Fear, Indeterminacy, and Policy Responses*

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Abstract

The global, stochastic dynamics of the New Keynesian model reveal novel insights about various policies as equilibrium selection tools. First, we unveil a new class of equilibria, characterized by self-fulfilled beliefs about output volatility in recessions, which no conventional Taylor rule can eliminate. An enriched monetary rule targeting risk premia can restore determinacy but becomes infeasible in the presence of any lower bound to interest rates. Second, a large class of fiscal policies kills all such self-fulfilling volatility. This uniqueness holds in many contexts, including: under an interest rate peg or Taylor rule, with long-term debt, and with fiscal rules.

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To address topics of inflation, aggregate demand stimulus, and monetary policy, macroeconomists often look to the New Keynesian model for advice. Despite its role as the dominant policy paradigm, this model is plagued by well-known equilibrium multiplicities that influence its answers to those standard macro questions. Currently, there is no consensus on how equilibria are selected and which of the many survive. Among the many alternatives, two popular selection mechanisms are an aggressive monetary policy that responds sufficiently to output and inflation (e.g., the "Taylor principle") versus an active tax and spending policy (e.g., the "Fiscal Theory of the Price Level" or FTPL).

This paper sheds new light on this old controversy by studying the textbook New Keynesian model but with a simple twist: unlike standard practice, we refrain from linearizing the equilibrium around its steady state. Instead, we study the model in its true nonlinear, stochastic form. The model's global dynamics reveal several insights about the nature of the multiplicities and which policies can eliminate them. Our main result is that the Taylor principle and FTPL are not alternatives to equilibrium selection: the Taylor principle permits a rich class of self-fulfilling dynamics, specifically related to recessionary risk premia, that are killed by FTPL. And hence, the equilibria under the Taylor principle are generally not "observationally equivalent" to the FTPL equilibria, in contrast to conventional wisdom (Cochrane, 2023, Chapter 16.6).

To start, we construct a new class of volatile equilibria that no Taylor rule, no matter how aggressive, can completely eliminate. These equilibria are non-explosive, so they represent a very different multiplicity than the explosive-inflation indeterminacies raised by Cochrane (2011). In fact, by focusing on non-explosive equilibria, a sharp distinction emerges: there exist sufficiently-aggressive interest rate rules that kill all deterministic multiplicities but leave our volatile equilibria intact. The key novelty that arises in a nonlinear, stochastic equilibrium is the presence of a risk premium in agents' Euler equation ("IS curve"). This risk premium, not volatility *per se*, is the source of multiplicity. We emphasize that our equilibria form a large class, in the sense that the economic primitives place very few restrictions on the volatility functions that can emerge in equilibrium.

All these volatile equilibria share some common properties. First, volatility-based indeterminacy is a recessionary phenomenon: it arises only if output is below potential. In a boom, a higher risk premium induces savings that raises agents' consumption growth rate in a way that is unsustainable and thus ruled out as an equilibrium. But in a recession with below-average demand, risk premia sustainably pushes demand back toward steady state and thus helps self-justify the initial uncertainty. Second, because the risk premium is a real object, this indeterminacy is a real rather than nominal phenomenon, in contrast to several theories of self-fulfilling inflations or deflations. In particular, all of these results hold even in (but not exclusively in) the rigid-price limit. The recessionary and real nature of our multiplicity thus further distinguishes it from the literature.

After examining conventional Taylor rules targeting the output gap and inflation, we proceed to study unconventional monetary rules that directly target the self-fulfilling risk premium. We find that this can work: sufficiently-strong risk-premium targeting (i.e., an interest rate that responds more than one-for-one to the risk premium) restores determinacy. That being said, risk-premium targeting prescribes interest rates to become arbitrarily negative. Why? Volatility induces self-fulfilling precautionary savings, which the central bank wants to undo by reducing interest rates; but volatility can always rise ever higher, requiring arbitrarily low rates to undo. Our indeterminacies thus survive these unconventional rules if there is *any* lower bound on policy rates.

We turn next to the FTPL, which has been studied mostly in linearized models. We prove that, in a variety of different settings—including arbitrary exogenous surplus-to-output ratios, long-term or short-term debt, some fiscal "rules" that respond to inflation or the output gap, and different utility functions—FTPL kills the volatile equilibria we discovered. Why? The basic intuition comes from the basic debt valuation equation:

$$\frac{\text{Nominal debt value}}{\text{Price level}} = \text{Current surplus} \times \text{Present value of real surplus growth} \tag{1}$$

This equation must hold at every point in time. If there is hypothetically any sentiment-driven output fluctuation that moves current surpluses, this shock must be "absorbed" by either the price level, the nominal debt valuation, or the present-value of surplus growth (via changes to future surpluses or their discount rates). Sticky prices say that prices cannot jump arbitrarily, and so the price level cannot absorb such a shock. Can the nominal debt value absorb the shock? In the baseline case with short-term debt, the debt price is fixed at 1, and so the quantity of debt is simply determined by the flow government budget constraint; thus, the nominal value of short-term debt is pinned down and cannot absorb the shock either. In the extended case with long-term debt, the bond price is an additional forward-looking variable that could potentially respond to shocks. But with the additional variable comes an additional constraint: the bond-pricing equation. This pricing equation puts severe restrictions on how the bond price can move; we show that these restrictions are so severe that they can never be consistent with the originally conjectured output shock. Consequently, sunspot demand volatility cannot be self-justified under the FTPL. A very similar logic applies to the present-value of real surplus growth on the right-hand-side of equation (1): this is a forward-looking variable that must obey a particular dynamic equation and cannot move arbitrarily with sentiment shocks. We conclude that active fiscal policies, in contrast to monetary policies, sharply trim the indeterminacies endemic to New Keynesian models.

A key takeaway from the discussion above is that FTPL works very differently than the Taylor rule as a selection device. Equation (1) holds at every point in time, essentially steering output volatility at a high frequency. By contrast, an active Taylor rule works by infusing an economy with unstable dynamics, which selects among equilibria by causing all but a subset to explode in the long run.

We think of FTPL's high-frequency steering as "aggregate demand management," because it works by pinning down real demand volatility. One way to understand the demand management interpretation transparently is to consider the rigid-price limit; even in this limit without inflation, FTPL succeeds as a selection device. The rigid-price limit case effectively corresponds to inflation-indexed government debt, which the FTPL literature typically regards as ineffectual for equilibrium selection. Nevertheless, demand volatility is pinned down. We thus advance an interpretation of FTPL as a theory of aggregate demand management, rather than just a theory of the price level.

Related literature. This paper relates to two vast literatures: (a) on New Keynesian indeterminacies and (b) on FTPL as equilibrium selection.

A well-developed literature exposits indeterminacies in monetary models. Going back to Sargent and Wallace (1975), we know that exogenous interest rate paths do not pin down the equilibrium. In related work, several papers have established indeterminacies in New Keynesian models related to the zero lower bound (ZLB)—e.g., Benhabib, Schmitt-Grohé, and Uribe (2001) identify "deflationary trap" equilibria related to inflation expectations, while Benigno and Fornaro (2018) expose "stagnation trap" equilibria related to R&D and growth expectations. There is additional scope for indeterminacy in heterogeneous-agent versions of the New Keynesian model when income risk is countercyclical (Acharya and Dogra, 2020; Ravn and Sterk, 2021; Bilbiie, 2024; Acharya and Benhabib, 2024). Relative to this literature, our multiplicity is novel in two ways: (i) unlike these papers' multiplicity of deterministic equilibria, ours fundamentally depends on aggregate risk and risk premia; and (ii) our multiplicity does not rely on the ZLB or constrained monetary policy and in fact holds for any conventional Taylor rule.

Our equilibria are nonlinear and stochastic by nature. In an important contribution, Caballero and Simsek (2020) study a nonlinear, stochastic version of the New Keynesian model and illustrate how risk premia are critical to aggregate demand dynamics, but restricting attention to the "fundamental equilibrium." We directly connect to their setting in Appendix F, where we unveil a class of sunspot equilibria that trap the economy at the

ZLB. Closely related to our study, contemporaneous work by Lee and Dordal i Carreras (2024) also studies a nonlinear IS curve with risk premia driving the multiplicity. Like us, they also argue that standard "active" Taylor rules do not prune this type of volatility. Our results are more general in proving that self-fulfilling volatility can survive any Taylor rule and that an entire class of risk-premium targeting can ensure determinacy. The most important difference between our papers is our exploration of FTPL.

There are two key differences between our FTPL analysis and the extant literature.[1] First, we emphasize real indeterminacies rather than self-fulfilling inflation. Second, unlike most of the literature, we analyze the fully nonlinear, stochastic, global dynamics of the model and provide formal uniqueness results in several environments.

Whereas some papers have examined FTPL in stochastic nonlinear environments with *flexible prices*, a benchmark nonlinear analysis of FTPL in the sticky-price New Keynesian model does not exist. In particular, Bassetto and Cui (2018), Brunnermeier, Merkel, and Sannikov (2023), and Brunnermeier, Merkel, and Sannikov (2024) study stochastic and nonlinear flexible-price models, focusing on the determinacy of a bubble and/or liquidity-service term in government debt valuation.[2] Two exceptions that do allow sticky prices, but effectively sidestep our determinacy questions, are Mehrotra and Sergeyev (2021) and Li and Merkel (2025). Mehrotra and Sergeyev (2021) studies fiscal sustainability with real debt, which is thus similar to the rigid-price special case of our model; however, they consider exogenous output, which abstracts from self-fulfilling demand fluctuations. Li and Merkel (2025) study FTPL in a New Keynesian model with idiosyncratic risk, which can induce a government debt bubble; however, they avoid determinacy questions by assuming that endogenous objects like inflation and the output gap are Markovian in exogenous states and government bonds outstanding (putting the economy into its minimum state variable solution). Overall, ours is the first paper to provide a formal nonlinear FTPL analysis in textbook sticky-price models.

Finally, we unveil novel limitations of Taylor rules as equilibrium selection devices. Following Cochrane (2011)'s results, Neumeyer and Nicolini (2025) have recently shown, in a precise sense, that destabilizing Taylor rules are not credible. A related message conveyed by our results is that fiscal policies are better suited than monetary policies to trim New Keynesian indeterminacies. As an alternative, future research could investigate

[1] Seminal contributions to the FTPL include Leeper (1991), Sims (1994), Woodford (1994), Woodford (1995), Kocherlakota and Phelan (1999), and Cochrane (2001). Cochrane (2023) synthesizes many results.

[2] Other recent papers studying the FTPL in nonlinear, but deterministic, environments with "liquidity premia" include Berentsen and Waller (2018), Williamson (2018), and Andolfatto and Martin (2018). Also see Miao and Su (2024), which studies determinacy with sticky prices and bubbly liquidity, but applying linearized analysis to a model which is deterministic on aggregate.

the common knowledge perturbation of Angeletos and Lian (2023), which that paper applied to the linearized New Keynesian model, in a nonlinear setting.

1 Model

We present a canonical New Keynesian economy with complete markets and nominal rigidities. The setup is a continuous-time version of the model exposited in Galí (2015), which the reader can consult for additional details.

Sunspot shocks. Our baseline model features no fundamental uncertainty in preferences or technologies. Nevertheless, we want to allow the possibility that economic objects evolve stochastically due to coordinated behavior. To do this, we introduce a standard Brownian motion Z that is extrinsic to all economic primitives. All random processes will be adapted to Z.

Preferences. The representative agent has rational expectations and time-separable utility with discount rate ρ, unitary EIS, and labor disutility parameter φ:

$$\mathbb{E}\Big[\int_0^\infty e^{-\rho t}\Big(\log(C_t) - \frac{L_t^{1+\varphi}}{1+\varphi}\Big)dt\Big]. \tag{2}$$

Consumption C_t has the nominal price P_t and labor L_t earns the nominal wage W_t.

Technology. The consumption good is produced by a linear technology $Y_t = L_t$. We abstract from fundamental uncertainty (e.g., productivity shocks) for maximal clarity.

Behind the aggregate production function is a structure common to most of the New Keynesian literature. In particular, there are a continuum of firms who produce intermediate goods using labor in a linear technology. These intermediate goods are aggregated by a competitive final goods sector. The elasticity of substitution across intermediate goods is a constant ϵ. The intermediate-goods firms behave monopolistically competitively and set prices strategically, described next.

Price setting. Intermediate-goods firms set prices strategically, taking into consideration the impact prices have on their demand. Price setting is not frictionless: firms changing their prices are subject to quadratic adjustment costs, a la Rotemberg (1982). (For simplicity, we assume these adjustment costs are non-pecuniary, so that resource constraints are not directly affected by price adjustments.) In the interest of exposition, we relegate the statement of and solution to this standard problem to Appendix B.

Definition: inflation and output gap. Let P_t denote the aggregate price level and $\pi_t := \dot{P}_t / P_t$ its inflation rate. Note also that the flexible-price level of output is given by $Y^* = (\frac{\varepsilon-1}{\varepsilon})^{\frac{1}{1+\varphi}}$. Following the literature, define the output gap $x_t := \log(Y_t/Y^*)$. Conjecture that x_t and π_t have dynamics of the form

$$dx_t = \mu_{x,t} dt + \sigma_{x,t} dZ_t \tag{3}$$

$$d\pi_t = \mu_{\pi,t} dt + \sigma_{\pi,t} dZ_t \tag{4}$$

for some $\mu_x, \sigma_x, \mu_\pi, \sigma_\pi$ to be determined in equilibrium.

Monetary policy. Let ι_t denote the nominal short-term interest rate, which is controlled by the central bank. Monetary policy follows a Taylor rule that targets the output gap and inflation with

$$\iota_t = \bar{\iota} + \Phi(x_t, \pi_t) \tag{MP}$$

for some target rate $\bar{\iota}$ and some response function that satisfies $\Phi(0,0) = 0$. A common linear example that we will use sometimes is

$$\iota_t = \bar{\iota} + \phi_x x_t + \phi_\pi \pi_t. \tag{linear MP}$$

In the main paper, we abstract from the zero lower bound (ZLB), which introduces well-known indeterminacy issues, and analyze it in Appendix F. For now, think of negative interest rates as a proxy for unconventional monetary policy that can work even when the short rate is zero.

Financial markets. Financial markets are complete. Let M_t be the real stochastic discount factor induced by the real interest rate $r_t := \iota_t - \pi_t$ and the equilibrium price of risk h_t associated to the sunspot shock Z_t. The risk-free bond market is in zero net supply—this will be generalized in Section 4 when we introduce fiscal policies. The equity market is a claim on the profits of the intermediate-goods producers. Alternatively, we can think of these profits as being rebated to the consumers lump-sum.

Definition 1. An *equilibrium* is processes $(C_t, Y_t, L_t, W_t, P_t, M_t, B_t, \iota_t, r_t, \pi_t)_{t \geq 0}$, such that

(i) Taking (M_t, W_t, P_t) as given, consumers choose $(C_t, L_t)_{t\geq 0}$ to maximize (2) subject to their lifetime budget and No-Ponzi constraints

$$\frac{B_0}{P_0} + \Pi_0 + \mathbb{E}\Big[\int_0^\infty M_t \frac{W_t L_t}{P_t} dt\Big] \geq \mathbb{E}\Big[\int_0^\infty M_t C_t dt\Big] \tag{5}$$

$$\lim_{T\to\infty} M_T \frac{B_T}{P_T} \geq 0, \tag{6}$$

where Π represents the real present-value of producer profits and B represents the bond-holdings of the consumer.[3]

(ii) Firms set prices optimally, subject to their quadratic adjustment costs.

(iii) Markets clear, namely $C_t = Y_t = L_t$ and $B_t = 0$.

(iv) The central bank follows the interest rate rule (MP) for some target rate $\bar{\imath}$ and some response function $\Phi(\cdot)$.

In what follows, we refer to a *deterministic equilibrium* as an equilibrium with no real volatility, $\sigma_x \equiv 0$. A *sunspot equilibrium* is an equilibrium with real volatility, $\sigma_x \neq 0$.

Equilibrium characterization. We first provide a summary characterization of all equilibria. Labor supply and consumption decisions satisfy the following optimality conditions:

$$e^{-\rho t} L_t^{\varphi} = \lambda M_t \frac{W_t}{P_t} \tag{7}$$

$$e^{-\rho t} C_t^{-1} = \lambda M_t, \tag{8}$$

where λ is the Lagrange multiplier on the lifetime budget constraint (5).

On the firm side, Appendix B shows that optimal firm price setting gives rise to aggregate inflation dynamics that satisfy

$$\mu_{\pi,t} = \rho \pi_t - \eta \varepsilon \frac{W_t}{P_t} + \eta(\varepsilon - 1), \tag{9}$$

where η is each firm's degree of price flexibility. Notice that as $\eta \to 0$ (prices changes become infinitely costly), one possible equilibrium is to have $\pi_t \to 0$ for all times. We will assume this "rigid-price limit" is the equilibrium that obtains as $\eta \to 0$.

We use these conditions to obtain an "IS curve" and a "Phillips curve." Applying Itô's formula to (8), we obtain the consumption Euler equation, which may be rewritten

[3] In addition, to prevent arbitrages like "doubling strategies" that can arise in continuous time, we must impose a uniform lower bound on borrowing, e.g., $M_t B_t / P_t \geq -\underline{b}$, although $\underline{b}$ can be arbitrarily large.

in terms of the output gap as

$$\mu_{x,t} = \iota_t - \pi_t - \rho + \frac{1}{2}\sigma_{x,t}^2. \tag{IS}$$

Equation (IS) is the IS curve. Next, divide the FOCs (7)-(8), and use goods and labor market clearing $C_t = Y_t = L_t$ to get $Y_t^{1+\varphi} = \frac{W_t}{P_t}$. Substitute this expression into (9) to obtain

$$\mu_{\pi,t} = \rho\pi_t - \kappa\Big(\frac{e^{(1+\varphi)x_t} - 1}{1+\varphi}\Big), \tag{PC}$$

where $\kappa := \eta(\varepsilon - 1)(1+\varphi)$. Equation (PC) is the Phillips curve. These IS and Phillips curves are written in their fully nonlinear form.

The most important novelty in our paper is the presence of precautionary savings due to aggregate risk. This force is captured by the term $\frac{1}{2}\sigma_x^2$ in (IS). We will often refer to this term as a "risk premium" because σ_x^2 is exactly the equilibrium risk premium on the aggregate consumption claim.[4] When writing the IS curve in terms of log consumption, as is typically done, the Jensen correction of $\frac{1}{2}$ also shows up.

Together with the monetary policy rule (MP), equations (IS) and (PC) form the nonlinear "three equation model" in standard New Keynesian models. An equilibrium is completely characterized by these three equations, along with conditions that ensure that any output gap or inflation explosions are consistent with optimization behavior. For example, since $C_t = e^{x_t}Y^* = L_t$, the representative agent would obtain minus infinite utility if $x_t = \pm\infty$ in finite time, or even if $x_t \to \pm\infty$ too quickly. Clearly, this is not compatible with optimizing behavior if the agent has an alternative that delivers finite utility. Consumers would be individually better off ignoring signals to coordinate, unravelling such a proposed allocation. (A straightforward example is the case when $x_t \to \infty$, since this implies that real wages are diverging towards plus infinity, and agents may obtain finite utility simply by working a finite amount forever.) Similarly, we show in Appendix B that firms' optimization rules out situations when $\pi_t \to \pm\infty$ too quickly, because that would induce an infinite amount of price adjustment costs.

In order to emphasize that the multiplicity unveiled later does not rely on explosive behavior (unlike the multiplicity exposited in Cochrane, 2011) and to streamline the analysis, we only consider equilibria that satisfy a simple condition that conforms with

[4] The aggregate risk premium is σ_x^2 for the following reason. Aggregate consumption growth volatility is σ_x. Due to log utility, agents optimally set their individual consumption growth volatility equal to the market price of risk (i.e., the sensitivity of $d\log M$ to dZ). Hence, the quantity of risk and price of risk are both equal to σ_x, implying the risk premium (as the product of risk quantity and risk price) is σ_x^2.

most of the existing literature and rules out both finite-time and asymptotic explosions:

Condition 1. *A non-explosive allocation has* $\mathbb{P}\{|x_t| < \infty, |\pi_t| < \infty; \forall t \geq 0\} = 1$,

$$\limsup_{T\to\infty} \mathbb{E}|x_T| < \infty, \quad \limsup_{T\to\infty} \mathbb{E}|\pi_T| < \infty, \quad \textit{and} \quad \limsup_{T\to\infty} \mathbb{E}[e^{(1+\varphi)x_T - \rho T}] = 0. \tag{10}$$

We summarize our characterization in the following lemma.

Lemma 1. *Suppose processes* $(x_t, \pi_t, \iota_t)_{t\geq 0}$ *satisfy (IS), (PC), (MP), and Condition 1. Then,* $(x_t, \pi_t, \iota_t)_{t\geq 0}$ *corresponds to an equilibrium of Definition 1.*

The proof of this lemma is standard except for a careful treatment of potentially explosive behavior. Condition 1 not only ensures that utility for the representative consumer and firm are finite but also, together with the other equations in Lemma 1, is sufficient to verify that their transversality conditions hold. See Appendix A.1 for these arguments. Going forward, we will want to make reference only to equilibria which satisfy Condition 1. For that reason, we include the following definition.

Definition 2. A *non-explosive equilibrium* is an equilibrium in which Condition 1 holds.

Linearized Phillips curve approximation. We will occasionally use a linearized Phillips curve in place of (PC). Since $e^{(1+\varphi)x_t} - 1 \approx (1+\varphi)x_t$, the Phillips curve to first order is

$$\mu_{\pi,t} = \rho\pi_t - \kappa x_t. \tag{linear PC}$$

We will occasionally work with (linear PC) instead of (PC), because, as will become clear, the nonlinearity in (IS) is the critically novel element, and not so much the nonlinearity in (PC). (We do some analysis with the nonlinear Phillips curve in Appendix C.) In this approximation, we will sometimes refer to non-explosive equilibrium as $(x_t, \pi_t, \iota_t)_{t\geq 0}$ that satisfy (IS), (linear PC), (MP), and Condition 1.

2 Review: Determinacy and the Taylor Principle

As a warm up, and to distinguish our main results that come after, we begin by reviewing the standard deterministic multiplicities in New Keynesian models (i.e., equilibria with $\sigma_x = 0$). We illustrate how aggressive monetary policy rules can eliminate this indeterminacy. In the appendix, we generalize these existing results to nonlinear Phillips curves and nonlinear Taylor rules as well.

First, note that there is always an equilibrium with $x = \pi = 0$ forever, provided monetary policy sets the target rate at $\bar{\imath} = \rho$.

Can there exist other equilibria? As is well known, the answer to this question hinges on the stability/instability properties of the equilibrium dynamical system for (x_t, π_t). We will review this analysis here. First, we specialize to the policy rule (linear MP) with the target rate $\bar{\imath} = \rho$. Combining (linear MP) with (IS), the dynamics of x_t are given by

$$\dot{x}_t = \phi_x x_t + (\phi_\pi - 1)\pi_t. \tag{11}$$

The IS curve is linear in a deterministic equilibrium with a linear Taylor rule. We also consider here the linear Philips curve (linear PC) as in most of the existing literature.

