Dynamic Self-Fulfilling Fire Sales*

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Abstract

Why do fire sales occur if many risks are hedgeable? We study a version of Brunnermeier and Sannikov (2014) in which all fundamental risks can be hedged frictionlessly. Our analysis shows that fire sales are inherently self-fulfilling. Whereas fundamental shocks can never cause fire sales, there exist equilibria in which agents coordinate their sales on non-fundamental shocks. A simple refinement based on vanishingly-small perceived fundamental risk selects the fire sale equilibrium as the unique outcome.

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Why do fire sales occur if many risks are hedgeable? Why is risk concentrated on the balance sheets of intermediaries and productive experts? Which types of risks are concentrated? Our paper provides on new, self-fulfilling perspective on these questions.

We study a canonical macro-finance model, similar to Kiyotaki and Moore (1997) and Brunnermeier and Sannikov (2014), that has launched a large subsequent literature. The core mechanism in such models is sometimes referred to as a "financial accelerator" because the economic impact of fundamental shocks is amplified by imperfect risk-sharing.[1] Our departure from these models: we allow agents to frictionlessly hedge all fundamental risks. This feature brings our analysis closer to the model of Di Tella (2017).

As a general principle, when a risk is shared, the accelerator mechanism breaks down. To re-open the door to accelerator-type fluctuations, we limit risk-sharing in a novel way. While our model features markets to hedge all *fundamental* shocks, markets do not exist for hedging certain endogenous risks that emerge in a self-fulfilling manner, which we will refer to as "sunspot shocks." Sunspot shocks are non-fundamental, but agents could potentially coordinate on them. Such coordination affects equilibrium precisely because sunspot shocks are not directly hedgeable.

What emerges are self-fulfilling fire sales. More productive agents ("experts") manage a disproportionate share of productive capital and share its fundamental risks perfectly with everyone else ("households"). In this world, fundamental shocks cannot cause amplification. But equilibrium is not necessarily efficient or smooth. If a sunspot shock arrives, experts may coordinate to sell capital to households. If such coordinated selling occurs, the capital price would fall in equilibrium, due to the lower productive efficiency of households. From the perspective of an individual expert, the selling by other experts impinges her wealth and motivates her to also sell capital.

Our main result proves and constructs such an equilibrium. We then go on to demonstrate how our sunspot equilibrium is a limiting case of the conventional equilibrium. Imagine fundamental risks were not hedgeable, and denote their volatility σ. As $\sigma \to 0$, the conventional financial accelerator equilibrium converges to an equilibrium with volatility (formalizing the "volatility paradox" by Brunnermeier and San-

[1]Extensions of this framework have been used to study idiosyncratic uncertainty shocks (Di Tella, 2017); shadow banking (Moreira and Savov, 2017); bank capital regulation (Phelan, 2016; Klimenko et al., 2017); monetary policy (Drechsler et al., 2018); quantitative easing (Silva, 2017); optimal policy (Di Tella, 2019); the quantitative frequency and severity of crises (He and Krishnamurthy, 2019); bank runs (Gertler and Kiyotaki, 2015; Gertler et al., 2020; Mendo, 2020); extrapolative sentiments (Krishnamurthy and Li, 2020; Maxted, 2023); time-varying diversification (Khorrami, 2021); and long-run risks (Hansen et al., 2024). On the asset-pricing side, this literature is often referred to as "intermediary asset pricing" (He and Krishnamurthy, 2012, 2013). For a survey, see Brunnermeier and Sannikov (2016).

nikov, 2014). Agents continue to coordinate on the fundamental shock, even though it has zero volatility, and the result is identical to our sunspot equilibrium.

Finally, we provide a very simple refinement that ensures uniqueness. Imagine the sunspot shock, which is not hedgeable, is perceived by agents to have a vanishingly-small fundamental impact ς. We show that, as $\varsigma \to 0$, the unique limit is our fire-sale equilibrium, and the safe "fundamental equilibrium" can never prevail.

These results clarify that the financial accelerator mechanism is robust to the inclusion of financial markets for hedging fundamental shocks. So long as agents cannot hedge *every conceivable shock*, the door remains open to accelerator-type fluctuations.

1 Model

The model structure is the same as in Khorrami and Mendo (2024), which is a simplified version of Brunnermeier and Sannikov (2014) that does not include capital investment.

Information structure. There are two types of uncertainty in the economy, modeled as two independent Brownian motions (W, Z). The *fundamental shock* W directly impacts production possibilities, whereas the second shock Z is a *sunspot shock* that is extrinsic to economic primitives.

Technology and markets. There are two goods, non-durable consumption and durable capital that produces consumption. When an individual agent i holds capital $k_{i,t}$, it grows exogenously as

$$dk_{i,t} = k_{i,t}[g dt + \sigma dW_t], \tag{1}$$

where g, $\sigma > 0$ are exogenous constants. The capital-quality shock σdW introduces fundamental randomness in technology. The relative price of capital is denoted by q_t and is determined in equilibrium. (Note that (1) excludes the effect of capital trades.)

There are two agent types, experts and households, who differ in their productivity. Experts produce a_e units of output per unit of capital, whereas households' productivity is $a_h \in (0, a_e)$. Because all agents of the same type will ultimately behave as scaled versions of each other, we index agents $i \in \{e, h\}$ simply by their type.

Financial markets consist of a short-term, risk-free bond in zero net supply that pays interest rate r_t and a financial market for contracting on the fundamental shock, which offers expected return $\pi_t dt$ per unit of exposure to dW_t. The financial friction is that agents cannot issue equity when managing capital.[2]

Preferences and optimization. Given the stated assumptions, we can write the dynamic budget constraint of any agent of type $i \in \{e, h\}$ as

$$dn_{i,t} = \underbrace{\Big[(n_{i,t} - q_t k_{i,t}) r_t - c_{i,t}\Big] dt}_{\text{consumption-savings}} + \underbrace{q_t k_{i,t} \Big[\frac{a_i}{q_t} dt + \frac{d(q_t k_{i,t})}{q_t k_{i,t}}\Big]}_{\text{capital returns}} + \underbrace{x_{i,t} \big[\pi_t dt + dW_t\big]}_{\text{financial hedges}}, \tag{2}$$

where n is the agent's net worth, c is consumption, k is capital holdings, and x denotes hedging positions. Brunnermeier and Sannikov (2014) effectively imposes $x \equiv 0$ as a constraint.

Experts and households have logarithmic utility, with discount rates ρ_e and $\rho_h < \rho_e$, respectively. In the online appendix, we generalize to CRRA utility with alternative levels of risk aversion. Experts' higher discount rate ensures a stationary wealth distribution. Agents solve

$$\sup_{c \geq 0, k \geq 0, n \geq 0, x \in \mathbb{R}} \mathbb{E}\Big[\int_0^\infty e^{-\rho_i t} \log(c_{i,t}) dt\Big] \tag{3}$$

subject to (2). The solvency constraint $n_{i,t} \geq 0$ is the natural borrowing limit, given the absence of labor income. Optimization problem (3) is homogeneous in (c, k, n, x), so we can think of the expert and household as representative agents within their class.

1.1 Equilibrium definition

The definition of competitive equilibrium is standard.

Definition 1. For initial endowments $k_{e,0}, k_{h,0}$ such that $k_{e,0} + k_{h,0} = K_0$, an *equilibrium* consists of stochastic processes—adapted to the filtered probability space generated by $\{W_t, Z_t : t \geq 0\}$—for capital price q_t, interest rate r_t, risk price π_t, capital holdings $(k_{e,t}, k_{h,t})$, hedges $(x_{e,t}, x_{h,t})$, consumptions $(c_{e,t}, c_{h,t})$, and net worths $(n_{e,t}, n_{h,t})$, such that:

(i) initial net worths satisfy $n_{e,0} = q_0 k_{e,0}$ and $n_{h,0} = q_0 k_{h,0}$;

(ii) taking processes (q, r, π) as given, agents solve (3) subject to (2);

[2] Partial equity issuance, as long as there is some limit, will generate similar results.

(iii) consumption, capital, and hedging markets clear at all dates, i.e.,

$$c_{e,t} + c_{h,t} = a_e k_{e,t} + a_h k_{h,t} \tag{4}$$

$$k_{e,t} + k_{h,t} = K_t \tag{5}$$

$$x_{e,t} + x_{h,t} = 0, \tag{6}$$

where K_t follows the same dynamics as those given in (1).

1.2 Equilibrium characterization

We start with a useful equilibrium characterization. First, conjecture the following form for capital price dynamics:

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t}dW_t + \varsigma_{q,t}dZ_t]. \tag{7}$$

There are two potential avenues for random fluctuations. The standard term σ_q represents amplification (or dampening) of fundamental shocks, as in Brunnermeier and Sannikov (2014) and others. By contrast, ς_q measures sunspot volatility that only exists because agents believe in it.

Given log utility and the scale-invariance of agents' budget sets, individual optimization problems are readily solvable. Optimal consumption satisfies the standard formula $c_{i,t} = \rho_i n_{i,t}$. Capital holdings and financial hedges are determined via a mean-variance problem:

$$\max_{k\geq 0, x\in\mathbb{R}} \Big\{ \mathbb{E}[\frac{dn}{n}] - \frac{1}{2}\mathrm{Var}[\frac{dn}{n}] \Big\}$$

Plugging in capital and price dynamics in the dynamic wealth equation (2), and rearranging, this problem becomes

$$\max_{\tilde{k}\geq 0, \tilde{x}\in\mathbb{R}} \Big\{ \tilde{k}\Big(\frac{a}{q} + g + \mu_q + \sigma\sigma_q - (\sigma+\sigma_q)\pi\Big) + \tilde{x}\pi - \frac{1}{2}\big(\tilde{k}\varsigma_q\big)^2 - \frac{1}{2}\tilde{x}^2 \Big\},$$

where $\tilde{k} := \frac{qk}{n}$ and $\tilde{x} := \frac{x}{n} + \frac{qk}{n}(\sigma + \sigma_q)$ are the agent's per-unit-of-wealth exposures to the sunspot shock $\varsigma_q dZ$ and fundamental shock dW, respectively. Note that $\tilde{x}$ is unconstrained because x is unconstrained. The optimality conditions are

$$\frac{a_e}{q} + g + \mu_q + \sigma\sigma_q - r = \frac{qk_e}{n_e}\varsigma_q^2 \tag{8}$$

$$\frac{a_h}{q} + g + \mu_q + \sigma\sigma_q - r \leq \frac{qk_h}{n_h}\varsigma_q^2 \quad \text{(with equality if } k_h > 0) \tag{9}$$

for capital holdings and

$$\pi - \frac{qk_e}{n_e}(\sigma + \sigma_q) = \frac{x_e}{n_e} \tag{10}$$

$$\pi - \frac{qk_h}{n_h}(\sigma + \sigma_q) = \frac{x_h}{n_h} \tag{11}$$

for hedges. (Note that experts' capital optimality condition (8) assumes the solution is interior, i.e., $k_e > 0$. But this is clearly required in any equilibrium given experts earn a strictly higher expected return than households.) These conditions fully summarize optimality.[3]

Next, we aggregate. Due to financial frictions and productivity heterogeneity, both the distribution of wealth and capital holdings will matter in equilibrium. Define experts' wealth and capital shares:

$$\eta := \frac{n_e}{n_e + n_h} = \frac{n_e}{qK} \quad \text{and} \quad \kappa := \frac{k_e}{K}.$$

Given agents' solvency and capital short-sales constraints, we must have $\eta \in [0, 1]$ and $\kappa \in [0, 1]$ in equilibrium. Substitute optimal consumption into goods market clearing (4), divide by aggregate capital K, and use the definitions of η and κ, to obtain

$$q\bar{\rho} = \kappa a_e + (1-\kappa)a_h, \tag{PO}$$

where $\bar{\rho}(\eta) := \eta\rho_e + (1-\eta)\rho_h$ is the wealth-weighted average discount rate. Equation (PO) connects asset price q to output efficiency κ, which we call a *price-output* relation.

