 always increase the drift μ_x and causes output to rise over time. In a recession (i.e., $x < 0$), this pushes x back toward zero, thereby preventing economic collapse and ensuring that the stochastic path remains admissible as an equilibrium. But

[11]To compute statistics at the annual frequency, we simulate the model at a daily frequency over a 5,000-year horizon and aggregate the data to the annual level.

[12]Khorrami and Mendo (2024) provides a similar global "stochastic stability" analysis of a financial accelerator model, based on boundary behavior of the relevant dynamical system.

in a boom (i.e., $x > 0$), risk premia would send the economy further away from steady state, which violates non-explosiveness conditions.

For simplicity, Theorem 1 is proved with only sunspot shocks. This is inessential: our results are, at their core, about an indeterminacy in uncertainty within NK models. Theorem B.1 in Appendix B thus presents a very similar result in an economy with monetary policy shocks, so that the uncertainty is actually fundamental. There, we show existence of a large class of equilibria with different sensitivities of demand to monetary shocks and in which all equilibria with "excess volatility" are necessarily recessionary.

For tractability, Theorem 1 is proved in the rigid-price limit. However, the same intuition carries over to a world with partially-flexible prices. Indeed, Theorem C.1 in Appendix C constructs a similar recessionary equilibrium in which both inflation and the output gap are stochastic. Figure 3 displays a numerical example of such an equilibrium in which $\pi_t = \pi(x_t)$ for some function $\pi(\cdot)$.

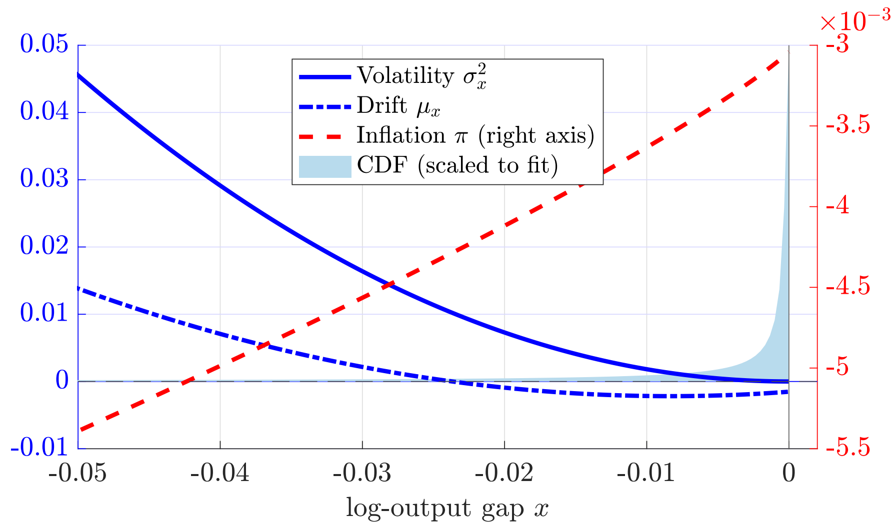

Figure 3: Equilibrium with partially-flexible prices ($\kappa > 0$), a linear Taylor rule, a linearized Phillips curve, but a nonlinear IS curve. The inflation and volatility functions are described in Appendix C, which describes a family of equilibria indexed by (i) the point $\bar{x}$ in the state space where the inflation and volatility functions have a kink (in this construction, we set $\bar{x} = -0.15$) and (ii) the slope of the inflation function in the region $\{x < \bar{x}\}$, we set $\bar{\pi} = 0.06$. The stationary CDF is computed via a discretized Kolmogorov Forward equation. Parameters: $\rho = 0.04$, $\kappa = 0.075$, $\bar{\iota} = \rho$, $\phi_x = 0.125$, $\phi_\pi = 1.5$.

This equilibrium has the following properties. First, there is *persistent deflation*. This is natural, since demand-driven fluctuations typically generate recessions jointly with below-average inflation (or even deflation), although we have not established that this property holds generally. Second, as in our earlier rigid-price example, uncertainty is recessionary and permanent, with $\mathbb{P}\{x_t < 0\} = 1$. Third, extreme recessions are rarely visited in equilibrium. In this example, output is 5% below potential only 0.41%

of the time in the ergodic distribution. This reflects the fact that μ_x becomes positive long before boundary behavior becomes relevant, as illustrated in Figure 3. And finally, the construction yields plausible quantitative results. The standard deviation of annual output growth is 2.69%, reinforcing that uncertainty traps need not be associated with counterfactually large aggregate volatility. Meanwhile, the annual inflation rate averages -0.32% with a standard deviation of 0.23%.

2.4 Extensions and quantitative relevance

A key question is whether or not uncertainty traps are plausible. One gauge of their plausibility is their quantitative implications, most importantly for equilibrium output volatility. As our numerical examples above have shown, there do exist specifications with quantitatively reasonable predictions—low aggregate volatility on average, infrequent episodes of high volatility. Here, we push this point further by presenting three simple and natural extensions that all work to reduce the amount of volatility needed in an uncertainty trap.

2.4.1 Zero lower bound

Realistically, monetary policy is constrained by a lower bound on interest rates. Here, we analyze uncertainty traps under a zero lower bound (ZLB). We proceed by an example equilibrium construction. In this example, not only can volatility be smaller than in our baseline model, but volatility remains *bounded* even as the economy is pushed into exceptionally deep downturn. Thus, realistic policy constraints eliminate the worry that uncertainty traps rely on very high volatility in extreme recessions.

We assume that prices are rigid ($\kappa \to 0$) for simplicity. In that case, policy rules with the ZLB constraint take the form

$$\iota_t = \max\left\{0, \bar{\iota} + \Phi(x_t)\right\}. \tag{20}$$

We assume $\Phi' > 0$, so that monetary policy is "active" when it is unconstrained. Thus, there is some threshold $\bar{x} < 0$, satisfying $0 = \bar{\iota} + \Phi(\bar{x})$, such that $\iota_t = 0$ whenever $x_t \leq \bar{x}$. (In Appendix D, we generalize policy by allowing for *optimal discretionary monetary policy*, following the work of Caballero and Simsek (2020), and the results are the same as here.) We prove, with an example construction, the following.

Proposition 1. *Suppose prices are rigid ($\kappa \to 0$). Consider a ZLB-constrained Taylor rule (20) with Φ increasing in x. Then, there exist non-explosive sunspot equilibria with bounded volatility that remain permanently below potential and visit the ZLB recurrently.*

Example 4. *Consider the following example volatility function*

$$\sigma_x = \nu \max\left[0, 1 - \epsilon - e^x\right] \tag{21}$$

for some $\epsilon > 0$ and $\nu > 0$ satisfying $\nu^2(1-\epsilon)^2 > 2\rho$. Notice that this volatility is bounded above by $\nu(1-\epsilon)$. With this volatility, the dynamics of x_t are given by

$$dx_t = \left[\left(\bar{\imath} + \Phi(x_t)\right)^+ - \rho + \frac{1}{2}\left[\left(1-\epsilon-e^{x_t}\right)^+\right]^2 \nu^2\right]dt + \left(1-\epsilon-e^{x_t}\right)^+ \nu dZ_t. \tag{22}$$

If $\nu^2(1-\epsilon)^2 > 2\rho$, there exists a non-explosive equilibrium that experiences recurrent episodes of depression at the ZLB (this is what the proof of Proposition 1 shows). In this case, the uncertainty trap resembles a liquidity trap (Benhabib et al., 2001), in the sense that some force (here, volatility) pushes the economy to the ZLB and then keeps it trapped there for some time. Figure 4 plots a numerical illustration in which the economy visits the ZLB 43% of the time.

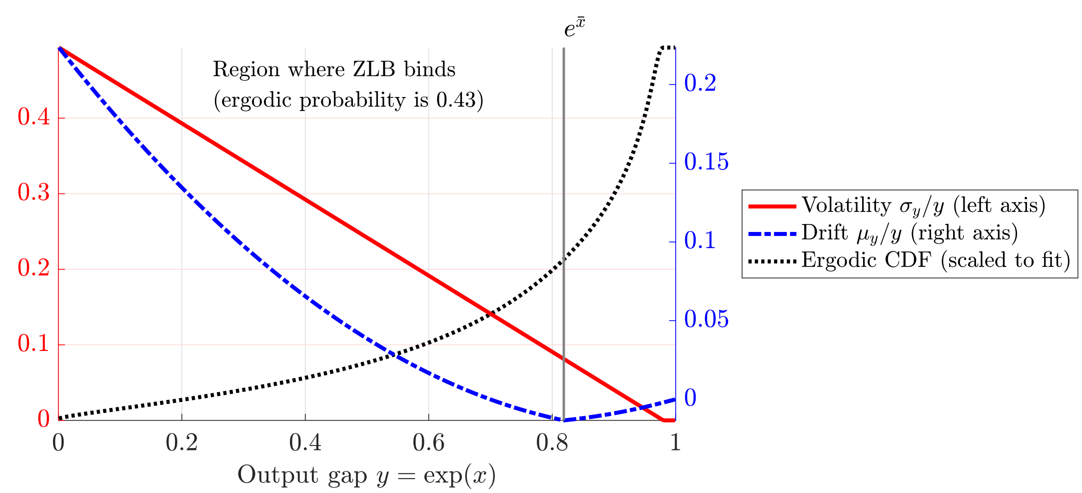

Figure 4: Output gap dynamics with the ZLB. The construction uses volatility function (21) and a linear reaction function $\Phi(x) = \phi_x x$. Parameters: $\rho = 0.02$, $\bar{\imath} = 0.02$, $\phi_x = 0.1$, $\nu = 0.504$, and $\epsilon = 0.02$.

2.4.2 Higher prudence with non-log preferences

Our baseline model adopts log utility for simplicity. If we use preferences with higher risk aversion or a lower elasticity of intertemporal substitution, the requirements for uncertainty traps become even weaker, as we now show.

For this section, start by letting the representative agent have a general time-additive utility function $u(c)$. The consumption FOC is now $e^{-\rho t}u'(C_t) \propto M_t$. Writing consumption dynamics as $dC_t = C_t[\mu_{C,t}dt + \sigma_{C,t}dZ_t]$, this then implies the Euler equation

$$\mu_{C,t} = \underbrace{\Big(-C_t\frac{u''(C_t)}{u'(C_t)}\Big)^{-1}}_{:=EIS_t}(\iota_t - \pi_t - \rho) + \frac{1}{2}\underbrace{\Big(-C_t\frac{u'''(C_t)}{u''(C_t)}\Big)}_{:=RP_t}\sigma^2_{C,t}.$$

The EIS_t modulates the consumption growth sensitivity to real rates $\iota_t - \pi_t$, while RP_t modulates the consumption growth sensitivity to risk $\sigma^2_{C,t}$. The dynamics of the output gap x_t, which coincide with those of $\log(C_t)$, are

$$\mu_{x,t} = EIS_t \times (\iota_t - \pi_t - \rho) + \frac{1}{2}(RP_t - 1)\sigma^2_{x,t}. \tag{23}$$

Preferences with higher prudence RP_t feature more precautionary savings, meaning that each unit of risk σ^2_x creates more growth μ_x. Preferences with lower substitution elasticity EIS_t are less responsive to interest rates, meaning that monetary policy must act more aggressively to affect μ_x. Both mechanisms permit uncertainty traps with less risk.

The most popular preferences in macroeconomics work in both of these directions. For example, with CRRA preferences $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$, we have $RP_t = 1+\gamma$ and $EIS_t = \gamma^{-1}$. Higher γ facilitates our uncertainty trap equilibria on two fronts: it increases RP_t, thus allowing risk to bring stability, and it reduces EIS_t, thus reducing the power of monetary policy to steer consumption through real rates. Substituting the expressions $RP_t = 1+\gamma$ and $EIS_t = \gamma^{-1}$ into (23), we can obtain the following result.

Proposition 2. *Assume the representative agent has CRRA utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ with $\gamma > 1$. Suppose prices are rigid ($\kappa \to 0$). Consider monetary policy rules with target rate $\bar{\iota} = \rho$. If the monetary rule $\Phi(x)$ and volatility function $\sigma^2_x(x)$ comprise a non-explosive equilibrium with log utility, then the more-aggressive monetary rule $\tilde{\Phi}(x) := \gamma\Phi(x)$ and smaller volatility $\tilde{\sigma}^2_x(x) := \gamma^{-1}\sigma^2_x(x)$ correspond to a non-explosive equilibrium with CRRA utility.*

2.4.3 Idiosyncratic risk

Idiosyncratic risk, which is quantitatively an order of magnitude larger than aggregate volatility, is a final reason why aggregate volatility need not be so high in uncertainty traps. Indeed, idiosyncratic risk itself induces precautionary savings, meaning that, all else equal, less aggregate risk is needed to sustain an uncertainty trap. Here, we pursue the more nuanced point that *countercyclicality* of idiosyncratic risk—extensively documented in the literature (Storesletten et al., 2004; Guvenen et al., 2014; Bloom, 2014)—reduces the burden of aggregate volatility for our equilibria. We demonstrate this point with a particularly simple and tractable modeling device that may serve as a useful technique for the literature.[13]

The details are as follows. We introduce an uninsurable idiosyncratic Brownian motion $\tilde{Z}_t$ that redistributes consumption between two types of consumers—types A and B. Specifically, the flow $Y_t \sqrt{\omega_t(1-\omega_t)}\nu(x_t)d\tilde{Z}_t$ is transferred from type-B to type-A agents, where ω_t is the equilibrium consumption share of the type-A agents. This exact functional form for redistribution is chosen in a way that permits a representative-agent characterization. The key function $\nu(x) > 0$, which depends on the output gap, governs the overall volatility of the uninsurable shocks. Note that agents cannot choose their consumption exposure to $d\tilde{Z}_t$—rather, they are endowed with this exposure. Instead, agents optimally choose their savings as well as their exposure to the sunspot shock dZ_t. Finally, we assume the monetary target rate is set to $\bar{\iota} = \rho - \nu(0)^2$, which appropriately adjusts for the presence of idiosyncratic risk at steady state.

As shown in Appendix E, equilibrium output-gap dynamics satisfy

$$dx_t = \Big(\Phi(x_t, \pi_t) - \pi_t + \nu(x_t)^2 - \nu(0)^2 + \frac{1}{2}\sigma_{x,t}^2\Big)dt + \sigma_{x,t}dZ_t. \tag{24}$$

Idiosyncratic shocks influence aggregate dynamics only through precautionary behavior, and countercyclical idiosyncratic volatility ($\nu'(x) < 0$) raises precautionary savings during recessions, boosting aggregate demand growth. Since the Phillips curve is unaffected by our extension, there are no other modifications to the equilibrium. This proves the following result, which says that idiosyncratic volatility substitutes exactly one-for-one with aggregate volatility.

Proposition 3. *Let $\nu'(x) < 0$. Consider equilibria in which inflation and volatilities take the form $\pi_t = \pi(x_t)$ and $\sigma_{x,t} = \sigma_x(x_t)$. If $\sigma_x^2(x)$ is volatility in a non-explosive equilibrium of the baseline model, then $\tilde{\sigma}_x^2(x) := \sigma_x^2(x) - 2[\nu(x)^2 - \nu(0)^2]$ is a non-explosive equilibrium volatility in the model with idiosyncratic risk. Thus, $\tilde{\sigma}_x^2(x) - \sigma_x^2(x) < 0$ for all $x < 0$, with the gap becoming more negative as x falls.*

[13] It is notoriously difficult to solve models with both idiosyncratic and aggregate uncertainty. We extend the model in a particularly simple, albeit reduced-form, way to tractably make our main points. Given the generic insight that idiosyncratic risk creates precautionary savings, we do not believe our insights would change with a more traditional modeling of idiosyncratic risk.

3 The Limits of Monetary Policy

Uncertainty traps are robust to all conventional monetary policies. Can *unconventional* rules help? We first demonstrate that a different type of policy rule, which targets the risk premium, can eliminate uncertainty traps. Unfortunately, we then show that risk premium targeting is limited if any effective lower bound exists for interest rates. Our argument applies more generally to any interest rate rule, and so effective lower bounds dramatically limit the efficacy of monetary policy in this regard.

3.1 Risk premium targeting

There is one type of monetary rule that can restore determinacy. Following the suggestion in Lee and Dordal i Carreras (2024), suppose we replace the plain-vanilla Taylor rule (MP) with a rule that explicitly targets the risk premium. However, we will provide a much broader proposition regarding the efficacy of such rules. We use

$$\iota_t = \rho + \Phi(x_t, \pi_t) - (\alpha_- \mathbf{1}_{\{x_t<0\}} + \alpha_+ \mathbf{1}_{\{x_t>0\}}) \times \left(\frac{1}{2}\sigma_{x,t}^2\right). \tag{MP-vol}$$

Although conventional wisdom would suggest that targeting an asset price—which maps one-to-one into the output gap—suffices to target the risk premium, that is not true here. Intuitively, because uncertainty $\sigma_{x,t}$ is the mechanism that self-fulfils non-fundamental demand recessions, monetary policy must target this quantity, not only the level of demand x_t. Rule (MP-vol) directly targets this uncertainty.

Risk premium targeting restores determinacy. Substitute (MP-vol) into (IS) to get

$$dx_t = \left[\Phi(x_t, \pi_t) - \pi_t + \frac{1}{2}(1 - \alpha(x_t))\sigma_{x,t}^2\right]dt + \sigma_{x,t} dZ_t, \tag{25}$$

where $\alpha(x) := \alpha_- \mathbf{1}_{\{x<0\}} + \alpha_+ \mathbf{1}_{\{x>0\}}$ is the state-dependent responsiveness to the risk premium. If $\alpha_- = \alpha_+ = 1$, then the risk premium vanishes from the drift, and we are back in the standard linearized situation where an aggressive reaction function Φ can restore determinacy. If $\alpha_+ < 1 < \alpha_-$, then the risk premium itself becomes destabilizing: higher levels of $\sigma_{x,t}^2$ induce a strong monetary policy reaction that actually *reduces* savings on net. Thus, our core mechanism whereby volatility leads to precautionary savings is broken by this aggressive policy. For this reason, a modified Taylor rule like (MP-vol), with more aggressive risk premium targeting in bad times, can eliminate uncertainty traps. Again, for analytical purposes, we state this result in the rigid-price limit.

Proposition 4. *Suppose prices are rigid ($\kappa \to 0$). With sufficiently strong recessionary risk premium targeting ($\alpha_+ \leq 1 \leq \alpha_-$) and sufficient responsiveness to the output gap, the modified Taylor rules (MP-vol) ensure that the unique non-explosive equilibrium is $x_t = 0$.*

3.2 Effective lower bounds

While risk-premium targeting can work to ensure determinacy, it relies strongly on an unconstrained monetary policy. To understand this, recall that all the rules advocated above share the property that $\iota_t \to -\infty$ as $x_t \to -\infty$. If interest rates are lower bounded, this cannot work. In this section, we explore here a situation where monetary policy is constrained: suppose ι_t must respect the lower bound $\iota_t \geq \underline{\iota}$. This lower bound does not need to be zero—it can be any level (thus generalizing our earlier results on the ZLB).

With any effective lower bound, uncertainty traps cannot be eliminated. To see this, consider the rigid-price equilibria and inspect output gap dynamics when ι_t is at its lower bound:

$$dx_t = \left[\underline{\iota} - \rho + \frac{1}{2}\sigma_{x,t}^2\right]dt + \sigma_{x,t}dZ_t, \quad \text{when} \quad x_t < 0 \quad \text{and} \quad \iota_t = \underline{\iota}. \tag{26}$$

A sufficiently high level of uncertainty can raise the drift and create stable dynamics by "pushing" x_t back toward steady state. Thus, high enough uncertainty once again permits non-fundamental demand recessions, since they will not lead to explosions. Using this logic, it is easy to prove the following.

Proposition 5. *Suppose prices are rigid ($\kappa \to 0$). Let ι_t be any interest rate process that obeys $\iota_t \geq \underline{\iota}$ and is "at target" in steady state (i.e., $\iota_t = \bar{\iota} = \rho$ when $x_t = \sigma_{x,t} = 0$). Then, any $x_0 \leq 0$ corresponds to at least one valid non-explosive equilibrium with volatility.*

With *any* lower bound, uncertainty traps exist. This is not simply a reflection of monetary policy being generically constrained; upper bounds on interest rates would not stymie monetary policy at all. The asymmetry arises because uncertainty traps are *recessionary and self-sustained by high risk premia*. In a hypothetical demand recession, the central bank would want to set low interest rates, but the lower bound $\iota_t \geq \underline{\iota}$ prevents such a force from being too strong. Risk premia can be so high as to overwhelm such efforts by monetary policy and permit an uncertainty trap.

4 Fiscal Policy

We extend the model to include fiscal policy, formulated via lump-sum taxation and government transfers to the representative household. The real primary surplus of the government is the difference between these taxes and transfers, which we denote by S_t. Fiscal policy is characterized by a specification for S_t, to be described shortly.

Surpluses may be non-zero, so the government borrows by issuing short-term nominally riskless bonds B_t. Later we will generalize to long-term debt. The flow budget constraint of the government is

$$\dot{B}_t = \iota_t B_t - P_t S_t. \tag{27}$$

The nominal interest rate ι_t is controlled by monetary policy.

With non-zero debt, equilibria must obey the household transversality condition for government debt holdings,

$$0 = \lim_{T \to \infty} \mathbb{E}_0 \Big[M_T \frac{B_T}{P_T} \Big], \tag{TVC}$$

where recall M denotes the real stochastic discount factor process. In this section, the transversality condition will be used extensively as a condition to trim equilibria. As is well known, the transversality condition implies the present-value formula for government debt:

$$\frac{B_t}{P_t} = \mathbb{E}_t \Big[\int_t^\infty \frac{M_u}{M_t} S_u du \Big]. \tag{GD}$$

The real value of debt must equal the real present value of surpluses. In some arguments, it will be easier to use (GD) than (TVC).

Surplus rule. For our main results in the text, we assume that surpluses follow a rule which features an exogenous component and a feedback from outstanding government debt back into surpluses. We specify the surplus-to-GDP ratio $s_t := S_t / Y_t$ as a function of an exogenous component $\hat{s}_t$ and a feedback from the real debt-to-GDP ratio $b_t := B_t / P_t Y_t$. The exogenous component $\hat{s}_t = \hat{s}(\Omega_t)$ is a bounded continuous function of Ω_t, which is an arbitrary vector of exogenous stationary state variables driven by a multivariate Brownian motion $\mathcal{Z}$ that is independent of the shock Z:

$$d\Omega_t = \mu_\Omega(\Omega_t) dt + \varsigma_\Omega(\Omega_t) \cdot d\mathcal{Z}_t.$$

Thus, all of our baseline fiscal rules take the form

$$s_t = \hat{s}(\Omega_t) + \alpha_t b_t, \tag{28}$$

where α_t is a potentially time- and state-dependent policy parameter, to be specified in examples below. In the language of Leeper (1991), fiscal policies with $\alpha_t > 0$ are typically regarded as "passive" and do not constrain equilibrium dynamics, whereas policies with $\alpha_t \leq 0$ are considered "active" and can prune equilibria. We also use this language. Given our α_t is time-varying, however, we will generalize the logic of how equilibrium selection depends on the sign of α_t.

Equilibrium definition. The dynamics of x and π now take the form

$$dx_t = \Big[\iota_t - \pi_t - \rho + \frac{1}{2}\sigma_{x,t}^2 + \frac{1}{2}|\varsigma_{x,t}|^2\Big]dt + \sigma_{x,t} dZ_t + \varsigma_{x,t} d\mathcal{Z}_t \tag{29}$$

$$d\pi_t = \Big[\rho\pi_t - \kappa x_t\Big]dt + \sigma_{\pi,t} dZ_t + \varsigma_{\pi,t} d\mathcal{Z}_t. \tag{30}$$

Other than the generalization that these dynamics now include potential sensitivities to the surplus shocks $d\mathcal{Z}$, these dynamics are identical to the baseline IS curve (IS) and Phillips curve (PC). The reason for this is that fiscal policy is neutral. Taxes and transfers are levied lump-sum, and a representative agent exists. While the representative household holds the government bonds B_t, it also owes the government future taxes and is owed future transfers.[14] Therefore, there is no direct effect of fiscal policy. The only role of fiscal policy here is to select among equilibria.

Together with a nominal interest rate rule for ι_t and a surplus rule for s_t, equilibrium is fully characterized by the dynamical system (29)-(30), combined with government transversality (TVC), or equivalently the valuation equation (GD). We continue to require the non-explosion Condition 1. Proofs for this section are contained in Appendix A.5.

4.1 Baseline: the simplest example of passive fiscal policy

Sections 1-2 did not have government debt or taxes/transfers. However, everything we have said until now still holds with fiscal policies, so long as those policies are "passive."

[14] The lifetime budget constraint of the representative household is

$$\mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t}\frac{W_u L_u}{P_u} du\Big] + \frac{B_t}{P_t} = \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} S_u du\Big] + \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} C_u du\Big].$$

By (GD), the lifetime budget constraint is equivalent to the budget constraint without any debt at all. And so the household optimality conditions are unchanged. This leads to (29)-(30).

What is a passive fiscal policy? As a leading example, consider a surplus rule with $\alpha_t = \bar{\alpha} > 0$. This policy raises surpluses when debt-to-GDP rises, which helps pay back the debt. Unsurprisingly, such a fiscal policy is sustainable in any equilibrium. To verify this, use the fact that the real SDF is $M_t = e^{-\rho t} Y_t^{-1}$ in order to write the scaled finite-horizon present-value equation as[15]

$$\mathbb{E}_0[e^{-\rho T} b_T] = e^{-\bar{\alpha} T}\Big(b_0 - \mathbb{E}_0\Big[\int_0^T e^{-(\rho-\bar{\alpha})t} \hat{s}_t dt\Big]\Big).$$

The terms on the right-hand-side vanish as $T \to \infty$, which implies that the transversality condition $\lim_{T\to\infty} \mathbb{E}_0[e^{-\rho T} b_T] = 0$ automatically holds, hence the government debt valuation equation automatically holds regardless of the equilibrium dynamics.[16] Passive fiscal policies do not trim equilibria, and all of our sentiment equilibria in Section 2 could emerge under such fiscal policies. Section 4.3 shows how a broader set of policies, beyond $\alpha_t = \bar{\alpha}$, also qualify as passive in the sense that no equilibria are trimmed.

Is passive fiscal policy realistic? On the surface, it sounds reasonable to assume that governments will act responsibly and raise surpluses when their debts rise. But at the same time, such policies permit our sentiment equilibria. In those equilibria, the dynamics of debt-to-GDP can be driven to extremes by the denominator, GDP.

Proposition 6. *Suppose the monetary rule (MP) is "active" in the sense that $\partial_x \Phi > 0$ and $\partial_\pi \Phi > 1$ for all (x, π). Consider a surplus rule (28) with α_t bounded and with $\hat{s}_t$ not identically zero. Then, b_t is unbounded in any non-explosive sunspot equilibrium.*

While boundedness of debt-to-GDP is not a requirement for equilibrium per se, if one imagines taxes and spending are capped by GDP, then the absolute value of the surplus-to-GDP ratio cannot exceed one. But with the surplus rule $s_t = \hat{s}_t + \bar{\alpha} b_t$, an unbounded debt-to-GDP would imply an unbounded surplus-to-GDP, an uncomfortable reality.

We thus interpret Proposition 6 as a result indicating a permanent active-money passive-fiscal policy may be unreasonable or unsustainable. Very natural constraints on taxes and spending imply we should explore alternatives until we find policies that eliminate our uncertainty trap equilibria and therefore keep debts and surpluses bounded.[17]

[15] To derive this equation, apply Itô's formula to $e^{\bar{\alpha} t} b_t := e^{(\bar{\alpha}-\rho)t} B_t / P_t Y_t$, using the flow government budget constraint (27), the price level dynamics $dP_t/P_t = \pi_t dt$, and the dynamics of x_t in (29).

[16] To see that $\lim_{T\to\infty} e^{-\bar{\alpha} T}\big(b_0 - \mathbb{E}_0\big[\int_0^T e^{-(\rho-\bar{\alpha})t} \hat{s}_t dt\big]\big) = 0$, use the boundedness of surpluses $|\hat{s}_t| \leq \hat{s}_{max}$ to see that

$$\Big|\mathbb{E}_0\Big[\int_0^T e^{\bar{\alpha}(t-T)} e^{-\rho t} \hat{s}_t dt\Big]\Big| \leq \hat{s}_{max} \int_0^T e^{\bar{\alpha}(t-T)} e^{-\rho t} dt = \frac{\hat{s}_{max}}{\bar{\alpha} - \rho}\big(e^{-\rho T} - e^{-\bar{\alpha} T}\big) \to 0.$$

[17] We do not take this result as a reason to just discard uncertainty trap equilibria, because agents don't

4.2 The simplest example of active fiscal policy

The simplest example of active fiscal policy sets $\alpha_t = 0$ forever. The surplus-to-GDP ratio is then exogenous. This is referred to as "active" because real surplus levels are chosen in a way that does not automatically ensure the government debt valuation equation holds (e.g., $S_t = \hat{s}_t Y_t$ is independent of the price level).

With this policy, and using (GD), we have

$$\frac{B_t}{P_t} = Y_t \times \mathbb{E}_t \Big[\int_t^\infty e^{-\rho(u-t)} \hat{s}(\Omega_u) du \Big]. \tag{31}$$

Hidden in this valuation equation is the equilibrium selection result. To see it, start by assuming the surplus-to-GDP ratio is constant, $\hat{s}(\cdot) \equiv \bar{s}$, so that the right-hand-side of this equation is $Y^* \frac{\bar{s}}{\rho} e^{x_t}$. In that case, if you apply Itô's formula to both sides of (31), using the fact that $dP_t/P_t = \pi_t dt$, you get

$$\Big[\frac{B_t}{P_t}\iota_t - \bar{s}Y_t - \frac{B_t}{P_t}\pi_t\Big]dt = \frac{\bar{s}}{\rho}e^{x_t}Y^*[\iota_t - \pi_t - \rho + \sigma_{x,t}^2]dt + \frac{\bar{s}}{\rho}e^{x_t}Y^*\sigma_{x,t}dZ_t.$$

Matching the "dZ" terms on both sides, we find $\sigma_{x,t} = 0$. Hence, active fiscal policy disciplines the response of aggregate demand to shocks, closing the door to the volatility patterns required to sustain uncertainty traps. Then, matching the "dt" terms on both sides, we find an identity: given $\sigma_{x,t} = 0$, and using equation (31), the "dt" terms match for any ι_t, π_t, and x_t. In other words, fiscal policy selects $\sigma_{x,t} = 0$, and that is all it does after the initial date $t = 0$. This argument holds for any finite speed of price adjustments κ and amount of inflation volatility $\sigma_{\pi,t}$ (if any).

This same logic holds even if the surplus-to-GDP ratio is not constant but *almost any* exogenous process. The key: the expectation on the right-hand-side of (31) is exogenous and therefore only depends on Ω_t, which itself is independent of sunspot shocks.

Theorem 2. *The economy with $\alpha_t = 0$ necessarily has $\sigma_{x,t} = 0$ and pins down $\varsigma_{x,t}$ uniquely. Conversely, if $\sigma_{x,t} = 0$ and $\varsigma_{x,t}$ takes a particular value, then the government debt valuation equation (GD) automatically holds at every date, given it holds at $t = 0$.*

Remark 1. *Under the policy $\alpha_t = 0$, what does the equilibrium look like? We first note that fiscal policy determines demand and exposes it to fiscal shocks via ς_x—this is known as "fiscal dominance." Fiscal policy eliminates self-fulfilling volatility but introduces fiscal volatility. More*

care directly about whether or not debt-to-GDP explodes. Instead, encoded laws and rules that prevent debt-to-GDP explosions imply that policy *must*, at some point, deviate from the passive fiscal benchmark.

specifically, once debt-to-GDP b_t is pinned down by the expectation on the right-hand-side of equation (31), then demand is pinned down by $x_t = \log(\frac{B_t/P_t}{b_t Y^*})$.

Second, with an appropriately coordinated monetary policy (i.e., passive money needs to be paired with active fiscal), it turns out the entire equilibrium is unique even beyond $\sigma_{x,t}$ and $\varsigma_{x,t}$. Proposition I.1 in Appendix I provides this familiar result, whereby inflation π_t is a function of x_t that depends on the parameters of the monetary policy rule.