The typical determinacy analysis picks an aggressive Taylor rule that renders the above system unstable. The system (11) and (linear PC) can be written in matrix form as

$$\begin{bmatrix} \dot{x}_t \\ \dot{\pi}_t \end{bmatrix} = \mathcal{A} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}, \quad \text{where} \quad \mathcal{A} := \begin{bmatrix} \phi_x & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}. \tag{12}$$

The eigenvalues of $\mathcal{A}$ are both strictly positive, and the system unstable, if $\phi_x > -\rho$ and $\phi_\pi > 1 - \rho\phi_x/\kappa$. This is the continuous-time version of the eigenvalue conditions in Blanchard and Kahn (1980).

By contrast, if either parameter condition is violated, then the system has one or two stable eigenvalues. In such case, there are a continuum of non-explosive equilibria, which one can index by the initial conditions (x_0, π_0). As the explicit parameter conditions make clear, instability occurs when monetary policy is sufficiently aggressive (i.e., "active") in responding to the output gap and inflation, whereas stability occurs when the response function is less aggressive (i.e., "passive"). The proof of the following proposition and all subsequent results in Sections 2-3 is contained in Appendix A.3.

Proposition 1. *Consider the linearized Phillips curve (linear PC) and monetary policy rule (linear MP) with $\bar{\imath} = \rho$. If $\phi_x > -\rho$, and $\phi_\pi > 1 - \rho\phi_x/\kappa$, the only non-explosive equilibrium is $(x_t, \pi_t) = (0, 0)$ forever. If either condition is violated, then a continuum of linear non-explosive equilibria exist.*

Remark 1 (Nonlinear Phillips curve)**.** *We have used the linearized Phillips curve here for simplicity and exposition. We analyze the nonlinear Phillips curve in Appendix C, and the conclusion is similar to Proposition 1 but the proof is more complicated.*

Remark 2 (Explosive equilibria)**.** *A key clause is the requirement that equilibria satisfy Condition 1, ruling out explosions. What if asymptotic explosions were permitted, while finite-time* *explosions were ruled out? It turns out that, by adopting an aggressive nonlinear Taylor rule, monetary policy can force an explosion in finite-time, and hence select a unique equilibrium. We analyze this situation in Appendix D. In that sense, the spirit of Proposition 1 is preserved even under broader notions of equilibrium.*

3 A New Class of Sunspot Equilibria

Now, we demonstrate several new results pertaining to volatility in New Keynesian models. Motivated by the results from Section 2, we will assume a sufficiently aggressive monetary policy, so that the deterministic equilibrium is unique. Nevertheless, we will show that a large class of volatile equilibria exist, due to the presence of risk premia. Moreover, such volatile equilibria persist for *any* aggressive Taylor rule. As we will then show, a different type of policy rule, which targets the risk premium, is required to eliminate stochastic multiplicities. Finally, we argue that risk premium targeting may be limited if an effective lower bound exists for interest rates.

3.1 Constructing volatile equilibria

For concreteness, assume that prices are permanently rigid, i.e., $\kappa \to 0$. This clarifies that we are focusing on real indeterminacy rather than inflation indeterminacy. An additional advantage is that we only need to study the dynamics of the output gap, rather than a two-dimensional stochastic system. Start with an example policy rule with target rate $\bar{\iota} = \rho$ and nonlinear response function

$$\Phi(x, \pi) = \phi_x(e^x - e^{-x}), \quad \phi_x \geq 0. \tag{13}$$

Rule (13) brings theoretical clarity to the discussion. This is a super-aggressive policy rule, evidently more aggressive than its linear approximation $2\phi_x x$. It would send the deterministic economy to a finite-time explosion unless $x_t = 0$ forever (see Appendix D), thus selecting a unique deterministic equilibrium.

Nevertheless, stochastic indeterminacy exists. Combining (MP) with (13) and (IS), the drift of x_t is given by

$$\mu_x = \phi_x(e^x - e^{-x}) + \frac{1}{2}\sigma_x^2.$$

Building off of the previous analysis, the question is whether the dynamical system characterized by (μ_x, σ_x) keeps x_t finite forever. But here, the volatility σ_x is determined purely via coordination, and some choices will lead to stability. The basic idea is that $\frac{1}{2}\sigma_x^2$

augments the drift μ_x, so if σ_x^2 rises sufficiently when x is low (i.e., in recessions), then μ_x will rise high enough to "push" the equilibrium back towards $x = 0$. Mathematically, the dynamical system will be stable. Economically, the rise in uncertainty creates *precautionary savings* that slowly pushes demand (hence output) back up over time. Thus, one can think of savings as precisely the mechanism of self-fulfillment that justifies risk. Both the economic intuition and mathematics also suggest that sunspot volatility should necessarily be recessionary and not expansionary. We revisit this point below and establish that recessionary volatility is a general characteristic of our equilibria.

Formally, to see how multiplicity is possible, examine instead the dynamics of the level output gap $y_t := e^{x_t}$ and verify that $0 < y_t < \infty$ forever (which more or less suffices to ensure that x_t is non-explosive). By Itô's formula, the drift and diffusion of y_t are

$$\mu_y = \phi_x(y^2 - 1) + y\sigma_x^2 \quad \text{and} \quad \sigma_y = y\sigma_x \tag{14}$$

Right away, we see that stability is possible, if agents coordinate on sufficiently high volatility. For example, suppose for some constant $\nu > 0$,

$$\sigma_x^2 = \begin{cases} \left(\frac{\nu}{y}\right)^2 + \phi_x \frac{1-y^2}{y}, & \text{if } y < 1; \\ 0, & \text{if } y \geq 1. \end{cases} \tag{15}$$

Putting these equations together, the dynamics for y_t would be

$$dy_t = \begin{cases} \frac{\nu^2}{y_t}dt + \sqrt{\nu^2 + \phi_x y_t(1 - y_t^2)}dZ_t, & \text{if } y_t < 1; \\ \phi_x(y_t^2 - 1)dt & \text{if } y_t \geq 1. \end{cases} \tag{16}$$

It is relatively intuitive to see that $y_t > 0$ for all t in this example: if y ever approached 0, the drift $\frac{\nu^2}{y}$ would explode fast enough to push y back up. Formalizing this mathematically, the process in (16) behaves asymptotically (as $y \to 0$) like a Bessel(3) process, which never hits 0 (more precisely, 0 is an "entrance boundary" for this process). And consequently, $x_t = \log(y_t)$ does not explode negatively.[5] Provided $y_0 \leq 1$, the process also does not explode positively: there is no volatility for $y_t \geq 1$, so the process will eventually converge to and stay stuck at the efficient level $y_t = 1$ (i.e., the sunspot volatility is temporary in this example). This entire construction works for any $\nu > 0$, so ν is a

[5] A Bessel(n) process corresponds to the solution of $dX_t = \frac{n-1}{2}X_t^{-1}dt + dZ_t$ where dZ_t is a one-dimensional Brownian motion. A Bessel(n) process is also equivalent to the Euclidean norm of a n-dimensional Brownian motion and therefore it satisfies $X_t > 0$ for all $t > 0$, provided $n \geq 2$. Taking the limit of the dy_t evolution equation as $y \to 0$, we can see that $\nu^{-1}y$ behaves exactly as a Bessel(3) process.

parameter indexing an continuum of possible sunspot equilibria. In summary, despite the super-aggressive response function (13), many equilibria exist with different σ_x.

The key reason for multiplicity is the risk premium term $\frac{1}{2}\sigma_x^2$, not volatility per se. To see this, contrast the *linearized* version of the Euler equation, which is $\mu_x = \iota - \pi - \rho$. In this linearized world, there can be risk (that is, x can have volatility), but it is as if the representative agent is risk-neutral, and so there are no risk premia or precautionary savings. Repeating the above analysis in this linearized world, (16) would be replaced by

$$dy_t = \begin{cases} \phi_x(y_t^2 - 1)dt + \sqrt{\nu^2 + \phi_x y_t(1 - y_t^2)}dZ_t, & \text{if } y_t < 1; \\ \phi_x(y_t^2 - 1)dt, & \text{if } y_t \geq 1. \end{cases} \tag{17}$$

The process in (17) behaves like an arithmetic Brownian motion with negative drift for $y_t \approx 0$. Consequently, one would conclude from the linearized model that $y_t \to 0$ in finite time with positive probability—in violation of Condition 1. Thus, the only possible linearized non-explosive equilibrium can be $y_t = 1$ at all times. A very aggressive Taylor rule trims equilibria in this linearized stochastic world, exactly as in the deterministic equilibria. It is easy to verify that a similar analysis applies for arbitrary choices of σ_x.

Figure 1 visualizes the difference between the actual dynamics and the linearized dynamics. The solid blue line is the drift μ_y in our baseline example from equation (16). Notice that μ_y rises sufficiently fast as y falls, which is enough to prevent $y \to 0$. This is a case of a strongly countercyclical risk premium. By contrast, the dotted red line is the linearized drift from equation (17). Because the linearization omits the risk premium term, it looks dramatically different: as y falls, the drift becomes more and more negative, due to the aggressiveness of monetary policy in lowering interest rates. Policy thus destabilizes the economy and leads to a unique equilibrium, indicated by the solid black dot, which is equivalent to the deterministic steady state. The difference between the nonlinear and linearized economy appears only in bad times, when $y < 1$, by construction. But it is clear from the figure that volatility could not arise in good times, when $y > 1$. Indeed, even without volatility, the drifts are all positive when $y > 1$, i.e., dynamics are unstable; adding a risk premium only makes them more unstable and cannot be part of a non-explosive equilibrium.

Finally, we emphasize one subtle issue regarding the global dynamics. To support multiplicity, *the risk premium must be sufficiently countercyclical*. This is because, in a stochastic system, stability is not only about the drift μ_y but depends on the ratio of the drift relative to the variance, i.e., μ_y/σ_y^2. The idea is that the forces impacting dynamics in expectation (via μ_y) must outweigh the shocks hitting the system (via σ_y^2). Going back to equation (14), we see that our policy rule implies the following general form for this ratio:

$$\frac{\mu_y}{\sigma_y^2} = \underbrace{\frac{1}{y}}_{\substack{\text{risk premium} \\ \text{effect (stabilizing)}}} + \underbrace{\frac{\phi_x(y^2-1)}{y^2\sigma_x^2}}_{\substack{\text{aggressive monetary} \\ \text{policy effect (destabilizing)}}} . \tag{18}$$

There is a race between these two terms. The first term arises due to the risk premium and always works to buoy the output gap, via the precautionary savings discussed above. The second term in equation (18), assuming $\phi_x > 0$, is negative: monetary policy is reducing savings with lower interest rates. As we show in the appendix (Lemma A.1), a sufficient condition for $y_t > 0$ forever and for the global dynamics to be stable is that $\theta := \lim_{y\to 0} y\mu_y/\sigma_y^2 > 1/2$. In other words, the risk premium effect must be sufficiently strong to dominate the destabilizing monetary policy effect, meaning σ_x^2 must be sufficiently countercyclical. Under the baseline construction in equation (15), such sufficient countercyclicality holds. Imagine, in contrast, that the risk premium took the less countercyclical form $\sigma_x^2 = \nu^2 + \phi_x(1-y^2)/y$ when $y < 1$. Then, the second term in equation (18) becomes $\frac{\phi_x(y^2-1)}{y^2\sigma_x^2} = -\frac{\phi_x(1-y^2)}{y^2\nu^2+\phi_x y(1-y^2)}$, which behaves like $-\frac{1}{y}$ as $y \to 0$. This is enough to cancel the first term, the risk premium effect, and so $y\mu_y/\sigma_y^2 \to 0$ as $y \to 0$. Coordination on this proposed volatility would lead to explosive dynamics and could not constitute an equilibrium. The dotted-dashed yellow line in Figure 1, labeled "insufficiently countercyclical risk premium", plots the drift for this case.

As mentioned, the particular construction in (15)-(16) features transitory volatility. That was only to develop an initial understanding and is easily generalized. For example, suppose agents coordinate on the following volatility process for some $\delta \in (0,1)$:

$$\sigma_x^2 = \begin{cases} \left(\frac{\nu}{y}\right)^2 + \phi_x \frac{1-y^2}{y}, & \text{if } y < 1-\delta; \\ 0, & \text{if } y \geq 1-\delta. \end{cases} \tag{19}$$

Now, agents only expect risk premia to arise in a deep enough recession. The induced dynamics of $y_t = e^{x_t}$ are

$$dy_t = \begin{cases} \frac{\nu^2}{y_t}dt + \sqrt{\nu^2 + \phi_x y_t(1-y_t^2)}\,dZ_t, & \text{if } y_t < 1-\delta \\ \phi_x(y_t^2-1)dt & \text{if } y_t \geq 1-\delta. \end{cases} \tag{20}$$

Provided $y_0 < 1$, this process will eventually exit the deterministic region, enter the

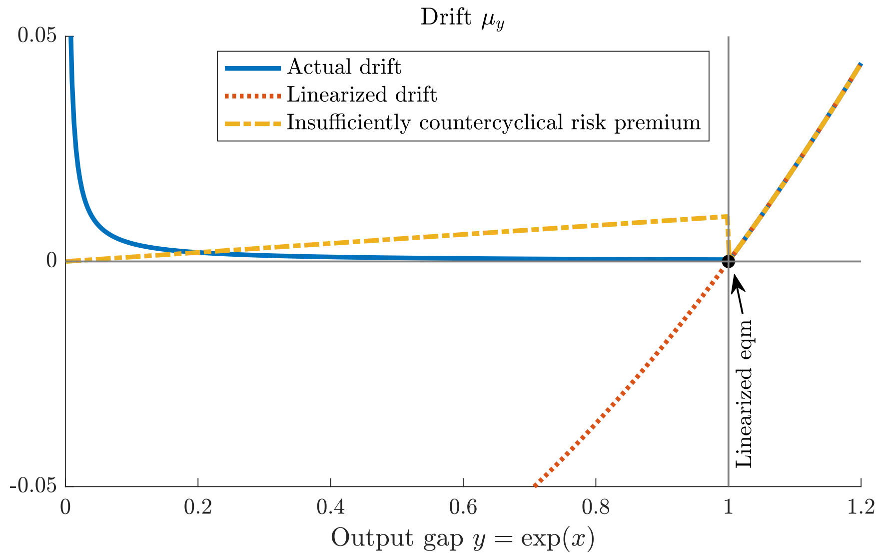

Figure 1: Output gap drifts in different scenarios with rigid prices ($\kappa \to 0$). The solid blue line is the drift in equation (16). The dotted red line is the linearized drift, given by equation (17). The dotted-dashed yellow line is the drift if, instead of (15), the volatility is alternatively given by $\sigma_x^2 = [(5\nu)^2 + \phi_x y^{-1}(1-y^2)]\mathbf{1}_{\{y<0\}}$. Parameters: $\rho = 0.02$, $\nu = 0.02$, $\phi_x = 0.1$.

volatile region, and remain inefficiently volatile for an infinite amount of time.[6] Figure 2 presents a numerical construction of this example, showing that the economy is permanently inefficient ($y < 1$ forever), volatility is not transitory, and nevertheless Condition 1 holds and there exists a stationary distribution for $y = e^x$.

3.2 General results on multiplicity with volatility

The analysis so far is confined to a particular example with a specific monetary policy. But perhaps monetary policy could act even more aggressively and eliminate the risk premium effect. Is there some Taylor rule that can kill these equilibria? No. Agents can always coordinate on a level of volatility that keeps the dynamics "stable" for *any* level of aggression in the Taylor rule satisfying the following mild regularity assumption (which all rules considered in this paper satisfy):

Assumption 1. *There exists $\beta > 0$ such that $\Phi(x)$ satisfies $\lim_{x \to -\infty} e^{\beta x}\Phi(x) > -\infty$.*

Proposition 2. *Suppose prices are rigid ($\kappa \to 0$). Consider any Taylor rule (MP) with $\bar{\iota} = \rho$ that is increasing in x and satisfies Assumption 1. Then,*

[6] To see all these points, note that the process has zero volatility and negative drift when $y \in (1-\delta, 1)$; therefore, the process exits the region $(1-\delta, 1)$ and enters $(0, 1-\delta)$ in finite time almost-surely. Upon entering the volatile region $(0, 1-\delta)$, the process can move around but will never reach $y = 0$, by the same Bessel(3) argument established in the text. Finally, the stationary distribution will additionally have a mass point at $y = 1-\delta$, because the dynamics induce y_t to visit the point $1-\delta$ infinitely often.

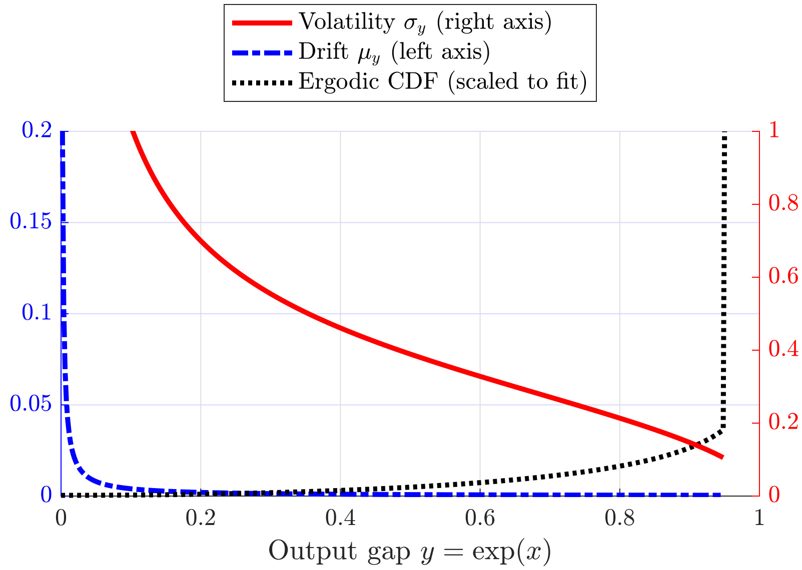

Figure 2: Equilibrium with rigid prices ($\kappa \to 0$) and dynamics given by equations (19)-(20). The stationary CDF is computed via a discretized Kolmogorov Forward equation. The resulting stationary CDF features a mass point at $y = 1 - \delta$. Parameters: $\rho = 0.02$, $\nu = 0.02$, $\delta = 0.05$, $\phi_x = 0.1$.

(i) There exist a continuum of non-explosive sunspot equilibria indexed by $x_0 < 0$ and the volatility function $\sigma_x(x)$. The volatility can be any mapping $\sigma_x : \mathbb{R} \mapsto \mathbb{R}$ that is finite for all $x \in (-\infty, 0)$ and satisfies suitable boundary conditions as $x \to -\infty$ and $x \to 0$.

(ii) If $\inf_x \Phi'(x) > 0$, then all non-explosive equilibria have $x_t \leq 0$ forever, and hence any sunspot equilibrium is recessionary.

Intuitively, the idea behind statement (i) of Proposition 2 is contained in the example construction above. For any Taylor rule, agents can coordinate on a level of volatility that "undoes" the effect of interest rates on output gap dynamics. The central bank tries to destabilize the economy, and agents coordinate on a risk premium that stabilizes it. We also emphasize a point regarding the fact that $\sigma_x(x)$ can essentially be any function satisfying suitable "boundary conditions": when one cares about global stability as we do, all that matters are boundary conditions on the equilibrium dynamics, rather than a local analysis around the fundamental equilibrium $(x, \pi) = (0, 0)$.

Statement (ii) of Proposition 2 says that self-fulfilling volatility is *recessionary*. Risk premia σ_x^2 always provide a positive force that increases the drift μ_x and buoys the output gap. In a recession (i.e., $x < 0$), this stabilizes the economy, pushing x back toward zero, and provides the needed dynamic self-justification. But in a boom (i.e., $x > 0$), risk premia would send the economy further and further away from steady state, which is destabilizing.

For analytical convenience, Proposition 2 is proved in the rigid price limit. However, the same intuition carries over to a world with partially-flexible prices. Indeed, Proposition E.1 in Appendix E constructs a similar recessionary sunspot equilibrium in which both inflation and the output gap are stochastic. Figure 3 displays a numerical example of such an equilibrium in which $\pi_t = \pi(x_t)$ for some function $\pi(\cdot)$. This equilibrium has the following properties. First, as in the rigid-price construction, volatility is recessionary and permanent, with a stationary distribution of the economy living in the region $x_t < 0$. Second, there is *persistent deflation* here. This is reasonable to expect, since demand-based fluctuations usually induce deflation and recession to occur together, although we have not proven that this property is general.

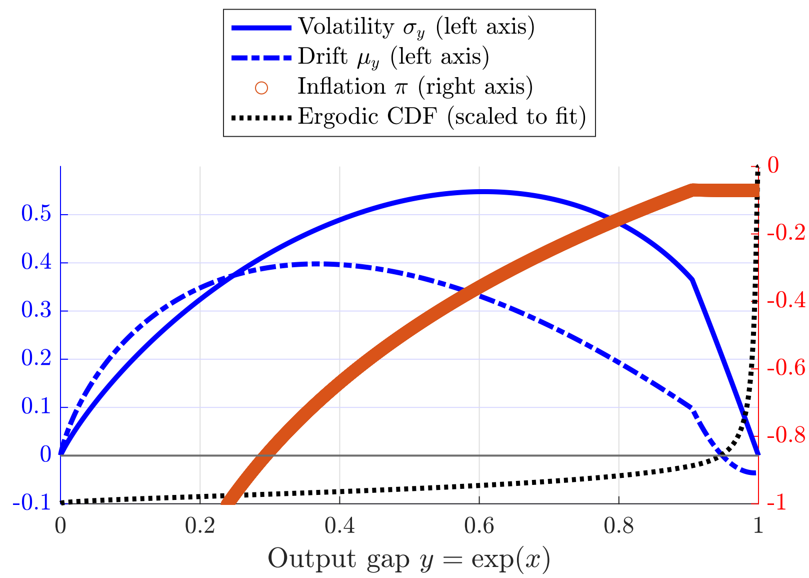

Figure 3: Equilibrium with partially-flexible prices ($\kappa > 0$), a linear Taylor rule, a linearized Phillips curve, but a nonlinear IS curve. The inflation and volatility functions are described in Appendix E, which describes a family of equilibria indexed by (i) the point $\bar{x}$ in the state space where the inflation and volatility functions have a kink (in this construction, we set $\bar{x} = \log 0.9$) and (ii) the slope of the inflation function in the region $\{x < \bar{x}\}$. The stationary CDF is computed via a discretized Kolmogorov Forward equation. Parameters: $\rho = 0.02$, $\kappa = 0.2$, $\bar{\iota} = \rho$, $\phi_x = 0.2$, $\phi_\pi = 1.5$.