Using the definitions of η and κ, experts' and households' portfolio shares can be written $\frac{qk_e}{n_e} = \frac{\kappa}{\eta}$ and $\frac{qk_h}{n_h} = \frac{1-\kappa}{1-\eta}$. Then, differencing the optimal portfolio conditions

[3] The only additional optimality conditions are the transversality conditions $\lim_{T\to\infty} \mathbb{E}[e^{-\rho_i T}\frac{1}{c_{i,T}} n_{i,T}] = 0$. However, using $c_i = \rho_i n_i$, we see that transversality automatically holds.

(8)-(9), we obtain the *risk-balance* condition

$$0 = \min\left[1-\kappa,\ \frac{a_e - a_h}{q} - \frac{\kappa-\eta}{\eta(1-\eta)}\varsigma_q^2\right]. \tag{RB}$$

Either experts manage the entire capital stock ($\kappa = 1$) or the differential return of experts over households, $(a_e - a_h)/q$, represents fair compensation for differential risk exposure, $\frac{\kappa-\eta}{\eta(1-\eta)}\varsigma_q^2$. Summing portfolio conditions (8)-(9), weighted by κ and $1-\kappa$, yields an equation for r:

$$r = \frac{\kappa a_e + (1-\kappa)a_h}{q} + g + \mu_q + \sigma\sigma_q - \Big(\frac{\kappa^2}{\eta} + \frac{(1-\kappa)^2}{1-\eta}\Big)\varsigma_q^2. \tag{12}$$

Combining optimal financial hedges (10)-(11) with the zero net supply condition (6), we obtain a full risk-sharing result for the equilibrium risk price

$$\pi = \sigma + \sigma_q. \tag{13}$$

Finally, applying Itô's formula to η, and using net worth dynamics (2), wealth share dynamics are given by

$$d\eta_t = \mu_{\eta,t}dt + \sigma_{\eta,t}dW_t + \varsigma_{\eta,t}dZ_t, \quad \text{given} \quad \eta_0, \tag{14}$$

where

$$\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + (\kappa - 2\eta\kappa + \eta^2)\frac{\kappa-\eta}{\eta(1-\eta)}\varsigma_q^2 \tag{15}$$

$$\sigma_\eta = 0 \tag{16}$$

$$\varsigma_\eta = (\kappa-\eta)\varsigma_q. \tag{17}$$

Note the full risk-sharing result $\sigma_\eta = 0$ on fundamental shocks. Also, the initial wealth distribution $\eta_0 = \frac{n_{e,0}}{q_0 K_0} = \frac{k_{e,0}}{K_0}$ is pinned down by the initial capital endowments.

1.3 Fundamental Equilibrium

We focus on equilibria which are Markov in the state variable η. (Khorrami and Mendo, 2024, provides a rich analysis of equilibria which permit additional "sentiment variables" beyond η.) Among those, categorize equilibria into fundamental or sunspot, depending on whether dZ matters.

Definition 2. A *Fundamental Equilibrium* is an equilibrium in which $\varsigma_q \equiv 0$. Any other equilibrium is called a *Brownian Sunspot Equilibrium* (BSE).

The Fundamental Equilibrium is unique, efficient, and deterministic. Allocative efficiency is immediately evident: if $\varsigma_q \equiv 0$, then (RB) implies $\kappa = 1$ forever. Risk-sharing is also efficient, captured by deterministic relative wealth dynamics: from (16)-(17), and using $\varsigma_q = 0$, we have that $\sigma_\eta = 0$ and $\varsigma_\eta = (\kappa - \eta)\varsigma_q = 0$. In fact, the equilibrium obeys the following deterministic dynamics:

$$\frac{\dot{q}_t}{q_t} = \eta_t(1-\eta_t)\frac{(\rho_e - \rho_h)^2}{\eta_t\rho_e + (1-\eta_t)\rho_h} \tag{18}$$

$$\dot{\eta}_t = -\eta_t(1-\eta_t)(\rho_e - \rho_h) \tag{19}$$

(To obtain these equations, substitute $\kappa = 1$ into equation (PO) to get $q_t = a_e/\bar{\rho}(\eta_t)$, and combine with the previous results.) Finally, because η_0 is given and $q = a_e/\bar{\rho}(\eta)$ is purely a function of η, this Fundamental Equilibrium is *unique*: its initial condition is pinned down, and its dynamics are given uniquely by (18)-(19). Therefore, none of our results arise due to a multiplicity without sunspot shocks, unlike many classical sunspot equilibrium constructions.

Lemma 1 (Fundamental Equilibrium)**.** *There exists a unique Fundamental Equilibrium in which experts manage all capital, $\kappa = 1$, and its price $q_t = a_e/\bar{\rho}(\eta_t)$ evolves deterministically.*

2 Brownian Sunspot Equilibrium (BSE)

Our main results construct and characterize a Brownian Sunspot Equilibrium (BSE).

2.1 BSE: existence, uniqueness, and properties

To construct a BSE, start from the conjecture that the capital price is only a function of η, i.e., $q_t = q(\eta_t)$ for some function q. By Itô's formula, $\sigma_q = \frac{q'}{q}\sigma_\eta$ and $\varsigma_q = \frac{q'}{q}\varsigma_\eta$. Combining this with equations (16)-(17), we have $\sigma_q = 0$ and

$$\Big[1 - (\kappa - \eta)\frac{q'}{q}\Big]\varsigma_q = 0. \tag{20}$$

There are two possibilities: either (i) $\varsigma_q = 0$; or (ii) $1 = (\kappa - \eta)\frac{q'}{q}$, in which case ς_q can be non-zero.

Consider first the situation where $\kappa = 1$. It must be that $\varsigma_q = 0$ in this region. Indeed, if not then (20) implies $1 = (1-\eta)\frac{q'}{q}$, which requires q to be an increasing function of η. On the other hand, (PO) implies that $q = a_e/\bar{\rho}(\eta)$ is a decreasing function of η. Thus, no solution can exist; it must be that $\varsigma_q = 0$ instead.

Next, consider the more interesting situation where capital is inefficiently allocated: $\kappa < 1$. In this situation, $\varsigma_q = 0$ cannot hold (to see this, plug $\varsigma_q = 0$ into the risk-balance condition (RB) to see that $\kappa = 1$ would be required). And so equation (20) can only hold if $1 = (\kappa - \eta)\frac{q'}{q}$. Substituting $\kappa = \frac{q\bar{\rho} - a_h}{a_e - a_h}$ from (PO), we obtain a first-order ODE for q:

$$q' = \frac{(a_e - a_h)q}{q\bar{\rho} - \eta a_e - (1-\eta)a_h}, \quad \text{if} \quad \kappa < 1. \tag{21}$$

Consider boundary condition $\kappa(0) = 0$, which translates via (PO) to $q(0) = a_h/\rho_h$. The online appendix justifies this choice of boundary condition, which says that experts fully de-lever as their wealth shrinks.[4] Then, ODE (21) is solved on the endogenous region $(0, \eta^*)$ where households manage some capital, i.e., $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$. Given a solution for (q, κ), the risk-balance equation (RB) yields capital price variance

$$\varsigma_q^2 = \frac{\eta(1-\eta)}{\kappa - \eta}\frac{a_e - a_h}{q}, \quad \text{if} \quad \kappa < 1. \tag{22}$$

Since $\varsigma_q \neq 0$ in (22), a BSE exists as long as ODE (21) has a solution. Unfortunately, the singularity $\lim_{\eta\to 0} q'(\eta) = \lim_{\eta \to 0} q(\eta)(\kappa(\eta) - \eta)^{-1} = +\infty$ forces us to go beyond standard ODE existence/uniqueness results. Instead, we build a monotonic sequence of auxiliary economies that converge to the BSE.

Proposition 1 (BSE)**.** *There exists a unique BSE with $\kappa(0) = 0$, in which $\varsigma_q(\eta) \neq 0$ on $(0, \eta^*)$ for $\eta^* > 0$, and $\varsigma_q(\eta) = 0$ on $(\eta^*, 1)$.*

Figure 1 displays a numerical example with the capital price q and volatility ς_q as functions of η. The left region where q is upward sloping corresponds to the inefficient region where $\kappa < 1$. This region induces a non-trivial amount of volatility (middle panel). Volatility can be so high because of the large productivity gap $a_e - a_h$; this gap makes fire sales impact asset prices significantly. Finally, notice that the equilibrium is

[4] The existing literature universally applies the boundary condition $\kappa(0) = 0$. In Online Appendix C.1, we show that this is not necessary in principle. There are actually a continuum of BSEs indexed by $\kappa_0 = \kappa(0) \in [0,1]$, which one can think of as agents' "disaster belief", i.e., what happens in the worst-case scenario. Nevertheless, there are good reasons to select $\kappa_0 = 0$. As we go on to show in Online Appendix C.2, if managing capital involves any amount of idiosyncratic risk, even if vanishingly-small, any equilibrium must feature $\kappa \to 0$ as $\eta \to 0$. Intuitively, even a tiny amount of additional capital risk implies an infinite amount of return-on-wealth risk as $\eta \to 0$ unless experts fully liquidate all their capital.

stationary (the right panel plots the stationary CDF of η). This is fairly easy to understand. Since $\rho_e > \rho_h$, experts consume at a higher rate than households, so they will never control all wealth in the economy. On the other hand, expert wealth will not vanish because they hold a disproportionate share of capital, which delivers a risk premium in equilibrium due to the volatility ς_q.[5]

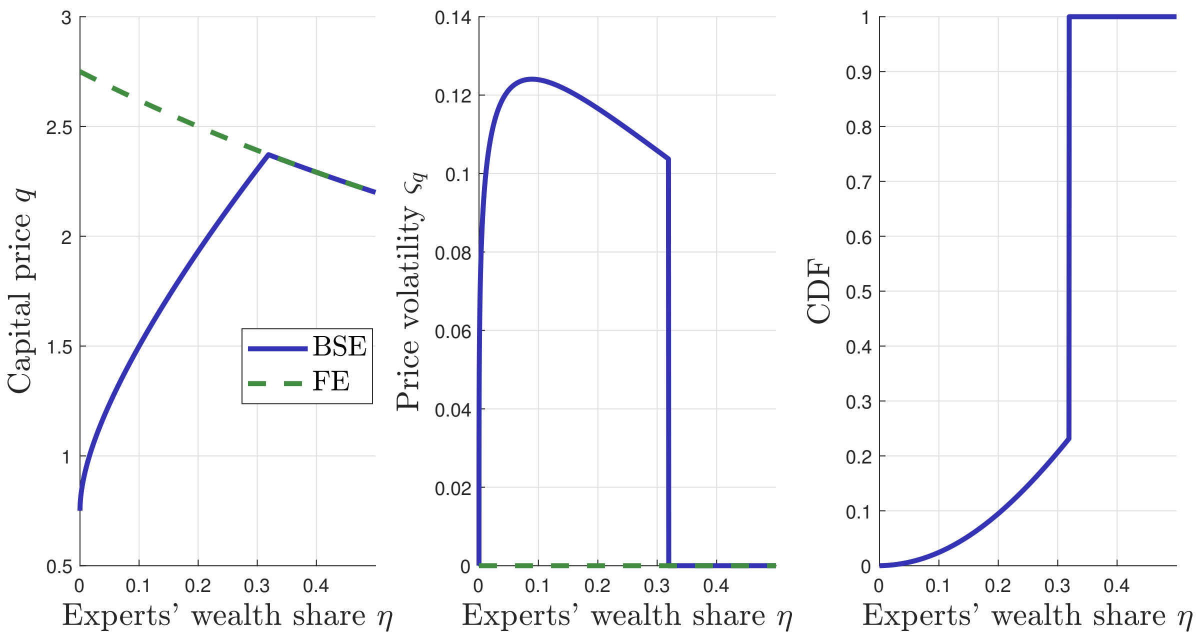

Figure 1: Capital price q, volatility ς_q, and stationary CDF of η in the Brownian Sunspot Equilibrium (BSE) and Fundamental Equilibrium (FE). Parameters: $\rho_e = 0.06$, $\rho_h = 0.04$, $a_e = 0.11$, $a_h = 0.03$. (Note that g and σ are irrelevant to the solution.)