Active fiscal policy eliminates self-fulfilling demand volatility σ_x but generically makes fiscal shocks matter via ς_x. In that sense, there is a trade-off introduced by active fiscal policy: it kills one type of volatility but introduces another that would otherwise be absent. Before proceeding to additional results, we discuss the intuition for Theorem 2 and explain why it is robust to some natural extensions of the model.

Intuition: nominal rigidities. Theorem 2 says that fiscal policy (i) pins down real demand shocks and (ii) does nothing else besides pin down real demand shocks.

The mathematical reasoning for why policy selects these equilibria is quite simple in this case: the aggregate real debt balance B_t/P_t evolves "locally deterministically" (meaning it only has drift and no diffusion over small time intervals dt), and so its present value $\mathbb{E}_t[\int_t^\infty \frac{M_u}{M_t} S_u du]$ must also not have any diffusion. This no-diffusion result immediately implies that Y_t must have no sunspot volatility.

At first glance, the fact that B_t/P_t evolves locally deterministically seems critical but potentially fragile. The fact that B_t evolves locally deterministically is due to short-term debt, which we generalize in a long-term debt extension discussed below. In principle, the price level could feature a diffusive component, i.e., $dP_t/P_t = \pi_t dt + \sigma_{P,t} dZ_t$ for some $\sigma_{P,t}$ to be determined. There are two problems with this line of reasoning. First, in typical continuous-time models of price stickiness, such a diffusion does not arise. For example, in our world with Rotemberg price stickiness, we have proved that $\sigma_{P,t} = 0$ (see Appendix G). If firms had such fast-moving prices, they would incur too many price adjustment costs, and this is not optimal. Similarly, in a world with Calvo price stickiness, where price-setting opportunities arrive idiosyncratically at some rate ω, a fraction ωdt of firms may change their price over a short time interval dt. This also implies $\sigma_{P,t} = 0$. In other words, the fact that P_t evolves locally deterministically is a standard outcome of sticky price models. Beyond being an implication of the modeling, $\sigma_{P,t} = 0$ is also deeply reasonable: nominal rigidities should mean that there is some high-enough frequency at which prices don't adjust; in continuous time, that high frequency is the Brownian one. The second problem with this reasoning: even if $\sigma_{P,t} \neq 0$ somehow, it could not be some arbitrary equilibrium object that allowed the government debt valuation equation to hold; it would need to be consistent with firms' pricing strategies.

Fiscal policy also does *nothing else besides pin down real demand shocks*. For this reason, we advance an interpretation of fiscal policy as providing "aggregate demand management" in sticky price models. One way to develop an intuition is to consider the rigid price limit $\kappa \to 0$. Nothing about the analysis above hinges on the value of κ, and so fiscal policy still selects $\sigma_x = 0$. Why? If aggregate demand were suddenly lower, the real present value of surpluses would fall. Since the price level cannot adjust, the real value of government debt would be too high, and households would view it as net wealth. Wanting to spend this extra wealth, they increase consumption enough so that aggregate demand does not fall in the first place. A version of this aggregate demand logic extends to any finite $\kappa > 0$, because the price level cannot adjust on impact to shocks.

So far, we have emphasized how $\sigma_x = 0$ relies critically on the (natural) property that sticky prices cannot absorb sunspot shocks at high frequency. One wonders how equilibrium selection changes if there were *some variable* in the debt valuation equation that *could absorb* sunspot volatility in demand. That is the point of several extensions we pursue in Appendix I, which we briefly outline next.

Extensions: debt prices, surplus rules, and discount rate variation. In the baseline case, the unit price of debt is fixed at 1 (given it is short-term debt), surplus-to-GDP is exogenous, and the equilibrium SDF is exactly reciprocal to GDP (due to log utility). Because of these assumptions, there is no channel that can potentially absorb self-fulfilling demand shocks. The generalizations in Appendix I relax these assumptions.

In a general model with long-term debt, a potentially endogenous surplus-to-output ratio, and CRRA utility with risk aversion γ, the present-value formula for aggregate government debt is

$$\frac{Q_t B_t}{P_t} = Y_t \mathbb{E}_t \Big[\int_t^\infty e^{-\rho(u-t)} s_u \Big(\frac{Y_u}{Y_t}\Big)^{1-\gamma} du \Big]. \tag{32}$$

Suppose Y_t has some volatility via $\sigma_{x,t} \neq 0$. Equation (32) illustrates three possible channels that can absorb this volatility, and thus permit it to exist. First, the long-term debt price Q_t can adjust to shocks; in the baseline model, $Q_t = 1$, and so this was not possible. Second, future surplus-to-GDP $(s_u)_{u \geq t}$ can be endogenous, through a rule that responds to output and inflation, which allows the present value of surplus-to-GDP ratios to adjust to shocks. Third, the term $e^{-\rho(u-t)}\big(\frac{Y_u}{Y_t}\big)^{1-\gamma} = \frac{M_u}{M_t}\frac{Y_u}{Y_t}$ represents the net variation of discount rates (i.e., marginal utility growth M_u/M_t) and economic growth (i.e., output growth Y_u/Y_t); in the baseline model, $\gamma = 1$, and this net variation was zero.

Overall, these three extensions are ways in which terms besides Y_t can have diffusive variation. Nevertheless, the key conclusion of Theorem 2 continues to hold, suggesting the logic for $\sigma_x = 0$ runs deeper than timing assumptions or mathematical artifacts.

What is the general intuition for equilibrium selection even in these more complex environments? Although bond prices and the present-value of surpluses can absorb demand shocks *in principle*, these objects are forward-looking. For instance, the per-unit bond price Q must satisfy an asset-pricing equation that constrains it; the bond price is not free to take the form required to absorb any and all demand shocks. The present-value of surpluses is also essentially a long-dated asset and also satisfies an asset-pricing equation, so a similar logic applies to the models with surplus rules and CRRA utility. In all cases, the extensions add a degree of freedom that might absorb demand shocks, but they also add a constraint, namely an asset-pricing equation, that forbids such absorption.

4.3 General classification results: what is passive and what is active?

The two example policies discussed above are over-simplified: the baseline passive fiscal policy features a constant reaction to debt ($\alpha_t = \bar{\alpha}$), while the baseline active fiscal policy never features any reaction to debt ($\alpha_t = 0$). This section is a bit of a detour, providing some tools that allow us in principle to go beyond these simple extremes, while preserving the main insights. Here, we provide two novel general tools to diagnose whether a more complex policy profile is ultimately passive or active. Our first such tool characterizes passive fiscal policies, in the sense that the government remains solvent irrespective of the economic dynamics.

Lemma 2 (Passive fiscal)**.** *Suppose $\alpha_t \geq 0$ and*

$$\mathbb{P}\Big\{ \int_t^\infty \alpha_u du = +\infty, \quad \forall t \geq 0 \Big\} = 1. \tag{33}$$

Then, transversality condition (TVC) holds, irrespective of the path of (x_t, π_t).

Lemma 2 is used to "rule in" equilibria. If $\alpha_t \geq 0$ and condition (33) holds within a conjectured equilibrium, then the fiscal policy profile does nothing to rule out the conjecture, precisely because the debt valuation equation is redundant to the other equations. Clearly, Lemma 2 covers the initial example with $\alpha_t = \bar{\alpha} > 0$. But it can also cover more complex policies with time-varying α_t. Some examples follow at the end of this section.

Whereas Lemma 2 is a tool to "rule in" equilibria, we also provide a converse result that will be sufficient to "rule out" equilibria. This is a characterization of what active fiscal policy looks like, in the sense that an additional substantive condition arises—a debt valuation-like equation—that ultimately constrains the economic dynamics.

Lemma 3 (Active fiscal)**.** *Suppose* $\sup_t \alpha_t < \rho$ *and*

$$\mathbb{P}\Big\{ \int_t^{\infty} \alpha_u du < +\infty, \quad \forall t \geq 0 \Big\} = 1. \tag{34}$$

Then, transversality condition (TVC) implies debt-to-GDP is given by

$$b_t = \mathbb{E}_t \Big[\int_t^{\infty} e^{-\rho(u-t)} e^{\int_t^u \alpha_z dz} \hat{s}_u du \Big]. \tag{35}$$

The critical valuation-like equation (35) looks much like the generic debt valuation equation (GD), but it is not exactly the same for two reasons. First, it is not always true that the generic valuation equation (GD), which holds in every equilibrium, coincides with the valuation-like equation (35). These two conditions are only equivalent under the assumptions of Lemma 3. Second, the valuation-like equation has purged the surpluses of their endogenous response to debt, making it easier to analyze.

We now consider several examples to illustrate the power of Lemmas 2-3. The first example is very simple and involves time-dependent regime-switching.

Example 5. *Let* $\bar{\alpha} \in (0, \rho)$*. Consider the two fiscal policies* α_t^A *and* α_t^B*, defined by*

Dates	α_t^A	α_t^B
$t < T$	0	$\bar{\alpha}$
$t \geq T$	$\bar{\alpha}$	0

Policy A is the type of policy discussed in Angeletos et al. (2024). It is clear that, regardless of the value of T, $\int_0^{\infty} \alpha_t^A dt = +\infty$*, and so Lemma 2 applies. Therefore, this is passive fiscal, even though government here refuses to accommodate its debt increases for an arbitrarily long time. Angeletos et al. (2024) use this to argue that FTPL equilibrium selection is not robust to very far in the future revisions in the policy regime.*

But one can easily make the exact opposite argument about policy B. Regardless of how large T is, $\int_0^{\infty} \alpha_t^B dt < +\infty$*, and so Lemma 3 applies. This policy is active. In that sense, the FTPL is indeed robust to arbitrarily long deviations from active fiscal policy.*

The problem here is the strangeness of the limit $T \to \infty$ *for when a permanent regime-switch occurs. No matter how large T is, there is infinite time afterward. That is why the two examples A and B are not helpful to determine whether or not FTPL's conclusions are fragile or not.*

As the next example illustrates, our lemmas can easily address even more complex policy profiles that involve state-dependent regime-switching. This example fiscal policy will be unsuccessful in providing equilibrium selection, despite appearing to have a substantial degree of "fiscal activism."

Example 6 (Recessionary switching)**.** *Recall that our self-fulfilling equilibria are always recessionary and depend on beliefs about what happens in extreme states when x is very low. Motivated by this, it is natural to consider a fiscal rule that incorporates a state-dependent switch in extreme recessions. Consider, for some threshold $\chi < 0$,*

$$\alpha_t = \begin{cases} \bar{\alpha} > 0, & \text{if } x_t \geq \chi; \\ 0, & \text{if } x_t < \chi. \end{cases} \tag{36}$$

This government does "active fiscal" whenever x_t falls low enough (i.e., in extreme recessions). This somewhat resembles real-world policies: in normal times, governments responsibly pay back debts by raising taxes and/or reducing spending; but in emergencies, governments abandon their fiscal responsibilities in favor of "stimulus" to lift the economy out of crisis.

Can uncertainty traps survive this policy? Consider a sunspot equilibrium with a stationary distribution for $x_t \in (-\infty, x_{max}]$. Assume without loss of generality that policy (36) has picked the threshold χ such that that fiscal policy is not always active, i.e., they pick $\chi < x_{max}$. In that case, we necessarily have $\mathbb{P}\{x_t \geq \chi\} > 0$. Using the ergodic theorem, we then have that

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \alpha_t dt \xrightarrow{a.s.} \mathbb{E}[\alpha_t] = \bar{\alpha}\mathbb{P}\{x_t \geq \chi\} > 0.$$

This implies $\int_0^\infty \alpha_t dt = +\infty$, so government transversality holds by Lemma 2. Consequently, the conjectured sunspot equilibrium survives such a fiscal policy.

The final example strengthens Example 6 in two ways: (i) fiscal policy is more "aggressive" in deep recessions; and (ii) the intervention threshold is "adaptive" to ensure regimes switch often enough. Because this example are rather involved, we simply sketch it and state the results, with more details in Appendix H.

Example 7 (Fiscal backstops)**.** *We make two changes to the previous example. For some $\underline{\alpha} < 0 < \bar{\alpha}$ and some intervention threshold χ_q, fiscal policy is described by*

$$\alpha_t = \begin{cases} \bar{\alpha}, & \text{if } x_t \geq \chi_q; \\ \underline{\alpha}, & \text{if } x_t < \chi_q. \end{cases} \tag{37}$$

The first change is that fiscal policy is sufficiently aggressive in the active regime: with $\underline{\alpha} < 0$, surpluses decline as debt-to-GDP rises. In some sense, this is beyond irresponsible because it allows debt to spiral out of control. But it reflects some aspects of real-world policies: debt-to-GDP increases are often due to negative GDP shocks to which the government responds by spending even more, as in stimulus packages, which further raises the debt-to-GDP ratio.

The second change is the design of the intervention threshold χ_q. Our design ensures the aggressively-active regime occurs with positive probability over the long run. Let $p(x)$ denote the (marginal) stationary density of x_t in equilibrium, and let

$$\chi_q := \inf\left\{\chi : \int_{-\infty}^{\chi} p(x)dx \geq q\right\} \tag{38}$$

denote the q^{th} percentile of p. In other words, policy conditions its switching point on the stationary distribution in equilibrium, such that the probability of the active regime is always at least q. Appendix H provides more discussion about why this design of χ_q is reasonable.

Under (37)-(38), fiscal policy imposes a non-redundant debt valuation equation on every equilibrium. To see this, recall that an adaptive threshold χ_q means that $\mathbb{P}\{x_t < \chi_q\} \geq q$, regardless of the equilibrium being played. Consequently, we have by the ergodic theorem that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \alpha_t dt \xrightarrow{a.s.} \mathbb{E}[\alpha_t] = \bar{\alpha}\mathbb{P}\{x_t \geq \chi_q\} + \underline{\alpha}\mathbb{P}\{x_t < \chi_q\} \leq (1-q)\bar{\alpha} + q\underline{\alpha}.$$

So long as $\underline{\alpha} < -\bar{\alpha}\frac{1-q}{q}$, we have $\mathbb{E}[\alpha_t] < 0$ and hence condition (34) of Lemma 3 is satisfied. If policy is either active sufficiently often (i.e., q is high enough) or sufficiently aggressive when active (i.e., $\underline{\alpha}$ is negative enough), we thus obtain a non-redundant debt valuation equation. This is enough to eliminate uncertainty traps, as proved in Theorem H.1 in Appendix H.

4.4 Fiscal volatility targeting

Our benchmark fiscal policies, while transparent, may be unsatisfactory. The always-passive fiscal policy with $\alpha_t = \bar{\alpha} > 0$ has the issue that it permits some self-fulfilling equilibria in which surplus-to-GDP ratios violate any bound (Proposition 6). The always-active fiscal policy with $\alpha_t = 0$ is also a highly committed policy. While it selects a unique equilibrium (Theorem 2), a large literature debates whether the resulting fiscal equilibrium is reasonable empirically. Does eliminating uncertainty traps necessitate this fiscal equilibrium?

Our final exercise illustrates, using a novel fiscal policy that we refer to as *fiscal volatility targeting*, that the answer is no. The policy specification builds on Cochrane (2023) in distinguishing between the debt-to-GDP path b_t that emerges under an arbitrary sequence of shocks versus a latent version of debt-to-GDP b_t^* that ignores contributions from certain subsets of shocks. In particular, we design b_t^*, which works like a debt-to-GDP target, by ignoring contributions from self-fulfilling demand and volatility shocks. Surplus policy depends on b_t^* rather than b_t, which intuitively disallows fiscal accommodation to arbitrary self-fulfilling shocks.

The details are as follows. Actual debt-to-GDP $b_t = \frac{B_t}{P_t Y_t}$ evolves as

$$db_t = (\rho b_t - s_t)dt - b_t\Big(\sigma_{x,t} dZ_t + \varsigma_{x,t} d\mathcal{Z}_t\Big) \tag{39}$$

where we have used the flow government budget constraint (27), the price level dynamics $dP_t/P_t = \pi_t dt$, and the dynamics of x_t in (29). By contrast, consider a latent variable that tracks debt-to-GDP without the non-fundamental demand shocks (i.e., with neither sunspot shocks nor fiscal shocks):

$$db_t^* = (\rho b_t^* - s_t)dt. \tag{40}$$

We assume that surpluses-to-GDP are given by the rule

$$s_t = \hat{s}_t + \bar{\alpha} b_t^*, \quad \bar{\alpha} > 0. \tag{41}$$

(Of course, using Lemma 2 in the previous section, we can easily generalize this from a constant $\bar{\alpha}$ to a time-varying specification.) Thus, fiscal policy has traits of being passive, in the sense that it raises taxes or reduces spending in response to rising debt paths—but not if debt-to-GDP is driven by arbitrary non-fundamental changes to the denominator, GDP. Faced with non-fundamental shocks, fiscal policy commits to maintain its active tax/spending plan and refuses to accommodate. But since it turns out that $b_t = b_t^*$ in equilibrium, fiscal policy does, in fact, look passive on the equilibrium path.

Theorem 3. *Under fiscal policy (41), all equilibria have $\sigma_{x,t} = 0$, $\varsigma_{x,t} = 0$, and $b_t = b_t^*$. Conversely, if $b_t = b_t^*$, then the transversality condition (TVC) automatically holds.*

The new fiscal policy rules out self-fulfilling dynamics. This includes sunspot dynamics ($\sigma_x = 0$) as well as fiscal-related dynamics ($\varsigma_x = 0$). The latter are also, in some sense, a self-fulfilling fluctuation, because fiscal shocks are not part of the "minimum state variables" that matter for consumers. Why are all these shocks eliminated? The basic idea is that fiscal policy, by responding only to b_t^* and committing to not accommodate shocks to $b_t - b_t^*$, is essentially *targeting* zero self-fulfilling fluctuations.

The next question is what the resulting equilibrium looks like. The usual result when fiscal policy selects equilibria is that demand becomes anchored by fiscal considerations via the debt valuation equation. One symptom of this anchoring is that demand becomes susceptible to fiscal shocks.

But here, the fiscal anchor is not present. Formally, Theorem 3 shows that $b = b^*$ implies the transversality condition automatically holds. Consequently, once fiscal policy establishes the equivalence $b = b^*$, the debt valuation equation becomes redundant and provides no anchor. Fiscal policy's entire impact here is to eliminate non-fundamental fluctuations.

To fully determine the equilibrium, we can add an assumption on monetary policy. If monetary policy is active, following the Taylor principle, then the MSV solution is the unique equilibrium. Alternatively, if monetary policy is passive, all non-fundamental fluctuations are eliminated (even for inflation), but the initial condition remains indeterminate.

Proposition 7. *Suppose fiscal policy follows (41), while monetary policy follows the linear rule (linear MP) with $\bar{\imath} = \rho$. With active monetary policy (i.e., $\rho + \phi_x > 0$, $\rho\phi_x + \kappa(\phi_x - 1) > 0$), the unique equilibrium is $x_t = \pi_t = 0$. With passive monetary policy (i.e., $\rho\phi_x + \kappa(\phi_x - 1) < 0$), the equilibrium is deterministic and features $\pi_t = \frac{1}{2(\phi_\pi - 1)}\left[\rho - \phi_x - \sqrt{(\rho - \phi_x)^2 - 4\kappa(\phi_\pi - 1)}\right]x_t$, with x_0 indeterminate.*

We conclude this section with two remarks about how to broaden these results.

Remark 2. *If monetary policy is active, then it is possible to obtain the same results with fiscal volatility targeting* only in deep recessions*. However, the trade-off is that the fiscal volatility targeting does not eliminate volatility all on its own. In particular, suppose the fiscal rule (41) is deployed only when $x_t < \chi < 0$. Otherwise, fiscal policy is simply passive. In that case, active monetary policy implies that $x_t < \chi$ will eventually occur with probability 1 in any equilibrium that is not $(x_t, \pi_t) = (0, 0)$.*[18] *Once $x_t < \chi$ occurs, fiscal volatility targeting kicks in, rendering the dynamics deterministic. But because monetary policy is active, these deterministic dynamics are explosive, and so the entire equilibrium path is ruled out.*

Remark 3. *If, in addition to sunspot and fiscal shocks, the model featured fundamental shocks, one must accommodate those shocks in the fiscal volatility targeting. For instance, if the economy featured TFP shocks, then natural output would be stochastic Y_t^*. Suppose its log dynamics*

[18]The proof of this claim is rather involved. First of all, we need only consider stochastic equilibria, since all deterministic paths that are not $(x_t, \pi_t) = (0, 0)$ are explosive, by the proof of Proposition 7. Then, Lemma A.3 proves that x_t is unbounded below in all stochastic equilibria, meaning that x_t will cross χ with probability 1.

featured diffusion $\sigma_{y^*,t} dZ_t$. Suppose actual output featured the diffusion $(\sigma_{y^*,t} + \sigma_{x,t}) dZ_t$, where $\sigma_{x,t}$ is a non-fundamental exposure coming from the output gap. Then, in order to eliminate only the non-fundamental risk in the economy, the design of the debt target b_t^* should be

$$db_t^* = (\rho b_t^* - s_t)dt - b_t^* \sigma_{y^*,t} dZ_t.$$

By just slightly modifying the proof of Theorem 3, we can see that this policy would steer output to have volatility $\sigma_{y^*,t}$, thereby selecting $\sigma_{x,t} = 0$.

Summarizing, different policies that succeed in eliminating uncertainty traps may lead to completely different equilibrium dynamics. It is possible to eliminate uncertainty traps with a typical active fiscal policy, which then puts us in a "fiscal equilibrium" (Theorem 2 and Proposition I.1). But it is equally possible, with a slightly modified fiscal policy, to escape uncertainty traps and be in a conventional "monetary equilibrium," the MSV solution (Theorem 3 and Proposition 7). From the perspective of quantitative and empirical researchers, it may be useful to know that either possibility—the fiscal or monetary equilibrium—remains on the table, because that flexibility provides the best chance to fit the data. From the perspective of theorists, the fact that we have a range of policies that eliminate uncertainty traps allows us to pick an optimal fiscal-monetary mix from among them.

5 Conclusion

Uncertainty traps are ultimately a point about indeterminacy in New Keynesian models, novel primarily in the sense that they are rooted in nonlinearity and risk. The results of this paper suggest that monetary policy is limited in its ability to restore determinacy once the model's inherent nonlinearity—agents' optimal response to risk—is taken seriously. Monetary policy needs help. By contrast, fiscal policies can ensure determinacy even when monetary policy is constrained (e.g., by an effective lower bound). We also provide a novel analysis of a policy—fiscal volatility targeting—that can deliver equilibrium uniqueness without fiscal dominance.

Going beyond a technical focus on equilibrium selection, these policy remedies have practical implications. For any of our policies to work, the government must have commitment power. For instance, we have argued that our policies resemble governments that prioritize debt sustainability in normal times but switch their focus to stimulus when non-fundamental shocks arise. While this sounds reasonable, it can be fragile too. Absolutely critical to such an enterprise is a public trust that such an intervention can, in fact, be financed in recessions and that the government will follow through on those intervention promises. An interesting question is to what extent an imperfect government with only partial commitment power can tame uncertainty traps and related phenomena.

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Online Appendix:
Fear, Indeterminacy, and Policy Responses

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December 9, 2025

A Proofs of Main Results

A.1 Non-explosion and transversality conditions

For completeness, we briefly document the non-explosion requirements imposed by consumer and firm optimality. We then show that our non-explosion Condition 1 suffices to ensure these requirements hold. Thus, besides the standard derivations in the text, this will complete the proof of Lemma 1.

For the consumer side, note that the representative agent's utility can be written

$$U_0 = \rho^{-1}\Big(\log Y^* - \frac{(Y^*)^{1+\varphi}}{1+\varphi}\Big) + \int_0^\infty e^{-\rho t}\mathbb{E}\Big[x_t - \frac{e^{(1+\varphi)x_t}}{1+\varphi}\Big]dt$$

We need to ensure the consumer obtains finite utility and that his transversality condition holds. To ensure $U_0 > -\infty$, we require

$$\lim_{T\to\infty} \mathbb{E}_t[e^{-\rho T}x_T] = 0 \tag{A.1}$$

$$\lim_{T\to\infty} \mathbb{E}_t[e^{(1+\varphi)x_T-\rho T}] = 0 \tag{A.2}$$

Requirement (A.1) rules out $\mathbb{E}x_T$ diverging to $-\infty$ faster than rate ρ. Requirement (A.2) rules out $\mathbb{E}e^{(1+\varphi)x_T}$ diverging to $+\infty$ faster than rate ρ. It is clear that if Condition 1 holds, then both (A.1)-(A.2) are satisfied.

The consumer's transversality condition holds if and only if the lifetime budget constraint (5) holds with equality. Now, note that since price adjustment costs are non-pecuniary, the real present value of aggregate profits are $\Pi_t = \mathbb{E}_t[\int_t^\infty \frac{M_s}{M_t}(Y_s - \frac{W_s L_s}{P_s})ds]$. Using the resource constraint $C_t = Y_t$ and $B_0 = 0$, we therefore have that the consumer lifetime budget constraint (5) holds with equality, so long as all these integrals converge. Convergence of the integrals can be evaluated using the FOCs. The consumption FOC (7) implies $\mathbb{E}_0[\int_0^\infty M_t C_t dt] = (\rho\lambda)^{-1}$, so this integral converges. The labor FOC (6) and market clearing $C_t = L_t$ imply $\mathbb{E}_0[\int_0^\infty M_t \frac{W_t L_t}{P_t} dt] = \lambda^{-1}\mathbb{E}_0[\int_0^\infty e^{-\rho t} C_t^{1+\varphi} dt]$, so this integral converges so long as $\mathbb{E}[C_t^{1+\varphi}]$ grows slower than $e^{\rho t}$, which is exactly identical to requirement (A.2) that has already been verified.

For the firm side, note that Appendix G derives the optimality conditions from the firm's price setting problem. There, we show that the firm's transversality conditions are, in a symmetric equilibrium in which firms charge identical prices,

$$\lim_{T\to\infty} \mathbb{E}_t[e^{(1+\varphi)x_T-\rho T}] = 0 \tag{A.3}$$

$$\lim_{T\to\infty} \mathbb{E}_t[e^{-\rho T}\pi_T^2] = 0 \tag{A.4}$$

Notice that requirement (A.3) is identical to (A.2), which we have already verified. Requirement (A.4) avoids nominal explosions that imply an infinite present value of adjustment costs. Note that under Condition 1, this automatically holds.

A.2 A tool to check for non-explosive equilibria

Here, we provide an abstract non-explosiveness lemma that is general enough to cover all the candidate stochastic equilibria in this paper, at least after an appropriate change-of-variables.

Let Z be a one-dimensional Brownian motion. Let $\mathcal{E}_t \in (e_L, e_H)$ be one-dimensional diffusion, following

$$d\mathcal{E}_t = \mu_e(\mathcal{E}_t)dt + \sigma_e(\mathcal{E}_t)dZ_t, \tag{A.5}$$

where $-\infty < e_L < e_H < \infty$. Assume μ_e, σ_e are continuous on $[e_L, e_H]$. Let $V_t \geq 0$ satisfy

$$dV_t = \mu_v(V_t, \mathcal{E}_t)dt + \sigma_v(V_t, \mathcal{E}_t)dZ_t. \tag{A.6}$$

Lemma A.1. *Consider the setting above. Assume that*

(H1) There is a strictly positive, bounded, and continuous function $\bar{v} : [e_L, e_H] \mapsto \mathbb{R}_+$ such that $V_t \leq \bar{v}(\mathcal{E}_t)$ at all times.

(H2) $\mu_v(v,\mathcal{E})$ and $\sigma_v(v,\mathcal{E})$ are finite on $\{(v,\mathcal{E}) : 0 < v \leq \bar{v}(\mathcal{E})\}$.

(H3) $\sigma_v^2(v,\mathcal{E})$ is strictly positive for all $v > 0$ and $v < \bar{v}(\mathcal{E})$.

(H4) $\lim_{v\to 0}(\frac{\sigma_v(v,\mathcal{E})}{v})^2 > 0$ for all $\mathcal{E}$ (i.e., σ_v^2 vanishes at most quadratically).

(H5) $\theta > 1$, where

$$\theta := \inf_{\mathcal{E}\in(e_L,e_H)} \lim_{v\to 0} \frac{2v\mu_v(v,\mathcal{E})}{\sigma_v^2(v,\mathcal{E})}. \tag{A.7}$$

Then, $\liminf_{t\to\infty} \mathbb{E}[\log(V_t)] > -\infty$.

Proof of Lemma A.1. Let $\mathcal{D} := \{(v, \mathcal{E}) : 0 \le v \le \bar{v}(\mathcal{E}),\, e_L < \mathcal{E} < e_H\}$ denote the domain of the state dynamics. By assumption (H1), the process $(V_t, \mathcal{E}_t)$ remains in $\mathcal{D}$ with probability 1. The crux of the proof is to essentially show that V_t does not concentrate probability near the lower boundary $v = 0$.

Let θ be defined by (A.7). Choose $\alpha \in (0, \frac{\theta-1}{2})$. Define the Lyapunov function

$$f(v, \mathcal{E}) = v^{-\alpha}.$$

Letting $\mathcal{L}$ denote the infinitesimal generator of $(V, \mathcal{E})$, we have

$$\mathcal{L}f(v, \mathcal{E}) = -\alpha \mu_v(v, \mathcal{E}) v^{-\alpha-1} + \frac{1}{2}\alpha(1+\alpha)\sigma_v^2(v, \mathcal{E}) v^{-\alpha-2}. \tag{A.8}$$

By assumption (H5), there exists $v_- < \min_{\mathcal{E}} \bar{v}(\mathcal{E})$ such that for all $v \le v_-$,

$$\mu_v(v, \mathcal{E}) \ge \frac{1}{2}(1+2\alpha)\frac{\sigma_v^2(v, \mathcal{E})}{v}. \tag{A.9}$$

Using (A.9) in (A.8), we have

$$\mathcal{L}f(v, \mathcal{E}) \le -\frac{1}{2}\alpha^2 v^{-\alpha-2}\sigma_v^2(v, \mathcal{E}), \quad \text{for } v \le v_- \tag{A.10}$$

By assumptions (H3) and (H4), we have that σ_v^2/v^2 above is strictly positive for all $v \le v_-$ and all $\mathcal{E}$. Thus, $c := \inf_{\mathcal{E}} \inf_{v \le v_-} \frac{1}{2}\alpha^2 v^{-2}\sigma_v^2(v, \mathcal{E})$ is strictly positive, and so

$$\mathcal{L}f(v, \mathcal{E}) \le -cf(v, \mathcal{E}), \quad \text{with } c > 0, \quad \text{for } v \le v_-. \tag{A.11}$$

On $\{v_- \le v \le v_+\}$, where $v_+ := \max_{\mathcal{E} \in [e_L, e_H]} \bar{v}(\mathcal{E})$, compactness and assumption (H2) implies that for some $b < \infty$, we have

$$\mathcal{L}f(v, \mathcal{E}) \le b - cf(v_-, \mathcal{E}), \quad \text{for } v_- \le v \le v_+ \tag{A.12}$$

Therefore, combining (A.11)-(A.12), and noting that $f(v, \mathcal{E}) < f(v_-, \mathcal{E})$ for all $v > v_-$, we obtain

$$\mathcal{L}f(v, \mathcal{E}) \le b - cf(v, \mathcal{E}), \quad \text{on } \mathcal{D}. \tag{A.13}$$

By Lemma A.2 below, (A.13) implies

$$\sup_{t\geq 0} \mathbb{E}[V_t^{-\alpha}] = \sup_{t\geq 0} \mathbb{E}[f(V_t, \mathcal{E}_t)] < \infty. \tag{A.14}$$

Letting $x_t := \log(V_t)$ and applying Jensen's inequality on the convex map $u \mapsto e^{-\alpha u}$ gives

$$e^{-\alpha \mathbb{E}[x_t]} \leq \mathbb{E}[e^{-\alpha x_t}] = \mathbb{E}[V_t^{-\alpha}] < \infty,$$

so $\mathbb{E}[x_t] \geq -\frac{1}{\alpha} \log \mathbb{E}[V_t^{-\alpha}]$. Finally, take the infimum of both sides, using (A.14), to obtain $\liminf_{t\to\infty} \mathbb{E}[x_t] > -\infty$. This completes the proof. □

Lemma A.2. *Let $(X_t)_{t\geq 0}$ be a time-homogeneous diffusion process on a state space $\mathcal{X}$ with extended generator $\mathcal{L}$. Suppose a C^2 function $f : \mathcal{X} \mapsto \mathbb{R}_+$ satisfies*

$$\mathcal{L}f(x) \leq -cf(x) + b, \quad c > 0,\ b < \infty.$$

Then, for all $t \geq 0$,

$$\mathbb{E}^x[f(X_t)] \leq \big(f(x) - b/c\big)e^{-ct} + \frac{b}{c}.$$

In particular, $\sup_{t\geq 0} \mathbb{E}^x[f(X_t)] < \infty$.