Comparing Propositions 1-2, our analysis sharply distinguishes stochastic sunspot equilibria from deterministic ones. Typically, they are tightly linked: one often constructs the sunspot equilibria as a "lottery" over deterministic equilibria (Azariadis, 1981). Here instead, the presence of risk premia means that stochastic equilibria can have a markedly different character than their deterministic counterparts: they work through the risk premium and are necessarily recessionary.

This result that stability properties can flip in the nonlinear stochastic model is in contrast to the conventional wisdom regarding such models. For example, Cochrane

(2023) writes

> this book is really about the broad determinacy and stability properties of monetary models. In one sense, the conclusions of these simple models are likely to be robust, because stability and determinacy depend on which eigenvalues are greater or less than 1. As long as a model modification does not move an eigenvalue across that boundary, the stability and determinacy conclusions are not changed. (Chapter 5.8)

While stability properties may be of some theoretical interest, a practical question we ask next is whether some policies can help trim or eliminate such sunspot equilibria.

3.3 Risk premium targeting

There is one type of rule that can restore determinacy. Following the suggestion in Lee and Dordal i Carreras (2024), suppose we replace the plain-vanilla Taylor rule (MP) with a rule that explicitly targets the risk premium. However, we will provide a much broader proposition regarding the efficacy of this rule. We use

$$\iota_t = \rho + \Phi(x_t, \pi_t) - (\alpha_- \mathbf{1}_{\{x_t<0\}} + \alpha_+ \mathbf{1}_{\{x_t>0\}}) \times \left(\frac{1}{2}\sigma_{x,t}^2\right). \tag{MP-vol}$$

Although conventional wisdom would suggest that targeting an asset price—which maps one-to-one into the output gap—suffices to target the risk premium, that is not true here, intuitively because coordination on a fearful equilibrium can raise uncertainty $\sigma_{x,t}$ independently, i.e., without affecting x_t in the short run. Rule (MP-vol) directly targets the uncertainty that generates risk premia.

Risk premium targeting restores determinacy. Substitute (MP-vol) into (IS) to get

$$dx_t = \left[\Phi(x_t, \pi_t) - \pi_t + \frac{1}{2}(1 - \alpha(x_t))\sigma_{x,t}^2\right]dt + \sigma_{x,t} dZ_t, \tag{21}$$

where $\alpha(x) := \alpha_- \mathbf{1}_{\{x<0\}} + \alpha_+ \mathbf{1}_{\{x>0\}}$ is the state-dependent responsiveness to the risk premium. If $\alpha_- = \alpha_+ = 1$, then the risk premium vanishes from the drift, and we are back in a situation where an aggressive response function Φ can trim equilibria by destabilizing the economy. If $\alpha_+ < 1 < \alpha_-$, then the risk premium itself becomes destabilizing: higher levels of $\sigma_{x,t}^2$ make the drift push x_t further away from zero. Therefore, a modified Taylor rule like (MP-vol), with more aggressive risk premium targeting in bad times, can always eliminate equilibrium multiplicity. Again, for analytical purposes, we state this result in the rigid price limit, with the proof in Appendix A.

Proposition 3. *Suppose prices are rigid ($\kappa \to 0$). With sufficiently strong risk premium targeting ($\alpha_+ \leq 1 \leq \alpha_-$) and sufficiently aggressive responsiveness to the output gap, the modified Taylor rules (MP-vol) ensure that the unique non-explosive equilibrium is $x_t = 0$.*

The deep difference between the multiplicity of sunspot equilibria and the multiplicity of deterministic equilibria was the presence of a stabilizing risk premium. And this manifests in a qualitatively distinct policy response to restore determinacy: by targeting the risk premium, with the interest rate moving more than one-for-one in bad times, the central bank can use it as a destabilizing nuclear threat.

3.4 Effective lower bounds

While risk-premium targeting can work to ensure determinacy, it requires an unconstrained monetary policy. To understand this, recall that all the rules advocated above share the property that $\iota_t \to -\infty$ as $x_t \to -\infty$. If interest rates are lower bounded, this cannot work. In this section, we explore here a situation where monetary policy is constrained: suppose ι_t must respect the lower bound $\iota_t \geq \underline{\iota}$.

With an effective lower bound, volatile equilibria cannot be trimmed. To see this, consider the rigid-price equilibria and inspect output gap dynamics when ι_t is at its lower bound:

$$dx_t = \left[\underline{\iota} - \rho + \frac{1}{2}\sigma_{x,t}^2\right]dt + \sigma_{x,t}dZ_t, \quad \text{when} \quad x_t < 0 \quad \text{and} \quad \iota_t = \underline{\iota}. \tag{22}$$

A sufficiently high level of uncertainty can raise the drift and create stable dynamics by "pushing" x_t to stay away from $-\infty$. Using this logic, it is easy to prove the following.

Proposition 4. *Suppose prices are rigid ($\kappa \to 0$). Let ι_t be any interest rate process that obeys the lower bound $\iota_t \geq \underline{\iota}$ and is "at target" in steady state (i.e., $\iota_t = \bar{\iota} = \rho$ when $x_t = \sigma_{x,t} = 0$). Then, any $x_0 \leq 0$ corresponds to at least one valid non-explosive equilibrium with volatility.*

With a zero lower bound (ZLB), or *any* lower bound, certain monetary threats are not credible. The reason is precisely that our sunspot equilibria are *recessionary and self-sustained by high risk premia*. In particular, suppose we are in a hypothetical recession (i.e., low x). An active monetary policy seeking to eliminate this hypothetical recession would want to set interest rates very low, thereby impounding a very negative drift to consumption growth. But the lower bound $\iota_t \geq \underline{\iota}$ prevents such a force from being too strong. In that case, risk premia can be so high as to stabilize the economy, outweighing the destabilizing effect of monetary policy.

We further analyze the ZLB case in Appendix F. There, we even generalize policy by allowing for *optimal discretionary monetary policy*, following the work of Caballero and Simsek (2020), and yet a tremendous amount of equilibrium multiplicity remains, precisely because policy is constrained at the ZLB. The mechanics at play are well-described as a "volatility trap": volatility rises, pushes the economy to the ZLB, and then keeps it trapped there, because of the stabilizing effect of risk premia.

3.5 Prudence and the generality of risk-based multiplicity

In this section, we briefly elaborate on the key mechanism and connect it to prudence. Doing so will also clarify that our volatile equilibria are substantially more general than our particular log utility model would suggest.

For this section, we suppose the representative agent has a general time-additive utility function $u(c)$. The consumption FOC is now $e^{-\rho t}u'(C_t) \propto M_t$. Writing consumption dynamics as $dC_t = C_t[\mu_{C,t}dt + \sigma_{C,t}dZ_t]$, this then implies the Euler equation

$$\mu_{C,t} = \underbrace{\Big(- C_t \frac{u''(C_t)}{u'(C_t)}\Big)^{-1}}_{:=EIS_t} (\iota_t - \pi_t - \rho) + \frac{1}{2} \underbrace{\Big(- C_t \frac{u'''(C_t)}{u''(C_t)}\Big)}_{:=RP_t} \sigma^2_{C,t}.$$

The elasticity of intertemporal substitution EIS_t modulates the consumption growth sensitivity to real rates $\iota_t - \pi_t$, while the relative prudence RP_t modulates the consumption growth sensitivity to risk $\sigma^2_{C,t}$. What is clear is that higher prudence RP_t strengthens our channel, namely the mechanism that risk creates precautionary savings that raises future consumption and "stabilizes" the equilibrium.

In this paper, as in all the New Keynesian literature, we study the output gap x_t whose dynamics coincide with $\log(C_t)$. These dynamics are

$$\mu_{x,t} = EIS_t \times (\iota_t - \pi_t - \rho) + \frac{1}{2}(RP_t - 1)\sigma^2_{x,t} \tag{23}$$

Risk induces stable dynamics in recessions—namely, higher σ^2_x raises μ_x—if and only if $RP_t > 1$. Sufficiently high prudence is the key assumption needed for our equilibria.

But the most popular preferences in macroeconomics satisfy this assumption. For example, with CRRA preferences $u(C) = \frac{C^{1-\gamma}-1}{1-\gamma}$, we have $RP_t = 1 + \gamma$ which is larger than one for any degree of risk aversion $\gamma > 0$. In fact, because these CRRA preferences impose a tight inverse link between $EIS_t = \gamma^{-1}$ and $RP_t = 1 + \gamma$, an easy observation is that higher γ facilitates our self-fulfilling equilibria on two fronts: it increases RP_t, thus allowing risk to bring stability, and it reduces EIS_t, thus reducing the power of monetary policy to affect consumption through real rates.

4 Fiscal Theory

Let us now explore a version of "Fiscal Theory of the Price Level" (FTPL). The idea here is to propose some fiscal policies that can prune equilibria. Our contribution to the literature is analysis of FTPL in a nonlinear stochastic monetary model.

We formulate fiscal policy in a particularly transparent situation: lump-sum taxation with government transfers to the representative household. Denote the lump-sum taxes levied by τ_t^+ and the transfers by τ_t^-, both in real terms. The real primary surplus of the government is then

$$S_t := \tau_t^+ - \tau_t^-.$$

Since the government can pick both taxes and transfers, it can effectively choose S_t, and we no longer make reference to τ^+ or τ^-.

Taxes and transfers do not necessarily offset, so the government borrows by issuing short-term nominally riskless bonds B_t. Later we will generalize to long-term debt. The flow budget constraint of the government is

$$\dot{B}_t = \iota_t B_t - P_t S_t. \tag{24}$$

The nominal interest rate ι_t will be controlled by monetary policy.

Because of the lump-sum nature of the taxes and transfers, there is no impact on the household optimality conditions. Essentially, Ricardian equivalence holds. Indeed, the present-value formula for government debt is

$$\frac{B_t}{P_t} = \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} S_u du\Big], \tag{GD}$$

where M denotes the real stochastic discount factor process (this is because the transversality condition $\lim_{T\to\infty} \mathbb{E}_t[M_T B_T / P_T] = 0$ holds in our representative agent setup). While the representative household holds the government bonds B_t, it also owes the government future taxes and is owed future transfers. Therefore, the lifetime budget constraint of the representative household is

$$\mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t}\frac{W_u L_u}{P_u} du\Big] + \frac{B_t}{P_t} = \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} S_u du\Big] + \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} C_u du\Big].$$

By (GD), the lifetime budget constraint is equivalent to the budget constraint without any debt at all. And so the household consumption FOC is unchanged.

For reference, let us restate the IS curve (IS) and Phillips curve (PC) as the following dynamical system in terms of (x_t, π_t):

$$dx_t = \Big[\iota_t - \pi_t - \rho + \frac{1}{2}\sigma_{x,t}^2\Big]dt + \sigma_{x,t}dZ_t \tag{25}$$

$$d\pi_t = \Big[\rho\pi_t - \kappa\Big(\frac{e^{(1+\varphi)x_t} - 1}{1+\varphi}\Big)\Big]dt + \sigma_{\pi,t}dZ_t. \tag{26}$$

(When we allow surplus shocks or non-logarithmic utility, the IS and Phillips curves change slightly; see Appendix A.4.) Together with some nominal interest rate rule for ι_t and some surplus rule for S_t, equilibrium is fully characterized by the government debt valuation (GD) and the dynamical system (25)-(26). We continue to require the non-explosion Condition 1.

Our previous results did not have government debt or taxes/transfers. However, everything we have said until now still holds with fiscal policies, so long as those policies are "passive" in the language of Leeper (1991). In particular, suppose fiscal policies are chosen so that equation (GD) always holds. Then, government debt valuation plays no role in the analysis, and by the Ricardian equivalence property shown above, the equilibria must be identical to those in Sections 2-3. Next, we explore what happens when fiscal policies are "active," as opposed to passive.

4.1 FTPL as equilibrium selection: the key argument

Consider, as a first example, a fiscal policy with real primary surpluses given by

$$S_t = \bar{s}Y_t, \quad \text{with} \quad \bar{s} > 0. \tag{27}$$

This policy is "active" because its real levels are chosen in a way that does not automatically ensure the government debt valuation equation holds (e.g., S_t is independent of the price level). Such proportional surpluses are also quite natural, in that they arise in the real world the case of proportional taxes and transfers—although we abstract from the distortionary effects of such policies.

With this policy, and using (GD) along with the consumption FOC (8), we have

$$\frac{B_t}{P_t} = \bar{s}\mathbb{E}_t \int_t^\infty e^{-\rho(u-t)}\frac{Y_t}{Y_u}Y_u du = \rho^{-1}\bar{s}e^{x_t}Y^*. \tag{28}$$

Now, apply Itô's formula to (28), using the fact that $dP_t/P_t = \pi_t dt$, to get

$$\Big[\frac{B_t}{P_t}\iota_t - \bar{s}Y_t - \frac{B_t}{P_t}\pi_t\Big]dt = \rho^{-1}\bar{s}e^{x_t}Y^*[\iota_t - \pi_t - \rho + \sigma_{x,t}^2]dt + \rho^{-1}\bar{s}e^{x_t}Y^*\sigma_{x,t}dZ_t.$$

Matching the "dZ" terms on both sides, we find $\sigma_{x,t} = 0$. Then, matching the "dt" terms on both sides, we find an identity: given $\sigma_{x,t} = 0$, and using equation (28), the "dt" terms match for any ι_t, π_t, and x_t. In other words, the FTPL selects $\sigma_{x,t} = 0$, and that is all it does after the initial date $t = 0$. This argument is completely independent of the level of price stickiness κ and amount of self-fulfilling inflation volatility $\sigma_{\pi,t}$ (if any).

It turns out this same logic holds even if the surplus-to-output ratio is not constant but *almost any* exogenous process. In particular, let Ω_t be an exogenous vector Markov diffusion, driven by a multivariate Brownian motion $\mathcal{Z}$ that is independent of the sunspot shock Z. Ω is the state vector describing fiscal policy. Let $s_t = s(\Omega_t)$ for some function $s(\cdot)$, and suppose

$$S_t = s(\Omega_t)Y_t. \tag{29}$$

Of course, allowing surplus shocks through Ω_t does alter the IS curve, which we take into account in the appendix. Even in this more general specification, the following theorem holds. The proof of all results in this section can be found in Appendix A.5, with some important preliminary characterizations provided in Appendix A.4.

Theorem 1. *The economy with fiscal policy following (29) necessarily has $\sigma_{x,t} = 0$. Conversely, if $\sigma_{x,t} = 0$, and if dx_t takes a particular loading on the surplus shocks $d\mathcal{Z}_t$, then the government debt valuation equation (GD) automatically holds at every date, given it holds at $t = 0$.*

Theorem 1 says that FTPL (i) *selects equilibria without real self-fulfilling volatility* and (ii) *does nothing else besides pin down real demand shocks*. This is surprising because we usually think of FTPL as selecting inflation or the price level. We elaborate on (i)-(ii) in turn.

Why FTPL eliminates volatility. The mathematical reasoning for why FTPL selects these equilibria is quite simple in this case: the aggregate real debt balance B_t/P_t evolves "locally deterministically" (meaning it only has drift and no diffusion over small time intervals dt), and so its present value $\mathbb{E}_t[\int_t^\infty \frac{M_u}{M_t}S_u du]$ must also not have any diffusion. This then implies x_t must not have any sunspot volatility. Indeed, in all the surplus specifications considered so far, $S_t/Y_t = s_t = s(\Omega_t)$ is an exogenous process. And so you can break up the real present value of surpluses into two components:

$$\mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} S_u du\Big] = Y_t \times \underbrace{\mathbb{E}_t\Big[\int_t^\infty e^{-\rho(u-t)} s_u du\Big]}_{\text{exogenous for now}}, \tag{30}$$

where we have used the consumption FOC (8) to replace the SDF with $M_u = e^{-\rho u} Y_u^{-1}$. The second term is exogenous and only driven by the surplus shocks $d\mathcal{Z}$. For the overall expression to have no diffusion, the first term Y_t must offset those surplus shocks (implying a particular loading of dx on $d\mathcal{Z}$) and *must not have any sunspot volatility*. Hence, $\sigma_{x,t} = 0$. This result holds for any degree of price stickiness (any κ) and any inflation volatility (any $\sigma_{\pi,t}$). We revisit the determination of σ_π in Section 4.5.

At first glance, the fact that B_t/P_t evolves locally deterministically seems critical but potentially fragile. In principle, the price level itself could feature a diffusive component, i.e., $dP_t/P_t = \pi_t dt + \sigma_{P,t} dZ_t$ for some $\sigma_{P,t}$ to be determined. However, in typical continuous-time models of price stickiness, such a diffusion does not arise. For example, in our world with Rotemberg price stickiness, we have proved that $\sigma_{P,t} = 0$ (see Appendix B). If firms had such fast-moving prices, they would incur too many price adjustment costs, and this is not optimal. Similarly, in a world with Calvo price stickiness, where price-setting opportunities arrive idiosyncratically at some rate χ, a fraction χdt of firms may change their price over a short time interval dt. This also implies $\sigma_{P,t} = 0$. In other words, the fact that P_t evolves locally deterministically is a standard outcome of sticky price models. Beyond being an implication of the modeling, $\sigma_{P,t} = 0$ is also deeply reasonable: nominal rigidities should mean that there is some high-enough frequency at which prices don't adjust; in continuous time, that high frequency is the Brownian one. (Furthermore, even if $\sigma_{P,t} \neq 0$ somehow, it could not be some arbitrary equilibrium object that allowed the government debt valuation equation to hold; it would need to be consistent with firms' pricing strategies.)

Debt prices, surplus rules, and discount rate variation. More broadly, one wishes to generalize the insights above to avoid the idea that our result is "knife-edge." In the baseline case, the unit price of debt is fixed at 1 (given it is short-term debt), surplus-to-output ratios are exogenous, and the equilibrium SDF is exactly reciprocal to surpluses (due to log utility). Because of these assumptions, there is no channel that can potentially absorb self-fulfilling demand shocks. The generalizations we pursue in the next subsections relax each of these assumptions one-by-one.

Just to motivate briefly why these extensions matter, let us preview a general model with long-term debt, a potentially endogenous surplus-to-output ratio, and CRRA utility with risk aversion γ. In that case, we show in Appendix A.4 that the present-value formula for aggregate government debt is

$$\frac{Q_t B_t}{P_t} = Y_t \mathbb{E}_t \Big[\int_t^\infty e^{-\rho(u-t)} s_u \Big(\frac{Y_u}{Y_t}\Big)^{1-\gamma} du \Big], \tag{31}$$

paralleling equation (1) in the introduction. Suppose Y_t has some self-fulfilling volatility, via $\sigma_{x,t} \neq 0$. Equation (31) illustrates three possible channels that can absorb this volatility, and thus permit it to exist. First, the long-term debt price Q_t can adjust to shocks; in the baseline model, $Q_t = 1$, and so this was not possible. Second, future surplus-to-output ratios $(s_u)_{u \geq t}$ can be endogenous, through a rule that responds to output and inflation, which allows the present value of surplus-to-output ratios to adjust to shocks. Third, the term $e^{-\rho(u-t)}\big(\frac{Y_u}{Y_t}\big)^{1-\gamma} = \frac{M_u}{M_t}\frac{Y_u}{Y_t}$ represents the net variation of discount rates (i.e., marginal utility growth M_u/M_t) and economic growth (i.e., output growth Y_u/Y_t); in the baseline model, $\gamma = 1$, and this net variation was zero.

Overall, these three extensions are ways in which terms besides Y_t can have diffusive variation. Nevertheless, we will demonstrate in subsequent sections that the key conclusion of Theorem 1 continues to hold, suggesting the logic of why FTPL selects $\sigma_x = 0$ runs deeper than timing assumptions or mathematical artifacts.

What is the general intuition for why FTPL selects equilibria even in these more complex environments? Although bond prices and the present-value of surpluses can absorb self-fulfilling demand shocks *in principle*, these objects are forward-looking. For instance, in the case of the bond price, it must satisfy an asset-pricing equation that constrains the bond pricing function; the bond price is not free to take the form required to absorb any and all demand shocks. The present-value of surpluses is essentially a long-dated asset and also satisfies an asset-pricing equation, so a similar logic applies to the models with surplus rules and CRRA utility. In all cases, the extensions add a degree of freedom that might absorb demand shocks, but they also add a constraint, namely an asset-pricing equation, that forbids such absorption.

A more technical but potentially revealing intuition follows from the uniqueness of solutions to dynamic asset-pricing equations. Suppose (x_t, π_t) are evolving according to some sunspot equilibrium with some volatilities σ_x and σ_π. In such an environment, under relatively weak conditions, one can solve for the present-value of anything else uniquely; for instance, consider solving for Q_t as the present-value of a bond's cash flows. The solution implies a particular sensitivity σ_Q of Q to the sunspot shock. But, except in knife-edge situations, this sensitivity σ_Q is not the precise level needed to absorb the demand fluctuations via σ_x. And so the only legitimate equilibrium under FTPL has $\sigma_x = 0$.

Formalizing the generalizations: a Markovian class of equilibria. Next, we generalize the key result that $\sigma_x = 0$. We explore (i) long-term debt; (ii) fiscal "rules" rather than exogenous surpluses; and (iii) more general CRRA utility. Because these settings can become substantially more complex, the proofs become unwieldy in the general case. For that reason, we introduce these extensions one-by-one and specialize our analysis in two ways. First, going forward, we will not consider surplus shocks (i.e., no exogenous states Ω with shocks $d\mathcal{Z}$), as they are mostly a distraction that complicates expressions. That said, the reader can find the general equations including surplus shocks in Appendix A.4 (and see Lemma A.3 for a generic result including surplus shocks and all the extensions simultaneously). Second, we restrict attention to the following class of Markovian equilibria and prove our claims within this class.

Definition 3. An *x-Markov equilibrium* is a non-explosive equilibrium such that inflation π_t and the volatility $\sigma_{x,t}$ are functions of x_t.

For our purposes, the equilibria covered by Definition 3 constitute a sufficiently general class. Indeed, all the sunspot equilibria constructed in this paper fall under the x-Markov type. This is clear for the rigid-price limit examples of Section 3, since $\pi_t = 0$ and $\sigma_{x,t} = \sigma_x(x_t)$ in those cases. The construction with non-trivial inflation in Appendix E is also of the x-Markov type: there, we construct a class of equilibria with $\pi_t = \pi(x_t)$. Thus, if FTPL can induce $\sigma_x = 0$ within the class of x-Markov equilibria, then it will have ruled out all the sunspot equilibria constructed in this paper. In this sense, we think the x-Markov class is rich enough to be useful in contrasting the equilibrium selection properties of Taylor rules versus FTPL.

4.2 FTPL with long-term debt

One important generalization replaces short-term debt with long-term debt. This is naturally of interest because short-term debt prices can never respond to shocks. This may lead one to think that short-term debt mechanically, in a knife-edge sense, rules out self-fulfilling demand volatility.

