

Rational Sentiments and Financial Frictions*

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Abstract

We discover sentiment-driven equilibria in popular models of imperfect risk sharing. In these equilibria, sentiment dynamics behave like uncertainty shocks, in the sense that self-fulfilled beliefs about volatility drive aggregate fluctuations. Because such fluctuations can decouple from the wealth distribution, rational sentiment helps resolve two puzzles plaguing models emphasizing balance sheets: (i) financial crises emerge suddenly, featuring large volatility spikes and asset-price declines; (ii) asset-price booms, with below-average risk premia, predict busts and financial crises. Methodologically, our contribution is using stochastic stability theory to establish existence of sunspot equilibria.

JEL Codes: E00, E44, G01.

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It has by now become commonplace, especially after the 2008 global financial crisis, for macroeconomic models to prominently feature banks, limited participation, imperfect risk-sharing, and other such “financial frictions.” Incorporating these features allows macroeconomists to speak meaningfully about financial crises and desirable policy responses. Despite the dramatic growth in this literature, there remain two major sources of disconnect between these models and actual data. For one, standard models have difficulty reproducing the observed severity and suddenness of economic downturns and asset-price dislocations. Secondly, standard models struggle to generate booms that are inherently fragile and prone to bust. To address these shortcomings, some recent contributions add large and sudden bank runs¹ while others deviate from rational expectations to model booms as episodes of over-optimism.²

We embrace *rational sentiment* as a complementary approach. This paper makes two main contributions. First, we uncover a wide variety of novel sentiment-driven sunspot equilibria supported by standard financial friction models. The fluctuations in these equilibria are self-fulfilling: they only occur because agents coordinate on them. Second, we demonstrate how sentiment fluctuations alleviate some of the empirical shortcomings for this class of models. Rational sentiment can generate both (i) large and sudden fluctuations, similar to bank runs (footnote 1), and (ii) booms that breed fragility, similar to the “behavioral sentiment” adopted by some recent papers (footnote 2).

Model and mechanism. We study a simple stripped-down model with financial frictions, similar to Kiyotaki and Moore (1997), Brunnermeier and Sannikov (2014), and many others.³ There are two types of agents (“experts” and “households”) with identical preferences but different levels of productivity when managing capital. Heterogeneous productivity means the identity of capital holders matters for aggregate output. But incomplete markets prevent agents from sharing risks associated to their capital holdings, so optimal capital holdings depend to some degree on these risks and not only on productivities. There are no other features: no borrowing/collateral constraints, no default externalities, and no irrational beliefs. And yet, this basic model can feature a tremendous amount of multiplicity that has been overlooked in the literature.

¹For example, Gertler and Kiyotaki (2015) and Gertler et al. (2020) attempt to integrate bank runs into a conventional financial accelerator model, in order to capture additional amplification and non-linearity. Without runs or panic-like behavior, financial accelerator models have a difficult time inducing the financial intermediary leverage needed to generate large amounts of amplification.

²For example, Krishnamurthy and Li (2020) and Maxted (2020) build an extrapolative sentiment process on top of a relatively standard financial accelerator model. Agents’ excessive optimism in booms lowers risk premia, erodes bank balance sheets, and creates fragility.

³We work in continuous time, contributing to a burgeoning literature (He and Krishnamurthy, 2012, 2013, 2019; Moreira and Savov, 2017; Klimenko et al., 2017; Caballero and Simsek, 2020).

At a very general level, indeterminacy in this model comes from the combination of incomplete financial markets and a causal impact of asset prices on the real side of the economy. With this combination of features, asset prices today are not pinned down by “fundamentals” and also depend on the future distribution of asset prices. This dependence allows beliefs about tomorrow’s asset prices to influence today’s prices, so long as beliefs are not “explosive” in a sense we make precise. Of particular importance in our specific model is the perceived dispersion in future asset prices, or today’s conditional price volatility.

The following story clarifies the mechanism. Suppose there is a sudden rise in *fear*, purely as a sunspot phenomenon. Fear manifests as higher perceived asset-price volatility, which results in a *fire sale*: first-best capital users (experts) sell to less-efficient users (households). The reason is that, with financial frictions, both risk-sharing and productive-efficiency considerations matter for the capital distribution; higher volatility propels risk-sharing considerations to the forefront. Because productive efficiency falls, asset prices fall as well. Thus, the new allocation features a less efficient capital allocation, lower asset prices, and higher volatility.

This high-volatility, low-price allocation will only be an equilibrium if it does not lead to explosive paths, mirroring the conventional idea that dynamic stability of equilibrium supports indeterminacy. Here, stability means that asset prices must eventually mean-revert, or “bounce back” from extreme values. Supposing the future distribution of asset prices q is characterized by a first and second moment (μ_q, σ_q^2) , then the rise in σ_q (fear) must be accompanied by an eventual rise in μ_q (bounce-back beliefs). In our continuous-time setup, bounce-back beliefs are just boundary conditions on μ_q at the extreme states. Such boundary restrictions are both analytically-convenient and mild; very rich dynamics are possible away from extreme states.

In summary, a sunspot rise in fear creates a self-fulfilled decline in asset prices, through coordinated fire sales. Conversely, sunspot bravery (decline in fear) raises asset prices, through coordinated purchases. These fear-driven dynamics are sustainable so long as they are expected to eventually subside. A distinctive feature is that sentiment dynamics are always characterized by time-varying endogenous uncertainty.

Overview of paper. While explaining our mechanism above, we abstracted from the wealth distribution between experts and households. Typically in the financial frictions literature, this wealth distribution is the key state variable modulating the dynamics. We demonstrate how restricting attention to these type of equilibria—equilibria which are Markovian solely in the wealth distribution—precludes essentially all interesting self-

fulfilling dynamics (Appendix A).

Our main results pertain to a richer class of self-fulfilling equilibria (Section 2). Mathematically, we dispense with the assumption that equilibria must be Markovian in the wealth distribution, which can be understood as removing an ad-hoc restriction on agents’ beliefs. This generalization considerably complicates the analysis, and our contribution here is to provide an explicit construction and characterization of a broad class of such equilibria. We also discuss how policies that affect beliefs (e.g., speeches and promises) can improve equilibria.

This richer class of equilibria engender several new insights, related to the shortcomings in such models (Section 3). First, fundamentals-based recessions are primarily about expert balance sheet impairment in our model, so they feature small volatility increases and very slow recoveries; sentiment-driven crises feature far larger volatility spikes and fast recoveries. In fact, we prove that arbitrary capital price volatility and recovery speeds can be justified by sunspot equilibria. Second, whereas fundamentals-based booms always reduce the prospect of crisis, sentiment-driven booms can actually increase crisis probabilities. Relatedly, in the years before large busts, an economy with sentiment tends to feature asset-price and output booms, low volatility, and below-average risk premia. We argue all of these properties of sentiment-driven fluctuations better resemble real-world financial cycles.

Related literature. The theoretical focus on financial frictions and sunspots is not new to this paper. Several studies show how multiplicity emerges through the interaction between asset valuations and borrowing constraints.⁴ Relative to these papers, we study different and more primitive financial frictions (equity-issuance constraints) that do not feature any mechanical link between prices and constraints.⁵

Bank runs, financial panics, and sudden stops are related to, but distinct from, our self-fulfilled fluctuations.⁶ All of these phenomena rely on financial frictions, are outcomes of coordination, and produce large fluctuations relative to fundamentals. However, whereas bank runs and its cousins are liability-side phenomena, self-fulfilled fire

⁴For instance, bubbles can relax credit constraints, allowing greater investment and thus justifying the existence of the bubble (Scheinkman and Weiss, 1986; Kocherlakota, 1992; Farhi and Tirole, 2012; Miao and Wang, 2018; Liu and Wang, 2014). Self-fulfilling credit dynamics can also arise with *unsecured* lending as opposed to collateralized (Gu et al., 2013; Azariadis et al., 2016).

⁵We call equity-issuance constraints “more primitive” because they are present (either explicitly or implicitly) even in models with borrowing constraints. With unlimited outside equity, perfect risk-sharing could always be achieved and the effects of borrowing constraints circumvented.

⁶In a setup close to ours, Mendo (2020) studies self-fulfilled panics that induce collapse of the financial sector, but remaining within the context of Markov equilibria. Gertler and Kiyotaki (2015) and Gertler et al. (2020) study bank runs in a similar class of models.

sales are pure asset-side phenomena.⁷ Furthermore, unlike runs, our mechanism does not require asset-market illiquidity or maturity mismatch. Finally, whereas runs are almost exclusively about large downside risk, our sentiment fluctuations also generate interesting boom-bust cycles.

Given our results hold even without borrowing constraints or runs, we illustrate that a much broader class of financial crisis models are subject to sunspots. We also do not rely on the more traditional multiplicity-inducing assumptions, such as overlapping generations,⁸ non-convexities or externalities in technology,⁹ asymmetric/imperfect information,¹⁰ or multiple assets.¹¹

Our focus on fear and volatility as drivers of self-fulfilling fluctuations closely relates to the “self-fulfilling risk panics” of [Bacchetta et al. \(2012\)](#). [Benhabib et al. \(2020\)](#) obtain a similar type of fluctuation by examining economies with either collateral or liquidity constraints, rather than an OLG setup as in [Bacchetta et al. \(2012\)](#). Although we do not rely on common multiplicity-inducing features like OLG or borrowing constraints, we expound on the deeper connection to these papers in Section 1.4.

Finally, our equilibrium construction differs somewhat from the literature. Sunspot equilibria are often constructed by essentially randomizing over a multiplicity of deterministic transition paths to a stable steady state. By contrast, the deterministic version of our model features an unstable steady state; critically, the introduction of volatility flips the stability properties of equilibrium. This distinction is likely why our sunspot equilibria have gone unnoticed despite the framework being so widespread. Methodologically, we prove our existence results with tools from the “stochastic stability” literature (the stochastic differential equation analog of Lyapunov stability for ODE systems). As one might expect from deterministic models, the existence of sunspot equilibria is tied directly to stability properties. Stochastic stability tools are ideally suited for this issue.

⁷When selling assets, investors simultaneously deleverage, which clarifies our mechanism as a “funding demand” decline rather than the “funding supply” decline that characterizes a run. It is not that investors cannot obtain financing, just that they do not want to.

⁸The classic studies on OLG and multiplicity are [Azariadis \(1981\)](#) and [Cass and Shell \(1983\)](#).

⁹For example, see [Azariadis and Drazen \(1990\)](#) for multiplicity under threshold investment behavior. See [Farmer and Benhabib \(1994\)](#) for a multiplicity under increasing returns to scale.

¹⁰In a macro context, [Piketty \(1997\)](#) and [Azariadis and Smith \(1998\)](#) for self-fulfilling dynamics in the presence of screened/rationed credit. In a finance context, [Benhabib and Wang \(2015\)](#) and [Benhabib et al. \(2016, 2019\)](#) generate sunspot fluctuations in dispersed information models. Like us, [Benhabib et al. \(2015\)](#) pins down volatility by certain fundamentals of the economy. However, whereas their mechanism is static in nature, ours is intrinsically dynamic—this is why the “fundamentals” that determine our volatility include asset prices themselves, whereas their volatility is fully determined by deep structural parameters. For this reason, our self-fulfilling volatility is naturally time-varying.

¹¹[Hugonnier \(2012\)](#), [Gârleanu and Panageas \(2021\)](#), and [Khorrami and Zentefis \(2020\)](#) all build “redistributive” sunspots that shift valuations among multiple positive-net-supply assets.

1 Model

Information structure. Time $t \geq 0$ is continuous. There are two types of uncertainty in the economy, modeled as two independent Brownian motions $Z := (Z^{(1)}, Z^{(2)})$. All random processes will be adapted to Z .¹² As will be clear below, the first shock $Z^{(1)}$ represents a *fundamental shock* in the sense that it directly impacts production possibilities, whereas the second shock $Z^{(2)}$ is a *sunspot shock* that is extrinsic to any economic primitives but nevertheless may impact endogenous objects. At the end of the paper, we will also consider extrinsic Poisson jumps as part of the information structure.

Technology and markets. There are two goods, a non-durable good (the numéraire, “consumption”) and a durable good (“capital”) that produces the consumption good. The aggregate supply of capital grows exogenously as

$$dK_t = K_t[gdt + \sigma dZ_t^{(1)}], \quad (1)$$

where g and $\sigma \geq 0$ are exogenous constants. The capital-quality shock $\sigma dZ^{(1)}$ is a standard way to introduce fundamental randomness in technology. Individual capital holdings evolve identically, except that capital may be traded frictionlessly between agents in the market.¹³ The relative price denoted by q_t , determined in equilibrium.

There are two types of agents, experts and households, who differ in their production technologies. Experts produce a_e units of the consumption good per unit of capital, whereas households’ productivity is $a_h \in (0, a_e)$.

Financial markets consist solely of an instantaneously-maturing, risk-free bond that pays interest rate r_t is in zero net supply. The key financial friction: agents cannot issue equity when managing capital. It is inconsequential that the constraint be this extreme. Partial equity issuance, as long as there is some limit, will generate identical results on sunspot volatility.¹⁴

¹²In the background, the Brownian motion Z exists on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$, equipped with all the “usual conditions.” All equalities and inequalities involving random variables are understood to hold almost-everywhere and/or almost-surely.

¹³Individual capital is thus a choice variable for any agent. In particular, if any individual holds capital k_t at time t , its law of motion is

$$dk_t = gk_t dt + \sigma k_t dZ_t^{(1)} + d\Omega_t,$$

where the term $d\Omega_t$ corresponds to net purchases.

¹⁴We analyze this situation in Online Appendix D.1. We show that the presence of *any* equity-issuance constraint permits some amount of sunspot volatility. However, in the limit as equity issuance becomes unconstrained, sunspot volatility becomes impossible, and the unique equilibrium is one with perfect capital allocation and perfect risk-sharing. Equity-issuance restrictions, sometimes called “skin-in-the-

Preferences and optimization. Given the stated assumptions, we can write the dynamic budget constraint of an agent of type ℓ (expert or household) as

$$dn_{\ell,t} = \left[(n_{\ell,t} - q_t k_{\ell,t}) r_t - c_{\ell,t} + a_{\ell} k_{\ell,t} \right] dt + q_t k_{\ell,t} dR_t, \quad (2)$$

where n_{ℓ} is the agent's net worth, c_{ℓ} is consumption, and k_{ℓ} is capital holdings. The last term $dR_t := \frac{d(q_t K_t)}{q_t K_t} = g dt + \sigma dZ_t^{(1)} + \frac{dq_t}{q_t} + \sigma \text{Cov}_t[dZ_t^{(1)}, \frac{dq_t}{q_t}]$ represents the capital and price appreciation that accrues while holding capital.

Experts and households have time-separable logarithmic utility, with discount rates ρ_e and $\rho_h \leq \rho_e$, respectively. All agents have rational expectations and solve

$$\sup_{c_{\ell} \geq 0, k_{\ell} \geq 0, n_{\ell} \geq 0} \mathbb{E} \left[\int_0^{\infty} e^{-\rho_{\ell} t} \log(c_{\ell,t}) dt \right] \quad (3)$$

subject to (2). The constraint $n_t \geq 0$ is the standard solvency constraint (natural borrowing limit). Everything in optimization problem (3) is homogeneous in (c, k, n) , so we can think of the expert and household as representative agents within their class.

Let us briefly discuss the constraint $n_t \geq 0$. It is important that this be the *natural* borrowing limit, as we have claimed our results have nothing to do with binding borrowing constraints. In the presence of fundamental risk ($\sigma > 0$), it is clear that the natural borrowing limit is $n_t \geq 0$, because a sequence of negative shocks can completely destroy an agent's capital stock leaving them without any assets to repay their debts; hence, a net worth buffer must be maintained to assure debt repayments in the worst-case scenario. Without fundamental risk ($\sigma = 0$), the natural borrowing limit is less obvious, yet we derive $n_t \geq 0$ in Appendix B.1, assuming a No-Ponzi condition $\liminf_{T \rightarrow \infty} n_T \geq 0$ and a lower bound on net worth $n_t \geq -\underline{n}$ (where \underline{n} can be arbitrarily large, but finite). Intuitively, because negative shocks can still hit the capital price q in our sunspot equilibria, trading strategies that allow negative net worth cannot simultaneously satisfy the No-Ponzi condition and lower-bounded net worth.

Finally, to guarantee a stationary wealth distribution, we also allow a type-switching structure: experts retire and become households at rate δ_e , while households retire and become experts at rate δ_h . Technically, the presence of type-switching alters the objective function from (3), but this is irrelevant under the assumption of log utility, as optimal behavior will be as if solving (3)—we show this in Appendix B.2. To acknowledge the fact that type-switching shifts wealth across agent groups, which does not affect agents'

game" constraints, often arise as the optimal contract in a moral hazard problem, though this micro-foundation is not important for our purposes here.

individual net worth evolution, let N_e and N_h denote aggregate expert and household net worth. The dynamic evolutions of N_e and N_h will mirror (2), with additional terms capturing switching.¹⁵ We reiterate that type-switching is unnecessary for our sunspot results and only serves to obtain stationarity in case we set $\rho_e = \rho_h$ (if $\rho_e > \rho_h + \sigma^2$, the wealth distribution will automatically be stationary even without type-switching).

1.1 Equilibrium definition

The definition of competitive equilibrium is standard, following Brunnermeier and San-nikov (2014). To write a formal definition, denote the set of experts by the interval $\mathbb{I} = [0, \nu]$, for some $\nu \in (0, 1)$ and index individual experts by $i \in \mathbb{I}$. Similarly, denote the set of households by $\mathbb{J} = (\nu, 1]$ with index j . If a type-switching structure exists, we necessarily have $\nu = \frac{\delta_h}{\delta_e + \delta_h}$ (i.e., the population size of experts is pinned down by switching rates) and the indexes of retiring experts/households are implicitly swapped with newly entering experts/households.

Definition 1. For any initial capital endowments $\{k_{e,0}^i, k_{h,0}^j : i \in \mathbb{I}, j \in \mathbb{J}\}$ such that $\int_{\mathbb{I}} k_{e,0}^i di + \int_{\mathbb{J}} k_{h,0}^j dj = K_0$, an *equilibrium* consists of stochastic processes—adapted to the filtered probability space generated by $\{Z_t : t \geq 0\}$ —for capital price q_t , interest rate r_t , capital holdings $(k_{e,t}^i, k_{h,t}^j)$, consumptions $(c_{e,t}^i, c_{h,t}^j)$, and net worths $(n_{e,t}^i, n_{h,t}^j)$, such that:

- (i) initial net worths satisfy $n_{e,0}^i = q_0 k_{e,0}^i$ and $n_{h,0}^j = q_0 k_{h,0}^j$ for $i \in \mathbb{I}$ and $j \in \mathbb{J}$;
- (ii) taking processes for q and r as given, each expert $i \in \mathbb{I}$ and household $j \in \mathbb{J}$ solves (3) subject to (2);
- (iii) consumption and capital markets clear at all dates, i.e.,

$$\int_{\mathbb{I}} c_{e,t}^i di + \int_{\mathbb{J}} c_{h,t}^j dj = a_e \int_{\mathbb{I}} k_{e,t}^i di + a_h \int_{\mathbb{J}} k_{h,t}^j dj \quad (4)$$

$$\int_{\mathbb{I}} k_{e,t}^i di + \int_{\mathbb{J}} k_{h,t}^j dj = K_t, \quad (5)$$

where K_t follows (1).

Note that the riskless bond market clears automatically by Walras' Law, which is why this condition is not included above.

¹⁵That is, $dN_e = N_e \frac{dn_e}{n_e} - \delta_e N_e dt + \delta_h N_h dt$ and $dN_h = N_h \frac{dn_h}{n_h} - \delta_h N_h dt + \delta_e N_e dt$.

1.2 Equilibrium characterization

We present a useful equilibrium characterization that aids all future analysis. First, conjecture the following form for capital price dynamics:

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t} \cdot dZ_t]. \quad (6)$$

There are two potential avenues for random fluctuations. The standard term $\sigma_q \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ represents amplification (or dampening) of fundamental shocks, as in [Brunnermeier and Sannikov \(2014\)](#) and others. By contrast, the second element $\sigma_q \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ measures sunspot volatility that only exists because agents believe in it.

Given log utility and the scale-invariance of agents' budget sets, individual optimization problems are readily solvable. Optimal consumption satisfies the standard formula $c_\ell = \rho_\ell n_\ell$. Optimality conditions for capital holding by experts and households are given by¹⁶

$$\frac{a_e}{q} + g + \mu_q + \sigma\sigma_q \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - r = \frac{qk_e}{n_e} |\sigma_R|^2 \quad (7)$$

$$\frac{a_h}{q} + g + \mu_q + \sigma\sigma_q \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - r \leq \frac{qk_h}{n_h} |\sigma_R|^2 \quad (\text{with equality if } k_h > 0), \quad (8)$$

where

$$\sigma_{R,t} := \sigma \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sigma_{q,t} \quad (9)$$

denotes the shock exposure of capital returns. From these optimality conditions, notice that agents' capital holdings decisions are uniquely determined given the price process for q . The only additional optimality conditions are the transversality conditions

$$\lim_{T \rightarrow \infty} \mathbb{E}[e^{-\rho_\ell T} \frac{1}{c_{\ell,T}} n_{\ell,T}] = 0. \quad (10)$$

However, using $c_\ell = \rho_\ell n_\ell$, we see that (10) automatically holds. As a consequence of (10), our equilibria will always be bubble-free.¹⁷

¹⁶Note that experts' optimality condition (7) assumes the solution is interior, i.e., $k_e > 0$. But this is clearly required in any equilibrium given experts earn a strictly higher expected return and will always want to take a strictly greater position in capital than households, whose position is lower-bounded by 0.

¹⁷Using transversality (10) and individual optimality (which equates the type- ℓ SDF to marginal utility as $M_{\ell,t} = e^{-\rho_\ell t} \frac{1}{c_{\ell,t}}$), one can show that $q_t K_t = \mathbb{E}_t[\int_t^\infty \frac{M_s}{M_t} Y_s ds]$, where M is a consumption-weighted-average of expert and household SDFs. Thus, capital is valued according to a present-value equation, and no bubbles exist.

Next, we aggregate. Due to financial frictions and productivity heterogeneity, both the distribution of wealth and capital holdings will matter in equilibrium. However, because all experts (and households) make the same scaled consumption c_ℓ/n_ℓ and portfolio choices k_ℓ/n_ℓ , the wealth and capital distributions may be summarized by experts' wealth share

$$\eta := \frac{N_e}{N_e + N_h} = \frac{N_e}{qK}$$

and experts' capital share

$$\kappa := \frac{\int_{\mathbb{I}} k_e^i di}{K}.$$

Given agents' solvency and capital short-sales constraints, we must have $\eta \in [0, 1]$ and $\kappa \in [0, 1]$ in equilibrium. Substitute optimal consumption into goods market clearing (4), divide by aggregate capital K , and use the definitions of η and κ , to obtain

$$q\bar{\rho} = \kappa a_e + (1 - \kappa)a_h, \quad (\text{PO})$$

where $\bar{\rho}(\eta) := \eta\rho_e + (1 - \eta)\rho_h$ is the wealth-weighted average discount rate. Equation (PO) connects asset price q to output efficiency κ , which we call a *price-output* relation for short.

Using the definitions of η and κ , experts' and households' portfolio shares can be written $\frac{qk_e}{n_e} = \frac{\kappa}{\eta}$ and $\frac{qk_h}{n_h} = \frac{1-\kappa}{1-\eta}$. Then, differencing the optimal portfolio conditions (7)-(8), we obtain the *risk-balance* condition

$$0 = \min \left[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1 - \eta)} |\sigma_R|^2 \right]. \quad (\text{RB})$$

Either experts manage the entire capital stock ($\kappa = 1$) or the excess return experts obtain over households, $(a_e - a_h)/q$, represents fair compensation for differential risk exposure, $\frac{\kappa - \eta}{\eta(1 - \eta)} |\sigma_R|^2$. On the other hand, summing portfolio conditions (7)-(8), weighted by κ and $1 - \kappa$, yields an equation for the riskless rate:

$$r = \frac{\kappa a_e + (1 - \kappa)a_h}{q} + g + \mu_q + \sigma\sigma_q \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \left(\frac{\kappa^2}{\eta} + \frac{(1 - \kappa)^2}{1 - \eta} \right) |\sigma_R|^2. \quad (11)$$

Finally, by applying Itô's formula to experts' wealth share $\eta = N_e/(N_e + N_h)$, and using agents' net worth dynamics (2) along with contributions from type-switching, wealth share dynamics are given by

$$d\eta_t = \mu_{\eta,t} dt + \sigma_{\eta,t} \cdot dZ_t, \quad \text{given } \eta_0, \quad (12)$$

where

$$\mu_\eta = \eta(1 - \eta)(\rho_h - \rho_e) + (\kappa - 2\eta\kappa + \eta^2) \frac{\kappa - \eta}{\eta(1 - \eta)} |\sigma_R|^2 + \delta_h - (\delta_e + \delta_h)\eta \quad (13)$$

$$\sigma_\eta = (\kappa - \eta)\sigma_R. \quad (14)$$

The initial wealth distribution $\eta_0 = \frac{\int \mathbb{1} n_{e,0}^i di}{q_0 K_0} = \frac{\int \mathbb{1} k_{e,0}^i di}{K_0}$ is given due being solely a function of the initial endowments of capital.

Lemma 1. *Given $\eta_0 \in (0, 1)$, consider a process $(\eta_t, q_t, \kappa_t, r_t)_{t \geq 0}$ with dynamics for q_t and η_t described by (6) and (12), respectively. If $\eta_t \in [0, 1]$, $\kappa_t \in [0, 1]$, and equations (PO), (RB), (11), (13) and (14) hold for all $t \geq 0$, then $(\eta_t, q_t, \kappa_t, r_t)_{t \geq 0}$ corresponds to an equilibrium of Definition 1. Moreover, any distinct pair of such processes corresponds to distinct equilibria.*

Lemma 1 summarizes the full set of conditions characterizing equilibrium and is proved in Appendix B.3. In the rest of the paper, we use this lemma as a tool to simplify our search for equilibria.

Lastly, we make some mild parameter restrictions that will be applicable in the remainder of the paper.

Assumption 1. *Parameters satisfy (i) $0 < \frac{a_h}{\rho_h} < \frac{a_e}{\rho_e} < +\infty$; (ii) $\sigma^2 < \rho_e(1 - a_h/a_e)$; and (iii) either $\sigma^2 < \rho_e - \rho_h$, or $\delta_e, \delta_h > 0$.*

Assumption 1 part (i), only for convenience, makes the very modest assumption that the capital price is higher if experts control 100% of wealth than if households control 100% of wealth. Part (ii), meant to make the problem interesting, ensures experts sometimes hold all capital, i.e., $\kappa = 1$, and sometimes do not, $\kappa < 1$. Part (iii) guarantees experts do not asymptotically hold all wealth.

1.3 Types of equilibria

We categorize our equilibria into two types: fundamental and sunspot. Fundamental equilibria have two properties: (i) the sunspot shock $Z^{(2)}$ plays no role; and (ii) only the minimal set of state variables affects observables. Because of financial frictions and productivity heterogeneity, the expert wealth share η is a necessary state variable to summarize economic conditions. Other objects (e.g., q, r, κ) are either prices or control variables, so there is a sense in which η is the minimal state variable needed in this class of models. In other words, a fundamental equilibrium should only depend on η . Sunspot equilibria constitute all other equilibria, which we further categorize into two types depending on whether or not they are Markov in η .