A BSE is a self-fulfilling fire sale. The mechanics are as follows. If agents believe sunspots can affect asset prices, then the actual arrival of such a shock triggers trading of capital between experts and households. Why? While experts are able to share risks from the fundamental shock dW, they cannot hedge the sunspot dZ. Anticipating a hit to their balance sheet from a decline in capital valuations, experts rush to sell capital.

Let us understand further. To sustain a self-fulfilling fire sale, it must be the case that (i) asset prices fall, and (ii) this asset price decline reduces expert wealth. Asset prices are connected to capital holdings because experts are more productive than households and are willing to pay more for capital. Equation (PO) captures this idea via the positive relationship between q and κ. Thus, if all experts coordinate to sell capital, its price will fall.

[5]As Lemma B.1 in Online Appendix B shows, this BSE possesses a stationary distribution on $[0, \eta^*]$ if and only if $\rho_e > \rho_h$. Note that there is a mass point at $\eta = \eta^*$ in the numerical example of Figure 1. This is a general feature of the BSE.

Falling asset prices do not necessarily damage expert balance sheets; for instance, if asset prices fall for fundamental reasons, this shock would be perfectly shared between experts and households. But if asset prices decline due to a non-hedgeable sunspot shock, experts' relative wealth would fall—see equation (17). This wealth effect coordinates trade. An individual expert, fearing an unhedgeable decline in asset prices, will want to pre-empt this by selling some of their own capital. Thus, the fear of a non-fundamental fire sale can kick off such a fire sale.

Ultimately, a fire sale causes a decline in q and η. The magnitudes of decline are driven by a relative effect and an absolute effect. The co-dependence of these objects is captured by the fact that $q_t = q(\eta_t)$. The function $q(\cdot)$ captures *relative* responses: how much asset prices move per unit of wealth share fluctuation. It is pinned down by ensuring consistency of $\varsigma_\eta = (\kappa - \eta)\varsigma_q$ with $\varsigma_q = \frac{q'}{q}\varsigma_\eta$. The risk-balance condition (RB) captures *absolute* response sizes. If both experts and households are marginal in capital markets, their relative risk exposure $\frac{\kappa}{\eta}\varsigma_q^2 - \frac{1-\kappa}{1-\eta}\varsigma_q^2$ must be matched by their relative expected returns $\frac{a_e}{q} - \frac{a_h}{q}$. The absolute size of the shock, ς_q, is thus pinned down.

While the analysis is simplest and cleanest in the case of log utility and Brownian shocks, the intuition above suggests that nothing is special about log nor Brownian motion. To confirm this, we perform two extensions. First, Online Appendix D shows how BSEs can be obtained with CRRA preferences and risk aversion $\gamma \neq 1$. Second, Online Appendix E illustrates a sunspot equilibrium with Poisson jumps instead of Brownian sunspots.

2.2 Self-fulfilling sunspots as a limiting case: equivalence

In this paper, we allow agents to perfectly hedge fundamental shocks. To compare our results to the literature, and provide further interpretation, consider what happens if no market existed for hedging dW_t. Equilibria for this "conventional situation" are studied extensively, with the defining feature that fundamental shocks are amplified by endogenous wealth dynamics.

Let us briefly recount the details for a "conventional equilibrium" in which dW_t is non-hedgeable and the sunspot shock dZ_t is absent. The key modification is that non-hedgeable return volatility is now $\sigma + \sigma_q$ rather than ς_q from the BSE. Thus, the entire set of equilibrium equations is as before, except ς_q is replaced by $\sigma + \sigma_q$ and ς_η by σ_η in all cases.

Follow a similar analysis that led to the critical equation (20). Solving the two-way feedback between the Itô condition $\sigma_q = \frac{q'}{q}\sigma_\eta$ and wealth volatility $\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q)$, we obtain

$$\sigma_q = \frac{(\kappa - \eta)q'/q}{1 - (\kappa - \eta)q'/q}\sigma. \tag{23}$$

Equation (23) is often interpreted as *amplification*, because $\frac{(\kappa-\eta)q'/q}{1-(\kappa-\eta)q'/q}$ takes the form of a convergent geometric series. In words, a negative fundamental shock reduces experts' wealth share η directly through $(\kappa - \eta)\sigma$, which reduces asset prices through q'/q. This explains the numerator of (23). But the reduction in asset prices has an indirect effect: a one percent drop in capital prices reduces experts' wealth share by $(\kappa - \eta)$, which feeds back into a $(\kappa - \eta)q'/q$ percent further reduction capital prices, which then triggers the loop again. The second-round impact is $[(\kappa - \eta)q'/q]^2$, and so on. This infinite series is convergent if $(\kappa - \eta)q'/q < 1$, such that incremental amplification is reduced in each successive round of the feedback loop.

In the BSE, recall that $(\kappa - \eta)q'/q = 1$ (equation (20)). This BSE has no dampening in successive rounds of the feedback loop, leading to infinite amplification! Despite this contrast, it turns out that the BSE is "close" to these conventional equilibria. As σ shrinks, amplification rises because falling exogenous volatility incentivizes expert leverage, which raises endogenous volatility. As σ vanishes, amplification rises explosively and equilibria become sunspot-like.[6]

Proposition 2 (Observational equivalence)**.** *Consider an alternative economy without sunspots and in which fundamental shocks W are non-hedgeable. Suppose a Markov equilibrium in η exists for each $\sigma > 0$ small enough, with $\kappa(0) = 0$, and suppose these equilibria vary continuously with σ for $\sigma > 0$. Then, as $\sigma \to 0$, the equilibrium converges to the BSE in distribution.*

It is relatively clear that taking $\sigma \to 0$ in equation (23) yields $[1 - (\kappa - \eta)q'/q]\sigma_q = 0$, analogous to the critical equation (20) from the benchmark model. Yet it is not clear why the solution $\sigma_q \equiv 0$ (hence $\kappa \equiv 1$, i.e., the safe Fundamental Equilibrium) is ruled out as a limiting equilibrium. Our formal proof rules this out and shows that fire sales remain non-negligible in the limit.

The observational equivalence result of Proposition 2 formalizes how our BSE "looks similar" to the conventional equilibria that have been studied in the financial accelerator literature. There are two take-aways. Theoretically, our finding demonstrates how

[6] Brunnermeier and Sannikov (2014) provide a related limiting result, arguing numerically that asset-price volatility does not vanish as $\sigma \to 0$, also known as the "volatility paradox." They also provide an analytical result that $\lim_{\eta\to 0} \frac{\sigma_\eta}{\eta} = \frac{a_e - a_h}{a_h}\frac{\rho_h}{\sigma} + O(\sigma)$. We go further in proving that the entire equilibrium converges, as $\sigma \to 0$, to a sunspot equilibrium. Related results can be found in Manuelli and Peck (1992) and Bacchetta et al. (2012), in which sunspot equilibria could be seen as limits of fundamental equilibria when fundamental uncertainty vanishes.

the self-fulfilling nature of fire sales is core to the economics of the financial accelerator. Practically, our finding can also be viewed as a robustness result: the dynamics of conventional financial accelerator equilibria are robust to the inclusion of markets for hedging fundamental risks.

2.3 Self-fulfilling sunspots as a limiting case: selection

So far, we have demonstrated that the BSE is a *possibility*. But our model inherently permits multiple equilibria. Agents may just as well coordinate on the Fundamental Equilibrium, which has no volatility nor fire sales. In this section, we provide a very simple rationale for selecting the BSE as the *unique equilibrium*. The idea: suppose agents perceive sunspot shocks as having a small fundamental impact. Then, even as this perceived fundamental impact vanishes, equilibrium requires sunspots to matter.

Suppose agents believe that dZ is a second *fundamental* shock that affects capital. But unlike dW, there are no hedging markets for dZ. Mathematically, introduce parameter ς in the perceived capital evolution:

$$dk_{i,t} = k_{i,t}[gdt + \sigma dW_t + \varsigma dZ_t]. \tag{24}$$

In reality, $\varsigma = 0$ so that dZ is a sunspot that does not affect capital evolution at all. Think of ς as small, since we will eventually take $\varsigma \to 0$.[7]

This perceived risk model is tractable because all the equations are either identical to or limiting versions of (as $\varsigma \to 0$) those that arise when the risk is real. For example, portfolio choice depends only on perceived risks, so the risk-balance equation is

$$0 = \min\left[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}(\varsigma + \varsigma_q)^2\right]. \tag{25}$$

This equation would be the same whether ς was actually a risk or not. Similarly, consumption-wealth ratios remain constant at ρ_i under log utility, so price-output relation (PO) still holds. Finally, agents' net worth dynamics depend on the actual risks, not the perceived ones, and so given dZ is really a sunspot, we recover equation (20) from the benchmark BSE. Equation (20) is also the limiting result as fundamental risk vanishes. Thus, all the relevant equilibrium equations in this perceived risk model converge to the BSE equations as $\varsigma \to 0$.

[7] Note that, in a diffusion model, misperceptions about volatility are extreme in the sense in that such beliefs are singular with respect to the objective probability—data at infinitely-high frequency could detect the true volatility. That said, we will take misperception $\varsigma \to 0$ in this argument. And so if investors receive data at anything less than infinitely-high frequency, the belief distortion can be interpreted as trivial.

Consequently, we may use the exact same argument as Proposition 2 to show that as $\varsigma \to 0$, the equilibrium converges to the BSE. Most importantly, *the safe equilibrium cannot emerge* in the limit $\varsigma \to 0$. Essentially, all we are doing here is using the previous limiting results in a different way, as an equilibrium refinement.

Proposition 3 (Refinement)**.** *Consider an alternative economy where capital is perceived to follow (24), but where $\varsigma = 0$ in reality. Suppose Z is non-hedgeable while W is hedgeable. Suppose a Markov equilibrium in η exists for each $\varsigma > 0$ small enough, with $\kappa(0) = 0$, and suppose these equilibria vary continuously with ς for $\varsigma > 0$. As $\varsigma \to 0$, the equilibrium converges to the BSE.*

3 Conclusion

We have studied a canonical macro-finance model and constructed an equilibrium with self-fulfilling fire sales. The key innovation is that, while all fundamental risks are perfectly shared, not every conceivable shock is hedgeable. Fundamentals-based fire sales are no longer possible, but endogenously-emerging risks are unhedgeable and could matter. If agents coordinate on selling capital, its price falls, which feeds back into net worth and self-justifies the initial fire sale. The resulting dynamics are familiar, resembling the conventional financial accelerator equilibria in a sense we make precise, but can only emerge out of non-fundamental shocks. For example, consider the emergence of new types of shocks for which hedging markets have not yet developed; these are the shocks likely to encourage coordination and fire sale behavior. Finally, despite the presence of multiple equilibria, we provide a simple trembling-hand-style refinement, based on agents mistaking the sunspot shock to have a vanishingly-small fundamental impact, justifying selecting the fire sale equilibrium and neglecting the safe equilibrium.

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Appendix:
Dynamic Self-Fulfilling Fire Sales

Paymon Khorrami and Fernando Mendo

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A Proofs

Proof of Lemma 1. The construction is contained in the text leading up to the Lemma. To confirm that this is an equilibrium, note that all equations are satisfied: (PO) by $\kappa = 1$ and $q = a_e/\bar{\rho}(\eta)$; (RB) by $\kappa = 1$ and $\varsigma_q = 0$; and r and π can be set by (12) and (13), respectively. Finally, the dynamics in (18)-(19) are consistent with equations (15)-(17) and Itô's formula applied to $q = a_e/\bar{\rho}(\eta)$. □

Proof of Proposition 1. As stated in the text, the existence of a BSE boils down to proving the existence of a solution q to ODE (21) with boundary condition $\kappa(0) = 0$, or equivalently $q(0) = a_h/\rho_h$. However, because $\kappa(0) = 0$ implies that $q'(0) = +\infty$, we cannot apply standard results to this problem and must argue differently.