Proof of Lemma A.2. Using the generator bound, for any fixed $t > 0$, one gets

$$\mathbb{E}^x[f(X_t)] = f(x) + \int_0^t \mathbb{E}^x\big[\mathcal{L}f(X_s)\big]ds \leq f(x) - c\int_0^t \mathbb{E}^x[f(X_s)]ds + bt.$$

Set $F(t) = \mathbb{E}^x[f(X_t)]$. Then,

$$F(t) \leq f(x) - c\int_0^t F(s)ds + bt.$$

Note that F is time-differentiable, by Itô's formula, since f is a C^2 function. Differentiating yields

$$F'(t) \leq -cF(t) + b, \quad F(0) = f(x).$$

Solve this linear differential inequality by integrating against the factor e^{ct}:

$$\frac{d}{dt}[e^{ct}F(t)] \leq be^{ct} \implies e^{ct}F(t) - F(0) \leq \frac{b}{c}(e^{ct} - 1).$$

Rearranging gives the stated bound. □

A.3 Proofs for Section 2

Proof of Theorem 1. Let us denote the fundamental equilibrium by $x^* = 0$. In this setting with $\bar{\imath} = \rho$, we have $x^* = 0$. (But if the target rate $\bar{\imath}$ differed from ρ, then x^* would be a non-zero constant; nevertheless, the entire proof below remains valid, because we proceed by examining the dynamics of the "gap" $x_t - x^*$).

Proof of statement (i). First, construct the volatility function $\sigma_x(x)$. Let $\omega > 0$ and $\delta > 0$ be arbitrary. Let $\beta > 0$ be a constant such that Assumption 1 holds. Define $\bar{x} := \min[0, x^* - \delta]$. Let the volatility function be given by

$$\sigma_x^2(x) := \begin{cases} 2\big[e^{-2\beta(x-x^*)}\omega^2 - \Phi(x) + \Phi(x^*)\big], & \text{if } x < \bar{x}; \\ 0, & \text{if } x \geq \bar{x}. \end{cases} \tag{A.15}$$

Notice that $\sigma_x^2(x) > 0$ for all $x < \bar{x}$. Since this only defines the squared volatility, we also pick the construction such that $\sigma_x > 0$ in this region.

Next, consider the change of variables:

$$V_t = \exp[\beta(x_t - x^*)].$$

Given any $\bar{\imath}$, the dynamics of x_t and x_t^* are

$$\begin{aligned} dx_t &= \Big[\bar{\imath} + \Phi(x_t) + \sigma\mathcal{E}_t + \frac{1}{2}\sigma_x^2(\mathcal{E}_t, x_t)\Big]dt + \sigma_x(\mathcal{E}_t, x_t)dZ_t \\ dx_t^* &= \Big[\bar{\imath} + \Phi(x_t^*)\Big]dt \end{aligned}$$

(Of course, $x_t^* = \Phi^{-1}(\bar{\imath})$ is constant in equilibrium, but we write the dynamics in their general form for now.) Therefore, the dynamics of V_t are

$$\begin{aligned} dV_t &= \mu_v(V_t)dt + \sigma_v(V_t)dZ_t, \quad \text{where} \\ \mu_v(v) &:= \beta v\Big[\Phi(\chi(v)) - \Phi(x^*) + \frac{(1+\beta)\sigma_x^2(\chi(v))}{2}\Big] \\ \sigma_v(v) &:= \beta v \sigma_x(\chi(v)) \end{aligned}$$

where $\chi(v) := x^* + \beta^{-1}\log(v)$. The plan is to show that V_t satisfies the hypotheses of Lemma A.1. Doing so will show that $\liminf_t \mathbb{E}[\log(V_t)] > -\infty$, which ensures $\liminf_t \mathbb{E}[x_t] > -\infty$ (i.e., Condition 1 holds).

We obviously have that $V_t \geq 0$ forever. Defining $\bar{v} := e^{-\beta\max[x^*,\delta]}$, let us show that

$V_t \leq \bar{v}$ forever if V_0 starts below $\bar{v}$. For x_t such that $\bar{x} \leq x_t < x^*$, or equivalently $\bar{v} \leq V_t < 1$, we have that $\sigma_v = 0$, and so $dV_t = [\Phi(x_t) - \Phi(x_t^*)]dt < 0$, by the fact that Φ is increasing and that $x < x^*$. Therefore, V_t enters the region below $\bar{v}$ in finite time when starting from any point $V_0 < 1$. And if $V_t \leq \bar{v}$, it can never exit this region. So V_t satisfies hypothesis (H1) of Lemma A.1.

It is easy to see that μ_v and σ_v^2 are continuous on $\{v < \bar{v}\}$ and are only potentially infinite when $v = 0$. So V_t satisfies hypothesis (H2) of Lemma A.1.

Next, we showed earlier that $\sigma_x^2(x) > 0$ for all $x < \bar{x}$, and so $\sigma_v^2(v) > 0$ for all $v < \bar{v}$. This verifies that V_t satisfies assumption (H3) of Lemma A.1.

Next, analyze the diffusion σ_v^2 asymptotically as $v \to 0$. We have that,

$$\begin{aligned}
\lim_{v\to 0} \sigma_v^2(v) &= \beta^2 \lim_{x\to -\infty} e^{2\beta(x-x^*)}\sigma_x^2(x) \\
&= 2\beta^2 \lim_{x\to -\infty} e^{2\beta(x-x^*)}[\Phi(x^*) - \Phi(x)] + 2\beta^2\omega^2 \\
&= 2\beta^2\omega^2 - 2\beta^2 \lim_{x\to -\infty} e^{2\beta(x-x^*)}\Phi(x) \\
&= 2\beta^2\omega^2.
\end{aligned}$$

The first line uses the definition of v. The second line uses the expression for σ_x^2 in (B.7). The third line uses that x^* is bounded. The fourth line uses Assumption 1. Hence, we have proved that σ_v^2 converges to a finite constant as $v \to 0$, implying V_t satisfies hypothesis (H4) of Lemma A.1.

We next do a similar limiting analysis for the drift. Plugging (B.7) into the expression for μ_v, we see that $v\mu_v(v) = \beta\omega^2 + \frac{1}{2}\sigma_v^2(v)$. Hence, we use the previous limiting results to show that

$$\lim_{v\to 0} v\mu_v(v) = \beta\omega^2 + \frac{1}{2}\lim_{v\to 0}\sigma_v^2(v) = \beta(1+\beta)\omega^2.$$

Combining the limits for the diffusion and the drift, we have that

$$\theta := \lim_{v\to 0} \frac{2v\mu_v(v)}{\sigma_v^2(v)} = \frac{1+\beta}{\beta} > 1$$

Consequently, hypothesis (H5) of Lemma A.1 holds.

This verifies all the hypotheses of Lemma A.1, proving that the construction is a valid non-explosive equilibrium.

Finally, we prove the claim that any volatility function is valid if it satisfies suitable boundary conditions. Instead of the σ_x^2 function in (B.7), consider any alternative function $\tilde{\sigma}_x^2$, which (a) coincides with σ_x^2 for $x \notin (-K, \bar{x} - K^{-1})$ for K arbitrarily large; and (b) is finite and strictly positive on $x \in (-K, \bar{x} - K^{-1})$. By inspection, the entire proof above remains valid. This proves statement (i) of the proposition.

Proof of statement (ii). We now prove that all equilibria are recessionary. Consider a general rule $\Phi(x)$ satisfying $\phi_x := \inf_x \Phi'(x) > 0$. Let $\sigma^2_{x,t}$ be any volatility process such that $\sigma^2_{x,t} \geq 0$. Again, we study the "gap process" $\Delta_t := x_t - x^*_t$ (again, note that x^*_t is constant over time in any equilibrium, and note also that the specific result in the stated proposition is the special case with $x^* = 0$). Suppose, leading to contradiction that $\Delta_0 > 0$ was part of a non-explosive equilibrium.

The dynamics of Δ_t are

$$d\Delta_t = \Big[\Phi(\Delta_t + x^*_t) - \Phi(x^*_t) + \frac{1}{2}\sigma^2_{x,t}\Big]dt + \sigma_{x,t}dZ_t, \quad \Delta_0 > 0.$$

Consider the alternative process $\tilde{\Delta}_t$ which drops the variance from the drift:

$$d\tilde{\Delta}_t = \Big[\Phi(\tilde{\Delta}_t + x^*_t) - \Phi(x^*_t)\Big]dt + \sigma_{x,t}dZ_t, \quad \tilde{\Delta}_0 = \Delta_0 > 0.$$

Define the stopping time

$$\tau_0 := \inf\{t > 0 : \tilde{\Delta}_t = 0\},$$

and put $T_0 := T \wedge \tau_0$. Then,

$$\mathbb{E}_0[e^{-\phi_x T_0}\tilde{\Delta}_{T_0}] = \tilde{\Delta}_0 + \mathbb{E}_0\Big[\int_0^{T_0} e^{-\phi_x t}\Big(\Phi(\tilde{\Delta}_t + x^*_t) - \Phi(x^*_t) - \phi_x\tilde{\Delta}_t\Big)dt\Big] \geq \tilde{\Delta}_0 = \Delta_0,$$

since $\Phi(\tilde{\Delta} + x^*) - \Phi(x^*) \geq \phi_x\tilde{\Delta}$ for all $\tilde{\Delta} \geq 0$, because $\phi_x > 0$ is the minimal slope of the general rule $\Phi(x)$. The left-hand-side can be written $e^{-\phi_x T}\mathbb{E}_0[\tilde{\Delta}_T \mathbf{1}_{\{T<\tau_0\}}]$, since $\tilde{\Delta}_{\tau_0} = 0$. Thus,

$$\mathbb{E}_0[\tilde{\Delta}_T \mathbf{1}_{\{T<\tau_0\}}] \geq e^{\phi_x T}\Delta_0,$$

which by taking $T \to \infty$ proves that $\limsup_{T\to\infty} \mathbb{E}_0[\tilde{\Delta}_T] = +\infty$ with positive probability. This implies that $\limsup_{T\to\infty} \mathbb{E}_0[\Delta_T] = +\infty$, since standard diffusion comparison theorems imply that $\Delta_T \geq \tilde{\Delta}_T$ almost-surely. Finally, this then implies that $\limsup_{T\to\infty} \mathbb{E}_0[x_T] = +\infty$, in violation of Condition 1, which proves the result. □

Proof of Proposition 1. It suffices to prove that Example 4 permits an uncertainty trap equilibrium with the desired properties. Suppose the economy begins in recession, $x_0 <$

0. This economy never collapses (i.e., x_t never reaches $-\infty$) and never fully recovers either (i.e., x_t never reaches 0). As $x_t \to -\infty$, parameter condition $\nu^2(1-\epsilon)^2 > 2\rho$ implies x_t behaves like an arithmetic Brownian motion with positive drift, thus preventing an explosion. On the other hand, as $x_t \to 0$, its volatility vanishes (since $\epsilon > 0$) while its drift is strictly negative. This proves that Condition 1 holds for the conjectured path, and so we have a non-explosive equilibrium. Furthermore, this proves that output remains forever below potential. □

Proof of Proposition 2. Into (23), we plug the volatility function $\tilde{\sigma}_x$, relative prudence $RP_t = 1+\gamma$, elasticity of intertemporal substitution $EIS_t = \gamma^{-1}$, monetary policy rule $\iota = \rho + \tilde{\Phi}(x)$, and impose the rigid-price limit $\pi = 0$. The result is

$$\mu_x = \gamma^{-1}\tilde{\Phi}(x) + \frac{1}{2}\gamma\tilde{\sigma}_x^2(x).$$

This implies that the drift and diffusion of $y_t = e^{x_t}$ are given by

$$\begin{aligned} \mu_y &= y\Big[\gamma^{-1}\tilde{\Phi}(x) + \frac{1}{2}(1+\gamma)\tilde{\sigma}_x^2(x)\Big] \\ \sigma_y &= y\tilde{\sigma}_x^2(x) \end{aligned}$$

For these dynamics, we compute the tail statistic in equation (A.7) of Lemma A.1 to be

$$\tilde{\theta} := \lim_{y\to 0} \frac{2y\mu_y}{\sigma_y^2} = 1 + \gamma + 2\lim_{x\to-\infty} \frac{\gamma^{-1}\tilde{\Phi}(x)}{\tilde{\sigma}_x^2(x)}.$$

Repeat the same analysis for log utility version with reaction function $\Phi(x)$ and volatility function $\sigma_x(x)$ to obtain the tail statistic

$$\theta := \lim_{y\to 0} \frac{2y\mu_y}{\sigma_y^2} = 2 + 2\lim_{x\to-\infty} \frac{\Phi(x)}{\sigma_x^2(x)}.$$

Assume $\Phi(x)$ and $\sigma_x^2(x)$ are an equilibrium with log utility, meaning $\theta > 1$. Using $\tilde{\Phi}(x) = \gamma\Phi(x)$ and $\tilde{\sigma}_x^2(x) = \gamma^{-1}\sigma_x^2(x)$, we then obtain that

$$\tilde{\theta} = 1 + \gamma + 2\gamma \lim_{x\to-\infty} \frac{\Phi(x)}{\sigma_x^2(x)} = \theta + (\gamma-1)(\theta-1)$$

Since $\theta > 1$ and $\gamma > 1$, we have $\tilde{\theta} > 1$, implying $\tilde{\Phi}(x)$ and $\tilde{\sigma}_x^2(x)$ constitute an equilibrium with CRRA utility. □

Proof of Proposition 3. Given the dynamics of x_t in (24), and given the Phillips curve is unchanged, the result is immediate. $\square$

Proof of Proposition 4. Let the enriched monetary rule be such that $\alpha_+ \leq 1 \leq \alpha_-$ and assume a linear output response function $\Phi(x) = \phi_x x$ for $\phi_x > 0$. Suppose, leading to contradiction, that a non-zero non-explosive equilibrium exists, and in particular $x_0 \neq 0$.

By the Itô-Tanaka formula, the dynamics of $|x_t|$ are

$$d|x_t| = \text{sign}(x_t)\Big[\phi_x x_t + \frac{1}{2}(1-\alpha(x_t))\sigma_{x,t}^2\Big]dt + \text{sign}(x_t)\sigma_{x,t}dZ_t + dL_t^0,$$

where $\alpha(x) := \alpha_+ \mathbf{1}_{\{x>0\}} + \alpha_- \mathbf{1}_{\{x<0\}}$ is the state-dependent risk premium response, and L_t^0 is the local time of x_t at 0 (note that L_t^0 is a non-negative, non-decreasing process). Integrating, taking expectations, and using the facts that $\text{sign}(x)x = |x|$, that $L_T^0 \geq 0$, that $\sigma_x^2 \geq 0$, and that $\text{sign}(x)(1-\alpha(x)) \geq 0$, we obtain

$$\begin{aligned}
\mathbb{E}_0|x_T| &= |x_0| + \mathbb{E}_0 \int_0^T \Big[\phi_x |x_t| + \text{sign}(x_t)(1-\alpha(x_t))\sigma_{x,t}^2\Big]dt + \mathbb{E}_0 L_T^0 \\
&\geq |x_0| + \phi_x \int_0^T \mathbb{E}_0|x_t|dt
\end{aligned}$$

Given $x_0 \neq 0$ and $\phi_x > 0$, this proves that $\lim_{T\to\infty} \mathbb{E}_0|x_T| > 0$. But if that is the case, then the integral on the right-hand-side does not converge, which then implies that $\lim_{T\to\infty} \mathbb{E}_0|x_T| = +\infty$, in violation of Condition 1. This contradicts the non-explosiveness of the proposed equilibrium, and so $x_0 = x_t = 0$ for all t must hold. $\square$

Proof of Proposition 5. To prove the proposition, we only need to provide an example construction for any $x_0 \leq 0$ starting point. The construction is as follows. Let ι_t be any interest rate process, subject to the lower bound $\iota_t \geq \underline{\iota}$. Consider some constant $a > 2(\rho - \underline{\iota})$ and set

$$\sigma_x^2 = \begin{cases} a, & \text{if } x < 0; \\ 0, & \text{if } x \geq 0. \end{cases}$$

Then, the dynamics of x_t are

$$dx_t = \begin{cases} [\iota_t - \rho + \frac{1}{2}a]dt + \sqrt{a}dZ_t, & \text{if } x_t < 0; \\ [\iota_t - \rho]dt, & \text{if } x_t \geq 0. \end{cases}$$

This will constitute a non-explosive equilibrium if x_t satisfies Condition 1. Consider the auxiliary process $\tilde{x}_t$, which satisfies $\tilde{x}_t = x_t$ whenever $x_t \geq 0$ and otherwise follows

$$d\tilde{x}_t = \left[\underline{\iota} - \rho + \frac{1}{2}a\right]dt + \sqrt{a}dZ_t, \quad \text{if} \quad x_t < 0.$$

Because the drift of x exceeds that of $\tilde{x}$, standard diffusion comparison theorems imply that $x_t \geq \tilde{x}_t$ forever. Furthermore, $\tilde{x}_t$ behaves like an arithmetic Brownian motion with positive drift when $x_t < 0$. By the well-known fact that a positive-drift arithmetic Brownian motion has $+\infty$ as its limit, we establish that $\liminf_{t\to\infty} \mathbb{E}[\tilde{x}_t] > -\infty$ almost-surely, hence by the inequality $x_t \geq \tilde{x}_t$ we have $\liminf_{t\to\infty} \mathbb{E}[x_t] > -\infty$. On the other hand, x_t is a path-continuous process and crosses 0 continuously. At $x_t = 0$, we have $\sigma_{x,t} = 0$, hence $\iota_t = \bar{\iota} = \rho$ is the interest rate, implying $dx_t = 0$. So $x = 0$ is an absorbing boundary, and $\limsup_{t\to\infty} \mathbb{E}[x_t] = 0 < +\infty$. Thus, Condition 1 is satisfied for x. □

A.4 Characterization of sunspot equilibria as $x \to -\infty$

Lemma A.3. *In the rigid-price limit ($\kappa \to 0$), assume that $\Phi'(x) > 0$ for all x. Then, any sunspot equilibrium has x_t unbounded below and $\sigma_{x,t}^2 \to +\infty$ as $x_t \to -\infty$. With partially-flexible prices ($\kappa > 0$), assume that $\partial_x \Phi > 0$ and $\partial_\pi \Phi > 1$ for all (x, π). Then, any sunspot equilibrium in which $\pi_t = \pi(x_t)$ and $\sigma_{x,t} = \sigma_x(x_t)$ has x_t unbounded below and $\lim_{x\to-\infty} \sigma_x^2(x) > 0$. Furthermore, if $\lim_{x\to-\infty} \sigma_x^2(x) < +\infty$, then $\lim_{x\to-\infty} \pi(x) = +\infty$.*

Proof of Lemma A.3. We prove separately the claims for rigid and flexible prices.

Proof for rigid-price limit ($\kappa \to 0$). The paper already proves that $x_t \leq 0$ is required, given $\Phi' > 0$, so we restrict attention to that case. Suppose that $x_t > \underline{x}$ with probability 1, where $0 > \underline{x} > -\infty$. We split the analysis into two possibilities: $\lim_{x\to\underline{x}} \sigma_x^2$ is finite or $\lim_{x\to\underline{x}} \sigma_x^2 = +\infty$.

If $\lim_{x\to\underline{x}} \sigma_x^2 < \infty$, analyze $y_t := e^{x_t}$ whose dynamics are

$$dy_t = \underbrace{y_t\left[\Phi(x_t) + \sigma_{x,t}^2\right]}_{:=\mu_{y,t}} dt + \underbrace{y_t \sigma_{x,t}}_{:=\sigma_{y,t}} dZ_t$$

The diffusion of y_t near its lower bound $\underline{y} := e^{\underline{x}}$ is $\sigma_{y,t}^2 \to \underline{y}^2 \sigma_{x,t}^2$, which is finite, while the drift is

$$\mu_{y,t} \to \underbrace{\underline{y}\Phi(\underline{x})}_{<0} + \underbrace{\underline{y}\sigma_{x,t}^2}_{\text{finite}}$$

If the diffusion is zero at $\underline{y}$ then the drift will be negative, implying y_t will cross $\underline{y}$. If the diffusion is positive at $\underline{y}$, then because the drift is finite, once again y_t will cross $\underline{y}$. This contradicts $x_t > \underline{x}$.

If $\lim_{x \to \underline{x}} \sigma_x^2 = +\infty$, we analyze instead $u_t := \log(e^{x_t} - e^{\underline{x}})$, whose dynamics satisfy

$$du_t = \frac{e^{x_t}}{e^{x_t} - e^{\underline{x}}}\Big[\Phi(x_t) + \Big(1 - \frac{e^{x_t}}{e^{x_t} - e^{\underline{x}}}\Big)\sigma_{x,t}^2\Big]dt + \frac{e^{x_t}}{e^{x_t} - e^{\underline{x}}}\sigma_{x,t} dZ_t$$

Notice that $\frac{e^x}{e^x - e^{\underline{x}}} \to \infty$ as $x \to \underline{x}$. Thus, as $x_t \to \underline{x}$ and hence $u_t \to -\infty$, the dynamics of u_t feature an infinite volatility and negative infinite drift, implying u_t will reach $-\infty$ and thus x_t will hit $\underline{x}$. This also contradicts $x_t > \underline{x}$.

Finally, we prove similarly that $\sigma_{x,t}^2 \to +\infty$ as $x_t \to -\infty$. Indeed, if not, then the dynamics of x_t would have finite diffusion and negative infinite drift as $x_t \to -\infty$, so it would necessarily explode.

Proof for partially-flexible prices ($\kappa > 0$). Consider partially-flexible prices ($\kappa > 0$) and a Markovian equilibrium: $\pi_t = \pi(x_t)$ and $\sigma_{x,t}^2 = \sigma_x^2(x_t)$. Applying Itô's formula to π and using the Phillips curve (linear PC) results in

$$\rho\pi - \kappa x = \big[\Phi(x, \pi) - \pi + \frac{1}{2}\sigma_x^2\big]\pi' + \frac{1}{2}\sigma_x^2 \pi''. \tag{A.16}$$

Let us try to construct an equilibrium for $x \in (\underline{x}, \overline{x})$, assuming only that $\underline{x} > -\infty$. We will prove that no function $\pi(\cdot)$ can satisfy (A.16) and coincide with dynamics for x_t that remain stable in this bounded region.

First, the dynamics need to be stable, meaning we need x_t to remain inside $(\underline{x}, \overline{x})$ forever. For this to hold, the diffusion $\sigma_x^2(x)$ must be some positive function that vanishes sufficiently rapidly as $x \to \underline{x}$ and $x \to \overline{x}$. Furthermore, the drift $\mu_x(x)$ must be pointing inward at these boundaries, i.e.,

$$\lim_{x \to \underline{x}} \Phi(x, \pi(x)) - \pi(x) \geq 0 \tag{A.17}$$

$$\lim_{x \to \overline{x}} \Phi(x, \pi(x)) - \pi(x) \leq 0 \tag{A.18}$$

Given assumptions that $\partial_x \Phi > 0$ and $\partial_\pi \Phi > 1$, this requires that

$$\pi(\underline{x}) > \pi(\overline{x}). \tag{A.19}$$

From these requirements, we can deduce the slope of π at the boundaries. Taking the limit $x \to \underline{x}$ or $x \to \overline{x}$ in (A.16), we have

$$\pi'(\underline{x}) = \lim_{x \to \underline{x}} \frac{\rho\pi(x) - \kappa x}{\Phi(x, \pi(x)) - \pi(x)} \tag{A.20}$$

$$\pi'(\overline{x}) = \lim_{x \to \overline{x}} \frac{\rho\pi(x) - \kappa x}{\Phi(x, \pi(x)) - \pi(x)} \tag{A.21}$$

We can prove that π is non-monotonic. Indeed, if not, then π must be monotonically decreasing by (A.19). But using (A.19) and (A.17)-(A.18) in (A.20)-(A.21), we have that if $\pi'(\underline{x}) < 0$, then $\pi'(\overline{x}) > 0$, a contradiction.

Given the non-monotonicity, there exists some intermediate point $x^* \in (\underline{x}, \overline{x})$ such that $\pi'(x^*) = 0$. At any such x^*, we have

$$\rho\pi(x^*) - \kappa x^* = \frac{1}{2}\sigma_x^2(x^*)\pi''(x^*) \tag{A.22}$$

There are three cases to consider: (i) $\pi'(\underline{x}) < 0 < \pi'(\overline{x})$; (ii) $\pi'(\underline{x}) > 0 > \pi'(\overline{x})$; (iii) $\pi'(\underline{x}), \pi'(\overline{x}) > 0$. (Recall that $\pi'(\underline{x}), \pi'(\overline{x}) < 0$ has just been ruled out above.) The cases where any of the boundary slopes equal zero are handled easily by extension.

- Case (i): $\pi'(\underline{x}) < 0 < \pi'(\overline{x})$. In this case, the function π is decreasing initially before increasing, so the first such point x^* must satisfy $\pi''(x^*) > 0$, and so $\rho\pi(x^*) - \kappa x^* > 0$ by (A.22). At the same time, $\pi'(\underline{x}) < 0$ implies $\rho\pi(\underline{x}) - \kappa\underline{x} < 0$ by (A.17) and (A.20). But this is a contradiction, since π decreasing between $\underline{x}$ and x^* implies

$$\rho\pi(\underline{x}) - \kappa\underline{x} > \rho\pi(x^*) - \kappa x^*.$$

- Case (ii) $\pi'(\underline{x}) > 0 > \pi'(\overline{x})$. This case is symmetric to case (i), but the argument starts from the upper boundary $\overline{x}$ instead. Indeed, the last such point x^* must satisfy $\pi''(x^*) < 0$, and so $\rho\pi(x^*) - \kappa x^* < 0$ by (A.22). At the same time, $\pi'(\overline{x}) < 0$ implies $\rho\pi(\overline{x}) - \kappa\overline{x} > 0$ by (A.18) and (A.21). But this is a contradiction, since π decreasing between x^* and $\overline{x}$ implies

$$\rho\pi(x^*) - \kappa x^* > \rho\pi(\overline{x}) - \kappa\overline{x}$$

- Case (iii): $\pi'(\underline{x}), \pi'(\overline{x}) > 0$. The fact that π is non-monotonic and has positive slopes at the boundaries implies that there exist two global extrema points $x^* < x^{**}$ such that $\pi'(x^*) = \pi'(x^{**}) = 0$ and $\pi''(x^*) < 0 < \pi''(x^{**})$—i.e., x^* is the global maximum, while x^{**} is the global minimum. By (A.22), $\pi''(x^*) < 0 < \pi''(x^{**})$

implies $\rho\pi(x^*) - \kappa x^* < 0 < \rho\pi(x^{**}) - \kappa x^{**}$. However, the fact that $\pi(x^*) > \pi(x^{**})$ implies the contradiction

$$\rho\pi(x^*) - \kappa x^* > \rho\pi(x^{**}) - \kappa x^{**}$$

Having ruled out all cases, and since $\underline{x}$ and $\overline{x}$ were arbitrary, there cannot exist any function $\pi(\cdot)$ corresponding to a bounded equilibrium. Thus, we have that $\underline{x} = -\infty$ must hold.

Finally, the claim that $\lim_{x\to-\infty} \sigma_x^2(x) > 0$ is proved using the same line of argument we just used to rule out $\underline{x} > -\infty$, since none of the arguments relied on $\underline{x}$ being finite—instead, looking closely, the arguments above rely only on the presumption that $\sigma_x^2(\underline{x}) = 0$ in order to obtain an expression for the drift $\mu_x(\underline{x})$. To then prove that $\lim_{x\to-\infty} \pi(x) = +\infty$ in any equilibrium with $\lim_{x\to-\infty} \sigma_x^2(x) < +\infty$, we assume not and proceed to contradiction. If $\lim_{x\to-\infty} \pi(x) < +\infty$ and $\lim_{x\to-\infty} \sigma_x^2(x) < +\infty$, then the asymptotic drift would be $\lim_{x\to-\infty} \mu_x(x) = -\infty$ by the fact that $\partial_x \Phi > 0$, and so the process for x_t would have finite volatility and negative infinite drift, which would thus explode to $-\infty$ with positive probability. □

Lemma A.4. *Let $y_t := e^{x_t}$ satisfy the assumptions of Lemma A.1, without the additional state variable $\mathcal{E}_t$. Then, y_t possesses a stationary distribution. For y near 0, this distribution has a density $p_y(y)$ that satisfies $\lim_{y\to 0} y^{-\theta}\sigma_y^2(y)p_y(y) = 1$, where θ is defined in (A.7).*

Proof of Lemma A.4. To prove the existence of and simultaneously to characterize the stationary distribution, let us guess-and-verify that the density p_y satisfies $P_y := \sigma_y^2 p_y \sim y^\zeta$ for some $\zeta > 1$. In that case, L'Hôpital's rule allows us to use $\frac{d}{dy}P_y \sim \zeta y^{\zeta-1}$ and $\frac{d^2}{dy^2}P_y \sim \zeta(\zeta-1)y^{\zeta-2}$. The Kolmogorov Forward Equation for the density is

$$0 = -\frac{d}{dy}[\mu_y p_y] + \frac{1}{2}\frac{d^2}{dy^2}[\sigma_y^2 p_y].$$

Using the change-of-variables $P_y := \sigma_y^2 p_y$, we have

$$0 = -\frac{d}{dy}[\frac{2y\mu_y}{\sigma_y^2} y^{-1} P_y] + \frac{d^2}{dy^2}[P_y].$$

Using the asymptotic form $P_y \sim y^\zeta$ and the assumption that $\frac{2y\mu_y}{\sigma_y^2} \sim \theta$ is finite and larger than one, we have for y small enough

$$0 = -\theta(\zeta - 1)y^{\zeta-2} + \zeta(\zeta - 1)y^{\zeta-2}.$$

This verifies the conjectured form of p_y if $\zeta = \theta > 1$. (The equation above also holds if $\zeta = 1$, but that would contradict the presumption that $\zeta > 1$.) Furthermore, since σ_y^2 vanishes at most quadratically, the resulting density $p_y \sim y^\theta \sigma_y^{-2}$ satisfies $p_y(y) = O(y^{\theta-2})$, where $f(y) = O(g(y))$ means that there exists a $C > 0$ such that $f(y) \leq Cg(y)$ for all y small enough. Since $\theta > 1$ by assumption, this proves that $p_y(y)$ is integrable, hence a valid density. □

Lemma A.5. *Consider the rigid-price limit ($\kappa \to 0$) and suppose $\inf_x \Phi'(x) > 0$. Let $\mathcal{S}$ be the set of sunspot equilibria possessing a stationary distribution. The tail probability can be arbitrarily small in stationary sunspot equilibria, i.e., for χ large enough,*

$$\inf_{\mathcal{S}} \mathbb{P}\{x_t < \chi\} = 0.$$

Proof of Lemma A.5. In rigid-price equilibria, the dynamics of $y := e^x$ are given by $\mu_y(y) = y[\Phi(x) + \sigma_x^2(x)]$ and $\sigma_y^2(y) = y^2\sigma_x^2(x)$. Therefore, we have that $\theta = 2 + \lim_{x\to-\infty} \frac{2\Phi(x)}{\sigma_x^2(x)}$, and so Lemma A.4 says

$$p_y \sim \sigma_x^{-2} y^{2\lim_{x\to-\infty} \Phi(x)\sigma_x^{-2}(x)}, \quad \text{as } y \to 0.$$

To prove the result, it suffices to find an example equilibrium in $\mathcal{S}$ that minimizes the tail probability. Consider the example from the paper, in which $\Phi(x) = \phi_x(e^x - e^{-x})$ and $\sigma_x^2 = (\frac{\nu}{y})^2 + \phi_x \frac{1-y^2}{y}$, for a free parameter ν. With this specification, we have $\Phi(x)\sigma_x^{-2}(x) \sim -\frac{\phi_x y}{\nu^2+\phi_x y}$ and so

$$p_y \sim \nu^{-2} y^{\frac{2\nu^2}{\nu^2+\phi_x y}} \sim (y/\nu)^2$$

Consequently, if χ is low enough, we can compute $\mathbb{P}\{x_t < \chi\} \approx \nu^{-2} \int_0^{e^\chi} y^2 dy = \frac{1}{3}e^{3\chi}/\nu^2$. This probability can be made arbitrarily small by taking ν large. □

A.5 Proofs for Section 4

Before proving the results in the text, we need a preliminary characterization of the equilibrium with fiscal policy. First, we derive the dynamics of the real debt-to-GDP ratio $b_t := \frac{B_t}{P_t Y_t}$. By Itô's formula,

$$
\begin{aligned}
db_t &= (r_t b_t - s_t)dt - b_t\Big(r_t - \rho + \sigma_{x,t}^2 + |\varsigma_{x,t}|^2\Big)dt + b_t\Big(\sigma_{x,t}^2 + |\varsigma_{x,t}|^2\Big)dt - b_t\Big(\sigma_{x,t}dZ_t + \varsigma_{x,t}d\mathcal{Z}_t\Big) \\
&= (\rho b_t - s_t)dt - b_t\Big(\sigma_{x,t}dZ_t + \varsigma_{x,t}d\mathcal{Z}_t\Big) \qquad \text{(A.23)}
\end{aligned}
$$

where we have used the flow government budget constraint (27), the price level dynamics $dP_t/P_t = \pi_t dt$, the dynamics of x_t in (29), and where $r_t = \iota_t - \pi_t$ is the real interest rate. In our model, the real SDF is $M_t = e^{-\rho t} Y_t^{-1}$, and so the transversality condition (TVC) can be written in terms of debt-to-GDP as

$$
\lim_{T\to 0} \mathbb{E}_0[e^{-\rho T} b_T] = 0. \qquad \text{(A.24)}
$$

Second, we purge surpluses of their endogenous component. So let us examine the dynamics of $e^{-\int_0^t (\rho - \alpha_u)du} b_t$. Using Itô's formula and the surplus rule (28), we have

$$
d\Big(e^{-\int_0^t (\rho-\alpha_u)du} b_t\Big) = e^{-\int_0^t (\rho-\alpha_u)du}\Big[-\hat{s}_t dt - b_t\Big(\sigma_{x,t}dZ_t + \varsigma_{x,t}d\mathcal{Z}_t\Big)\Big]
$$

Then,

$$
\mathbb{E}_0[e^{-\rho T} b_T] = \mathbb{E}_0\Big[e^{-\int_0^T \alpha_t dt}\Big(b_0 - \int_0^T e^{-\rho t} e^{\int_0^t \alpha_u du} \hat{s}_t dt\Big)\Big] \qquad \text{(A.25)}
$$

From here, we prove Lemmas 2-3, our classification results for passive and active fiscal policy, respectively.