To fix ideas and keep things tractable, let us assume that debt is coupon-free and has a constant exponential maturity structure. Per unit of time dt, a constant fraction βdt of outstanding debts mature, and their principal must be repaid. Denote the per-unit price of this debt by Q_t. The government's flow budget constraint is now

$$Q_t \dot{B}_t = \beta B_t - \beta B_t Q_t - P_t S_t. \tag{32}$$

This says that new net debt sales $\dot{B}_t + \beta B_t$, which garner price Q_t, plus primary surpluses $P_t S_t$ must be sufficient to pay back maturing debts βB_t. By standard no-arbitrage asset-pricing, the per-unit bond price is given by

$$Q_t = \mathbb{E}_t\Big[\int_t^\infty \frac{M_T}{M_t}\frac{P_t}{P_T}\beta e^{-\beta(T-t)}dT\Big]. \tag{33}$$

In the above, debt is nominal, so it is priced using the nominal SDF M/P (intuitively, dividing by P converts a nominal cash flow into a real cash flow). The total real value of debt is $Q_t B_t / P_t$, and so the government debt valuation equation is now

$$\frac{Q_t B_t}{P_t} = \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} S_u du\Big]. \tag{34}$$

In an equilibrium with long-term debt, all three equations (32), (33), and (34) must hold.

To develop the intuition, we first consider the special example where the interest rate is pegged $\iota_t = \bar{\iota}$. Recall the result that, with $s_t = \bar{s}$, the right-hand-side of (34) equals $\rho^{-1}\bar{s}e^{x_t}Y^*$. Equate this expression to $Q_t B_t / P_t$ and apply Itô's formula to both sides, recalling equation (32) for $\dot{B}_t$ and that $\dot{P}_t / P_t = \pi_t$. By matching the "dZ" terms, we obtain

$$\sigma_{Q,t} = \sigma_{x,t}, \tag{35}$$

where σ_Q denotes the sunspot loading of $\log(Q_t)$ on dZ_t. In other words, the self-fulfilling demand shocks must be absorbed by long-term debt prices. The key question is whether the pricing of long-term debt in (33) is consistent with this absorption.

Now, to price each bond, note that the nominal SDF in this setting is

$$\frac{M_t}{P_t} = \exp\Big[-\int_0^t \iota_u du - \frac{1}{2}\int_0^t \sigma_{x,u}^2 du - \int_0^t \sigma_{x,u} dZ_u\Big].$$

Using the notation $\tilde{\mathbb{E}}$ for the risk-neutral expectation (which absorbs the martingale $\frac{1}{2}\int_0^t \sigma_{x,u}^2 du - \int_0^t \sigma_{x,u} dZ_u$), the debt price from (33) is then

$$Q_t = \tilde{\mathbb{E}}_t\Big[\int_t^\infty \beta e^{-\int_t^T (\iota_u + \beta)du} dT\Big].$$

Finally, use the assumption of a pegged interest rate $\iota_t = \bar{\iota}$, which implies $Q_t = \frac{\beta}{\bar{\iota}+\beta}$. Debt prices are constant, so $\sigma_Q = 0$, and therefore equation (35) implies $\sigma_x = 0$. In fact, the risk-neutral bond pricing formula just above reveals that the *only way* self-fulfilling demand can enter Q_t is via the interest rate rule. But this suggests that the result is much more general than the peg example: monetary policy would need to follow a very particular rule in order to create fluctuations in the bond price that are consistent with self-fulfilling demand, which generically would not happen.

With unpegged interest rates, the debt price is no longer constant and can have volatility. However, the volatility implied by the bond pricing equation (33) is inconsistent with the bond price volatility required to support self-fulfilling demand in (34), unless all these volatilities are zero. To summarize the reasoning, the introduction of long-term debt allows for one extra degree of freedom, namely σ_Q, to absorb self-fulfilling demand shocks, but it also introduces an extra constraint, namely the no-arbitrage pricing equation for a single unit of debt. If σ_Q were some arbitrary process absorbing demand shocks, that would violate the pricing equation for debt.

Theorem 2. *Consider the economy with long-term debt and $s_t = \bar{s}$. Suppose equilibrium is x-Markov. Then, the economy generically has $\sigma_{x,t} = 0$.*

4.3 FTPL with fiscal rules

Our next generalization allows surpluses to respond to endogenous variables, similarly to the interest rate rule. Suppose again that $S_t = s_t Y_t$, where

$$s_t = s(x_t, \pi_t), \tag{36}$$

for some bounded function s that satisfies $s(0,0) = \bar{s} > 0$. In this environment, we will also specialize to the linear Taylor rule (linear MP) to keep the analysis tractable.

Repeating the debt valuation computation from (GD), we obtain

$$\frac{B_t}{P_t} = Y_t \Psi_t, \tag{37}$$

$$\text{where} \quad \Psi_t := \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(T-t)} s_T dT\Big] \tag{38}$$

In the class of x-Markov equilibria of Definition 3, we have the major simplification that $\Psi_t = \Psi(x_t)$ for some function Ψ that only depends on x_t. In that case, even without computing the function Ψ, by applying Itô's formula to (37) and examining the loading on the sunspot shock dZ, we can say that

$$0 = \sigma_{x,t}\big[\Psi(x_t) + \Psi'(x_t)\big] \tag{39}$$

One possibility is $\sigma_x = 0$, which is the natural case we hope to prove. On the other hand, if $\sigma_x \neq 0$, then the present-value of future surpluses needs to inherit any output gap volatility, implying a particular functional form for Ψ. What we show is that this functional form is generically inconsistent with equation (38), which provides a different equation for Ψ, unless inflation $\pi(x)$ and volatility $\sigma_x(x)$ take a particular form. Then, we show that this particular sunspot form, under some conditions on the policy rules, implies unstable dynamics, meaning that $\sigma_x = 0$ must hold.

Theorem 3. *Consider the economy with fiscal rule (36) and monetary rule (linear MP), with $\phi_x > 0$, $\phi_\pi < 1$, and $\frac{\phi_x}{1-\phi_\pi} > -\frac{s+\partial_x s}{\partial_\pi s}$. Suppose equilibrium is x-Markov. Then, the economy generically has $\sigma_{x,t} = 0$.*

4.4 FTPL with general CRRA utility

Finally, we replace log utility with general CRRA $u(c,l) = \frac{c^{1-\gamma}}{1-\gamma} - \frac{l^{1+\varphi}}{1+\varphi}$. This extension is of interest because log utility exhibits the knife-edge property that the present-value of future surplus growth can have no net fluctuations from "discount rates" in excess of "cash flows", since the log utility SDF is related to the inverse of output.

In the CRRA world, two changes arise from the new consumption FOC $M_t = e^{-\rho t} Y_t^{-\gamma}$. First, the IS curve now takes the slightly different form (A.11), and it depends on γ. Second, the present value of surpluses is now different: with a constant surplus-to-output ratio $s_t = \bar{s}$, we have

$$\mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t} S_u du\Big] = \bar{s} Y_t \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(u-t)}\Big(\frac{Y_u}{Y_t}\Big)^{1-\gamma} du\Big]$$

The important point relative to log utility is that the present-value of surpluses can now admit an additional type of fluctuation, because discount rates $\frac{M_u}{M_t} = e^{-\rho(u-t)}(\frac{Y_u}{Y_t})^{-\gamma}$ do not exactly offset surplus growth $\frac{S_u}{S_t} = \frac{Y_u}{Y_t}$. This potentially permits short-run volatility σ_x because it can be absorbed, leaving the present-value of surpluses unaffected. That said, we prove that our key result carries over to CRRA preferences in some cases. The key intuition is that the absorption of short-run volatility by future discount rates requires a very particular specification for the present-value of surpluses, which will generically not arise.

Theorem 4. *Consider the economy with CRRA utility and risk aversion γ. Suppose $s_t = \bar{s} > 0$ and monetary policy follows a linear Taylor rule (linear MP) with $\frac{\phi_x}{1-\phi_\pi} > 0$. Suppose equilibrium is x-Markov. Then, the economy generically has $\sigma_{x,t} = 0$.*

Remark 3 (Conditions on policy rules)**.** *One may notice that Theorems 3-4 included additional conditions on the ratio $\frac{\phi_x}{1-\phi_\pi}$ beyond what was required for the other cases. It is important to realize that these conditions are only needed in ruling out a single particular sunspot equilibrium. Specifically, the proofs show that FTPL generically rules out all sunspot equilibria except a particular one where $\sigma_x(x)$ and $\pi(x)$ are uniquely-determined functions. The conditions on the policy rules are needed to rule out this final sunspot equilibrium.*

4.5 Inflation determination

Theorems 1-4 only provide a "local" result, i.e., that $\sigma_x = 0$, without characterizing the full dynamic equilibrium. They also show that inflation is not determined from the debt valuation equation alone. Monetary policy is needed to pin inflation down.

For tractability, we now specialize to the following quasi-linear setting. This is no longer a major issue, given we have already determined that $\sigma_x = 0$—in other words, the key nonlinearity of this paper, namely via the IS curve, is no longer present. For the results of this section, we consider the linearized Phillips curve (linear PC), replacing $(1+\varphi)^{-1}[e^{(1+\varphi)x} - 1] \approx x$ in equation (26). We then specialize the surplus dynamics in (29) to $ds_t = \lambda_s(\bar{s} - s_t) + \varsigma_{s,t} \cdot d\mathcal{Z}_t$, which is a continuous-time version of an AR(1) but with an arbitrary volatility process $\varsigma_{s,t}$. We also assume debt is short-term as in the baseline specification. Finally, let us assume the linear Taylor rule (linear MP) with target rate $\bar{\imath}_t = \rho - \frac{1}{2}|\varsigma_{x,t}|^2$, where ς_x is the endogenous sensitivity of x to $d\mathcal{Z}$. Because Theorem 1 pins down $\varsigma_{x,t}$, its inclusion in the target rate is conceptually distinct from the "risk premium targeting" we studied in Section 3.3. The present target rate only serves as a normalization, so that the economy fluctuates around $(x, \pi) = (0, 0)$.

Writing the equilibrium dynamics in vector form, with $F_t := (x_t, \pi_t)'$, we have

$$dF_t = \mathcal{A}F_t dt + \mathcal{B}_t dZ_t + \mathcal{C}_t d\mathcal{Z}_t,$$

$$\text{where} \quad \mathcal{A} := \begin{bmatrix} \phi_x & \phi_\pi - 1 \\ -\kappa & \rho \end{bmatrix}, \quad \mathcal{B}_t := \begin{bmatrix} 0 \\ \sigma_{\pi,t} \end{bmatrix}, \quad \text{and} \quad \mathcal{C}_t := \begin{bmatrix} \varsigma'_{x,t} \\ \varsigma'_{\pi,t} \end{bmatrix}$$

Notice that the first entry of $\mathcal{B}_t$ is zero, because of Theorem 1. Theorem 1 also places some restrictions on the first entry of surplus shock loadings $\mathcal{C}_t$. We follow a relatively standard analysis by doing a spectral decomposition of the transition matrix

$\mathcal{A} = V\Lambda V^{-1}$, and analyzing the rotated system $\tilde{F}_t := V^{-1}F_t$. By integrating the system $d\tilde{F}_t = \Lambda\tilde{F}_t dt + V^{-1}\mathcal{B}_t dZ_t + V^{-1}\mathcal{C}_t \cdot d\mathcal{Z}_t$, we obtain

$$\mathbb{E}_0\tilde{F}_t = \exp(\Lambda t)\tilde{F}_0. \tag{40}$$

The rest is a familiar stability analysis of (40), in view of the non-explosion Condition 1. There are three cases: both eigenvalues have positive real parts, the eigenvalues have opposite signs, or both eigenvalues have negative real parts. Pursuing this analysis, we then obtain the following generalization of some familiar results, with the proof in Appendix A.6.

Proposition 5. *Consider the linearized Phillips curve (linear PC), the linear Taylor rule (linear MP) with target rate $\bar{\imath} = \rho + \frac{1}{2}|\varsigma_{x,t}|^2$, and the surplus dynamics $ds_t = \lambda_s(\bar{s} - s_t) + \varsigma_{s,t} \cdot d\mathcal{Z}_t$. A non-explosive equilibrium takes one of three forms, ignoring knife-edge cases for the parameters:*

1. *If $\rho + \phi_x > 0$ and $\rho\phi_x + \kappa(\phi_\pi - 1) > 0$, then equilibrium generically fails to exist.*
2. *If $\rho\phi_x + \kappa(\phi_\pi - 1) < 0$, the unique equilibrium features $\pi_t = \frac{1}{2}\frac{\rho-\phi_x-\sqrt{(\rho-\phi_x)^2-4\kappa(\phi_\pi-1)}}{\phi_\pi-1}x_t$ with $\sigma_{x,t} = \sigma_{\pi,t} = 0$.*
3. *If $\rho + \phi_x < 0$ and $\rho\phi_x + \kappa(\phi_\pi - 1) > 0$, then the equilibrium is not unique.*

In all cases, the output gap is pinned down by fiscal states $(\frac{B_t}{P_t}, s_t)$ as $x_t = \log(\frac{B_t/P_t}{\Psi(s_t)Y^*})$ for some function $\Psi(s)$.

Proposition 5 is reminiscent of the large literature of FTPL in linearized New Keynesian models. Equilibrium cannot exist with both "active fiscal" and "active money" regimes (case 1). Equilibrium exists and is unique when "passive money" is paired with "active fiscal" (case 2). This equilibrium has $\pi_t = \pi(x_t)$, so that it automatically falls into the x-Markov category proposed in Definition 3. These results echo Leeper (1991). A finding which differs slightly from the literature is our case 3: monetary policy that acts super aggressively against inflation but acts counterintuitively to output induces non-uniqueness, despite active fiscal policy. This case can have $\sigma_\pi \neq 0$ because monetary policy induces globally stable dynamics through its rule. This third case shows clearly that monetary policy remains, in fact, critical to inflation determination, even when FTPL is operative.

Of these three cases, the interesting case is the active-fiscal passive-money regime (case 2), which delivers a unique equilibrium. There are two important takeaways.

First, FTPL eliminates self-fulfilling fluctuations for a broad range of monetary policy rules and not some knife-edge rule. As long as $\rho\phi_x + \kappa(\phi_\pi - 1) < 0$ (monetary policy is not too aggressive), FTPL guarantees $\sigma_x = \sigma_\pi = 0$. For instance, an interest rate peg fits into this case, and delivers a unique equilibrium converging to steady state (absent surplus shocks) at rate $\frac{1}{2}[\sqrt{\rho^2 + 4\kappa} - \rho]$. The absence of self-fulfilling fluctuations for a broad range of monetary rules contrasts sharply with Section 3, where without active fiscal policy, sunspot volatility could exist *for any conventional monetary policy rule*.

The second takeaway is that the FTPL equilibrium is observationally distinct from the self-fulfilling stochastic equilibria under the Taylor principle. Under FTPL, the output gap x_t is pinned down uniquely as a function of the real debt balance B_t/P_t and the primary surplus level s_t. All output gap volatility is tied to these fiscal states. By contrast, the volatile equilibria possible under the Taylor principle feature output gap dynamics which are decoupled from fiscal states.

4.6 Summary and discussion: the roles of fiscal and monetary policies

Theorem 1 and Proposition 5 provide a clean breakdown of what fiscal and monetary policies do. Fiscal policy provides what we refer to as "aggregate demand management": it rules out real sunspot volatility (i.e., $\sigma_x = 0$) and induces surplus shocks to become demand shocks (i.e., ς_x is pinned down by ς_s), for any monetary policy rule. The monetary policy rule then connects inflation to output. For example, in the standard passive-money regime (case 2 of Proposition 5), monetary policy forces π_t to be a function of x_t, so that sunspot inflation volatility is zero (i.e., $\sigma_\pi = 0$) and inflation shocks are fiscally determined (i.e., ς_π is pinned down by ς_x, hence by ς_s).

Let us elaborate on our view of FTPL as "aggregate demand management." One way to develop an intuition is to consider the rigid price limit $\kappa \to 0$, where government debt becomes equivalent to real debt. Nothing about the analysis above hinges on the value of κ, and so FTPL still selects $\sigma_x = 0$. The key reason for this selection is that the government debt valuation equation (GD) becomes a "no-default" condition in a rigid-price world. Rather than determine the price level, or future inflation, equation (GD) says that surpluses must eventually be positive enough to justify the current debt value. But if the government's taxation and spending regime is exogenous, and lacks the flexibility that inflation provides, the only way a government can fulfill its no-default commitment is if demand takes a particular path. The government debt valuation equation (GD) thus constrains demand if $\kappa \to 0$.

Our argument is that, surprisingly, a version of this logic extends to any $\kappa > 0$, because FTPL is not really a theory of the price level *per se*, but a theory of aggregate demand management. In fact, demand management corresponds to the typical stories told about FTPL. Cochrane (2023), Chapter 2.3, writes

> What force pushes the price level to its equilibrium value? ...If the price level is too low, money may be left overnight. Consumers try to spend this money, raising aggregate demand. Alternatively, a too-low price level may come because the government soaks up too much money from bond sales. Consumers either consume too little today relative to the future or too little overall, violating intertemporal optimization or the transversality condition. Fixing these, consumers again raise aggregate demand, raising the price level.

The key margin of adjustment in these stories is aggregate demand. In a frictionless model, the equilibrium price reflecting this adjustment is the price level. But in sticky price models, the price level cannot jump, so equilibrium partly adjusts via output.

Here, there is a sense in which *entire adjustment to fiscal policy* comes via output. In particular, Theorem 1 says that FTPL eliminates self-fulfilling demand volatility, pins down the response of x to surplus shocks, and provides an "initial condition" that ultimately pins down x_0. Nothing about this story relates to inflation. Instead, as Proposition 5 shows, inflation is determined by monetary policy. There is a clean breakdown of what fiscal and monetary policies do.

5 Conclusion

We show that New Keynesian models inherently permit a novel type of sunspot volatility that appears only in the nonlinear version of the model. The distinguishing features of our volatility are that it is self-fulfilled by the presence of risk premia and is countercyclical, arising only in recessionary times. Conventional monetary policy that only targets output and inflation has almost no power to trim these volatile equilibria. An enriched monetary rule that directly targets risk premia, lowering rates aggressively when risk premia rise, can ensure determinacy. As an alternative to monetary policies, active fiscal policies ensure determinacy across a wide variety of settings. Our fiscal theory examples permit: any level of price stickiness, long-term debt, arbitrary exogenous surpluses, and some types of endogenous surpluses (surplus rules). In contrast to all interest rate rules, fiscal policies can ensure determinacy even when monetary policy is constrained (e.g., by an effective lower bound).

What are the implications of our results for current practices in monetary economics? Importantly, the standard New Keynesian paradigm of using the Taylor principle to select a unique equilibrium is not necessarily valid when considering risk and risk premia.

By contrast, the FTPL approach has merit, implying monetary and fiscal policies are not *alternatives* to each other in selecting a unique equilibrium. Rather, active fiscal policy provides determinacy in strictly more situations than active money. But here, FTPL operates differently than conventionally thought. In our nonlinear, stochastic solution, FTPL works by eliminating the self-fulfilling volatility in real variables (i.e., $\sigma_x = 0$), while the monetary policy rule is needed to determine inflation. Thus, our analysis provides a clean distinction between the role of fiscal and monetary policies, a distinction that is not evident in existing research that restricts attention to linear equilibria.

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Online Appendix:
Fear, Indeterminacy, and Policy Responses

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A Proofs

A.1 Non-explosion and transversality conditions

For completeness, we briefly document the non-explosion requirements imposed by consumer and firm optimality. We then show that our non-explosion Condition 1 suffices to ensure these requirements hold. Thus, besides the standard derivations in the text, this completes the proof of Lemma 1.

For the consumer side, note that the representative agent's utility can be written

$$U_0 = \rho^{-1}\Big(\log Y^* - \frac{(Y^*)^{1+\varphi}}{1+\varphi}\Big) + \int_0^\infty e^{-\rho t}\mathbb{E}\Big[x_t - \frac{e^{(1+\varphi)x_t}}{1+\varphi}\Big]dt$$

We need to ensure the consumer obtains finite utility and that his transversality condition holds. To ensure $U_0 > -\infty$, we require

$$\lim_{T\to\infty} \mathbb{E}_t[e^{-\rho T}x_T] = 0 \tag{A.1}$$

$$\lim_{T\to\infty} \mathbb{E}_t[e^{(1+\varphi)x_T-\rho T}] = 0 \tag{A.2}$$

Requirement (A.1) rules out $\mathbb{E}x_T$ diverging to $-\infty$ faster than rate ρ. Requirement (A.2) rules out $\mathbb{E}e^{(1+\varphi)x_T}$ diverging to $+\infty$ faster than rate ρ. It is clear that if Condition 1 holds, then both (A.1)-(A.2) are satisfied.

The consumer's transversality condition holds if and only if the lifetime budget constraint (5) holds with equality. Now, note that since price adjustment costs are non-pecuniary, the real present value of aggregate profits are $\Pi_t = \mathbb{E}_t[\int_t^\infty \frac{M_s}{M_t}(Y_s - \frac{W_s L_s}{P_s})ds]$. Using the resource constraint $C_t = Y_t$ and $B_0 = 0$, we therefore have that the consumer lifetime budget constraint (5) holds with equality, so long as all these integrals converge. Convergence of the integrals can be evaluated using the FOCs. The consumption FOC (8) implies $\mathbb{E}_0[\int_0^\infty M_t C_t dt] = (\rho\lambda)^{-1}$, so this integral converges. The labor FOC (7) and market clearing $C_t = L_t$ imply $\mathbb{E}_0[\int_0^\infty M_t \frac{W_t L_t}{P_t} dt] = \lambda^{-1}\mathbb{E}_0[\int_0^\infty e^{-\rho t}C_t^{1+\varphi}dt]$, so this integral converges so long as $\mathbb{E}[C_t^{1+\varphi}]$ grows slower than $e^{\rho t}$, which is exactly identical to requirement (A.2) that has already been verified.

For the firm side, note that Appendix B derives the optimality conditions from the firm's price setting problem. There, we show that the firm's transversality conditions are, in a symmetric equilibrium in which firms charge identical prices,

$$\lim_{T\to\infty} \mathbb{E}_t[e^{(1+\varphi)x_T-\rho T}] = 0 \tag{A.3}$$

$$\lim_{T\to\infty} \mathbb{E}_t[e^{-\rho T}\pi_T^2] = 0 \tag{A.4}$$

Notice that requirement (A.3) is identical to (A.2), which we have already verified. Requirement (A.4) avoids nominal explosions that imply an infinite present value of adjustment costs. Note that under Condition 1, this automatically holds.