In our first step, we replace the boundary condition $\kappa(0) = 0$ by any $\kappa(0) = \kappa_0 \in (0, 1)$. We prove existence of a solution to (21) with this modified boundary. In our second step, we take the limit $\kappa_0 \to 0$ and argue the limit satisfies the relevant equations. Our third step shows this limit is the unique solution to the BSE ODE.

Step 1: Existence (and uniqueness) for $\kappa_0 \in (0, 1)$. Consider the initial value problem

$$q' = F(\eta, q) := \frac{a_e - a_h}{q\bar{\rho}(\eta) - \eta a_e - (1-\eta)a_h} q \quad \text{s.t.} \quad q(0) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h}.$$

Notice that $q'(0+) = \frac{q(0)}{\kappa_0}$ is bounded, which is enough to ensure that F is bounded and uniformly Lipschitz on the domain $\mathcal{R}_\epsilon := \{(\eta, q) : 0 < \eta < 1,\ (\epsilon + \eta)a_e + (1 - \epsilon - \eta)a_h < q\bar{\rho}(\eta)\}$, for each $\epsilon \in (0, \kappa_0)$. This is the relevant domain because $\kappa'(0+) = \frac{\rho_e - \rho_h}{a_e - a_h} q(0) + \frac{\rho_h}{a_e - a_h} q'(0+) = 1 + \frac{a_h}{\kappa_0} + (\kappa_0 + a_h)(\frac{\rho_e - \rho_h}{\rho_h}) > 1$, so that the solution points into the interior of this region as long as $\epsilon \le \kappa_0$. Thus, the Picard-Lindelöf theorem implies that there exists a *unique* solution q^* to this initial value problem, for $\eta \in (0, b)$, some b. Standard continuation arguments can be used to extend the solution to either the entire domain $\mathcal{R} := \cup \mathcal{R}_\epsilon$ or until a point such that the solution or its generator F explodes. In other words, either:

(i) $b = 1$;

(ii) $q^*(\eta) \to +\infty$ as $\eta \to b$; or

(iii) b satisfies $ba_e + (1-b)a_h = q^*(b)\bar{\rho}(b)$.

Let us first rule out case (iii). Consider the pricing function $\underline{q}(\eta) = \frac{\kappa_0(a_e - a_h) + \eta a_e + (1-\eta)a_h}{\bar{\rho}(\eta)}$, which corresponds by equation (PO) to the expert capital share $\underline{\kappa}(\eta) = \kappa_0 + \eta$. Note that $\underline{q}$ uniquely solves the alternative ODE

$$q' = \underline{F}(\eta, q) := \frac{a_e - a_h - (\rho_e - \rho_h)q}{\bar{\rho}(\eta)} \quad \text{s.t.} \quad q(0) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h}.$$

Since $q^*(0) = \underline{q}(0)$ and since $F(\eta, q) > \underline{F}(\eta, q)$ on $\mathcal{R}$, the comparison theorem for ODEs implies that $q^*(\eta) > \underline{q}(\eta)$ for all $\eta > 0$. Because $\underline{q}(\eta) > \frac{\eta a_e + (1-\eta)a_h}{\bar{\rho}(\eta)}$, this proves that there cannot exist any b with $q^*(b) = \frac{ba_e + (1-b)a_h}{\bar{\rho}(b)}$. In passing, also note that this proves that the solution $q^*(\eta)$ is necessarily such that the associated capital share $\kappa^*(\eta) = \frac{q^*(\eta)\bar{\rho}(\eta) - a_h}{a_e - a_h}$ from equation (PO) is such that $\kappa^*(\eta) > \eta$.

We are left with cases (i) or (ii). In either case, set

$$\eta^* = \inf\{\eta \in (0, b) : q^*(\eta) = a_e/\bar{\rho}(\eta)\}$$

with the convention that $\eta^* = 1$ if the set is empty.

In case (ii), with $b < 1$ and $q^*(b-) = +\infty$, it is clear by the continuity of the solution q^* that $\eta^* < b < 1$.

In case (i) with $b = 1$, we may easily show by contradiction that $\eta^* < 1$. Indeed, if $\eta^* \geq 1$, then $q^*(1-)\bar{\rho}(1-) < a_e$, which implies $F(1-, q^*(1-)) < 0$. But by continuity of q^*, the only way F could have changed signs is that there exists an $\eta^\circ \in (0,1)$ such that $\eta^\circ a_e + (1-\eta^\circ)a_h = q^*(\eta^\circ)\bar{\rho}(\eta^\circ)$. This latter possibility was just ruled out (case (iii)). And so $\eta^* < 1$.

Consequently, in cases (i)-(ii), there exists $0 < \eta^* < 1$ such that $q^*(\eta^*) = a_e/\bar{\rho}(\eta^*)$.

Finally, define

$$q(\eta) := \begin{cases} q^*(\eta), & \text{if } \eta < \eta^*; \\ a_e/\bar{\rho}(\eta), & \text{if } \eta \geq \eta^*. \end{cases}$$

This function satisfies $q' = F(\eta, q)$ on $(0, \eta^*)$, with boundary values $q(0) = \frac{\kappa_0 a_e + (1-\kappa_0)a_h}{\rho_h}$ and $q(\eta^*) = a_e/\bar{\rho}(\eta^*)$. Thus, we have found a solution to the capital price satisfying all the desired relations. And as shown above, the capital share satisfies $\kappa(\eta) > \eta$.

Equation (22), plus the fact that $\kappa > \eta$, implies $\varsigma_q^2 > 0$ on $(0, \eta^*)$. Since $\eta^* > 0$, we thus have $\varsigma_q(\eta) \neq 0$ on a positive measure subset as desired.

Step 2: Limit as $\kappa_0 \to 0$. For each initial condition $\kappa(0) = \kappa_0$, let $(q_{\kappa_0}, \eta^*_{\kappa_0})$ be the associated equilibrium capital price and fire-sale threshold. Write the integral version of the ODE:

$$q_{\kappa_0}(\eta) = \frac{\kappa_0 a_e + (1-\kappa_0) a_h}{\rho_h} + \int_0^\eta F(x, q_{\kappa_0}(x))dx, \quad \eta < \eta^*_{\kappa_0}.$$

We first claim that $q_{\kappa_0}(x)$ is weakly increasing in κ_0, for each x. Indeed, $q_{\kappa_0}(0)$ is strictly increasing in κ_0. By continuity, we may consider $x^* := \inf\{x : q_{\tilde{\kappa}_0}(x) = q_{\kappa_0}(x)\}$ for some $\tilde{\kappa}_0 > \kappa_0$. In that case, since F does not depend on $\tilde{\kappa}_0$ or κ_0, we have $q_{\tilde{\kappa}_0}(x) = q_{\kappa_0}(x)$ for all $x \geq x^*$. This proves $q_{\tilde{\kappa}_0}(x) \geq q_{\kappa_0}(x)$ for all x. The monotonicity of q_{κ_0} in κ_0 also proves that $\eta^*_{\kappa_0}$, by its definition, is weakly decreasing in κ_0.

Because of these monotonicity results, the following limit $(q_0, \eta_0^*) := \lim_{\kappa_0 \to 0}(q_{\kappa_0}, \eta^*_{\kappa_0})$ exists. This limit is our candidate solutions for the BSE. It suffices to show that q_0 satisfies (a) $q_0' = F(\eta, q_0)$ on $(0, \eta_0^*)$, (b) $q_0(0) = a_h/\rho_h$, and (c) $q_0(\eta_0^*) = a_e/\bar{\rho}(\eta_0^*)$.

Combine the monotonicity result for $q_{\kappa_0}(x)$ with the fact that $\partial_y F < 0$ to see that $\{F(x, q_{\kappa_0}(x)) : \kappa_0 \in (0,1)\}$ is a sequence which is monotonically (weakly) decreasing in κ_0, for each x. Thus, applying the monotone convergence theorem to the integral version of the ODE, and recalling that $\eta_0^* \geq \eta^*_{\kappa_0}$, we have

$$q_0(\eta) = \frac{a_h}{\rho_h} + \int_0^\eta F(x, q_0(x))dx, \quad \eta < \eta_0^*$$

which proves (a), by differentiating, and (b), by substituting $\eta = 0$.

To prove (c), note that $q_{\kappa_0}(x)$ is a bounded, continuous function for κ_0. Because the space of bounded, continuous functions (equipped with the sup-norm) is a Banach space, it holds that $q_0(x)$ is also a bounded, continuous function. Therefore,

$$q_0(\eta_0^*) = \lim_{n\to 0} q_n\Big(\lim_{m \to 0} \eta_m^*\Big) = \lim_{n\to 0}\lim_{m\to 0} q_n(\eta_m^*) = \lim_{\kappa_0 \to 0} q_{\kappa_0}(\eta^*_{\kappa_0}) = \lim_{\kappa_0\to 0} \frac{a_e}{\bar{\rho}(\eta^*_{\kappa_0})} = \frac{a_e}{\bar{\rho}(\eta_0^*)}$$

which proves (c).

Step 3: Uniqueness. Suppose two solutions q and $\tilde{q}$ solved the ODE (21) with boundary conditions $\kappa(0) = \tilde{\kappa}(0) = 0$. Let η^* and $\tilde{\eta}^*$ denote the points where $\kappa(\eta)$ and $\tilde{\kappa}(\eta)$ reach 1. Without loss of generality, we may consider the situation $\tilde{q}(\eta) > q(\eta)$ for all $\eta < \bar{\eta}$. The reason: if the two solutions ever crossed at some value of $\bar{\eta}$, then they would necessarily coincide for all $\eta \geq \bar{\eta}$.

Since $q(0) = \tilde{q}(0)$, we have

$$q(\eta) - \tilde{q}(\eta) = \int_0^{\eta} \Big[F(x, q(x)) - F(x, \tilde{q}(x))\Big] dx, \quad \eta < \bar{\eta}$$

Recall that F is decreasing in its second argument. Therefore, $\tilde{q} > q$ on $(0, \bar{\eta})$ implies $F(x, q(x)) > F(x, \tilde{q}(x))$ for $x < \bar{\eta}$, which from the equation above implies $q(\eta) > \tilde{q}(\eta)$, a contradiction. □

Proof of Proposition 2. As mentioned in the text, the equilibrium equations for are the same as in the benchmark BSE, but with ς_q and ς_η replaced everywhere by $\sigma + \sigma_q$ and σ_η, respectively. We write these conditions here for completeness:

$$0 = \min\Big[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}(\sigma + \sigma_q)^2\Big]. \tag{A.1}$$

$$\sigma_q = \frac{(\kappa - \eta)q'/q}{1 - (\kappa - \eta)q'/q}\sigma \tag{A.2}$$

$$r = \frac{\kappa a_e + (1-\kappa)a_h}{q} + g + \mu_q + \sigma\sigma_q - \Big(\frac{\kappa^2}{\eta} + \frac{(1-\kappa)^2}{1-\eta}\Big)(\sigma + \sigma_q)^2. \tag{A.3}$$

$$\mu_\eta = \eta(1-\eta)(\rho_h - \rho_e) + (\kappa - 2\eta\kappa + \eta^2)\frac{\kappa - \eta}{\eta(1-\eta)}(\sigma + \sigma_q)^2 \tag{A.4}$$

$$\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q). \tag{A.5}$$

As there is no sunspot shock present, we also have $\varsigma_q = \varsigma_\eta = 0$ by assumption. In addition, recall that (PO) still holds.

Note that the key equations characterizing equilibrium are (PO), (A.1), and (A.2)—these equations determine (q, κ, σ_q) independently from $(r, \mu_\eta, \sigma_\eta)$, which can then be determined from (A.3), (A.4), and (A.5). Denote the equilibrium solution for $\sigma > 0$ by $(q_\sigma, \kappa_\sigma)$, which by assumption exist uniquely for all σ small enough.

Combine equations (A.1)-(A.2) and (PO) and rearrange terms to get

$$q' = F_\sigma(\eta, q) := \frac{(a_e - a_h)q}{\bar{\rho}(\eta)q - \eta a_e - (1-\eta)a_h}\Big[1 - \sigma\sqrt{\frac{(\bar{\rho}(\eta)q - \eta a_e - (1-\eta)a_h)q}{\eta(1-\eta)(a_e - a_h)^2}}\Big] \tag{A.6}$$

if $\kappa < 1$ (or equivalently $q < a_e/\bar{\rho}$). This is the ODE that fully characterizes the solution q_σ. It is solved with the boundary condition $q_\sigma(0) = a_h/\rho_h$.