Definition 2. A *Fundamental Equilibrium* (FE) is an equilibrium that is Markov in η and in which $\sigma_q \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \equiv 0$. Any other equilibrium is a *Brownian Sunspot Equilibrium* (BSE). A BSE that is Markov in η is called a *Wealth-driven BSE* (W-BSE). A BSE that is non-Markov in η is called a *Sentiment-driven BSE* (S-BSE).

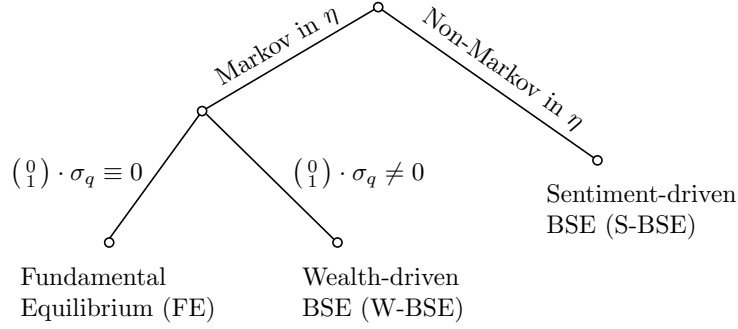


Figure 1: Types of equilibria.

Figure 1 displays the equilibrium taxonomy. The literature universally focuses on the FE of this model, e.g., Brunnermeier and Sannikov (2014).¹⁸ Our paper is devoted to the BSEs. For a few reasons that we formalize and expound on in Appendix A, we deem the W-BSEs “uninteresting” and proceed directly to S-BSEs. First, we show that W-BSEs cannot exist if fundamental uncertainty σ is non-zero (see Lemma A.2). Second, although a W-BSE can exist when $\sigma = 0$ (see Proposition A.1), this equilibrium is well-approximated by a FE in the sense that the FE converges to the W-BSE as $\sigma \rightarrow 0$ (see Lemma A.3).¹⁹ In summary, Markov equilibria in experts’ wealth share η are either (a) pure FE or (b) look very much like pure FE. Consequently, the remainder of the paper studies S-BSEs in the hopes of uncovering new insights relative to the literature.

¹⁸We discuss and analyze these fundamental equilibria in Online Appendix E. We provide some new results, including a multiplicity of fundamental equilibria when $\sigma > 0$. In particular, following the spirit of footnote 16 in Kiyotaki and Moore (1997), we show that there are two types of equilibria: a normal equilibrium in which negative shocks reduce asset prices (this is the one studied by the literature) and a “hedging equilibrium” in which, due to coordinated capital purchases/sales, asset prices respond oppositely to shocks.

¹⁹It may be surprising that a sunspot equilibrium exists if $\sigma = 0$. In such a model without fundamental uncertainty, the reader might think that the market is complete (recall: a riskless bond is traded with no borrowing/lending frictions) and so perhaps the welfare theorems should hold. If so, equilibrium can be solved via a planner’s problem, and the result is generically unique. However, one must keep in mind that no market exists to trade the endogenously-arising exposure of q_t to the sunspot shock $Z_t^{(2)}$, so if agents perceive this exposure as a possibility, the market will not necessarily be complete.

1.4 Benchmarks that preclude sentiment-driven equilibria

Before proceeding to the main analysis, we analyze two benchmarks that clarify the underpinnings of sentiment-driven equilibria. The fact that we require financial frictions and productivity heterogeneity is not surprising—these features are required even in the “financial accelerator” equilibria of [Kiyotaki and Moore \(1997\)](#) and [Brunnermeier and Sannikov \(2014\)](#). More interestingly, sentiment-driven equilibria require nothing more.

Frictionless equity issuance. Suppose any agent, when managing capital, could issue unlimited equity to the market. In exchange for taking some exposure to the risk σ_R in capital returns, these outside equity contracts promise an expected excess return $\sigma_R \cdot \pi$ (here, π is the equilibrium risk price vector associated to the two shocks in Z). All agents can participate as buyers in this market. Since equity-issuance is unconstrained, it is straightforward to see that any capital owner must equate her expected excess returns on capital to $\sigma_R \cdot \pi$. (If $\sigma_R \cdot \pi$ were below an agent’s expected excess capital returns, unlimited capital purchases financed by unlimited equity issuances would be an arbitrage; if $\sigma_R \cdot \pi$ were above, the agent would prefer to sell all their capital and invest solely in equity securities.) Experts always manage some capital, so

$$\frac{a_e}{q} + g + \mu_q + \sigma\sigma_q \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - r = \sigma_R \cdot \pi.$$

However, the analogous equation cannot hold for households, since their productivity is lower, $a_h < a_e$. Households will never manage capital in this economy, so $\kappa_t = 1$ at all times, hence $q_t = a_e / \bar{\rho}(\eta_t)$ by equation (PO). That q is solely a function of η rules out S-BSEs.²⁰ Thus, it is critical that capital is traded, i.e., κ varies.

For our main results, the friction in equity markets need not be as stark as the baseline model. Indeed, Online Appendix D.1 extends the baseline model to allow “partial equity issuance,” subject to a constraint parameterized by $\chi \in [0, 1]$. In particular, suppose any agent can issue some equity up to a limit: he/she can offload up to $1 - \chi$ fraction of the risk associated to their capital stock as equity to a competitive financial market. The baseline model corresponds to $\chi = 1$ (i.e., zero issuance), while the frictionless model outlined above corresponds to $\chi = 0$ (i.e., unlimited issuance). We show that self-fulfilling volatility is possible for any $\chi > 0$ and is eliminated only when $\chi \rightarrow 0$.

²⁰In fact, q cannot be stochastic at all. Indeed, experts and households share identical risk preferences, so households will purchase the outside equity of experts in an amount that is consistent with perfect risk-sharing, meaning $\sigma_\eta \equiv 0$. Since $q_t = a_e / \bar{\rho}(\eta_t)$ is solely a function of η , which is deterministic, we have $\sigma_q \equiv 0$ as well. Shocks can play no role with frictionless equity markets.

Homogeneous productivities. Consider our economy with $a_e = a_h = a$. Based on equation (PO), equal productivities immediately implies $q_t = a/\bar{\rho}(\eta_t)$. Again, q is solely a function of η , which rules out S-BSEs. Critically, sentiment-driven equilibria require real outcomes to depend on κ .

In fact, with equal productivities, equilibrium cannot support any endogenous dependence on shocks, i.e., one can show $\sigma_q \equiv 0$ when $a_e = a_h$.²¹ This unveils a more general point about the endogeneity of market incompleteness: one cannot necessarily add unspanned extrinsic shocks to an economy and declare markets incomplete. Even though this equal-productivity economy lacks insurance markets against $Z^{(2)}$ shocks, financial markets are *effectively complete*, in the sense that the economic structure imposes that $Z^{(2)}$ can have no impact on outcomes. What is required is a set of assumptions such that $Z^{(2)}$ has “real effects” in which case financial market incompleteness will have some bite. In our economy, all we require is $a_e > a_h$.

Discussion: imperfect risk-sharing and productivity heterogeneity. Based on the benchmarks above, let us explain the deep reasons why our model admits sentiment-driven equilibria. First, with limited equity issuance and lack of markets for insurance against sunspot shocks, capital is traded partly for risk-sharing purposes. In other words, risk can affect the capital ownership distribution (i.e., σ_R can affect κ). Second, productive heterogeneity permits “misallocation”: the capital distribution can affect aggregate output, which translates into capital prices (i.e., κ can affect q).

The way to visualize this chain of causality through the symbols of our model is

$$\sigma_R \implies \kappa \implies q. \quad (15)$$

Financial frictions modulate the first link ($\sigma_R \Rightarrow \kappa$), while productive heterogeneity modulates the second ($\kappa \Rightarrow q$). Current asset price q then depends on the distribution of future asset prices through σ_R . But what determines σ_R ? In general, nothing. S-BSEs, by removing the ad-hoc restriction that equilibria be Markov in η , remove an artificial anchor for σ_R and allow volatility to be coordination-driven.

Chain (15) also suggests a connection to the “self-fulfilling risk panics” of [Bacchetta et al. \(2012\)](#), further analyzed by [Benhabib et al. \(2020\)](#). [Bacchetta et al. \(2012\)](#) emphasize a negative relationship between asset prices and volatility, effectively collapsing the

²¹Plugging $a_e = a_h$ into equation (RB) implies either $\kappa = \eta$ or $|\sigma_R| = 0$. Either way, $\sigma_\eta = (\kappa - \eta)\sigma_R = 0$. Then, applying Itô’s formula to $q_t = a/\bar{\rho}(\eta_t)$, and using the form of wealth share dynamics $d\eta_t = \mu_{\eta,t}dt + \sigma_{\eta,t} \cdot dZ_t$, we obtain the diffusion of q_t as $q\sigma_q = -\frac{\rho_e - \rho_h}{\bar{\rho}(\eta)}q\sigma_\eta$, which equals zero.

causal chain in equation (15) to $\sigma_R \Rightarrow q$.²² But digging deeper, Benhabib et al. (2020) explain that the key to risk panic equilibria is a causal dependence of the stochastic discount factor (SDF) on asset prices. Bacchetta et al. (2012) obtain a price-SDF link via OLG; Benhabib et al. (2020) show how a price-SDF link can also arise due to collateral or liquidity constraints. Our results are deeply connected—our price-output link (PO) necessarily implies a price-SDF link—but distinguished by the fact we do not rely on the common multiplicity-inducing features of OLG or borrowing constraints.

2 Sentiment-driven equilibria

We endeavor here to analyze a rich class of equilibria that are not Markov in η , the S-BSEs. Below, we construct and provide detailed characterization of such equilibria.

Because the capital price q is the critical endogenous object (one may think of q as the “co-state” variable), equilibria which are not Markov in η share the defining characteristic that a variety of different asset prices can prevail for a given wealth distribution. Since η captures all fundamental information in this economy, one can think of “sentiment” as responsible for generating the multiplicity of asset prices corresponding to the same η . This is why Definition 2 refers to this class of equilibria as Sentiment-driven BSEs.

The usual approach to constructing sunspot equilibria is to first analyze the non-stochastic equilibria of a model, identify a fundamental indeterminacy, and then add sunspot shocks that essentially randomize over the multiplicity of fundamental equilibria. Before diving into the details, we remark on how and why our construction must differ from this usual approach.

Remark 1 (Stability and multiplicity: connection to literature). *Stability is the critical property enabling sunspots in deterministic dynamical systems. For example, recall the neoclassical growth model, in which capital and consumption are the state and co-state variables, respectively, and only one value of initial consumption is consistent with a non-explosive equilibrium. By contrast, OLG versions of the growth model can feature a stable steady state, to which many alternative values of initial consumption would converge (Azariadis, 1981; Cass and Shell, 1983). This literature generates stochastic sunspot equilibria by basically randomizing over the multiplicity of transition paths.*

S-BSEs will also feature a type of stability, whereby for a fixed initial wealth distribution η_0 , many initial values of the co-state q_0 can be consistent with non-explosive behavior. But the analogy to deterministic models breaks down in an important sense: Online Appendix D.3

²²Our model also possesses a negative price-volatility link—see equation (20) below.

shows that the deterministic steady state of our class of models is only saddle-path stable. In other words, in the deterministic equilibrium of our model, q is pinned down to be a function of η . Given η_0 , there is a single transition path to steady state, so we cannot obtain volatility by randomizing over a multiplicity of deterministic transition paths. For the same reason, we cannot hard-wire arbitrary amounts of volatility for any combination (η, q) . Rather, as will soon be clear, our model uniquely determines return volatility $|\sigma_R|$ for each (η, q) , reminiscent of the endogenously-determined sentiment distribution in [Benhabib et al. \(2015\)](#).

2.1 Construction of S-BSEs

Now, we provide a sketch of an explicit construction of an S-BSE. Remember the goal from Lemma 1: we want to find a process $(\eta_t, q_t, \kappa_t, r_t)_{t \geq 0}$ satisfying equations (PO), (RB), (11), and (13)-(14) for all $t \geq 0$ and such that $\eta_t, \kappa_t \in [0, 1]$.

First, let us count the number of equations and unknowns. The equations are (PO), (RB), (11), (13), and (14)—these are 6 equations (recall that (14) involves two equations) that hold at each time t . Given η_t at a particular point in time, the unknowns are the wealth share dynamics (μ_η, σ_η) , the level and dynamics of capital prices (q, μ_q, σ_q) , the capital share κ , and the interest rate r —these are 9 unknowns (recall σ_η and σ_q are 2-by-1 vectors). Thus, we seem to have 3 degrees of freedom in constructing equilibrium. A Fundamental Equilibrium, universally studied by the literature, additionally imposes that equilibria be Markov in η . Such a Markovian restriction eliminates the 3 degrees of freedom: applying Itô's formula to $q(\eta)$ delivers 3 additional conditions for σ_q and μ_q . But in an S-BSE, q_t is not simply a function of η_t , so the 3 Itô conditions are dropped. Instead, (σ_q, μ_q) are determined by coordination.

The specific construction we outline below has the property that all equilibrium objects are functions of (η_t, q_t) . We are using one degree of freedom in making q a “state variable” in the equilibrium. It will turn out that the relevant domain for (η, q) is

$$\mathcal{D} := \{(\eta, q) : 0 < \eta < 1, q^L(\eta) < q \leq q^H(\eta)\}, \quad (16)$$

where

$$\begin{aligned} q^H(\eta) &:= a_e / \bar{\rho}(\eta) \\ q^L(\eta) &:= [\eta a_e + (1 - \eta) a_h] / \bar{\rho}(\eta). \end{aligned}$$

From the price-output relation (PO), notices that q^H corresponds to the capital price when $\kappa = 1$, whereas q^L corresponds to the capital price when $\kappa = \eta$. Equilibrium must

have $\kappa \leq 1$ (Lemma 1) and $\kappa > \eta$, the latter because a solution to equation (RB) will not exist otherwise. These restrictions are captured by ensuring (η, q) remains in \mathcal{D} .

The first step in the construction is to reduce the system. Imagine we know the values of $(\eta, q, \sigma_q, \mu_q)$. Price-output relation (PO) determines κ as a function of (η, q) and nothing else, given by

$$\kappa(\eta, q) := \frac{q\bar{p}(\eta) - a_h}{a_e - a_h}. \quad (17)$$

Substituting this result for κ , equation (11) then fully determines r . Equations (13)-(14), after plugging in the result for κ , fully determine (σ_η, μ_η) . At this point, given (η, q) , the remaining unknowns are (σ_q, μ_q) and the remaining equation is (RB).

When capital is efficiently allocated (i.e., $\kappa = 1$), we have $q = q^H(\eta)$ as an explicit function of η . Hence, both σ_q and μ_q are determined by Itô's formula. But when $q < q^H(\eta)$ (i.e., $\kappa < 1$), we have much more flexibility. Equation (RB) requires

$$|\sigma_R| = \sqrt{\frac{\eta(1-\eta)}{\kappa(\eta, q) - \eta} \frac{a_e - a_h}{q}}, \quad \text{if } q < q^H(\eta). \quad (18)$$

In other words, given (η, q) , the level of return volatility is pinned down. But notice that this only restricts the norm of $\sigma_q = \sigma_R - \sigma(\frac{1}{0})$, not each of its components separately. We will revisit this indeterminacy in the components of σ_q below.

Similarly, there is as yet no restriction on μ_q despite using all the equilibrium equations. All that remains is to show that $(\eta_t, q_t)_{t \geq 0}$ remains in \mathcal{D} almost-surely, and this will provide some mild restrictions on μ_q . The importance of proving that $(\eta_t, q_t)_{t \geq 0}$ remains in \mathcal{D} is to ensure that no optimality or market clearing conditions are violated along the proposed equilibrium path. For example, equation (18) is only well-defined for $\kappa_t > \eta_t$, or equivalently $q_t > q^L(\eta_t)$. Also, Lemma 1 requires $\kappa_t \leq 1$ and $\eta_t \in [0, 1]$, which only hold on \mathcal{D} .

The idea is that, when $q < q^H(\eta)$, we can set μ_q to any C^1 function with some boundary conditions that prevent q from hitting $q^L(\eta)$. For example, it turns out that a condition like the following, which says that the drift of q diverges fast enough, is sufficient:

$$\inf_{\eta \in (0,1)} \lim_{q \searrow q^L(\eta)} [q - q^L(\eta)] \mu_q(\eta, q) = +\infty. \quad (19)$$

If a condition like (19) did not hold, shocks to the capital price could potentially push the economy into a region that would constitute a violation of some of the equilibrium

conditions. This lower boundary, in particular, has the technical issue that σ_q in (18) explodes near it, so the formal proof in Appendix B.4 actually imposes a slightly stronger condition whereby μ_q diverges slightly above $q^L(\eta)$. The conditions at the upper boundary $q^H(\eta)$ are slightly more complicated because the economy is actually allowed to visit this upper boundary—these technical details are all addressed in Appendix B.4. The important takeaway is that equilibrium only imposes boundary conditions on μ_q and leaves it indeterminate in the interior of \mathcal{D} .

Methodologically, our formal proof employs stochastic stability theory to show that this construction yields a non-degenerate stationary distribution for $(\eta_t, q_t)_{t \geq 0}$. Appendix B.5 states and proves the appropriate version of a stochastic stability lemma that we use. In particular, the key object is the infinitesimal generator \mathcal{L} of the joint process $(\eta_t, q_t)_{t \geq 0}$ induced by equilibrium. And the key task is to find a positive function v , which diverges at the boundaries of \mathcal{D} , such that $\mathcal{L}v \rightarrow -\infty$ at the boundaries of \mathcal{D} . This mathematical condition exactly captures the intuition that boundary conditions on the dynamics are sufficient for stationarity.

Theorem 1 (Existence). *Let Assumption 1 hold. Then, there exists an S-BSE in which $(\eta_t, q_t)_{t \geq 0}$ remains in \mathcal{D} almost-surely and possesses a non-degenerate stationary distribution.*

Theorem 1 is formally proved in Appendix B.4 with an explicit S-BSE construction that addresses several of the minor technical issues raised in the preceding discussion.

Figure 2 plots the admissible set of η and q , along with return volatility $|\sigma_R|$ (indicated by shading) at each point in the space \mathcal{D} . For reference, we also place the W-BSE (which has $\sigma = 0$) and a Fundamental Equilibrium (with $\sigma = 0.1$). These equilibria attain only 10-20% volatility, a tiny amount relative to what S-BSEs can do. In fact, we have the following formal result.

Corollary 1 (Volatility indeterminacy). *Given wealth share $\eta \in (0, 1)$, let $\mathcal{Q}(\eta)$ denote the set of possible S-BSE values of q , and let $\mathcal{V}(\eta)$ denote the associated set of possible S-BSE values of return variance $|\sigma_R(\eta, q)|^2$. Then, $\mathcal{Q}(\eta) = (q^L(\eta), q^H(\eta))$, and the set $\mathcal{V}(\eta)$ has*

$$\begin{aligned} \inf \mathcal{V}(\eta) &= \min \left[\eta \bar{\rho}(\eta) \frac{a_e - a_h}{a_e}, \sigma^2 (\bar{\rho}(\eta) / \rho_e)^2 \right] \\ \sup \mathcal{V}(\eta) &= +\infty. \end{aligned}$$

Finally, an S-BSE can be constructed such that, in the stationary distribution, positive probability is placed on all elements of $\mathcal{Q}(\eta)$.

In an S-BSE, return variance $|\sigma_R|^2$ is pinned down once we know both η and q together; see equation (18). But q can take any value in the interval $\mathcal{Q}(\eta)$ for each η , which

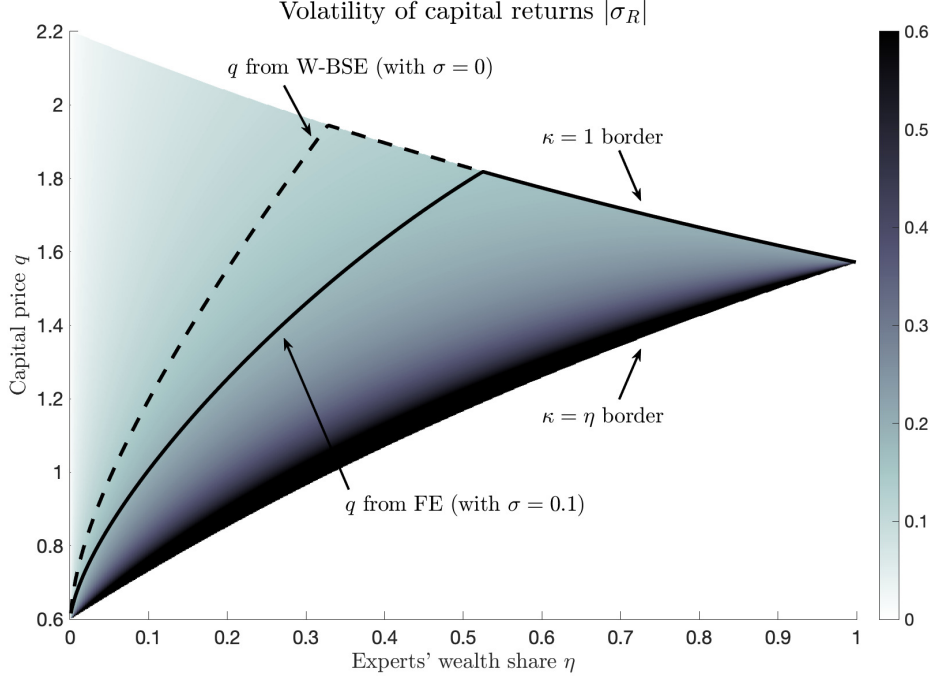


Figure 2: Colormap of volatility $|\sigma_R|$ as a function of (η, q) , in the region $\mathcal{D} := \{(\eta, q) : \eta \in (0, 1) \text{ and } \eta a_e + (1 - \eta)a_h < q\bar{\rho}(\eta) \leq a_e\}$. Volatility is truncated for aesthetic purposes (because $|\sigma_R| \rightarrow \infty$ as $\kappa \rightarrow \eta$). For reference, also included are the W-BSE with $\sigma = 0$ and the Fundamental Equilibrium (FE) with $\sigma = 0.1$. Parameters: $\rho_e = 0.07$, $\rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$.

implicitly defines a set $\mathcal{V}(\eta)$ of values for $|\sigma_R|^2$. Corollary 1 shows that the range of possible return volatilities is large, in fact unbounded above.

2.2 Economic intuition behind S-BSEs

Next, we explain our S-BSEs more intuitively. We first offer an interpretation of our equilibrium as driven by *uncertainty shocks*. Then, we take a dynamical-system perspective to understand why self-fulfilling volatility is possible.

Uncertainty shocks. Given a wealth distribution η and a level of return volatility $|\sigma_R|$, the capital market is equilibrated at each time via the risk-balance condition (RB) and the price-output relation (PO), restated here for convenience:

$$0 = \min \left[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1 - \eta)} |\sigma_R|^2 \right] \quad (\text{RB})$$

$$q\bar{\rho} = \kappa a_e + (1 - \kappa)a_h. \quad (\text{PO})$$

The left panel of Figure 3 shows how the intersection of these two curves determines the

capital allocation κ and the capital price q . The downward-sloping risk-balance (RB) can be thought of as experts' relative capital demand: for a fixed level of wealth η and return volatility $|\sigma_R|$, experts will only hold more capital if it is cheaper, thereby offering a higher expected return. The upward-sloping price-output (PO) is a capital supply curve: experts' capital provision raises allocative efficiency and capital valuations.

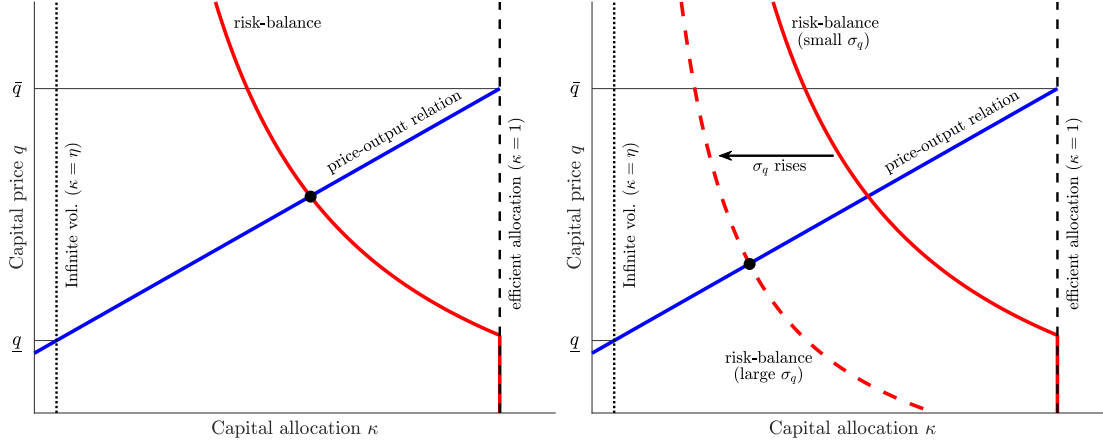


Figure 3: An uncertainty shock. Both panels plot the risk-balance condition (RB) and price-output relation (PO) for a fixed level of $\eta = 0.2$. The horizontal lines labeled \bar{q} and \underline{q} refer to maximal and minimal possible values of the capital price, respectively, corresponding to an efficient capital allocation ($\kappa = 1$) and an infinite-volatility allocation ($\kappa = \eta$). *Left panel*: equilibrium with $|\sigma_R| = 0.13$. *Right panel*: equilibrium after a shift to $|\sigma_R| = 0.20$. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, and $\sigma = 0.10$.

But whereas η is a state variable that can be rightly treated as fixed in this static sense, return volatility $|\sigma_R|$ is not. The right panel of Figure 3 shows what changes if there is a sudden rise in *fear*, manifested as higher perceived volatility $|\sigma_R|$. Experts, being risk-averse, are less willing to hold capital when volatility is high. This is illustrated as a leftward shift in the risk-balance curve from the solid to the dashed line. After this “fire sale,” capital is allocated less efficiently, and asset prices are lower.

So far, nothing rules out this arbitrary rise in fear, and $|\sigma_R|$ appears indeterminate. Mathematically, fixing η , equations (RB) and (PO) constitute two equations in the three unknowns $(\kappa, q, |\sigma_R|)$. The indeterminacy in $|\sigma_R|$ translates into an indeterminacy in q , which can be seen by combining (RB) and (PO) to eliminate κ and obtain the negative price-variance association:

$$|\sigma_R|^2 = \frac{\eta(1-\eta)(a_e - a_h)^2}{q\bar{\rho}(\eta) - \eta a_e - (1-\eta)a_h} \frac{1}{q} \quad \text{when } \kappa < 1. \quad (20)$$

In our construction leading up to Theorem 1, we treated (η, q) as state variables and determined all other equilibrium objects as functions of (η, q) . The preceding story about fear suggests that one can also think of S-BSEs as being driven by uncertainty

shocks—time-varying beliefs about volatility $|\sigma_R|$ —an interpretation which is supported by the one-to-one mapping between q and $|\sigma_R|$ in equation (20).

Bounce-back beliefs and dynamic stability. Based on the short-run conditions (RB) and (PO), equilibrium seems to support a multiplicity of prices q for a fixed η . To understand the long-run beliefs that sustain this multiplicity, it will be helpful to take a dynamical-systems perspective, as suggested in Remark 1.

Let us think of $(\eta_t, q_t)_{t \geq 0}$ as a (stochastic) dynamical system. As in deterministic dynamical systems, a pair (η_t, q_t) will only be an equilibrium if it does not lead to explosive paths. Thus, beliefs must be such that (η_t, q_t) will mean-revert, or bounce back, from extreme states. What does this entail?