Proof of Lemma 2. By $\alpha_t \geq 0$, we have $e^{-\int_0^T \alpha_t dt} \leq 1$. Also, recall that $\hat{s}_t$ is bounded. These assumptions imply we can use the dominated convergence theorem to take $T \to \infty$ inside the expectation on the right-hand-side of (A.25). Then, using condition (33), we get

$$
\lim_{T\to\infty} \mathbb{E}_0\Big[e^{-\int_0^T \alpha_t dt}\Big(b_0 - \int_0^T e^{-\rho t} e^{\int_0^t \alpha_u du}\hat{s}_t dt\Big)\Big] = \mathbb{E}_0\Big[e^{-\int_0^\infty \alpha_t dt}\Big(b_0 - \int_0^\infty e^{-\rho t} e^{\int_0^t \alpha_u du}\hat{s}_t dt\Big)\Big] = 0.
$$

Therefore, the left-hand-side of (A.25) must also vanish as $T \to \infty$. □

Proof of Lemma 3. We will prove that equation (35) at $t = 0$ since the same argument will hold at any $t > 0$. Similar to the derivation leading to equation (A.25), we can obtain

$$\mathbb{E}_0[e^{-\rho T} b_T A_T] = \mathbb{E}_0[I_T], \tag{A.26}$$
$$\text{where} \quad A_T := e^{\int_0^T \alpha_t dt}$$
$$\text{and} \quad I_T := b_0 - \int_0^\infty e^{-\rho t} e^{\int_0^t \alpha_u du} \hat{s}_t dt$$

In this compact notation, our aim is to prove that $0 = \mathbb{E}_0[I_\infty]$, where $I_\infty := \lim_{T\to\infty} I_T$ is the pointwise limit of I_T (it will be shown that this limit exists).

To proceed with our proof, we will guess and then verify ex-post that $(e^{-\rho T} b_T A_T)_{T>0}$ is uniformly integrable (UI).

To begin, we prove $\mathbb{E}_0[e^{-\rho T} b_T A_T]$ converges to zero along a subsequence. Given the transversality condition $\lim_{T\to 0} \mathbb{E}_0[e^{-\rho T} b_T] = 0$, there exists a subsequence of times $(T_j)_{j=1}^\infty$ with $T_j \to \infty$ such that $e^{-\rho T_j} b_{T_j} \to 0$. Given condition (34), we also have $A_T \to A_\infty < \infty$ and so $A_{T_j} \to A_\infty < \infty$. Combining these conditions, we have that $\lim_{j\to\infty} e^{-\rho T_j} b_{T_j} A_{T_j} = 0$. Given that $(e^{-\rho T} b_T A_T)_{T>0}$ is UI, we can conclude by Vitali's convergence theorem that $\lim_{j\to\infty} \mathbb{E}_0[A_{T_j} e^{-\rho T_j} b_{T_j}] = \mathbb{E}_0[\lim_{j\to\infty} A_{T_j} e^{-\rho T_j} b_{T_j}] = 0$.

Next, we have that $(I_{T_j})_{j>0}$ are UI. Indeed, we have proven $\lim_{j\to\infty} \mathbb{E}_0[A_{T_j} e^{-\rho T_j} b_{T_j}] = 0$, so by (A.26), we have that $\lim_{j\to\infty} \mathbb{E}_0[I_{T_j}] = 0$. This convergence-in-mean implies UI.

Next, we establish that $\lim_{j\to\infty} I_{T_j} = \lim_{T\to\infty} I_T =: I_\infty$ (i.e., convergence of I_T is the same along any subsequence). To do this, start by noting that $\hat{s}_t$ is uniformly bounded. Hence, the assumption $\sup_t \alpha_t < \rho$ implies that $\int_T^\infty e^{-\rho t} e^{\int_0^t \alpha_u du} \hat{s}_t dt$ converges to zero as $T \to \infty$, implying that $I_\infty := b_0 - \int_0^\infty e^{-\rho t} e^{\int_0^t \alpha_u du} \hat{s}_t dt$ is well-defined as the limit.

These facts allow us to conclude. The pointwise convergence $I_T \to I_\infty$ (hence convergence in probability), plus the fact that $(I_{T_j})_{j>0}$ are UI, implies that $\lim_{j\to\infty} \mathbb{E}_0[I_{T_j}] = \mathbb{E}_0[\lim_{j\to\infty} I_{T_j}] = \mathbb{E}_0[I_\infty]$ by Vitali's convergence theorem, as desired. As mentioned at the beginning, this implies that the valuation-like equation (35) holds for all $t > 0$.

Finally, we verify that $(e^{-\rho T} b_T A_T)_{T>0}$ is indeed UI as conjectured. Using equation (35), combined with the assumptions that $\sup_t \alpha_t < \rho$ and $\hat{s}_t$ is uniformly bounded, we immediately have that b_t is also uniformly bounded. Using again that $\sup_t \alpha_t < \rho$, we thus have that $e^{-\rho T} A_T b_T$ is bounded, hence UI. □

We now develop a characterization of the debt-to-GDP ratio in a class of Markovian environments. The following lemma is a key tool used in the unboundedness argument of Proposition 6, as well as the equilibrium selection argument of Theorem H.1.

Lemma A.6. *Assume an equilibrium in which inflation and volatilities take the form* $\pi_t = \pi(x_t, \Omega_t)$, $\sigma_{x,t} = \sigma_x(x_t, \Omega_t)$, *and* $\varsigma_{x,t} = \varsigma_x(x_t, \Omega_t)$, *and in which* $\alpha_t = \alpha(x_t, \Omega_t)$. *Suppose*

$\sigma_x(x,\Omega) \neq 0$ on a connected set $\mathcal{D}$. Suppose $b_t = b(x_t,\Omega_t)$ for some function $b(\cdot)$. Then, $b(x,\Omega) = \bar{b}e^{-x}$ on $\mathcal{D}$ for some constant $\bar{b}$. Additionally, $\bar{b} = 0$ if and only if $\hat{s}(\Omega) \equiv 0$.

Proof of Lemma A.6. Following the statement of the lemma, let us now consider a hypothetical sunspot equilibrium in which $b_t = b(x_t,\Omega_t)$, $\pi_t = \pi(x_t,\Omega_t)$, $\sigma_{x,t} = \sigma_x(x_t,\Omega_t)$, $\varsigma_{x,t} = \varsigma_x(x_t,\Omega_t)$. Under these assumptions, the process (x_t,Ω_t) is a Markov diffusion. We consider the domain $\mathcal{D}$ on which $\sigma_x \neq 0$. Then, by Itô's formula, the following hold on $\mathcal{D}$:

$$\sigma_x \partial_x b = -\sigma_x b \tag{A.27}$$

$$\varsigma_x' \partial_x b + \varsigma_\Omega' \partial_\Omega b = -\varsigma_x' b \tag{A.28}$$

$$\begin{aligned}(\rho - \alpha(x,\Omega))b - \hat{s} = \Big[\Phi(x,\pi) - \pi + \frac{1}{2}\sigma_x^2 + \frac{1}{2}|\varsigma_x|^2\Big]\partial_x b + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx} b \\ + \mu_\Omega' \partial_\Omega b + \frac{1}{2}\mathrm{trace}[\varsigma_\Omega \varsigma_\Omega'(\partial_{\Omega\Omega'} b)] + \varsigma_x \varsigma_\Omega' \partial_{x\Omega} b.\end{aligned} \tag{A.29}$$

We do not need the drift condition (A.29) for this argument. From the first two equations (A.27)-(A.28), assuming $\sigma_x \neq 0$, we have that

$$\partial_x b = -b$$
$$\varsigma_\Omega' \partial_\Omega b = 0$$

These conditions imply $b(x,\Omega) = e^{-x}\bar{b}(\Omega)$ for some function $\bar{b}(\cdot)$. It also turns out that $\bar{b}$ must, in fact, be a constant, so that

$$b(x,\Omega) = \hat{b}e^{-x}.$$

Indeed, there are two cases: (i) fiscal states Ω are present and (ii) Ω are absent. If Ω are present, then $\varsigma_\Omega \neq 0$. From $\varsigma_\Omega' \partial_\Omega b = 0$, this then implies $\partial_\Omega b = 0$, so $\bar{b}(\Omega) \equiv \bar{b}$ is a constant independent of Ω. If Ω are absent, then obviously $\bar{b}$ is also a constant. □

Proof of Proposition 6. Assume a sunspot equilibrium, so $\sigma_{x,t} \neq 0$. There are two possibilities that can be analyzed separately: (i) in the Markovian case, $b_t = b(x_t,\Omega_t)$ for some function $b(\cdot)$; (ii) in the non-Markovian case, b_t is not a function of (x_t,Ω_t). (Obviously, case (i) requires that certain equilibrium objects like σ_x^2 and π be solely functions of (x,Ω), but that is not important for the argument here.)

In case (i), Lemma A.6 says that $b_t = \bar{b}e^{-x_t}$ whenever $\sigma_{x,t} \neq 0$. Next, given the conditions on the monetary policy being "active," Lemma A.3 says that x_t is unbounded below and that $\sigma_{x,t}^2 > 0$ for all x_t low enough, which implies $b_t = \bar{b}e^{-x_t}$ is unbounded above unless $\bar{b} = 0$. But $\bar{b} = 0$ if and only if $\hat{s} \equiv 0$, which is assumed away in the proposition.

In case (ii), we suppose, leading to contradiction, that b_t had the upper bound $b_{max} > 0$. This implies an upper bound for debt-to-GDP during the subset of times when sunspot volatility is present, so without loss of generality we may assume that b_{max} is the upper bound during those times. Recall the dynamics of b_t:

$$db_t = \big[(\rho - \alpha_t)b_t - \hat{s}_t\big]dt - b_t\sigma_{x,t}dZ_t - b_t\varsigma_{x,t}d\mathcal{Z}_t.$$

While sunspot volatility is present ($\sigma_{x,t} \neq 0$), the process for b_t, as $b_t \to b_{max}$, has a finite drift and non-zero volatility, and so it will violate the upper bound if it ever approaches it. The only way b_t will not approach b_{max} is if $b_t\sigma_{x,t} = 0$ identically, which can only happen in a sunspot equilibrium if $b_t = 0$ identically. However, given $\hat{s}_t \neq 0$, the dynamics db_t show that $b_t = 0$ identically is impossible. □

Proof of Theorem 2. The theorem specializes to the fiscal rule (28) with $\alpha_t = 0$, so that $s_t = \hat{s}(\Omega_t)$ is exogenous. Then, scaling (GD) by Y_t, we obtain

$$b_t = \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(u-t)}\hat{s}(\Omega_u)du\Big]$$

This implies that b_t is purely determined by Ω_t, i.e., there exists a deterministic function $\hat{b}(\cdot)$ such that $b_t = \hat{s}(\Omega_t)$. In that case, b_t has no loading on the sunspot shock dZ_t, which implies by (A.23) that $b_t\sigma_{x,t} = 0$. Thus, $\sigma_{x,t} = 0$ for almost all t (except at the times when $b_t = 0$, which are zero Lebesgue measure almost-surely). For the statement about (GD) holding for every $t > 0$, given it holds at $t = 0$, simply note that the dynamics of db_t in (A.23) are derived without reference to the valuation equation (GD). □

Remark A.1 (Non-Markovian surpluses)**.** *From the proof of Theorem 2, it is clear that the same arguments hold even in the more general non-Markovian case where $(s_t)_{t\geq 0}$ is independent of $(Z_t)_{t\geq 0}$, because in that case db_t still must not have any loading on dZ_t.*

Proof of Theorem 3. Integrating the dynamics of b^* from (40) and taking expectations, we obtain

$$e^{-\rho T}\mathbb{E}_0[b_T^*] = e^{-\bar{\alpha}T}\Big(b_0^* - \mathbb{E}_0\Big[\int_0^T e^{-(\rho-\bar{\alpha})t}\hat{s}_t dt\Big]\Big)$$

Given that $\hat{s}_t$ is exogenous and bounded, we have that the right-hand-side converges to zero as $T \to \infty$, provided $\bar{\alpha} > 0$. Therefore, $\lim_{T\to\infty} e^{-\rho T}\mathbb{E}_0[b_T^*] = 0$. At this point, we can already prove the final statement of the theorem: if $b = b^*$, then we have just proved that the transversality condition automatically holds.

Next, we may apply Itô's formula to $e^{-\rho t}(b_t - b_t^*)$ to obtain

$$b_0 - b_0^* = e^{-\rho T}\mathbb{E}_t[b_T] - e^{-\rho T}\mathbb{E}_t[b_T^*] \underset{T\to\infty}{\longrightarrow} 0,$$

where the limit uses $\lim_{T\to\infty} e^{-\rho T}\mathbb{E}_0[b_T^*] = 0$, proved above, along with the transversality condition (TVC), which says $\lim_{T\to\infty} e^{-\rho T}\mathbb{E}_0[b_T] = 0$. This proves that $b_0 = b_0^*$. The same holds for any arbitrary initial date, and so $b_t = b_t^*$.

Using this result and comparing the dynamics db_t with db_t^* in (39)-(40), we have that $\sigma_{x,t} = 0$ and $\varsigma_{x,t} = 0$. □

Proof of Proposition 7. From Theorem 3, we have that $\sigma_{x,t} = 0$ and $\varsigma_{x,t} = 0$. Furthermore, we know that the debt valuation equation is redundant after $b_t = b_t^*$ is chosen. Thus, under fiscal policy (41), the necessary and sufficient conditions for equilibrium are

$$\begin{aligned} dx_t &= \big[\phi_x x_t + (\phi_\pi - 1)\pi_t\big]dt \\ d\pi_t &= \big[\rho\pi_t - \kappa x_t\big]dt + \sigma_{\pi,t}dZ_t + \varsigma_{\pi,t}d\mathcal{Z}_t. \end{aligned}$$

Clearly, one solution is $(x_t, \pi_t) = (0, 0)$ forever. To diagnose the solution set, notice that the drifts are linear in (x_t, π_t). Thus, we may follow the standard practice by computing the eigenvalue decomposition of the transition matrix (i.e., the infinitesimal generator) to determine whether or not the dynamics are stable or unstable under the assumed monetary policy.

This is done in the proof of Proposition I.1 (set $\sigma = 0$ to eliminate the monetary policy shocks that are also considered in that proof). Case 1 of Proposition I.1 covers active money, while case 2 covers passive money. Under active money, the result is that if $(x_t, \pi_t) \neq (0, 0)$, then $\mathbb{E}_0[x_T]$ and $\mathbb{E}_0[\pi_T]$ explode exponentially as $T \to \infty$, in violation of Condition 1. Under passive money, the result is that $\pi_t \propto x_t$. Since x_t evolves deterministically, this implies that π_t also does. Hence, the only remaining indeterminacy is x_0. □

B Monetary Policy Shocks (rather than sunspots)

Our baseline model features only sunspot uncertainty, for theoretical clarity. However, this is not necessary. In environments in which sunspot equilibria arise, there is often also indeterminacy in the response of the economy to fundamental shocks, since agents can use the fundamental shock as a source of coordination. Here, we demonstrate this by adding fundamental uncertainty to the monetary policy rule.

The model is the same, except now the interest rate rule is given by

$$\iota_t = \bar{\iota} + \Phi(x_t, \pi_t) + \sigma \mathcal{E}_t, \tag{B.1}$$

where $\mathcal{E}_t$ is an exogenous "monetary shock" process that is driven by Z_t and is mean-reverting to zero. The baseline model can be thought of as the $\sigma \to 0$ limit of this rule.

B.1 Benchmark: MSV Equilibrium

We start by expositing the benchmark "minimum state variable" (MSV) solution to the model. The sole state variable is $\mathcal{E}_t$, the monetary policy shock. An MSV solution assumes all other endogenous variables are purely a function of $\mathcal{E}_t$. Let us write the dynamics of $\mathcal{E}_t$ in the general form

$$d\mathcal{E}_t = \mu_e(\mathcal{E}_t)dt + \sigma_e(\mathcal{E}_t)dZ_t \tag{B.2}$$

for some drift and diffusion coefficients $\mu_e(\mathcal{E})$ and $\sigma_e(\mathcal{E})$.

Assume $x_t = x(\mathcal{E}_t)$ and $\pi_t = \pi(\mathcal{E}_t)$ for some functions $x(\cdot)$ and $\pi(\cdot)$. By Itô's formula, we may calculate $\mu_{x,t} = \mu_e(\mathcal{E}_t)x'(\mathcal{E}_t) + \frac{1}{2}\sigma_e(\mathcal{E}_t)^2 x''(\mathcal{E}_t)$ and $\sigma_{x,t} = \sigma_e(\mathcal{E}_t)x'(\mathcal{E}_t)$. Plugging these into (IS), along with the monetary policy rule (MP), and dropping time subscripts, we have

$$\mu_e(\mathcal{E})x'(\mathcal{E}) + \frac{\sigma_e(\mathcal{E})^2 x''(\mathcal{E})}{2} = \bar{\iota} + \Phi(x(\mathcal{E}), \pi(\mathcal{E})) + \sigma\mathcal{E} - \pi(\mathcal{E}) - \rho + \frac{(\sigma_e(\mathcal{E})x'(\mathcal{E}))^2}{2} \tag{B.3}$$

Similarly, substituting $\mu_{\pi,t} = \mu_e(\mathcal{E}_t)\pi'(\mathcal{E}_t) + \frac{1}{2}\sigma_e(\mathcal{E}_t)^2\pi''(\mathcal{E}_t)$ into (PC), we have

$$\mu_e(\mathcal{E})\pi'(\mathcal{E}) + \frac{\sigma_e(\mathcal{E})^2\pi''(\mathcal{E})}{2} = \rho\pi(\mathcal{E}) - \kappa\Big(\frac{e^{(1+\varphi)x(\mathcal{E})} - 1}{1+\varphi}\Big) \tag{B.4}$$

Equations (B.3)-(B.4) is a system of second-order ODEs that characterizes the MSV solution. We assume an MSV solution exists. As will become clear, all the equilibria unveiled in this paper constitute deviations from any MSV solution, since x and π will not be solely functions of $\mathcal{E}$.

Example B.1 (Linear monetary policy)**.** *To illustrate an example analytically, we adopt the following specification. The monetary policy shock is an Ornstein-Uhlenbeck process, i.e.,* $\mu_e(\mathcal{E}) =$

$-\zeta\mathcal{E}$ *and* $\sigma_e(\mathcal{E}) = 1$. *This is the continuous-time equivalent of an AR(1) process. In addition, the economy features a linearized Phillips curve (linear PC) and a linear monetary policy rule (linear MP). Substituting these linear functional forms into (B.3)-(B.4), we obtain*

$$-\zeta\mathcal{E}x'(\mathcal{E}) + \frac{1}{2}x''(\mathcal{E}) = \bar{\imath} + \phi_x x(\mathcal{E}) + (\phi_\pi - 1)\pi(\mathcal{E}) + \sigma\mathcal{E} - \rho + \frac{1}{2}x'(\mathcal{E})^2 \tag{B.5}$$

$$-\zeta\mathcal{E}\pi'(\mathcal{E}) + \frac{1}{2}\pi''(\mathcal{E}) = \rho\pi(\mathcal{E}) - \kappa x(\mathcal{E}) \tag{B.6}$$

Let us guess a linear equilibrium $x(\mathcal{E}) = a_x + b_x\mathcal{E}$ *and* $\pi(\mathcal{E}) = a_\pi + b_\pi\mathcal{E}$. *Plugging these into (B.5)-(B.6), we verify the form of the solution as long as*

$$b_x = -\frac{(\zeta+\rho)\sigma}{(\zeta+\rho)(\zeta+\phi_x) + \kappa(\phi_\pi - 1)} \quad \text{and} \quad b_\pi = -\frac{\kappa\sigma}{(\zeta+\rho)(\zeta+\phi_x) + \kappa(\phi_\pi - 1)}$$

and

$$a_x = -\rho\frac{\bar{\imath} - \rho + \frac{1}{2}b_x^2}{\phi_x\rho + (\phi_\pi - 1)\kappa} \quad \text{and} \quad a_\pi = -\kappa\frac{\bar{\imath} - \rho + \frac{1}{2}b_x^2}{\phi_x\rho + (\phi_\pi - 1)\kappa}$$

So long as $(\zeta+\rho)(\zeta+\phi_x) + \kappa(\phi_\pi - 1) > 0$, *the MSV solution has a particularly simple and intuitive form where* x *and* π *respond negatively to monetary tightening. If, furthermore, monetary policy picks the "natural" target rate of* $\bar{\imath} = \rho - \frac{1}{2}b_x^2$, *then* $a_x = a_\pi = 0$, *so that the time-series average values of* x_t *and* π_t *will be zero.*

Example B.2 (No monetary policy uncertainty)**.** *Next, consider the* $\sigma \to 0$ *limiting case where monetary shocks vanish, so that* $\mathcal{E}_t$ *is an extrinsic process. Consequently, the unique MSV solution has* $x_t = x^*$ *and* $\pi_t = \pi^*$ *both equal to some constants, independent of* $\mathcal{E}$. *Plugging this into (B.3)-(B.4), we obtain the system of equations*

$$0 = \bar{\imath} - \rho + \Phi(x^*, \pi^*) - \pi^*$$

$$0 = \rho\pi^* - \kappa\Big(\frac{e^{(1+\varphi)x^*} - 1}{1+\varphi}\Big)$$

If the target rate is set appropriately at $\bar{\imath} = \rho$, *then the MSV solution becomes* $x^* = \pi^* = 0$. *Thus, this* $\sigma \to 0$ *limit indeed corresponds to our benchmark model.*

B.2 Indeterminacy in Uncertainty

In the context of this model, we prove a general result about indeterminacy in the economy's sensitivity to monetary shocks. This generalization of Theorem 1 shows that our results are, at their core, about an indeterminacy in uncertainty within NK models, not about sunspot volatility *per se*. It could, instead, simply be the case that agents coordinate on a responsiveness to fundamental shocks in a manner which differs from the MSV solution. This may be important if one wonders about the practical relevance of our indeterminacy, because it means that no extrinsic sources of uncertainty are needed to induce coordination in NK models.

Theorem B.1. *Suppose prices are rigid ($\kappa \to 0$) and monetary shocks are present ($\sigma > 0$). Consider any Taylor rule (MP) with $\bar{\imath} = \rho$, increasing in x, and satisfying Assumption 1. Let $x^*(\mathcal{E})$ denote the MSV solution for this economy, and let $\sigma_{x^*}(\mathcal{E})$ denote its volatility. Then,*

(i) If $\mathcal{E}$ is bounded, there exist a continuum of non-explosive equilibria indexed by $x_0 < x^*(\mathcal{E}_0)$ with a volatility function $\sigma_x(\mathcal{E}, x)$ possessing "excess volatility" in the sense that $\sigma_x^2(\mathcal{E}, x) \geq \sigma_{x^*}^2(\mathcal{E})$. The volatility can be any mapping $\sigma_x : \mathbb{R}^2 \mapsto \mathbb{R}$ that is finite on $\{(\mathcal{E}, x) : -\infty < x \leq x^*(\mathcal{E})\}$ and satisfies suitable boundary conditions as $x \to -\infty$ and $x \to x^*(\mathcal{E})$.

(ii) If $\inf_x \Phi'(x) > 0$, then all non-explosive equilibria possessing excess volatility have $x_t \leq x^*(\mathcal{E}_t)$ forever, and hence excess volatility is recessionary.

Proof of Theorem B.1. Let $x_t^* = x^*(\mathcal{E}_t)$ denote the MSV solution. Denote demand volatility in the MSV solution by $\sigma_{x^*}(\mathcal{E}_t) = \frac{d}{d\mathcal{E}} x^*(\mathcal{E}_t)$. Note that both x^* and σ_{x^*} must be bounded, since the MSV solution requires the second derivative of x^* to exist, and since the monetary state $\mathcal{E}$ is bounded.

Proof of statement (i). First, construct the volatility function $\sigma_x(\mathcal{E}, x)$. Let $\omega > 0$ and $\delta > 0$ be arbitrary. Let $\beta > 0$ be a constant such that Assumption 1 holds. Define $\bar{x}(\mathcal{E}) := \min[0, x^*(\mathcal{E}) - \delta]$. Let the volatility function be given by

$$\sigma_x^2(\mathcal{E}, x) := \sigma_{x^*}^2(\mathcal{E}) + \begin{cases} 2\big[e^{-2\beta(x - x^*(\mathcal{E}))}\omega^2 - \Phi(x) + \Phi(x^*(\mathcal{E}))\big], & \text{if } x < \bar{x}(\mathcal{E}); \\ 0, & \text{if } x \geq \bar{x}(\mathcal{E}). \end{cases} \tag{B.7}$$

Since this only defines the squared volatility, we also pick the construction such that σ_x and σ_{x^*} always have the same sign. Since $\Phi(\cdot)$ is increasing and continuous, we have that $\sigma_x^2(\mathcal{E}, x) \geq \sigma_{x^*}^2(\mathcal{E})$, as required for there to be "excess volatility." For the same reason, notice that $\sigma_x^2(\mathcal{E}, x) > \sigma_{x^*}^2(\mathcal{E})$ for all $x < \bar{x}(\mathcal{E})$.

Next, consider the change of variables:

$$V_t = \exp[\beta(x_t - x_t^*)].$$

Given $\bar{\iota} = \rho$, the dynamics of x_t and $x_t^* = x^*(\mathcal{E}_t)$ are

$$
\begin{aligned}
dx_t &= \Big[\Phi(x_t) + \sigma\mathcal{E}_t + \frac{1}{2}\sigma_x^2(\mathcal{E}_t, x_t)\Big]dt + \sigma_x(\mathcal{E}_t, x_t)dZ_t \\
dx_t^* &= \Big[\Phi(x_t^*) + \sigma\mathcal{E}_t + \frac{1}{2}\sigma_{x^*}^2(\mathcal{E}_t)\Big]dt + \sigma_{x^*}(\mathcal{E}_t)dZ_t
\end{aligned}
$$

Therefore, the dynamics of V_t are

$$
\begin{aligned}
&dV_t = \mu_v(\mathcal{E}_t, V_t)dt + \sigma_v(\mathcal{E}_t, V_t)dZ_t, \quad \text{where} \\
&\mu_v(\mathcal{E}, v) := \beta v\Big[\Phi(\chi(\mathcal{E}, v)) - \Phi(x^*(\mathcal{E})) + \frac{\sigma_x^2(\mathcal{E}, \chi(\mathcal{E}, v)) - \sigma_{x^*}^2(\mathcal{E})}{2} + \frac{\beta}{2}\Big(\sigma_x(\mathcal{E}, \chi(\mathcal{E}, v)) - \sigma_{x^*}(\mathcal{E})\Big)^2\Big] \\
&\sigma_v(\mathcal{E}, v) := \beta v\Big(\sigma_x(\mathcal{E}, \chi(\mathcal{E}, v)) - \sigma_{x^*}(\mathcal{E})\Big)
\end{aligned}
$$

where $\chi(\mathcal{E}, v) := x^*(\mathcal{E}) + \beta^{-1}\log(v)$. The plan is to show that V_t satisfies the hypotheses of Lemma A.1. Doing so will show that $\liminf_t \mathbb{E}[\log(V_t)] > -\infty$, which ensures $\liminf_t \mathbb{E}[x_t] > -\infty$ (i.e., Condition 1 holds) because $x^*(\mathcal{E}_t)$ is bounded.

We obviously have that $V_t \geq 0$ forever. Defining $\bar{v}(\mathcal{E}) := e^{-\beta\max[x^*(\mathcal{E}),\delta]}$, let us show that $V_t \leq \bar{v}(\mathcal{E}_t)$ forever if V_0 starts below $\bar{v}(\mathcal{E}_0)$. For x_t such that $\bar{x}(\mathcal{E}_t) \leq x_t < x^*(\mathcal{E}_t)$, or equivalently $\bar{v}(\mathcal{E}_t) \leq V_t < 1$, we have that $\sigma_v = 0$, and so $dV_t = [\Phi(x_t) - \Phi(x_t^*)]dt < 0$, by the fact that Φ is increasing and that $x < x^*(\mathcal{E})$. Therefore, V_t enters the region below $\bar{v}(\mathcal{E}_t)$ in finite time when starting from any point $V_0 < 1$. And if $V_t \leq \bar{v}(\mathcal{E}_t)$, it can never exit this region. So V_t satisfies hypothesis (H1) of Lemma A.1.

It is easy to see that μ_v and σ_v^2 are continuous on $\{v < \bar{v}(\mathcal{E})\}$ and are only potentially infinite when $v = 0$. So V_t satisfies hypothesis (H2) of Lemma A.1.

Next, we showed earlier that $\sigma_x^2(\mathcal{E}, x) > \sigma_{x^*}(\mathcal{E})$ for all $x < \bar{x}(\mathcal{E})$, and so $\sigma_v^2(\mathcal{E}, v) > 0$ for all $v < \bar{v}(\mathcal{E})$. This verifies that V_t satisfies assumption (H3) of Lemma A.1.