A.2 Sufficient condition for non-explosive equilibria

Here, we provide a single abstract non-explosiveness lemma that is general enough to cover all the candidate stochastic equilibria in this paper, at least after an appropriate change-of-variables. Let $V_t \in [0, \bar{v}]$ be a one-dimensional diffusion process that follows

$$dV_t = \mu_v(V_t)dt + \sigma_v(V_t)dZ_t, \tag{A.5}$$

where Z is a one-dimensional Brownian motion. We assume throughout that the dynamics μ_v and σ_v are such that $\bar{v}$ is not an absorbing boundary. Hence, the determination of the stationarity of V_t and characterization of its stationary distribution depend only on the boundary behavior of V_t near 0. In fact, the key object is

$$\theta := \lim_{v\to 0} \frac{2v\mu_v(v)}{\sigma_v^2(v)} \tag{A.6}$$

We have the following general result.

Lemma A.1. *Assume that $\sigma_v^2 : [0, \bar{v}] \mapsto \mathbb{R}^+$ is bounded and strictly positive away from the boundaries. Assume in addition that $\lim_{v\to 0}(\frac{\sigma_v(v)}{v})^2 > 0$ (i.e., σ_v^2 vanishes at most quadratically). Assume finally that θ from (A.6) is finite and larger than 1. Then, the following hold.*

(i) The boundary $v = 0$ is inaccessible, i.e., $\mathbb{P}\{V_t > 0, \forall t\} = 1$.

(ii) The process V_t possesses a stationary distribution. For v near 0, this distribution has a density $p_v(v)$ that satisfies $\lim_{v\to 0} v^{-\theta}\sigma_v^2(v)p_v(v) = 1$.

(iii) Defining the variable $X_t := \frac{1}{\beta}\log(V_t)$ for $\beta > 0$, we have $\liminf_{t\to\infty} \mathbb{E}[X_t] > -\infty$.

Proof of Lemma A.1. To streamline notation, we will use $f(v) \sim g(v)$ here to always mean $\lim_{v\to 0} \frac{f(v)}{g(v)} = 1$.

Proof of statement (i). We may apply Feller's boundary classification to decide whether $v = 0$ is inaccessible or accessible. To do so, let ϵ and v_0 be arbitrary numbers within interval $(0, \bar{v})$. Define $s(y) := \exp(-\int_{v_0}^{y} \frac{2\mu_v(u)}{\sigma_v^2(u)} du)$ and $m(x) := \frac{2}{s(x)\sigma_v^2(x)}$. Boundary $v = 0$ is inaccessible if and only if (see Karatzas and Shreve, 1991, Section 5.5C)

$$I := \int_0^\epsilon m(x) \Big(\int_0^x s(y) dy \Big) dx = +\infty.$$

For v sufficiently small, we have that $\frac{2\mu_v(v)}{\sigma_v^2(v)} \sim \frac{\theta}{v}$ and so $s(v) \sim v^{-\theta}$. Therefore, for x sufficiently small,

$$S(x, \theta) := \int_0^x \frac{s(y)}{s(x)} dy = \int_0^x \Big(\frac{x}{y}\Big)^\theta dy = x^\theta \lim_{z \downarrow 0} \frac{x^{1-\theta} - z^{1-\theta}}{1 - \theta} = +\infty,$$

where the last equality to $+\infty$ comes from the assumption that $\theta > 1$. Since σ_v^2 is bounded, this proves that

$$I = \int_0^\epsilon \frac{2}{\sigma_v^2(x)} S(x, \theta) dx = +\infty,$$

implying that the boundary $v = 0$ is inaccessible. (Note that this statement does not require any condition on the rate at which σ_v^2 vanishes at zero.)

Proof of statement (ii). To prove the existence of and simultaneously to characterize the stationary distribution, let us guess-and-verify that the density p_v satisfies $P_v := \sigma_v^2 p_v \sim v^\zeta$ for some $\zeta > 1$. In that case, L'Hôpital's rule allows us to use $\frac{d}{dv} P_v \sim \zeta v^{\zeta - 1}$ and $\frac{d^2}{dv^2} P_v \sim \zeta(\zeta - 1) v^{\zeta - 2}$. The Kolmogorov Forward Equation for the density is

$$0 = -\frac{d}{dv}[\mu_v p_v] + \frac{1}{2}\frac{d^2}{dv^2}[\sigma_v^2 p_v].$$

Using the change-of-variables $P_v := \sigma_v^2 p_v$, we have

$$0 = -\frac{d}{dv}[\frac{2v\mu_v}{\sigma_v^2} v^{-1} P_v] + \frac{d^2}{dv^2}[P_v].$$

Using the asymptotic form $P_v \sim v^\zeta$ and the assumption that $\frac{2v\mu_v}{\sigma_v^2} \sim \theta$ is finite and larger than one, we have for v small enough

$$0 = -\theta(\zeta - 1) v^{\zeta - 2} + \zeta(\zeta - 1) v^{\zeta - 2}.$$

This verifies the conjectured form of p_v if $\zeta = \theta > 1$. (The equation above also holds if $\zeta = 1$, but that would contradict the presumption that $\zeta > 1$.) Furthermore, since σ_v^2 vanishes at most quadratically, the resulting density $p_v \sim v^\theta \sigma_v^{-2}$ satisfies $p_v(v) = O(v^{\theta-2})$, where $f(v) = O(g(v))$ means that there exists a $C > 0$ such that $f(v) \leq Cg(v)$ for all v small enough. Since $\theta > 1$ by assumption, this proves that $p_v(v)$ is integrable, hence a valid density.

Proof of statement (iii). Consider $X_t := \frac{1}{\beta}\log(V_t)$ for $\beta > 1$. The stationary density of X_t, by the change-of-variables formula, is

$$p_x(x) = \beta e^{\beta x} p_v(e^{\beta x})$$

Using part (ii), that $p_v(v) \sim v^\theta \sigma_v(v)^{-2}$, and using the fact that the diffusion coefficient of X satisfies $\sigma_x(x) = \frac{1}{\beta}\frac{\sigma_v(v)}{v}$, we have

$$p_x(x) \sim \beta e^{\beta x} e^{\beta\theta x} (\beta e^{\beta x} \sigma_x(x))^{-2} = e^{\beta(\theta-1)x} \frac{1}{\beta \sigma_x^2(x)}$$

By assumption, we have $\lim_{x\to-\infty} \sigma_x^2(x) > 0$ and $\theta > 1$, which implies that for $\bar{x}$ large enough, there exists a finite constant $K > 0$ such that

$$\int_{-\infty}^{-\bar{x}} x p_x(x) dx \geq K \int_{-\infty}^{-\bar{x}} x e^{\beta(\theta-1)x} dx = -\frac{e^{-\beta(\theta-1)\bar{x}}}{\beta(\theta-1)} \left(\bar{x} + \frac{1}{\beta(\theta-1)}\right) > -\infty$$

Consequently, we have $\mathbb{E}[\mathbf{1}_{\{X_t \leq -\bar{x}\}} X_t] > -\infty$, which implies

$$\mathbb{E}[X_t] = \mathbb{E}[\mathbf{1}_{\{X_t \leq -\bar{x}\}} X_t] + \mathbb{E}[\mathbf{1}_{\{X_t > -\bar{x}\}} X_t] \geq \mathbb{E}[\mathbf{1}_{\{X_t \leq -\bar{x}\}} X_t] - \bar{x} > -\infty.$$

Since this calculation uses the stationary distribution, it holds identically for all t, and so $\liminf_{t\to\infty} \mathbb{E}[X_t] > -\infty$. □

A.3 Proofs for Sections 2-3

Proof of Proposition 1. See the proof of Proposition 5 in Appendix A.6 for the spectral decomposition of the matrix $\mathcal{A}$. Asymptotic instability of this system is guaranteed if $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) > 0$. This holds if and only if $\det(\mathcal{A}) > 0$ and $\mathrm{tr}(\mathcal{A}) > 0$, which is equivalent to $\phi_x > -\rho$ and $\phi_\pi > 1 - \rho\phi_x/\kappa$. □

Proof of Proposition 2. Since inflation is rigid, consider any Taylor rule with target rate $\bar{\imath} = \rho$ and response function $\Phi(x)$ that is increasing in x and satisfies $\lim_{x\to-\infty} e^{\hat{\beta}x}\Phi(x) >$

$-\infty$ for some $\hat{\beta} > 0$, as required by Assumption 1. Define $\beta := \max(1, \hat{\beta})$. Obviously, we also have $\lim_{x \to -\infty} e^{\beta x} \Phi(x) > -\infty$.

Proof of statement (i). A similar argument applies as in the text but for the process $V_t := y_t^{\beta} = e^{\beta x_t}$ rather than y_t. Let $\omega > 0$ and $x_{\max} < 0$ be arbitrary. Specify volatility by

$$\sigma_x^2(x) = \begin{cases} \frac{2}{1+\beta}\left[e^{-2\beta x}\omega^2 - \Phi(x)\right], & \text{if } x < x_{\max}; \\ 0, & \text{if } x \geq x_{\max}. \end{cases} \tag{A.7}$$

Since $\Phi(\cdot)$ is increasing and continuous, and $\Phi(0) = 0$, we have that $e^{-2\beta x}\omega^2 > \Phi(x)$ for all $x < x_{\max}$, as needed to ensure $\sigma_x^2 > 0$ in that region. From here, we will verify that the assumptions of Lemma A.1 hold for $V_t = e^{\beta x_t}$.

By Itô's formula, the dynamics of $V_t := y_t^{\beta}$ are

$$dV_t = \beta V_t \Big(\Phi(\beta^{-1} \log(V_t)) + \frac{\beta + 1}{2} \sigma_{x,t}^2 \Big) dt + \beta V_t \sigma_{x,t} dZ_t$$

Plugging in (A.7), we obtain

$$\begin{aligned} dV_t &= \mu_v(V_t)dt + \sigma_v(V_t)dZ_t, \\ \text{where} \quad \mu_v(v) &:= \beta\Big[v\Phi(\beta^{-1}\log(v))\mathbf{1}_{\{v \geq v_{\max}\}} + \frac{\omega^2}{v}\mathbf{1}_{\{v < v_{\max}\}}\Big] \\ \sigma_v(v) &:= \beta v \sigma_x(\beta^{-1}\log(v)), \end{aligned}$$

and where $v_{\max} := \exp[\beta x_{\max}] < 1$.

First, considering the non-volatile region, namely $v \in [v_{\max}, 1)$, we have $dV_t = \mu_v(V_t)dt < 0$, by the fact that $\Phi(0) = 0$ and $\Phi(x)$ is increasing. Therefore, V_t enters the region $(-\infty, v_{\max})$ in finite time when starting from any point $v_0 \leq 1$. Consequently, we may consider $v_{\max}$ to be a non-absorbing upper boundary of the state space.

Next, note that $\sigma_v(v)$ is strictly positive and finite on $v \in (0, v_{\max})$, and furthermore

$$\lim_{v \to 0} \sigma_v^2(v) = \frac{2\beta^2}{1+\beta}\Big(\omega^2 - \lim_{v \to 0} v^2 \Phi(\beta^{-1}\log(v))\Big) = \frac{2\beta^2\omega^2}{1+\beta},$$

the latter equality because $v\Phi(\beta^{-1}\log(v))$ stays finite as $v \to 0$ (Assumption 1), implying that $v^2\Phi(\beta^{-1}\log(v)) \to 0$. In addition, note that $\theta := \lim_{v \to 0} \frac{2v\mu_v(v)}{\sigma_v^2(v)} = (1+\beta)\frac{2\beta\omega^2}{2\beta^2\omega^2} = \frac{1+\beta}{\beta} > 1$. Thus, all the assumptions of Lemma A.1 hold for $V_t := y_t^{\beta}$, and so (i) the lower boundary $v = 0$ is inaccessible for V_t; (ii) V_t possesses a non-degenerate stationary distribution; and (iii) $x_t = \beta^{-1}\log(V_t)$ satisfies $\liminf_{T \to \infty} \mathbb{E}[X_T] > -\infty$. Thus, we have proved that the non-explosion Condition 1 holds, implying the candidate dynamics correspond to an equilibrium.

Finally, we prove the claim that any volatility function is valid if it satisfies suitable boundary conditions. Instead of the σ_x function in (A.7), consider any alternative function $\tilde{\sigma}_x$, which (a) coincides with σ_x for $x \notin (-C, -C^{-1})$ for C arbitrarily large; and (b) is bounded on $x \in (-C, -C^{-1})$. By inspection, the entire proof above remains valid. This proves statement (i) of the proposition.

Proof of statement (ii). To prove statement (ii), namely that all volatile equilibria are recessionary, consider a general rule $\Phi(x)$ satisfying $\phi_x := \inf_x \Phi'(x) > 0$. To prove that $x_t \leq 0$ in any non-explosive equilibrium, suppose $x_0 > 0$, leading to contradiction. Suppose $\sigma_{x,t}$ is any arbitrary non-zero equilibrium volatility process. Then, the output gap follows

$$dx_t = \Big(\Phi(x_t) + \frac{1}{2}\sigma_{x,t}^2\Big)dt + \sigma_{x,t}dZ_t, \quad x_0 > 0.$$

Without loss of generality, we may assume that $\sigma_{x,t} > 0$ on the set of time-points $\{t : x_t > 0\}$. Indeed, if $\sigma_{x,t} = 0$ when $x_t > 0$, then $\dot{x}_t = \Phi(x_t) > 0$, and so x_t drifts upwards until some time when $\sigma_{x,t} > 0$. (In other words, x_t would explode upward asymptotically unless some shocks bring it back down.) Therefore, the remainder of the proof assumes $\sigma_{x,t} > 0$ on $\{t : x_t > 0\}$.

Consider the auxiliary process

$$d\tilde{x}_t = \Phi(\tilde{x}_t)dt + \sigma_{x,t}dZ_t, \quad \tilde{x}_0 = x_0,$$

which features an identical initial condition and shock exposure as x but without the risk premium in the drift. Because the drift of the true output gap is larger than the auxiliary process, standard comparison theorems for diffusions imply $\tilde{x}_T \leq x_T$ almost-surely. Thus, if we show that $\tilde{x}$ explodes upward, we will have proven that x does, in violation of Condition 1, meaning that the candidate process is not an equilibrium.

Let us define the stopping time

$$\tau_0 := \inf\{t > 0 : \tilde{x}_t = 0\},$$

and put $T_0 := T \wedge \tau_0$. Then,

$$\mathbb{E}_0[e^{-\phi_x T_0}\tilde{x}_{T_0}] = \tilde{x}_0 + \mathbb{E}_0\Big[\int_0^{T_0} e^{-\phi_x t}\Big(\Phi(\tilde{x}_t) - \phi_x\tilde{x}_t\Big)dt\Big] \geq \tilde{x}_0 = x_0,$$

since $\Phi(x) \geq \phi_x x$ for all $x \geq 0$ (recall that $\phi_x > 0$ is the minimal slope of the general rule $\Phi(x)$). The left-hand-side can be written $e^{-\phi_x T}\mathbb{E}_0[\tilde{x}_T \mathbf{1}_{\{T<\tau_0\}}]$, since $\tilde{x}_{\tau_0} = 0$. Thus,

$$\mathbb{E}_0[\tilde{x}_T \mathbf{1}_{\{T<\tau_0\}}] \geq e^{\phi_x T} x_0,$$

which by taking $T \to \infty$ proves that $\limsup_{T\to\infty} \mathbb{E}_0[\tilde{x}_T] = +\infty$ with positive probability. This explosiveness proves the result. □

Proof of Proposition 3. Let the enriched monetary rule be such that $\alpha_+ \leq 1 \leq \alpha_-$ and assume a linear output response function $\Phi(x) = \phi_x x$ for $\phi_x > 0$. Suppose, leading to contradiction, that a non-zero non-explosive equilibrium exists, and in particular $x_0 \neq 0$.

By the Itô-Tanaka formula, the dynamics of $|x_t|$ are

$$d|x_t| = \mathrm{sign}(x_t)\Big[\phi_x x_t + \frac{1}{2}(1-\alpha(x_t))\sigma_{x,t}^2\Big]dt + \mathrm{sign}(x_t)\sigma_{x,t} dZ_t + dL_t^0,$$

where $\alpha(x) := \alpha_+ \mathbf{1}_{\{x>0\}} + \alpha_- \mathbf{1}_{\{x<0\}}$ is the state-dependent risk premium response, and L_t^0 is the local time of x_t at 0 (note that L_t^0 is a non-negative, non-decreasing process). Integrating, taking expectations, and using the facts that $\mathrm{sign}(x)x = |x|$, that $L_T^0 \geq 0$, that $\sigma_x^2 \geq 0$, and that $\mathrm{sign}(x)(1-\alpha(x)) \geq 0$, we obtain

$$\begin{aligned}\mathbb{E}_0|x_T| &= |x_0| + \mathbb{E}_0 \int_0^T \Big[\phi_x |x_t| + \mathrm{sign}(x_t)(1-\alpha(x_t))\sigma_{x,t}^2\Big]dt + \mathbb{E}_0 L_T^0 \\ &\geq |x_0| + \phi_x \int_0^T \mathbb{E}_0|x_t| dt\end{aligned}$$

Given $x_0 \neq 0$ and $\phi_x > 0$, this proves that $\lim_{T\to\infty} \mathbb{E}_0|x_T| > 0$. But if that is the case, then the integral on the right-hand-side does not converge, which then implies that $\lim_{T\to\infty} \mathbb{E}_0|x_T| = +\infty$, in violation of Condition 1. This contradicts the non-explosiveness of the proposed equilibrium, and so $x_0 = x_t = 0$ for all t must hold. □

Proof of Proposition 4. To prove the proposition, we only need to provide an example construction for any $x_0 \leq 0$ starting point. The construction is as follows. Let ι_t be any interest rate process, subject to the lower bound $\iota_t \geq \underline{\iota}$. Consider some constant $b > 2(\rho - \underline{\iota})$ and set

$$\sigma_x^2 = \begin{cases} b, & \text{if } x < 0; \\ 0, & \text{if } x \geq 0. \end{cases}$$

Then, the dynamics of x_t are

$$dx_t = \begin{cases} [\iota_t - \rho + \frac{1}{2}b]dt + \sqrt{b}dZ_t, & \text{if } x_t < 0; \\ [\iota_t - \rho]dt, & \text{if } x_t \geq 0. \end{cases}$$

This will constitute a non-explosive equilibrium if x_t satisfies Condition 1. Consider the auxiliary process $\tilde{x}_t$, which satisfies $\tilde{x}_t = x_t$ whenever $x_t \geq 0$ and otherwise follows

$$d\tilde{x}_t = \left[\underline{\iota} - \rho + \frac{1}{2}b\right]dt + \sqrt{b}dZ_t, \quad \text{if} \quad x_t < 0.$$

Because the drift of x exceeds that of $\tilde{x}$, standard diffusion comparison theorems imply that $x_t \geq \tilde{x}_t$ forever. Furthermore, $\tilde{x}_t$ behaves like an arithmetic Brownian motion with positive drift when $x_t < 0$. By the well-known fact that a positive-drift arithmetic Brownian motion has $+\infty$ as its limit, we establish that $\liminf_{t\to\infty} \mathbb{E}[\tilde{x}_t] > -\infty$ almost-surely, hence by the inequality $x_t \geq \tilde{x}_t$ we have $\liminf_{t\to\infty} \mathbb{E}[x_t] > -\infty$. On the other hand, x_t is a path-continuous process and crosses 0 continuously. At $x_t = 0$, we have $\sigma_{x,t} = 0$, hence $\iota_t = \bar{\iota} = \rho$ is the interest rate, implying $dx_t = 0$. So $x = 0$ is an absorbing boundary, and $\limsup_{t\to\infty} \mathbb{E}[x_t] = 0 < +\infty$. Thus, Condition 1 is satisfied for x. □

A.4 Characterization of FTPL in a general setting

Before proving the theorems from the text, we derive a useful characterization of the government debt valuation equation that holds in a general environment nesting all the cases in the text. The environment below will feature a general surplus process, long-term debt, and CRRA utility.

We first set up a general surplus dynamic that nests all cases of interest. Let $\mathcal{Z}_t$ be a k-dimensional Brownian motion independent of the sunspot shock Z_t. Let Ω follow a Markov diffusion driven by $\mathcal{Z}$. Let $s_t := S_t/Y_t$ be a rule of the form

$$s_t = s(\Omega_t, x_t, \pi_t) \tag{A.8}$$

$$d\Omega_t = \mu_\Omega(\Omega_t)dt + \varsigma_\Omega(\Omega_t) \cdot d\mathcal{Z}_t \tag{A.9}$$

For now, we let the rule $s(\cdot)$ and dynamics μ_Ω, ς_Ω be arbitrary functions. This is more general than what we need going forward. Note that we obtain exogenous surpluses, following the description in the text before equation (29), if we impose that s only depends on Ω. Furthermore, we obtain surplus rules if we pick the dependence of s on (x, π) appropriately.

Second, we generalize the model to the CRRA utility $u(c,l) = \frac{c^{1-\gamma}}{1-\gamma} + \frac{l^{1+\varphi}}{1+\varphi}$ as in Section 4.4. In that case, the consumption FOC says

$$M_t = e^{-\rho t} C_t^{-\gamma}. \tag{A.10}$$

The labor-consumption margin is unaffected. Applying Itô's formula to (A.10), and noting that $C_t = Y_t = Y^* e^{x_t}$ and $-\frac{1}{dt}\mathbb{E}[\frac{dM_t}{M_t}] = r_t = \iota_t - \pi_t$, the IS curve generalizes to

$$dx_t = \Big[\frac{\iota_t - \pi_t - \rho}{\gamma} + \frac{1}{2}\gamma\sigma_{x,t}^2 + \frac{1}{2}\gamma|\varsigma_{x,t}|^2\Big]dt + \sigma_{x,t}dZ_t + \varsigma_{x,t}\cdot d\mathcal{Z}_t. \tag{A.11}$$

The dynamics of $Y_t = Y^* e^{x_t}$ can be derived from (A.11). When $\gamma \neq 1$, the Phillips curve is also different and requires an additional approximation to obtain a form similar to that used throughout the paper. Indeed, the derivation of the Phillips curve in Appendix B relies on $M_t Y_t \propto e^{-\rho t}$, which is no longer true with general CRRA utility. We make this approximation, which is tantamount to approximating around steady-state where $Y_t/Y_0 \approx 1$. With this approximation, the Phillips curve (PC) is replaced by

$$\mu_{\pi,t} = \rho\pi_t - \kappa\Big(\frac{e^{(\gamma+\varphi)x_t} - 1}{\gamma+\varphi}\Big), \tag{A.12}$$

where $Y^* := (\frac{\varepsilon-1}{\varepsilon})^{\frac{1}{\gamma+\varphi}}$ is the flexible-price output level and $\kappa := \eta(\varepsilon-1)(\gamma+\varphi)$ is the composite price-stickiness parameter.