To fix ideas, consider the following explosive path. Suppose a fear shock raises volatility $|\sigma_q|$ and lowers asset prices q . Under higher volatility, any subsequent fear shocks would have a larger direct impact on q , further raise volatility $|\sigma_q|$, and so on, ad infinitum. Thus, with enough such fear shocks, we will have $q \searrow q^L(\eta)$ and $|\sigma_q| \nearrow +\infty$ (see equation (RB) and take $\kappa \rightarrow \eta$).

For this fear-driven path to be an equilibrium, agents must believe that, at least eventually, q will recover and $|\sigma_q|$ will fall.²³ In other words, agents must believe μ_q will increase enough to buoy q if prices ever fall too low. This is an example of what we term *bounce-back beliefs*.

Bounce-back beliefs can be justified, because μ_q is not pinned down by any other equilibrium considerations. Importantly, optimal capital holdings are a function of the *risk premium*. This is clearly visible in the optimal portfolio FOCs (7)-(8), where only the spread $\mu_q - r$ appears. Consequently, even given a price q and diffusion σ_q , only the spread $\mu_q - r$ is pinned down in equilibrium, as equation (11) shows; μ_q and r are not separately determined.

Translating agents' bounce-back beliefs into specific mathematical conditions on μ_q is straightforward. Because $(\eta_t, q_t)_{t \geq 0}$ evolves in a diffusive fashion, stability criteria conveniently boil down to boundary behavior of the dynamical system. Imposing conditions on μ_q at the boundaries of the domain \mathcal{D} (i.e., the triangle in Figure 2) is thus sufficient to ensure a stochastically stable system. For example, we can impose that $\mu_q \rightarrow +\infty$ if q falls too low, and $\mu_q \rightarrow -\infty$ if q rises too high.

In a sense, the mean-reversion embedded in bounce-back beliefs is precisely the mechanism of self-fulfillment in our model. Fear can push asset prices very low pre-

²³Indeed, with unbounded volatility, there is an imminent violation of the equilibrium conditions. For example, a negative shock would push q below $q^L(\eta)$, and (RB) could not hold. Rational forward-looking agents would rule this out at the beginning, and the entire fear-driven sequence would not be admissible.

cisely because a recovery is expected. Prices can rise in a sentiment-driven boom precisely because agents know the boom will eventually subside.

2.3 The three indeterminacies

Recall that there are three indeterminacies in S-BSEs:

- (i) The level of volatility $|\sigma_R|$ is only pinned down by (η, q) but not by η alone;
- (ii) The two components of σ_q are indeterminate given (η, q) ;
- (iii) The drift μ_q is indeterminate given (η, q) , except at the boundaries of \mathcal{D} .

The first indeterminacy was covered by Corollary 1. These second and third indeterminacies are formalized and explained in the next two corollaries.

Corollary 2 (Decoupling). *The economy can be arbitrarily coupled or decoupled from fundamentals in the following sense. Let $\gamma(\eta, q) \in [0, 1]$ be any C^1 function. An equilibrium exists such that when $\kappa < 1$, a fraction $\gamma(\eta, q)$ of return variance $|\sigma_R|^2$ is due to the fundamental shock.*

S-BSEs do not pin down the fraction of volatility stemming from the fundamental and sunspot shocks, $Z^{(1)}$ and $Z^{(2)}$, respectively. The reason: when trading, agents only care about total return variance, not its source. Mathematically, the return volatility $|\sigma_R|$ is pinned down in (18), but σ_R itself has two components that can make indeterminate contributions to equilibrium. Asset prices and economic activity can be either closely linked to fundamentals, or completely decoupled from them, and this decoupling can be time-varying in arbitrary ways. Nevertheless, Section 3 presents perhaps the most natural example of an S-BSE, in which volatility and fundamentals must decouple as total volatility rises.²⁴

The theoretical possibility that $\gamma = 1$ in Corollary 2 helps illustrate that our multiplicity does not require any extrinsic force. Even with $Z^{(2)}$ playing no role, it is possible for agents to coordinate purely on endogenous objects in a self-fulfilling way.

Corollary 3 (Drift indeterminacy). *The economy can feature any degree of persistence or transience in the following sense. Let $m(\eta, q)$ be any C^1 function. An equilibrium exists with $\mathbb{P}[\mu_{q,t} = m(\eta_t, q_t) \mid \kappa_t < 1]$ arbitrarily close to one. Furthermore, the inefficiency probability $\mathbb{P}[\kappa_t < 1]$ can take any value between zero and one.*

²⁴We implement the example next section with a sentiment state variable s_t whose innovations depend only on the sunspot shock $Z^{(2)}$, which we think of as a natural case. More generally, letting ϕ be the (fixed) correlation between ds and $dZ^{(1)}$, one can show that as volatility rises ($|\sigma_R| \rightarrow \infty$), the fraction of return variance attributable to the fundamental shock approaches ϕ^2 .

As suggested earlier, the proof of Theorem 1 only imposes certain boundary conditions on μ_q , which allows almost any behavior in the interior of the state space. For example, asset prices could almost always behave like a random walk (corresponding to $\mu_q \approx 0$ in the interior), with just enough mean-reversion in extreme states to keep things stationary; in such a design, extreme states become arbitrarily close to reflecting boundaries. Alternatively, fluctuations could be much more transitory in nature. In Section 3, we harness the indeterminacy in μ_q to address predictability of busts and speed of recovery.

Remark 2 (Dynamics and indeterminacies). *Indeterminacies arise because beliefs about capital price dynamics influence real outcomes such as capital allocation. In this model we have two prices—capital price q and interest rate r —and two (non-redundant) market clearing conditions. However, we need to solve not only for current prices but also for future capital price behavior, which is summarized by the diffusion $\sigma_q \in \mathbb{R}^2$ and drift $\mu_q \in \mathbb{R}$ terms.²⁵ Optimality imposes a tight (negative) link between q and $|\sigma_q|$, while long-run stability imposes some mild conditions on μ_q in extreme states. Besides those restrictions, (σ_q, μ_q) are indeterminate. We map these three indeterminacies to total capital return volatility (Corollary 1), the source of volatility (Corollary 2), and the persistence or transience of sentiment fluctuations (Corollary 3).*

2.4 Manipulating beliefs with policy

At this point, it should be clear that beliefs about the future behavior of asset prices are indeterminate in our model. Here, we ask: assuming policymakers can manipulate these beliefs, in what way could outcomes improve? Future research might investigate how policies affect beliefs, which policies are most effective in doing so, and which types of commitment devices are required.

For simplicity, we consider a policymaker that pledges to support asset prices at some lower level $\underline{q}(\eta)$. One could think about policy pledges to make future asset purchases, because of the intuitive idea that asset purchases introduce demand pressure that pushes up prices. To be concrete, suppose agents perceive $\underline{q}(\eta)$ as a *reflecting boundary* for asset prices, i.e., beliefs are such that $q_t \geq \underline{q}(\eta_t)$. One can interpret this as sufficient credibility attached to the policymaker’s ability to affect asset prices. By rational expectations,

²⁵The logic in a discrete time model is analogous: the indeterminacies will be associated to the distribution of capital price tomorrow. This distribution is an infinite dimensional object, though it is usually the case that only a some moments are relevant for equilibrium conditions, e.g. mean and variance. The degrees of freedom will be associated to those moments. Nevertheless, it is more challenging to prove the existence of our sentiment equilibria in discrete time models because the dynamic stability requirement does not translate into tractable boundary conditions.

prices will in fact always obey this lower bound, but no intervention need occur. Instead, the policy promise induces self-fulfilling dynamics: agents believe prices will be reflected at $\underline{q}(\eta)$ and trade capital to make it so. In this sense, the reflecting boundary is an extreme case of the bounce-back beliefs described earlier.

Reflection introduces a new term to price dynamics:

$$dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t} \cdot dZ_t + dP_t],$$

where P is the barrier process that increases only to keep $q_t \geq \underline{q}(\eta_t)$. Absence of arbitrage requires the riskless bond return to be $r_t dt + dP_t$, such that the excess return on capital is unaffected by dP_t (c.f., Appendix B of [Karatzas and Shreve, 1998](#)). Consequently, agents' FOCs on capital holding remain unaffected, and both the risk-balance condition (RB) and equation (11) for r_t still hold.

Finally, the policy has no impact on the dynamics of η_t , which still take the diffusive form (12). Indeed, excess capital returns feature no dP_t component, so expert and household return-on-wealth contain identical contributions from dP_t , implying $d\eta_t$ contains no dP_t term. This is a clear indication that our policy has no “fundamental impact.” [He and Krishnamurthy \(2013\)](#), by contrast, analyze policies with only fundamental effects and no belief effects (they study borrowing subsidies, asset purchases, and equity injections). For them, policy effects on wealth distribution dynamics are critical.

We have thus constructed an equilibrium with $q_t \geq \underline{q}(\eta_t)$ at all times, for an arbitrarily designed lower boundary \underline{q} . For example, $\underline{q}(\eta)$ could be designed to keep volatility below some threshold, e.g., $|\sigma_R| \leq \Sigma^*$, and volatility would sometimes approach but never exceed that threshold. Away from the boundary $\underline{q}(\eta)$, equilibrium is identical to the one constructed in Theorem 1; this property is an artifact of log utility, for which only the local dynamics of asset prices matter. Policies that truncate the lower tail of asset prices are clearly helpful, but with log utility the truncation is the entirety of their impact.²⁶ With more general utility functions, how much can policy promises removing tomorrow's tail risk affect today's asset price dynamics by “calming the market?” We leave this question for future research.

²⁶Given the freedom to manipulate beliefs costlessly, an optimal policy would be much more aggressive in keeping κ as close to 1 as possible and volatility $|\sigma_R|$ as low as possible. Of course, perhaps it is costly to manipulate beliefs (or impossible to be credible) with very frequent interventions.

3 Resolving puzzles with sentiment

We have just demonstrated that sunspot equilibria, which are endemic to this class of models, in principle can support rich dynamics. Now, we solve some concrete examples to illustrate several substantive results along these lines.

3.1 Explicit construction with a sentiment state variable

In contrast to the previous section, where (η, q) was the state variable, here we implement our sentiment-driven equilibria with an auxiliary state variable s and with q as a function of η and s . Being explicit about a sentiment state variable is useful for several reasons. First, the Markov equilibrium construction will be pedagogically more familiar to the literature on sunspots. Second, the sentiment state dynamics can be modeled as locally uncorrelated with fundamental shocks, which brings some clarity. Third, this setting happens to facilitate building sunspot equilibria in which experts fully de-lever as their wealth shrinks, i.e., $\kappa \rightarrow 0$ as $\eta \rightarrow 0$, for which there are natural justifications.

Let s be a pure sunspot that is irrelevant to economic fundamentals and loads on only the second shock (recall $Z^{(1)}$ affects capital and $Z^{(2)}$ does not):²⁷

$$ds_t = \mu_{s,t}dt + \sigma_{s,t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot dZ_t, \quad s_t \in \mathcal{S}. \quad (21)$$

We will also find some use in introducing auxiliary state variables that can (only) affect the drift $\mu_{s,t}$. This is possible to do in a very flexible way, due to the drift indeterminacy result of Corollary 3. Let $x_t \in \mathcal{X}$ be an arbitrary bounded diffusion (perhaps allowing reflections at the boundaries of the state space \mathcal{X}),

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t) \cdot dZ_t,$$

which affects the sentiment drift, through $\mu_{s,t} = \mu_s(\eta_t, s_t, x_t)$.

Definition 3. A Markov S-BSE in states $(\eta, s, x) \in (0, 1) \times \mathcal{S} \times \mathcal{X}$ consists of functions $(q, \kappa, r, \sigma_\eta, \mu_\eta, \sigma_s) : (0, 1) \times \mathcal{S} \mapsto \mathbb{R}$, and $\mu_s : (0, 1) \times \mathcal{S} \times \mathcal{X} \mapsto \mathbb{R}$, all C^2 almost-everywhere, such that the process $(\eta_t, q(\eta_t, s_t), \kappa(\eta_t, s_t), r(\eta_t, s_t))_{t \geq 0}$ is an S-BSE.

²⁷This is a natural choice that also brings clarity. We have also solved examples with sentiment correlated to fundamentals, i.e., with $ds = \mu_s dt + \sigma_s^{(1)} dZ^{(1)} + \sigma_s^{(2)} dZ^{(2)}$. An additional feature that emerges relative to what we show here is that $\sigma_s^{(1)}$ can work to reduce asset price volatility at times, unlike $\sigma_s^{(2)}$. See Online Appendix D.6 for details.

Remark 3 (Endogenous sentiment dynamics). *Note that the statement of Definition 3 allows (σ_s, μ_s) to be endogenous, in the sense that they could depend on the wealth distribution η . Our examples in this section purposefully entertain this endogeneity, partly because we think of this as the more interesting and realistic situation. Why? As shown in Section 2, dynamics depend explicitly on q in an S-BSE. Thus, it is completely sensible for agents in our S-BSEs to use asset prices directly in forecasting; in particular, sentiment dynamics (σ_s, μ_s) —which are nothing but belief dynamics—themselves should condition on q . But q will depend on both s and η , implying sentiment dynamics (σ_s, μ_s) depend on η too, through q . That said, Online Appendix D.7 verifies that similar types of sunspot equilibria can be constructed with exogenous sentiment dynamics, i.e., (σ_s, μ_s) are only functions of s , not η .²⁸*

The Markov assumption in Definition 3 allows us to specialize equilibrium conditions. By applying Itô’s formula to $q(\eta, s)$, we obtain the capital price volatility σ_q in terms of σ_η , namely

$$q\sigma_q = \sigma_\eta \partial_\eta q + \sigma_s \partial_s q.$$

From equation (14), we also have σ_η in terms of σ_q . Solving this two-way feedback, we obtain

$$\sigma_q = \frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix}(\kappa - \eta)\sigma \partial_\eta \log q + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sigma_s \partial_s \log q}{1 - (\kappa - \eta)\partial_\eta \log q}. \quad (22)$$

Using (22) in (RB), we obtain the following equation linking capital prices, the capital distribution, and sentiment volatility:

$$0 = \min \left[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1 - \eta)} \left(\frac{\sigma^2 + (\sigma_s \partial_s \log q)^2}{(1 - (\kappa - \eta)\partial_\eta \log q)^2} \right) \right]. \quad (23)$$

Our strategy to find a Markov S-BSE is to guess a capital price function $q(\eta, s)$ and then use equation (23) to “back out” the sunspot volatility σ_s that justifies it. We will perform a construction such that sunspots only increase volatility relative to the fundamental equilibrium, to highlight their potential for resolving puzzles. For this reason, we sometimes refer to s as *rational pessimism*.

²⁸Because it may be more natural to think that agents coordinate on variables that have direct real effects, Online Appendix D.7 also entertains the situation that s impacts fundamental volatility σ . In this environment, we illustrate two main results. First, depending on coordination on s , equilibrium can either behave more “intuitively” or less. Intuition usually says that prices move inversely to fundamental volatility, but coordination can easily reverse this relationship. Second, as the contribution of s to fundamental volatility becomes small, equilibrium can converge to one where s still matters, i.e., a sunspot equilibrium. Thus, sunspot equilibria can be thought of as limits of fundamental equilibria, with a vanishingly-small real contribution of s .

More specifically, suppose a fundamental equilibrium, where sunspots do not matter, exists with equilibrium capital price q^{FE} (see Online Appendix E for details on the fundamental equilibria). We will think of q^{FE} as the “best-case” capital price, because despite featuring amplification, q^{FE} inherits no sunspot volatility. Conversely, think of the capital price q^∞ associated to an infinite-volatility equilibrium as the “worst-case” capital price (substitute $|\sigma_R| \rightarrow \infty$ into (20) to find $q^\infty := \frac{\eta a_c + (1-\eta)a_h}{\bar{\rho}}$).

Our strategy is essentially to treat the sentiment variable s as a device to shift continuously between the best-case q^{FE} and the worst-case q^∞ . Mathematically, we conjecture a capital price that is approximately a weighted average of q^{FE} and q^∞ , with weights $1 - s$ and s .²⁹ The novelty of our approach here is to then use equation (23) to solve for sunspot volatility σ_s , which will generically depend on experts’ wealth share η . In terms of Figure 2, the economy will live in the sub-region bounded by the solid FE line and the $\kappa = \eta$ border (and notice this implies the full-deleveraging condition $\kappa \rightarrow 0$ as $\eta \rightarrow 0$). In the proposition below, we verify that such a construction is indeed an equilibrium.

Proposition 1. *Let Assumption 1 hold, and assume a fundamental equilibrium exists for each $\sigma \geq 0$ small enough. Then, for all $\sigma \geq 0$ small enough, there exists a Markov S-BSE with capital prices arbitrarily close to $(1 - s)q^{FE}(\eta) + sq^\infty(\eta)$. In this equilibrium, μ_s is indeterminate except near the boundaries of $(0, 1) \times \mathcal{X} \times \mathcal{S}$.*

We construct a numerical example closely following Proposition 1, which we will use in subsequent sections. Online Appendix C.3 provides details. The left panel of Figure 4 shows the capital price function. A rise in rational pessimism s reduces the capital price q , independently of wealth share η (although η will also endogenously respond to s -shocks).

The middle panel of Figure 4 displays capital return volatility, which can be substantially greater than in the fundamental equilibrium. Implied by capital return volatility is an underlying sunspot shock size σ_s , which is displayed in the right panel of Figure 4. Sunspot dynamics become more volatile both as experts become poor (η shrinks) and as the economy approaches the worst-case equilibrium (s rises). The dependence of σ_s on η is the notion of endogenous beliefs that can occur in S-BSEs.

²⁹In this particular equilibrium, capital prices can never literally achieve the “worst-case” capital price q^∞ , for two technical reasons, both of which ensure that sunspot volatility stays σ_s bounded: (i) to ensure $\kappa(0, s) = 0$, we need $q(\eta, s)$ to behave like the fundamental solution $q^{FE}(\eta)$ for η close enough to zero, and all s ; (ii) we need $q(\eta, s) > q^\infty(\eta)$, so that $\kappa(\eta, s) > \eta$ for all (η, s) . Thus, in the proof of Proposition 1, we actually construct q^∞ as a close approximation to the worst-case price, such that (i) and (ii) are satisfied.

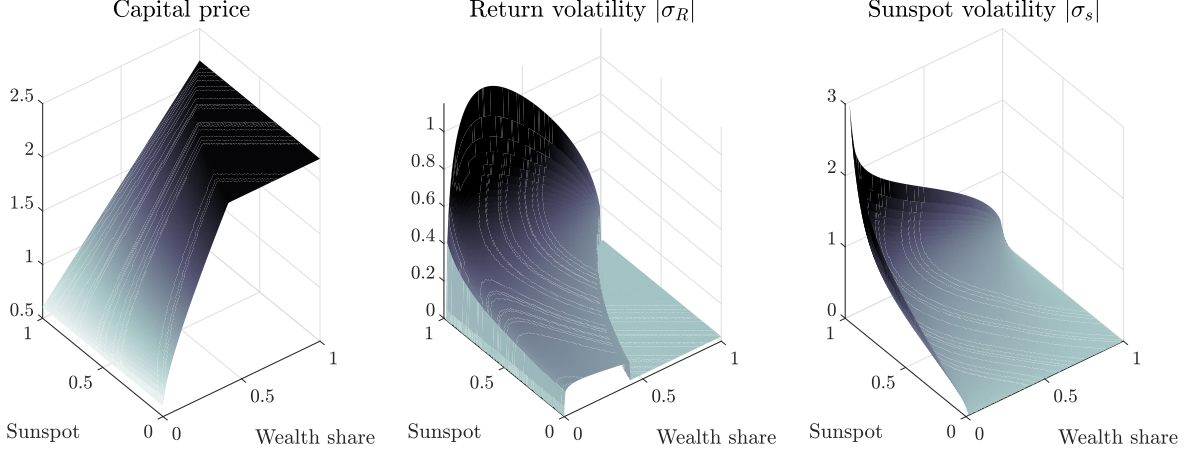


Figure 4: Capital price q , volatility of capital returns $|\sigma_R|$, and sunspot shock volatility σ_s . Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$.

3.2 Non-fundamental crises and large amplification

We now show how our model with sentiment shocks naturally resolves some empirical issues related to financial crises and recoveries.

First, Figure 5 compares impulse responses to a large negative balance-sheet shock (i.e., decline in η) versus a wave of pessimism (i.e., increase in s). The shock sizes are chosen so that the initial drop in capital price $q_0 - q_{0-}$ is roughly the same. “Balance-sheet recessions” (decline in η) feature a modest increase in volatility followed by relatively slow recoveries, as experts can only rebuild their balance sheets by earning profits over time. By contrast, “pessimism crises” (increase in s) feature large temporary volatility spikes and can have accelerated recoveries (depending on the choice of μ_s). The dynamics after a pessimism shock—both the rise in volatility and speed of recovery—are closer to empirical evidence.³⁰ Our results on recovery speeds are related to Maxted (2020), who shows how extrapolative beliefs can help this class of models match such evidence, but with our rational sentiment in place of his behavioral sentiment.

To establish some more confidence in these results, we present the following two propositions which together show that amplification can be arbitrarily high (Proposition 2) as long as sentiment shocks are the source (Proposition 3). Given the literature’s strug-

³⁰During the 2008 financial crisis and 2020 COVID-19 episode in the US, implied volatility from option markets spiked by magnitudes on the order of 60%. For a rough idea of what the data says about crisis recoveries, see Jordà et al. (2013) and Reinhart and Rogoff (2014) for output, and see Muir (2017) and Krishnamurthy and Muir (2017) for credit spreads and stock prices. Across these many measures, and using broad-based international panels, crisis recovery times tend to range from 4-6 years on average.

Of course, note that η responds to s -shocks, i.e., σ_η has a non-zero second component. Thus, a true sentiment-driven crisis features dynamics that are a blend of the two IRFs in Figure 5. Figure 5 shows a pure shock to s , without the endogenous co-movement in η , for theoretical clarity.

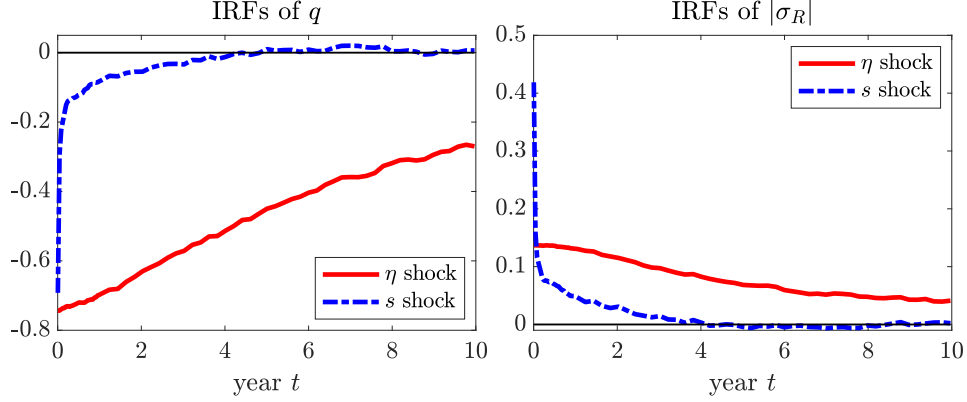


Figure 5: Bust IRFs of capital price q and return volatility $|\sigma_R|$. The IRFs labeled “ η shock” are responses to a decrease in η from $\eta_{0-} = 0.5$ to $\eta_0 = 0.2$, holding s_0 fixed at 0.1. The IRFs labeled “ s shock” are responses to an increase in s from $s_{0-} = 0.1$ to $s_0 = 0.9$, holding η_0 fixed at 0.5. These shock sizes are chosen such that the initial response of q are approximately equal. Note that η_0 would respond to an “ s shock,” since σ_η has a non-zero second element, but we keep it fixed here. IRFs are computed as averages across 500 simulations at daily frequency. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. Type-switching parameters: $\delta_h = 0.004$ and $\delta_e = 0.036$. In this example, we set the sunspot drift $\mu_s = 0.0002s^{-1.5} - 0.0002(s_{\max} - s)^{-1.5}$, where $s_{\max} = 0.95$. This choice ensures $s_t \in (0, s_{\max})$ with probability 1.

gle to identify a “smoking gun” (e.g., TFP shocks, capital efficiency shocks) for financial crises, we view this set of results as a helpful insight. The importance of sentiment s , relative to experts’ wealth share η , also echoes the empirical results suggesting financial crises are not associated with pre-crisis levels of bank capital (Jordà et al., 2021).

Proposition 2 (Arbitrary volatility). *Given a target variance $\Sigma^* > 0$ and any parameters satisfying the assumptions of Proposition 1, there exists a Markov S-BSE with stationary average return variance exceeding the target, i.e., $\mathbb{E}[|\sigma_R|^2] > \Sigma^*$.*

Proposition 3 (Volatility decoupling). *In the Markov S-BSEs of Proposition 1, both the fraction of return volatility due to sentiments $|\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \sigma_R| / |\sigma_R|$ and total return volatility $|\sigma_R|$ increase with s .*

3.3 Booms predict crises

We now use the same framework to cast light on empirical findings suggesting that financial crises are predictable, in particular by large credit and asset price booms (Reinhart and Rogoff, 2009; Jordà et al., 2011, 2013, 2015a,b; Mian et al., 2017) that feature below-average credit spreads (Krishnamurthy and Muir, 2017; López-Salido et al., 2017; Baron and Xiong, 2017).

To do this, we make use of the auxiliary variable x that can affect the sentiment drift. Following some models of extrapolative beliefs (Barberis et al., 2015; Maxted, 2020),

define an exponentially-declining weighted average of sentiment shocks:

$$x_t := x_0 + \sigma_x \int_0^t e^{-\beta_x(t-u)} dZ_u^{(2)}. \quad (24)$$

The variable x measures the stock of past pessimism. Assume the drift of s depends on x via

$$\mu_{s,t} = b_x x_t + \hat{\mu}_s(s_t) \quad \text{with} \quad b_x \leq 0.$$

Similar to Section 3.2, the term $\hat{\mu}_s$ will be designed to prevent non-stationarity in s_t . The new term $b_x x$ induces the following dynamics: after a wave of optimism ($dZ_t^{(2)} < 0$), s_t and x_t will be low, but this raises $\mu_{s,t}$ and shifts up the conditional distributions of future pessimism s_{t+h} . If the constant b_x is large enough, the shift can generate dynamics reminiscent of “overshooting,” in which an optimism-driven boom raises bust probabilities. Differently from the extrapolative belief literature, the beliefs implied by these sentiment dynamics are completely rational.

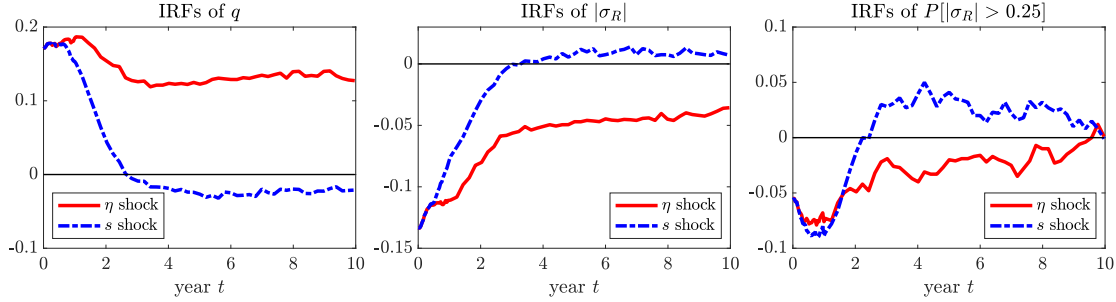


Figure 6: Boom IRFs of capital price q , return volatility $|\sigma_R|$, and the probability that $|\sigma_R| > 0.25$ at some point over future years. The IRFs labeled “ η shock” are responses to an increase in η from $\eta_{0-} = 0.5$ to $\eta_0 = 0.7$, holding s_0 fixed at 0.4. The IRFs labeled “ s shock” are responses to a decrease in s from $s_{0-} = 0.4$ to $s_0 = 0.1$, holding η_0 fixed at 0.5. These shock sizes are chosen such that the initial response of q are approximately equal. Note that η_0 would respond to an “ s shock,” since σ_η has a non-zero second element, but we keep it fixed here. IRFs are computed as averages across 1000 simulations at daily frequency. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. Type-switching parameters: $\delta_h = 0.004$ and $\delta_e = 0.036$. In this example, we set the sunspot drift $\mu_s = b_x x + 0.0002s^{-1.5} - 0.0002(s_{\max} - s)^{-1.5}$, where $s_{\max} = 0.95$, $b_x = -10$, $\beta_x = 0.1$, and $\sigma_x = 0.025$. The parameters (β_x, σ_x) are approximately the values used for the mean-reversion and volatility of the diagnostic belief process in [Maxted \(2020\)](#).