The proof proceeds as follows. We first prove that, for any $\sigma > 0$, fire sales happen and so ODE (A.6) applies in some region. Second, we prove the fire sale region does not vanish as $\sigma \to 0$. Third, we prove that the limiting equilibrium is the BSE.

Step 1: fire sales happen for every $\sigma > 0$. We first show that it is not possible to have $\kappa_\sigma \equiv 1$ across the state space as an equilibrium. Indeed, if so then (PO) gives $q(\eta) = a_e/\bar{\rho}(\eta)$, implying from (A.2) that $\sigma + \sigma_q = \frac{1}{1+(1-\eta)(\rho_e-\rho_h)}\sigma > 0$. Plugging this result into (A.1) along with the guess $\kappa = 1$, we see that (A.1) is violated for all η close enough to zero. Thus, there is a positive measure region on which $\kappa_\sigma < 1$ is required. Let us denote $\eta^*_\sigma := \inf\{\eta : \kappa_\sigma(\eta) = 1\} > 0$ as the upper-bound of the fire-sale region.

Step 2: the fire sale region does not vanish as $\sigma \to 0$. First, we establish that the relevant limits exist. Note that the ODE generator F_σ in (A.6) is decreasing in σ uniformly, which implies that the solution q_σ is monotonically (weakly) decreasing in σ. By the monotone convergence theorem, the limit $q_0 := \lim_{\sigma\to 0} q_\sigma$ exists, and by association $\eta^*_0 := \lim_{\sigma \to 0} \eta^*_\sigma$ exists also.

Furthermore, $q_\sigma : [0,1] \mapsto \mathbb{R}$ is bounded and continuous. Since the space of bounded, continuous functions is a Banach space, the limit $q_0 : [0,1] \mapsto \mathbb{R}$ is also a bounded, continuous function.

Continuity immediately implies that $\eta^*_0 > 0$. Indeed, if we had $\eta^*_0 = 0$, then continuity implies $q_0(0) = q_0(0+) = a_e/\rho_h$, which contradicts the boundary value $q_0(0) = a_h/\rho_h$. This proves that the fire-sale region does not vanish as $\sigma \to 0$.

Step 3: the limiting equilibrium is the BSE. Recall that q_σ is decreasing in σ. This implies that η^*_σ is increasing in σ, and in particular $\eta^*_0 = \inf_\sigma \eta^*_\sigma$. Thus, the entire family $(q_\sigma)_{\sigma>0}$ of solutions satisfy

$$q_\sigma(\eta) = \frac{a_h}{\rho_h} + \int_0^\eta F_\sigma(x, q_\sigma(x))dx, \quad \eta < \eta^*_0 \tag{A.7}$$

To take the limit as $\sigma \to 0$, we make use of the following. First, F_σ is continuous in σ for all $\sigma \geq 0$. Second, q_σ is continuous in σ for all $\sigma > 0$ by the proposition's assumption. This continuity is extended to all $\sigma \geq 0$ by the definition of q_0 as the limit. Third, $F_\sigma(\eta, q)$ is continuous in q on the domain $\mathcal{R} := \{(\eta, q) : 0 < \eta < 1, \eta a_e + (1-\eta)a_h < \bar{\rho}(\eta)q\}$. And fourth, the solution graph $\{(\eta, q_\sigma(\eta)) : 0 < \eta < 1\}$ is a subset of $\mathcal{R}$. Using these results, take $\sigma \to 0$ in (A.7) to get

$$q_0(\eta) = \frac{a_h}{\rho_h} + \int_0^\eta F_0(x, q_0(x))dx, \quad \eta < \eta^*_0$$

which proves that q_0 solves the BSE ODE (21) on $\eta \in (0, \eta^*_0)$. By uniqueness of the BSE (Proposition 1), this limiting equilibrium thus coincides with the BSE.

Finally, the qualification about "convergence in distribution" is only needed because the BSE is driven by the sunspot shock Z, while the present limiting equilibrium is driven by the fundamental shock W. These shocks have the same distribution but are not pointwise identical. $\square$

Proof of Proposition 3. The proof is identical to Proposition 2, but with σ and σ_q replaced by ς and ς_q everywhere, except in (A.6) where σ is replaced by 0. $\square$

Online Appendix:
Dynamic Self-Fulfilling Fire Sales

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B Stationarity of the BSE

Lemma B.1. *In any BSE, the dynamics prevent η from reaching zero with probability one. Moreover, if $\rho_e > \rho_h$, then $(\eta_t)_{t\geq 0}$ has a non-degenerate stationary distribution on $(0, \eta^*]$, and when $\eta_t \in (\eta^*, 1)$, it follows a deterministic path towards η^*.*

Proof of Lemma B.1. We consider the baseline model of Section 2.1 with boundary condition $\kappa(0+) = \kappa_0 \in [0, 1)$. As shown in Proposition 1, a BSE that is Markov in η exists uniquely given this boundary condition. For reference, we re-state the dynamics of η in such an equilibrium:

$$\mu_\eta = (\rho_e - \rho_h)\,\eta + \frac{a_e - a_h}{q}[\kappa - 2\kappa\eta + \eta^2]\mathbf{1}_{\eta<\eta^*} + (\rho_e - \rho_h)\,\eta^2 \tag{B.1}$$

$$\varsigma_\eta^2 = \eta(1-\eta)(\kappa-\eta)\frac{a_e - a_h}{q}\mathbf{1}_{\eta<\eta^*}, \tag{B.2}$$

where equation (B.2) follows from $\varsigma_\eta = (\kappa - \eta)\varsigma_q$ in (17) and $\varsigma_q^2 = \frac{\eta(1-\eta)}{\kappa-\eta}\frac{a_e-a_h}{q}\mathbf{1}_{\eta<\eta^*}$ in (22). We proceed in 3 steps, examining dynamics of η above η^*, in a neighborhood just below η^*, and in a neighborhood just above 0.

Step 1: Dynamics for $\eta > \eta^*$. Equation (B.2) shows that $\varsigma_\eta(\eta) = 0$ for all $\eta \geq \eta^*$. Thus, η it follows a deterministic path towards η^* if $\mu_\eta(\eta) < 0$ for all $\eta \in [\eta^*, 1)$. Substituting $\kappa = 1$ into (B.1) and using $\rho_e > \rho_h$ delivers the result immediately. Given the deterministic transition toward η^*, we can ignore the sub-interval $(\eta^*, 1)$ in our state space and instead consider only $(0, \eta^*)$.

In general, consider a one-dimensional process $(X_t)_{t\geq 0}$ with $dX_t = \mu_x(X_t)dt + \sigma_x(X_t)dZ_t$ that is a regular diffusion on interval $(e_1, e_2) \subset \mathbb{R}$ (i.e., the dynamics of X depend only on X itself, and imply that it reaches every point in (e_1, e_2) with positive probability). Our process $(\eta_t)_{t\geq 0}$ satisfies these conditions for $e_1 = 0$ and $e_2 = \eta^*$.

In such case, we may apply Feller's boundary classification to decide whether boundaries e_1 and e_2 are inaccessible (avoided forever with probability 1) or accessible. To do so, first define $s(y) := \exp(-\int_{x_0}^{y} \frac{2\mu_x(u)}{\sigma_x^2(u)}du)$, $m(x) := \frac{2}{s(x)\sigma_x^2(x)}$, and let ϵ and x_0 be arbitrary numbers within interval (e_1, e_2). Boundary e_1 is inaccessible if and only if

$$I_1 := \int_{e_1}^{\epsilon} m(x)\Big(\int_{e_1}^{x} s(y)dy\Big)dx = +\infty.$$

Boundary e_2 is accessible if and only if

$$I_2 := \int_{\epsilon}^{e_2} m(x)\Big(\int_{x}^{e_2} s(y)dy\Big)dx < +\infty.$$

We will prove these results in the next two steps.

Step 2: Dynamics near $e_2 = \eta^*$. Compute

$$\begin{aligned}\mu_\eta(\eta^*-) &= -\eta^*(1-\eta^*)(\rho_e - \rho_h) + (1-\eta^*)\bar{\rho}(\eta^*)\frac{a_e - a_h}{a_e} \\ \varsigma_\eta^2(\eta^*-) &= \eta^*(1-\eta^*)^2\bar{\rho}(\eta^*)\frac{a_e - a_h}{a_e}.\end{aligned}$$

Since $\varsigma_\eta^2(\eta^*-)$ is bounded away from zero and $\mu_\eta(\eta^*-)$ is finite, it is easy to check that $I_2 < +\infty$, meaning $e_2 = \eta^*$ is an accessible boundary that is hit in finite time with positive probability. Furthermore, we may also show

$$J_2 := \int_{\epsilon}^{e_2} m(x)\Big(\int_{\epsilon}^{x} s(y)dy\Big)dx < +\infty,$$

which implies $e_2 = \eta^*$ is a so-called "regular boundary" that must be included in the state space.

We must establish what occurs when η_t hits boundary point $e_2 = \eta^*$. Recall from step 1 that $\mu_\eta(\eta) < 0$ and $\varsigma_\eta(\eta) = 0$ for all $\eta \geq \eta^*$. This implies that η_t can never enter the region $(\eta^*, 1)$ from η^* and that η_t will not stay at point η^* for an infinite amount of time. Consequently, the region $(0, \eta^*]$ is the ergodic set.

Step 3a: General analysis of dynamics near $e_1 = 0$. First, suppose our diffusion satisfied the following near $e_1 = 0$ (the notation $f(x) \sim g(x)$ means $\lim_{x\to 0} f(x)/g(x) = 1$):

$$\begin{aligned}\sigma_x^2(x) &\sim \phi x^\beta \quad \phi > 0, \quad \beta \geq 0 \\ \frac{\mu_x(x)}{\sigma_x^2(x)} &\sim \theta x^{-\alpha}, \quad \alpha \geq 1, \quad \theta > 0.\end{aligned}$$

As we will show below in step 3b, this asymptotic description is flexible enough to cover all cases within our model.

If $\alpha = 1$, we have, for x sufficiently small,

$$S_1(x,\theta) := \int_0^x \frac{s(y)}{s(x)}dy = \int_0^x \exp\Big[2\theta(\log(x) - \log(y))\Big]dy \tag{B.3}$$
$$= x^{2\theta} \lim_{z\downarrow 0} \frac{x^{1-2\theta} - z^{1-2\theta}}{1-2\theta},$$

so letting ϵ be sufficiently small, we obtain

$$I_1 = \int_0^\epsilon \frac{2x^{2\theta-\beta}}{\phi} \lim_{z\downarrow 0} \frac{x^{1-2\theta} - z^{1-2\theta}}{1-2\theta} dx.$$

If $2\theta \geq 1$ (note that $2\theta = 1$ corresponds to $\frac{z^{1-2\theta}}{1-2\theta}$ being replaced by $\log(z)$ in the expression above), then the interior limit is $+\infty$ for all $x > 0$ and therefore $I_1 = +\infty$. This holds independently of the value of β. If $2\theta < 1$, then

$$I_1 = \int_0^\epsilon \frac{2}{(1-2\theta)\phi} x^{1-\beta} dx = \frac{2}{(1-2\theta)\phi}\Big(\frac{\epsilon^{2-\beta}}{2-\beta} - \lim_{x\downarrow 0} \frac{x^{2-\beta}}{2-\beta}\Big).$$

So, in this case, $I_1 = +\infty$ only if $\beta \geq 2$ (for $\beta = 2$, $\frac{x^{2-\beta}}{2-\beta}$ is replaced by $\log(x)$).