Next, analyze the diffusion σ_v^2 asymptotically as $v \to 0$. We have that, for every $\mathcal{E}$ in its bounded space,

$$
\begin{aligned}
\lim_{v\to 0}\sigma_v(\mathcal{E}, v)^2 &= \beta^2 \lim_{x\to-\infty} e^{2\beta(x - x^*(\mathcal{E}))}\sigma_x(\mathcal{E}, x)^2 \\
&= \beta^2 \lim_{x\to-\infty} e^{2\beta(x - x^*(\mathcal{E}))}[\sigma_x^*(\mathcal{E})^2 - 2\Phi(x) + 2\Phi(x^*(\mathcal{E}))] + 2\beta^2\omega^2 \\
&= 2\beta^2\omega^2 - 2\beta^2 \lim_{x\to-\infty} e^{2\beta(x - x^*(\mathcal{E}))}\Phi(x) \\
&= 2\beta^2\omega^2.
\end{aligned}
$$

The first line uses the fact that x^* and σ_{x^*} are bounded. The second line uses the expression for σ_x^2 in (B.7). The third line uses again that x^* and σ_x^* bounded. The fourth line uses Assumption 1. Hence, we have proved that σ_v^2 converges to a finite constant as $v \to 0$, for every $\mathcal{E}$, implying V_t satisfies hypothesis (H4) of Lemma A.1.

We next do a similar limiting analysis for the drift. Plugging (B.7) into the expression for μ_v, we see that $v\mu_v(\mathcal{E}, v) = \beta\omega^2 + \frac{1}{2}\sigma_v(\mathcal{E}, v)^2$. Hence, we use the previous limiting results to show that

$$\lim_{v \to 0} v\mu_v(\mathcal{E}, v) = \beta\omega^2 + \frac{1}{2}\lim_{v \to 0} \sigma_v(\mathcal{E}, v)^2 = \beta(1+\beta)\omega^2,$$

for every $\mathcal{E}$. Combining the limits for the diffusion and the drift, we have that

$$\theta := \lim_{v \to 0} \frac{2v\mu_v(\mathcal{E}, v)}{\sigma_v(\mathcal{E}, v)^2} = \frac{1+\beta}{\beta} > 1$$

for every $\mathcal{E}$. Consequently, hypothesis (H5) of Lemma A.1 holds.

This verifies all the hypotheses of Lemma A.1, proving that the construction is a valid non-explosive equilibrium.

Finally, we prove the claim that any volatility function is valid if it satisfies suitable boundary conditions. Instead of the σ_x^2 function in (B.7), consider any alternative function $\tilde{\sigma}_x^2$, which (a) coincides with σ_x^2 for $x \notin (-K, \bar{x}(\mathcal{E}) - K^{-1})$ for K arbitrarily large; and (b) is finite and exceeds $\sigma_{x^*}^2$ on $x \in (-K, \bar{x}(\mathcal{E}) - K^{-1})$. By inspection, the entire proof above remains valid. This proves statement (i) of the proposition.

Proof of statement (ii). We now prove that all equilibria with "excess volatility" are recessionary. Consider a general rule $\Phi(x)$ satisfying $\phi_x := \inf_x \Phi'(x) > 0$. Let $\sigma_{x,t}^2$ be any volatility process such that $\sigma_{x,t}^2 \geq \sigma_x^*(\mathcal{E}_t)^2$. We will study the "gap process" $\Delta_t := x_t - x_t^*$. Suppose, leading to contradiction that $\Delta_0 > 0$ was part of a non-explosive equilibrium.

The dynamics of Δ_t are

$$d\Delta_t = \left[\Phi(\Delta_t + x_t^*) - \Phi(x_t^*) + \frac{1}{2}\Big(\sigma_{x,t}^2 - \sigma_x^*(\mathcal{E}_t)^2\Big)\right]dt + \Big(\sigma_{x,t} - \sigma_x^*(\mathcal{E}_t)\Big)dZ_t, \quad \Delta_0 > 0.$$

Consider the alternative process $\tilde{\Delta}_t$ which drops the excess variance from the drift:

$$d\tilde{\Delta}_t = \left[\Phi(\tilde{\Delta}_t + x_t^*) - \Phi(x_t^*)\right]dt + \Big(\sigma_{x,t} - \sigma_x^*(\mathcal{E}_t)\Big)dZ_t, \quad \tilde{\Delta}_0 = \Delta_0 > 0.$$

Define the stopping time

$$\tau_0 := \inf\{t > 0 : \tilde{\Delta}_t = 0\},$$

and put $T_0 := T \wedge \tau_0$. Then,

$$\mathbb{E}_0[e^{-\phi_x T_0}\tilde{\Delta}_{T_0}] = \tilde{\Delta}_0 + \mathbb{E}_0\Big[\int_0^{T_0} e^{-\phi_x t}\Big(\Phi(\tilde{\Delta}_t + x_t^*) - \Phi(x_t^*) - \phi_x\tilde{\Delta}_t\Big)dt\Big] \geq \tilde{\Delta}_0 = \Delta_0,$$

since $\Phi(\tilde{\Delta} + x^*) - \Phi(x^*) \geq \phi_x\tilde{\Delta}$ for all $\tilde{\Delta} \geq 0$ (recall that $\phi_x > 0$ is the minimal slope of the general rule $\Phi(x)$). The left-hand-side can be written $e^{-\phi_x T}\mathbb{E}_0[\tilde{\Delta}_T \mathbf{1}_{\{T<\tau_0\}}]$, since $\tilde{\Delta}_{\tau_0} = 0$. Thus,

$$\mathbb{E}_0[\tilde{\Delta}_T \mathbf{1}_{\{T<\tau_0\}}] \geq e^{\phi_x T}\Delta_0,$$

which by taking $T \to \infty$ proves that $\limsup_{T\to\infty} \mathbb{E}_0[\tilde{\Delta}_T] = +\infty$ with positive probability. This implies that $\limsup_{T\to\infty} \mathbb{E}_0[\Delta_T] = +\infty$, since standard diffusion comparison theorems imply that $\Delta_T \geq \tilde{\Delta}_T$ almost-surely. Finally, this then implies that $\limsup_{T\to\infty} \mathbb{E}_0[x_T] = +\infty$, in violation of Condition 1, which proves the result. $\square$

B.3 Numerical examples

While multiplicity naturally challenges the robustness of any model's predictions, this issue is often downplayed when only one equilibrium yields empirically plausible outcomes. We illustrate that this is not the case using two numerical examples. First, we use our baseline framework, which featured only sunspot uncertainty, to analyze responses to a negative unanticipated demand shock (which could be, for example, a monetary policy shock). This exercise establishes an indeterminacy in responses to demand shocks, many of which can be reasonable. Second, we study the extended version of the model—described in this appendix—driven by monetary policy shocks rather than sunspot shocks. We provide an example to illustrate the indeterminacy in the economy's response to those monetary policy shocks. Thus, both exercises demonstrate an indeterminacy in responsiveness to monetary policy, but they are different in the sense that the first exercise is a fully unanticipated shock, whereas households are aware of these shocks in the second exercise.

Unanticipated demand shock. Consider the rigid-price limit of our baseline model and assume that the output gap is initially zero. Let a negative unanticipated demand shock affect the IS curve according to

$$\mu_{x,t} = \Phi(x_t) + \tfrac{1}{2}\sigma_{x,t}^2 + \sigma_\epsilon \epsilon_t, \tag{B.8}$$

where $\epsilon_t = \mathbf{1}_{\{0 \le t < \tau\}}$. The left panel of Figure B.1 summarizes the response of the output gap under two scenarios. The solid blue line depicts first scenario: the conventional deterministic response (i.e., in the equilibrium with $\sigma_{x,t} = 0$). Output contracts on impact and then reverts, returning to potential at $t = \tau$. This deterministic impulse response is uniquely pinned down by the non-explosion condition.

Alternatively, an equally valid theoretical prediction is that the economy is propelled into a volatile regime. Specifically, we consider a case where, for $t \geq \tau$, the volatility function $\sigma_x^2(x,t)$ equals the stationary one from Example 3, illustrated in Figure 2.[19] In this case, the initial response x_0 is not fully disciplined by equilibrium conditions, and any $x_0 \leq x_0^{det}$ is admissible, where x_0^{det} denotes the deterministic counterpart. For this exercise, we assume the impact response is $x_0 = \mathbb{E}[x]$, where the expectation is taken with respect to the ergodic distribution associated to this volatile regime.

The average response (dotted–dashed green line in Figure B.1) is both larger and more persistent than its deterministic counterpart. Moreover, this average masks three qualitatively distinct paths, depending on the shocks realized during the depressed-demand phase ($t < \tau$). If shocks are sufficiently favorable, the economy escapes the volatile regime and recovers as fast as in the deterministic case (dashed orange line). Moderately favorable shocks allow a temporary escape, followed by a gradual return to the volatile regime (dotted red line). When shocks are unfavorable, the economy remains permanently volatile. These alternative paths are not only theoretically valid, but also reasonable as descriptions of how an economy might respond to a depressed-demand period.

Aggregate uncertainty driven by monetary policy. Now, we illustrate a numerical example in the economy with anticipated monetary policy shocks. The equilibrium responsiveness to monetary policy is the volatility function $\sigma_x^2(\mathcal{E}, x)$, which recall is indeterminate. The lack of discipline over $\sigma_x^2(\mathcal{E}, x)$ implies that qualitatively distinct patterns can emerge in the economy's response to monetary policy shocks. For instance, there exist equilibria in which monetary policy has a stronger impact on output as the policy stance becomes tighter ($\partial \sigma_x^2 / \partial \mathcal{E} > 0$), and others in which the opposite pattern holds ($\partial \sigma_x^2 / \partial \mathcal{E} < 0$). Similarly, the strength of monetary policy can be procyclical in some equilibria ($\partial \sigma_x^2 / \partial x > 0$) and countercyclical in others ($\partial \sigma_x^2 / \partial x < 0$). The right panel of Figure B.1 illustrates one example in which monetary policy becomes more effective as its stance

[19] For $t < \tau$, the volatility function $\sigma_x^2(x,t)$ is also given by the same stationary function, with the exception that we additionally impose $\sigma_x^2(x,t) = 0$ whenever $x_t \geq x_t^{det}$, where x_t^{det} denotes the deterministic impulse response. This restriction prevents the possibility of positive explosions for certain realizations of uncertainty.

tightens ($\partial\sigma_x^2/\partial\mathcal{E} > 0$) or as the output gap deviates far below its MSV-equilibrium level ($\partial\sigma_x^2/\partial x < 0$).

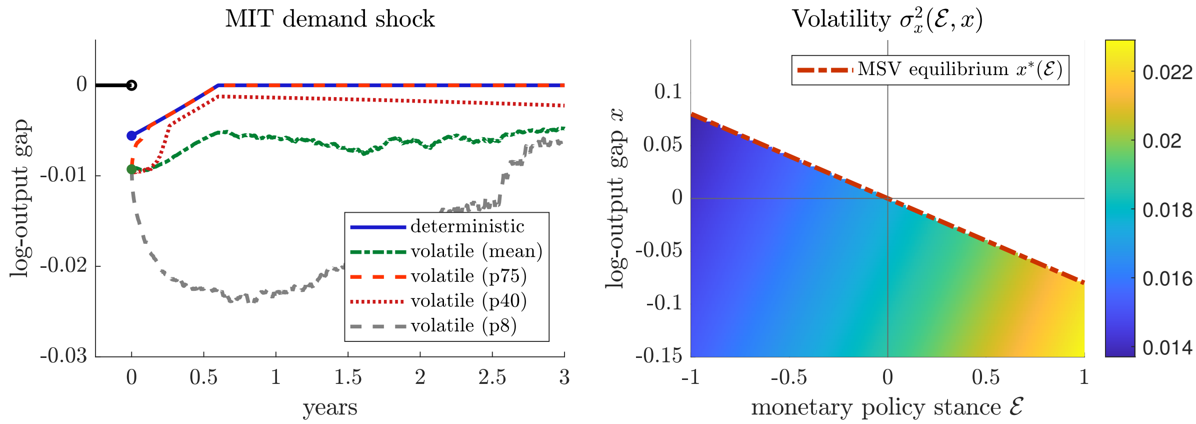

Figure B.1: Responses to fundamental shocks. Left panel: Impulse responses to a negative unanticipated demand shock. The solid blue line shows the deterministic equilibrium response, while the other lines report the mean and percentile statistics from simulations under a volatile equilibrium, conditional on $x_0 = \mathbb{E}[x]$. Model parameters and the stationary volatility function $\sigma_x^2(x)$ follow Example 3 and Figure 2. Shock parameters are $\sigma_\epsilon = 0.01$ and $\tau = 0.6$. We simulate 1,000 response paths. Right panel: Sensitivity of the output gap to an expected monetary policy shock (σ_x^2). Model details and the MSV solution are described in Example B.1. The volatility function $\sigma_x^2(\mathcal{E}, x)$ is an arbitrary function increasing in $\mathcal{E}$ and decreasing in x that vanishes as $x \to x^*(\mathcal{E})$ and $x \to -\infty$, as required by Theorem B.1. Parameters: $\rho = 0.04$, $\phi_x = 0.125$, $\sigma = 0.01$, $\zeta = 0.5$, and $\kappa = 0$ (rigid-price limit).

C Sunspot Equilibria with Inflation

In the sunspot equilibrium constructions of Theorem 1, we work in the rigid price limit ($\kappa \to 0$) for analytical tractability. Here, we provide one class of equilibrium constructions where prices are partially flexible, so inflation is present. For this example, we will assume the linearly approximated Phillips curve (linear PC) and utilize a linear Taylor rule (linear MP) that is sufficiently aggressive. In particular, we will assume $\phi_x > 0$ and $\phi_\pi > 1$, so the Taylor principle is satisfied and deterministic multiplicities (as well as linearized stochastic multiplicities) are ruled out.

To maintain tractability, we assume a type of Markovian equilibrium where inflation is a function of the output gap. In particular, suppose $\pi_t = \pi(x_t)$ for some function $\pi(\cdot)$, to be determined. Obviously, this must be supported by a volatility process σ_x which is solely a function of x. These restrictions imply only one dimension of multiplicity, but the set of equilibria can still be relatively rich. By Theorem 1, part (ii), we need only consider sunspot equilibria with $x \le 0$. A numerical illustration of the equilibrium constructed in the following proposition is contained in the text (Figure 3).

Theorem C.1. *Consider an economy with the linearly approximated Phillips curve (linear PC) and a linear Taylor rule (linear MP) with $\phi_x > 0$, $\phi_\pi > 1$, $\bar{\iota} = \rho$, and without monetary shocks ($\sigma \to 0$). Then, there exists a family of non-explosive sunspot equilibria, indexed by constants $\bar{x} < 0$ and $\bar{\pi}$ satisfying $0 < \bar{\pi} < \kappa/\rho$. In particular, define the functions*

$$\pi(x) := \begin{cases} \bar{\pi}x, & \text{if } x \leq \bar{x} \\ f(x), & \text{if } x > \bar{x} \end{cases} \tag{C.1}$$

$$\sigma_x^2(x) := \begin{cases} 2\Big(\rho - \kappa/\bar{\pi} - \phi_x - (\phi_\pi - 1)\bar{\pi}\Big)x, & \text{if } x \leq \bar{x} \\ \bar{\sigma}^2 x^2, & \text{if } x > \bar{x}, \end{cases} \tag{C.2}$$

where $\bar{\sigma}^2$ is such that $\sigma_x(x)^2$ is continuous at $\bar{x}$, and where the function f solves the ODE

$$\rho f(x) - \kappa x = \Big[\bar{\iota} + \phi_x x + (\phi_\pi - 1)f(x) - \rho + \frac{1}{2}\bar{\sigma}^2 x^2\Big] f'(x) + \frac{1}{2}\bar{\sigma}^2 x^2 f''(x) \tag{C.3}$$

on $x \in (\bar{x}, 0)$, subject to the boundary conditions $f(\bar{x}) = \bar{\pi}\bar{x}$ and $f(0) < 0$, assuming such solution exists. Then, a non-explosive sunspot equilibrium exists, which is stationary and ergodic on $\{x_t < 0\}$, in which inflation is given by $\pi_t = \pi(x_t)$ and volatility by $\sigma_{x,t}^2 = \sigma_x^2(x_t)$.

Proof of Theorem C.1. Let us conjecture an equilibrium of the form described in the proposition. By Itô's formula, we may derive the dynamics of π, which when combined with the linear Phillips curve (linear PC) yields the equation

$$\rho\pi(x) - \kappa x = \Big[\bar{\iota} + \phi_x x + (\phi_\pi - 1)\pi(x) - \rho + \frac{1}{2}\sigma_x^2\Big]\pi'(x) + \frac{1}{2}\sigma_x^2 \pi''(x) \tag{C.4}$$

First, consider the lower region $\{x < \bar{x}\}$. Plug in the guess $\pi(x) = \bar{\pi}x$, the target rate $\bar{\iota} = \rho$, and then rearrange the equation for σ_x^2 to obtain

$$\frac{1}{2}\sigma_x^2 = \Big(\rho - \frac{\kappa}{\bar{\pi}} - \phi_x - (\phi_\pi - 1)\bar{\pi}\Big)x$$

This equation clearly coincides with (C.2), provided the right-hand-side is non-negative. Notice that the right-hand-side is in fact non-negative precisely when $x < 0$, by the restrictions $\phi_x > 0$, $\phi_\pi > 1$, and the fact that $0 < \bar{\pi} < \kappa/\rho$.

Next, consider the upper region $\{x > \bar{x}\}$. Substitute the volatility function $\sigma_x^2 = \bar{\sigma}^2 x^2$ from (C.2) into the Phillips curve (C.4) to obtain the ODE (C.3). As stated, we assume a solution f exists to this ODE, subject to the boundary condition $f(\bar{x}) = \bar{\pi}\bar{x}$. If so, then the resulting inflation function $\pi(x)$ is continuous at $\bar{x}$ (i.e., inflation does not jump). In that case, the Phillips curve holds for almost all x, the exception being $x = \bar{x}$, where

$d\pi_t$ can include a local time. However, so long as the resulting stationary distribution places no point mass at $\bar{x}$, then firm optimality still holds, because firms' FOCs hold for almost all times. Given the continuity of the volatility function $\sigma_x^2(x)$, and given the continuity of $\pi(x)$, hence $\mu_x(x)$, the resulting stationary distribution (if it exists) cannot have a point mass at $\bar{x}$.

Thus, in the conjectured equilibrium, which presumably has $x_t < 0$ forever (to be verified), we will have $\sigma_x^2(x_t)$ well-defined and positive, and $\pi(x_t)$ satisfying the Phillips curve at almost all times. This proves that the equilibrium is valid, subject to the non-explosion Condition 1. The rest of the proof is dedicated to verifying this non-explosion.

First, let us prove that x_t never visits the upper boundary $\{x = 0\}$. For $x_t \in (\bar{x}, 0)$, the output gap dynamics are

$$\begin{aligned} \mu_x &= \phi_x x + (\phi_\pi - 1) f(x) + \frac{1}{2}\bar{\sigma}^2 x^2 \\ \frac{1}{2}\sigma_x^2 &= \frac{1}{2}\bar{\sigma}^2 x^2 \end{aligned}$$

Study $\tilde{x} := -x$, which has $\tilde{x}\mu_{\tilde{x}} = x\mu_x$ and $\sigma_{\tilde{x}}^2 = \sigma_x^2$. Both the drift and diffusion vanish as $x \to 0$, but dividing them we obtain

$$\theta_0 := \lim_{\tilde{x} \searrow 0} \frac{2\tilde{x}\mu_{\tilde{x}}}{\sigma_{\tilde{x}}^2} = \lim_{x \nearrow 0} \frac{2x\mu_x}{\sigma_x^2} = \frac{2\phi_x}{\bar{\sigma}^2} + \frac{2(\phi_\pi - 1)}{\bar{\sigma}^2} \lim_{x \nearrow 0} \frac{f(x)}{x}$$

Given the assumption that $f(0) < 0$, we have $\lim_{x \nearrow 0} \frac{f(x)}{x} = +\infty$, and so $\theta_0 = +\infty$. Applying an analogous logic to Lemma A.1, we find that $x_t < 0$ for all t almost-surely.

Next, we prove that $x_t > -\infty$ almost-surely. To do this, compute the dynamics of $y_t = e^{x_t}$ on $\{x_t < \bar{x}\}$ by Itô's formula as

$$\begin{aligned} \mu_y &= y\mu_x + \frac{1}{2} y \sigma_x^2 = \Big(2\Big(\rho - \frac{\kappa}{\bar{\pi}}\Big) - \phi_x - (\phi_\pi - 1)\bar{\pi}\Big) y \log(y) \\ \frac{1}{2}\sigma_y^2 &= \frac{1}{2} y^2 \sigma_x^2 = \Big(\rho - \frac{\kappa}{\bar{\pi}} - \phi_x - (\phi_\pi - 1)\bar{\pi}\Big) y^2 \log(y) \end{aligned}$$

Notice that both the drift and diffusion vanish as $y \to 0$ (i.e., as $x \to -\infty$). However, dividing these results in

$$\theta_{-\infty} := \frac{2y\mu_y}{\sigma_y^2} = 1 + \frac{\rho - \kappa/\bar{\pi}}{\rho - \kappa/\bar{\pi} - \phi_x - (\phi_\pi - 1)\bar{\pi}}$$

Given the parameter assumptions made, $\theta > 1$. Furthermore, the other hypotheses of Lemma A.1 all hold—σ_y^2 is strictly positive, bounded, and vanishes slower than quadratically as $y \to 0$. Consequently, $x_t = \log(y_t)$ satisfies $\liminf_{T\to\infty} \mathbb{E}[x_T] > -\infty$. This verifies all the parts of Condition 1. □

Remark C.1. *Theorem C.1 presumes the existence of a solution f to the ODE (C.3) that satisfies $f(\bar{x}) = \bar{\pi}\bar{x}$ and $f(0) < 0$. While the right boundary condition may seem unusual, there is conceptually no issue, as we now show. Taking $x \nearrow 0$ in the ODE (C.3), and using $\bar{\iota} = \rho$, we obtain $\rho f(0) = (\phi_\pi - 1) f(0) f'(0-)$. If $f(0) < 0$, we must have that $f'(0-) = \rho/(\phi_\pi - 1)$. Consequently, it is equivalent to think of solving (C.3) subject to $f(\bar{x}) = \bar{\pi}\bar{x}$ and $f'(0) = \rho/(\phi_\pi - 1)$, which is a more conventional situation with one Dirichlet and one Neumann boundary condition.*

D ZLB and Optimal Discretionary Monetary Policy

In the main text, we have discussed how a zero lower bound (ZLB) constrains monetary policy. Here, we demonstrate that the optimal discretionary monetary policy will in fact hit the ZLB in recessions, thus proving that our sunspot equilibria are robust to this type of monetary policy as well. To simplify the exposition, we work exclusively in the rigid-price limit $\kappa \to 0$, and so inflation is zero ($\pi_t = 0$) and the nominal rate is equal to the real rate ($\iota_t = r_t$).

First, let us describe the resulting policy. Imagine monetary policy aims to achieve the flexible-price allocation whenever possible, but they are subject to the ZLB. If so, monetary authorities set the nominal rate (hence the real rate r_t) to implement $x_t = 0$ whenever possible, subject to $r_t \geq 0$. This is the same idea behind the policy in Caballero and Simsek (2020), who consider a version of the NK model with risky capital. Under this policy description, zero output gap prevails whenever the real rate is positive, and a negative output gap must arise at the ZLB (because recall raising the interest rate will lower output):

$$0 = \min[-x_t, r_t]. \tag{D.1}$$

In Lemma D.1 below, we show that within the class of equilibria we study, (D.1) is the outcome of optimal discretionary monetary policy (i.e., monetary policy without commitment to future policies).

Lemma D.1. *Optimal discretionary monetary policy—which maximizes (2) subject to $r_t \geq 0$, optimal household and firm decisions, and its own future decisions—implements (D.1).*

Proof of Lemma D.1. Since there is no upper bound on interest rates, the central bank can always threaten r_t high enough to ensure that $x_t \leq 0$. Since positive output gaps are undesirable, they will implement this. Then, we can restate the problem as: optimal discretionary monetary policy seeks to pick a r_t to maximize (2), subject to (IS), $x_t \leq 0$, the ZLB $r_t \geq 0$, and subject to its own future decisions.

We will discretize the problem to time intervals of length Δ and later take $\Delta \to 0$. Noting that $C_t = e^{x_t} Y^*$, the time-t household utility is proportional to

$$\begin{aligned}\mathbb{E}_t\Big[\int_0^\infty \rho e^{-\rho s} x_{t+s} ds\Big] &\approx \rho x_t \Delta + \mathbb{E}_t\Big[\int_\Delta^\infty \rho e^{-\rho s} x_{t+s} ds\Big] \\ &\approx -\rho\Delta \mathbb{E}_t[x_{t+\Delta} - x_t] + \underbrace{\mathbb{E}_t\Big[\int_\Delta^\infty \rho e^{-\rho s} x_{t+s} ds\Big] + \rho\Delta\mathbb{E}_t[x_{t+\Delta}]}_{\text{taken as given by discretionary central bank}}.\end{aligned}$$

The term with brackets underneath is taken as given by the time-t discretionary central bank, because it involves expectations of future variables that the future central bank can influence.

Thus, taking $\Delta \to 0$, the time-t central bank solves

$$\min_{r_t \geq 0} \mathbb{E}_t[dx_t]$$

subject to the constraints

$$\begin{aligned}&r_t = \rho + \mu_{x,t} - \frac{1}{2}\sigma_{x,t}^2 \\ &x_t \leq 0 \quad \text{and if} \quad x_t = 0 \quad \text{then} \quad \mu_{x,t} = \sigma_{x,t} = 0.\end{aligned}$$

Note that $\sigma_{x,t}$ is independent of policy when $x_t < 0$. There are two cases. If $x_t = 0$, then the constraints imply that $r_t = \rho$. If $x_t < 0$, we may substitute the dynamics of x_t (replacing μ_x from the first constraint) to re-write the problem as

$$\min_{r_t \geq 0}[r_t - \rho + \frac{1}{2}\sigma_{x,t}^2].$$

Since σ_x is taken as given, the optimal solution is $r_t = 0$. Thus, the discretionary central bank optimally sets

$$r_t = \rho \mathbf{1}_{\{x_t = 0\}}.$$

In other words, the complementary slackness condition $x_t r_t = 0$ holds, which together with $r_t \geq 0$ and $x_t \leq 0$ implies (D.1). □

E Extension with Idiosyncratic Risk

We flesh out some details of the idiosyncratic risk extension discussed in Section 2.4.3. There is a continuum of agents of two types, labeled A and B. Recall the Brownian motion shock $d\tilde{Z}_t$ directly redistributes consumption across types. When $d\tilde{Z}_t > 0$, an amount $Y_t \, \nu(\omega_t, x_t) \, d\tilde{Z}_t$ of consumption is transferred from type-B to type-A agents; when $d\tilde{Z}_t < 0$ the transfer operates in the opposite direction. Here, C_t denotes aggregate output or consumption, while $\nu(\omega_t, x_t)$ is an exogenous function of the consumption share of type-A agents ω_t and the output gap x_t. We consider the following functional form

$$\nu(\omega_t, x_t) = \sqrt{\omega_t(1-\omega_t)}\nu(x_t) \tag{E.1}$$

where $x_t = \log(Y_t/Y^*)$ denotes output gap. Consumption goods received through this redistribution must be consumed immediately. Within each type, the charge or transfer is allocated proportionally to the individual agents. Finally, there are no assets that allow agents to insure against $\tilde{Z}_t$ uncertainty.

We also allow a shift in the monetary policy target, described explicitly later, to keep consistency of the monetary policy rule with full stabilization, i.e., $x_t = \pi_t = 0$. Apart from this, the rest of the model is identical to the baseline specification, including the presence of the non-fundamental Brownian motion dZ_t.

Equilibrium characterization. The model aggregates within type, so we can work with a representative agent for each type. Write agent type i's consumption dynamics

$$\frac{dc_{i,t}}{c_{i,t}} = \mu_{c_i,t}dt + \sigma_{c_i,t}dZ_t + \left(\mathbf{1}_{\{i=A\}} - \mathbf{1}_{\{i=B\}}\right)\frac{\nu(\omega_t, x_t)Y_t}{c_{i,t}}d\tilde{Z}_t. \tag{E.2}$$

The transfers due to $\tilde{Z}$ are exogenous, and the agent is only allowed to optimally pick μ_{c_i} and σ_{c_i}. To solve this problem, we must consider a different "shadow SDF" for each type of agent given that markets are incomplete with respect to $\tilde{Z}_t$. The shadow SDF for agent type i follows

$$\frac{dM_{i,t}}{M_{i,t}} = -r_t dt - h_t dZ_t + \nu_{m_i,t} d\tilde{Z}_t \tag{E.3}$$

where r_t is the risk-free real interest rate, h_t is the risk price associated to Z_t, and $\nu_{m_i,t}$ is a shadow risk price associated to the transfer shock $\tilde{Z}_t$.

Given logarithmic preferences, the FOC is $M_{i,t} = \lambda e^{-\rho t} c_{i,t}^{-1}$. By Itô's lemma,

$$\mu_{c_i,t} = r_t - \rho + h_t^2 + \nu(\omega_t, x_t)^2 \tag{E.4}$$

$$\sigma_{c_i,t} = h_t \tag{E.5}$$

$$\Big(\mathbf{1}_{\{i=A\}} - \mathbf{1}_{\{i=B\}}\Big)\nu(\omega_t, x_t)(Y_t/c_{t,i}) = -\nu_{m_i,t} \tag{E.6}$$

Market clearing for goods is given by

$$Y_t = c_{A,t} + c_{B,t} \tag{E.7}$$

and so

$$\frac{dY_t}{Y_t} = \omega_t \frac{dc_{A,t}}{c_{A,t}} + (1-\omega_t)\frac{dc_{B,t}}{c_{B,t}} \tag{E.8}$$

The dynamics of Y_t take the generic form

$$\frac{dY_t}{Y_t} = \mu_{y,t}dt + \sigma_{y,t}dZ_t t + \nu_{y,t}d\tilde{Z}_t \tag{E.9}$$

for some drift and diffusion coefficients. Substituting in the optimal consumption dynamics derived above into (E.8), we obtain

$$\begin{aligned}
\mu_{y,t} &= r_t - \rho + h_t^2 + \omega_t\Big(\frac{\nu(\omega_t, x_t)}{\omega_t}\Big)^2 + (1-\omega_t)\Big(\frac{\nu(\omega_t, x_t)}{1-\omega_t}\Big)^2 \\
&= r_t - \rho + \sigma_{y,t}^2 + \nu(x_t)^2 \\
\sigma_{y,t} &= h_t \\
\nu_{y,t} &= \nu(\omega_t, x_t) - \nu(\omega_t, x_t) = 0
\end{aligned}$$

which verifies that the transfer shock does not affect aggregate dynamics directly. Given $x_t = \log(Y_t/Y^*)$, we can write the dynamic IS curve as

$$\mu_{x,t} = r_t - \rho + \nu(x_t)^2 + \frac{1}{2}\sigma_{x,t}^2 \tag{E.10}$$

We consider the following monetary policy rule

$$\iota_t = \rho - \nu(0)^2 + \Phi(x_t, \pi_t) \tag{E.11}$$

which has the target rate $\bar{\iota} = \rho - \nu(0)^2$. Then, using (E.11) inside (E.10), we have the final result

$$\mu_{x,t} = \Phi(x_t, \pi_t) - \pi_t + \nu(x_t)^2 - \nu(0)^2 + \frac{1}{2}\sigma_{x,t}^2 \tag{E.12}$$

This proves equation (24) in the main text. Together with the Phillips curve, equation (E.12) characterizes the equilibrium.

F Taylor Principle in the Non-Stochastic Model

To distinguish our main results, we also review the standard *deterministic* multiplicity in NK models and how aggressive monetary policy rules can eliminate this indeterminacy. For these results, we start by using a linear Taylor rule and the linearized Phillips curve (as in the literature). After that, we generalize these existing results to nonlinear Phillips curves and nonlinear Taylor rules as well.

F.1 Linearized Phillips Curve

We study a deterministic setting. In this setting, recall that the MSV equilibrium is $x = \pi = 0$ forever, provided monetary policy sets the target rate at $\bar{\imath} = \rho$.