Third, we generalize to the long-term debt setup described in Section 4.2. Let Q_t denote the per-unit bond price, which has dynamics of the form

$$dQ_t = Q_t\Big[\mu_{Q,t}dt + \sigma_{Q,t}dZ_t + \varsigma_{Q,t}\cdot d\mathcal{Z}_t\Big] \tag{A.13}$$

for some μ_Q, σ_Q, and ς_Q to be determined. With long-term debt, the flow government budget constraint is (32), the per-unit bond pricing equation is (33), and the aggregate government debt valuation equation is (34), all repeated here for convenience:

$$Q_t\dot{B}_t = \beta B_t - \beta B_t Q_t - P_t S_t. \tag{A.14}$$

$$Q_t = \mathbb{E}_t\Big[\int_t^\infty \frac{M_T}{M_t}\frac{P_t}{P_T}\beta e^{-\beta(T-t)}dT\Big] \tag{A.15}$$

$$\frac{Q_t B_t}{P_t} = \mathbb{E}_t\Big[\int_t^\infty \frac{M_u}{M_t}S_u du\Big]. \tag{A.16}$$

Substituting the consumption FOC (A.10) into (A.16), we may rewrite the aggregate debt valuation equation as

$$\frac{Q_t B_t}{P_t} = Y_t^{\gamma} \Psi_t \quad \text{where} \quad \Psi_t := \mathbb{E}_t\Big[\int_t^{\infty} e^{-\rho(u-t)} s_u Y_u^{1-\gamma} du\Big] \tag{A.17}$$

The next steps are to derive the dynamics of the two key present values Q_t and Ψ_t. These are essentially two "asset-pricing equations."

Starting from the per-unit bond pricing equation (A.15), we have that the object

$$e^{-\beta t} \frac{Q_t M_t}{P_t} + \int_0^t \frac{M_u}{P_u} \beta e^{-\beta u} du$$

is a local martingale and has zero drift. Note that, from the consumption FOC (A.10), the nominal SDF M_t/P_t has dynamics

$$d(M_t/P_t) = -(M_t/P_t)\Big[\iota_t dt + \gamma \sigma_{x,t} dZ_t + \gamma \varsigma_{x,t} \cdot d\mathcal{Z}_t\Big] \tag{A.18}$$

Then, by applying Itô's formula to the previous expression, and setting the resulting drift to zero, we have

$$\mu_{Q,t} = \beta - \frac{\beta}{Q_t} + \iota_t + \gamma \sigma_{x,t} \sigma_{Q,t} + \gamma \varsigma_{x,t} \cdot \varsigma_{Q,t} \tag{A.19}$$

From the definition of Ψ_t, we have

$$e^{-\rho t} \Psi_t + \int_0^t e^{-\rho u} s_u Y_u^{1-\gamma} du = \mathbb{E}_t\Big[\int_0^{\infty} e^{-\rho u} s_u Y_u^{1-\gamma} du\Big],$$

which is a local martingale. By the martingale representation theorem, we have that

$$d\Big(e^{-\rho t} \Psi_t + \int_0^t e^{-\rho u} s_u Y_u^{1-\gamma} du\Big) = e^{-\rho t}\Big(\sigma_{\Psi,t} dZ_t + \varsigma_{\Psi,t} \cdot d\mathcal{Z}_t\Big)$$

for some $\sigma_{\Psi,t}$ and some $\varsigma_{\Psi,t}$. On the other hand, we also have by applying Itô's formula to the left-hand-side,

$$d\Big(e^{-\rho t} \Psi_t + \int_0^t e^{-\rho u} s_u Y_u^{1-\gamma} du\Big) = \Big[-\rho e^{-\rho t} \Psi_t + e^{-\rho t} s_t Y_t^{1-\gamma}\Big] dt + e^{-\rho t} d\Psi_t$$

Equating these last two results, and rearranging for $d\Psi_t$, we have

$$d\Psi_t = (\rho \Psi_t - s_t Y_t^{1-\gamma}) dt + \sigma_{\Psi,t} dZ_t + \varsigma_{\Psi,t} \cdot d\mathcal{Z}_t \tag{A.20}$$

We now state and prove a useful characterization lemma.

Lemma A.2. *In the setting above with general surpluses, long-term debt, and CRRA utility,*

$$\Psi_t \gamma \sigma_{x,t} = \Psi_t \sigma_{Q,t} - \sigma_{\Psi,t} \tag{A.21}$$

$$\Psi_t \gamma \varsigma_{x,t} = \Psi_t \varsigma_{Q,t} - \varsigma_{\Psi,t} \tag{A.22}$$

Conversely, if the asset-pricing equations (A.19)-(A.20) hold, and the diffusion-matching equations (A.21)-(A.22) hold, then the government debt valuation equation (A.17) holds at every date, provided it holds at the initial date.

Proof of Lemma A.2. We apply Itô's formula to both sides of (A.17), using the flow government budget constraint (A.14), the price level dynamics $dP_t/P_t = \pi_t dt$, the dynamics of x_t in (A.11), the dynamics of Ψ_t in (A.20), and the dynamics of Q_t in (A.13) and (A.19). Matching drift and diffusion coefficients, we obtain

$$\begin{aligned}
[dt]\ :\ \frac{Q_t B_t}{P_t}\left[\iota_t - \pi_t + \gamma \sigma_{x,t}\sigma_{Q,t} + \gamma \varsigma_{x,t} \cdot \varsigma_{Q,t}\right] - s_t Y_t \\
= Y_t^{\gamma}(\rho \Psi_t - s_t Y_t^{1-\gamma}) + \gamma Y_t^{\gamma} \Psi_t \left[\frac{\iota_t - \pi_t - \rho}{\gamma} + \frac{1}{2}(\gamma+1)(\sigma_{x,t}^2 + |\varsigma_{x,t}|^2)\right] \\
+ \frac{1}{2}\gamma(\gamma-1) Y_t^{\gamma}\Psi_t(\sigma_{x,t}^2 + |\varsigma_{x,t}|^2) + \gamma Y_t^{\gamma}\sigma_{x,t}\sigma_{\Psi,t} + \gamma Y_t^{\gamma}\varsigma_{x,t} \cdot \varsigma_{\Psi,t} \\
[dZ]\ :\ \frac{Q_t B_t}{P_t}\sigma_{Q,t} = \gamma Y_t^{\gamma}\Psi_t \sigma_{x,t} + Y_t^{\gamma}\sigma_{\Psi,t} \\
[d\mathcal{Z}]\ :\ \frac{Q_t B_t}{P_t}\varsigma_{Q,t} = \gamma Y_t^{\gamma}\Psi_t \varsigma_{x,t} + Y_t^{\gamma}\varsigma_{\Psi,t}
\end{aligned}$$

Equations $[dZ]$ and $[d\mathcal{Z}]$, combined with (A.17), imply (A.21)-(A.22).

Conversely, plugging (A.21)-(A.22) into the first equation $[dt]$, using (A.17), and simplifying, we obtain an identity. Therefore, the $[dt]$ equation holds automatically, given the other equations all hold. This means that, provided (A.17) holds at $t = 0$, it will hold at every future date $t > 0$. □

We next provide a generalization of Definition 3 that allows exogenous state variables in the fiscal rule. This permits a general analysis that nests all special cases in Section 4.

Definition 4. An (x, Ω)-*Markov equilibrium* is a non-explosive equilibrium such that inflation π_t and the volatilities $\sigma_{x,t}, \varsigma_{x,t}$ are functions of (x_t, Ω_t).

Lemma A.3. *The generalized model above has no (x, Ω)-Markov sunspot equilibria "generically," in the sense that the $2 + dim(\mathcal{Z})$ endogenous variables $\pi(x, \Omega)$, $\sigma_x(x, \Omega)$, and $\varsigma_x(x, \Omega)$*

have only dim$(\mathcal{Z})$ degrees of freedom whenever $\sigma_x \neq 0$. In particular, if dim$(\mathcal{Z}) = 0$ (no fiscal shocks), then $\pi(x)$ and $\sigma_x(x)$ are pinned down uniquely.

Proof of Lemma A.3. We start by using the (x, Ω)-Markov assumption, which implies all dynamics are fully Markovian in (x_t, Ω_t). Hence, the bond price Q_t and the present-value Ψ_t solely functions of x_t and Ω_t, i.e., $Q_t = Q(x_t, \Omega_t)$ and $\Psi_t = \Psi(x_t, \Omega_t)$ for some functions $Q(\cdot)$ and $\Psi(\cdot)$ to be determined.[7]

Let us define the differential operator $\mathscr{L}$ that acts on C^2 functions g of (x, Ω) by

$$\mathscr{L}g = \Big(\mu_x \partial_x + \mu_\Omega' \partial_\Omega + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx} + \frac{1}{2}\mathrm{tr}(\varsigma_\Omega' \varsigma_\Omega \partial_{\Omega\Omega'}) + \varsigma_x' \varsigma_\Omega \partial_{\Omega x}\Big) g \tag{A.23}$$

This operator produces drifts of any process which is a function of (x, Ω). Apply Itô's formula to Q and Ψ to obtain (after dropping t subscripts)

$$Q\sigma_Q = \sigma_x \partial_x Q \tag{A.24}$$
$$Q\varsigma_Q = \varsigma_x \partial_x Q + \varsigma_\Omega \partial_\Omega Q \tag{A.25}$$
$$Q\mu_Q = \mathscr{L}Q \tag{A.26}$$
$$\sigma_\Psi = \sigma_x \partial_x \Psi \tag{A.27}$$
$$\varsigma_\Psi = \varsigma_x \partial_x \Psi + \varsigma_\Omega \partial_\Omega \Psi \tag{A.28}$$
$$\mu_\Psi = \mathscr{L}\Psi \tag{A.29}$$

Combining these results with equations (A.19), (A.20), (A.21), and (A.22), we obtain

$$\gamma\sigma_x = \sigma_x \partial_x Q / Q - \sigma_x \partial_x \Psi / \Psi \tag{A.30}$$
$$\gamma\varsigma_x = \varsigma_x \partial_x Q / Q - \varsigma_x \partial_x \Psi / \Psi + \varsigma_\Omega \partial_\Omega Q / Q - \varsigma_\Omega \partial_\Omega \Psi / \Psi \tag{A.31}$$

[7] Indeed, in an (x, Ω)-Markov equilibrium, we have that (x_t, Ω_t) is a bivariate Markov diffusion. Now, recall the bond pricing equation (33), which after plugging in the nominal SDF from (A.18) says

$$Q_t = \mathbb{E}_t\Big[\int_t^\infty e^{-\int_t^u (\iota_\tau + \frac{1}{2}\gamma^2(\sigma_{x,\tau}^2 + |\varsigma_{x,\tau}|^2))d\tau - \int_t^u \gamma\sigma_{x,\tau} dZ_\tau - \int_t^u \gamma\varsigma_{x,\tau} \cdot d\mathcal{Z}_\tau} \beta e^{-\beta(u-t)} du\Big].$$

Since $\iota_t = \bar{\iota} + \Phi(x_t, \pi_t) = \bar{\iota} + \Phi(x_t, \pi(x_t, \Omega_t))$ is purely a function of (x_t, Ω_t), as are $\sigma_{x,t}$ and $\varsigma_{x,t}$, the bond pricing equation above implies that Q_t is purely a function of (x_t, Ω_t). Similarly, we have that surpluses s_t are solely a function of (x_t, Ω_t). Given the definition of Ψ_t in (A.17), i.e., $\Psi_t := \mathbb{E}_t\big[\int_t^\infty e^{-\rho(u-t)} s_u Y_u^{1-\gamma} du\big]$, and given that $Y_t = Y^* e^{x_t}$, we obtain that Ψ_t is a function of (x_t, Ω_t) alone.

and

$$(\beta+\bar{\imath}+\Phi(x,\pi))Q-\beta+\gamma\sigma_x^2\partial_x Q+\gamma|\varsigma_x|^2\partial_x Q+\gamma\varsigma_x\cdot\varsigma_\Omega\partial_\Omega Q=\mathscr{L}Q \tag{A.32}$$

$$\rho\Psi-s(\Omega,x,\pi)(Y^*)^{1-\gamma}e^{(1-\gamma)x}=\mathscr{L}\Psi \tag{A.33}$$

The equations above hold in all the particular specifications considered in the paper. Note that in the short-term debt case, which can be derived by taking $\beta\to\infty$, equation (A.32) implies $Q\to 1$ uniformly, as we have imposed in the paper. In addition, after taking this limit we have $\lim_{\beta\to\infty}\partial_x Q=0$ and $\lim_{\beta\to\infty}\partial_\Omega Q=0$, and so $\lim_{\beta\to\infty}(\frac{\beta}{Q}-\beta)=\bar{\imath}+\Phi(x,\pi)$. This limiting result is also consistent with taking the $\beta\to\infty$ in the flow budget constraint (A.14) in order to recover (24).

Now, suppose $\sigma_x\neq 0$. In that case, equation (A.30) says that $\gamma=\partial_x Q/Q-\partial_x\Psi/\Psi$, and equation (A.31) says that $\partial_\Omega Q/Q=\partial_\Omega\Psi/\Psi$. The first equation implies that $Q(x,\Omega)=\Psi(x,\Omega)G(\Omega)e^{\gamma x}$ for some function $G(\cdot)$. The second equation implies that $G(\Omega)=G$ constant. Thus,

$$Q(x,\Omega)=G\Psi(x,\Omega)e^{\gamma x}. \tag{A.34}$$

Note that then G is pinned down by equation (A.17) at time $t=0$, since combining that equation with (A.34) says $\frac{B_0}{P_0}G=(Y^*)^\gamma$. Thus, (A.34) pins down Q given Ψ. Substitute (A.34) into equation (A.32) and then subtract equation (A.33) to get

$$\frac{s(\Omega,x,\pi)(Y^*)^{1-\gamma}}{\Psi}e^{(1-\gamma)x}-\rho+\beta+\bar{\imath}+\Phi(x,\pi)-\frac{\beta}{G\Psi}e^{-\gamma x}=\gamma\mu_x-\frac{1}{2}\gamma^2(\sigma_x^2+|\varsigma_x|^2)$$

Now, plug in μ_x from the IS curve (A.11) to get

$$0=\pi+\frac{s(\Omega,x,\pi)(Y^*)^{1-\gamma}}{\Psi}e^{(1-\gamma)x}+\beta-\frac{\beta}{G\Psi}e^{-\gamma x} \tag{A.35}$$

Note that, for the short-term debt case, $\lim\beta(1-\frac{1}{G\Psi}e^{-\gamma x})=-(\bar{\imath}+\Phi(x,\pi))$ as argued above. Equation (A.35) thus pins down Ψ given π. Since we only used so far the difference between equations (A.32) and (A.33), we still need to ensure that one of them holds in isolation. Thus, consider equation (A.33), after plugging in μ_x from (A.11):

$$\begin{aligned}
&\rho\Psi - s(\Omega, x, \pi)(Y^*)^{1-\gamma} e^{(1-\gamma)x} \qquad (A.36)\\
&= \Big[\frac{\bar{\imath} + \Phi(x,\pi) - \pi - \rho}{\gamma} + \frac{1}{2}\gamma(\sigma_x^2 + |\varsigma_x|^2)\Big]\partial_x\Psi + \mu_\Omega'\partial_\Omega\Psi + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx}\Psi\\
&+ \frac{1}{2}\mathrm{tr}(\varsigma_\Omega'\varsigma_\Omega\partial_{\Omega\Omega'}\Psi) + \varsigma_x'\varsigma_\Omega\partial_{\Omega x}\Psi
\end{aligned}$$

Given $(\pi, \sigma_x, \varsigma_x)$, equation (A.36) is a PDE for Ψ. Finally, recall the Phillips curve (A.12), apply Itô's formula to a generic inflation function $\pi(x, \Omega)$ to replace μ_π, and then plug in μ_x from (A.11):

$$\begin{aligned}
&\rho\pi - \kappa f(x) \qquad (A.37)\\
&= \Big[\frac{\bar{\imath} + \Phi(x,\pi) - \pi - \rho}{\gamma} + \frac{1}{2}\gamma(\sigma_x^2 + |\varsigma_x|^2)\Big]\partial_x\pi + \mu_\Omega'\partial_\Omega\pi + \frac{1}{2}(\sigma_x^2 + |\varsigma_x|^2)\partial_{xx}\pi\\
&+ \frac{1}{2}\mathrm{tr}(\varsigma_\Omega'\varsigma_\Omega\partial_{\Omega\Omega'}\pi) + \varsigma_x'\varsigma_\Omega\partial_{\Omega x}\pi
\end{aligned}$$

where $f(x) := \frac{e^{(\gamma+\varphi)x} - 1}{\gamma+\varphi}$. Given (σ_x, ς_x), equation (A.37) is a PDE for π.

At this point, consider the following experiment. Suppose $\pi(x, \Omega)$ is any function. Then, equation (A.35) pins down $\Psi(x, \Omega)$ uniquely, and equation (A.34) pins down $Q(x, \Omega)$ uniquely. Given π and Ψ, we can compute all their derivatives, and so equations (A.36) and (A.37) pin down 2 dimensions of the $1 + \dim(\mathcal{Z})$ dimensional vector (σ_x, ς_x). In other words, we must pick σ_x and/or ς_x in order to ensure equations (A.36) and (A.37) hold.

Thus, if $\dim(\mathcal{Z}) = 0$, then either $\sigma_x = 0$, or $\pi(x)$ and $\sigma_x(x)^2$ must take a particular form. Note also that these functions are independent of Ω since there are no sunspot shocks (hence Ω is not a state variable for any object in the case $\dim(\mathcal{Z}) = 0$). □

Corollary A.1. *Without fiscal state variables Ω, in an x-Markov equilibrium, the model above requires the following equations to hold whenever $\sigma_x \neq 0$:*

$$Q(x) = G\Psi(x)e^{\gamma x} \qquad (A.38)$$

$$\Psi(x) = \frac{G^{-1}\beta e^{-\gamma x} - s(x, \pi(x))(Y^*)^{1-\gamma}e^{(1-\gamma)x}}{\beta + \pi(x)} \qquad (A.39)$$

$$\sigma_x(x)^2 = 2\frac{\rho\pi(x) - \kappa f(x) - \frac{\bar{\imath} + \Phi(x,\pi(x)) - \pi(x) - \rho}{\gamma}\pi'(x)}{\gamma\pi'(x) + \pi''(x)} \qquad (A.40)$$

and

$$\rho\Psi(x) - s(x,\pi(x))(Y^*)^{1-\gamma}e^{(1-\gamma)x} = \frac{\gamma\Psi'(x)+\Psi''(x)}{\gamma\pi'(x)+\pi''(x)}(\rho\pi(x)-\kappa f(x)) \tag{A.41}$$
$$= \Big[\frac{\bar{\iota}+\Phi(x,\pi(x))-\pi(x)-\rho}{\gamma}\Big]\frac{\Psi'(x)\pi''(x)-\Psi''(x)\pi'(x)}{\gamma\pi'(x)+\pi''(x)}$$

Thus, the objects (Q,Ψ,σ_x^2,π) are all pinned down in an x-Markov equilibrium when $\sigma_x \neq 0$.

A.5 Proofs of FTPL Theorems 1-4

Proof of Theorem 1. We specialize the result of Lemma A.2 as follows. (We use Lemma A.2 as opposed to Lemma A.3 because this theorem does not specialize to the x-Markov class of equilibria.) First, with log utility ($\gamma = 1$), the present value Ψ_t in (A.17) becomes

$$\Psi_t := \mathbb{E}_t\Big[\int_t^\infty e^{-\rho(u-t)}s_u du\Big]$$

Second, with the exogenous Markovian surplus process $s_t = s(\Omega_t)$, we have that Ψ_t is purely determined by Ω_t, i.e., there exists a deterministic function $\Psi(\cdot)$ such that $\Psi_t = \Psi(\Omega_t)$. In that case, we have by Itô's formula and (A.9) that $\sigma_{\Psi,t} = 0$. Third, we have instantaneously-maturing debt, which is nested in the above formulas by setting $Q_t = 1$. This implies $\sigma_{Q,t} = 0$. Using these results, (A.21) holds if and only if $\Psi_t\sigma_{x,t} = 0$. Thus, $\sigma_{x,t} = 0$ for almost all t (except at the times when $\Psi_t = 0$, which are zero Lebesgue measure almost-surely). For the statement about (GD) holding for every $t > 0$, given it holds at $t = 0$, see the final statement of Lemma A.2. □

Remark A.1 (Non-Markovian surpluses)**.** *From the proof of Theorem 1, it is clear that the same arguments hold even in the more general non-Markovian case where $(s_t)_{t\geq 0}$ is independent of $(Z_t)_{t\geq 0}$, because in that case $\sigma_{\Psi,t} = 0$ still holds.*

Proof of Theorem 2. The combination of log utility ($\gamma = 1$) and constant surplus-to-output ratio $s_t = \bar{s}$ implies that $\Psi_t = \bar{s}/\rho$ is constant for any $\pi(x)$ and any $\sigma_x(x)$ functions. We then specialize the results of Corollary A.1 as follows. Equation (A.39) pins down inflation as

$$\pi(x) = \frac{\rho\beta}{G\bar{s}}e^{-x} - \beta - \rho, \quad \text{when} \quad \sigma_x \neq 0. \tag{A.42}$$

Note that $\pi'(x) + \pi''(x) = 0$. Then, equation (A.40) implies that, after plugging in the derivatives of π from (A.42),

$$e^{-x}\frac{\rho\beta}{G\bar{s}}\left(\bar{\iota}+\Phi(x,\pi)-\pi\right)-\kappa f(x)=\rho(\rho+\beta), \quad \text{when} \quad \sigma_x \neq 0. \tag{A.43}$$

But everything is pinned down in equation (A.43). The result cannot be consistent with the solution for π in (A.42) unless the monetary policy rule Φ takes a knife-edge form, and so generically we reach a contradiction. Thus, $\sigma_x = 0$ must hold. □

Proof of Theorem 3. We specialize the results of Corollary A.1 as follows. Using log utility ($\gamma = 1$) and short-term debt ($\beta \to \infty$) in equation (A.39) implies that

$$\Psi(x)=\bar{\Psi}e^{-x}, \quad \text{when} \quad \sigma_x \neq 0,$$

for $\bar{\Psi} = 1/G$. Notice that $\Psi'(x)+\Psi''(x) = 0$ in this solution. Thus, equation (A.41), after plugging in the solution for Ψ and its derivatives, says that

$$s(x,\pi)=(\bar{\iota}+\Phi(x,\pi)-\pi)\bar{\Psi}e^{-x}, \quad \text{when} \quad \sigma_x \neq 0. \tag{A.44}$$

Equation (A.44) pins down π uniquely when $\sigma_x \neq 0$, unless the rules $s(\cdot), \Phi(\cdot)$ take a knife-edge form. Finally, equation (A.40) specializes to

$$\sigma_x^2=\tilde{\sigma}_x^2 := 2\frac{\rho\pi-\kappa f(x)-[\bar{\iota}+\Phi(x,\pi)-\pi-\rho]\pi'}{\pi'+\pi''}, \quad \text{when} \quad \sigma_x \neq 0. \tag{A.45}$$

Given the solution for π, this pins down σ_x^2 uniquely when it is non-zero.