If $\alpha > 1$ instead, we will show that $I_1 = +\infty$ independent of any other parameters. We have

$$S_\alpha(x,\theta) := \int_0^x \frac{s(y)}{s(x)}dy = \int_0^x \exp\Big[\frac{2\theta}{1-\alpha}(x^{1-\alpha} - y^{1-\alpha})\Big]dy \tag{B.4}$$

The corresponding expression for the case with $\alpha = 1$ is $S_1(x,\theta)$ in (B.3). We showed above that for $\tau < 1/2$, we have $S_1(x,\tau) = +\infty$. Fix such a τ. We now show that $S_\alpha(x,\theta) \geq S_1(x,\tau)$ for all x sufficiently small and all θ.

Fix any $x > 0$, and define $f(y) := 2\tau(\log(x) - \log(y))$ and $g(y) := \frac{2\theta}{1-\alpha}(x^{1-\alpha} - y^{1-\alpha})$. Since both functions are strictly positive for $y < x$, and since $\lim_{y\to 0} g(y)/f(y) = \lim_{y\to 0}(\theta/\tau)y^{1-\alpha} = +\infty$, there exists $\bar{y} \in (0,x)$ such that $g(y) > f(y)$ for all $y \in (0,\bar{y})$. From this comparison, we conclude $S_\alpha(\bar{y},\theta) = \int_0^{\bar{y}} \exp(g(y))dy \geq \int_0^{\bar{y}} \exp(f(y))dy = S_1(\bar{y},\tau) = +\infty$. Since this argument is independent of (β,θ,ϕ), this proves that $I_1 = +\infty$ if $\alpha > 1$.

Step 3b: Model-specific analysis of dynamics near $e_1 = 0$. Now, we map our model dynamics into the setup of step 3a. If $\kappa(0+) = \kappa_0 > 0$, then in the limit as $\eta \to 0$, equations

(B.1)-(B.2) become

$$\mu_\eta = \frac{a_e - a_h}{q(0+)}\kappa_0 - \left(\rho_e - \rho_h + 2\frac{a_e - a_h}{q(0+)}\kappa_0\right)\eta + o(\eta)$$
$$\sigma_\eta^2 = \frac{a_e - a_h}{q(0+)}\kappa_0\eta + o(\eta).$$

Hence, in terms of the notation in step 3a, we have $\alpha = 1$, $\beta = 1$ and $\theta = 1 > \frac{1}{2}$. Thus, η avoids zero with probability one.

If $\kappa(0+) = 0$, we need to know the rate at which $\kappa \to 0$ as $\eta \to 0$. Guess, and verify after, that $\kappa = \varphi\eta^\omega + o(\eta^\omega)$ in the limit as $\eta \to 0$. Differentiating the price-output condition (PO), we have

$$q' = \frac{1}{\bar{\rho}}\left[(a_e - a_h)\kappa' - (\rho_e - \rho_h)q\right]$$

Combining this with the sunspot differential equation for q, equation (20), we obtain

$$\left[(a_e - a_h)\kappa' - (\rho_e - \rho_h)q\right](\kappa - \eta) = \bar{\rho}q.$$

Taking the limit as $\eta \to 0$, we have

$$(a_e - a_h)\lim_{\eta\to 0}(\kappa')(\kappa - \eta) = a_h$$

Hence, the guess is verified if $\omega = 1/2$ and $\varphi^2 = 2a_h/(a_e - a_h) > 0$. Substituting this asymptotic behavior into equations (B.1)-(B.2), we have

$$\mu_\eta = \sqrt{\frac{2(a_e - a_h)}{a_h}}\rho_h\eta^{1/2} + o(\eta^{1/2})$$
$$\sigma_\eta^2 = \sqrt{\frac{2(a_e - a_h)}{a_h}}\rho_h\eta^{3/2} + o(\eta^{3/2}).$$

These dynamics match step 3a with $\alpha = 1$, $\beta = 3/2$, and $\theta = 1$. In that case, we have shown that η cannot reach zero with probability one.

In summary, $(\eta_t)_{t\geq 0}$ possesses a non-degenerate stationary distribution with support $(0, \eta^*]$, the boundary $\{0\}$ is inaccessible, and the boundary η^* is accessible but non-absorbing. $\square$

C Disaster beliefs and equilibrium refinement

C.1 Beliefs about disaster states

In this section, we outline a richer class of BSEs. The entire set of BSEs studied here will be indexed by agents' beliefs about the "tail scenario" in the economy, i.e., what happens when experts are severely undercapitalized.

Mathematically, recall that we previously have assumed $\kappa(0) = 0$; in other words, experts fully deleverage as their wealth vanishes. Some intuitive refinements like a small amount of idiosyncratic risk (Section C.2 below) can justify the assumption $\kappa(0) = 0$. However, strictly speaking, $\kappa(0) = 0$ turns out to not be necessary without these refinements, and it will be interesting to relax this assumption.

Consider any $\kappa_0 \in (0, 1)$ and put $\kappa(0) = \kappa_0$. We will call κ_0 the *disaster belief* in the economy. The sunspot equilibrium is similar to Proposition 1, with the generalization that the boundary condition to the ODE (21) is now $\kappa(0) = \kappa_0$ rather than $\kappa(0) = 0$. Along the way toward proving Proposition 1, we actually showed that there is a unique solution to this problem, hence a sunspot equilibrium for each κ_0.[8]

In that proof, we also showed that the BSE is the result of taking the limit $\kappa_0 \to 0$. Similarly, one can show that as $\kappa_0 \to 1$, the equilibrium converges to the Fundamental Equilibrium of Lemma 1 Therefore, one can view both the BSE and the Fundamental Equilibrium as outcomes of coordination on experts' deleveraging. If experts never sell any capital, there can be no price volatility, with $\varsigma_q = 0$ at all times. If agents expect $\kappa_0 = 0$, which translates to full deleveraging and large capital fire sales, then the benchmark BSE of the paper prevails. But for any $\kappa_0 \in (0, 1)$, an intermediate sunspot equilibrium will prevail, with a self-fulfilling amount of expert deleveraging and associated price dynamics. In this simple way, the boundary condition $\kappa_0 \in [0, 1]$ spans an entire range of sunspot equilibria from more to less volatile. An illustration is in Figure C.1.[9]

This result clearly illustrates of the central property that the degree of fire sales is indeterminate in these models. Intuitively, greater optimism about other experts' ability to retain capital in the tail scenario induces smaller capital fire sales in response to

[8]There is a closed-form solution when $\rho_h = \rho_e = \rho$, which is

$$q(\eta) = \frac{1}{\rho}\Big[(a_e - a_h)\eta + a_h + \sqrt{((a_e - a_h)\eta + a_h)^2 - a_h^2 + (a_e - a_h)^2\kappa_0^2}\Big], \quad \text{for} \quad \eta < \eta^* = \frac{1}{2}\frac{a_e - a_h}{a_e}(1 - \kappa_0^2).$$

As κ_0 decreases, the slope $q'(\eta)$ increases, consistent with the idea that pessimism about the disaster state raises the sensitivity of equilibrium to sunspot shocks away from disaster.

[9]This result is also convenient in some numerical situations. Since the BSE is just the limit of equilibria as $\kappa_0 \to 0$, we can construct an approximate numerical solution with κ_0 very small (but not quite 0).

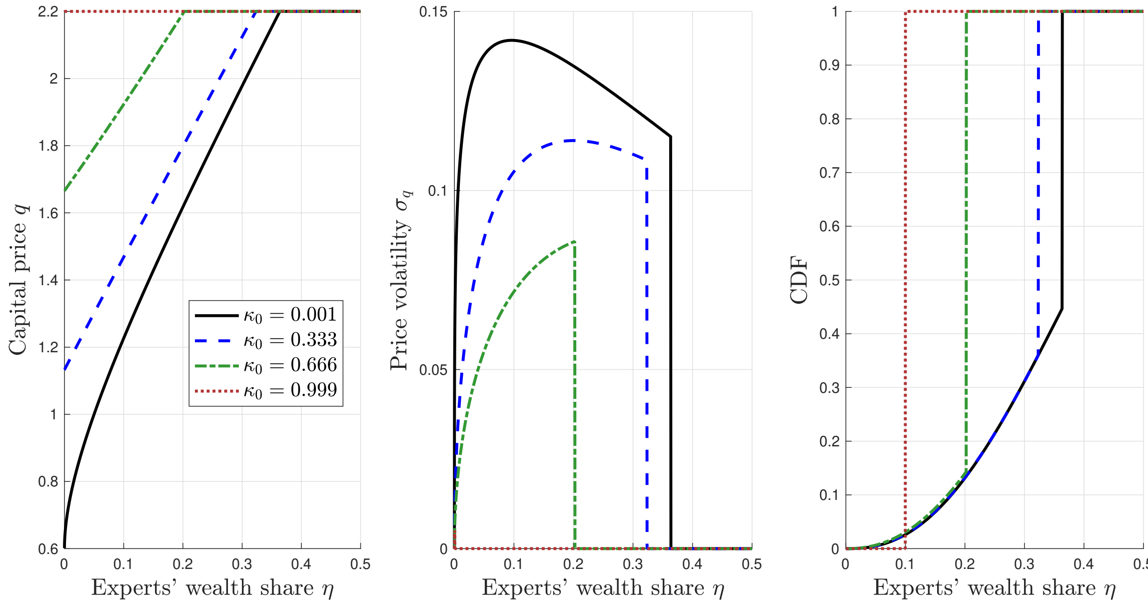

Figure C.1: Capital price q, sunspot volatility ς_q, and stationary CDFs of η for different levels of disaster belief κ_0. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$. To keep the wealth distribution stationary in an economy with symmetric discount rates, we augment benchmark assumptions with random type-switching at rates δ_e and δ_h for experts and households, respectively. This does not change any consumption or portfolio decision, but augments the η dynamics with an additional drift of $(1-\eta)\delta_h - \eta\delta_e$. We set $\delta_h = 0.004$ and $\delta_e = 0.036$.

sunspot shocks, which keeps volatility low, asset prices high, and justifies the optimism.

C.2 Idiosyncratic risk as an equilibrium refinement

In the main text, we have assumed the disaster belief $\kappa_0 := \kappa(0) = 0$, i.e., full deleveraging by experts as their wealth vanishes. However, Section C.1 shows that, strictly speaking, this is not required. Any disaster belief $\kappa_0 \in [0,1]$ can be justified and a corresponding BSE can exist. This section provides a very simple refinement to justify selecting $\kappa_0 = 0$.

The refinement we consider involves adding a vanishingly small amount of idiosyncratic risk to capital. In particular, suppose individual capital now evolves as

$$dk_{i,t} = k_{i,t}[g dt + \sigma dW_t + \tilde{\sigma} d\tilde{W}_{i,t}], \tag{C.1}$$

where $(\tilde{W}_i)_{i\in[0,1]}$ is a continuum of independent Brownian motions. Agents with indexes $i \in [0,\nu]$ are experts, and those with $i \in [\nu, 1]$ are households. As in Section 1, the aggregate stock of capital $K_t := \int_0^1 k_{i,t} di$ grows as $dK_t = K_t[g dt + \sigma dW_t]$. Also as before, the shock dZ_t is a sunspot shock, independent of dW_t and all the idiosyncratic shocks. Besides this addition of idiosyncratic uncertainty, the definition of equilibrium is the same as Definition 1. Conjecture that capital prices follow a process of the form $dq_t = q_t[\mu_{q,t}dt + \varsigma_{q,t}dZ_t]$ in equilibrium.

The most important feature is that only the aggregate fundamental shock dW is hedgeable. Neither the idiosyncratic shocks $d\tilde{W}_i$, nor the aggregate sunspot shock dZ are hedgeable. In such a world, $\tilde{\sigma}^2 + \varsigma_q^2$ is the total amount of unhedgeable return-on-capital risk. Thus, the risk-balance condition (RB), which arose from the combination of expert and household capital FOCs, is now modified to read

$$0 = \min\left[1-\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1-\eta)}(\tilde{\sigma}^2 + \varsigma_q^2)\right]. \tag{C.2}$$

It turns out that *any* equilibrium must feature $\lim_{\eta\to 0}\kappa = 0$. To see this, take the limit $\eta \to 0$ in equation (C.2), making note that $a_e > a_h$ and $\tilde{\sigma}^2 + \varsigma_q^2 > 0$, so that $\kappa \to 0$ is required. In words, experts fully de-lever as they become poor, simply as a consequence of portfolio optimality.