Figure 6 displays IRFs consistent with this overshooting logic. An optimism shock raises asset prices and lowers volatility for 2-3 years, but predicts mildly lower prices and higher volatility afterward. The right panel shows a tail risk measure, namely the probability that volatility exceeds 0.25 in the future, which emphasizes the overshooting logic is particularly salient for the extreme adverse outcomes. This is the sense in which a sentiment-driven boom has predictive power for a future bust and particularly a future crisis. By contrast, a boom driven by expert wealth counterfactually predicts high prices,

lower volatility, and lower fragility at all horizons.

To connect more directly to the empirical literature, we conduct a financial crisis event study in Figure 7. We simulate our model (which thus features contributions from both fundamental and sunspot shocks), identify crises in the simulated data, and plot average outcomes in the several years before and after crisis. Crises are identified using the worst 3rd percentile of monthly output drops, but any other tail outcome will produce similar graphs. We see that conditions are improving up to 2 years before the crisis, with risk premia below average and *declining*. The crisis emerges suddenly and features spikes in all variables. Although we do not report it here, such dynamics cannot be produced in the non-sunspot equilibria of the model.

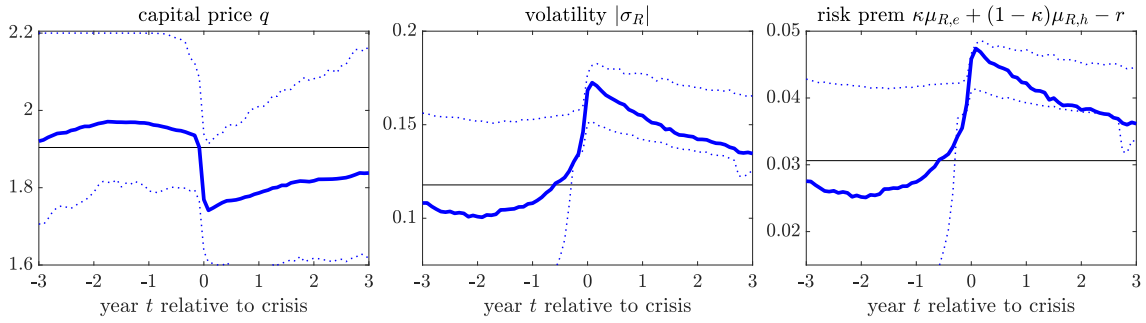


Figure 7: Event studies around financial crises. Crises are defined as the bottom 3rd percentile of month-to-month log output declines. Data is generated via a 10,000 year simulation at the daily frequency, with the outcomes above then averaged to the monthly level. The solid blue line is the mean path, and the dotted blue lines represent the 25th and 75th percentiles. The thin horizontal line represents the unconditional average. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. Type-switching parameters: $\delta_h = 0.004$ and $\delta_e = 0.036$. In this example, we set the sunspot drift $\mu_s = b_x x + 0.0002s^{-1.5} - 0.0002(s_{\max} - s)^{-1.5}$, where $s_{\max} = 0.95$, $b_x = -1$, $\beta_x = 0.1$, and $\sigma_x = 0.025$. The parameters (β_x, σ_x) are approximately the values used for the mean-reversion and volatility of the diagnostic belief process in [Maxted \(2020\)](#).

3.4 Sentiment-based jumps

In our final exercise, we show how similar substantive results—large and sudden crises that are preceded by booms featuring low volatility and risk premia—also hold in alternative equilibria with sentiment-based jumps. There are three reasons why jump-type fluctuations are an interesting avenue to explore vis á vis the puzzles in this literature. First, jumps are large and sudden by definition, helping resolve the trouble with limited amplification. Second, the larger jumps that characterize a financial crisis can only happen from a moderate or good state that characterizes a boom. Third, introducing jumps reveals an additional indeterminacy that can be useful in exacerbating the previous point, namely the jump probability can be rigged so that it is more likely in good times.

Consider a broader class of solutions for the baseline model where capital price can also respond to an extrinsic jump shock, i.e.,

$$\frac{dq_t}{q_{t-}} = \mu_{q,t-} dt + \sigma_{q,t-} \cdot dZ_t - \ell_{q,t-} dJ_t,$$

where J is a Poisson process with intensity λ . For simplicity, we restrict attention to equilibria where the jump size ℓ_q is pre-determined, in particular a function of (η, q) , and we focus on adverse jumps with $\ell_q \geq 0$.

We sketch the solution of a jumpy equilibrium (with more details in Online Appendix C.4). The risk-balance condition (RB) is modified to read

$$0 = \min \left[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1 - \eta)} \left(|\sigma_R|^2 + \frac{\lambda \ell_q^2}{\left(1 - \frac{\kappa}{\eta} \ell_q\right) \left(1 - \frac{1 - \kappa}{1 - \eta} \ell_q\right)} \right) \right]. \quad (\text{RBJ})$$

The additional terms involving ℓ_q arise because there is a jump risk premium. The price-output relation remains (PO).

By adding a new source of risk, we have an additional degree of freedom. The risk-balance condition disciplines overall risk—the term in parentheses of (RBJ) is pinned down given (η, q) —but the split between the Brownian and Poisson shocks is indeterminate. We have tremendous flexibility in our choice of ℓ_q .

It is easy to avoid stability concerns: just set $\ell_q = 0$ near the boundaries of the equilibrium region (i.e., the triangle in the (η, q) space in Figure 2). Doing this, the stability analysis remains unchanged from Theorem 1, since near the boundaries the economy behaves as if it is only hit by Brownian shocks.

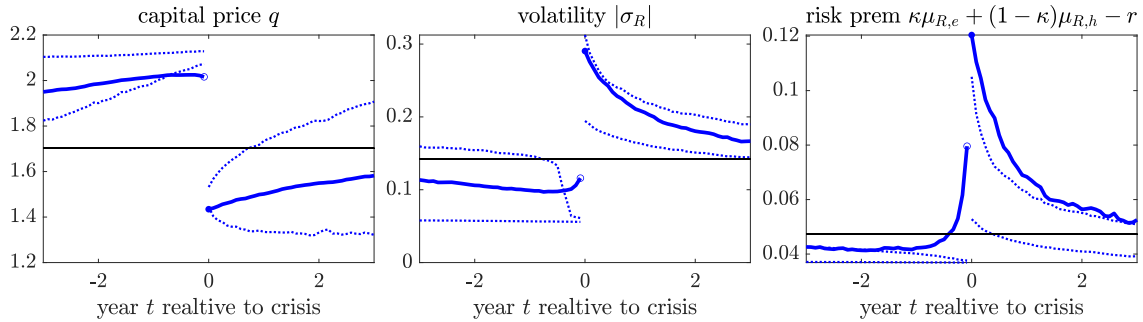


Figure 8: Event studies around financial crises in the jump-diffusion model. Crises are defined as the bottom 3rd percentile of month-to-month log output declines. Data is generated via a 100,000 year simulation at monthly frequency. The solid blue line is the mean, and dotted blue lines represent 25th and 75th percentiles. The horizontal black line is the unconditional mean. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. Type-switching parameters: $\delta_h = 0.004$ and $\delta_e = 0.036$. We reflect (η, q) near boundaries of $\mathcal{D} := \{(\eta, q) : 0 < \eta < 1 \text{ and } \eta a_e + (1 - \eta) a_h < q \bar{\rho}(\eta) \leq a_e\}$. Away from the boundaries, we set $\mu_q = 0.1(q^{\text{mid}}(\eta) - q)$, where q^{mid} corresponds to $\kappa(q^{\text{mid}}, \eta) = 0.8$.

Figure 8 shows a financial crisis event study from simulated data of the jump model. The solution and simulation method is described in detail in Online Appendix C.4. We make the following choice for jump sizes

$$\ell_q = \begin{cases} 0.95\ell_q^{\max}, & \text{if } \kappa > 0.9 \text{ and } 0.95\ell_q^{\max} > 0.2 \\ 0, & \text{otherwise,} \end{cases}$$

where ℓ_q^{\max} is the maximum allowable jump consistent with equilibrium (derived in the appendix). Thus, we focus attention on an economy with large jumps (greater than 20%) that are additionally only realized from efficient high- κ states.³¹

Because we focus on large jumps and only allow them in high- κ states, crises tend to arrive after a sequence of positive fundamental Brownian shocks. Accordingly, in the years before the crisis, asset prices are high, and both volatility and risk premia are below their usual level. Similar to Figure 7, volatility and risk premia tend to decline in the years prior to crisis. Crises arrive suddenly—with only a few months “warning” in terms of rising volatility and risk premia—and generate large movements in observables, because simulated crises often coincide with realizations of a jump.

4 Conclusion

We have shown that macroeconomic models with financial frictions may inherently permit sunspot volatility. The types of models we study are extremely common in macroeconomics, so this phenomenon cannot be ignored.

On the bright side, our paper demonstrates how a fully-rational notion of “sentiments” can be a powerful input into macro-finance dynamics. Time-varying uncertainty drives all dynamics in our sentiment-driven fluctuations. Sharp volatility spikes and belief-driven boom-bust cycles are among the many interesting possibilities raised by our framework. While ours is not a full-blown quantitative analysis, we aim to show that rational sentiment can help on these dimensions.

On the hazier side, our results suggest a modicum of caution. Many researchers employ numerical techniques to solve and analyze DSGE models that are built upon the core assumptions in our paper—these procedures implicitly select an equilibrium, without any explicit justification. In Online Appendix D, we have considered some simple

³¹In unreported results, we also solved an example without the $\kappa > 0.9$ restriction, i.e., where we set $\ell_q = 0.95\ell_q^{\max}\mathbf{1}_{\{0.95\ell_q^{\max} > 0.2\}}$. The results are similar to Figure 8—because large jumps still tend to happen from good states—but slightly muted.

refinements, based on small amounts of idiosyncratic risk and limited commitment, but these refinements only stipulate the full-deleveraging boundary condition $\lim_{\eta \rightarrow 0} \kappa = 0$, which barely trims indeterminacies. A deeper analysis of refinements, perhaps leveraging global-games approaches or adaptive learning, still remains to be done.

What about policy?³² Caveated by the need for further study on refinements, we can offer some initial thoughts. Some traditional policies become less effective in sunspot equilibria. For example, deposit insurance has less bite because run-like behavior can occur solely due to fire-sale coordination, i.e., on the asset side rather than the liability side. Sunspot equilibria also decouple financial crises from bank balance sheets and wealth, which defangs capital requirements, bailouts, and the like. On the other hand, policies that manipulate beliefs can be effective (Section 2.4 briefly investigates this possibility). Future research might better explain which policy designs have the power to manipulate beliefs in this way. Given the framework we study relies on fire sales, asset purchases (or future commitments to them) are one interesting candidate.

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³²Many studies in the recent literature have moved toward policy analysis (Phelan, 2016; Dávila and Korinek, 2018; Drechsler et al., 2018; Di Tella, 2019; Silva, 2017; Elenev et al., 2021; Begenau, 2020; Begenau and Landvoigt, 2021; Klimenko et al., 2016).

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Online Appendix:

Rational Sentiments and Financial Frictions

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A Wealth-driven BSEs

Universally, papers studying this class of models restrict attention to Markov equilibria in which η is the only state variable. This section illustrates how these equilibria, even if sunspot shocks can matter, are too restrictive for our purposes. We first present a W-BSE (Section A.1) and then argue it is inherently uninteresting (Sections A.2-A.3). Mathematical proofs of results in this section are at the end (Section A.4).

A.1 W-BSE: existence and properties

In this section, we study the version of the model without fundamental shocks, i.e., $\sigma = 0$. Hence, both shocks represent extrinsic uncertainty, and we dispense with $Z^{(1)}$ to simplify the exposition.³³ Without any intrinsic uncertainty, there always exists a deterministic Fundamental Equilibrium (FE).

Lemma A.1 (Fundamental Equilibrium). *If $\sigma = 0$, there exists an equilibrium in which experts manage all capital, $\kappa = 1$, and its price $q_t = a_e / \bar{\rho}(\eta_t)$ evolves deterministically.*

But there is also another equilibrium, a Wealth-driven Brownian Sunspot Equilibrium (W-BSE), which is Markov in η and has volatility. In this W-BSE, the capital price will depend only on η , i.e., $q_t = q(\eta_t)$ for some function q . By Itô's formula, we then have $\sigma_q = \frac{q'}{q} \sigma_\eta$. On the other hand, equations (9) and (14) with $\sigma = 0$ imply $\sigma_\eta = (\kappa - \eta) \sigma_q$. Solving this two-way feedback between σ_q and σ_η ,

$$\left[1 - (\kappa - \eta) \frac{q'}{q}\right] \sigma_q = 0. \quad (\text{A.1})$$

There are two possibilities: either (i) $\sigma_q = 0$, which corresponds to the FE of Lemma A.1; or (ii) $1 = (\kappa - \eta) \frac{q'}{q}$, in which case σ_q can be non-zero. We pursue the latter.

³³This also allows us to maintain consistency with Definition 2, which says that a W-BSE should have non-zero loading on the second shock.

Substituting $\kappa < 1$ from (PO), we obtain a first-order ODE for q :

$$q' = \frac{(a_e - a_h)q}{q\bar{\rho} - \eta a_e - (1 - \eta)a_h}, \quad \text{if } \kappa < 1. \quad (\text{A.2})$$

Consider boundary condition $\kappa(0) = 0$, which translates via (PO) to $q(0) = a_h/\rho_h$. The appendix justifies this choice of boundary condition, which says that experts fully de-lever as their wealth shrinks.³⁴ Then, ODE (A.2) is solved on the endogenous region $(0, \eta^*)$ where households manage some capital, i.e., $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$.³⁵ Given a solution for (q, κ) , the risk-balance equation (RB) yields capital price variance as

$$\sigma_q^2 = \frac{\eta(1 - \eta)}{\kappa - \eta} \frac{a_e - a_h}{q}, \quad \text{if } \kappa < 1. \quad (\text{A.3})$$

Since $\sigma_q \neq 0$ in (A.3), a W-BSE exists as long as a solution exists to ODE (A.2).

Proposition A.1 (W-BSE). *If $\sigma = 0$, there exists a W-BSE with $\kappa(0) = 0$, in which $\sigma_q(\eta) \neq 0$ on $(0, \eta^*)$ and $\sigma_q(\eta) = 0$ on $(\eta^*, 1)$.*

Figure A.1 displays a numerical example with the capital price q and volatility σ_q as functions of the expert wealth share. Notice that the equilibrium is stationary (the right panel of Figure A.1 plots the stationary CDF of η).³⁶ While the model of this section is solved under log utility, Online Appendix D.5 shows how W-BSEs can be obtained with more general CRRA preferences.

The intuition communicated by the W-BSE equations above is as follows. If agents believe the sunspot shock can affect asset prices, then the actual arrival of such a shock

³⁴We use the boundary condition $\kappa(0) = 0$ in accordance with the literature. In Online Appendix D.2, we show that this is not necessary in principle. There are actually a continuum of W-BSEs indexed by $\kappa_0 = \kappa(0) \in [0, 1]$, which one can think of as agents' "disaster belief", i.e., what happens in the worst-case scenario. Nevertheless, there are good reasons to select $\kappa_0 = 0$. First, as we show in Online Appendix D.3, if managing capital involves any amount of idiosyncratic risk, even if vanishingly-small, any equilibrium must feature $\kappa \rightarrow 0$ as $\eta \rightarrow 0$. Second, Online Appendix D.4 shows how adding any amount of limited commitment frictions, even if vanishingly-small, automatically restricts equilibrium to feature $\kappa \rightarrow 0$ as $\eta \rightarrow 0$. Despite this discussion, several numerical examples in the paper use κ_0 slightly above zero, to aid numerical stability. This approximation is not problematic, given we show in Proposition D.2 that equilibria are continuous in κ_0 , even at $\kappa_0 = 0$.

³⁵When $\rho_h = \rho_e$, there is a closed form solution for capital price

$$q(\eta) = \frac{1}{\rho} \left[(a_e - a_h)\eta + a_h + \sqrt{((a_e - a_h)\eta + a_h)^2 - a_h^2} \right], \quad \text{for } \eta < \eta^* = \frac{1}{2} \frac{a_e - a_h}{a_e}.$$

³⁶This economy possesses a stationary distribution on $(0, \eta^*]$ under mild parameter restrictions, for example if experts are more impatient than households ($\rho_e > \rho_h$) or the economy has a type-switching structure with sufficiently few experts ($\delta_e, \delta_h > 0$ and $\delta_h < \eta^*(\delta_h + \delta_e)$). Note that $\eta_t = \eta^*$ about 55% of the time in the numerical example of Figure A.1, i.e., there is a mass point at η^* .

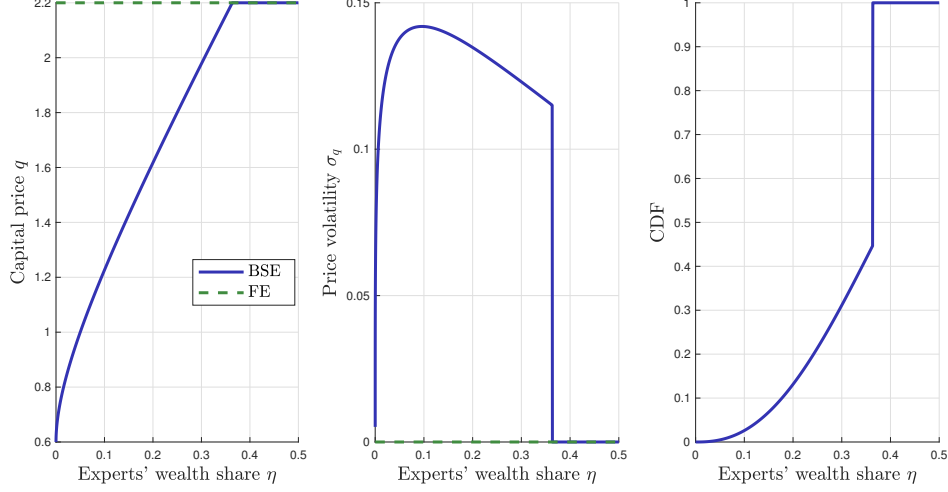


Figure A.1: Capital price q , volatility σ_q , and stationary CDF of η . Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$. Type-switching parameters (for the CDF): $\delta_h = 0.004$ and $\delta_e = 0.036$.

triggers trading of capital between experts and households. Since experts are more productive than households, capital transfers have real effects and move asset prices. But it does not end there: asset-price fluctuations feed back into the wealth distribution, which initiates another round of capital transfers, and so on. The question “does there exist an initial belief about asset prices that can be self-justified by this process?” is tantamount to solving the ODE (A.2).

A.2 W-BSEs are inconsistent with fundamental shocks

The previous section shows how a W-BSE can arise without fundamental shocks ($\sigma = 0$). But with fundamental shocks ($\sigma \neq 0$), we obtain the stark result that, in an equilibrium that is Markov in η , capital prices must be completely insensitive to the sunspot shock $Z^{(2)}$. In this sense, W-BSEs are not robust to the inclusion of fundamental uncertainty.

To see this, solve for the shock loadings σ_η and σ_q . Following the same analysis leading to equation (A.1), we obtain an equation for σ_q :

$$\left[1 - (\kappa - \eta) \frac{q'}{q}\right] \sigma_q = \begin{pmatrix} 1 \\ 0 \end{pmatrix} (\kappa - \eta) \sigma \frac{q'}{q}. \quad (\text{A.4})$$

Equation (A.4) is really two equations stacked. Given $\sigma \neq 0$, the first equation can only hold if $(\kappa - \eta) \frac{q'}{q} \neq 1$. This is inconsistent with the second equation, unless $\sigma_q \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$. Thus, we have proved

Lemma A.2. *If $\sigma \neq 0$, any Markov equilibrium in η is insensitive to sunspot shocks.*

Remark 4 (Hedging markets). Lemma A.2 is false if financial markets exist for agents to share fundamental risks. In particular, suppose that while equity issuance is still prohibited, all agents can frictionlessly access a market to trade claims on the fundamental shock $dZ^{(1)}$. No such market for the sunspot shock $dZ^{(2)}$ exists (one can debate whether it is realistic that such markets exist for $dZ^{(1)}$ and not $dZ^{(2)}$). A sunspot equilibrium re-emerges, which is exactly the same as the W-BSE, in the sense that $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \sigma_q$ coincides with the capital price volatility of Proposition A.1. Intuitively, perfect sharing of $dZ^{(1)}$ risks converts an economy with $\sigma \neq 0$ into one which looks like a fictitious economy with $\sigma = 0$. Thus, the more precise version of Lemma A.2 states that non-traded fundamental uncertainty eliminates the W-BSE. We thank Stavros Panageas for this observation.

A.3 The W-BSE is approximately a fundamental equilibrium

Lemma A.2 shows that, in the presence of fundamental shocks, a Markovian equilibrium in η must be a Fundamental Equilibrium (FE). These FE are studied extensively in the literature, with the defining feature that fundamental shocks are amplified by endogenous wealth dynamics (Brunnermeier and Sannikov, 2014). We analyze and discuss these FE in Online Appendix E.³⁷

To briefly recap these FE, rearrange equation (A.4) to obtain

$$\sigma_q = \frac{(\kappa - \eta)q'/q}{1 - (\kappa - \eta)q'/q} \sigma. \quad (\text{A.5})$$

Equation (A.5) is often interpreted as *amplification*, because $\frac{(\kappa - \eta)q'/q}{1 - (\kappa - \eta)q'/q}$ takes the form of a convergent geometric series. In words, a negative fundamental shock reduces experts' wealth share η directly through $(\kappa - \eta)\sigma$, which reduces asset prices through q'/q . This explains the numerator of (A.5). But the reduction in asset prices has an indirect effect: a one percent drop in capital prices reduces experts' wealth share by $(\kappa - \eta)$, which feeds back into a $(\kappa - \eta)q'/q$ percent further reduction capital prices, which then triggers the loop again. The second-round impact is $[(\kappa - \eta)q'/q]^2$, and so on. This infinite series is convergent if $(\kappa - \eta)q'/q < 1$, such that incremental amplification is reduced in each successive round of the feedback loop.

In the W-BSE, recall that $(\kappa - \eta)q'/q = 1$ (equation (A.1)). This BSE has no dampening in successive rounds of the feedback loop, leading to infinite amplification!

³⁷As a new but tangential result, this online appendix also demonstrates the multiplicity of FEs along two dimensions, κ_0 and $\text{sgn}(\sigma_R)$, neither of which have been documented in the literature.

Despite this contrast, it turns out that the W-BSE is “close” to these FE. As σ shrinks, amplification rises because falling exogenous volatility incentivizes expert leverage, which raises endogenous volatility. As σ vanishes, amplification rises explosively and equilibria become sunspot-like.³⁸

Lemma A.3. *Suppose a Markov equilibrium in η exists for each $\sigma > 0$ small enough, with $\kappa(0) = 0$. As $\sigma \rightarrow 0$, the equilibrium converges to the W-BSE.*

Thus, even if fundamentals are truly deterministic, our W-BSE “looks similar” to the FEs that have been studied in the literature. This approximate observational equivalence implies the W-BSE cannot possibly generate the type of novel dynamics promised in the introduction.

A.4 Proofs for W-BSEs

PROOF OF LEMMA A.1. Suppose $\kappa = 1$, $q = a_e/\bar{\rho}$, and $\sigma_q = 0$. Set μ_η and σ_η by (13)-(14), and set r by (11). By inspection, both (PO) and (RB) are satisfied. Furthermore, the Itô condition $\sigma_q = \frac{q'}{q}\sigma_\eta$ is trivially satisfied. Thus, Lemma 1 is satisfied. \square

PROOF OF PROPOSITION A.1. Consequence of Proposition D.2 (take $\kappa_0 \rightarrow 0$). \square

PROOF OF LEMMA A.2. In the text leading up to the statement of the lemma. \square

PROOF OF LEMMA A.3. Note that the other equations characterizing equilibrium, beyond (A.5), are (PO) and (RB), the latter repeated here for convenience:

$$0 = \min \left[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1 - \eta)}(\sigma + \sigma_q)^2 \right]. \quad (\text{A.6})$$

Denote the equilibrium solution for $\sigma > 0$ by $(q^{(\sigma)}, \kappa^{(\sigma)})$. Define $q^{(0)} := \lim_{\sigma \rightarrow 0} q^{(\sigma)}$ and $\kappa^{(0)} := \lim_{\sigma \rightarrow 0} \kappa^{(\sigma)}$. Combine equations (A.5) and (A.6) and rearrange terms to get

$$\left(1 - (\kappa^{(\sigma)} - \eta) \frac{(q^{(\sigma)})'}{q^{(\sigma)}} \right)^2 = \frac{(\kappa^{(\sigma)} - \eta)q^{(\sigma)}}{\eta(1 - \eta)(a_e - a_h)}\sigma, \quad \text{if } \kappa^{(\sigma)} < 1. \quad (\text{A.7})$$

³⁸Brunnermeier and Sannikov (2014) provide a related limiting result, arguing that asset-price volatility does not vanish as $\sigma \rightarrow 0$, also known as the “volatility paradox.” Related results can be found in Manuelli and Peck (1992) and Bacchetta et al. (2012), in which sunspot equilibria could be seen as limits of fundamental equilibria when fundamental uncertainty vanishes.

Note that this implies $\kappa^{(\sigma)} > \eta$. Furthermore, continuity of $\kappa^{(\sigma)}(\eta)$ and $\kappa_0 = \kappa^{(\sigma)}(0+) < 1$ imply $\kappa^{(\sigma)}(\eta) < 1$ for all η close enough to 0. Using these facts, and writing (A.7) instead as an integral equation, we obtain

$$\frac{q^{(\sigma)}(\eta_2)}{q^{(\sigma)}(\eta_1)} = \exp \left\{ \int_{\eta_1}^{\eta_2} \frac{1}{\kappa^{(\sigma)}(x) - x} \left[1 \pm \sqrt{\frac{(\kappa^{(\sigma)}(x) - x)q^{(\sigma)}(x)}{x(1-x)(a_e - a_h)}\sigma} \right] dx \right\}, \quad 0 < \eta_1 < \eta_2,$$

where η_2 is chosen small enough. Because the right-hand-side is continuous in both $q^{(\sigma)}$ and $\kappa^{(\sigma)}$, and both are bounded, taking the limit as $\sigma \rightarrow 0$ implies

$$\frac{q^{(0)}(\eta_2)}{q^{(0)}(\eta_1)} = \exp \left\{ \int_{\eta_1}^{\eta_2} \frac{1}{\kappa^{(0)}(x) - x} dx \right\}.$$

Differentiate this equation with respect to η_2 to obtain

$$\frac{d}{d\eta} \log q^{(0)} = \frac{1}{\kappa^{(0)} - \eta},$$

for all η small enough. Rearranging this equation delivers the ODE characterizing the W-BSE, i.e., selecting the solution $(\kappa - \eta)q'/q = 1$ in equation (A.1). Since $\kappa^{(\sigma)}(0+) = \kappa_0$ is fixed for all $\sigma > 0$, we also have the desired boundary condition $\kappa^{(0)}(0+) = \kappa_0$, for any $\kappa_0 \in [0, 1)$. Finally, all the other equations of the W-BSE can be verified by simply taking limits as $\sigma \rightarrow 0$. \square

B Proofs for Sections 1-2

B.1 Natural borrowing limit

Here, we derive the natural borrowing limit in general. The idea of a natural borrowing limit is that agents can borrow at most the present-value of their future income if they want to consume non-negative amounts and also not run a Ponzi scheme (see, e.g., [Aiyagari, 1994](#)). In our context, the only asset is capital, and the stream of its future dividends represents future income. Thus, if the income stream is valued at $q_t k_t$ for k_t units of capital holdings, it is sensible that an agent should be able to borrow at most this amount, implying net worth must be non-negative. We formalize this intuition below.