Can there exist other deterministic equilibria? As is well known, the answer to this question hinges on the stability/instability properties of the equilibrium dynamical system for (x_t, π_t). We will review this analysis here. First, we specialize to the policy rule (linear MP) with the target rate $\bar{\imath} = \rho$. Combining (linear MP) with (IS), the dynamics of x_t are given by

$$\dot{x}_t = \phi_x x_t + (\phi_\pi - 1)\pi_t. \tag{F.1}$$

The IS curve is linear in a deterministic equilibrium with a linear Taylor rule. We also consider here the linear Philips curve (linear PC) as in most of the existing literature.

The typical determinacy analysis picks an aggressive Taylor rule that renders the above system unstable. The system (F.1) and (linear PC) can be written in matrix form as

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \mathcal{A}_{\text{linear}} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}, \quad \text{where} \quad \mathcal{A}_{\text{linear}} := \begin{bmatrix} \phi_x & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}. \tag{F.2}$$

The eigenvalues of $\mathcal{A}_{\text{linear}}$ are both strictly positive, and the system unstable, if $\phi_x > -\rho$ and $\phi_\pi > 1 - \rho\phi_x/\kappa$. This is the continuous-time version of the eigenvalue conditions in Blanchard and Kahn (1980).

By contrast, if either parameter condition is violated, then the system has one or two stable eigenvalues. In such case, there are a continuum of non-explosive equilibria, which one can index by the initial conditions (x_0, π_0). As the explicit parameter conditions make clear, instability occurs when monetary policy is sufficiently aggressive (i.e., "active") in responding to the output gap and inflation, whereas stability occurs when the response function is less aggressive (i.e., "passive").

Proposition F.1. *Consider the linearized Phillips curve (linear PC) and monetary policy rule (linear MP) with $\bar{\imath} = \rho$. Restrict attention to deterministic equilibria ($\sigma_x = \sigma_\pi = 0$). If $\phi_x > -\rho$ and $\phi_\pi > 1 - \rho\phi_x/\kappa$, the only deterministic non-explosive equilibrium is $(x_t, \pi_t) = (0, 0)$ forever—the MSV solution. If either condition is violated, then a continuum of deterministic non-explosive equilibria exist.*

Proof of Proposition F.1. This result is a special case of the proof of Proposition I.1 in Appendix I.3. The eigenvalues (λ_1, λ_2) of $\mathcal{A}_{\text{linear}}$ are the key. Asymptotic instability of this system is guaranteed if $\text{Re}(\lambda_1), \text{Re}(\lambda_2) > 0$. This holds if and only if $\det(\mathcal{A}_{\text{linear}}) > 0$ and $\text{tr}(\mathcal{A}_{\text{linear}}) > 0$, which is equivalent to $\phi_x > -\rho$ and $\phi_\pi > 1 - \rho\phi_x/\kappa$. $\square$

Remark F.1 (Nonlinear Phillips curve)**.** *We have used the linearized Phillips curve here for simplicity and exposition. We analyze the nonlinear Phillips curve in Appendix F.2, and the conclusion is similar to Proposition F.1 but the proof is more complicated.*

Remark F.2 (Explosive equilibria)**.** *A key clause is the requirement that equilibria satisfy Condition 1, ruling out explosions. What if asymptotic explosions were permitted, while finite-time explosions were ruled out? It turns out that, by adopting an aggressive nonlinear Taylor rule, monetary policy can force an explosion in finite-time, and hence select a unique equilibrium. We analyze this situation in Appendix F.3. In that sense, the spirit of Proposition F.1 is preserved even under broader notions of equilibrium.*

F.2 Nonlinear Phillips Curve

This section briefly explores the stability properties of the nonlinear Phillips curve, in contrast to the linearized version used oftentimes in the paper. We will do this only in the context of deterministic equilibria, for simplicity. For convenience, we repeat this nonlinear equation here:

$$\dot{\pi}_t = \rho \pi_t - \kappa \Big(\frac{e^{(1+\varphi)x_t} - 1}{1+\varphi} \Big). \tag{F.3}$$

We also repeat the IS curve after substituting the linear Taylor rule with target rate $\bar{\imath} = \rho$:

$$\dot{x}_t = \phi_x x_t + (\phi_\pi - 1)\pi_t. \tag{F.4}$$

A deterministic non-explosive equilibrium in this environment is (x_t, π_t) that satisfy (F.3)-(F.4) and asymptotic non-explosion Condition 1.

The nonlinearity of the Phillips curve does not change the basic determinacy result of Proposition F.1, as we show next (although our proof requires stronger assumptions on the Taylor rule to ensure global determinacy).

Proposition F.2. *Consider the system (F.3)-(F.4) with $\phi_x > \rho$ and $\phi_\pi > 1$. Then, the only initial pair (x_0, π_0) consistent with a deterministic non-explosive equilibrium is $(x_0, \pi_0) = (0, 0)$. Any other initial pair diverges, but only asymptotically (i.e., not in finite time).*

Proof of Proposition F.2. Define $f(x) := \frac{e^{(1+\varphi)x} - 1}{1+\varphi}$. From (F.3)-(F.4), the steady state solves

$$-\phi_x x = (\phi_\pi - 1)\kappa\rho^{-1} f(x)$$

The two sides of this equation have opposite slopes in x, so the unique solution is $x = 0$, proving the unique steady state is $(x, \pi) = (0, 0)$. The steady state is locally unstable, by the same linearized eigenvalue analysis leading to Proposition F.1. By the local stable manifold theorem, we have that the unique stable solution to the dynamics is in fact this steady state. We now prove that any non-explosive equilibrium (satisfying Condition 1) must have $(x_t, \pi_t) = (0, 0)$ for all t. Assume not, i.e., assume, leading to contradiction, that $x_t \in [\underline{x}, \overline{x}]$ for all $t > 0$, where $\underline{x} < 0 < \overline{x}$.

First, from (F.3),

$$e^{-\rho t}\pi_t - \pi_0 = -\kappa \int_0^t e^{-\rho s} f(x_s) ds \tag{F.5}$$

Substituting (F.5) into (F.4), we have

$$\dot{x}_t = \phi_x x_t + (\phi_\pi - 1)\Big[e^{\phi_x t}\pi_0 - \frac{\kappa}{\rho} \int_0^t \rho e^{\rho(t-u)} f(x_u) du \Big] \tag{F.6}$$

Under the boundedness assumption, we may bound $f(\underline{x}) \le f(x_t) \le f(\overline{x})$, which when plugging into (F.6) leads to

$$\underbrace{\phi_x x_t + (\phi_\pi - 1)\left[e^{\phi_x t}\pi_0 - \frac{\kappa}{\rho}(e^{\rho t} - 1)f(\overline{x})\right]}_{:=L_t} \leq \dot{x}_t \leq \underbrace{\phi_x x_t + (\phi_\pi - 1)\left[e^{\phi_x t}\pi_0 - \frac{\kappa}{\rho}(e^{\rho t} - 1)f(\underline{x})\right]}_{:=U_t}$$

If $\pi_0 > 0$, then $L_t, U_t \to +\infty$ as $t \to \infty$ for every possible value of $x_t \in [\underline{x}, \overline{x}]$. On the other hand, if $\pi_0 < 0$, then $L_t, U_t \to -\infty$ as $t \to \infty$ for every possible value of $x_t \in [\underline{x}, \overline{x}]$. Hence, $\pi_0 > 0$ implies $x_T > \overline{x}$ for some $T > 0$, while $\pi_0 < 0$ implies $x_T < \underline{x}$ for some $T > 0$. This contradicts the bounded set $x_t \in [\underline{x}, \overline{x}]$, which implies $\pi_0 = 0$ is required.

However, since time 0 is arbitrary in this analysis, and the entire argument could be shifted forward in time, we in fact require $\pi_t = 0$ for all $t \geq 0$. Going back to equation (F.3), we then have that $x_t = 0$ for all $t \geq 0$. □

F.3 Very Nonlinear Taylor Rules and Finite-Time Explosions

Suppose we would like to allow deterministic equilibria that explode asymptotically, in violation of Condition 1. For instance, Cochrane (2011) considers some types of asymptotically exploding equilibria in his argument for non-uniqueness. In that case, is the spirit of Proposition F.1 still true, i.e., do there exist Taylor rules which can eliminate indeterminacies? The answer is yes, but a "nuclear Taylor rule" is required to force explosion in finite time.

In particular, let us dispense with the linear rule (linear MP). Suppose the response function (MP) takes the nonlinear form

$$\Phi(x, \pi) = \frac{\phi_x}{2}(e^x - e^{-x}) + \pi \tag{F.7}$$

with $\phi_x > 0$ and suppose the target rate is again the natural rate $\bar{\imath} = \rho$. Note that the log-linearized version of (F.7) renders the linear Taylor rule (linear MP) with $\phi_\pi = 1$.

Combining (F.7) with (IS), the dynamics of x_t are given by

$$\dot{x}_t = \frac{\phi_x}{2}(e^{x_t} - e^{-x_t}) \tag{F.8}$$

This ODE has solution

$$x_t = \log\left(\frac{1 - Ke^{\phi_x t}}{1 + Ke^{\phi_x t}}\right)$$

where $K = \frac{1-e^{x_0}}{1+e^{x_0}}$. This process diverges in *finite time* for any $x_0 \neq 0$: it explodes at time

$T = -\phi_x^{-1} \log(|K|)$. Hence, we have proved by construction the following result.

Proposition F.3. *Taylor rules exist such that any deterministic equilibrium has $x_t = 0$ forever.*

The analysis above abstracts from any feedback effects from inflation to output gap by setting a monetary policy rule with $\phi_\pi = 1$. This serves two purposes. First, it emphasizes the focus on self-fulfilling demand and not inflation per se. Equilibrium characterization requires the output gap to remain bounded for any finite horizon. There is no such requirement for inflation (e.g., hyperinflation might be an equilibrium outcome). Second, it simplifies the analysis and illustrates the point with examples that permit closed form solutions. As an additional benefit, Proposition F.3 holds for either the linearized or non-linear Phillips curves.

Determinacy extends beyond the particular response function (F.7) that has exactly a one-for-one inflation response. In particular, consider inflation sensitivities of more than one-for-one, such as

$$\Phi(x, \pi) = \frac{\phi_x}{2}(e^x - e^{-x}) + \phi_\pi \pi, \quad \phi_x > 0, \, \phi_\pi > 1. \tag{F.9}$$

While more challenging technically to analyze, this rule also selects the zero output gap equilibrium $x_t = 0$. We demonstrate this result formally next.

Under rule (F.9), the dynamical system for (x_t, π_t) is

$$\dot{\pi}_t = \rho \pi_t - \kappa f(x_t) \tag{F.10}$$

$$\dot{x}_t = \frac{\phi_x}{2}(e^{x_t} - e^{-x_t}) + (\phi_\pi - 1)\pi_t \tag{F.11}$$

where $f(x) := (1+\varphi)^{-1}[e^{(1+\varphi)x} - 1]$.

Proposition F.4. *Consider the system (F.10)-(F.11) with $\phi_x > 0$ and $\phi_\pi > 1$. Then, $(x_t, \pi_t) = (0, 0)$ is the unique equilibrium that does not explode in finite time.*

Proof of Proposition F.4. Suppose the solution $(x_t(\phi_\pi), \pi_t(\phi_\pi))_{t\geq 0}$ associated to some $\phi_\pi > 1$ (which is unique prior to an explosion by the standard ODE uniqueness theorem) did not explode in finite time. In that case, because the solution is continuous in ϕ_π (again, standard ODE theorems ensure this), it follows that the solution $(x_t(\tilde{\phi}_\pi), \pi_t(\tilde{\phi}_\pi))_{t\geq 0}$ associated with $\tilde{\phi}_\pi < \phi_\pi$ also does not explode in finite time. Continuity requires this: otherwise, the two solutions would be infinitely far apart at some finite time T when one of the solutions does explode. But Proposition F.3 has already shown that $(x_t(1), \pi_t(1))_{t\geq 0}$ is explosive in finite time, a contradiction. □

G Inflation Dynamics under Rotemberg

Here, we generalize the sticky-price model of Rotemberg (1982) to our environment. Since firms in our economy are ex-ante identical, they will have identical utilization and price-setting incentives, allowing us to study a representative firm's problem and a symmetric equilibrium.

To set up the representative intermediate-goods-producer problem, let l_t denote the firm's hired labor, at some equilibrium wage W_t. The firm produces $y_t = l_t$. The firm makes its price choice p_t, internalizing its demand $y_t = (p_t/P_t)^{-\varepsilon} Y_t$, where P_t and Y_t are the aggregate price and output. This demand curve comes from an underlying Dixit-Stiglitz structure with CES preferences (with substitution elasticity $\varepsilon > 1$) and monopolistic competition in the intermediate goods sector.

Letting M_t denote the real SDF process, the representative firm solves

$$\sup_{p,l} \mathbb{E}\Big[\int_0^\infty M_t\Big(\frac{p_t}{P_t}y_t - \frac{W_t l_t}{P_t} - \frac{1}{2\eta}\big(\frac{1}{dt}\frac{dp_t}{p_t}\big)^2 Y_t\Big)dt\Big] \tag{G.1}$$

$$\text{subject to} \quad y_t = (p_t/P_t)^{-\varepsilon} Y_t \tag{G.2}$$

$$y_t = l_t \tag{G.3}$$

The quadratic price adjustment cost in (G.1) has a penalty parameter η. As $\eta \to 0$ ($\eta \to \infty$), prices become permanently rigid (flexible). We assume that this price adjustment cost is purely non-pecuniary for simplicity (this means that adjustment costs do not affect the resource constraint). Alternatively, we could redistribute these adjustment costs lump-sum to the representative household.

Before solving the problem, we can immediately note the following property: price changes are necessarily absolutely continuous ("order dt"). Indeed, the adjustment cost per unit of time is a function of price changes per unit of time, i.e., $\frac{1}{dt}\frac{dp_t}{p_t}$. If prices were to change faster than dt, say with the Brownian motion dZ_t, then $\frac{1}{dt}\frac{dp_t}{p_t}$ would be unbounded almost-surely (because Brownian motion is nowhere-differentiable), leading to infinite adjustment costs. Consequently, we know that $\frac{1}{dt}\frac{dp_t}{p_t} = \frac{\dot{p}_t}{p_t}$ for some $\dot{p}_t$.

The firm's optimal price sequence solves a dynamic optimization problem. Substituting the demand curve from (G.2) and the production function from (G.3), we may rewrite problem (G.1) as

$$\sup_{\dot{p}} \mathbb{E}_t\Big[\int_t^\infty \frac{M_s Y_s}{M_t Y_t}\Big(\big(\frac{p_s}{P_s}\big)^{1-\varepsilon} - \frac{W_s}{P_s}\big(\frac{p_s}{P_s}\big)^{-\varepsilon} - \frac{1}{2\eta}\big(\frac{\dot{p}_s}{p_s}\big)^2\Big)ds\Big].$$

Furthermore, note that in the log utility model used in the text, we have $M_t Y_t = e^{-\rho t}$. Letting J denote this firm's value function, the HJB equation is

$$0 = \sup_{\dot{p}_t} \left\{ (\frac{p_t}{P_t})^{1-\varepsilon} - \frac{W_t}{P_t}(\frac{p_t}{P_t})^{-\varepsilon} - \frac{1}{2\eta}(\frac{\dot{p}_t}{p_t})^2 - \rho J_t + \frac{1}{dt}\mathbb{E}_t[dJ_t] \right\}$$

The firm value function follows a process of the form

$$dJ_t = [\mu_{J,t} + \dot{p}_t \frac{\partial}{\partial p} J_t]dt + \sigma_{J,t} dZ_t,$$

where $\mu_{J,t}$ and $\sigma_{J,t}$ are only functions of aggregate states (not the individual price). The only part that the firm can affect is $\dot{p}_t \frac{\partial}{\partial p} J_t$. Plugging these results back into the HJB equation and taking the FOC, we have

$$0 = -\frac{1}{\eta}(\frac{\dot{p}_t}{p_t})\frac{1}{p_t} + \frac{\partial}{\partial p} J_t \tag{G.4}$$

Differentiating the HJB equation with respect to the state variable p_t, we have the envelope condition

$$(\varepsilon - 1)(\frac{p_t}{P_t})^{-\varepsilon}\frac{1}{P_t} - \varepsilon \frac{W_t}{P_t}(\frac{p_t}{P_t})^{-\varepsilon-1}\frac{1}{P_t} = \frac{1}{\eta}(\frac{\dot{p}_t}{p_t})^2 \frac{1}{p_t} - \rho \frac{\partial}{\partial p} J_t + \frac{1}{dt}\mathbb{E}_t[d(\frac{\partial}{\partial p} J_t)], \tag{G.5}$$

where the last term uses the stochastic Fubini theorem. Combining equations (G.4) and (G.5), we have

$$\eta(\varepsilon - 1)(\frac{p_t}{P_t})^{-\varepsilon}\frac{1}{P_t} - \eta\varepsilon \frac{W_t}{P_t}(\frac{p_t}{P_t})^{-\varepsilon-1}\frac{1}{P_t} = (\frac{\dot{p}_t}{p_t})^2 \frac{1}{p_t} - \rho(\frac{\dot{p}_t}{p_t})\frac{1}{p_t} + \frac{1}{dt}\mathbb{E}_t[d((\frac{\dot{p}_t}{p_t})\frac{1}{p_t})] \tag{G.6}$$

At this point, define the firm-level inflation rate $\pi_t := \dot{p}_t / p_t$, note that $\mathbb{E}_t[d(\pi_t \frac{1}{p_t})] = \frac{1}{p_t}\mathbb{E}_t[d\pi_t] - \frac{1}{p_t}\pi_t^2 dt$, and use the symmetry assumption $p_t = P_t$ in (G.6) to get

$$\eta(\varepsilon - 1) - \eta\varepsilon \frac{W_t}{P_t} = -\rho\pi_t + \frac{1}{dt}\mathbb{E}_t[d\pi_t]. \tag{G.7}$$

Equation (G.7) is the continuous-time stochastic Phillips curve, with π_t interpreted also as the aggregate inflation rate (given a symmetric equilibrium).

Finally, note that the firm's optimization problem also requires the following transversality condition (see Theorem 9.1 of Fleming and Soner (2006)):

$$\lim_{T\to\infty} \mathbb{E}_t[M_T Y_T J_T] = 0.$$

In a symmetric equilibrium ($p = P$), and using the log utility result $M_t Y_t = e^{-\rho t}$, we have that

$$M_T Y_T J_T = \mathbb{E}_T\Big[\int_T^\infty e^{-\rho t}\Big(1 - (Y^*)^{1+\varphi} e^{(1+\varphi)x_t} - \frac{1}{2\eta}\pi_t^2\Big)dt\Big]$$

Take expectations and the limit $T \to \infty$. Sufficient conditions for the result to be zero are

$$\lim_{T\to\infty} \mathbb{E}_t[e^{(1+\varphi)x_T - \rho T}] = 0 \tag{G.8}$$

$$\lim_{T\to\infty} \mathbb{E}_t[e^{-\rho T}\pi_T^2] = 0 \tag{G.9}$$

Equation (G.8) is identical to the one of the requirements for the consumer's problem to be well-defined (see Appendix A.1). Equation (G.9) avoids nominal explosions that imply an infinite present value of adjustment costs. Note that under Condition 1, both of these equations automatically hold.

H Example with Fiscal Backstops

We provide the details and results for Example 7 in the text. This class of fiscal policies successfully eliminates uncertainty traps. We refer to these policies as *fiscal backstops* because they are less invasive than the baseline always-active fiscal policy from Theorem 2, in the sense that an active fiscal regime only emerges sometimes. The details are as follows.

First, when in the active regime, fiscal policy is sufficiently aggressive. We model α_t as a two-state process, with states $\bar{\alpha} > 0$ and $\underline{\alpha} < 0$. The regime with $\bar{\alpha} > 0$ is the standard passive one. By contrast, $\underline{\alpha} < 0$ captures a fiscal policy regime that is very aggressively active: as debt-to-GDP rises, surpluses decline (e.g., spending rises and/or taxes fall). In some sense, this is beyond irresponsible because it allows debt to spiral out of control. But it reflects some aspects of real-world policies: debt-to-GDP increases are often due to negative GDP shocks to which the government responds by spending even more, as in stimulus packages, which further raises the debt-to-GDP ratio. An important consideration is the size of $\underline{\alpha}$.

Second, we design the intervention threshold in such a way that the aggressively-active regime occurs with positive probability over the long run. To achieve this, we assume that regimes switch at a threshold that is endogenous to the equilibrium being played—we refer to this as an *adaptive backstop* and explain this name shortly. What we specifically assume is that policy can condition its switching point on the stationary distribution in equilibrium. Let $p(x)$ denote the (marginal) stationary density of x_t in equilibrium, and let

$$\chi_q := \inf\Big\{\chi : \int_{-\infty}^{\chi} p(x)dx \geq q\Big\} \tag{H.1}$$

denote the q^{th} percentile of p. For some $q > 0$, an adaptive backstop is characterized by

$$\alpha_t = \begin{cases} \bar{\alpha}, & \text{if } x_t \geq \chi_q; \\ \underline{\alpha}, & \text{if } x_t < \chi_q. \end{cases} \tag{H.2}$$

The feature of this policy is that χ_q is chosen such that the probability of the active regime is, in every equilibrium, at least q. Notice that α_t is purely a function of x_t at equilibrium. (Technically, the intervention threshold χ_q depends on the endogenous distribution, so it is also a "function of the equilibrium" and solves a fixed point problem. But, as mentioned, *at equilibrium* χ_q is merely a constant, so α_t depends only on x_t.)

Is it realistic for policy to condition on the equilibrium stationary distribution? We think so. Imagine policy promises to intervene in the worst 10% of recessions. To achieve this, policy can start by setting an initial threshold $\tilde{\chi}^{(1)}$. Over time, perhaps policymakers see that sunspot equilibria are persisting and they are intervening less than promised, maybe only 5% of the time. They can promise more intervention in the future by tightening their threshold, $\tilde{\chi}^{(2)} > \tilde{\chi}^{(1)}$. As they observe equilibrium dynamics, policymakers continue to adjust the thresholds $\tilde{\chi}^{(n)}, \tilde{\chi}^{(n+1)}, \dots$ until they converge to something approximating their desired 10th percentile threshold $\chi_{0.10}$. Dynamic adjustment is one way to think about achieving our so-called adaptive backstops.

With the combination of a very aggressive active regime and an adaptive backstop, fiscal policy imposes a non-redundant debt valuation equation on every equilibrium. To see this, recall that an adaptive threshold χ_q means that $\mathbb{P}\{x_t < \chi_q\} \geq q$, regardless of the equilibrium being played. Consequently, we have by the ergodic theorem that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \alpha_t dt \xrightarrow{a.s.} \mathbb{E}[\alpha_t] = \bar{\alpha}\mathbb{P}\{x_t \geq \chi_q\} + \underline{\alpha}\mathbb{P}\{x_t < \chi_q\} \leq (1-q)\bar{\alpha} + q\underline{\alpha}.$$

So long as

$$\underline{\alpha} < -\bar{\alpha}\frac{1-q}{q}, \tag{H.3}$$

we have $\mathbb{E}[\alpha_t] < 0$ and hence condition (34) of Lemma 3 is satisfied. If policy is either active sufficiently often (i.e., q is high enough) or sufficiently aggressive when active (i.e., $\underline{\alpha}$ is negative enough), we thus obtain a non-redundant debt valuation equation.[20]

In all designs, we require that the active-fiscal regime is paired with a passive-money regime, which is the standard coordination in the literature (Leeper, 1991). Write the interest rate rule as the linear rule (linear MP) with regime-specific coefficients,

$$\Phi(x, \pi; \alpha) = \phi_x(\alpha)x + \phi_\pi(\alpha)\pi, \tag{H.4}$$

allowing the monetary regime to shift in tandem with the fiscal regime. We assume that

$$\begin{aligned} &\text{(Active fiscal / passive money):} \quad \phi_x(\underline{\alpha}) \leq 0 \quad \text{and} \quad \phi_\pi(\underline{\alpha}) < 1 \\ &\text{(Passive fiscal / arbitrary money):} \quad \phi_x(\bar{\alpha}) \text{ free} \quad \text{and} \quad \phi_\pi(\bar{\alpha}) \text{ free} \end{aligned} \tag{H.5}$$

Appropriate fiscal-monetary coordination guarantees that our equilibrium selection results do not stem from "inconsistent or overdetermined policies" as critiqued by Cochrane (2011). For the purpose of eliminating sunspot volatility, perhaps surprisingly, we do not need any assumption about monetary policy when fiscal policy is active ($\alpha = \bar{\alpha}$).

We now state a formal result. Under the adaptive backstop policy, there cannot exist any uncertainty trap within the class discovered in our paper, even extending this class to the case where the dynamics include a dependence on fiscal shocks Ω.

Theorem H.1. *Consider fiscal policy (H.1)-(H.2) that satisfies $\bar{\alpha} < \rho$ and condition (H.3), paired with monetary policy (H.4)-(H.5). Within the class of equilibria in which inflation and volatilities take the form $\pi_t = \pi(x_t, \Omega_t)$, $\sigma_{x,t} = \sigma_x(x_t, \Omega_t)$, and $\varsigma_{x,t} = \varsigma_x(x_t, \Omega_t)$, any equilibrium must have zero sunspot volatility, i.e., $\sigma_{x,t} = 0$.*

The equilibrium with fiscal backstops bears many similarities to the conventional "fiscal equilibrium" that emerges under always-active fiscal policy. In particular, the reason $\sigma_{x,t} = 0$ must hold is that demand is anchored by the valuation-like equation,

[20] A natural question is why the backstop needs to be *adaptive*. Imagine the adaptive threshold χ_q were replaced by a fixed threshold χ. The problem: there are stationary sunspot equilibria in which the tail probability $\mathbb{P}\{x_t < \chi\}$ is vanishingly small (see Lemma A.5). In these equilibria, fiscal policy would be almost-always passive and provide no discipline to the dynamics. To rule out all sunspot equilibria with a fixed threshold, policy would either need to increase the threshold (i.e., put $\chi = 0$ as in Theorem 2) or explode the level of aggression (i.e., take $\underline{\alpha} \to -\infty$), neither of which is particularly appealing.

which recall says that

$$b_t = \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(u-t)} e^{\int_t^u \alpha_z dz} \hat{s}_u du\Big].$$

This not only eliminates sunspot volatility but pins down overall demand volatility as follows. Since $\hat{s}_t$ is an exogenous function of Ω_t, while α_t in our example policies is a function of (Ω_t, x_t), the valuation-like equation implies that b_t is a function of (Ω_t, x_t). From its definition, debt-to-GDP has a sensitivity to fiscal shocks $d\mathcal{Z}_t$ of $-b_t\varsigma_{x,t}$. On the other hand, given we just showed that debt-to-GDP is a function of (Ω_t, x_t), its sensitivity to fiscal shocks must also be $\varsigma_{x,t}\partial_x b_t + \varsigma_{\Omega,t}\partial_\Omega b_t$, by Itô's formula. Equating the two and rearranging for ς_x implies demand inherits fiscal shocks according to

$$\varsigma_x(x,\Omega) = -\frac{\varsigma_\Omega(\Omega)\partial_\Omega b(x,\Omega)}{b(x,\Omega) + \partial_x b(x,\Omega)} \tag{H.6}$$

Fiscal-based volatility emerges even during regimes in which fiscal policies appears to behave passively. Thus, a general conclusion about fiscal backstops is that they substitute self-fulfilling demand volatility (σ_x) for fiscal volatility (ς_x).

On the other hand, fiscal backstops permit some flexibility which is absent under the always-active fiscal policy. This flexibility is that, as (H.5) makes clear, monetary policy may take any reaction function during times when a passive fiscal regime emerges. From the perspective of quantitative and empirical researchers, this is an important degree of freedom, since it allows them to pick the monetary rule that best fits the data. From the perspective of theorists, the flexibility allows us to pick an optimal monetary rule unfettered by any equilibrium selection constraints—by contrast, typical linearized analyses often require active money be paired with passive fiscal (e.g., they often require $\phi_x(\underline{\alpha}) > 0$ and $\phi_\pi(\underline{\alpha}) > 1$). If, for example, the best policy involves passive fiscal regimes to be paired with an interest rate peg, that is totally permissible here.

Proof of Theorem H.1. We proceed in steps. First, we verify the sufficient conditions for "active fiscal" to obtain a valuation equation for b_t. Second, assuming a non-explosive sunspot equilibrium, we obtain the unique solution for b_t in the Markovian environment assumed. Third, we use this solution to pin down inflation and volatility uniquely. Fourth, we demonstrate that this unique solution cannot be consistent with "stability," namely the non-explosive Condition 1 is violated, a contradiction to the assumption that sunspot volatility is present.

Step 1: valuation equation. First, by the definition of χ_q in (H.1), every equilibrium satisfies

$\mathbb{P}\{x_t < \chi_q\} \geq q$. Given that condition (H.3) holds and that $\bar{\alpha} < \rho$, we have that the assumptions of Lemma 3 hold, so we obtain valuation equation (35) for b_t.

Step 2: solution for b. We assume a non-explosive sunspot equilibrium. Let the ergodic set of the sunspot equilibrium be $\mathcal{X} := \{(x, \Omega) : x \in (x_{min}(\Omega), x_{max}(\Omega))\}$, where

$$x_{min}(\Omega) := \inf\{x : \sigma_x^2(x, \Omega) > 0\}$$
$$x_{max}(\Omega) := \sup\{x : \sigma_x^2(x, \Omega) > 0\}.$$

Define $\mathcal{X}^\circ := \{(x, \Omega) : \sigma_x^2(x, \Omega) > 0\}$ to be the sub-domain with sunspot volatility. Notice that all (x, Ω) close enough to the boundary of $\mathcal{X}$ lie inside $\mathcal{X}^\circ$.

Now, since π_t, $\sigma_{x,t}$, and $\varsigma_{x,t}$ are functions of (x_t, Ω_t), we have that (x_t, Ω_t) is a Markov process. Since α_t is also a function of (x_t, Ω_t), equation (35) implies that $b_t = b(x_t, \Omega_t)$ for some function b. Applying Lemma A.6, we then know that for some constant $\bar{b}$,

$$b(x, \Omega) = \bar{b}e^{-x}, \quad \text{on } \mathcal{X}^\circ. \tag{H.7}$$

Step 3a: pinning down π, σ_x^2. Substituting (H.7) into the differential equation (A.29) and rearranging, we have that

$$\phi_x(\alpha)x + (\phi_\pi(\alpha) - 1)\pi = \frac{\hat{s}(\Omega)}{\bar{b}}e^x + \alpha - \rho, \quad \text{on } \mathcal{X}^\circ. \tag{H.8}$$

This equation cannot hold for all $x \in \mathcal{X}^\circ$, except if inflation takes a particular knife-edge functional form—let this solution for inflation be denoted by

$$\pi_0(x, \Omega) := \frac{\hat{s}(\Omega)e^x/\bar{b} + \alpha - \rho - \phi_x(\alpha)x}{\phi_\pi(\alpha) - 1}. \tag{H.9}$$

On the other hand, the function π_0 must also be consistent with the Phillips curve (30). Applying Itô's formula to π_0, the result is

$$\begin{aligned}\rho\pi_0 - \kappa x = \Big[\frac{\hat{s}}{\bar{b}}e^x + \alpha - \rho + \frac{1}{2}\sigma_x^2 + \frac{1}{2}|\varsigma_x|^2\Big]\partial_x\pi_0 + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx}\pi_0 \\ + \mu_\Omega'\partial_\Omega\pi_0 + \frac{1}{2}\text{trace}[\varsigma_\Omega\varsigma_\Omega'(\partial_{\Omega\Omega'}\pi_0)] + \varsigma_x\varsigma_\Omega'\partial_{x\Omega}\pi_0, \quad \text{on } \mathcal{X}^\circ.\end{aligned} \tag{H.10}$$

Given that π_0 is determined, this equation pins down $\frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2) = \Sigma_0$ on $\mathcal{X}^\circ$, where

$$\Sigma_0 := \frac{\rho\pi_0 - \kappa x - (\frac{\hat{s}}{\bar{b}}e^x + \alpha - \rho)\partial_x\pi_0 - \big(\mu_\Omega'\partial_\Omega\pi_0 + \frac{1}{2}\mathrm{trace}[\varsigma_\Omega\varsigma_\Omega'(\partial_{\Omega\Omega'}\pi_0)] + \varsigma_x\varsigma_\Omega'\partial_{x\Omega}\pi_0\big)}{\partial_x\pi_0 + \partial_{xx}\pi_0} \tag{H.11}$$

Step 3b: useful properties of π_0, Σ_0. We note, for later, some key properties. First, π_0 and its derivatives are finite for all x finite. Second, its limiting values are given by

$$\lim_{x\to-\infty}\pi_0 = \begin{cases} +\infty, & \text{if } \phi_x(\underline{\alpha}) < 0; \\ \frac{\underline{\alpha}-\rho}{\phi_\pi(\underline{\alpha})-1}, & \text{if } \phi_x(\underline{\alpha}) = 0. \end{cases}$$
$$\lim_{x\to-\infty}\partial_x\pi_0 = -\frac{\phi_x(\underline{\alpha})}{\phi_\pi(\underline{\alpha})-1} \leq 0$$
$$\lim_{x\to-\infty}\partial_{xx}\pi_0 = 0$$

Here, we have used the key fact that $\alpha(x) \to \underline{\alpha}$ as $x \to -\infty$, by definition of the intervention threshold χ_q. Using these properties for π_0, the total diffusion Σ_0 inherits the following properties:

(P1) $\lim_{x\to-\infty}\Sigma_0 \neq 0$.