Intuitively, non-hedgeable idiosyncratic risk gives experts an additional motive to sell capital. This motive is magnified as experts become relatively poorer, because the risk is embedded in the capital stock: holding any capital becomes infinitely risky, per unit of wealth, as net worth vanishes. Thus, even a small amount idiosyncratic risk is enough to force coordination on maximal selling in response to negative shocks.[10]

We can use this as a refinement in the following sense. By taking $\tilde{\sigma} \to 0$, our equilibrium equations converge back to those in Section 1. However, since the full-deleveraging property $\lim_{\eta\to 0}\kappa = 0$ holds for any $\tilde{\sigma} > 0$, we retain this property as we take $\tilde{\sigma} \to 0$. Thus, among BSEs with disaster belief κ_0, the unique one that survives this refinement is the one in the main text corresponding to $\kappa_0 = 0$.

Proposition C.1. *Any equilibrium with idiosyncratic capital risk $\tilde{\sigma} > 0$ must feature $\kappa(0) = 0$. As $\tilde{\sigma} \to 0$, any stochastic equilibrium converges to the baseline BSE.*

Proof of Proposition C.1. We have already proven the first statement that any equilibrium with $\tilde{\sigma} > 0$ features $\kappa(0) = 0$. We prove the second statement with the more general claim that any stochastic equilibrium is unique for any $\tilde{\sigma} > 0$. Importantly, note that the ODE (21) still holds in this environment if $\varsigma_q \neq 0$. This is because of the following points. First, the aggregate wealth share exposure $\varsigma_\eta = (\kappa - \eta)\varsigma_q$ holds as before (idiosyncratic risk washes out when aggregated to the expert sector level). Second, the Itô condition $\varsigma_q = \varsigma_\eta q'/q$ still holds as before. Together, these two imply that condition

[10] In passing, also note that this argument justifies our original choice of $\kappa(0) = 0$ as the boundary condition for ODE (21) rather than any other value $\kappa(0) > 0$.

(20) still holds. If $\varsigma_q \neq 0$, it must therefore be that $1 = (\kappa - \eta)q'/q$. And third, because the introduction of idiosyncratic risk scales with capital holdings, all log utility agents still consume ρ_i fraction of their net worth, meaning that the market clearing condition (PO) still holds. Combining (PO) with $1 = (\kappa - \eta)q'/q$ leads to ODE (21), as before.

Proposition 1, by its construction, characterizes the entire set of solutions to ODE (21) in which $\{\eta : \kappa(\eta) < 1\}$ is an interval of the form $(0, \eta^*)$. Each solution is indexed by its boundary condition $\kappa(0) = \kappa_0 \in [0, 1]$, and given this boundary condition the function $\kappa(\eta)$ is uniquely determined. Thus, the uniqueness of the entire equilibrium is proved given $\kappa_0 = 0$.

Finally, the fact that all the equilibrium equations converge to those of the BSE as $\tilde{\sigma} \to 0$ is obvious and is omitted for brevity. Combining this convergence with the uniqueness result above, we have proved the proposition. □

D General CRRA preferences

We modify the model by generalizing preferences to the CRRA type. In particular, we replace the $\log(c)$ term in utility specification (3) with the flow consumption utility $c^{1-\gamma}/(1-\gamma)$. We impose no fundamental volatility, $\sigma = 0$, to simplify the expressions.

Equilibrium. The key equation (20) still holds, repeated here for convenience, but in terms of ς_η rather than ς_q:

$$\left[1 - (\kappa - \eta)\frac{q'}{q}\right]\varsigma_\eta = 0. \tag{D.1}$$

The sunspot equilibrium is associated with the term in brackets being equal to zero. Unlike with logarithmic preferences, this condition does not pin down $q(\eta)$ function, because we can no longer write $\kappa(q, \eta)$ from the goods market clearing condition: the consumption to wealth ratio is not constant anymore, and depends on agents' value functions.

The value function can be written as $V_i = v_i(\eta)K^{1-\gamma}/(1-\gamma)$ where $v_i(\eta)$ is determined in equilibrium. Then, consumption is $c_i/n_i = (\eta_i q)^{1/\gamma - 1}/v_i^{1/\gamma}$ where η_i corresponds to the wealth share of sector i. Then, goods market clearing becomes

$$q^{1/\gamma}\left[\left(\frac{\eta}{v_e}\right)^{1/\gamma} + \left(\frac{1-\eta}{v_h}\right)^{1/\gamma}\right] = (a_e - a_h)\kappa + a_h. \tag{D.2}$$

Optimal portfolio decisions imply that

$$0 = \min\left[1-\kappa,\, \frac{a_e - a_h}{q} - \left(\frac{v_h'}{v_h} - \frac{v_e'}{v_e} + \frac{1}{\eta(1-\eta)}\right)(\kappa-\eta)\varsigma_q^2\right]. \tag{D.3}$$

The HJB equation for $i \in \{e, h\}$ has the familiar form $\rho_i V_i = u(c) + \mathbb{E}[\frac{dV_i}{dt}]$, which becomes

$$\rho_i = \frac{(\eta_i q)^{1/\gamma - 1}}{v_i^{1/\gamma}} + \frac{v_i'}{v_i}\mu_\eta + \frac{1}{2}\frac{v_i''}{v_i}\varsigma_\eta^2 + (1-\gamma)g. \tag{D.4}$$

The dynamics of η satisfy

$$\varsigma_\eta = (\kappa - \eta)\varsigma_q \tag{D.5}$$

$$\mu_\eta = \eta(1-\eta)\left(\pi_e \frac{\kappa}{\eta}\varsigma_q - \pi_h \frac{1-\kappa}{1-\eta}\varsigma_q + \frac{c_h}{n_h} - \frac{c_e}{n_e}\right) - \varsigma_\eta \varsigma_q \tag{D.6}$$

and agent-specific risk prices satisfy

$$\pi_e = -\frac{v_e'}{v_e}\varsigma_\eta + \frac{\varsigma_\eta}{\eta} + \varsigma_q \tag{D.7}$$

$$\pi_h = -\frac{v_h'}{v_h}\varsigma_\eta - \frac{\varsigma_\eta}{1-\eta} + \varsigma_q. \tag{D.8}$$

A Markov equilibrium is a set of prices $\{q, \sigma_q, \pi_e, \pi_h\}$, allocation $\{\kappa\}$, value functions $\{v_h, v_e\}$ and aggregate state dynamics $\{\varsigma_\eta, \mu_\eta\}$ that solve the system (D.1)-(D.8).

The Fundamental Equilibrium corresponds to the solution for (D.1) where $\varsigma_\eta = 0$, which implies deterministic economic dynamics. Then, the capital price has no volatility ($\varsigma_q = 0$), risk prices are zero ($\pi_e = \pi_h = 0$), and experts hold the entire capital stock ($\kappa = 1$). The capital price is then solved from (D.2), and the value functions satisfy

$$\rho_i = \frac{(\eta_i q)^{1/\gamma - 1}}{v_i^{1/\gamma}} + \frac{v_i'}{v_i}\underbrace{\eta(1-\eta)\left(\frac{c_h}{n_h} - \frac{c_e}{n_e}\right)}_{=\mu_\eta} + (1-\gamma)g.$$

Conversely, the sunspot equilibrium corresponds to the solution for (D.1) with $\frac{q'}{q} = (\kappa - \eta)^{-1}$ (and potentially $\varsigma_\eta \neq 0$).

Disaster belief. With logarithmic preferences, we proved that any sunspot equilibrium must satisfy $\varsigma_q(0+) = 0$. This allowed us, in Appendix C.1, to construct sunspot equilibria with $\kappa(0+) = \kappa_0$ for any $\kappa_0 \in [0, 1)$. With CRRA preferences, we attempt to construct the same class of equilibria, with $\varsigma_q(0+) = 0$ and $\kappa_0 \in [0, 1)$.

In order to have a non-degenerate stationary distribution, we have the following requirements. Since $\varsigma_\eta(0+) = \kappa_0 \varsigma_q(0+) = 0$, the state variable avoids the boundary $\{0\}$ if $\mu_\eta(0+) > 0$. Using (D.3) for $\kappa < 1$, we have[11]

$$\frac{a_e - a_h}{q(0+)} = (\pi_e(0+) - \pi_h(0+))\varsigma_q(0+)$$

which allows us to show that[12]

$$\mu_\eta(0+) = \kappa_0 \frac{a_e - a_h}{q(0+)} > 0.$$

In addition, we need $\mu_\eta(\eta^*+) < 0$ where $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$. This requirement should be satisfied for $\rho_e - \rho_h$ sufficiently large.

Numerical solution. We do not provide an existence proof—which involves the existence of a solution to the ODE system—but construct numerical examples. For numerical stability, the examples are constructed for $\kappa_0 > 0$, which keeps $q'(0+) = q(0+)/\kappa_0$ bounded.[13]

The numerical strategy is the following. Construct a grid $\{\eta_1, \dots, \eta_N\}$ with limit points arbitrarily close to but bounded away from zero and one. Conjecture value functions $v_h(\eta)$ and $v_e(\eta)$. Impose $\kappa(\eta_1) = \kappa_0$ and use (D.2) to solve for $q(\eta_1)$. At each interior grid point, use $q' = q/(\kappa - \eta)$ and (D.2) to solve for $\kappa(\eta)$ and $q(\eta)$ until $\kappa(\eta^*) = 1$. In this region, recover ς_q from (D.3). For $\eta \in (\eta^*, 1]$ impose $\kappa(\eta) = 1$ and $\varsigma_q = 0$, and solve capital price from (D.2). The rest of equilibrium objects are calculated directly from the system above. The guesses of the value functions are updated by augmenting the HJBs (D.4) with a time derivative and moving a small time-step backward, as in Brunnermeier and Sannikov (2016). The procedure terminates when the value functions converge to time-independent functions.

In Figure D.1, we plot the equilibrium objects as functions of η, for different levels of risk aversion γ. In Figure D.2, we make the same plots, for different levels of the disaster belief κ_0. Higher risk aversion (higher γ) or more pessimism about disasters (lower κ_0)

[11] Note that this implies $\pi_e(0+) - \pi_h(0+)$ diverges.

[12] This expression also assumes that $\pi_h(0+)$ remains bounded. This is a mild assumption that is always confirmed numerically when we solve for the value functions.

[13] With logarithmic utility, we obtain a limiting result in Proposition 1, that as $\kappa_0 \to 0$, the equilibrium converges to the BSE with $\kappa(0) = 0$. With CRRA, we do not prove such a result analytically, but we do observe numerically what looks like convergence as κ_0 becomes small.