We assume the following. First, we impose a No-Ponzi condition

$$\liminf_{T \rightarrow \infty} n_T \geq 0 \quad \text{almost-surely,} \quad (\text{B.1})$$

which states that an individual must be able to pay back his/her debts asymptotically. Second, we impose a lower bound on net worth, i.e.,

$$n_t \geq -\underline{n}, \quad (\text{B.2})$$

where \underline{n} can be arbitrarily large but finite—portfolios satisfying (B.2) are so-called “tame” (see Karatzas and Shreve, 1998, Chapter 1, Definition 2.4). Portfolio tameness is a requirement in dynamic trading models, in order to rule out certain trivial arbitrage opportunities. For example, as is well known, without tameness, “doubling strategies” can generate positive gains almost-surely in finite time with zero outlays (c.f., Karatzas and Shreve, 1998, Chapter 1, Example 2.3).

We will now prove the following. Note that the condition $\sigma_R \neq 0$ below implies we are either in an economy with fundamental risk $\sigma > 0$ or a sunspot equilibrium of an economy without fundamental risk, so that we can cover all cases of interest.

Lemma B.1 (Natural borrowing limit). *Let (B.1) and (B.2) hold, and consider an equilibrium in which capital returns are stochastic, i.e., $\sigma_{R,t} \neq 0$. Then, every agent must obey $n_t \geq 0$.*

PROOF OF LEMMA B.1. Consider any consumption and trading strategy $\{c_t, k_t\}_{t \geq 0}$ that satisfies appropriate mild integrability conditions. The dynamic budget constraint of any agent takes the form

$$dn_t = \left[r_t n_t - c_t + q_t k_t (\mu_{R,t} - r_t) \right] dt + q_t k_t \sigma_{R,t} \cdot dZ_t, \quad n_0 \text{ given,} \quad (\text{B.3})$$

where $\mu_{R,t}$ is that agent’s expected return on capital (which differs between experts and households). Consider the state price density M_t perceived by this agent, which is

$$M_t = \exp \left[- \int_0^t \left(r_s + \frac{1}{2} |\pi_s|^2 \right) ds - \int_0^t \pi_s \cdot dZ_s \right], \quad (\text{B.4})$$

$$\text{where } \sigma_{R,t} \cdot \pi_t = \mu_{R,t} - r_t. \quad (\text{B.5})$$

Note that equation (B.5) defines π_t as the market price of risk process, which will actually be agent-specific in our model. Using Itô’s formula on $M_t n_t$, and using equation (B.5)

for π_t , we obtain

$$M_T n_T - M_t n_t = - \int_t^T M_s c_s ds + \int_t^T M_s (q_s k_s \sigma_{R,s} - n_s \pi_s) \cdot dZ_s.$$

Because consumption must be non-negative, and because of the lower bound (B.2), we have that

$$\begin{aligned} \int_t^T M_s (q_s k_s \sigma_{R,s} - n_s \pi_s) \cdot dZ_s &= M_T n_T - M_t n_t + \int_t^T M_s c_s ds \\ &\geq -M_T \underline{n} - M_t n_t, \end{aligned}$$

which shows that $T \mapsto \int_t^T M_s (q_s k_s \sigma_{R,s} - n_s \pi_s) \cdot dZ_s$ is a lower-bounded local martingale, hence a super-martingale. Consequently,

$$\mathbb{E}_t \left[M_T n_T + \int_t^T M_s c_s ds \right] \leq M_t n_t.$$

Now, by Fatou's lemma (which applies because $n_T \geq -\underline{n}$ for all T), we can take the limit $T \rightarrow \infty$ to obtain $\liminf_{T \rightarrow \infty} \mathbb{E}_t [M_T n_T] \geq \mathbb{E}_t [\liminf_{T \rightarrow \infty} M_T n_T] \geq 0$, where the latter inequality is due to the No-Ponzi condition (B.1). By the monotone convergence theorem (recall consumption is non-negative), we can also take $T \rightarrow \infty$ inside the expectation for $\mathbb{E}_t [\int_t^T M_s c_s ds]$. Thus,

$$\mathbb{E}_t \left[\int_t^\infty M_s c_s ds \right] \leq M_t n_t. \tag{B.6}$$

Equation (B.6) is the usual "static" budget constraint. From (B.6), the fact that consumption is non-negative, and the fact that the state-price density is strictly positive, we immediately obtain $n_t \geq 0$. Since time t was arbitrary, this must hold for all times. \square

B.2 Irrelevance of type-switching for optimal behavior

The objective function with type-switching technically differs from (3), because agents understand that at a future exponentially-distributed time, they will switch occupations.

Mathematically, the objective functions and indirect utilities satisfy the recursions

$$V_{e,t} = \sup_{c_e \geq 0, k_e \geq 0, n_e \geq 0} \mathbb{E} \left[\int_0^{T_e} e^{-\rho_e s} \log(c_{e,t+s}) ds + e^{-\rho_e T_e} V_{h,t+T_e} \right], \quad T_e \sim \exp(\delta_e)$$

$$V_{h,t} = \sup_{c_h \geq 0, k_h \geq 0, n_h \geq 0} \mathbb{E} \left[\int_0^{T_h} e^{-\rho_h s} \log(c_{h,t+s}) ds + e^{-\rho_h T_h} V_{e,t+T_h} \right], \quad T_h \sim \exp(\delta_h).$$

Standard homogeneity arguments imply that indirect utilities take the additively-separable form $V_{e,t} = \rho_e^{-1} \log(n_{e,t}) + \xi_{e,t}$ and $V_{h,t} = \rho_h^{-1} \log(n_{h,t}) + \xi_{h,t}$, for processes $\xi_{e,t}$ and $\xi_{h,t}$ that only depend on aggregates (i.e., not on individual net worth). Write $d\xi_{e,t} = \mu_{\xi,e,t} dt + \sigma_{\xi,e,t} \cdot dZ_t$ and $d\xi_{h,t} = \mu_{\xi,h,t} dt + \sigma_{\xi,h,t} \cdot dZ_t$. Then, the HJB equations associated with these equations are

$$\rho_e V_e = \max_{c,k \geq 0} \log(c) + (\partial_n V_e)[rn - c + qk(\mu_{R,e} - r)] + \frac{1}{2}(\partial_{nn} V_e)(qk)^2 |\sigma_R|^2 + \mu_{\xi,e} + \delta_e[V_h - V_e]$$

$$\rho_h V_h = \max_{c,k \geq 0} \log(c) + (\partial_n V_h)[rn - c + qk(\mu_{R,h} - r)] + \frac{1}{2}(\partial_{nn} V_h)(qk)^2 |\sigma_R|^2 + \mu_{\xi,h} + \delta_h[V_e - V_h],$$

where $\mu_{R,e}$ and $\mu_{R,h}$ are the expected returns on capital for experts and households, respectively. Using the form of V_e and V_h , these HJB equations become

$$\log(n) + \rho_e \xi_e = \max_{c,k \geq 0} \log(c) + \rho_e^{-1} \left[r - \frac{c}{n} + \frac{qk}{n}(\mu_{R,e} - r) \right] - \frac{1}{2} \left(\frac{qk}{n} \right)^2 |\sigma_R|^2 + \mu_{\xi,e} + \delta_e[\xi_h - \xi_e]$$

$$\log(n) + \rho_h \xi_h = \max_{c,k \geq 0} \log(c) + \rho_h^{-1} \left[r - \frac{c}{n} + \frac{qk}{n}(\mu_{R,h} - r) \right] - \frac{1}{2} \left(\frac{qk}{n} \right)^2 |\sigma_R|^2 + \mu_{\xi,h} + \delta_h[\xi_e - \xi_h].$$

Optimal choices take the familiar log-utility forms: consumptions are $c_e = \rho_e n_e$ and $c_h = \rho_h n_h$; portfolios are $\frac{qk_e}{n_e} = \left[\frac{\mu_{R,e} - r}{|\sigma_R|^2} \right]^+$ and $\frac{qk_h}{n_h} = \left[\frac{\mu_{R,h} - r}{|\sigma_R|^2} \right]^+$. Most importantly, these choices are independent of the switching parameters δ_e, δ_h . To fully verify that this is correct, we must substitute the optimality conditions back into the HJB equations and check that we recover equations for ξ_e and ξ_h that only depend on aggregate variables (e.g., capital price q , interest rate r , etc.). Doing this, we obtain

$$\rho_e \xi_e = \log(\rho_e) + \rho_e^{-1} \left[r - \rho_e + \frac{1}{2} \left(\frac{[\mu_{R,e} - r]^+}{|\sigma_R|} \right)^2 \right] + \mu_{\xi,e} + \delta_e[\xi_h - \xi_e]$$

$$\rho_h \xi_h = \log(\rho_h) + \rho_h^{-1} \left[r - \rho_h + \frac{1}{2} \left(\frac{[\mu_{R,h} - r]^+}{|\sigma_R|} \right)^2 \right] + \mu_{\xi,h} + \delta_h[\xi_e - \xi_h],$$

which verifies the conjecture, as all terms either pertain to the ξ processes or aggregate variables. \square

B.3 Proof of Lemma 1

We need to check conditions (i)-(iii) of Definition 1, given η_0 and conditions (PO), (RB), (11), and (13)-(14). Condition (i) holds by the definition of η_0 .

For condition (ii), note that standard martingale techniques can be applied to verify that individual optimality, subject to the dynamic budget constraint (2), is equivalent to the following conditions holding: $c_\ell = \rho_\ell n_\ell$; the portfolio conditions (7)-(8); and the transversality and No-Ponzi conditions in (10). Given q_t, η_t, κ_t , and individual net worths $n_{e,t}^i$ and $n_{h,t}^j$, let us set

$$c_{e,t}^i = \rho_e n_{e,t}^i \quad \text{and} \quad k_{e,t}^i = \frac{\kappa_t}{q_t \eta_t} n_{e,t}^i, \quad \text{for } i \in \mathbb{I} \quad (\text{B.7})$$

$$c_{h,t}^j = \rho_h n_{h,t}^j \quad \text{and} \quad k_{h,t}^j = \frac{1 - \kappa_t}{q_t(1 - \eta_t)} n_{h,t}^j, \quad \text{for } j \in \mathbb{J}. \quad (\text{B.8})$$

If we do this, then clearly the optimal consumption-wealth ratio holds. Similarly, after substituting the suggesting capital holdings in (B.7)-(B.8), the optimal portfolio conditions (7)-(8) become a linear transformation of equations (RB) and (11), so they hold as well. Finally, after substituting the proposals in (B.7)-(B.8) into the transversality conditions in (10), we see that these hold automatically.

For condition (iii), note that $\kappa \in [0, 1]$ automatically implies capital market clearing (5). Similarly, substituting $c_\ell = \rho_\ell n_\ell$ and the definitions of κ and η into (PO), we obtain goods market clearing (4).

Thus, we have constructed an equilibrium of Definition 1. Note that (13)-(14) have not been used in this construction, but they are direct consequences (via Itô's formula) of the definition of η .

The final statement is clearly true in the sense that the prices (q_t, r_t) are directly involved in Definition 1, while the objects (η_t, κ_t) constitute two summary statistics of the distribution of net worth and capital $\{n_{e,t}^i, n_{h,t}^j, k_{e,t}^i, k_{h,t}^j : i \in \mathbb{I}, j \in \mathbb{J}\}$. Thus, two distinct values of $(\eta_t, q_t, \kappa_t, r_t)_{t \geq 0}$ cannot correspond to the same equilibrium of Definition 1. \square

B.4 Proof of Theorem 1

Step 0: Reduce the system. We will start by eliminating $(r, \kappa, \sigma_\eta, \mu_\eta)$ from the system of endogenous objects, given $(\eta, q, \sigma_q, \mu_q)$. First, price-output relation (PO) determines κ as

a function of (η, q) and nothing else, given by

$$\kappa(\eta, q) := \frac{q\bar{\rho}(\eta) - a_h}{a_e - a_h}. \quad (\text{B.9})$$

Second, substituting this result for κ , equation (11) fully determines r , given knowledge of $(\eta, q, \sigma_q, \mu_q)$. Third, equations (13)-(14), after plugging in the result for κ , fully determine (σ_η, μ_η) , given knowledge of (η, q, σ_q) . Thus, given (η, q) , it suffices to determine (σ_q, μ_q) from the remaining conditions, namely (RB).

Step 1: Define perturbed domain. To facilitate analysis, it will be convenient to analyze a slightly modified system instead of (η, q) , and on a perturbed domain.

First, define the following auxiliary functions. Fix $\epsilon \in (0, \frac{a_e - a_h}{\rho_h})$. Let $\beta(\cdot)$ be a strictly increasing, continuously differentiable function such that $\beta(1) = -\beta(0) = \epsilon$, and $\beta(\eta_\beta^*) = 0$, where $\eta_\beta^* \in (\eta^*, 1)$ and

$$\eta^* := \frac{\rho_h}{\rho_e} \left(\frac{1 - a_h/a_e}{\sigma^2} \rho_e - 1 + \frac{\rho_h}{\rho_e} \right)^{-1}. \quad (\text{B.10})$$

Note that $\eta^* < 1$ by Assumption 1, part (ii). Let $\alpha(\cdot)$ be an increasing, continuously differentiable function such that $\alpha(0) = 0$, $\alpha'(0) \in (0, \infty)$, and $\alpha(1) = \epsilon/2$.

Next, define the following functions,

$$\begin{aligned} q^H(\eta) &:= a_e / \bar{\rho}(\eta) \\ q^L(\eta) &:= \bar{a}(\eta) / \bar{\rho}(\eta), \end{aligned}$$

where $\bar{a}(\eta) := \eta\rho_e + (1 - \eta)\rho_h$. Using (B.9), one notices that q^H corresponds to the capital price when $\kappa = 1$, whereas q^L corresponds to the capital price when $\kappa = \eta$. Put

$$\begin{aligned} q_\beta^H(\eta) &:= q^H(\eta) + \beta(\eta) \\ q_\alpha^L(\eta) &:= q^L(\eta) + \alpha(\eta). \end{aligned}$$

Using these functions, define the perturbed domain (which is an open set)

$$\mathcal{X} := \left\{ (\eta, x) : \eta \in (0, 1) \text{ and } q_\alpha^L(\eta) < x < q_\beta^H(\eta) \right\}.$$

Note that, boundaries aside, \mathcal{X} will coincide with \mathcal{D} as $\epsilon \rightarrow 0$. For reference, the perturbed domain \mathcal{X} is displayed in Figure B.1.

We will define a stochastic process x_t such that the capital price q coincides with x

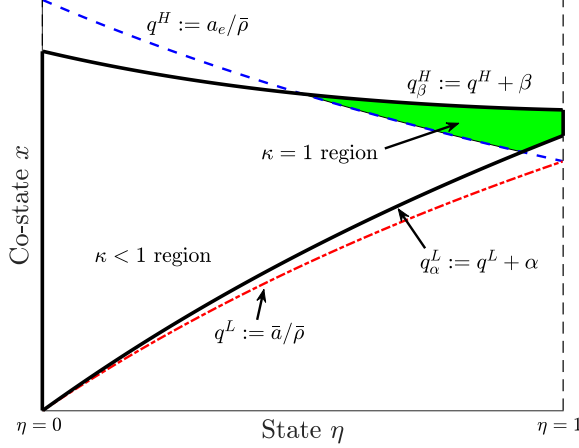


Figure B.1: The perturbed domain \mathcal{X} is shown as the region surrounded by solid black lines. The original domain \mathcal{D} is the region defined by the dashed lines. The perturbation functions α and β are chosen to be linear functions, with $\epsilon = 0.2$. Parameters: $\rho_e = 0.07$, $\rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$.

when it lies below q^H , i.e.,

$$q_t = \min \left[x_t, q^H(\eta_t) \right]. \quad (\text{B.11})$$

By (B.11), we may analyze the dynamical system $(\eta_t, x_t)_{t \geq 0}$ rather than $(\eta_t, q_t)_{t \geq 0}$. Furthermore, to prove the claim that $(\eta_t, q_t)_{t \geq 0}$ remains in \mathcal{D} almost-surely, it suffices to prove $(\eta_t, x_t)_{t \geq 0}$ remains in \mathcal{X} almost-surely (Step 4 below).

Step 2: Construct σ_q so that (RB) is satisfied. First consider $\{x < q^H(\eta)\}$ so that $q = x$. Note that this case corresponds to $\kappa < 1$. Let $\gamma(\eta, x) : \mathcal{X} \mapsto (0, 1)$ be any C^1 function. Put

$$\sigma_q = \begin{bmatrix} \sqrt{\gamma \frac{\eta(1-\eta)}{\kappa-\eta} \frac{a_e - a_h}{q}} - \sigma \\ \sqrt{(1-\gamma) \frac{\eta(1-\eta)}{\kappa-\eta} \frac{a_e - a_h}{q}} \end{bmatrix}, \quad \text{if } x < q^H(\eta). \quad (\text{B.12})$$

Substituting (B.12), one can verify that the second term of condition (RB) is zero. Importantly, the definitions of q_α^L and q_β^H imply that σ_q is bounded on $\mathcal{X} \cap \{x < q^H(\eta)\}$. Indeed, because of $\alpha'(0) > 0$, the slowest possible rate that $\kappa \rightarrow 0$ as $\eta \rightarrow 0$ is lower-bounded away from 1, i.e., $\liminf_{\eta \rightarrow 0, (\eta, x) \in \mathcal{X}} \kappa/\eta > 1$. And because $\alpha(1) > 0$, we have $\kappa = 1$ for all η near enough to 1; thus η is bounded away from 1 on $\{x < q^H(\eta)\}$.

Next consider $\{x \geq q^H(\eta)\}$ so that $q = q^H(\eta)$. Note that this case corresponds to $\kappa = 1$. Since q is an explicit function of η , we use Itô's formula to compute $(\frac{1}{0}) \cdot \sigma_q = -\sigma_\eta \bar{\rho}' / \bar{\rho}$, which after substituting equation (14) for σ_η delivers

$$\sigma_q = \begin{bmatrix} -\frac{(1-\eta)(\rho_e - \rho_h)/\bar{\rho}}{1 + (1-\eta)(\rho_e - \rho_h)/\bar{\rho}} \sigma \\ 0 \end{bmatrix}, \quad \text{if } x \geq q^H(\eta). \quad (\text{B.13})$$

Note that (B.13) will be consistent with (RB) as long as $(\eta_t, x_t)_{t \geq 0}$ remains in \mathcal{X} almost-surely, which will be verified in Step 4.³⁹

Note finally that σ_q defined in (B.12)-(B.13) is solely a function of (η, x) , so sometimes we will write $\sigma_q(\eta, x)$. Similarly, with σ_q in hand, we now have μ_η and σ_η as functions of (η, x) alone.

Step 3: Construct μ_q . Similar to σ_q , separately consider $\{x < q^H(\eta)\}$ and $\{x \geq q^H(\eta)\}$. On $\{x \geq q^H(\eta)\}$, since $q = q^H(\eta)$ is an explicit function of η , we set μ_q via Itô's formula. On $\{x < q^H(\eta)\}$, we have no equilibrium considerations restricting μ_q . Thus, we will put $\mu_q = m_q$, where m_q is a function in class \mathcal{M} , defined as follows. A function $m : \mathcal{X} \mapsto \mathbb{R}$ is a member of \mathcal{M} if m is C^1 and possesses the following boundary conditions:

$$\inf_{\eta \in (0,1)} \lim_{x \searrow q_\alpha^L(\eta)} (x - q_\alpha^L(\eta)) m(\eta, x) = +\infty \quad (\text{B.14})$$

$$\sup_{\eta \in (0,1)} \lim_{x \nearrow q_\beta^H(\eta)} (q_\beta^H(\eta) - x) m(\eta, x) = -\infty \quad (\text{B.15})$$

$$\text{for any } x \in (q_\alpha^L(0), q_\beta^H(0)), \quad \lim_{\eta \searrow 0} |m(\eta, x)| < +\infty \quad (\text{B.16})$$

$$\text{for any } x \in (q_\alpha^L(1), q_\beta^H(1)), \quad \lim_{\eta \nearrow 1} |m(\eta, x)| < +\infty. \quad (\text{B.17})$$

Collecting these results

$$\mu_q(\eta, x) = \begin{cases} m_q(\eta, x), & \text{if } x < q^H(\eta); \\ \frac{\rho_e - \rho_h}{\bar{\rho}(\eta)^2} [-\bar{\rho}(\eta) \mu_\eta(\eta, x) + |\sigma_\eta(\eta, x)|^2], & \text{if } x \geq q^H(\eta). \end{cases} \quad (\text{B.18})$$

Step 4: Verify stationarity. We demonstrate the time-paths $(\eta_t, x_t)_{t \geq 0}$ remain in \mathcal{X} almost-surely and admit a stationary distribution.

The dynamics of x_t are specified as follows. Denote its diffusion and drift coefficients by $(x\sigma_x, x\mu_x)$, where σ_x and μ_x are functions of (η, x) to be specified shortly. By (B.11), when $q_\alpha^L(\eta) < x < q^H(\eta)$, we must put $\sigma_x = \sigma_q$ and $\mu_x = \mu_q$. Outside of this region, we put σ_x and μ_x to preserve stationarity.

³⁹Plugging $q = a_e/\bar{\rho}$ into the second term of equation (RB), we require $|\sigma_R|^2 \leq \eta \bar{\rho}(\eta)(1 - a_h/a_e)$. Substituting (B.13), we obtain $|\sigma_R|^2 = \sigma^2(\bar{\rho}/\rho_e)^2$. Combining these, we require. $\eta \geq \eta^*$ when $x \geq q^H(\eta)$, where η^* is defined in (B.10). Therefore, for all $\eta < \eta^*$, we insist $x < q^H(\eta)$. As long as $(\eta, x) \in \mathcal{X}$, this will hold, because $q_\beta^H(\eta) < q^H(\eta)$ for all $\eta < \eta^*$, and $x < q_\beta^H(\eta)$ for all η .

To this end, let $\tilde{\sigma}_x : \mathcal{X} \mapsto \mathbb{R}_+$ be any positive, bounded, C^1 function.⁴⁰ Put

$$\sigma_x(\eta, x) = \begin{cases} \sigma_q(\eta, x), & \text{if } x < q^H(\eta); \\ \tilde{\sigma}_x(\eta, x), & \text{if } x \geq q^H(\eta). \end{cases}$$

Note that σ_x is bounded (recall σ_q is bounded, and $\tilde{\sigma}_x$ is assumed bounded).

Similarly, for the drift, let $m_x : \mathcal{X} \mapsto \mathbb{R}$ be any function in class \mathcal{M} defined above (note: m_x need not coincide with m_q above). Put

$$\mu_x(\eta, x) = \begin{cases} \mu_q(\eta, x), & \text{if } x < q^H(\eta); \\ m_x(\eta, x), & \text{if } x \geq q^H(\eta). \end{cases}$$

Thus, μ_x satisfies boundary conditions (B.14)-(B.17) on all boundaries of \mathcal{X} .

Corresponding to the SDEs induced by $(\sigma_\eta, \sigma_x, \mu_\eta, \mu_x)$, define the infinitesimal generator \mathcal{L} , where for any C^2 function f ,

$$\mathcal{L}f = \mu_\eta \partial_\eta f + (x\mu_x) \partial_x f + \frac{1}{2} |\sigma_\eta|^2 \partial_{\eta\eta} f + \frac{1}{2} |x\sigma_x|^2 \partial_{xx} f + x\sigma_x \cdot \sigma_\eta \partial_{\eta x} f.$$

Let $\{\mathcal{X}_n\}_{n \geq 1}$ be an increasing sequence of open sets, whose closures are contained in \mathcal{X} , such that $\cup_{n \geq 1} \mathcal{X}_n = \mathcal{X}$. Note that $(\sigma_\eta, \sigma_x, \mu_\eta, \mu_x)$ are bounded on \mathcal{X}_n for each n . Consequently, despite the (potential) discontinuity in $(\sigma_\eta, \sigma_x, \mu_\eta, \mu_x)$ at the one-dimensional subset $\{x = q^H(\eta)\}$, there exists a unique weak solution $(\tilde{\eta}_t^n, \tilde{x}_t^n)_{0 \leq t \leq \tau_n}$, up to first exit time $\tau_n := \inf\{t : (\eta_t, x_t) \notin \mathcal{X}_n\}$, to the SDEs defined by the infinitesimal generator \mathcal{L} . See Krylov (1969, 2004) for weak existence and uniqueness in the presence of drift and diffusion discontinuities.

Letting $\tau := \lim_{n \rightarrow \infty} \tau_n$, we thus define $(\eta_t, x_t)_{0 \leq t \leq \tau}$ by piecing $(\tilde{\eta}_t^n, \tilde{x}_t^n)_{0 \leq t \leq \tau_n}$ together for successive n . In other words, $(\eta_t, x_t) = (\tilde{\eta}_t^n, \tilde{x}_t^n)$ for $0 \leq t \leq \tau_n$, each n . Our goal is to show (a) $\tau = +\infty$ a.s.; and (b) the resulting stochastic process $(\eta_t, x_t)_{t \geq 0}$ possesses a non-degenerate stationary distribution on \mathcal{X} . These will be proved if we can obtain a function v satisfying Lemma B.2 below.

Define the positive function v by

$$v(\eta, x) := \frac{1}{\eta^{1/2}} + \frac{1}{1 - \eta} + \frac{1}{x - q_\alpha^L(\eta)} + \frac{1}{q_{\beta, \lambda}^H(\eta) - x}.$$

⁴⁰Note that $\tilde{\sigma}_x$ need not vanish at the boundary of \mathcal{X} , but if it does some of the boundary conditions on m_x , to follow, can be relaxed.

Note that v diverges to $+\infty$ at the boundaries of \mathcal{X} , so assumption (i) of Lemma B.2 is proved. Next, if assumption (iii) of Lemma B.2 holds (which we will prove below), then there exists N such that $\mathcal{L}v < 0$ on $\mathcal{X} \setminus \mathcal{X}_n$ for all $n > N$. Furthermore, for each given n , $\mathcal{L}v$ is bounded on \mathcal{X}_n . Consequently, we can find a constant c large enough such that $\mathcal{L}v \leq cv$ on all of \mathcal{X} , which verifies part (ii) of Lemma B.2.

It remains to prove assumption (iii) of Lemma B.2, namely that $\mathcal{L}v \rightarrow -\infty$ as $(\eta, x) \rightarrow \partial\mathcal{X}$. We will examine the boundaries of \mathcal{X} one-by-one. In the following, we use the notation $g(x) = o(f(x))$ if $g(x)/f(x) \rightarrow 0$ as $x \rightarrow 0$, and the notation $g(x) = O(f(x))$ if $g(x)/f(x) \rightarrow C$ as $x \rightarrow 0$, where C is a finite constant.