[Proof: Because of the fact that π_0 and its derivatives are bounded for all finite x, we have that $\Sigma_0 = 0$ if and only if the numerator in (H.11) vanishes. Plugging in the expression for π_0 from (H.9), and then taking the limit $x \to -\infty$ in the numerator of (H.11), we find that $\Sigma_0 \not\to 0$ unless $\phi_x(\underline{\alpha}) = -\rho$ and $\rho\phi_x(\underline{\alpha}) + \kappa(\phi_\pi(\underline{\alpha}) - 1) = 0$. But these conditions cannot hold under monetary policy (H.5), since $\phi_\pi(\underline{\alpha}) < 1$.]

(P2) $\lim_{x\to-\infty}\Sigma_0 < +\infty$ for all Ω if $\phi_x(\underline{\alpha}) \neq 0$ and on $\{\Omega : \hat{s}(\Omega) > 0\}$ if $\phi_x(\underline{\alpha}) = 0$.

[Proof: Indeed, by the fact that π_0 and its derivatives are finite for all finite x, Σ_0 can only explode if $x \to -\infty$ and if the following condition holds:

$$\lim_{x\to-\infty}\Sigma_0 = +\infty \iff \lim_{x\to-\infty}\frac{\rho\pi_0 - \kappa x}{\partial_x\pi_0 + \partial_{xx}\pi_0} = +\infty \tag{H.12}$$

If $\phi_x(\underline{\alpha}) \neq 0$, (H.12) is equivalent to $\lim_{x\to-\infty}[\rho + \frac{\kappa(\phi_\pi(\underline{\alpha})-1)}{\phi_x(\underline{\alpha})}]x = +\infty$, which cannot hold by the fact that monetary policy (H.5) uses $\phi_\pi(\underline{\alpha}) < 1$. If $\phi_x(\underline{\alpha}) = 0$, condition (H.12) is equivalent to $\lim_{x\to-\infty}\frac{\bar{b}\kappa(\phi_\pi(\underline{\alpha})-1)x}{\hat{s}(\Omega)e^x} = -\infty$, which cannot hold for any Ω such that $\hat{s}(\Omega) > 0$, again by the fact that monetary policy (H.5) uses $\phi_\pi(\underline{\alpha}) < 1$.]

Step 4: explosiveness of dynamics. Under the restriction (H.8), the drift of x is given by

$$\begin{aligned}\mu_x &:= \Phi(x, \pi; \alpha) - \pi + \frac{1}{2}\sigma_x^2 + \frac{1}{2}|\varsigma_x|^2 \\ &= \hat{s}e^x/\bar{b} + \alpha - \rho + \Sigma_0, \quad \text{on } \mathcal{X}^\circ.\end{aligned} \tag{H.13}$$

For the domain to be valid as part of an equilibrium, we must have that the dynamics are such that $(x_t, \Omega) \in \mathcal{X}$ forever—we refer to this as "stability." Recall that all points near the boundary of $\mathcal{X}$ are inside $\mathcal{X}^\circ$, and so the drift above is the relevant expression near the boundaries. There are two possibilities which require different analyses: $x_{min}(\Omega) > -\infty$ and $x_{min}(\Omega) = -\infty$.

If $x_{min}(\Omega) > -\infty$, stability requires that the volatilities vanish there, because the drift μ_x in (H.13) is also finite there. So we require $\Sigma_0(x_{min}(\Omega), \Omega) = 0$. (This restricts the boundary $x_{min}(\Omega)$, meaning it cannot be arbitrary; however, this is not critical for the argument below.) Furthermore, stability requires the drift to point "inwards" into the domain $\mathcal{X}$, i.e., $\mu_x(x_{min}(\Omega), \Omega) \geq 0 \geq \mu_x(x_{max}(\Omega), \Omega)$ for all Ω. However, given $\alpha(x)$ defined in (H.2) is weakly increasing in x, we have the following for all Ω such that $\hat{s}(\Omega) > 0$:

$$\begin{aligned}\mu_x(x_{min}(\Omega), \Omega) &= e^{x_{min}(\Omega)}\hat{s}(\Omega)/\bar{b} + \alpha(x_{min}(\Omega)) - \rho \\ &< e^{x_{max}(\Omega)}\hat{s}(\Omega)/\bar{b} + \alpha(x_{max}(\Omega)) - \rho \\ &\leq e^{x_{max}(\Omega)}\hat{s}(\Omega)/\bar{b} + \alpha(x_{max}(\Omega)) - \rho + \Sigma_0(x_{max}(\Omega), \Omega) = \mu_x(x_{max}(\Omega), \Omega),\end{aligned}$$

which contradicts the stability requirement on the drifts.

On the other hand, if $x_{min}(\Omega) = -\infty$, then stability requires that volatilities explode asymptotically. Indeed, property (P1) above shows that $\lim_{x \to -\infty} \Sigma_0 \neq 0$. If $\lim_{x \to -\infty} \Sigma_0$ were a finite value, then the drift μ_x in (H.13) would also be finite, and so the process x_t would hit $-\infty$ in finite time with positive probability. Thus, the only way to obtain a non-explosive solution is $\lim_{x \to -\infty} \Sigma_0 = +\infty$. Property (P2) above rules this out.

Thus, we have shown that, regardless of how x_{min} is constructed, the dynamics cannot be non-explosive. This contradiction completes the proof. □

I Fiscal Policy: Generalizations and Extensions

We consider a series of extensions to our baseline model with fiscal policy. We explore (i) long-term debt; (ii) fiscal rules depending on (x, π); and (iii) more general CRRA utility. To facilitate the analysis, we first present a general nesting framework that accommodates all the extensions (Section I.1). This allows us to provide a general characterization. Next, we analyze the three specific extensions (Section I.2). Then, we explore how inflation is determined under active fiscal policies, after the sunspot volatility has been eliminated (Section I.3). The proofs of the theorems for all extensions and analyses are contained in the final section, for ease of readability (Section I.4).

I.1 Characterization of fiscal policy in a general setting

We first set up a general surplus dynamic that nests all cases of interest. Let $\mathcal{Z}_t$ be a k-dimensional Brownian motion independent of the monetary/sunspot shock Z_t. Let Ω follow a Markov diffusion driven by $\mathcal{Z}$. Let $s_t := S_t/Y_t$ be a rule of the form

$$s_t = s(\Omega_t, x_t, \pi_t) \tag{I.1}$$

$$d\Omega_t = \mu_\Omega(\Omega_t)dt + \varsigma_\Omega(\Omega_t) \cdot d\mathcal{Z}_t \tag{I.2}$$

For now, we let the rule $s(\cdot)$ and dynamics μ_Ω, ς_Ω be arbitrary functions. This is more general than what we need going forward.

Second, we generalize the model to the CRRA utility $u(c,l) = \frac{c^{1-\gamma}}{1-\gamma} + \frac{l^{1+\varphi}}{1+\varphi}$. In that case, the consumption FOC says

$$M_t = e^{-\rho t} C_t^{-\gamma}. \tag{I.3}$$

The labor-consumption margin is unaffected. Applying Itô's formula to (I.3), and noting that $C_t = Y_t = Y^* e^{x_t}$ and $-\frac{1}{dt}\mathbb{E}[\frac{dM_t}{M_t}] = r_t = \iota_t - \pi_t$, the IS curve generalizes to

$$dx_t = \Big[\frac{\iota_t - \pi_t - \rho}{\gamma} + \frac{1}{2}\gamma\sigma_{x,t}^2 + \frac{1}{2}\gamma|\varsigma_{x,t}|^2\Big]dt + \sigma_{x,t}dZ_t + \varsigma_{x,t} \cdot d\mathcal{Z}_t. \tag{I.4}$$

The dynamics of $Y_t = Y^* e^{x_t}$ can be derived from (I.4). When $\gamma \neq 1$, the Phillips curve is also different and requires an additional approximation to obtain a form similar to that used throughout the paper. Indeed, the derivation of the Phillips curve in Appendix G relies on $M_t Y_t \propto e^{-\rho t}$, which is no longer true with general CRRA utility. We make this approximation, which is tantamount to approximating around steady-state where $Y_t/Y_0 \approx 1$. With this approximation, the Phillips curve (PC) is replaced by

$$\mu_{\pi,t} = \rho\pi_t - \kappa\Big(\frac{e^{(\gamma+\varphi)x_t} - 1}{\gamma + \varphi}\Big), \tag{I.5}$$

where $Y^* := (\frac{\varepsilon-1}{\varepsilon})^{\frac{1}{\gamma+\varphi}}$ is the flexible-price output level and $\kappa := \eta(\varepsilon-1)(\gamma+\varphi)$ is the composite price-stickiness parameter. A linearized Phillips curve replaces $f(x) := \frac{e^{(\gamma+\varphi)x}-1}{\gamma+\varphi}$ in (I.5) with its first-order approximation x.

Third, we generalize to long-term debt. To keep things tractable, let us assume that debt is coupon-free and has a constant exponential maturity structure. Per unit of time dt, a constant fraction βdt of outstanding debts mature, and their principal must be repaid. Denote the per-unit price of this debt by Q_t. The government's flow budget constraint is now

$$Q_t \dot{B}_t = \beta B_t - \beta B_t Q_t - P_t S_t. \tag{I.6}$$

This says that new net debt sales $\dot{B}_t + \beta B_t$, which garner price Q_t, plus primary surpluses $P_t S_t$ must be sufficient to pay back maturing debts βB_t. By standard no-arbitrage asset-pricing, the per-unit bond price is given by

$$Q_t = \mathbb{E}_t\Big[\int_t^\infty \frac{M_T}{M_t}\frac{P_t}{P_T}\beta e^{-\beta(T-t)}dT\Big]. \tag{I.7}$$

In the above, debt is nominal, so it is priced using the nominal SDF M/P (intuitively, dividing by P converts a nominal cash flow into a real cash flow). The total real value of debt is $Q_t B_t / P_t$, and so the government debt valuation equation is now

$$\frac{Q_t B_t}{P_t} = \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} S_u du\Big]. \tag{I.8}$$

In an equilibrium with long-term debt, all three of (I.6), (I.7), and (I.8) must hold. Substituting the consumption FOC (I.3) into (I.8), we may rewrite the aggregate debt valuation equation as

$$\frac{Q_t B_t}{P_t} = Y_t^\gamma \Psi_t \quad \text{where} \quad \Psi_t := \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(u-t)} s_u Y_u^{1-\gamma} du\Big] \tag{I.9}$$

The next steps are to derive the dynamics of the two key present values Q_t and Ψ_t. These are essentially two "asset-pricing equations." The bond price Q_t has dynamics of the form

$$dQ_t = Q_t\Big[\mu_{Q,t}dt + \sigma_{Q,t}dZ_t + \varsigma_{Q,t}\cdot d\mathcal{Z}_t\Big] \tag{I.10}$$

for some μ_Q, σ_Q, and ς_Q to be determined. Starting from the per-unit bond pricing equation (I.7), we have that the object

$$e^{-\beta t}\frac{Q_t M_t}{P_t} + \int_0^t \frac{M_u}{P_u}\beta e^{-\beta u} du$$

is a local martingale and has zero drift. Note that, from the consumption FOC (I.3), the nominal SDF M_t/P_t has dynamics

$$d(M_t/P_t) = -(M_t/P_t)\Big[\iota_t dt + \gamma\sigma_{x,t} dZ_t + \gamma\varsigma_{x,t}\cdot d\mathcal{Z}_t\Big] \tag{I.11}$$

Then, by applying Itô's formula to the previous expression, and setting the resulting drift to zero, we have

$$\mu_{Q,t} = \beta - \frac{\beta}{Q_t} + \iota_t + \gamma\sigma_{x,t}\sigma_{Q,t} + \gamma\varsigma_{x,t}\cdot\varsigma_{Q,t} \tag{I.12}$$

This characterizes the drift of Q.

From the definition of Ψ_t, we have

$$e^{-\rho t}\Psi_t + \int_0^t e^{-\rho u} s_u Y_u^{1-\gamma} du = \mathbb{E}_t\Big[\int_0^\infty e^{-\rho u} s_u Y_u^{1-\gamma} du\Big],$$

which is a local martingale. By the martingale representation theorem, we have that

$$d\Big(e^{-\rho t}\Psi_t + \int_0^t e^{-\rho u} s_u Y_u^{1-\gamma} du\Big) = e^{-\rho t}\Big(\sigma_{\Psi,t} dZ_t + \varsigma_{\Psi,t}\cdot d\mathcal{Z}_t\Big)$$

for some $\sigma_{\Psi,t}$ and some $\varsigma_{\Psi,t}$. On the other hand, we also have by applying Itô's formula to the left-hand-side,

$$d\Big(e^{-\rho t}\Psi_t + \int_0^t e^{-\rho u} s_u Y_u^{1-\gamma} du\Big) = \Big[-\rho e^{-\rho t}\Psi_t + e^{-\rho t} s_t Y_t^{1-\gamma}\Big]dt + e^{-\rho t} d\Psi_t$$

Equating these last two results, and rearranging for $d\Psi_t$, we have

$$d\Psi_t = (\rho\Psi_t - s_t Y_t^{1-\gamma})dt + \sigma_{\Psi,t} dZ_t + \varsigma_{\Psi,t}\cdot d\mathcal{Z}_t \tag{I.13}$$

This characterizes the drift of Ψ.

We have the following characterization lemma that can be used as a tool to evaluate the government debt valuation equation.

Lemma I.1. *In the setting above with general surpluses, long-term debt, and CRRA utility,*

$$\Psi_t \gamma \sigma_{x,t} = \Psi_t \sigma_{Q,t} - \sigma_{\Psi,t} \tag{I.14}$$

$$\Psi_t \gamma \varsigma_{x,t} = \Psi_t \varsigma_{Q,t} - \varsigma_{\Psi,t} \tag{I.15}$$

Conversely, if the asset-pricing equations (I.12)-(I.13) hold, and the diffusion-matching equations (I.14)-(I.15) hold, then the government debt valuation equation (I.9) holds at every date, provided it holds at the initial date.

To prove our key result, we need to restrict attention to a class of of equilibria in which everything is a function of (x, Ω). This still permits a somewhat general analysis that nests all the equilibria developed in this paper.

Definition 3. An (x, Ω)-*Markov equilibrium* is a non-explosive equilibrium such that inflation π_t and volatilities $\sigma_{x,t}, \varsigma_{x,t}$ are functions of (x_t, Ω_t).

For our purposes, the equilibria covered by Definition 3 constitute a sufficiently general class. Indeed, all the sunspot equilibria constructed in this paper fall into this class. This is clear for the rigid-price limit examples of Section 2, since $\pi_t = 0$ and $\sigma_{x,t} = \sigma_x(x_t)$ in those cases. The construction with non-trivial inflation in Appendix C is also of this class: there, we construct a class of equilibria with $\pi_t = \pi(x_t)$. Thus, if fiscal policy can induce $\sigma_x = 0$ within the class of (x, Ω)-Markov equilibria, then it will have ruled out all the sunspot equilibria constructed in this paper. In this sense, we think the (x, Ω)-Markov class is rich enough to be useful in contrasting the equilibrium selection properties of Taylor rules versus active fiscal policies.

Lemma I.2. *The generalized model above has no (x, Ω)-Markov sunspot equilibria "generically," in the sense that, without imposing Condition 1, the $2 + dim(\mathcal{Z})$ endogenous variables $\pi(x, \Omega)$, $\sigma_x(x, \Omega)$, and $\varsigma_x(x, \Omega)$ have only $dim(\mathcal{Z})$ degrees of freedom whenever $\sigma_x \neq 0$. In particular, if $dim(\mathcal{Z}) = 0$ (no fiscal shocks), then $\pi(x)$ and $\sigma_x(x)$ are pinned down uniquely.*

Lemma I.2 shows that there essentially cannot be sunspot equilibria. The reason is that the objects are severely restricted in their degrees of freedom, which then implies that the resulting dynamics could generically not satisfy the additional non-explosion requirements that are needed. This is seen most transparently in the case without fiscal shocks, because then π and σ_x are pinned down uniquely in a way that, we will show, can definitively not be consistent with non-explosion. As a corollary, the result below displays the uniquely determined functional forms of all the key objects, in the case without fiscal shocks.

Corollary I.1. *Without fiscal state variables (no Ω), the model above requires the following to hold whenever $\sigma_x \neq 0$:*

$$Q(x) = G\Psi(x)e^{\gamma x} \tag{I.16}$$

$$\Psi(x) = \frac{G^{-1}\beta e^{-\gamma x} - s(x,\pi(x))(Y^*)^{1-\gamma}e^{(1-\gamma)x}}{\beta + \pi(x)} \tag{I.17}$$

$$\sigma_x(x)^2 = 2\frac{\rho\pi(x) - \kappa f(x) - \frac{\bar{\imath}+\Phi(x,\pi(x))-\pi(x)-\rho}{\gamma}\pi'(x)}{\gamma\pi'(x) + \pi''(x)} \tag{I.18}$$

and

$$\begin{aligned}\rho\Psi(x) - s(x,\pi(x))(Y^*)^{1-\gamma}e^{(1-\gamma)x} &= \frac{\gamma\Psi'(x) + \Psi''(x)}{\gamma\pi'(x) + \pi''(x)}(\rho\pi(x) - \kappa f(x)) \\ &= \Big[\frac{\bar{\imath} + \Phi(x,\pi(x)) - \pi(x) - \rho}{\gamma}\Big]\frac{\Psi'(x)\pi''(x) - \Psi''(x)\pi'(x)}{\gamma\pi'(x) + \pi''(x)}\end{aligned} \tag{I.19}$$

Thus, the objects $(Q, \Psi, \sigma_x^2, \pi)$ are all pinned down in an equilibrium with $\sigma_x \neq 0$.

I.2 Extensions with long-term debt, fiscal rules, and CRRA utility

Now, we explore our three key extensions: (i) long-term debt; (ii) fiscal rules depending on (x, π); and (iii) more general CRRA utility. Because these settings can become complex, the proofs become unwieldy in the general case. For that reason, we introduce these extensions one-by-one and specialize our analysis by dropping fiscal shocks (i.e., no exogenous states Ω with shocks $d\mathcal{Z}$). These fiscal shocks are mostly a distraction that complicates expressions and introduces extra state variables. That said, the reader can find the general equations including surplus shocks in Appendix I.1 (and see Lemma I.2 for a generic result including surplus shocks and all the extensions simultaneously). Given the absence of Ω, we can specialize Definition 3 to the following class of equilibria.

Definition 4. An *x-Markov equilibrium* is a non-explosive equilibrium in which inflation π_t and volatility $\sigma_{x,t}$ are functions of x_t.

I.2.1 Long-term debt

One important generalization replaces short-term debt with long-term debt. This is naturally of interest because short-term debt prices can never respond to shocks. This may lead one to think that short-term debt mechanically, in a knife-edge sense, rules out self-fulfilling demand volatility.

To develop the intuition, we first consider the special example where the interest rate is pegged $\iota_t = \bar{\iota}$. Recall the result that, with $s_t = \bar{s}$, the right-hand-side of (I.8) equals $\rho^{-1}\bar{s}e^{x_t}Y^*$. Equate this expression to $Q_t B_t / P_t$ and apply Itô's formula to both sides, recalling equation (I.6) for $\dot{B}_t$ and that $\dot{P}_t / P_t = \pi_t$. By matching the "dZ" terms, we obtain

$$\sigma_{Q,t} = \sigma_{x,t}, \tag{I.20}$$

where σ_Q denotes the sunspot loading of $\log(Q_t)$ on dZ_t. In other words, the self-fulfilling demand shocks must be absorbed by long-term debt prices. The key question is whether the pricing of long-term debt in (I.7) is consistent with this absorption.

Now, to price each bond, note that the nominal SDF in this setting is

$$\frac{M_t}{P_t} = \exp\Big[- \int_0^t \iota_u du - \frac{1}{2}\int_0^t \sigma_{x,u}^2 du - \int_0^t \sigma_{x,u} dZ_u \Big].$$

Using the notation $\tilde{\mathbb{E}}$ for the risk-neutral expectation (which absorbs the martingale $\frac{1}{2}\int_0^t \sigma_{x,u}^2 du - \int_0^t \sigma_{x,u} dZ_u$), the debt price from (I.7) is then

$$Q_t = \tilde{\mathbb{E}}_t \Big[\int_t^\infty \beta e^{-\int_t^T (\iota_u + \beta) du} dT \Big].$$

Finally, use the assumption of a pegged interest rate $\iota_t = \bar{\iota}$, which implies $Q_t = \frac{\beta}{\bar{\iota}+\beta}$. Debt prices are constant, so $\sigma_Q = 0$, and therefore equation (I.20) implies $\sigma_x = 0$. In fact, the risk-neutral bond pricing formula just above reveals that the *only way* self-fulfilling demand can enter Q_t is via the interest rate rule. But this suggests that the result is much more general than the peg example: monetary policy would need to follow a very particular rule in order to create fluctuations in the bond price that are consistent with self-fulfilling demand, which generically would not happen.

With unpegged interest rates, the debt price is no longer constant and can have volatility. However, the volatility implied by the bond pricing equation (I.7) is inconsistent with the bond price volatility required to support self-fulfilling demand in (I.8), unless all these volatilities are zero. To summarize the reasoning, the introduction of long-term debt allows for one extra degree of freedom, namely σ_Q, to absorb self-fulfilling demand shocks, but it also introduces an extra constraint, namely the no-arbitrage pricing equation for a single unit of debt. If σ_Q were some arbitrary process absorbing demand shocks, that would violate the pricing equation for debt.

Theorem I.1. *Consider the economy with long-term debt and $s_t = \bar{s}$. Suppose equilibrium is x-Markov. Then, the economy generically has $\sigma_{x,t} = 0$.*

I.2.2 Fiscal rules

Our next generalization allows surpluses to respond to endogenous variables, similarly to the interest rate rule. Suppose again that $S_t = s_t Y_t$, where

$$s_t = s(x_t, \pi_t), \tag{I.21}$$

for some bounded function s that satisfies $s(0,0) = \bar{s} > 0$. In this environment, we will also specialize to the linear Taylor rule (linear MP) to keep the analysis tractable.

Repeating the debt valuation computation from (GD), we obtain

$$\frac{B_t}{P_t} = Y_t \Psi_t, \tag{I.22}$$

$$\text{where} \quad \Psi_t := \mathbb{E}_t \Big[\int_t^\infty e^{-\rho(T-t)} s_T dT \Big] \tag{I.23}$$

In the class of x-Markov equilibria of Definition 4, we have the major simplification that $\Psi_t = \Psi(x_t)$ for some function Ψ that only depends on x_t. In that case, even without computing the function Ψ, by applying Itô's formula to (I.22) and examining the loading on the sunspot shock dZ, we can say that

$$0 = \sigma_{x,t} \big[\Psi(x_t) + \Psi'(x_t) \big] \tag{I.24}$$

One possibility is $\sigma_x = 0$, which is the natural case we hope to prove. On the other hand, if $\sigma_x \neq 0$, then the present-value of future surpluses needs to inherit any output gap volatility, implying a particular functional form for Ψ. What we show is that this functional form is generically inconsistent with equation (I.23), which provides a different equation for Ψ, unless inflation $\pi(x)$ and volatility $\sigma_x(x)$ take a particular form. Then, we show that this particular sunspot form, under some conditions on the policy rules, implies unstable dynamics, meaning that $\sigma_x = 0$ must hold.

Theorem I.2. *Consider the economy with fiscal rule (I.21) and monetary rule (linear MP), with $\phi_x > 0$, $\phi_\pi < 1$, and $\frac{\phi_x}{1-\phi_\pi} > -\frac{s+\partial_x s}{\partial_\pi s}$. Suppose equilibrium is x-Markov. Then, the economy generically has $\sigma_{x,t} = 0$.*

I.2.3 General CRRA utility

Finally, we replace log utility with general CRRA $u(c,l) = \frac{c^{1-\gamma}}{1-\gamma} - \frac{l^{1+\varphi}}{1+\varphi}$. This extension is of interest because log utility exhibits the knife-edge property that the present-value of future surplus growth can have no net fluctuations from "discount rates" in excess of "cash flows", since the log utility SDF is related to the inverse of output.

In the CRRA world, two changes arise from the new consumption FOC $M_t = e^{-\rho t} Y_t^{-\gamma}$. First, the IS curve now takes the slightly different form (I.4), and it depends on γ. Second, the present value of surpluses is now different: with a constant surplus-to-output ratio $s_t = \bar{s}$, we have

$$\mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} S_u du\Big] = \bar{s} Y_t \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(u-t)}\Big(\frac{Y_u}{Y_t}\Big)^{1-\gamma} du\Big]$$

The important point relative to log utility is that the present-value of surpluses can now admit an additional type of fluctuation, because discount rates $\frac{M_u}{M_t} = e^{-\rho(u-t)}(\frac{Y_u}{Y_t})^{-\gamma}$ do not exactly offset surplus growth $\frac{S_u}{S_t} = \frac{Y_u}{Y_t}$. This potentially permits short-run volatility σ_x because it can be absorbed, leaving the present-value of surpluses unaffected. That said, we prove that our key result carries over to CRRA preferences in some cases. The key intuition is that the absorption of short-run volatility by future discount rates requires a very particular specification for the present-value of surpluses, which will generically not arise.

Theorem I.3. *Consider the economy with CRRA utility. Suppose $s_t = \bar{s} > 0$ and monetary policy follows a linear Taylor rule (linear MP) with $\frac{\phi_x}{1-\phi_\pi} > 0$. Suppose equilibrium is x-Markov. Then, the economy generically has $\sigma_{x,t} = 0$.*

Remark I.1 (Conditions on policy rules)**.** *One may notice that Theorems I.2-I.3 included additional conditions on the ratio $\frac{\phi_x}{1-\phi_\pi}$ beyond what was required for the other cases. It is important to realize that these conditions are only needed in ruling out a single particular sunspot equilibrium. Specifically, the proofs show that active fiscal policy generically rules out all sunspot equilibria except a particular one where $\sigma_x(x)$ and $\pi(x)$ are uniquely-determined functions. The conditions on the policy rules are needed to rule out this final sunspot equilibrium.*

I.3 Inflation determination with active fiscal

Theorems I.1-I.3 only provide a "local" result, i.e., that $\sigma_x = 0$, without characterizing the full dynamic equilibrium. They also show that inflation is not determined from the debt valuation equation alone. Monetary policy is needed to pin inflation down.

For tractability, we specialize here to a quasi-linear setting that corresponds to Example B.1 in Section B.1. In particular, this section assumes a linearized Phillips curve (linear PC), the linear Taylor rule (linear MP), and an Ornstein-Uhlenbeck process for monetary shocks $d\mathcal{E}_t = -\zeta \mathcal{E}_t dt + dZ_t$, which is the continuous-time version of an AR(1). In Section B.1, we obtained a closed-form solution for the MSV equilibrium in this setting.

Here with active fiscal policy, we will obtain a different solution. For the fiscal side, we assume the surplus-to-output has dynamics following $ds_t = \lambda_s(\bar{s} - s_t) + \varsigma_{s,t} \cdot d\mathcal{Z}_t$, for some arbitrary volatility process $\varsigma_{s,t}$. We also assume debt is short-term as in the baseline specification. Finally, to keep the analysis simple, we assume monetary policy adopts the time-varying target rate $\bar{\iota}_t = \rho - \frac{1}{2}|\varsigma_{x,t}|^2$, where ς_x is the endogenous sensitivity of x to the fiscal shocks $d\mathcal{Z}$. Because Theorem 2 pins down $\varsigma_{x,t}$ uniquely, its inclusion in the target rate is conceptually distinct from the "risk premium targeting" we studied in Section 3.1. The present target rate only serves as a normalization, so that the economy fluctuates around $(x, \pi) = (0, 0)$ rather than inheriting a non-zero steady-state.

Writing the equilibrium dynamics in vector form, with $F_t := (x_t, \pi_t, \mathcal{E}_t)'$, we have

$$dF_t = \mathcal{A}F_t dt + \mathcal{B}_t dZ_t + \mathcal{C}_t d\mathcal{Z}_t,$$

$$\text{where} \quad \mathcal{A} := \begin{bmatrix} \phi_x & \phi_\pi - 1 & \sigma \\ -\kappa & \rho & 0 \\ 0 & 0 & -\zeta \end{bmatrix}, \quad \mathcal{B}_t := \begin{bmatrix} 0 \\ \sigma_{\pi,t} \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathcal{C}_t := \begin{bmatrix} \varsigma'_{x,t} \\ \varsigma'_{\pi,t} \\ 0 \end{bmatrix}$$

Notice that the first entry of $\mathcal{B}_t$ is zero, because of Theorem 2. Theorem 2 also restricts the first entry of surplus shock loadings $\mathcal{C}_t$. We follow a relatively standard analysis by doing a spectral decomposition of the transition matrix $\mathcal{A} = V\Lambda V^{-1}$, and analyzing the rotated system $\tilde{F}_t := V^{-1}F_t$. By integrating the system $d\tilde{F}_t = \Lambda \tilde{F}_t dt + V^{-1}\mathcal{B}_t dZ_t + V^{-1}\mathcal{C}_t \cdot d\mathcal{Z}_t$, we obtain

$$\mathbb{E}_0 \tilde{F}_t = \exp(\Lambda t)\tilde{F}_0. \tag{I.25}$$

The rest is a familiar stability analysis of (I.25), in view of the non-explosion Condition 1. One eigenvalue of $\mathcal{A}$ is $-\zeta$, corresponding to the exogenous monetary shock. There are three cases regarding the other two eigenvalues that correspond to the dynamics of (x, π): both eigenvalues have positive real parts, the eigenvalues have opposite signs, or both eigenvalues have negative real parts. Pursuing this analysis, we then obtain the following generalization of some familiar results.

Proposition I.1. *Consider the economy with: the linearized Phillips curve (linear PC); the linear Taylor rule (B.1) with target rate $\bar{\iota}_t = \rho - \frac{1}{2}|\varsigma_{x,t}|^2$ and monetary shocks $\sigma\mathcal{E}_t$, where $\mathcal{E}_t$ follows the process $d\mathcal{E}_t = -\zeta\mathcal{E}_t dt + dZ_t$; and the surplus dynamics $ds_t = \lambda_s(\bar{s} - s_t) + \varsigma_{s,t} \cdot d\mathcal{Z}_t$. A non-explosive equilibrium takes one of three forms, ignoring knife-edge cases for the parameters:*

1. *If $\rho + \phi_x > 0$ and $\rho\phi_x + \kappa(\phi_\pi - 1) > 0$, then equilibrium generically fails to exist.*

2. *If $\rho\phi_x + \kappa(\phi_\pi - 1) < 0$, the unique equilibrium features*

$$\pi_t = \frac{1}{\phi_\pi - 1}\Big[\beta x_t + \frac{\beta(\rho+\zeta) - \kappa(\phi_\pi - 1)}{(\phi_x + \zeta)(\rho+\zeta) + \kappa(\phi_\pi - 1)}\sigma\mathcal{E}_t\Big],$$

   where $\beta := \frac{1}{2}\big(\rho - \phi_x - \sqrt{(\rho - \phi_x)^2 - 4\kappa(\phi_\pi - 1)}\big)$.