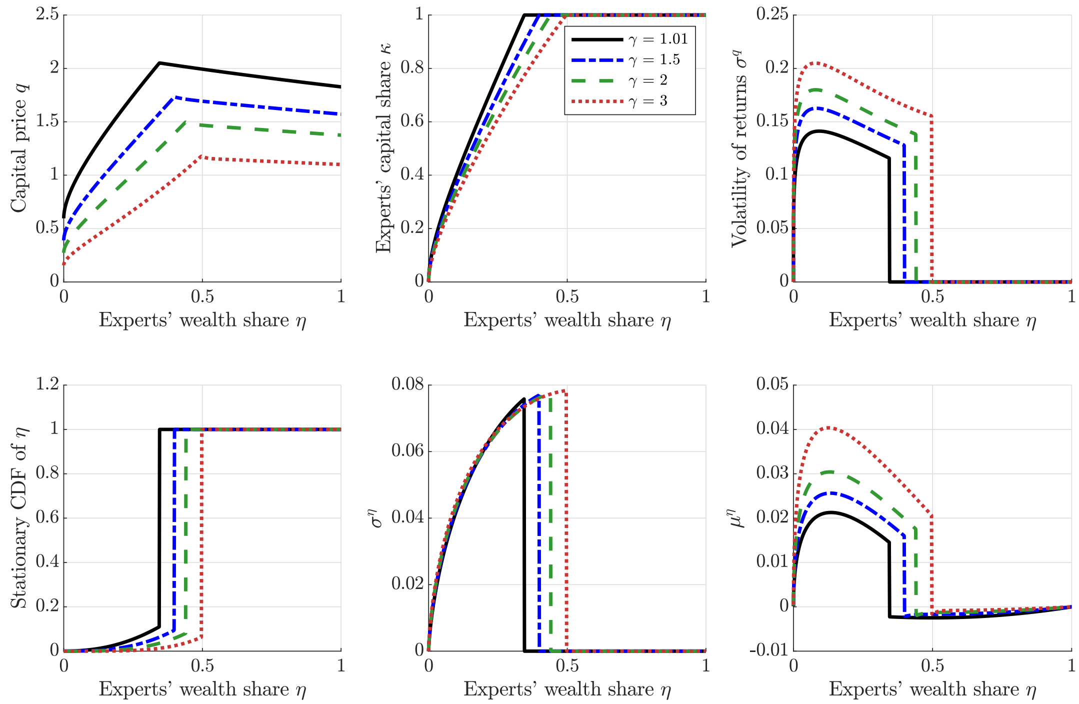

Figure D.1: Sunspot equilibrium for different risk aversion γ. The disaster belief is set to $\kappa_0 = 0.001$. Other parameters: $a_e = 0.11$, $a_h = 0.03$, $\rho_e = 0.06$, $\rho_h = 0.05$, $g = 0.02$.

generates sunspot equilibria featuring lower capital prices and higher volatility.

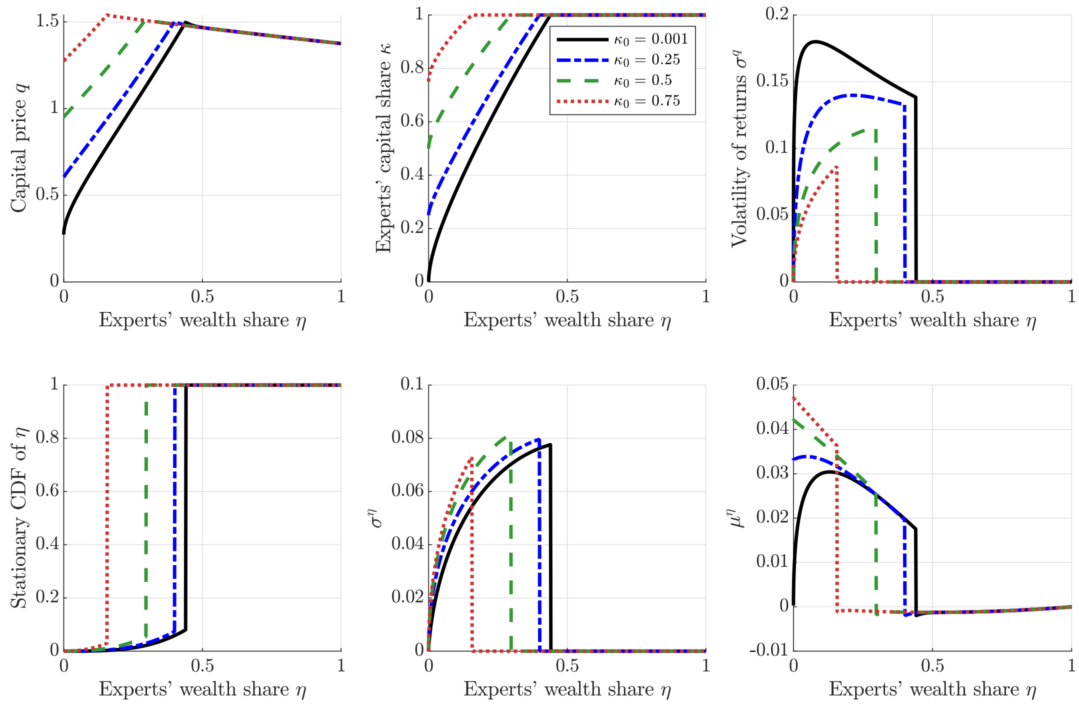

Figure D.2: Sunspot equilibrium for different disaster beliefs κ_0. Risk aversion is set to $\gamma = 2$. Other parameters: $a_e = 0.11$, $a_h = 0.03$, $\rho_e = 0.06$, $\rho_h = 0.05$, $g = 0.02$.

E Poisson Sunspot Equilibrium (PSE)

Rather than model sunspots as the Brownian shock dZ, here we conjecture capital prices can "jump" for non-fundamental reasons. Mathematically, write

$$dq_t = q_{t-}[\mu_{q,t-}dt + \zeta_{q,t-}dJ_t],$$

where J is a Poisson process with intensity λ that does not affect physical capital at all. An equilibrium in which ζ_q is not identically zero will be called the *Poisson Sunspot Equilibrium* (PSE).

In a Markov equilibrium, the sole state variable will still be experts' wealth share η, which follows a jump process

$$d\eta_t = \mu_{\eta,t-}dt + \zeta_{\eta,t-}dJ_t.$$

Note that $\zeta_{\eta,t-} := \eta_t - \eta_{t-}$ by definition. Because agents' portfolios (capital and bonds) are predetermined, we can determine the wealth share jump from the jump in q, with the result being[14]

$$\zeta_\eta = (\kappa - \eta)\frac{\zeta_q}{1+\zeta_q}. \tag{E.1}$$

On the other hand, once the post-jump wealth share is known, the capital price is also known, since η is the sole state variable, i.e., we have $q_t = q(\eta_t)$ for some function q. Thus, if we denote the post-jump wealth share by $\hat{\eta}$,

$$\zeta_q = \frac{q(\hat{\eta}) - q}{q}. \tag{E.2}$$

This is the way to solve the two-way feedback between the wealth distribution and capital prices, similar to the Brownian model. Combining (E.1)-(E.2) yields $\hat{\eta} - \eta = (\kappa - \eta)\frac{q(\hat{\eta})-q}{q(\hat{\eta})}$, which is analogous to the sunspot differential equation of the BSE. Indeed, as $\hat{\eta} \to \eta$, this system converges exactly to $q'/q = (\kappa - \eta)^{-1}$ as in equation (20).

Because we do not model bankruptcy procedures, we must also make sure the jump renders experts solvent, meaning $\zeta_{\eta,t-} > -\eta_{t-}$, to preserve the risk-free status of the bond. If solvency cannot be ensured, then no self-fulfilling jump can take place.

Portfolio choices are still relatively simple, because the jump size is locally predictable, i.e., ζ_q is known just before the jump actually occurs. Ultimately, one can show that the equations characterizing an equilibrium of this model are given by the following simple lemma.

Lemma E.1 (Equilibrium with Jumps)**.** *A Markov equilibrium with jumps requires functions $(q, \kappa, \hat{\eta}, \zeta_q, \zeta_\eta)$ of η to satisfy price-output relation (PO), equations (E.1)-(E.2), and*

$$\hat{\eta} = \eta + (\kappa - \eta)\frac{\zeta_q}{1+\zeta_q} > 0$$

$$0 = \min\left[1-\kappa,\ \eta(1-\eta)\frac{a_e - a_h}{q} - (\kappa - \eta)\frac{\lambda\zeta_q^2}{(1+\frac{\kappa}{\eta}\zeta_q)(1+\frac{1-\kappa}{1-\eta}\zeta_q)}\right].$$

To show what the PSE looks like, we provide a numerical solution and plot some

[14] The derivation is as follows. Let variables with hats, e.g., "$\hat{x}$", denote post-jump variables. Note $\hat{N}_e = \hat{q}\hat{K}\kappa - B$ and $\hat{N}_h = \hat{q}\hat{K}(1-\kappa) + B$, where B is expert borrowing (and household lending, by bond market clearing). Then, $\hat{\eta} = \hat{N}_e/(\hat{q}\hat{K}) = \kappa - B/(\hat{q}\hat{K})$ and by similar logic the pre-jump wealth share is $\eta = \kappa - B/qK$. Thus, $\zeta_\eta = \hat{\eta} - \eta = B[1/(qK) - 1/(\hat{q}\hat{K})] = qK(\kappa - \eta)[1/(qK) - 1/(\hat{q}\hat{K})]$. Using the fact that $\hat{K} = K$ and the definition $\zeta_q := \hat{q}/q - 1$, we arrive at $\zeta_\eta = (\kappa - \eta)[1 - (1+\zeta_q)^{-1}]$. This derivation assumes the presumably risk-free bond price does not jump when capital prices jump. Conceptually, there is no reason why this needs to be true, but it preserves its risk-free conjecture. If bond prices are allowed to jump at the same time, we would find different expressions.

aspects below. The first panel of Figure E.1 displays one simulation of the PSE, also comparing it with a simulation of the BSE. We use the boundary condition $\kappa(0) = 0$ in both the BSE and PSE. The second panel plots the stationary capital price densities (although note that the BSE "density" in fact has a point mass at $\eta = \eta^*$). Capital prices in the PSE tend to remain at lower levels than in the BSE for our example.

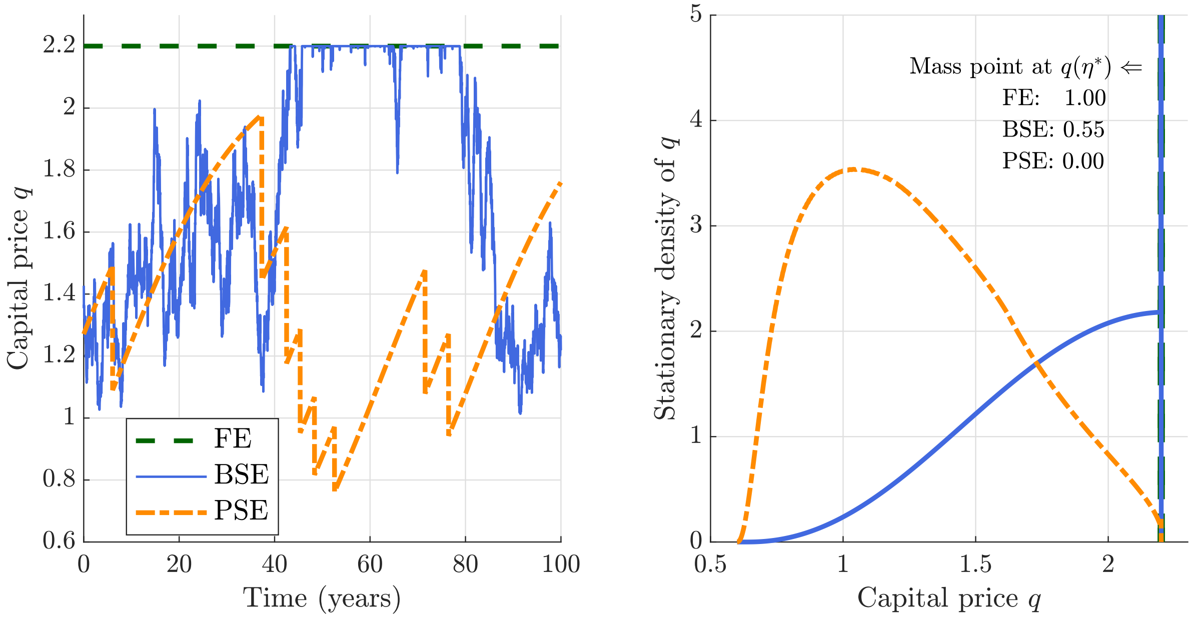

Figure E.1: Time series and stationary density of capital price q in a PSE, BSE, and Fundamental Equilibrium (FE). Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$. For the PSE, we use $\lambda = 0.1$ as the arrival rate of Poisson jumps. To keep the wealth distribution stationary in an economy with symmetric discount rates, we augment benchmark assumptions with random type-switching at rates δ_e and δ_h for experts and households, respectively. This does not change any consumption or portfolio decision, but augments the η dynamics with an additional drift of $(1 - \eta)\delta_h - \eta\delta_e$. We set $\delta_h = 0.004$ and $\delta_e = 0.036$.