As $\eta \rightarrow 0$ (and x bounded away from $q_\alpha^L(0)$ and $q_\beta^H(0)$, such that κ is bounded away from 0 and 1, the latter due to the definition of q_β^H), we have

$$\begin{aligned} \mu_\eta &= \delta_h + \frac{a_e - a_h}{x} \kappa + \eta[\rho_h - \rho_e - \delta_e - \delta_h] + o(\eta) \quad \text{and} \quad |\sigma_\eta|^2 = \eta(\kappa - \eta) \frac{a_e - a_h}{x} + o(\eta) \\ \mu_x &= O(1) \quad \text{and} \quad |\sigma_x|^2 = O(1). \end{aligned}$$

We used condition (B.16) to obtain μ_x bounded. Thus,

$$\mathcal{L}v = -\frac{1}{2\eta^{3/2}} \left[\delta_h + \frac{1}{4} \frac{a_e - a_h}{x} \kappa \right] + o(\eta^{-3/2}) \rightarrow -\infty,$$

irrespective of $\delta_h > 0$ or $\delta_h = 0$.

As $\eta \rightarrow 1$ (and x bounded away from $q_\alpha^L(1)$ and $q_\beta^H(1)$; note that $\kappa = 1$ at this boundary), we have

$$\begin{aligned} \mu_\eta &= -\delta_e - (\rho_e - \rho_h)(1 - \eta) + o(1 - \eta) \quad \text{and} \quad |\sigma_\eta|^2 = (1 - \eta)^2 \sigma^2 \\ \mu_x &= O(1) \quad \text{and} \quad |\sigma_x|^2 = O(1). \end{aligned}$$

We used condition (B.17) to obtain μ_x bounded. Thus,

$$\mathcal{L}v = -(1 - \eta)^{-2} \delta_e - (1 - \eta)^{-1} [\rho_e - \rho_h - \sigma^2] + o((1 - \eta)^{-1}) \rightarrow -\infty,$$

irrespective of δ_e , due to Assumption 1 part (iii).

We separately calculate the limit $x \rightarrow q_\alpha^L(\eta)$ (with η bounded away from 0) in the two cases $\{x < q^H(\eta)\}$ and $\{x \geq q^H(\eta)\}$, since $\kappa < 1$ in the first case, and $\kappa = 1$ in the

second case. Still, we find that in both cases,

$$\begin{aligned}\mu_\eta &= O(1) \quad \text{and} \quad |\sigma_\eta|^2 = O(1) \\ \mu_x &= o((x - q_\alpha^L)^{-1}) \quad \text{and} \quad |\sigma_x|^2 = O(1).\end{aligned}$$

We used condition (B.14) to obtain the order of μ_x . Thus,

$$\mathcal{L}v = -(x - q_\alpha^L)^{-2}x\mu_x + O((x - q_\alpha^L)^{-3}) \rightarrow -\infty.$$

Similarly, we separately calculate the limit $x \rightarrow q_\beta^H(\eta)$ (with η bounded away from 0) in the two cases $\{x < q^H(\eta)\}$ and $\{x \geq q^H(\eta)\}$. Again, we find that in both cases,

$$\begin{aligned}\mu_\eta &= O(1) \quad \text{and} \quad |\sigma_\eta|^2 = O(1) \\ \mu_x &= (-1) \times o((q_\beta^H - x)^{-1}) \quad \text{and} \quad |\sigma_x|^2 = O(1).\end{aligned}$$

We used condition (B.15) to obtain the order of μ_x . Thus,

$$\mathcal{L}v = (q_\beta^H - x)^{-2}x\mu_x + O((q_\beta^H - x)^{-3}) \rightarrow -\infty.$$

Finally, all the corners of \mathcal{X} can be analyzed in a straightforward way by combining the cases above, with the exception of $(\eta, x) = (0, q_\alpha^L(0)) = (0, a_h/\rho_h)$. Approaching this corner, we must take a particular path of $x \rightarrow a_h/\rho_h$ as $\eta \rightarrow 0$. Denote this path by $\hat{x}(\eta)$ and denote the asymptotic slope by $\hat{x}'(0) \in (\frac{d}{d\eta}q_\alpha^L(0), +\infty)$, where $\frac{d}{d\eta}q_\alpha^L(0) = [\frac{a_e}{a_h} - \frac{\rho_e}{\rho_h}]\frac{a_h}{\rho_h} + \alpha'(0) > 0$, by Assumption 1, part (i), and the fact that $\alpha'(0) > 0$. Denote the associated path of κ by $\hat{\kappa}(\eta)$ and the corresponding asymptotic slope by $\hat{\kappa}'(0) = \frac{1}{a_e - a_h}[\hat{x}'(0)\rho_h + (\rho_e - \rho_h)a_h/\rho_h]$. Substituting in, we find $\hat{\kappa}'(0) \in (1 + \frac{\alpha'(0)}{a_e - a_h}, +\infty)$. When computing $\mathcal{L}v$, we will take the supremum over all possible paths, meaning over $\hat{x}'(0)$ and $\hat{\kappa}'(0)$. Using similar calculations from the initial $\eta \rightarrow 0$ case, but using these paths, we obtain

$$\begin{aligned}\mu_\eta &= \delta_h + \eta[\frac{a_e - a_h}{\hat{x}}\hat{\kappa}' + \rho_h - \rho_e - \delta_e - \delta_h] + o(\eta) \quad \text{and} \quad |\sigma_\eta|^2 = \eta^2[\hat{\kappa}' - 1]\frac{a_e - a_h}{\hat{x}} + o(\eta) \\ \mu_x &= o((\hat{x} - q_\alpha^L)^{-1}) \quad \text{and} \quad |\sigma_x|^2 = O(1) \\ \text{and} \quad \sigma_x \cdot \sigma_\eta &= \eta[\frac{a_e - a_h}{\hat{x}} - \sigma(\gamma(\hat{\kappa}' - 1)\frac{a_e - a_h}{\hat{x}})^{1/2}] + o(\eta).\end{aligned}$$

Since $\hat{x} \geq O(\eta)$ and $\hat{\kappa} \geq O(\eta)$ (in the sense that both could be $+\infty$), we may treat terms like $(\hat{x} - q_\alpha^L)^{-1}$ as smaller than η^{-1} . This identifies the dominant terms as those

associated to μ_η , $|\sigma_\eta|^2$, and μ_x . Thus,

$$\begin{aligned}\mathcal{L}v &= -\frac{1}{2\eta^{3/2}}\delta_h + \frac{1}{2\eta^{1/2}}[\rho_e - \rho_h + \delta_e + \delta_h - \frac{a_e - a_h}{\hat{x}} - \frac{a_e - a_h}{\hat{x}}(\hat{\kappa}' - 1)/4] + o(\eta^{-3/2}) \\ &\quad - (\hat{x} - q_\alpha^L)^{-2}x\mu_x + O((\hat{x} - q_\alpha^L)^{-3}) \rightarrow -\infty,\end{aligned}$$

irrespective of δ_h , because $\rho_e - \rho_h - \frac{a_e - a_h}{a_h/\rho_h} = \rho_h[\rho_e/\rho_h - a_e/a_h] < 0$ by Assumption 1, part (i), and because $\inf\{\hat{\kappa}'(0)\} > 1$.

This completes the verification that $\mathcal{L}v \rightarrow -\infty$ as $(\eta, x) \rightarrow \partial\mathcal{X}$, which proves stationarity by Lemma B.2 below. This completes the proof. \square

B.5 Stochastic stability: a useful lemma

To prove the stationarity claims of Theorem 1 and Proposition 1, we need the following lemma, which is a slight generalization of Theorems 3.5 and 3.7 of [Khasminskii \(2011\)](#), in the sense that weaker conditions are imposed on the coefficients α and β . Indeed, any coefficients (α, β) are permissible as long as they admit existence of a weak solution to the SDE system. See also Remark 3.5 and Corollary 3.1 in [Khasminskii \(2011\)](#) which allow the arguments in \mathbb{R}^l to generalize to any open domain \mathcal{D} .

Lemma B.2. *Suppose $(X_t)_{0 \leq t \leq \tau}$ is a weak solution to the SDE $dX_t = \beta(X_t)dt + \alpha(X_t)dZ_t$ in an open connected domain $\mathcal{D} \subset \mathbb{R}^l$, where Z is a d -dimensional Brownian motion and $\tau := \inf\{t : X_t \notin \mathcal{D}\}$ is the first exit time from \mathcal{D} . Define the infinitesimal generator \mathcal{L} by (for any C^2 function f)*

$$\mathcal{L}f = \sum_{i=1}^n \beta_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^n (\alpha_i \cdot \alpha_j) \frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Suppose there is a non-negative C^2 function $v : \mathcal{D} \mapsto \mathbb{R}_+$ such that (i) $\liminf_{x \rightarrow \partial\mathcal{D}} v(x) = +\infty$; (ii) $\mathcal{L}v \leq cv$ for some constant $c \geq 0$; and (iii) $\limsup_{x \rightarrow \partial\mathcal{D}} \mathcal{L}v(x) = -\infty$. Then,

(a) $\tau = +\infty$ almost-surely;

(b) the distribution of X_0 can be chosen such that $(X_t)_{t \geq 0}$ is stationary.

PROOF OF LEMMA B.2. Let $\{\mathcal{D}_n\}_{n \geq 1}$ be an increasing sequence of open sets, whose closures are contained in \mathcal{D} , such that $\cup_{n \geq 1} \mathcal{D}_n = \mathcal{D}$. Let $\tau_n := \inf\{t : X_t \notin \mathcal{D}_n\}$, and note that $\tau = \lim_{n \rightarrow \infty} \tau_n$ is the monotone limit of these exit times. Define $w(t, x) := v(x) \exp(-ct)$, which satisfies $\mathcal{L}w \leq 0$ by assumption (ii). Using Itô's formula, we have

$$\mathbb{E}[v(X_{\tau_n \wedge t})e^{-c(\tau_n \wedge t)} - v(X_0)] = \mathbb{E} \int_0^{\tau_n \wedge t} \mathcal{L}w(u, X_u) du \leq 0.$$

Since $(\tau_n \wedge t) \leq t$ and $v \geq 0$, we obtain

$$\mathbb{E}[v(X_{\tau_n \wedge t})] \leq e^{ct} \mathbb{E}[v(X_0)].$$

Because $\mathbb{E}[v(X_{\tau_n \wedge t})] \geq \mathbb{P}[\tau_n \leq t] \inf_{x \in \mathcal{D} \setminus \mathcal{D}_n} v(x)$, we thus have

$$\mathbb{P}[\tau_n \leq t] \leq \frac{e^{ct} \mathbb{E}[v(X_0)]}{\inf_{x \in \mathcal{D} \setminus \mathcal{D}_n} v(x)}.$$

Taking the limit $n \rightarrow \infty$, we obtain

$$\mathbb{P}[\tau \leq t] \leq \frac{e^{ct} \mathbb{E}[v(X_0)]}{\liminf_{x \rightarrow \partial \mathcal{D}} v(x)} = 0.$$

Thus, taking $t \rightarrow \infty$, we prove (a).

Next, since $\tau = +\infty$ a.s., we may consider $(X_t)_{t \geq 0}$ that is now defined for all time. Using Itô's formula,

$$\mathbb{E}[v(X_{\tau_n \wedge t}) - v(X_0)] = \mathbb{E} \int_0^{\tau_n \wedge t} \mathcal{L}v(X_u) du.$$

Note that $\mathbb{E}[v(X_{\tau_n \wedge t}) - v(X_0)] \leq b_1$ for some constant b_1 . Also note that $\sup_{x \in \mathcal{D}} \mathcal{L}v(x) \leq b_2$ for some constant b_2 , given assumptions (i)-(iii) and the fact that v is C^2 . Using these bounds, plus the following obvious inequality

$$\mathcal{L}v(X_u) \leq \mathbf{1}_{\{X_u \in \mathcal{D} \setminus \mathcal{D}_k\}} \sup_{x \in \mathcal{D} \setminus \mathcal{D}_k} \mathcal{L}v(x) + \sup_{x \in \mathcal{D}} \mathcal{L}v(x),$$

we get

$$- \sup_{x \in \mathcal{D} \setminus \mathcal{D}_k} \mathcal{L}v(x) \mathbb{E} \int_0^{\tau_n \wedge t} \mathbf{1}_{\{X_u \in \mathcal{D} \setminus \mathcal{D}_k\}} du \leq tb_2 - b_1.$$

Given the proof of (a), we may take the limit $n \rightarrow \infty$ (so that $\tau_n \rightarrow +\infty$), then apply Fubini's theorem, and then rearrange to obtain

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}[X_u \in \mathcal{D} \setminus \mathcal{D}_k] du \leq \frac{b_2}{-\sup_{x \in \mathcal{D} \setminus \mathcal{D}_k} \mathcal{L}v(x)}.$$

Taking $k \rightarrow \infty$ and using assumption (iii), we obtain

$$\lim_{k \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{P}[X_u \in \mathcal{D} \setminus \mathcal{D}_k] du \leq 0.$$

Applying Theorem 3.1 of [Khasminskii \(2011\)](#), there exists a stationary initial distribution for X_0 . The process $(X_t)_{t \geq 0}$ augmented with this initial distribution is clearly stationary by definition. \square

B.6 Proofs of Corollaries 1-3

PROOF OF COROLLARY 1. Start from the construction of S-BSE in Theorem 1, and note that we can make ϵ arbitrarily small such that the boundaries $q_\alpha^L \rightarrow \bar{a}/\bar{\rho}$ and $q_\beta^H \rightarrow a_e/\bar{\rho}$. In addition, the limit can be taken such that $\eta_\beta^* \rightarrow \eta^*$, its minimal possible level. Hence, an S-BSE can be constructed such that the set of prices q matches $\mathcal{Q}(\eta)$ arbitrarily closely. The result on return variance comes from using (B.12) when $\kappa < 1$ (i.e., when $\eta < \eta^*$) and using (B.13) when $\kappa = 1$ (i.e., when $\eta \geq \eta^*$ and q is at its upper bound). Using the definition of η^* provides the form of \mathcal{V} with the minimum as the lower bound.

To show that an S-BSE can be constructed such that positive probability is placed on all elements of \mathcal{Q} and \mathcal{V} , we simply note that a construction exists such that $\sigma_q \neq 0$ on the entirety of $\text{int}(\mathcal{D})$. \square

PROOF OF COROLLARIES 2-3. These follow from the proof of Theorem 1. \square

C Proofs and analysis for Section 3

C.1 Proof of Proposition 1

We proceed by construction. Without loss of generality, let $\mathcal{S} = (0, 1)$ and ignore the auxiliary states \mathcal{X} , so that the domain of the state variables is $\mathcal{D} = (0, 1) \times (0, 1)$. The reason we can effectively ignore \mathcal{X} is everything below is that only the drift μ_s depends on x , and x_t is assumed to be a bounded process, thereby introducing no problems of non-stationarity. Recall that $\bar{\rho} := \eta\rho_e + (1 - \eta)\rho_h$. By analogy, define $\bar{a} := \eta a_e + (1 - \eta)a_h$.

Step 1: Fundamental equilibrium. Let $(\hat{q}^0, \hat{\kappa}^0)$ be the solution to the fundamental equilibrium (which exists by assumption), and let $\eta^0 := \inf\{\eta : \hat{q}^0 \geq a_e/\bar{\rho}\} = \inf\{\eta : \hat{\kappa}^0 \geq 1\}$. By part (v) of Lemma E.1, there exists $\bar{\sigma}_A > 0$ such that, if $\sigma < \bar{\sigma}_A$, then $\eta^0 < 1$. By part (iv) of Lemma E.1, there exists $\bar{\sigma}_B > 0$ such that, if $\sigma < \bar{\sigma}_B$, then $(\hat{q}^0)' > \frac{a_e - a_h}{\bar{\rho}}$ for $\eta \in (0, \eta^0)$. Only to assist with step 9 below, we also denote $\bar{\sigma}_C = \sqrt{\rho_e - \rho_h}\mathbf{1}_{\delta=0} + (+\infty)\mathbf{1}_{\delta>0}$. Assume $\sigma < \min(\bar{\sigma}_A, \bar{\sigma}_B, \bar{\sigma}_C)$. In particular, this implies $\frac{d}{d\eta}[\hat{q}^0 - \bar{a}/\bar{\rho}] > 0$ for $\eta \in (0, \eta^0)$.

Step 2: Two basis functions. We design two “extremal” functions that will assist our construction. First, let φ be a C^2 function with the properties $\varphi(\eta^0) = 0$ and $\varphi' >$

$(\bar{a}/\bar{\rho})' - (a_e/\bar{\rho})' = \frac{a_e - a_h}{\bar{\rho}} [1 - (1 - \eta)^{\frac{\rho_e - \rho_h}{\bar{\rho}}}]$ for all η . Define

$$q^0(\eta) := \begin{cases} \hat{q}^0(\eta), & \text{if } \eta < \eta^0; \\ \hat{q}^0(\eta) + \varphi(\eta), & \text{if } \eta \geq \eta^0. \end{cases} \quad (\text{C.1})$$

Note that q^0 is C^∞ except at $\eta = \eta^0$, due to part (vi) of Lemma E.1.

To construct the other basis function, fix some $\epsilon \in (0, \eta^0)$, let $\tilde{\epsilon} \in (\epsilon, \eta^0)$, and define a C^∞ (but necessarily non-analytic) function $\beta : (0, 1) \mapsto \mathbb{R}_+$ with the following properties

$$\begin{aligned} \beta(\epsilon) &= q^0(\epsilon) - \bar{a}(\epsilon)/\bar{\rho}(\epsilon) \\ \beta^{(k)}(\epsilon) &= \frac{d^k}{d\eta^k} [q^0 - \bar{a}(\eta)/\bar{\rho}(\eta)]|_{\eta=\epsilon} \quad \text{for each derivative of order } k \geq 1 \\ \beta'(\eta) &< \frac{d}{d\eta} [q^0 - \bar{a}(\eta)/\bar{\rho}(\eta)] \quad \text{for all } \eta > \epsilon \\ \beta(\eta) &= 0 \quad \text{for all } \eta > \tilde{\epsilon}. \end{aligned}$$

A particular consequence of $\sigma < \bar{\sigma}_B$ in step 1 is $\frac{d}{d\eta} [q^0 - \bar{a}/\bar{\rho}] > 0$ for $\eta \in (0, \eta^0)$. A consequence of $\varphi' > (\bar{a}/\bar{\rho})' - (a_e/\bar{\rho})'$ is $\frac{d}{d\eta} [q^0 - \bar{a}/\bar{\rho}] > 0$ for $\eta \in (\eta^0, 1)$. Together, these properties imply such a function β exists. Then, we put

$$q^1(\eta) := \begin{cases} \hat{q}^0(\eta), & \text{if } \eta \leq \epsilon; \\ \bar{a}(\eta)/\bar{\rho}(\eta) + \beta(\eta), & \text{if } \eta > \epsilon. \end{cases} \quad (\text{C.2})$$

Note that $\eta^1 := \inf\{\eta : q^1 \geq a_e/\bar{\rho}\} = 1$. By the properties of β and φ , note the following slope results:

$$(q^0)' > (q^1)' \quad \text{on } \eta \in (\epsilon, 1) \quad (\text{C.3})$$

$$(q^0)^{(k)}(\epsilon) = (q^1)^{(k)}(\epsilon) \quad \text{for all derivatives of order } k \geq 0. \quad (\text{C.4})$$

Step 3: Useful monotonicity results. Before continuing, we make the following claims:

$$\frac{\bar{a}}{\bar{\rho}} < q^1 = q^0 < \frac{a_e}{\bar{\rho}}, \quad \text{for } \eta \in (0, \epsilon); \quad (\text{C.5})$$

$$\frac{\bar{a}}{\bar{\rho}} \leq q^1 < q^0 < \frac{a_e}{\bar{\rho}}, \quad \text{for } \eta \in (\epsilon, \eta^0); \quad (\text{C.6})$$

$$\frac{\bar{a}}{\bar{\rho}} = q^1 < \frac{a_e}{\bar{\rho}} < q^0, \quad \text{for } \eta \in (\eta^0, 1). \quad (\text{C.7})$$

All inequalities in relationship (C.5), as well as the third inequality in relationship (C.6), hold by part (ii) of Lemma E.1. The first inequality in relationship (C.6) holds because $\beta \geq 0$, whereas the first equality in relationship (C.7) holds because $\beta = 0$ on that set. The second inequality in relationship (C.6) holds due to (C.3). The second inequality in relationship (C.7) holds by the definition of $\eta^1 = 1$. The second inequality in relationship (C.7) holds since $q^0(\eta^0) = a_e/\bar{\rho}(\eta^0)$ combined with $(q^0 - a_e/\bar{\rho})' > (q^1 - a_e/\bar{\rho})' > 0$, for $\eta > \eta^0$.

Step 4: Construct candidate (q, κ) . We proceed to combine our basis functions according to the following convex combination, where $\alpha \in (0, 1)$ is fixed:

$$\tilde{q}(\eta, s) := (1 - \alpha s)q^0(\eta) + \alpha s q^1(\eta), \quad (\eta, s) \in \mathcal{D} = (0, 1) \times \mathcal{S}. \quad (\text{C.8})$$

For each $s \in \mathcal{S}$, define $\eta^*(s) := \inf\{\eta : \tilde{q}(\eta, s) \geq a_e/\bar{\rho}\}$, which can be shown is strictly increasing.⁴¹ Put

$$q(\eta, s) := \begin{cases} \tilde{q}(\eta, s), & \text{if } \eta < \eta^*(s) \\ a_e/\bar{\rho}(\eta), & \text{if } \eta \geq \eta^*(s) \end{cases} \quad \text{and} \quad \kappa := \frac{\bar{\rho}q - a_h}{a_e - a_h}.$$

By construction, the pair (q, κ) satisfy equation (PO).

Step 5: Properties of (q, κ) . Let $\mathcal{D}^* := \{(\eta, s) : \eta \in (\epsilon, \eta^*(s)), s \in \mathcal{S}\}$. On this set, we have $\kappa > \eta$, or equivalently $\bar{\rho}q > \bar{a}$, by (C.6)-(C.7). In fact, κ is bounded away from η on \mathcal{D}^* , since $\alpha < 1$ in (C.8). We also have the following derivative conditions on \mathcal{D}^* :

$$\partial_s q = \alpha(q^1 - q^0) < 0 \quad (\text{C.9})$$

$$\partial_\eta q = (1 - \alpha s)(q^0)' + \alpha s(q^1)' > 0 \quad (\text{C.10})$$

$$\partial_\eta q < q/(\kappa - \eta). \quad (\text{C.11})$$

⁴¹Indeed, note that \tilde{q} is C^2 on $(\eta^0, \eta^1) \times \mathcal{S}$, which implies η^* is C^1 . Then, use the fact that η^* is C^1 to differentiate $\tilde{q}(\eta^*(s), s) = a_e/\bar{\rho}(\eta^*(s))$ with respect to s , and use the fact that $\partial_s \tilde{q} = q^1 - q^0$, and finally rearrange to obtain

$$(\eta^*)'(s) \left[\partial_\eta \tilde{q}(\eta^*(s), s) + \frac{a_e}{\bar{\rho}(\eta^*(s))} \frac{\rho_e - \rho_h}{\bar{\rho}(\eta^*(s))} \right] = q^0(\eta^*(s)) - q^1(\eta^*(s)).$$

If at any point s , we had $(\eta^*)'(s) = 0$, we would necessarily have $q^0(\eta^*(s)) = q^1(\eta^*(s))$. But this contradicts the fact from (C.6)-(C.7) that $q^0 > q^1$ for all $\eta > \epsilon$, since $\eta^*(s) \geq \eta^0 > \epsilon$ (the fact that $\eta^*(s) \geq \eta^0$ comes from (C.6), which shows that $\tilde{q}(\eta, s) < a_e/\bar{\rho}(\eta)$ on $(\epsilon, \eta^0) \times \mathcal{S}$). Thus, $(\eta^*)'(s) \neq 0$ for all s . We can also rule out $(\eta^*)'(s) < 0$ by the fact that $\eta^*(0+) = \eta^0$ and $\eta^*(s) \geq \eta^0$ for all s . Thus, $(\eta^*)'(s) > 0$ for all s .

Inequality (C.9) holds by (C.6)-(C.7). Inequality (C.10) holds by (C.3) and Assumption 1(ii), which implies $(q^1)' > 0$. Inequality (C.11) is proven as follows. First, note that the function $f(\eta, x) = \frac{(a_e - a_h)x}{\bar{\rho}(\eta)x - \bar{a}(\eta)}$ is strictly decreasing in x on $x > \bar{a}(\eta)/\bar{\rho}(\eta)$. Second, part (i) of Lemma E.1 implies

$$(q^0)' < \frac{(a_e - a_h)q^0}{\bar{\rho}q^0 - \bar{a}} = f(\cdot, q^0).$$

Given (C.3), we thus have $\partial_\eta q < f(\cdot, q^0)$ for any value of s . Finally, since f is decreasing in its second argument, and $q < q^0$ on \mathcal{D}^* , we have $\partial_\eta q < f(\cdot, q)$, which proves the claim.

We remark on one additional smoothness property that holds at $\eta = \epsilon$, due to condition (C.4):

$$\partial_\eta^{(k)} q(\epsilon, s) = (q^0)^{(k)}(\epsilon) \quad \forall s, \quad \text{for all derivatives of order } k \geq 0. \quad (\text{C.12})$$

Step 6: Construct candidate σ_s . Consider solving the following problem.

$$\begin{aligned} \text{Problem:} \quad & \text{for each } (\eta, s) \in \mathcal{D}^*, \text{ solve for } y \text{ in the equation} \\ & y(\partial_s \log q)^2 = G, \end{aligned} \quad (\text{C.13})$$

where

$$G := \frac{\eta(1-\eta)}{\kappa-\eta} \frac{a_e - a_h}{q} (1 - (\kappa - \eta)\partial_\eta \log q)^2 - \sigma^2.$$

Note that G is bounded, as κ is bounded away from η (step 5). Checking boundedness of the solution y thus boils down to checking $\partial_s q$ at the boundaries of \mathcal{D}^* . By (C.9), as $s \rightarrow 0$ or $s \rightarrow 1$, $\partial_s q \not\rightarrow 0$, so y remains bounded. To check the result as $\eta \rightarrow \epsilon$, we first claim that $\lim_{\eta \searrow \epsilon} \partial_\eta^{(k)} G = 0$ for all derivatives of order $k \geq 0$. This is a consequence of parts (i) and (vi) of Lemma E.1, whereby $\partial_\eta^{(k)} G = 0$ for all $k \geq 0$ on $\eta < \epsilon$, combined with result (C.12). Since we also have $\partial_s q \rightarrow 0$, we apply L'Hôpital's rule twice to compute $\lim_{\eta \searrow \epsilon} G/(\partial_s \log q)^2 = 0$, noting both times that $\partial_{s\eta} \log q = \frac{\alpha}{q} [(q^1)' - (q^0)'] < 0$ is non-zero. Therefore, the solution $y = G/(\partial_s \log q)^2$ is bounded on \mathcal{D}^* .