Given the functions $\Psi(x)$, $\pi(x)$, and $\sigma_x(x)^2$ are all pinned down assuming $\sigma_x \neq 0$, it remains to verify that the candidate sunspot equilibrium explodes, which then implies $\sigma_x = 0$. First, we want to show that the solution for $\tilde{\sigma}_x^2$ in (A.45) necessarily becomes negative at some $\underline{x} > -\infty$, which implies that non-exposiveness requires us to ensure $x_t \geq \underline{x}$ for all t. Using the equation (A.44) for π, notice that s bounded implies that, as $x \to -\infty$, we must have $\bar{\iota}+\Phi(x,\pi)-\pi \to 0$. This then implies that, using the linear Taylor rule $\Phi(x,\pi) = \phi_x x + \phi_\pi \pi$,

$$\lim_{x\to-\infty}\left(\pi(x)-\frac{\bar{\iota}+\phi_x x}{1-\phi_\pi}\right)=0.$$

As $x \to \infty$, we thus have $\pi \to -\infty$ (using the assumption that $\phi_x > 0$ and $\phi_\pi < 1$), $\pi' \to \frac{\phi_x}{1-\phi_\pi}$, and $\pi'' \to 0$. Plugging these into (A.45), we obtain $\lim_{x\to-\infty}\tilde{\sigma}_x^2 = -\infty$. Thus, let us define $\underline{x} := \inf\{x : \tilde{\sigma}_x^2 = 0\} > -\infty$.

Second, note that to ensure $x_t \geq \underline{x}$, we require

$$\mu_x(\underline{x}) = \bar{\imath} + \Phi(\underline{x}, \pi(\underline{x})) - \pi(\underline{x}) - \rho \geq 0.$$

Rather than prove this cannot hold, we instead prove that the function $\tilde{\mu}_x(x) := \bar{\imath} + \Phi(x, \pi(x)) - \pi(x) - \rho$ is strictly increasing when π is given by (A.44). This suffices, since it implies that $x_t \to +\infty$ under any parameters such that $x_t \geq \underline{x}$, in violation of the non-explosion condition. Indeed, $\tilde{\mu}_x(x)$ is the drift of x_t when volatility is zero, and the volatility only serves to increase the drift.

Differentiating $\tilde{\mu}_x(x)$, using the linear form of the Taylor rule, and substituting π' from implicitly differentiating (A.44), we obtain

$$\tilde{\mu}_x' = \phi_x + (\phi_\pi - 1)\pi' = \phi_x - (\phi_\pi - 1)\frac{\partial_x s + s - \bar{\Psi} e^{-x} \phi_x}{\partial_\pi s + (1 - \phi_\pi)\bar{\Psi} e^{-x}}$$

Using the assumptions of the theorem that $\phi_x > 0$, $\phi_\pi < 1$, and $\frac{\phi_x}{1-\phi_\pi} > -\frac{s + \partial_x s}{\partial_\pi s}$, and noting that $\bar{\Psi} = G^{-1} = \frac{B_0}{P_0}(Y^*)^{-1} > 0$, we have

$$\tilde{\mu}_x' = \phi_x \Big[1 - \frac{-\frac{\partial_x s + s}{\phi_x} + \bar{\Psi} e^{-x}}{\frac{\partial_\pi s}{1-\phi_\pi} + \bar{\Psi} e^{-x}} \Big] > \phi_x \Big[1 - \frac{\frac{\partial_\pi s}{1-\phi_\pi} + \bar{\Psi} e^{-x}}{\frac{\partial_\pi s}{1-\phi_\pi} + \bar{\Psi} e^{-x}} \Big] = 0.$$

This proves that x_t necessarily explodes, implying that $\sigma_x = 0$ generically. $\square$

Proof of Theorem 4. We specialize the results of Corollary A.1 as follows. Using the assumption of short-term debt ($\beta \to \infty$) in equation (A.39) implies that

$$\Psi(x) = \bar{\Psi} e^{-\gamma x}, \quad \text{when} \quad \sigma_x \neq 0,$$

where $\bar{\Psi} = 1/G$. Notice that $\gamma \Psi'(x) + \Psi''(x) = 0$ in this solution. Then, equation (A.41) implies, given $s = \bar{s}$ and the function Ψ,

$$(Y^*)^{1-\gamma} \bar{s} = \big(\bar{\imath} + \Phi(x, \pi) - \pi\big) \bar{\Psi} e^{-x}, \quad \text{when} \quad \sigma_x \neq 0, \tag{A.46}$$

which pins down π uniquely when $\sigma_x \neq 0$, unless the rule $\Phi(\cdot)$ takes a knife-edge form. Finally, equation (A.40) specializes to

$$\sigma_x^2 = \tilde{\sigma}_x^2 := 2 \frac{\rho \pi - \kappa f(x) - \frac{\bar{\imath} + \Phi(x,\pi) - \pi - \rho}{\gamma} \pi'}{\gamma \pi' + \pi''}, \quad \text{when} \quad \sigma_x \neq 0. \tag{A.47}$$

Given the solution for π, this pins down σ_x^2 uniquely when it is non-zero.

Given the functions $\Psi(x)$, $\pi(x)$, and $\sigma_x(x)^2$ are all pinned down assuming $\sigma_x \neq 0$, it remains to verify that the candidate sunspot equilibrium explodes, which then implies $\sigma_x = 0$. This step is almost identical to Theorem 3. First, we want to show that the solution for $\tilde{\sigma}_x^2$ in (A.45) necessarily becomes negative at some $\underline{x} > -\infty$, which implies that non-exposiveness requires us to ensure $x_t \geq \underline{x}$ for all t. Using the equation (A.46), and the linear Taylor rule $\Phi(x, \pi) = \phi_x x + \phi_\pi \pi$, we solve for π and its derivatives explicitly as

$$\begin{aligned}
\pi(x) &= \frac{\bar{\Psi}^{-1}(Y^*)^{1-\gamma}\bar{s}e^x - \bar{\imath} - \phi_x x}{\phi_\pi - 1} \\
\pi'(x) &= \frac{\bar{\Psi}^{-1}(Y^*)^{1-\gamma}\bar{s}e^x - \phi_x}{\phi_\pi - 1} \\
\pi''(x) &= \frac{\bar{\Psi}^{-1}(Y^*)^{1-\gamma}\bar{s}e^x}{\phi_\pi - 1}
\end{aligned}$$

Notice that, as $x \to -\infty$, given the assumption that $\frac{\phi_x}{1-\phi_\pi} > 0$, we have $\pi \to -\infty$, $\pi' \to \frac{\phi_x}{1-\phi_\pi}$, and $\pi'' \to 0$. Plugging these into (A.47), we obtain $\lim_{x \to -\infty} \tilde{\sigma}_x^2 = -\infty$. Thus, let us define $\underline{x} := \inf\{x : \tilde{\sigma}_x^2 = 0\} > -\infty$.

Second, note that to ensure $x_t \geq \underline{x}$, we require

$$\mu_x(\underline{x}) = \frac{1}{\gamma}\Big[\bar{\imath} + \Phi(\underline{x}, \pi(\underline{x})) - \pi(\underline{x}) - \rho\Big] \geq 0.$$

Rather than prove this cannot hold, we instead prove that the function $\tilde{\mu}_x(x) := \gamma^{-1}[\bar{\imath} + \Phi(x, \pi(x)) - \pi(x) - \rho]$ is strictly increasing when π is given by (A.46). This suffices, since it implies that $x_t \to +\infty$ under any parameters such that $x_t \geq \underline{x}$, in violation of the non-explosion condition. Indeed, $\tilde{\mu}_x(x)$ is the drift of x_t when volatility is zero, and the volatility only serves to increase the drift.

Differentiating $\tilde{\mu}_x(x)$, using the linear form of the Taylor rule, and substituting π' from above, we obtain

$$\gamma \tilde{\mu}_x' = \phi_x + (\phi_\pi - 1)\pi' = \phi_x + (\phi_\pi - 1)\frac{\bar{\Psi}^{-1}(Y^*)^{1-\gamma}\bar{s}e^x - \phi_x}{\phi_\pi - 1} = \bar{\Psi}^{-1}(Y^*)^{1-\gamma}\bar{s}e^x > 0,$$

where we have used the fact that $\bar{s} > 0$ and $\bar{\Psi} = G^{-1} = \frac{B_0}{P_0}(Y^*)^{-\gamma} > 0$. This proves that x_t necessarily explodes, implying that $\sigma_x = 0$ generically. □

A.6 Proof of Proposition 5

First, note that the spectral decomposition of $\mathcal{A} = V\Lambda V^{-1}$ is

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} v(\lambda_1) & v(\lambda_2) \end{bmatrix},$$

where the eigenvalues λ_1, λ_2 and the corresponding eigenvectors $v(\lambda_1), v(\lambda_2)$ are

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\left[\rho + \phi_x + \sqrt{(\rho - \phi_x)^2 - 4\kappa(\phi_\pi - 1)}\right] \\ \lambda_2 &= \frac{1}{2}\left[\rho + \phi_x - \sqrt{(\rho - \phi_x)^2 - 4\kappa(\phi_\pi - 1)}\right] \\ \text{and} \quad v(\lambda) &= \begin{pmatrix} \frac{\phi_\pi - 1}{\lambda - \phi_x} \\ 1 \end{pmatrix}. \end{aligned}$$

Recall equation (40) that

$$\mathbb{E}_0[\tilde{F}_t] = \exp(\Lambda t)\tilde{F}_0, \tag{A.48}$$

where $\tilde{F}_t = V^{-1}F_t$ is a rotated version of the state $F_t = (x_t, \pi_t)'$, and where

$$V^{-1} = \frac{1}{\lambda_2 - \lambda_1}\begin{bmatrix} \frac{(\lambda_1 - \phi_x)(\lambda_2 - \phi_x)}{\phi_\pi - 1} & -(\lambda_1 - \phi_x) \\ -\frac{(\lambda_1 - \phi_x)(\lambda_2 - \phi_x)}{\phi_\pi - 1} & \lambda_2 - \phi_x \end{bmatrix}.$$

In equation (A.48), $\exp(\Lambda t)$ refers to element-by-element exponentiation of Λ.

Let's consider the three cases of the proposition, using Condition 1 to kill explosive solutions to (A.48):

1. *Case 1:* $\rho + \phi_x > 0$ and $\rho\phi_x + \kappa(\phi_\pi - 1) > 0$.

   In this case, $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) > 0$. Therefore, the only non-explosive solution to (A.48) is $\tilde{F}_t = 0$, which implies $F_t = 0$, i.e., $x_t = \pi_t = 0$.

2. *Case 2:* $\rho\phi_x + \kappa(\phi_\pi - 1) < 0$.

   In this case, both eigenvalues are real and have opposite signs: $\lambda_1 > 0 > \lambda_2$. Therefore, all non-explosive solutions to (A.48) must satisfy $\tilde{F}_t^{(1)} = 0$, which using the expression for V^{-1} implies

   $$\pi_t = \frac{\lambda_2 - \phi_x}{\phi_\pi - 1} x_t.$$

Given $\sigma_{x,t} = 0$ from Theorem 1, this then implies $\sigma_{\pi,t} = 0$ as well.

3. *Case 3:* $\rho + \phi_x < 0$ and $\rho\phi_x + \kappa(\phi_\pi - 1) > 0$.

   In this case, $\mathrm{Re}(\lambda_1), \mathrm{Re}(\lambda_2) < 0$, meaning all initial conditions to (A.48) are non-explosive. Therefore, any $\tilde{F}_0$ corresponds to a valid equilibrium.

In all cases, we note that x_0 and $\varsigma_{x,t}$ are pinned down by (GD) at $t = 0$ and at $t > 0$, respectively. Indeed, using $ds_t = \lambda_s[s_t - \bar{s}]dt + \varsigma_{s,t} \cdot d\mathcal{Z}_t$ in equation (A.17), we obtain

$$\Psi_t = \Psi(s_t) := \frac{\bar{s}}{\rho} + \frac{s_t - \bar{s}}{\rho + \lambda_s},$$

which is exogenous. Using Ψ_t in (GD), we obtain

$$x_t = \log\left(\frac{B_t/P_t}{\Psi(s_t)Y^*}\right). \tag{A.49}$$

On the other hand, for $t > 0$, we have $\varsigma_\Psi = \frac{1}{\rho+\lambda_s}\varsigma_s$. Apply this in equation (A.22) of Lemma A.2, with $\gamma = 1$ and $Q \equiv 1$, to obtain

$$\varsigma_{x,t} = -\frac{\rho}{\lambda_s\bar{s} + \rho s_t}\varsigma_{s,t}. \tag{A.50}$$

Thus, x_t and $\varsigma_{x,t}$ are pinned down by B_t/P_t and s_t.

The remaining claims to prove are the existence/uniqueness statements. In Case 1, the equilibrium fails to exist generically, because $x_t = 0$ cannot be consistent with (A.49) and (A.50). In Case 2, the equilibrium is unique, because the initial conditions x_0 and $\pi_0 = \frac{\lambda_2 - \phi_x}{\phi_\pi - 1}x_0$ are pinned down by (A.49), and because the surplus shock exposures $\varsigma_{x,t}$ and $\varsigma_{\pi,t} = \frac{\lambda_2 - \phi_x}{\phi_\pi - 1}\varsigma_{x,t}$ are pinned down by (A.50). In Case 3, the equilibrium is not unique because, although x_0 is pinned down by (A.49), π_0 is not pinned down. Furthermore, π_t can have arbitrary sunspot volatility $\sigma_{\pi,t}$, despite the fact that $\sigma_{x,t} = 0$.

B Inflation Dynamics under Rotemberg

Here, we generalize the sticky-price model of Rotemberg (1982) to our environment. Since firms in our economy are ex-ante identical, they will have identical utilization and price-setting incentives, allowing us to study a representative firm's problem and a symmetric equilibrium.

To set up the representative intermediate-goods-producer problem, let l_t denote the firm's hired labor, at some equilibrium wage W_t. The firm produces $y_t = l_t$. The firm makes its price choice p_t, internalizing its demand $y_t = (p_t/P_t)^{-\varepsilon} Y_t$, where P_t and Y_t are the aggregate price and output. This demand curve comes from an underlying Dixit-Stiglitz structure with CES preferences (with substitution elasticity $\varepsilon > 1$) and monopolistic competition in the intermediate goods sector.

Letting M_t denote the real SDF process, the representative firm solves

$$\sup_{p,l} \mathbb{E}\Big[\int_0^\infty M_t\Big(\frac{p_t}{P_t}y_t - \frac{W_t l_t}{P_t} - \frac{1}{2\eta}\big(\frac{1}{dt}\frac{dp_t}{p_t}\big)^2 Y_t\Big)dt\Big] \tag{B.1}$$

$$\text{subject to} \quad y_t = (p_t/P_t)^{-\varepsilon} Y_t \tag{B.2}$$

$$y_t = l_t \tag{B.3}$$

The quadratic price adjustment cost in (B.1) has a penalty parameter η. As $\eta \to 0$ ($\eta \to \infty$), prices become permanently rigid (flexible). We assume that this price adjustment cost is purely non-pecuniary for simplicity (this means that adjustment costs do not affect the resource constraint). Alternatively, we could redistribute these adjustment costs lump-sum to the representative household.

Before solving the problem, we can immediately note the following property: price changes are necessarily absolutely continuous ("order dt"). Indeed, the adjustment cost per unit of time is a function of price changes per unit of time, i.e., $\frac{1}{dt}\frac{dp_t}{p_t}$. If prices were to change faster than dt, say with the Brownian motion dZ_t, then $\frac{1}{dt}\frac{dp_t}{p_t}$ would be unbounded almost-surely (because Brownian motion is nowhere-differentiable), leading to infinite adjustment costs. Consequently, we know that $\frac{1}{dt}\frac{dp_t}{p_t} = \frac{\dot{p}_t}{p_t}$ for some $\dot{p}_t$.

The firm's optimal price sequence solves a dynamic optimization problem. Substituting the demand curve from (B.2) and the production function from (B.3), we may rewrite problem (B.1) as

$$\sup_{\dot{p}} \mathbb{E}_t\Big[\int_t^\infty \frac{M_s Y_s}{M_t Y_t}\Big(\big(\frac{p_s}{P_s}\big)^{1-\varepsilon} - \frac{W_s}{P_s}\big(\frac{p_s}{P_s}\big)^{-\varepsilon} - \frac{1}{2\eta}\big(\frac{\dot{p}_s}{p_s}\big)^2\Big)ds\Big].$$

Furthermore, note that in the log utility model used in the text, we have $M_t Y_t = e^{-\rho t}$. Letting J denote this firm's value function, the HJB equation is

$$0 = \sup_{\dot{p}_t}\left\{\big(\frac{p_t}{P_t}\big)^{1-\varepsilon} - \frac{W_t}{P_t}\big(\frac{p_t}{P_t}\big)^{-\varepsilon} - \frac{1}{2\eta}\big(\frac{\dot{p}_t}{p_t}\big)^2 - \rho J_t + \frac{1}{dt}\mathbb{E}_t[dJ_t]\right\}$$

The firm value function follows a process of the form

$$dJ_t = [\mu_{J,t} + \dot{p}_t \frac{\partial}{\partial p} J_t] dt + \sigma_{J,t} dZ_t,$$

where $\mu_{J,t}$ and $\sigma_{J,t}$ are only functions of aggregate states (not the individual price). The only part that the firm can affect is $\dot{p}_t \frac{\partial}{\partial p} J_t$. Plugging these results back into the HJB equation and taking the FOC, we have

$$0 = -\frac{1}{\eta}\Big(\frac{\dot{p}_t}{p_t}\Big)\frac{1}{p_t} + \frac{\partial}{\partial p} J_t \tag{B.4}$$

Differentiating the HJB equation with respect to the state variable p_t, we have the envelope condition

$$(\varepsilon - 1)\Big(\frac{p_t}{P_t}\Big)^{-\varepsilon}\frac{1}{P_t} - \varepsilon \frac{W_t}{P_t}\Big(\frac{p_t}{P_t}\Big)^{-\varepsilon-1}\frac{1}{P_t} = \frac{1}{\eta}\Big(\frac{\dot{p}_t}{p_t}\Big)^2\frac{1}{p_t} - \rho\frac{\partial}{\partial p}J_t + \frac{1}{dt}\mathbb{E}_t\Big[d\Big(\frac{\partial}{\partial p}J_t\Big)\Big], \tag{B.5}$$

where the last term uses the stochastic Fubini theorem. Combining equations (B.4) and (B.5), we have

$$\eta(\varepsilon - 1)\Big(\frac{p_t}{P_t}\Big)^{-\varepsilon}\frac{1}{P_t} - \eta\varepsilon \frac{W_t}{P_t}\Big(\frac{p_t}{P_t}\Big)^{-\varepsilon-1}\frac{1}{P_t} = \Big(\frac{\dot{p}_t}{p_t}\Big)^2\frac{1}{p_t} - \rho\Big(\frac{\dot{p}_t}{p_t}\Big)\frac{1}{p_t} + \frac{1}{dt}\mathbb{E}_t\Big[d\Big(\Big(\frac{\dot{p}_t}{p_t}\Big)\frac{1}{p_t}\Big)\Big] \tag{B.6}$$

At this point, define the firm-level inflation rate $\pi_t := \dot{p}_t / p_t$, note that $\mathbb{E}_t\big[d\big(\pi_t \frac{1}{p_t}\big)\big] = \frac{1}{p_t}\mathbb{E}_t[d\pi_t] - \frac{1}{p_t}\pi_t^2 dt$, and use the symmetry assumption $p_t = P_t$ in (B.6) to get

$$\eta(\varepsilon - 1) - \eta\varepsilon\frac{W_t}{P_t} = -\rho\pi_t + \frac{1}{dt}\mathbb{E}_t[d\pi_t]. \tag{B.7}$$

Equation (B.7) is the continuous-time stochastic Phillips curve, with π_t interpreted also as the aggregate inflation rate (given a symmetric equilibrium).

Finally, note that the firm's optimization problem also requires the following transversality condition (see Theorem 9.1 of Fleming and Soner (2006)):

$$\lim_{T\to\infty} \mathbb{E}_t[M_T Y_T J_T] = 0.$$

In a symmetric equilibrium ($p = P$), and using the log utility result $M_t Y_t = e^{-\rho t}$, we have that

$$M_T Y_T J_T = \mathbb{E}_T\Big[\int_T^\infty e^{-\rho t}\Big(1-(Y^*)^{1+\varphi}e^{(1+\varphi)x_t}-\frac{1}{2\eta}\pi_t^2\Big)dt\Big]$$

Take expectations and the limit $T \to \infty$. Sufficient conditions for the result to be zero are

$$\lim_{T\to\infty}\mathbb{E}_t[e^{(1+\varphi)x_T-\rho T}] = 0 \tag{B.8}$$

$$\lim_{T\to\infty}\mathbb{E}_t[e^{-\rho T}\pi_T^2] = 0 \tag{B.9}$$

Equation (B.8) is identical to the one of the requirements for the consumer's problem to be well-defined (see Appendix A.1). Equation (B.9) avoids nominal explosions that imply an infinite present value of adjustment costs. Note that under Condition 1, both of these equations automatically hold.

C Nonlinear Phillips Curve

This section briefly explores the stability properties of the nonlinear Phillips curve, in contrast the linearized version used oftentimes in the paper. We will do this only in the context of deterministic equilibria, for simplicity. For convenience, we repeat this nonlinear equation here:

$$\dot{\pi}_t = \rho\pi_t - \kappa\Big(\frac{e^{(1+\varphi)x_t}-1}{1+\varphi}\Big). \tag{C.1}$$

We also repeat the IS curve after substituting the linear Taylor rule with target rate $\bar{\imath} = \rho$:

$$\dot{x}_t = \phi_x x_t + (\phi_\pi - 1)\pi_t. \tag{C.2}$$

A deterministic non-explosive equilibrium in this environment is (x_t, π_t) that satisfy (C.1)-(C.2) and asymptotic non-explosion Condition 1.

The nonlinearity of the Phillips curve does not change the basic determinacy result of Proposition 1, as we show next (although our proof requires stronger assumptions on the Taylor rule to ensure global determinacy).

Proposition C.1. *Consider the system (C.1)-(C.2) with $\phi_x > \rho$ and $\phi_\pi > 1$. Then, the only initial pair (x_0, π_0) consistent with a deterministic non-explosive equilibrium is $(x_0, \pi_0) = (0, 0)$. Any other initial pair diverges, but only asymptotically (i.e., not in finite time).*

Proof of Proposition C.1. Define $f(x) := \frac{e^{(1+\varphi)x}-1}{1+\varphi}$. From (C.1)-(C.2), the steady state solves

$$-\phi_x x = (\phi_\pi - 1)\kappa\rho^{-1} f(x)$$

The two sides of this equation have opposite slopes in x, so the unique solution is $x = 0$, proving the unique steady state is $(x, \pi) = (0, 0)$. The steady state is locally unstable, by the same linearized eigenvalue analysis leading to Proposition 1. By the local stable manifold theorem, we have that the unique stable solution to the dynamics is in fact this steady state. We now prove that any non-explosive equilibrium (satisfying Condition 1) must have $(x_t, \pi_t) = (0, 0)$ for all t. Assume not, i.e., assume, leading to contradiction, that $x_t \in [\underline{x}, \overline{x}]$ for all $t > 0$, where $\underline{x} < 0 < \overline{x}$.