3. *If $\rho + \phi_x < 0$ and $\rho\phi_x + \kappa(\phi_\pi - 1) > 0$, then the equilibrium is not unique.*

In all cases, the output gap is pinned down by fiscal states $(\frac{B_t}{P_t}, s_t)$ as $x_t = \log(\frac{B_t/P_t}{\Psi(s_t)Y^*})$ for some function $\Psi(s)$.

Proposition I.1 is reminiscent of the large literature of fiscal theory of the price level (FTPL) in linearized NK models. Equilibrium cannot exist with both “active fiscal” and “active money” regimes (case 1). Equilibrium exists and is unique when “passive money” is paired with “active fiscal” (case 2). Passive money just means that $\rho\phi_x + \kappa(\phi_\pi - 1) < 0$, which says that monetary policy is not too aggressive. These results echo Leeper (1991). A finding which differs slightly from the literature is our case 3: monetary policy that acts super aggressively against inflation but acts counterintuitively to output induces non-uniqueness, despite active fiscal policy. This case can have self-fulfilling inflation dynamics because monetary policy induces globally stable dynamics through its rule.

Of these three cases, the interesting case is the active-fiscal passive-money regime (case 2), which delivers a unique equilibrium. There are two important takeaways. First, FTPL ensures uniqueness for a broad range of monetary policy rules and not some knife-edge rule. A unique equilibrium is achieved even under an interest rate peg, which fits into case 2 by $\phi_x = \phi_\pi = \sigma = 0$. Second, the FTPL equilibrium is observationally distinct from the self-fulfilling stochastic equilibria under the Taylor principle. Under FTPL, the output gap x_t is pinned down as a function of the real debt balance B_t/P_t and the primary surplus level s_t. All output gap volatility is tied to fiscal states and not directly to monetary shocks. By contrast, the volatile equilibria possible under the Taylor principle feature an entire class of possible output gap dynamics which are all decoupled from fiscal states.

I.4 Proofs for Fiscal Policy Extensions

Proof of Lemma I.1. We apply Itô's formula to both sides of (I.9), using the flow government budget constraint (I.6), the price level dynamics $dP_t/P_t = \pi_t dt$, the dynamics of x_t in (I.4), the dynamics of Ψ_t in (I.13), and the dynamics of Q_t in (I.10) and (I.12). Matching drift and diffusion coefficients, we obtain

$$\begin{aligned}
[dt] \ : \ \frac{Q_t B_t}{P_t}\big[\iota_t - \pi_t + \gamma\sigma_{x,t}\sigma_{Q,t} + \gamma\varsigma_{x,t}\cdot\varsigma_{Q,t}\big] - s_t Y_t & \\
= Y_t^{\gamma}(\rho\Psi_t - s_t Y_t^{1-\gamma}) + \gamma Y_t^{\gamma}\Psi_t\Big[\frac{\iota_t - \pi_t - \rho}{\gamma} + \frac{1}{2}(\gamma+1)(\sigma_{x,t}^2 + |\varsigma_{x,t}|^2)\Big] & \\
+ \frac{1}{2}\gamma(\gamma-1)Y_t^{\gamma}\Psi_t(\sigma_{x,t}^2 + |\varsigma_{x,t}|^2) + \gamma Y_t^{\gamma}\sigma_{x,t}\sigma_{\Psi,t} + \gamma Y_t^{\gamma}\varsigma_{x,t}\cdot\varsigma_{\Psi,t} & \\
[dZ] \ : \ \frac{Q_t B_t}{P_t}\sigma_{Q,t} = \gamma Y_t^{\gamma}\Psi_t\sigma_{x,t} + Y_t^{\gamma}\sigma_{\Psi,t} & \\
[d\mathcal{Z}] \ : \ \frac{Q_t B_t}{P_t}\varsigma_{Q,t} = \gamma Y_t^{\gamma}\Psi_t\varsigma_{x,t} + Y_t^{\gamma}\varsigma_{\Psi,t} &
\end{aligned}$$

Equations $[dZ]$ and $[d\mathcal{Z}]$, combined with (I.9), imply (I.14)-(I.15).

Conversely, plugging (I.14)-(I.15) into the first equation $[dt]$, using (I.9), and simplifying, we obtain an identity. Therefore, the $[dt]$ equation holds automatically, given the other equations all hold. This means that, provided (I.9) holds at $t = 0$, it will hold at every future date $t > 0$. □

Proof of Lemma I.2. We start by using the (x, Ω)-Markov assumption, which implies all dynamics are fully Markovian in (x_t, Ω_t). Hence, the bond price Q_t and the present-value Ψ_t solely functions of x_t and Ω_t, i.e., $Q_t = Q(x_t, \Omega_t)$ and $\Psi_t = \Psi(x_t, \Omega_t)$ for some functions $Q(\cdot)$ and $\Psi(\cdot)$ to be determined.[21]

Let us define the differential operator $\mathscr{L}$ that acts on C^2 functions g of (x, Ω) by

$$\mathscr{L}g = \Big(\mu_x\partial_x + \mu_\Omega'\partial_\Omega + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx} + \frac{1}{2}\mathrm{tr}(\varsigma_\Omega'\varsigma_\Omega\partial_{\Omega\Omega'}) + \varsigma_x'\varsigma_\Omega\partial_{\Omega x}\Big)g \tag{I.26}$$

[21] Indeed, in an (x, Ω)-Markov equilibrium, we have that (x_t, Ω_t) is a bivariate Markov diffusion. Now, recall the bond pricing equation (I.7), which after plugging in the nominal SDF from (I.11) says

$$Q_t = \mathbb{E}_t\Big[\int_t^\infty e^{-\int_t^u(\iota_\tau + \frac{1}{2}\gamma^2(\sigma_{x,\tau}^2 + |\varsigma_{x,\tau}|^2))d\tau - \int_t^u \gamma\sigma_{x,\tau}dZ_\tau - \int_t^u \gamma\varsigma_{x,\tau}\cdot d\mathcal{Z}_\tau}\beta e^{-\beta(u-t)}du\Big].$$

Since $\iota_t = \bar{\iota} + \Phi(x_t, \pi_t) = \bar{\iota} + \Phi(x_t, \pi(x_t, \Omega_t))$ is purely a function of (x_t, Ω_t), as are $\sigma_{x,t}$ and $\varsigma_{x,t}$, the bond pricing equation above implies that Q_t is purely a function of (x_t, Ω_t). Similarly, we have that surpluses s_t are solely a function of (x_t, Ω_t). Given the definition of Ψ_t in (I.9), i.e., $\Psi_t := \mathbb{E}_t\big[\int_t^\infty e^{-\rho(u-t)}s_u Y_u^{1-\gamma}du\big]$, and given that $Y_t = Y^* e^{x_t}$, we obtain that Ψ_t is a function of (x_t, Ω_t) alone.

This operator produces drifts of any process which is a function of (x, Ω). Apply Itô's formula to Q and Ψ to obtain (after dropping t subscripts)

$$Q\sigma_Q = \sigma_x \partial_x Q \tag{I.27}$$

$$Q\varsigma_Q = \varsigma_x \partial_x Q + \varsigma_\Omega \partial_\Omega Q \tag{I.28}$$

$$Q\mu_Q = \mathscr{L} Q \tag{I.29}$$

$$\sigma_\Psi = \sigma_x \partial_x \Psi \tag{I.30}$$

$$\varsigma_\Psi = \varsigma_x \partial_x \Psi + \varsigma_\Omega \partial_\Omega \Psi \tag{I.31}$$

$$\mu_\Psi = \mathscr{L}\Psi \tag{I.32}$$

Combining these results with equations (I.12), (I.13), (I.14), and (I.15), we obtain

$$\gamma\sigma_x = \sigma_x \partial_x Q/Q - \sigma_x \partial_x \Psi/\Psi \tag{I.33}$$

$$\gamma\varsigma_x = \varsigma_x \partial_x Q/Q - \varsigma_x \partial_x \Psi/\Psi + \varsigma_\Omega \partial_\Omega Q/Q - \varsigma_\Omega \partial_\Omega \Psi/\Psi \tag{I.34}$$

and

$$(\beta + \bar{\iota} + \Phi(x,\pi))Q - \beta + \gamma\sigma_x^2 \partial_x Q + \gamma|\varsigma_x|^2 \partial_x Q + \gamma\varsigma_x \cdot \varsigma_\Omega \partial_\Omega Q = \mathscr{L} Q \tag{I.35}$$

$$\rho\Psi - s(\Omega, x, \pi)(Y^*)^{1-\gamma} e^{(1-\gamma)x} = \mathscr{L}\Psi \tag{I.36}$$

The equations above hold in all the particular specifications considered in the paper. Note that in the short-term debt case, which can be derived by taking $\beta \to \infty$, equation (I.35) implies $Q \to 1$ uniformly, as we have imposed in the paper. In addition, after taking this limit we have $\lim_{\beta\to\infty} \partial_x Q = 0$ and $\lim_{\beta\to\infty} \partial_\Omega Q = 0$, and so $\lim_{\beta\to\infty}(\frac{\beta}{Q} - \beta) = \bar{\iota} + \Phi(x,\pi)$. This limiting result is also consistent with taking the $\beta \to \infty$ in the flow budget constraint (I.6) in order to recover (27).

Now, suppose $\sigma_x \neq 0$. In that case, equation (I.33) says that $\gamma = \partial_x Q/Q - \partial_x \Psi/\Psi$, and equation (I.34) says that $\partial_\Omega Q/Q = \partial_\Omega \Psi/\Psi$. The first equation implies that $Q(x,\Omega) = \Psi(x,\Omega)G(\Omega)e^{\gamma x}$ for some function $G(\cdot)$. The second equation implies that $G(\Omega) = G$ constant. Thus,

$$Q(x,\Omega) = G\Psi(x,\Omega)e^{\gamma x}. \tag{I.37}$$

Note that then G is pinned down by equation (I.9) at time $t = 0$, since combining that equation with (I.37) says $\frac{B_0}{P_0}G = (Y^*)^\gamma$. Thus, (I.37) pins down Q given Ψ. Substitute

(I.37) into equation (I.35) and then subtract equation (I.36) to get

$$\frac{s(\Omega,x,\pi)(Y^*)^{1-\gamma}}{\Psi}e^{(1-\gamma)x} - \rho + \beta + \bar{\iota} + \Phi(x,\pi) - \frac{\beta}{G\Psi}e^{-\gamma x} = \gamma\mu_x - \frac{1}{2}\gamma^2(\sigma_x^2 + |\varsigma_x|^2)$$

Now, plug in μ_x from the IS curve (I.4) to get

$$0 = \pi + \frac{s(\Omega,x,\pi)(Y^*)^{1-\gamma}}{\Psi}e^{(1-\gamma)x} + \beta - \frac{\beta}{G\Psi}e^{-\gamma x} \tag{I.38}$$

Note that, for the short-term debt case, $\lim \beta(1 - \frac{1}{G\Psi}e^{-\gamma x}) = -(\bar{\iota} + \Phi(x,\pi))$ as argued above. Equation (I.38) thus pins down Ψ given π. Since we only used so far the difference between equations (I.35) and (I.36), we still need to ensure that one of them holds in isolation. Thus, consider equation (I.36), after plugging in μ_x from (I.4):

$$\begin{aligned} &\rho\Psi - s(\Omega,x,\pi)(Y^*)^{1-\gamma}e^{(1-\gamma)x} \\ &\quad= \Big[\frac{\bar{\iota} + \Phi(x,\pi) - \pi - \rho}{\gamma} + \frac{1}{2}\gamma(\sigma_x^2 + |\varsigma_x|^2)\Big]\partial_x\Psi + \mu_\Omega'\partial_\Omega\Psi + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx}\Psi \\ &\quad+ \frac{1}{2}\mathrm{tr}(\varsigma_\Omega'\varsigma_\Omega\partial_{\Omega\Omega'}\Psi) + \varsigma_x'\varsigma_\Omega\partial_{\Omega x}\Psi \end{aligned} \tag{I.39}$$

Given $(\pi,\sigma_x,\varsigma_x)$, equation (I.39) is a PDE for Ψ. Finally, recall the Phillips curve (I.5), apply Itô's formula to a generic inflation function $\pi(x,\Omega)$ to replace μ_π, and then plug in μ_x from (I.4):

$$\begin{aligned} &\rho\pi - \kappa f(x) \\ &\quad= \Big[\frac{\bar{\iota} + \Phi(x,\pi) - \pi - \rho}{\gamma} + \frac{1}{2}\gamma(\sigma_x^2 + |\varsigma_x|^2)\Big]\partial_x\pi + \mu_\Omega'\partial_\Omega\pi + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx}\pi \\ &\quad+ \frac{1}{2}\mathrm{tr}(\varsigma_\Omega'\varsigma_\Omega\partial_{\Omega\Omega'}\pi) + \varsigma_x'\varsigma_\Omega\partial_{\Omega x}\pi \end{aligned} \tag{I.40}$$

where $f(x) := \frac{e^{(\gamma+\varphi)x}-1}{\gamma+\varphi}$ if we are using a nonlinear Phillips curve and $f(x) = x$ if we are using a linearized Phillips curve. Given (σ_x,ς_x), equation (I.40) is a PDE for π.

At this point, consider the following experiment. Suppose $\pi(x,\Omega)$ is any function. Then, equation (I.38) pins down $\Psi(x,\Omega)$ uniquely, and equation (I.37) pins down $Q(x,\Omega)$ uniquely. Given π and Ψ, we can compute all their derivatives, and so equations (I.39) and (I.40) pin down 2 dimensions of the $1+\dim(\mathcal{Z})$ dimensional vector (σ_x,ς_x). In other words, we must pick σ_x and/or ς_x in order to ensure equations (I.39) and (I.40) hold.

Thus, if $\dim(\mathcal{Z}) = 0$, then either $\sigma_x = 0$, or $\pi(x)$ and $\sigma_x(x)^2$ must take a particular form. Note also that these functions are independent of Ω since there are no surplus shocks (hence Ω is not a state variable for any object in the case $\dim(\mathcal{Z}) = 0$). $\square$

Proof of Theorem I.1. The combination of log utility ($\gamma = 1$) and constant surplus-to-output ratio $s_t = \bar{s}$ implies that $\Psi_t = \bar{s}/\rho$ is constant for any $\pi(x)$ and any $\sigma_x(x)$ functions. We then specialize the results of Corollary I.1 as follows. Equation (I.17) pins down inflation as

$$\pi(x) = \frac{\rho\beta}{G\bar{s}} e^{-x} - \beta - \rho, \quad \text{when} \quad \sigma_x \neq 0. \tag{I.41}$$

Note that $\pi'(x) + \pi''(x) = 0$. Then, equation (I.18) implies that, after plugging in the derivatives of π from (I.41),

$$e^{-x} \frac{\rho\beta}{G\bar{s}} \big(\bar{\imath} + \Phi(x,\pi) - \pi\big) - \kappa f(x) = \rho(\rho+\beta), \quad \text{when} \quad \sigma_x \neq 0. \tag{I.42}$$

But everything is pinned down in equation (I.42). The result cannot be consistent with the solution for π in (I.41) unless the monetary policy rule Φ takes a knife-edge form, and so generically we reach a contradiction. Thus, $\sigma_x = 0$ must hold. $\square$

Proof of Theorem I.2. We specialize the results of Corollary I.1 as follows. Using log utility ($\gamma = 1$) and short-term debt ($\beta \to \infty$) in equation (I.17) implies that

$$\Psi(x) = \bar{\Psi} e^{-x}, \quad \text{when} \quad \sigma_x \neq 0,$$

for $\bar{\Psi} = 1/G$. Notice that $\Psi'(x) + \Psi''(x) = 0$ in this solution. Thus, equation (I.19), after plugging in the solution for Ψ and its derivatives, says that

$$s(x,\pi) = (\bar{\imath} + \Phi(x,\pi) - \pi)\bar{\Psi} e^{-x}, \quad \text{when} \quad \sigma_x \neq 0. \tag{I.43}$$

Equation (I.43) pins down π uniquely when $\sigma_x \neq 0$, unless the rules $s(\cdot), \Phi(\cdot)$ take a knife-edge form. Finally, equation (I.18) specializes to

$$\sigma_x^2 = \tilde{\sigma}_x^2 := 2\frac{\rho\pi - \kappa f(x) - [\bar{\imath} + \Phi(x,\pi) - \pi - \rho]\pi'}{\pi' + \pi''}, \quad \text{when} \quad \sigma_x \neq 0. \tag{I.44}$$

Given the solution for π, this pins down σ_x^2 uniquely when it is non-zero.

Given the functions $\Psi(x)$, $\pi(x)$, and $\sigma_x(x)^2$ are all pinned down assuming $\sigma_x \neq 0$, it remains to verify that the candidate sunspot equilibrium explodes, which then implies $\sigma_x = 0$. First, we want to show that the solution for $\tilde{\sigma}_x^2$ in (I.44) necessarily becomes negative at some $\underline{x} > -\infty$, which implies that non-exposiveness requires us to ensure

$x_t \geq \underline{x}$ for all t. Using the equation (I.43) for π, notice that s bounded implies that, as $x \to -\infty$, we must have $\bar{\imath} + \Phi(x,\pi) - \pi \to 0$. This then implies that, using the linear Taylor rule $\Phi(x,\pi) = \phi_x x + \phi_\pi \pi$,

$$\lim_{x \to -\infty} \left(\pi(x) - \frac{\bar{\imath} + \phi_x x}{1 - \phi_\pi} \right) = 0.$$

As $x \to \infty$, we thus have $\pi \to -\infty$ (using the assumption that $\phi_x > 0$ and $\phi_\pi < 1$), $\pi' \to \frac{\phi_x}{1-\phi_\pi}$, and $\pi'' \to 0$. Plugging these into (I.44), we obtain $\lim_{x \to -\infty} \tilde{\sigma}_x^2 = -\infty$. Thus, let us define $\underline{x} := \inf\{x : \tilde{\sigma}_x^2 = 0\} > -\infty$.

Second, note that to ensure $x_t \geq \underline{x}$, we require

$$\mu_x(\underline{x}) = \bar{\imath} + \Phi(\underline{x}, \pi(\underline{x})) - \pi(\underline{x}) - \rho \geq 0.$$

Rather than prove this cannot hold, we instead prove that the function $\tilde{\mu}_x(x) := \bar{\imath} + \Phi(x, \pi(x)) - \pi(x) - \rho$ is strictly increasing when π is given by (I.43). This suffices, since it implies that $x_t \to +\infty$ under any parameters such that $x_t \geq \underline{x}$, in violation of the non-explosion condition. Indeed, $\tilde{\mu}_x(x)$ is the drift of x_t when volatility is zero, and the volatility only serves to increase the drift.

Differentiating $\tilde{\mu}_x(x)$, using the linear form of the Taylor rule, and substituting π' from implicitly differentiating (I.43), we obtain

$$\tilde{\mu}_x' = \phi_x + (\phi_\pi - 1)\pi' = \phi_x - (\phi_\pi - 1)\frac{\partial_x s + s - \bar{\Psi} e^{-x} \phi_x}{\partial_\pi s + (1 - \phi_\pi)\bar{\Psi} e^{-x}}$$

Using the assumptions of the theorem that $\phi_x > 0$, $\phi_\pi < 1$, and $\frac{\phi_x}{1-\phi_\pi} > -\frac{s + \partial_x s}{\partial_\pi s}$, and noting that $\bar{\Psi} = G^{-1} = \frac{B_0}{P_0}(Y^*)^{-1} > 0$, we have

$$\tilde{\mu}_x' = \phi_x \left[1 - \frac{-\frac{\partial_x s + s}{\phi_x} + \bar{\Psi} e^{-x}}{\frac{\partial_\pi s}{1-\phi_\pi} + \bar{\Psi} e^{-x}} \right] > \phi_x \left[1 - \frac{\frac{\partial_\pi s}{1-\phi_\pi} + \bar{\Psi} e^{-x}}{\frac{\partial_\pi s}{1-\phi_\pi} + \bar{\Psi} e^{-x}} \right] = 0.$$

This proves that x_t necessarily explodes, implying that $\sigma_x = 0$ generically. □

Proof of Theorem I.3. We specialize the results of Corollary I.1 as follows. Using the assumption of short-term debt ($\beta \to \infty$) in equation (I.17) implies that

$$\Psi(x) = \bar{\Psi} e^{-\gamma x}, \quad \text{when} \quad \sigma_x \neq 0,$$

where $\bar{\Psi} = 1/G$. Notice that $\gamma \Psi'(x) + \Psi''(x) = 0$ in this solution. Then, equation (I.19) implies, given $s = \bar{s}$ and the function Ψ,

$$(Y^*)^{1-\gamma}\bar{s} = \big(\bar{\imath} + \Phi(x,\pi) - \pi\big)\bar{\Psi}e^{-x}, \quad \text{when} \quad \sigma_x \neq 0, \tag{I.45}$$

which pins down π uniquely when $\sigma_x \neq 0$, unless the rule $\Phi(\cdot)$ takes a knife-edge form. Finally, equation (I.18) specializes to

$$\sigma_x^2 = \tilde{\sigma}_x^2 := 2\frac{\rho\pi - \kappa f(x) - \frac{\bar{\imath}+\Phi(x,\pi)-\pi-\rho}{\gamma}\pi'}{\gamma\pi' + \pi''}, \quad \text{when} \quad \sigma_x \neq 0. \tag{I.46}$$

Given the solution for π, this pins down σ_x^2 uniquely when it is non-zero.

Given the functions $\Psi(x)$, $\pi(x)$, and $\sigma_x(x)^2$ are all pinned down assuming $\sigma_x \neq 0$, it remains to verify that the candidate sunspot equilibrium explodes, which then implies $\sigma_x = 0$. This step is almost identical to Theorem I.2. First, we want to show that the solution for $\tilde{\sigma}_x^2$ in (I.44) necessarily becomes negative at some $\underline{x} > -\infty$, which implies that non-exposiveness requires us to ensure $x_t \geq \underline{x}$ for all t. Using the equation (I.45), and the linear Taylor rule $\Phi(x,\pi) = \phi_x x + \phi_\pi \pi$, we solve for π and its derivatives explicitly as

$$\begin{aligned}
\pi(x) &= \frac{\bar{\Psi}^{-1}(Y^*)^{1-\gamma}\bar{s}e^x - \bar{\imath} - \phi_x x}{\phi_\pi - 1} \\
\pi'(x) &= \frac{\bar{\Psi}^{-1}(Y^*)^{1-\gamma}\bar{s}e^x - \phi_x}{\phi_\pi - 1} \\
\pi''(x) &= \frac{\bar{\Psi}^{-1}(Y^*)^{1-\gamma}\bar{s}e^x}{\phi_\pi - 1}
\end{aligned}$$

Notice that, as $x \to -\infty$, given the assumption that $\frac{\phi_x}{1-\phi_\pi} > 0$, we have $\pi \to -\infty$, $\pi' \to \frac{\phi_x}{1-\phi_\pi}$, and $\pi'' \to 0$. Plugging these into (I.46), we obtain $\lim_{x\to-\infty}\tilde{\sigma}_x^2 = -\infty$. Thus, let us define $\underline{x} := \inf\{x : \tilde{\sigma}_x^2 = 0\} > -\infty$.

Second, note that to ensure $x_t \geq \underline{x}$, we require

$$\mu_x(\underline{x}) = \frac{1}{\gamma}\Big[\bar{\imath} + \Phi(\underline{x}, \pi(\underline{x})) - \pi(\underline{x}) - \rho\Big] \geq 0.$$

Rather than prove this cannot hold, we instead prove that the function $\tilde{\mu}_x(x) := \gamma^{-1}[\bar{\imath} + \Phi(x,\pi(x)) - \pi(x) - \rho]$ is strictly increasing when π is given by (I.45). This suffices, since it implies that $x_t \to +\infty$ under any parameters such that $x_t \geq \underline{x}$, in violation of the non-explosion condition. Indeed, $\tilde{\mu}_x(x)$ is the drift of x_t when volatility is zero, and the volatility only serves to increase the drift.

Differentiating $\tilde{\mu}_x(x)$, using the linear form of the Taylor rule, and substituting π' from above, we obtain

$$\gamma\tilde{\mu}_x' = \phi_x + (\phi_\pi - 1)\pi' = \phi_x + (\phi_\pi - 1)\frac{\bar{\Psi}^{-1}(Y^*)^{1-\gamma}\bar{s}e^x - \phi_x}{\phi_\pi - 1} = \bar{\Psi}^{-1}(Y^*)^{1-\gamma}\bar{s}e^x > 0,$$

where we have used the fact that $\bar{s} > 0$ and $\bar{\Psi} = G^{-1} = \frac{B_0}{P_0}(Y^*)^{-\gamma} > 0$. This proves that x_t necessarily explodes, implying that $\sigma_x = 0$ generically. □

Proof of Proposition I.1. First, note that the spectral decomposition of $\mathcal{A} = V\Lambda V^{-1}$ is

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} v(\lambda_1) & v(\lambda_2) & v(\lambda_3) \end{bmatrix},$$

where the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and the corresponding eigenvectors $v(\lambda_1), v(\lambda_2), v(\lambda_3)$ are

$$\begin{aligned}
\lambda_1 &= \frac{1}{2}\left[\rho + \phi_x + \sqrt{(\rho - \phi_x)^2 - 4\kappa(\phi_\pi - 1)}\right] \\
\lambda_2 &= \frac{1}{2}\left[\rho + \phi_x - \sqrt{(\rho - \phi_x)^2 - 4\kappa(\phi_\pi - 1)}\right] \\
\lambda_3 &= -\zeta.
\end{aligned}$$

and

$$v(\lambda_1) = \begin{pmatrix} \frac{\phi_\pi - 1}{\lambda_1 - \phi_x} \\ 1 \\ 0 \end{pmatrix}, \quad v(\lambda_2) = \begin{pmatrix} \frac{\phi_\pi - 1}{\lambda_2 - \phi_x} \\ 1 \\ 0 \end{pmatrix}, \quad v(\lambda_3) = \begin{pmatrix} -(\rho + \zeta)\sigma \\ -\kappa\sigma \\ (\phi_x + \zeta)(\rho + \zeta) + \kappa(\phi_\pi - 1) \end{pmatrix}$$

Recall equation (I.25) that

$$\mathbb{E}_0[\tilde{F}_t] = \exp(\Lambda t)\tilde{F}_0, \tag{I.47}$$

where $\tilde{F}_t = V^{-1}F_t$ is a rotated version of the state $F_t = (x_t, \pi_t, \mathcal{E}_t)'$, and where

$$V^{-1} = \frac{1}{\lambda_2 - \lambda_1}\begin{bmatrix} \frac{(\lambda_1 - \phi_x)(\lambda_2 - \phi_x)}{\phi_\pi - 1} & -(\lambda_1 - \phi_x) & \frac{\lambda_1 - \phi_x}{\phi_\pi - 1}\frac{(\lambda_2 - \phi_x)(\rho + \zeta) - \kappa(\phi_\pi - 1)}{(\phi_x + \zeta)(\rho + \zeta) + \kappa(\phi_\pi - 1)}\sigma \\ -\frac{(\lambda_1 - \phi_x)(\lambda_2 - \phi_x)}{\phi_\pi - 1} & \lambda_2 - \phi_x & -\frac{\lambda_2 - \phi_x}{\phi_\pi - 1}\frac{(\lambda_1 - \phi_x)(\rho + \zeta) - \kappa(\phi_\pi - 1)}{(\phi_x + \zeta)(\rho + \zeta) + \kappa(\phi_\pi - 1)}\sigma \\ 0 & 0 & \frac{\lambda_2 - \lambda_1}{(\phi_x + \zeta)(\rho + \zeta) + \kappa(\phi_\pi - 1)} \end{bmatrix}.$$

In equation (I.47), $\exp(\Lambda t)$ refers to element-by-element exponentiation of Λ.

Let's consider the three cases of the proposition, using Condition 1 to kill explosive solutions to (I.47):

1. *Case 1:* $\rho + \phi_x > 0$ and $\rho\phi_x + \kappa(\phi_\pi - 1) > 0$.

   In this case, $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) > 0$. Therefore, all non-explosive solutions to (I.47) must satisfy $\tilde{F}_t^{(1)} = \tilde{F}_t^{(2)} = 0$. Using the expression for V^{-1}, this implies that

   $$x_t = -\frac{(\rho+\zeta)\sigma}{(\phi_x+\zeta)(\rho+\zeta)+\kappa(\phi_\pi-1)}\mathcal{E}_t \quad \text{and} \quad \pi_t = -\frac{\kappa\sigma}{(\phi_x+\zeta)(\rho+\zeta)+\kappa(\phi_\pi-1)}\mathcal{E}_t$$

2. *Case 2:* $\rho\phi_x + \kappa(\phi_\pi - 1) < 0$.

   In this case, both eigenvalues are real and have opposite signs: $\lambda_1 > 0 > \lambda_2$. Therefore, all non-explosive solutions to (I.47) must satisfy $\tilde{F}_t^{(1)} = 0$, which using the expression for V^{-1} implies

   $$\pi_t = \frac{\lambda_2 - \phi_x}{\phi_\pi - 1}x_t + \frac{(\lambda_2-\phi_x)(\rho+\zeta)-\kappa(\phi_\pi-1)}{(\phi_x+\zeta)(\rho+\zeta)+\kappa(\phi_\pi-1)}\frac{\sigma}{\phi_\pi-1}\mathcal{E}_t. \tag{I.48}$$

   Given $\sigma_{x,t} = 0$ from Theorem 2, this then implies $\sigma_{\pi,t} = 0$ as well.

3. *Case 3:* $\rho + \phi_x < 0$ and $\rho\phi_x + \kappa(\phi_\pi - 1) > 0$.

   In this case, $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) < 0$, meaning all initial conditions to (I.47) are non-explosive. Therefore, any $\tilde{F}_0$ corresponds to a valid equilibrium.

In all cases, we note that x_0 and $\varsigma_{x,t}$ are pinned down by (GD) at $t = 0$ and at $t > 0$, respectively. Indeed, using $ds_t = \lambda_s[s_t - \bar{s}]dt + \varsigma_{s,t} \cdot d\mathcal{Z}_t$ in equation (I.9), we obtain

$$\Psi_t = \Psi(s_t) := \frac{\bar{s}}{\rho} + \frac{s_t - \bar{s}}{\rho + \lambda_s},$$

which is exogenous. Using Ψ_t in (GD), we obtain

$$x_t = \log\Big(\frac{B_t/P_t}{\Psi(s_t)Y^*}\Big). \tag{I.49}$$

On the other hand, for $t > 0$, we have $\varsigma_\Psi = \frac{1}{\rho+\lambda_s}\varsigma_s$. Apply this in equation (I.15) of Lemma I.1, with $\gamma = 1$ and $Q \equiv 1$, to obtain

$$\varsigma_{x,t} = -\frac{\rho}{\lambda_s\bar{s} + \rho s_t}\varsigma_{s,t}. \tag{I.50}$$

Thus, x_t and $\varsigma_{x,t}$ are pinned down by B_t/P_t and s_t.

The remaining claims to prove are the existence/uniqueness statements. In Case 1, the equilibrium fails to exist generically, because $x_t \propto \sigma\mathcal{E}_t$ cannot be consistent with (I.49) and (I.50). In Case 2, the equilibrium is unique, because the initial condition $\mathcal{E}_0$ is given exogenously, x_0 is pinned down by (I.49), and so π_0 is pinned down by (I.48), and because the surplus shock exposures $\varsigma_{x,t}$ and $\varsigma_{\pi,t} = \frac{\lambda_2 - \phi_x}{\phi_\pi - 1}\varsigma_{x,t}$ are pinned down by (I.50). In Case 3, the equilibrium is not unique because, although x_0 is pinned down by (I.49), π_0 is not pinned down. Furthermore, π_t can have arbitrary non-fiscal volatility $\sigma_{\pi,t}$, despite the fact that $\sigma_{x,t} = 0$. $\square$