Clearly, \sqrt{y} will be a real number if and only if $G \geq 0$. To prove $G \geq 0$, note that $\lim_{s \searrow 0} G = 0$, meaning it suffices to prove $\partial_s G \geq 0$. Differentiating G , we get

$$\begin{aligned} \frac{\partial_s G}{\eta(1-\eta)} = & -\frac{a_e - a_h}{(\kappa - \eta)q} (1 - (\kappa - \eta)\partial_\eta \log q) \left[(1 - (\kappa - \eta)\partial_\eta \log q) \left(\frac{\partial_s \kappa}{\kappa - \eta} + \frac{\partial_s q}{q} \right) \right. \\ & \left. + 2 \frac{(\kappa - \eta)\bar{a}}{\bar{\rho}q - \bar{a}} (\partial_s \log q)(\partial_\eta \log q) + 2(\kappa - \eta)\alpha \frac{(q^1)' - (q^0)'}{q} \right]. \end{aligned}$$

By properties (C.9)-(C.11), and the fact that $\text{sgn}(\partial_s \kappa) = \text{sgn}(\partial_s q)$, we prove $\partial_s G > 0$ on \mathcal{D}^* . So not only is \sqrt{y} real, it is non-zero.

We set σ_s as follows:

$$\sigma_s(\eta, s) := \begin{cases} \sqrt{y(\eta, s)}, & \text{if } (\eta, s) \in \mathcal{D}^*; \\ \sqrt{y(\epsilon+, s)} = 0, & \text{if } (\eta, s) \in \{(\eta, s) : \eta \in (0, \epsilon), s \in \mathcal{S}\}; \\ \sqrt{y(\eta^*(s)-, s)}, & \text{if } (\eta, s) \in \{(\eta, s) : \eta > \eta^*(s), s \in \mathcal{S}\}. \end{cases} \quad (\text{C.14})$$

In passing, we note that we have also shown that $\sigma_s > 0$ on a positive-measure set, as required in a sunspot equilibrium.

Step 7: Verify equation (23) is satisfied. By the construction of σ_s , equation (23) is satisfied on \mathcal{D}^* . On $\{(\eta, s) : \eta \in (0, \epsilon), s \in \mathcal{S}\}$, recall $\partial_s q = 0$, so (23) holds by property (i) of Lemma E.1. On $\{(\eta, s) : \eta > \eta^*(s), s \in \mathcal{S}\}$, recall $\kappa = 1$, so (23) is satisfied if and only if the second term inside the minimum is non-negative. Substituting $\kappa = 1$ and $q = a_e/\bar{\rho}$, hence $\partial_s q = 0$, into this term shows the non-negativity requirement is

$$\sigma^2 \leq \eta \bar{\rho} \frac{a_e - a_h}{a_e} (1 + (1 - \eta) \partial_\eta \log \bar{\rho})^2 \quad \text{for } \eta > \eta^*(s), s \in \mathcal{S}. \quad (\text{C.15})$$

On the other hand, property (v) of Lemma E.1, combined with the fact that $\eta^*(s)$ is increasing, imply

$$\eta^*(s) \geq \frac{\rho_h}{\rho_e} \left(\frac{1 - a_h/a_e}{\sigma^2} \rho_e - 1 + \frac{\rho_h}{\rho_e} \right)^{-1}, \quad \forall s \in \mathcal{S}. \quad (\text{C.16})$$

Straightforward algebra demonstrates that (C.15) and (C.16) are equivalent, proving (23) holds.

Step 8: Finish equilibrium construction. Having determined q , κ , and σ_s , we define μ_η and σ_η by (13)-(14). It remains to determine μ_s . We will pick $\mu_s(\eta, s) = m(\eta, s)$, where m is a C^2 function with the following properties: $\partial_s m < 0$, and for some $0 \leq s^0 < s^1 \leq 1$ thresholds,

$$(\text{if } s^0 > 0) \quad \inf_{\eta \in (0,1)} \lim_{s \searrow s^0} (s - s^0) m(\eta, s) = +\infty \quad (\text{C.17})$$

$$(\text{if } s^0 = 0) \quad \inf_{\eta \in (0,1)} \lim_{s \searrow s^0} m(\eta, s) > 0 \quad (\text{C.18})$$

$$\sup_{\eta \in (0,1)} \lim_{s \nearrow s^1} (s^1 - s) m(\eta, s) = -\infty. \quad (\text{C.19})$$

Step 9: Verify stationarity. Finally, we should demonstrate the time-paths $(\eta_t, s_t)_{t \geq 0}$ remain in \mathcal{D} almost-surely and admit a stationary distribution. This step is very similar to Theorem 1 and is therefore omitted. \square .

C.2 Proofs of Propositions 2-3

PROOF OF PROPOSITION 2. Fix any $\Sigma^* > 0$. The proof is a simple consequence of the fact that σ_q must be unbounded as κ approaches η , which is as q approaches the worst-case price q^1 . We fill in the technical details below.

We construct a sequence of equilibria—indexed by $(\alpha, \epsilon, \zeta)$ —as follows. Recall the capital price construction in Proposition 1:

$$q = (1 - \alpha s)q^0 + \alpha s q^1, \quad \text{when } \kappa < 1,$$

where $\alpha < 1$ is a parameter, q^0 is the fundamental equilibrium price, and

$$q^1 = \begin{cases} q^0, & \text{if } \eta < \epsilon; \\ \bar{a}/\bar{\rho} + \beta, & \text{if } \eta \in (\epsilon, \tilde{\epsilon}); \\ \bar{a}/\bar{\rho}, & \text{if } \eta > \tilde{\epsilon}. \end{cases}$$

The function β is a positive mollifier that vanishes uniformly as $\epsilon, \tilde{\epsilon} \rightarrow 0$. We set $\tilde{\epsilon} = \epsilon(1 + \epsilon)$. Based on the discussion in the text, we may choose μ_s such that equilibrium concentrates on any particular value of s . Thus, pick μ_s such that $s_t \geq \zeta$ almost-surely. Clearly, the choice of μ_s depends on α and ϵ , but such a choice can always be made for any parameters.

Let $p_{\text{low}} > 0$, $p_{\text{high}} > 0$ be given with $p_{\text{low}} + p_{\text{high}} < 1$. First, note that there exist α^* , ζ^* , and ϵ^* such that $\mathbb{P}[\eta_t \leq \tilde{\epsilon} \cap \kappa_t < 1] < p_{\text{low}}$ and $\mathbb{P}[\eta_t \geq 1 - \tilde{\epsilon} \cap \kappa_t < 1] < p_{\text{high}}$ for all $\alpha > \alpha^*$, $\zeta > \zeta^*$, and $\epsilon < \epsilon^*$. This is a consequence of the fact that in any stationary distribution, we have $\lim_{x \rightarrow 0} \mathbb{P}[\eta_t < x] = \lim_{x \rightarrow 1} \mathbb{P}[\eta_t > x] = 0$ and the fact that $\lim_{\alpha \rightarrow 1} \lim_{s \rightarrow 1} \kappa(\eta, s) < 1$ for all η .

At this point, fix such an $\epsilon < \epsilon^*$. Let a constant $M > 0$ be given satisfying

$$M \leq (1 - p_{\text{low}} - p_{\text{high}}) \frac{(a_e - a_h)^2 \tilde{\epsilon}(1 - \tilde{\epsilon})}{a_e/\rho_h \Sigma^*}. \quad (\text{C.20})$$

Note that

$$\lim_{\alpha \rightarrow 1} \lim_{s \rightarrow 1} \sup_{\eta \in (\tilde{\epsilon}, 1 - \tilde{\epsilon})} |q(\eta, s) - \bar{a}(\eta)/\bar{\rho}(\eta)| = 0.$$

Consequently, we may pick $\alpha > \alpha^*$ close enough to 1 and $\zeta > \zeta^*$ close enough to 1 such that

$$\sup_{s \in (\zeta, 1)} \sup_{\eta \in (\tilde{\epsilon}, 1 - \tilde{\epsilon})} |q(\eta, s) - \bar{a}(\eta) / \bar{\rho}(\eta)| \leq M.$$

Finally, using equation (23) and substituting $\kappa < 1$ from (PO), we have $|\sigma(\frac{1}{0}) + \sigma_q|^2 = \frac{(a_e - a_h)^2}{q} \frac{\eta(1 - \eta)}{\bar{\rho}q - \bar{a}}$. Note also that $q \leq a_e / \rho_h$ is an upper bound. Then,

$$\mathbb{E}[|\sigma(\frac{1}{0}) + \sigma_{q,t}|^2] > (1 - p_{\text{low}} - p_{\text{high}}) \frac{(a_e - a_h)^2}{a_e / \rho_h} \frac{\tilde{\epsilon}(1 - \tilde{\epsilon})}{M}.$$

Using (C.20), we obtain $\mathbb{E}[|\sigma(\frac{1}{0}) + \sigma_{q,t}|^2] > \Sigma^*$. □

PROOF OF PROPOSITION 3. First, we prove that $|\sigma_R|$ is increasing in s . From (23), we obtain $|\sigma_R|^2 = \frac{(a_e - a_h)^2}{q} \frac{\eta(1 - \eta)}{\bar{\rho}q - \bar{a}}$ on $\{\kappa < 1\}$. Differentiating with respect to s , we obtain

$$\partial_s |\sigma_R|^2 = -\eta(1 - \eta) \frac{(a_e - a_h)^2}{q(\bar{\rho}q - \bar{a})} \left[\frac{1}{q} + \frac{\bar{\rho}}{\bar{\rho}q - \bar{a}} \right] \partial_s q > 0,$$

since $\partial_s q = \alpha(q^1 - q^0) < 0$ by (C.9).

Next, we show that $|(\frac{1}{0}) \cdot \sigma_R|$ is decreasing in s . Revisiting the proof of Proposition 1, we compute on $\{\kappa < 1\}$ and for each $\eta > \epsilon$,

$$\partial_s [(\kappa - \eta) \partial_\eta \log q] = \alpha \left[(\kappa - \eta) \frac{(q^1)' - (q^0)'}{q} + \frac{\bar{a}(q^1 - q^0)}{(a_e - a_h)q^2} \partial_\eta q \right] < 0.$$

The inequality uses (C.3) to say $(q^1)' - (q^0)' < 0$, and (C.6)-(C.7) to say $q^1 - q^0 < 0$, and (C.10) to say $\partial_\eta q > 0$. Therefore, $(1 - (\kappa - \eta) \partial_\eta \log q)^{-1}$ is decreasing in s on $\{\kappa < 1\}$ for each $\eta > \epsilon$. Since q and κ are independent of s on $\{\eta < \epsilon\}$, this proves $(1 - (\kappa - \eta) \partial_\eta \log q)^{-1}$ is weakly decreasing in s on $\{\kappa < 1\}$. Using $|(\frac{1}{0}) \cdot \sigma_R| = \frac{\sigma}{1 - (\kappa - \eta) \partial_\eta \log q}$, we obtain the result.

Using the two claims just proved, we see that $|(\frac{0}{1}) \cdot \sigma_R|$ is increasing in s on $\{\kappa < 1\}$, due to the identity $|\sigma_R|^2 = |(\frac{0}{1}) \cdot \sigma_R|^2 + |(\frac{1}{0}) \cdot \sigma_R|^2$. For the same reason, we have $|(\frac{0}{1}) \cdot \sigma_R| / |\sigma_R|$ increasing in s on $\{\kappa < 1\}$. □

C.3 Numerical method for Markov S-BSE of Section 3.1

Step 0: Let (i, j) index the state space $\mathcal{N} \times \mathcal{S} = (0, 1) \times (0, 1)$. Let $i \in \{1, \dots, I\}$ and $j \in \{1, \dots, J\}$.

Step 1: Construct the candidate capital price function $q(\eta_i, s_j)$.

- (a) Find the fundamental solution $q^0(\eta_i)$ to the economy from the ODE in part (i) of Lemma E.1. Our example uses this fundamental solution as an upper bound for the sunspot q function. Calculate also the unbounded fundamental solution $\tilde{q}^0(\eta_i)$, i.e., the solution to the ODE that ignores the restriction $\kappa < 1$.
- (b) Construct a lower bound $q^1(\eta_i)$ as follows. Let $q^1(\eta_i) = q^0(\eta_i)$ for $i = 1, 2$. This ensures that $(q^1)'(0+) = (q^0)'(0+)$. Then, set $q^1(\eta_I) = a_e/\rho_e$ and interpolate all other points, i.e., to obtain $q^1(\eta_i)$ for $i = 3, \dots, I-1$. Any interpolation method that delivers a monotonic function should work; we use a linear interpolation. Thus, the lower bound q^1 is approximately equal to the infinite-volatility solution $q^\infty = \frac{\eta a_e + (1-\eta)a_h}{\bar{\rho}(\eta)}$.
- (c) Set the boundary values by $q(\eta_i, s_1) = q^0(\eta_i)$ and $q(\eta_i, s_J) = q^1(\eta_i)$ for all i . For $j \in \{2, \dots, J-1\}$, put $q(\eta_i, s_j) = \min\{a_e/\bar{\rho}(\eta_i), (1-s_j)\tilde{q}^0(\eta_i) + s_j q^1(\eta_i)\}$ for all i .

Step 2: Solve for κ by plugging q into (PO).

Step 3: Compute σ_s as follows.

- (a) For points where $\kappa < 1$, the variance of capital returns $|\sigma_R|^2$ is recovered exactly from (23)—it is the term in parentheses. If $\partial_s \log q \neq 0$, then we solve for $\sigma_s^2 = |\sigma_R|^2(1 - (\kappa - \eta)(\partial_\eta \log q)^2 - \sigma^2)/(\partial_s \log q)^2$. If $\partial_s \log q = 0$ and $\sigma_R \neq 0$, we must have $\sigma_s \rightarrow \infty$ and this is not an equilibrium. This is only the case at $\eta \in \{\eta_1, \eta_2\}$, but as the proof of Proposition 1 shows, this is a non-issue. First, for η_1 , equation (23) will hold independently of the value for σ_s . Second, for η_2 , the proof shows a mollification (“smoothing out”) is possible in a small neighborhood around the point to eliminate this issue. Since we study a finite-difference approximation, we may therefore take the neighborhood smaller than the grid spacing and ignore the issue. Therefore, we may set $\sigma_s(\eta_1, s_j)$ and $\sigma_s(\eta_2, s_j)$ by any extrapolation of $\sigma_s(\eta_i, s_j)$, for $i \geq 3$ and each j .
- (b) For points where $\kappa = 1$, we have $\partial_s \log q = 0$, and σ_s can take any value (in our numerical example, we set $\sigma_s(\eta_I, s_j) = 0$ and interpolate the rest of the values).

C.4 Model with jumps in Section 3.4

Recall that our jumps ℓ_q are assumed to occur randomly but have a known size, given observables. Therefore, optimal portfolio conditions are

$$\begin{aligned}\frac{a_e}{q} + g + \mu_q + \sigma\left(\frac{1}{0}\right) \cdot \sigma_q - r &= \frac{\kappa}{\eta} |\sigma_R|^2 + \frac{\lambda \ell_q}{1 - \frac{\kappa}{\eta} \ell_q} \\ \frac{a_h}{q} + g + \mu_q + \sigma\left(\frac{1}{0}\right) \cdot \sigma_q - r &\leq \frac{1 - \kappa}{1 - \eta} |\sigma_R|^2 + \frac{\lambda \ell_q}{1 - \frac{1 - \kappa}{1 - \eta} \ell_q}.\end{aligned}$$

Combining these two equations, we obtain (RBJ).

We can determine the other equilibrium objects similarly to before. The riskless rate is given by

$$r = \frac{\kappa a_e + (1 - \kappa) a_h}{q} + g + \mu_q + \sigma\left(\frac{1}{0}\right) \cdot \sigma_q - \left(\frac{\kappa^2}{\eta} + \frac{(1 - \kappa)^2}{1 - \eta}\right) |\sigma_R|^2 - \lambda \ell_q \left(\frac{\kappa}{1 - \frac{\kappa}{\eta} \ell_q} + \frac{1 - \kappa}{1 - \frac{1 - \kappa}{1 - \eta} \ell_q}\right).$$

The dynamics of η are now given by $d\eta_t = \mu_{\eta,t} dt + \sigma_{\eta,t-} \cdot dZ_t - \ell_{\eta,t-} dJ_t$, where

$$\mu_\eta = \eta(1 - \eta)(\rho_h - \rho_e) + (\kappa - 2\eta\kappa + \eta^2) \frac{\kappa - \eta}{\eta(1 - \eta)} |\sigma_R|^2 + \delta_h - (\delta_e + \delta_h)\eta + \frac{(\kappa - \eta)\lambda \ell_q}{(1 - \frac{\kappa}{\eta} \ell_q)(1 - \frac{1 - \kappa}{1 - \eta} \ell_q)}$$

$$\sigma_\eta = (\kappa - \eta)\sigma_R.$$

The wealth share jump ℓ_η is derived by using knowledge of the jump size in q and noting that agents' portfolios (capital and bonds) are predetermined:⁴²

$$\ell_\eta = (\kappa - \eta) \frac{\ell_q}{1 - \ell_q}.$$

For a valid equilibrium, jumps cannot be so large as to send experts into bankruptcy, nor can they induce households' leverage to exceed experts' (as this would contradict (RBJ)). It turns out the no-bankruptcy condition, which says $\ell_q < \kappa/\eta$, is automatically satisfied given (RBJ) holds; intuitively, experts would never take so much risk that their wealth is wiped out. The other requirement, that jumps not send the economy into a region in

⁴²The derivation is as follows. Let variables with hats, e.g., " \hat{x} ", denote post-jump variables. Note $\hat{N}_e = \hat{q}\hat{K} - B$ and $\hat{N}_h = \hat{q}\hat{K}(1 - \kappa) + B$, where B is expert borrowing (and household lending, by bond market clearing). Then, $\hat{\eta} = \hat{N}_e/(\hat{q}\hat{K}) = \kappa - B/(\hat{q}\hat{K})$ and by similar logic the pre-jump wealth share is $\eta = \kappa - B/qK$. Thus, $\ell_\eta = \eta - \hat{\eta} = B[1/(\hat{q}\hat{K}) - 1/(qK)] = qK(\kappa - \eta)[1/(\hat{q}\hat{K}) - 1/(qK)]$. Using the fact that $\hat{K} = K$ and the definition $\ell_q := 1 - \hat{q}/q$, we arrive at $\ell_\eta = (\kappa - \eta)[(1 - \ell_q)^{-1} - 1]$. This derivation assumes the presumably risk-free bond price does not jump when capital prices jump. Conceptually, there is no reason why this needs to be true, but it preserves its risk-free conjecture. If bond prices are allowed to jump at the same time, we would find different expressions.

which $\eta \leq \kappa$, can be stated as

$$\bar{\rho}(\hat{\eta})(1 - \ell_q)q > (a_e - a_h)\hat{\eta} + a_h, \quad (\text{C.21})$$

where $\hat{\eta} := \eta - (\kappa - \eta)\frac{\ell_q}{1 - \ell_q}$ is the post-jump expert wealth share. Although it is obvious, (RBJ) implies another bound on ℓ_q that arises because of $|\sigma_R| \geq 0$, which is

$$\frac{a_e - a_h}{q} \geq \frac{\kappa - \eta}{\eta(1 - \eta)} \left[\frac{\lambda \ell_q^2}{\left(1 - \frac{\kappa}{\eta} \ell_q\right) \left(1 - \frac{1 - \kappa}{1 - \eta} \ell_q\right)} \right]. \quad (\text{C.22})$$

Condition (C.22) evaluated at equality implies that all risk is jump risk. With these equations in hand, we describe our simulation procedure.

Step 0. Given (η, q) solve for $\kappa(\eta, q)$ from (PO).

Step 1. Solve for the upper bound of $\ell_q(\eta, q)$ using (C.21)-(C.22).

Note that, fixing (η, q) , the RHS of (C.22) is strictly increasing in ℓ_q when $\ell_q \in (0, \frac{\eta}{\kappa})$ while the LHS is constant. Moreover, the inequality is satisfied for $\ell_q = 0$ and violated as $\ell_q \rightarrow \frac{\eta}{\kappa}$. Hence, this condition defines an upper bound $\ell_q^A(\eta, q)$, which can be solved by a bisection procedure.

Next, after some algebra, we can write condition (C.21) as

$$(1 - \ell_q)^2 - (1 - \ell_q) + \underbrace{\frac{(a_e - a_h)(\kappa - \eta)}{\bar{\rho}(\eta)q + q(\rho_e - \rho_h)(\kappa - \eta)}}_{:=\varphi(\eta, q)} > 0.$$

It is straightforward to notice that the condition holds for any $\ell_q \in (0, 1)$ if $\varphi(\eta, q) \geq 1/4$. When $\varphi(\eta, q) < 1/4$, then the condition holds for $\ell_q \in (0, \ell_q^{B, low}) \cup (\ell_q^{B, high}, 1)$, where

$$1 - \ell_q^{B, high} = \frac{1}{2} \left(1 - \sqrt{1 - 4\varphi} \right) \quad \text{and} \quad 1 - \ell_q^{B, low} = \frac{1}{2} \left(1 + \sqrt{1 - 4\varphi} \right).$$

Define

$$\ell_q^B := \mathbf{1}_{\{\varphi \geq 1/4\}} + \ell_q^{B, low} \mathbf{1}_{\{\varphi < 1/4\}}.$$

Then, an upper bound that ensures all required inequalities are satisfied is

$$\ell_q^{\max}(\eta, q) := \min\{\ell_q^A(\eta, q), \ell_q^B(\eta, q)\}.$$

Step 2. Choose a sub-region within the domain $\mathcal{D} := \{(\eta, q) : 0 < \eta < 1 \text{ and } \eta a_e + (1 - \eta)a_h < q\bar{\rho}(\eta) \leq a_e\}$ that is away from the upper and lower boundaries. For example, in our numerical exercise, we choose the sub-region $\mathcal{D}^\circ := \{(\eta, q) : \kappa < 0.98 \text{ and } \kappa > \eta + 0.02\}$. On $\mathcal{D} \setminus \mathcal{D}^\circ$, we will set $\ell_q = 0$ and choose μ_q to ensure the economy never escapes \mathcal{D} . In fact, we can choose μ_q in a way that the boundary of \mathcal{D}° acts arbitrarily close to a reflecting boundary, which is what we have done in for Figure 8. Pick an arbitrary function $\ell_q(\eta, q) \in [0, \ell_q^{\max}(\eta, q))$ and an arbitrary μ_q for the set \mathcal{D}° .

Step 3. Use risk-balance condition (RBJ) to solve for $|\sigma_R|^2$. For each (η, q) , assign $\gamma(\eta, q)$ fraction of the variance to the fundamental Brownian shock, and $1 - \gamma(\eta, q)$ to the sunspot Brownian shock. In constructing Figure 8, we set $\gamma \equiv 1$. Then, solve for other equilibrium objects from the equations above.

Online Appendix 2 (not for publication): Rational Sentiments and Financial Frictions

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D Model extensions and further analyses

D.1 Partial equity issuance

We extend the model to allow some equity issuance by capital holders, subject to a constraint. In particular, at any point of time, agents managing capital can issue some equity to the market, but the issuer must keep at least $\chi \in [0, 1]$ fraction of their capital risk—this is a so-called “skin-in-the-game” constraint. In other words, if experts and households retain χ_e and χ_h of their capital risk, respectively, it must be the case that

$$\chi_{\ell,t} \geq \chi, \quad \ell \in \{e, h\}. \quad (\text{D.1})$$

Thus, the frictionless model corresponds to $\chi = 0$, while our baseline model corresponds to $\chi = 1$. Outside equity contracts are risky, having risk exposure σ_R (the endogenous capital return volatility), so they must promise an excess return $\sigma_R \cdot \pi$, where π is the equilibrium risk price vector that applies to securities tradable by both experts and households.

Agents’ dynamic budget constraints are now given by

$$\begin{aligned} dn_{\ell,t} = & \left[(n_{\ell,t} - q_t k_{\ell,t}) r_t - c_{\ell,t} + a_{\ell} k_{\ell,t} \right] dt + d(q_t k_{\ell,t}) \\ & + [\theta_{\ell,t} n_{\ell,t} - (1 - \chi_{\ell,t}) q_t k_{\ell,t}] \sigma_{R,t} \cdot (\pi_t dt + dZ_t). \end{aligned} \quad (\text{D.2})$$

The second line of (D.2) contains the new terms pertaining to equity-issuance: $\theta_{\ell,t} \geq 0$ denotes purchases of equity contracts in the market, per unit of wealth, while $\chi_{\ell,t}$ denotes the fraction of capital risk. Notice that it will be without loss of generality to assume $\chi_{\ell,t} = \chi$ at all times and for all agents, because the purchase variable $\theta_{\ell,t}$ is available as a control. For example, an agent with a slack equity-issuance constraint ($\chi_{\ell} > \chi$) could issue equity to the constraint (D.1) and then buy back such equity by increasing their θ_{ℓ} control. Going forward, we simply assume $\chi_{e,t} = \chi_{h,t} = \chi$. The presence of a public equity market implies an additional market clearing condition for equity

securities, namely

$$\theta_{e,t}N_{e,t} + \theta_{h,t}N_{h,t} = (1 - \chi)q_tK_t. \quad (\text{D.3})$$

At this point, we may solve for equilibrium.

Model solution. The introduction of equity issuance changes nothing about optimal consumption choices, so the price-output relation (PO) still holds.

Optimal portfolio choice now implies the following four FOCs:

$$\mu_{R,e} - (1 - \chi)\sigma_R \cdot \pi - r = \chi \left(\frac{\chi q k_e}{n_e} + \theta_e \right) |\sigma_R|^2 \quad (\text{D.4})$$

$$\mu_{R,h} - (1 - \chi)\sigma_R \cdot \pi - r \leq \chi \left(\frac{\chi q k_h}{n_h} + \theta_h \right) |\sigma_R|^2, \quad \text{with equality if } k_h > 0 \quad (\text{D.5})$$

$$\left(\frac{\chi q k_e}{n_e} + \theta_e \right) |\sigma_R|^2 \geq \sigma_R \cdot \pi, \quad \text{with equality if } \theta_e > 0 \quad (\text{D.6})$$

$$\left(\frac{\chi q k_h}{n_h} + \theta_h \right) |\sigma_R|^2 \geq \sigma_R \cdot \pi, \quad \text{with equality if } \theta_h > 0 \quad (\text{D.7})$$

where $\mu_{R,\ell} := \frac{a_\ell}{q} + g + \mu_q + \sigma\sigma_q \cdot \left(\frac{1}{0}\right)$ is the expected return on capital for agent ℓ . Equations (D.4)-(D.5) are the FOCs for capital holdings, and (D.6)-(D.7) are the FOCs for equity purchases. Note that the equality in (D.4) assumes $k_e > 0$, which is easy to verify must always be the case in equilibrium, exactly as in the baseline model.

We can derive a new “risk-balance” condition, analogously to the baseline model. If in addition to $k_e > 0$ we have $k_h > 0$, then we cannot simultaneously have $\theta_e > 0$, as this would contradict $\mu_{R,e} > \mu_{R,h}$. Thus, $\theta_e = 0$ whenever $k_h > 0$, and so we may difference (D.4)-(D.5) and use the market clearing condition (D.3) to substitute $\theta_h = \frac{1-\chi}{1-\eta}$, which leads to

$$0 = \min \left[1 - \kappa, \frac{a_e - a_h}{q} - \chi \frac{\chi\kappa - \eta}{\eta(1 - \eta)} |\sigma_R|^2 \right]. \quad (\text{RBE})$$

In addition to (RBE), equation (D.7) must hold with equality and (D.6) with inequality when $\kappa < 1$. By (D.7) and the derived expression $\theta_h = \frac{1-\chi}{1-\eta}$, we have $\sigma_R \cdot \pi = \frac{1-\chi\kappa}{1-\eta} |\sigma_R|^2$, for which a viable solution is

$$\pi = \frac{1 - \chi\kappa}{1 - \eta} \sigma_R, \quad \text{if } \kappa < 1. \quad (\text{D.8})$$

Using this expression for π , (D.6) requires $\chi\kappa \geq \eta$, which holds by equation (RBE).