First, from (C.1),

$$e^{-\rho t}\pi_t - \pi_0 = -\kappa \int_0^t e^{-\rho s} f(x_s) ds \tag{C.3}$$

Substituting (C.3) into (C.2), we have

$$\dot{x}_t = \phi_x x_t + (\phi_\pi - 1)\left[e^{\phi_x t}\pi_0 - \frac{\kappa}{\rho}\int_0^t \rho e^{\rho(t-u)} f(x_u) du\right] \tag{C.4}$$

Under the boundedness assumption, we may bound $f(\underline{x}) \leq f(x_t) \leq f(\overline{x})$, which when plugging into (C.4) leads to

$$\underbrace{\phi_x x_t + (\phi_\pi - 1)\left[e^{\phi_x t}\pi_0 - \frac{\kappa}{\rho}(e^{\rho t} - 1) f(\overline{x})\right]}_{:=L_t} \leq \dot{x}_t \leq \underbrace{\phi_x x_t + (\phi_\pi - 1)\left[e^{\phi_x t}\pi_0 - \frac{\kappa}{\rho}(e^{\rho t} - 1) f(\underline{x})\right]}_{:=U_t}$$

If $\pi_0 > 0$, then $L_t, U_t \to +\infty$ as $t \to \infty$ for every possible value of $x_t \in [\underline{x}, \overline{x}]$. On the other hand, if $\pi_0 < 0$, then $L_t, U_t \to -\infty$ as $t \to \infty$ for every possible value of $x_t \in [\underline{x}, \overline{x}]$. Hence, $\pi_0 > 0$ implies $x_T > \overline{x}$ for some $T > 0$, while $\pi_0 < 0$ implies $x_T < \underline{x}$ for some $T > 0$. This contradicts the bounded set $x_t \in [\underline{x}, \overline{x}]$, which implies $\pi_0 = 0$ is required.

However, since time 0 is arbitrary in this analysis, and the entire argument could be shifted forward in time, we in fact require $\pi_t = 0$ for all $t \geq 0$. Going back to equation (C.1), we then have that $x_t = 0$ for all $t \geq 0$. □

D Nuclear Taylor Rules and Finite-Time Explosions

Suppose we would like to allow deterministic equilibria that explode asymptotically, in violation of Condition 1. For instance, Cochrane (2011) considers some types of asymptotically exploding equilibria in his argument for non-uniqueness. In that case, is the spirit of Proposition 1 still true, i.e., do there exist Taylor rules which can eliminate indeterminacies? The answer is yes, but a "nuclear Taylor rule" is required to force explosion in finite time.

In particular, let us dispense with the linear rule (linear MP). Suppose the response function (MP) takes the nonlinear form

$$\Phi(x, \pi) = \frac{\phi_x}{2}(e^x - e^{-x}) + \pi \tag{D.1}$$

with $\phi_x > 0$ and suppose the target rate is again the natural rate $\bar{\imath} = \rho$. Note that the log-linearized version of (D.1) renders the linear Taylor rule (linear MP) with $\phi_\pi = 1$.

Combining (D.1) with (IS), the dynamics of x_t are given by

$$\dot{x}_t = \frac{\phi_x}{2}(e^{x_t} - e^{-x_t}) \tag{D.2}$$

This ODE has solution

$$x_t = \log\left(\frac{1 - Ke^{\phi_x t}}{1 + Ke^{\phi_x t}}\right)$$

where $K = \frac{1-e^{x_0}}{1+e^{x_0}}$. This process diverges in *finite time* for any $x_0 \neq 0$: it explodes at time $T = -\phi_x^{-1}\log(|K|)$. Hence, we have proved by construction the following result.

Proposition D.1. *Taylor rules exist such that any deterministic equilibrium has $x_t = 0$ forever.*

The analysis above abstracts from any feedback effects from inflation to output gap by setting a monetary policy rule with $\phi_\pi = 1$. This serves two purposes. First, it emphasizes the focus on self-fulfilling demand and not inflation per se. Equilibrium characterization requires the output gap to remain bounded for any finite horizon. There is no such requirement for inflation (e.g., hyperinflation might be an equilibrium outcome). Second, it simplifies the analysis and illustrates the point with examples that permit closed form solutions. As an additional benefit, Proposition D.1 holds for either the linearized or non-linear Phillips curves.

Determinacy extends beyond the particular response function (D.1) that has exactly a one-for-one inflation response. In particular, consider inflation sensitivities of more than one-for-one, such as

$$\Phi(x,\pi) = \frac{\phi_x}{2}(e^x - e^{-x}) + \phi_\pi \pi, \quad \phi_x > 0,\, \phi_\pi > 1. \tag{D.3}$$

While more challenging technically to analyze, this rule also selects the zero output gap equilibrium $x_t = 0$. We demonstrate this result formally next.

Under rule (D.3), the dynamical system for (x_t, π_t) is

$$\dot{\pi}_t = \rho \pi_t - \kappa f(x_t) \tag{D.4}$$

$$\dot{x}_t = \frac{\phi_x}{2}(e^{x_t} - e^{-x_t}) + (\phi_\pi - 1)\pi_t \tag{D.5}$$

where $f(x) := (1+\varphi)^{-1}[e^{(1+\varphi)x} - 1]$.

Proposition D.2. *Consider the system (D.4)-(D.5) with $\phi_x > 0$ and $\phi_\pi > 1$. Then, $(x_t, \pi_t) = (0,0)$ is the unique equilibrium that does not explode in finite time.*

Proof of Proposition D.2. Suppose the solution $(x_t(\phi_\pi), \pi_t(\phi_\pi))_{t\geq 0}$ associated to some $\phi_\pi > 1$ (which is unique prior to an explosion by the standard ODE uniqueness theorem) did not explode in finite time. In that case, because the solution is continuous in ϕ_π (again, standard ODE theorems ensure this), it follows that the solution $(x_t(\tilde{\phi}_\pi), \pi_t(\tilde{\phi}_\pi))_{t\geq 0}$ associated with $\tilde{\phi}_\pi < \phi_\pi$ also does not explode in finite time. Continuity requires this: otherwise, the two solutions would be infinitely far apart at some finite time T when one of the solutions does explode. But Proposition D.1 has already shown that $(x_t(1), \pi_t(1))_{t\geq 0}$ is explosive in finite time, a contradiction. □

E Sunspot equilibria with inflation

In the sunspot equilibrium constructions of Proposition 2, we work in the rigid price limit ($\kappa \to 0$) for analytical tractability. Here, we provide one class of equilibrium constructions where prices are partially flexible, so inflation is present. For this example, we will assume the linearly approximated Phillips curve (linear PC) and utilize a linear Taylor rule (linear MP) that is sufficiently aggressive. In particular, we will assume $\phi_x > 0$ and $\phi_\pi > 1$, so the Taylor principle is satisfied and deterministic multiplicities (as well as linearized stochastic multiplicities) are ruled out.

To maintain tractability, we assume a type of Markovian equilibrium where inflation is a function of the output gap. In particular, suppose $\pi_t = \pi(x_t)$ for some function $\pi(\cdot)$, to be determined. Obviously, this must be supported by a volatility process σ_x which is solely a function of x. These restrictions imply only one dimension of multiplicity, but the set of equilibria can still be relatively rich. By Proposition 2, part (ii), we need only consider sunspot equilibria with $x \leq 0$. A numerical illustration of the equilibrium constructed in the following proposition is contained in the text (Figure 3).

Proposition E.1. *Consider an economy with the linearly approximated Phillips curve (linear PC) and a linear Taylor rule (linear MP) with $\phi_x > 0$, $\phi_\pi > 1$, and $\bar{\imath} = \rho$. Then, there exists a family of non-explosive sunspot equilibria, indexed by constants $\bar{x} < 0$ and $\bar{\pi}$ satisfying $0 < \bar{\pi} < \kappa/\rho$. In particular, define the functions*

$$\pi(x) := \begin{cases} \bar{\pi}x, & \text{if } x \leq \bar{x} \\ f(x), & \text{if } x > \bar{x} \end{cases} \tag{E.1}$$

$$\sigma_x^2(x) := \begin{cases} 2\Big(\rho - \kappa/\bar{\pi} - \phi_x - (\phi_\pi - 1)\bar{\pi}\Big)x, & \text{if } x \leq \bar{x} \\ \bar{\sigma}^2 x^2, & \text{if } x > \bar{x}, \end{cases} \tag{E.2}$$

where $\bar{\sigma}^2$ is such that $\sigma_x(x)^2$ is continuous at $\bar{x}$, and where the function f solves the ODE

$$\rho f(x) - \kappa x = \Big[\bar{\imath} + \phi_x x + (\phi_\pi - 1)f(x) - \rho + \frac{1}{2}\bar{\sigma}^2 x^2\Big] f'(x) + \frac{1}{2}\bar{\sigma}^2 x^2 f''(x) \tag{E.3}$$

on $x \in (\bar{x}, 0)$, subject to the boundary conditions $f(\bar{x}) = \bar{\pi}\bar{x}$ and $f(0) < 0$, assuming such solution exists. Then, a non-explosive sunspot equilibrium exists, which is stationary and ergodic on $\{x_t < 0\}$, in which inflation is given by $\pi_t = \pi(x_t)$ and volatility by $\sigma_{x,t}^2 = \sigma_x^2(x_t)$.

Proof of Proposition E.1. Let us conjecture an equilibrium of the form described in the proposition. By Itô's formula, we may derive the dynamics of π, which when combined with the linear Phillips curve (linear PC) yields the equation

$$\rho\pi(x) - \kappa x = \Big[\bar{\imath} + \phi_x x + (\phi_\pi - 1)\pi(x) - \rho + \frac{1}{2}\sigma_x^2\Big]\pi'(x) + \frac{1}{2}\sigma_x^2 \pi''(x) \tag{E.4}$$

First, consider the lower region $\{x < \bar{x}\}$. Plug in the guess $\pi(x) = \bar{\pi}x$, the target rate $\bar{\imath} = \rho$, and then rearrange the equation for σ_x^2 to obtain

$$\frac{1}{2}\sigma_x^2 = \Big(\rho - \frac{\kappa}{\bar{\pi}} - \phi_x - (\phi_\pi - 1)\bar{\pi}\Big)x$$

This equation clearly coincides with (E.2), provided the right-hand-side is non-negative. Notice that the right-hand-side is in fact non-negative precisely when $x < 0$, by the restrictions $\phi_x > 0$, $\phi_\pi > 1$, and the fact that $0 < \bar{\pi} < \kappa/\rho$.

Next, consider the upper region $\{x > \bar{x}\}$. Substitute the volatility function $\sigma_x^2 = \bar{\sigma}^2 x^2$ from (E.2) into the Phillips curve (E.4) to obtain the ODE (E.3). As stated, we assume a solution f exists to this ODE, subject to the boundary condition $f(\bar{x}) = \bar{\pi}\bar{x}$. If so, then the resulting inflation function $\pi(x)$ is continuous at $\bar{x}$ (i.e., inflation does not jump). In that case, the Phillips curve holds for almost all x, the exception being $x = \bar{x}$, where $d\pi_t$ can include a local time. However, so long as the resulting stationary distribution places no point mass at $\bar{x}$, then firm optimality still holds, because firms' FOCs hold for almost all times. Given the continuity of the volatility function $\sigma_x^2(x)$, and given the continuity of $\pi(x)$, hence $\mu_x(x)$, the resulting stationary distribution (if it exists) cannot have a point mass at $\bar{x}$.

Thus, in the conjectured equilibrium, which presumably has $x_t < 0$ forever (to be verified), we will have $\sigma_x^2(x_t)$ well-defined and positive, and $\pi(x_t)$ satisfying the Phillips curve at almost all times. This proves that the equilibrium is valid, subject to the non-explosion Condition 1. The rest of the proof is dedicated to verifying this non-explosion.

First, let us prove that x_t never visits the upper boundary $\{x = 0\}$. For $x_t \in (\bar{x}, 0)$, the output gap dynamics are

$$\begin{aligned}\mu_x &= \phi_x x + (\phi_\pi - 1) f(x) + \frac{1}{2}\bar{\sigma}^2 x^2 \\ \frac{1}{2}\sigma_x^2 &= \frac{1}{2}\bar{\sigma}^2 x^2\end{aligned}$$

Study $\tilde{x} := -x$, which has $\tilde{x}\mu_{\tilde{x}} = x\mu_x$ and $\sigma_{\tilde{x}}^2 = \sigma_x^2$. Both the drift and diffusion vanish as $x \to 0$, but dividing them we obtain

$$\theta_0 := \lim_{\tilde{x} \searrow 0} \frac{2\tilde{x}\mu_{\tilde{x}}}{\sigma_{\tilde{x}}^2} = \lim_{x \nearrow 0} \frac{2x\mu_x}{\sigma_x^2} = \frac{2\phi_x}{\bar{\sigma}^2} + \frac{2(\phi_\pi - 1)}{\bar{\sigma}^2} \lim_{x \nearrow 0} \frac{f(x)}{x}$$

Given the assumption that $f(0) < 0$, we have $\lim_{x \nearrow 0} \frac{f(x)}{x} = +\infty$, and so $\theta_0 = +\infty$. Applying an analogous logic to Lemma A.1, part (i), we find that $x_t < 0$ for all t almost-surely.

Next, we prove that $x_t > -\infty$ almost-surely. To do this, compute the dynamics of $y_t = e^{x_t}$ on $\{x_t < \bar{x}\}$ by Itô's formula as

$$\begin{aligned}\mu_y &= y\mu_x + \frac{1}{2}y\sigma_x^2 = \Big(2\Big(\rho - \frac{\kappa}{\bar{\pi}}\Big) - \phi_x - (\phi_\pi - 1)\bar{\pi}\Big) y \log(y) \\ \frac{1}{2}\sigma_y^2 &= \frac{1}{2}y^2\sigma_x^2 = \Big(\rho - \frac{\kappa}{\bar{\pi}} - \phi_x - (\phi_\pi - 1)\bar{\pi}\Big) y^2 \log(y)\end{aligned}$$

Notice that both the drift and diffusion vanish as $y \to 0$ (i.e., as $x \to -\infty$). However, dividing these results in

$$\theta_{-\infty} := \frac{2y\mu_y}{\sigma_y^2} = 1 + \frac{\rho - \kappa/\bar{\pi}}{\rho - \kappa/\bar{\pi} - \phi_x - (\phi_\pi - 1)\bar{\pi}}$$

Given the parameter assumptions made, $\theta > 1$. Furthermore, the other conditions of Lemma A.1—namely that σ_y^2 is positive, bounded, and vanishes slower than quadratically as $y \to 0$—all hold. Consequently, (i) the boundary $\{y = 0\}$ is inaccessible for y_t; (ii) y_t possesses a non-degenerate stationary distribution on $(0, 1)$; and (iii) $x_t = \log(y_t)$ satisfies $\liminf_{T\to\infty} \mathbb{E}[x_T] > -\infty$. This verifies all the parts of Condition 1. □

Remark E.1. *Proposition E.1 presumes the existence of a solution f to the ODE (E.3) that satisfies $f(\bar{x}) = \bar{\pi}\bar{x}$ and $f(0) < 0$. While the right boundary condition may seem unusual, there is conceptually no issue, as we now show. Taking $x \nearrow 0$ in the ODE (E.3), and using $\bar{\iota} = \rho$, we obtain $\rho f(0) = (\phi_\pi - 1) f(0) f'(0-)$. If $f(0) < 0$, we must have that $f'(0-) = \rho/(\phi_\pi - 1)$. Consequently, it is equivalent to think of solving (E.3) subject to $f(\bar{x}) = \bar{\pi}\bar{x}$ and $f'(0) = \rho/(\phi_\pi - 1)$, which is a more conventional situation with one Dirichlet and one Neumann boundary condition.*

F Zero Lower Bound

Let us address the fact that a zero lower bound (ZLB) constrains monetary policy. To simplify the exposition, we work exclusively in the rigid-price limit $\kappa \to 0$, and so inflation is zero ($\pi_t = 0$) and the nominal rate is equal to the real rate ($\iota_t = r_t$). To make matters interesting, we will assume that monetary policy aims to achieve the flexible-price allocation whenever possible, but they are subject to the ZLB $r_t \geq 0$.

In particular, monetary authorities set the nominal rate (hence the real rate) to implement $x_t = 0$ whenever possible, subject to the ZLB. This is the same idea behind the policy in Caballero and Simsek (2020), who consider a version of the New Keynesian model with risky capital. Under this policy rule, zero output gap prevails whenever the real rate is positive, and a negative output gap must arise at the ZLB (because recall raising the interest rate will lower output):

$$0 = \min[-x_t, r_t]. \tag{F.1}$$

In Lemma F.1 below, we show that within the class of equilibria we study, (F.1) is the outcome of optimal discretionary monetary policy (i.e., monetary policy without commitment to future policies). More deeply, the implementation of $x_t = 0$ "whenever possible" itself requires some kind of commitment to off-equilibrium threats, for instance to reduce interest rates if x_t ever fell below 0—this is the standard notion of "active" monetary policy that pervades the New Keynesian literature, but it becomes somewhat hidden by the outcome (F.1). In that sense, the rule (F.1) actually embeds some amount of commitment power.

Lemma F.1. *Optimal discretionary monetary policy—which maximizes (2) subject to $r_t \geq 0$, optimal household and firm decisions, and its own future decisions—implements (F.1).*

Proof of Lemma F.1. Since there is no upper bound on interest rates, the central bank can always threaten r_t high enough to ensure that $x_t \leq 0$. Since positive output gaps are undesirable, they will implement this. Then, we can restate the problem as: optimal discretionary monetary policy seeks to pick a r_t to maximize (2), subject to (IS), $x_t \leq 0$, the ZLB $r_t \geq 0$, and subject to its own future decisions.

We will discretize the problem to time intervals of length Δ and later take $\Delta \to 0$. Noting that $C_t = e^{x_t} Y^*$, the time-t household utility is proportional to

$$\mathbb{E}_t\Big[\int_0^\infty \rho e^{-\rho s} x_{t+s} ds\Big] \approx \rho x_t \Delta + \mathbb{E}_t\Big[\int_\Delta^\infty \rho e^{-\rho s} x_{t+s} ds\Big]$$
$$\approx -\rho\Delta \mathbb{E}_t[x_{t+\Delta} - x_t] + \underbrace{\mathbb{E}_t\Big[\int_\Delta^\infty \rho e^{-\rho s} x_{t+s} ds\Big] + \rho\Delta\mathbb{E}_t[x_{t+\Delta}]}_{\text{taken as given by discretionary central bank}}.$$

The term with brackets underneath is taken as given by the time-t discretionary central bank, because it involves expectations of future variables that the future central bank can influence.

Thus, taking $\Delta \to 0$, the time-t central bank solves

$$\min_{r_t \geq 0} \mathbb{E}_t[dx_t]$$

subject to the constraints

$$r_t = \rho + \mu_{x,t} - \frac{1}{2}\sigma_{x,t}^2$$
$$x_t \leq 0 \quad \text{and if} \quad x_t = 0 \quad \text{then} \quad \mu_{x,t} = \sigma_{x,t} = 0.$$

Note that $\sigma_{x,t}$ is independent of policy when $x_t < 0$. There are two cases. If $x_t = 0$, then the constraints imply that $r_t = \rho$. If $x_t < 0$, we may substitute the dynamics of x_t

(replacing μ_x from the first constraint) to re-write the problem as

$$\min_{r_t \geq 0} [r_t - \rho + \frac{1}{2}\sigma_{x,t}^2].$$

Since σ_x is taken as given, the optimal solution is $r_t = 0$. Thus, the discretionary central bank optimally sets

$$r_t = \rho \mathbf{1}_{\{x_t = 0\}}.$$

In other words, the complementary slackness condition $x_t r_t = 0$ holds, which together with $r_t \geq 0$ and $x_t \leq 0$ implies (F.1). □

The entire model dynamics are characterized by the IS curve (IS) with volatility when $r_t = 0$ and $x_t < 0$ and deterministic dynamics otherwise, i.e.,

$$\mu_{x,t} = (-\rho + \frac{1}{2}\sigma_{x,t}^2)\mathbf{1}_{\{x_t < 0\}}. \tag{F.2}$$

The entire previous analysis from Section 3 goes through with $\phi_x = 0$ and $\bar{\imath} = 0$.

However, just to see a different construction, let $y = e^x$ and suppose

$$\sigma_x = \begin{cases} \nu(1 - y), & \text{if } y < 1; \\ 0, & \text{if } y \geq 1. \end{cases} \tag{F.3}$$

(If we had set $\sigma_x = \nu/y$ when $y < 1$, then the argument would be identical to that in Section 3.) In this case, the dynamics of y_t are

$$dy_t = y_t \Big[- \rho + \nu^2(1 - y_t) \Big] \mathbf{1}_{\{y_t < 1\}} dt + y_t(1 - y_t)\nu \mathbf{1}_{\{y_t < 1\}} dZ_t. \tag{F.4}$$

This process never reaches $y = 0$, since it behaves asymptotically as a geometric Brownian motion as $y_t \to 0$. Thus, we have constructed a valid equilibrium with volatility at the ZLB.

If agents expect volatility to be sufficiently countercyclical, then the volatility is forever recurrent. To see this, suppose $\nu^2 > 2\rho$ so that $\log(y_t)$ has a positive drift as $y_t \to 0$. By standard arguments, y_t will not concentrate mass near $y = 0$ in the long run. On the other hand, the drift of $\log(y_t)$ is negative as $y_t \to 1$, and its volatility vanishes, so y_t will not ever reach $y = 1$ either. There will be a non-degenerate ergodic distribution of y_t, hence volatility $\sigma_{x,t}$. This economy is persistently demand-driven and stuck at the ZLB.

By adding coordinated jumps in σ_x, we believe we can make the equilibria even more realistic. Initially, volatility can be non-existent and the economy sitting at $x_t = 0$. All of a sudden, fear can rise sufficiently that x_t must jump to negative territory. Because of the ZLB, it is not possible for monetary policy to correct this fear-driven recession. The rise in volatility essentially forces r to the ZLB, similar to Caballero and Simsek (2020). Once $x_t < 0$, volatility can vary continuously, and sunspot shocks will be moving demand. Imagine at some later time T, demand reverts back to the flexible-price outcome $x_T = 0$. At some still later date, volatility can re-emerge. In this way, we can construct equilibria that alternate between efficiency and inefficient, self-fulfilling, volatile recessions.