By contrast, when $k_h = 0$ (so $\kappa = 1$), equations (D.6)-(D.7) imply

$$\pi = \min \left(1, \frac{1 - \chi}{1 - \eta} \right) \sigma_R, \quad \text{if } \kappa = 1. \quad (\text{D.9})$$

To prove this, combine the two possible cases:

- (i) Suppose $\theta_e > 0$. Note that $\theta_h = 0$ cannot occur, as $\theta_e > 0$ implies $\sigma_R \cdot \pi > 0$ while $k_h = \theta_h = 0$ implies the opposite. Thus, we may combine (D.6)-(D.7), both evaluated under equality, to obtain $\theta_h = \theta_e + \frac{\chi}{\eta}$. Plugging this result into market clearing (D.3) yields $\theta_e = 1 - \chi/\eta$ and $\theta_h = 1$. Using $\theta_h = 1$ back in (D.7), we obtain $\sigma_R \cdot \pi = |\sigma_R|^2$, for which a viable solution is $\pi = \sigma_R$. Note that $\theta_e = 1 - \chi/\eta > 0$ if and only if $\eta > \chi$.
- (ii) Suppose $\theta_e = 0$. Note that market clearing (D.3) implies $\theta_h = \frac{1 - \chi}{1 - \eta} > 0$ in this case. By (D.7), we have $\sigma_R \cdot \pi = \frac{1 - \chi}{1 - \eta} |\sigma_R|^2$, for which a viable solution is $\pi = \frac{1 - \chi}{1 - \eta} \sigma_R$. Using the expression for π , (D.6) requires $\eta \leq \chi$.

Putting the results of (D.8)-(D.9) together, we have that

$$\pi = \min \left(1, \frac{1 - \chi\kappa}{1 - \eta} \right) \sigma_R. \quad (\text{D.10})$$

Finally, the riskless interest rate can be derived as always, by summing a $(\kappa, 1 - \kappa)$ -weighted-average of equations (D.4)-(D.5) to get

$$\begin{aligned} r = & \frac{\kappa a_e + (1 - \kappa) a_h}{q} + g + \mu_q + \sigma \sigma_q \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - (1 - \chi) \sigma_R \cdot \pi \\ & - \chi \left[\kappa \left(\frac{\chi\kappa}{\eta} + \theta_e \right) + (1 - \kappa) \left(\frac{\chi(1 - \kappa)}{1 - \eta} + \theta_h \right) \right] |\sigma_R|^2. \end{aligned} \quad (\text{D.11})$$

We can simplify this equation using the following facts. First, from the discussion above, $\theta_h > 0$ always holds, so that (D.7) holds with equality, hence $\theta_h = \frac{\sigma_R \cdot \pi}{|\sigma_R|^2} - \frac{\chi(1 - \kappa)}{1 - \eta}$. Next, we may use the market clearing condition (D.3) to obtain $\theta_e = \frac{1 - \chi}{\eta} - \frac{1 - \eta}{\eta} \theta_h$. We use these two facts to eliminate θ_e and θ_h from (D.11), then we substitute the solution for π from (D.10), and finally we simplify the result to obtain

$$r = \frac{\kappa a_e + (1 - \kappa) a_h}{q} + g + \mu_q + \sigma \sigma_q \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} - |\sigma_R|^2 - \left(\frac{\chi\kappa}{\eta} - 1 \right) \max \left(0, \frac{\chi\kappa - \eta}{1 - \eta} \right). \quad (\text{D.12})$$

This completes the derivation of equilibrium.

Properties of equilibrium. For any $\chi > 0$, we can construct S-BSEs using a similar procedure as the baseline model, i.e., by solving equation (PO) for κ as a function of (η, q) , and then substituting this into (RBE) to also solve for $|\sigma_R|$ as a function of (η, q) . Importantly, any solution to equation (RBE) requires $\chi\kappa \geq \eta$, and so the effect of lower equity issuance frictions (lower χ) is to reduce the range of possible fluctuations of κ , hence q , for any given η . This effect is depicted in Figure D.1, which shows that the range of possible fluctuations for price q is unambiguously shrinking as χ falls.

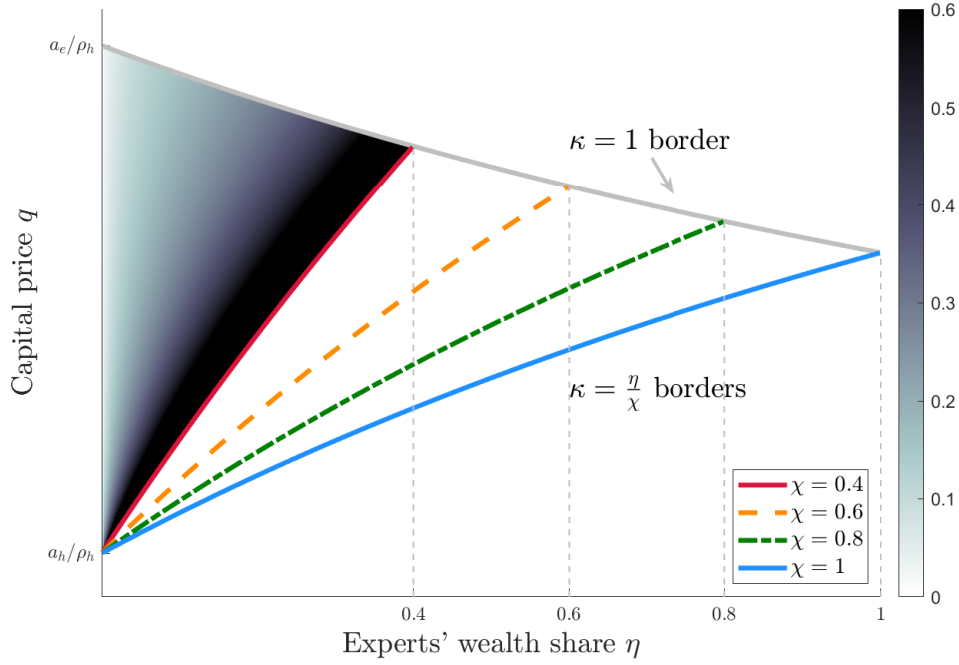


Figure D.1: Colormap of volatility $|\sigma_R|$ as a function of (η, q) , in the region $\mathcal{D} := \{(\eta, q) : \eta \in (0, 1) \text{ and } (\eta/\chi)a_e + (1 - \eta/\chi)a_h < q\bar{\rho}(\eta) \leq a_e\}$. Volatility is truncated for aesthetic purposes (because $|\sigma_R| \rightarrow \infty$ as $\kappa \rightarrow \eta/\chi$). Parameters: $\rho_e = 0.07$, $\rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$.

In particular, as $\chi \rightarrow 0$, no sunspot equilibrium can exist. This is very easy to see—a solution $\kappa < 1$ to (RBE) requires $\chi\kappa \geq \eta$, but as $\chi \rightarrow 0$ this becomes impossible for any $\eta > 0$. Thus, as $\chi \rightarrow 0$, capital misallocation converges to zero, and capital prices converge to their maximum $a_e/\bar{\rho}(\eta)$. Relatedly, taking $\chi \rightarrow 0$ in equation (D.10), we see that $\pi \rightarrow \sigma_R$ for each $\eta > 0$. This is the complete-markets risk price with log utility agents. Thus, as $\chi \rightarrow 0$, risk allocations converge to the frictionless solution. As both capital and risk are allocated frictionlessly in the limit, the First Welfare Theorem obtains.

Proposition D.1. *As $\chi \rightarrow 0$, the set of equilibria converges to a singleton, namely the non-stochastic Fundamental Equilibrium with $\kappa_t = 1$ and $q_t = a_e/\bar{\rho}(\eta_t)$.*

D.2 Beliefs about disaster states

In this section, we outline a richer class of W-BSE when $\sigma = 0$. The entire set of W-BSEs studied here will be indexed by agents' beliefs about the "tail scenario" in the economy, i.e., what happens when experts are severely undercapitalized.

Mathematically, recall that we previously have assumed $\kappa(0) = 0$; in other words, experts fully deleverage as their wealth vanishes. Some intuitive refinements like a small amount of idiosyncratic risk (Appendix D.3) or a small amount of commitment frictions (Appendix D.4) can justify the assumption $\kappa(0) = 0$. However, strictly speaking, $\kappa(0) = 0$ turns out to not be necessary without these refinements, and it will be interesting to relax this assumption.

Consider any $\kappa_0 \in (0, 1)$ and put $\kappa(0) = \kappa_0$. We will call κ_0 the *disaster belief* in the economy. The sunspot equilibrium is similar to Proposition A.1, with the generalization that the boundary condition to the ODE (A.2) is now $\kappa(0) = \kappa_0$ rather than $\kappa(0) = 0$.⁴³

Proposition D.2. *For $\sigma = 0$ and fixed tail belief $\kappa_0 \in (0, 1)$, there exists a W-BSE, with $\sigma_q(\eta) \neq 0$ on a positive measure subset of $(0, 1)$. As $\kappa_0 \rightarrow 0$, this equilibrium converges to the W-BSE of Proposition A.1. As $\kappa_0 \rightarrow 1$, the equilibrium converges to the FE of Lemma A.1.*

Based on Proposition D.2, proved at the end of this section, one can view both the W-BSE and the FE as outcomes of coordination on experts' deleveraging. If experts never sell any capital, there can be no price volatility, with $\sigma_q = 0$ at all times. If agents expect $\kappa_0 = 0$, which translates to full deleveraging and large capital fire sales, then the W-BSE prevails. But for any $\kappa_0 \in (0, 1)$, an intermediate sunspot equilibrium will prevail, with a self-fulfilling amount of expert deleveraging and associated price dynamics. In this simple way, the boundary condition $\kappa_0 \in [0, 1]$ spans an entire range of sunspot equilibria from more to less volatile. An illustration is in Figure D.2.⁴⁴

This result provides a clear illustration of the central property that the degree of capital fire sales is indeterminate in these models. Intuitively, greater optimism about other experts' ability to retain capital in the tail scenario induces smaller capital fire sales

⁴³As in footnote 35, there is a closed-form solution when $\rho_h = \rho_e$, which is

$$q(\eta) = \frac{1}{\rho} \left[(a_e - a_h)\eta + a_h + \sqrt{((a_e - a_h)\eta + a_h)^2 - a_h^2 + (a_e - a_h)^2 \kappa_0^2} \right], \quad \text{for } \eta < \eta^* = \frac{1}{2} \frac{a_e - a_h}{a_e} (1 - \kappa_0^2).$$

As κ_0 decreases, the slope $q'(\eta)$ increases, consistent with the idea that pessimism about the disaster state raises the sensitivity of equilibrium to sunspot shocks away from disaster. Clearly, this solution converges to the W-BSE solution in footnote 35 as $\kappa_0 \rightarrow 0$, and to the FE solution a_e/ρ as $\kappa_0 \rightarrow 1$.

⁴⁴This result is also convenient in some numerical situations. Since the W-BSE is just the limit of equilibria as $\kappa_0 \rightarrow 0$, we can construct an approximate numerical solution with κ_0 very small (but not quite 0).

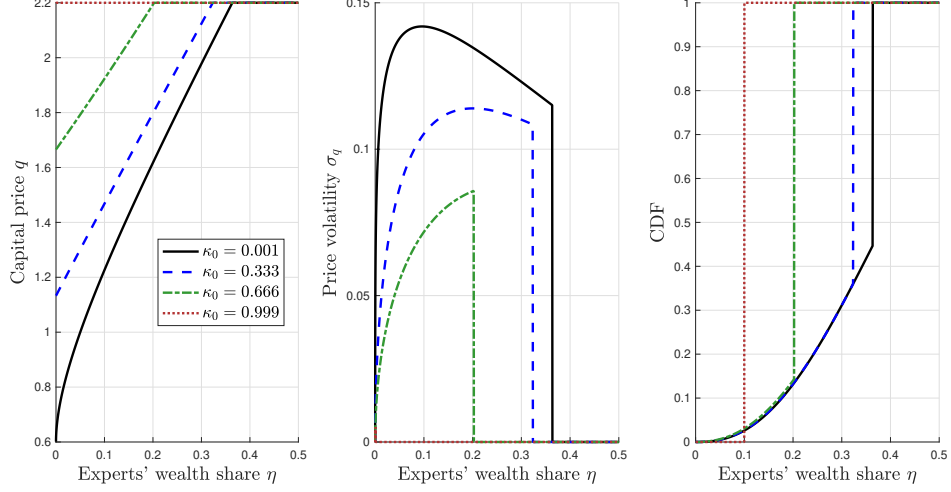


Figure D.2: Capital price q , volatility σ_q , and stationary CDFs of η for different levels of disaster belief κ_0 . Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$. Type-switching parameters (for the CDF): $\delta_h = 0.004$ and $\delta_e = 0.036$.

in response to sunspot shocks, which keeps volatility low, asset prices high, and justifies the optimism.

PROOF OF PROPOSITION D.2. In the first step, we prove existence of an equilibrium for fixed $\kappa_0 \in (0, 1)$. In the second step, we take the limits as $\kappa_0 \rightarrow 0$ and $\kappa_0 \rightarrow 1$.

Step 1: Existence. Let $F(x, y) := \frac{a_e - a_h}{y\bar{\rho}(x) - xa_e - (1-x)a_h}y$. Fix $\epsilon > 0$. Consider the initial value problem $y' = F(x, y)$, with $y(0) = (\kappa_0 a_e + (1 - \kappa_0)a_h)/\rho_h$. As discussed in the text, $y'(0+)$ is bounded, which is enough to ensure that F is bounded and uniformly Lipschitz on $\mathcal{R} := \{(x, y) : 0 < x < 1, xa_e + (1 - x)a_h < y\bar{\rho}(x)\}$. Thus, the standard Picard-Lidelföf theorem implies that there exists a unique solution q^* to this initial value problem, for $\eta \in (0, b)$, some b . Standard continuation arguments can be used to show that either (i) $b = 1$, (ii) $q^*(\eta)$ is unbounded as $\eta \rightarrow b$, or (iii) b satisfies $ba_e + (1 - b)a_h = q^*(b)\bar{\rho}(b)$. If case (ii) is true, since $F > 0$ on \mathcal{R} , we will in fact have $q^*(b-) = +\infty$. Case (iii) is ruled out by the fact that $F(b-, q^*(b-)) = +\infty$. We are left with cases (i) or (ii).

In case (i), we will set $\eta^* = \inf\{\eta \in (0, 1) : q^*(\eta) = a_e/\bar{\rho}(\eta)\}$, with the convention that $\eta^* = 1$ if this set is empty. Note that $\eta^* < 1$ in this case: otherwise $q^*(1-)\bar{\rho}(1-) < a_e$, which implies $F(1-, q^*(1-)) < 0$, which by continuity of q^* and F implies an $\eta^\circ \in (0, 1)$ such that $\eta^\circ a_e + (1 - \eta^\circ)a_h = q^*(\eta^\circ)$, which was just ruled out (case (iii)). In case (ii), we will set $\eta^* = \inf\{\eta \in (0, b) : q^*(\eta) = a_e/\bar{\rho}(\eta)\}$, with the convention that $\eta^* = 1$ if this set is empty. Note that we also clearly have $\eta^* < b < 1$ in this case.

Finally, set $q(\eta) = \mathbf{1}_{\eta < \eta^*} q^*(\eta) + \mathbf{1}_{\eta \geq \eta^*} a_e/\bar{\rho}(\eta)$. This function satisfies $q' = F(\eta, q)$ on $(0, \eta^*)$, $q(0) = (\kappa_0 a_e + (1 - \kappa_0)a_h)/\rho_h$, and $q(\eta^*) = a_e/\bar{\rho}(\eta)$. Thus, we have found a

solution to the capital price satisfying all the desired relations. As discussed in the text, finding such a capital price is enough to prove that a Markov sunspot equilibrium exists.

Since equation (A.3) implies $\sigma_q^2 > 0$ on $(0, \eta^*)$, in order to establish $\sigma_q(\eta) \neq 0$ on a positive measure subset, it suffices to show that $\eta^* > 0$. But this is automatically implied by the boundary condition $q(0) = (\kappa_0 a_e + (1 - \kappa_0) a_h) / \rho_h < a_e / \rho_h$ for $\kappa_0 < 1$, coupled with the continuity of the solution $q(\eta)$.

Step 2: W-BSE and FE as limiting equilibria. For each initial condition $\kappa(0) = \kappa_0$, let $(q_{\kappa_0}, \eta_{\kappa_0}^*)$ be the associated equilibrium capital price and misallocation threshold (at which point households begin purchasing capital).

Define the candidate solution for the W-BSE, $(q_0, \eta_0^*) := \lim_{\kappa_0 \rightarrow 0} (q_{\kappa_0}, \eta_{\kappa_0}^*)$. It suffices to show that q_0 satisfies (i) $q_0' = F(\eta, q_0)$ on $(0, \eta_0^*)$, (ii) $q_0(0) = a_h / \rho_h$, and (iii) $q_0(\eta_0^*) = a_e / \bar{\rho}(\eta_0^*)$. Write the integral version of the ODE:

$$q_{\kappa_0}(\eta) = \frac{\kappa_0 a_e + (1 - \kappa_0) a_h}{\rho_h} + \int_0^\eta F(x, q_{\kappa_0}(x)) dx.$$

Next, we claim that $q_{\kappa_0}(x)$ is weakly increasing in κ_0 , for each x . Indeed, $q_{\kappa_0}(0)$ is strictly increasing in κ_0 . By continuity, we may consider $x^* := \inf\{x : q_{\tilde{\kappa}_0}(x) = q_{\kappa_0}(x)\}$ for some $\tilde{\kappa}_0 > \kappa_0$. In that case, since F does not depend on $\tilde{\kappa}_0$ or κ_0 , we have $q_{\tilde{\kappa}_0}(x) = q_{\kappa_0}(x)$ for all $x \geq x^*$. This proves $q_{\tilde{\kappa}_0}(x) \geq q_{\kappa_0}(x)$ for all x . Combine this with the fact that $\partial_q F < 0$ to see that $\{F(x, q_{\kappa_0}(x)) : \kappa_0 \in (0, 1)\}$ is a sequence which is monotonically (weakly) decreasing in κ_0 , for each x . Thus, by the monotone convergence theorem,

$$q_0(\eta) = \frac{a_h}{\rho_h} + \int_0^\eta F(x, q_0(x)) dx,$$

which proves (i), by differentiating, and (ii), by substituting $\eta = 0$. Similarly,

$$\begin{aligned} q_{\kappa_0}(\eta_{\kappa_0}^*) &= \frac{a_e}{\bar{\rho}(\eta_{\kappa_0}^*)} \\ \xrightarrow{\kappa_0 \rightarrow 0} q_0(\eta_0^*) &= \frac{a_e}{\bar{\rho}(\eta_0^*)}, \end{aligned}$$

which proves (iii).

Define the candidate solution for the FE, $(q_1, \eta_1^*) := \lim_{\kappa_0 \rightarrow 1} (q_{\kappa_0}, \eta_{\kappa_0}^*)$. It suffices to show that $\eta_1^* = 0$, so that $q_1(\eta) = a_e / \bar{\rho}(\eta)$ for all η . Note that $q_{\kappa_0}(0) \rightarrow a_e / \rho_h$ as $\kappa_0 \rightarrow 1$. By continuity of $(q_{\kappa_0}, \eta_{\kappa_0}^*)$ in κ_0 , we also have $q_{\kappa_0}(0) \rightarrow q_1(0)$ as $\kappa_0 \rightarrow 1$. Thus, $q_1(0) = a_e / \rho_h$. By the definition of $\eta_1^* = \inf\{\eta : q(\eta) = a_e / \bar{\rho}(\eta)\}$, we must have $\eta_1^* = 0$. \square

D.3 Idiosyncratic uncertainty

Here, we add idiosyncratic risk to capital. Doing so raises 3 substantive points: (i) small idiosyncratic uncertainty can provide some equilibrium refinement, by selecting equilibria with the property $\lim_{\eta \rightarrow 0} \kappa = 0$; (ii) large idiosyncratic uncertainty eliminates sunspot equilibria where η is the sole state variable; (iii) the addition of idiosyncratic uncertainty allows us to study, in a non-trivial way, the stability properties of the “deterministic steady state” of our model.

Setting. In addition to the model assumptions listed in Section 1, individual capital now evolves as

$$dk_{i,t} = k_{i,t}[gdt + \tilde{\sigma}d\tilde{B}_{i,t}], \quad (\text{D.13})$$

where $(\tilde{B}_i)_{i \in [0,1]}$ is a continuum of independent Brownian motions. Agents with indexes $i \in [0, \nu]$ are experts, and those with $i \in [\nu, 1]$ are households. As in Section 1, the aggregate stock of capital $K_t := \int_0^1 k_{i,t} di$ grows as $dK_t = K_t[gdt + \sigma dZ_t^{(1)}]$. Also as before, the second shock $Z^{(2)}$ is the sunspot shock, independent of $Z^{(1)}$. The idiosyncratic Brownian motions are independent of Z . Besides this addition of idiosyncratic uncertainty, the definition of equilibrium is the same as Definition 1. Conjecture $dq_t = q_t[\mu_{q,t}dt + \sigma_{q,t} \cdot dZ_t]$.

Small uncertainty as equilibrium refinement. The first result in this environment is that *any* equilibrium must feature full deleveraging by experts, as they become poor, simply as a consequence of portfolio optimality. To see this, note that risk-balance condition (RB), the combination of expert and household capital FOCs, is now modified to read

$$0 = \min \left[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1 - \eta)}(\tilde{\sigma}^2 + |\sigma_R|^2) \right], \quad (\text{D.14})$$

where $\sigma_R := \sigma(\frac{1}{0}) + \sigma_q$ is the aggregate diffusion in capital returns, as before. Note that $a_e - a_h > 0$ and $\tilde{\sigma}^2 + |\sigma_R|^2 > 0$. Thus, as $\eta \rightarrow 0$, we must have $\kappa \rightarrow 0$. Since this holds for any arbitrarily small $\tilde{\sigma}$, we conclude that the equilibria with disaster beliefs $\kappa_0 > 0$ (see Section D.2) are not robust.

Lemma D.1. *Any equilibrium with $\tilde{\sigma} > 0$ has the property $\lim_{\eta \rightarrow 0} \kappa = 0$.*

Intuitively, idiosyncratic risk gives experts an additional motive to sell capital. This motive is magnified as experts become relatively poorer, because the risk is embedded in the capital stock, which is then amplified by leverage in affecting experts’ net worth. In fact, the selling motive is magnified infinitely, because experts that do not sell capital

will see their leverage grow unboundedly as their wealth shrinks. Thus, even a small amount idiosyncratic risk is enough to force coordination on maximal selling in response to negative shocks.

Large uncertainty eliminates W-BSEs. In Section A.2, we have demonstrated how sunspot equilibria with η as the sole state variable are incompatible with the presence of exogenous aggregate fundamental risk. Here, we show that the conclusion is similar if the exogenous risk is idiosyncratic rather than aggregate. For this point, we set the aggregate fundamental risk to zero, $\sigma = 0$, and evaluate how W-BSEs (sunspot equilibria that are Markov in η) are affected by $\tilde{\sigma}$.

Even with idiosyncratic risk $\tilde{\sigma}$, one may follow the same analysis as Section A.1 to show that equation (A.2) still determines q if $\sigma_q \neq 0$. In other words, the candidate sunspot equilibrium of this model has a solution (q, κ) , both as functions of η , which are independent of the amount of idiosyncratic risk $\tilde{\sigma}$ (i.e., the same as in the W-BSE). Denote $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$ the boundary point where households begin managing capital. This is also independent of $\tilde{\sigma}$.

Next, use equation (D.14) to solve for σ_q , given the solutions (q, κ) . We get

$$\sigma_q^2 = -\tilde{\sigma}^2 + \frac{\eta(1-\eta)}{\kappa-\eta} \frac{a_e - a_h}{q}, \quad \text{if } \kappa < 1.$$

Since $\sigma_q^2 \geq 0$ is required, an immediate consequence is that $\tilde{\sigma}$ high enough eliminates the existence of any sunspot volatility. We collect these results in the following lemma.

Lemma D.2. *Let (q, κ, η^*) be given by the W-BSE of Proposition A.1. If capital has idiosyncratic risk $\tilde{\sigma}$, and $\tilde{\sigma}^2 \geq \sup_{\eta < \eta^*} \frac{\eta(1-\eta)}{\kappa(\eta)-\eta} \frac{a_e - a_h}{q(\eta)}$, any Markov equilibrium in η requires $\sigma_q = 0$.*

Intuitively, there is a trade-off between endogenous volatility σ_q and exogenous volatility $\tilde{\sigma}$. With higher idiosyncratic volatility $\tilde{\sigma}$, amplification of the aggregate sunspot shock is necessarily reduced. To understand this, consider Merton's optimal capital portfolio when there is only idiosyncratic volatility

$$\frac{qk_j}{n_j} = \frac{a_j/q + g - r}{\tilde{\sigma}^2}, \quad j \in \{e, h\}.$$

As $\tilde{\sigma}$ increases the optimal capital demand becomes more inelastic to changes in the capital price q . Thus, for a given shift in the wealth distribution η and change in capital price q , the amount of capital that changes hands between experts and households will be dampened as $\tilde{\sigma}$ increases. But it is exactly such capital purchases/sales which are the

key ingredient to our sunspot volatility, allowing price fluctuations to be self-fulfilled. As $\tilde{\sigma}$ increases, this mechanism is weakened, leading to a decrease in σ_q . Eventually, the mechanism is severed altogether because $\sigma_q^2 < 0$ is not possible.

Steady state instability. In an attempt to differentiate ourselves from the literature, here we examine the traditional stability properties of this model. The addition of idiosyncratic risk provides a convenient environment for stability analysis, for the following reason. Stability properties are typically studied around the “steady state” of a deterministic equilibrium. In Section A.1 (with $\tilde{\sigma} = 0$), the volatile W-BSE precludes this type of analysis, and studying a deterministic equilibrium instead puts us in the FE, which trivially has $\kappa = 1$ always. With idiosyncratic risk, we can study a fundamental equilibrium in which capital prices evolve deterministically, even though $\kappa < 1$ in steady state. To do this, we set aggregate fundamental risk to zero, $\sigma = 0$, and study the properties of the non-stochastic equilibrium having $\sigma_q = 0$.

The crucial feature this model, as we show below, is that capital prices are determined by a function q such that $q_t = q(\eta_t)$. Supposing that to be true, a steady state is fully characterized by the value $\eta = \eta^{ss}$ such that all non-growing variables are constant over time. This steady state is thus determined by the equation $\dot{\eta} = 0$, where

$$\dot{\eta} = \eta(1 - \eta) \left[\rho_h - \rho_e + \tilde{\sigma}^2 \left(\left(\frac{\kappa}{\eta} \right)^2 - \left(\frac{1 - \kappa}{1 - \eta} \right)^2 \right) \right] + \delta_h - (\delta_e + \delta_h)\eta.$$

It is straightforward to show that equilibrium features stable state variable dynamics, in the sense that $\frac{\partial \dot{\eta}}{\partial \eta} \big|_{\eta = \eta^{ss}} < 0$. However, because the “co-state” q is determined explicitly as a function of η , the steady state is not “stable” in the usual sense required by the multiplicity literature. Technically, there is only one stable eigenvalue of the dynamical system (η_t, q_t) near steady state (η^{ss}, q^{ss}) .

Lemma D.3. *The steady state of the model with idiosyncratic risk is saddle path stable.*

PROOF OF LEMMA D.3. First, we show that q is a function of η , i.e., $q_t = q(\eta_t)$. Goods market clearing is still characterized by the price-output relation (PO). With idiosyncratic risk, the risk-balance condition (RB) is now (D.14). The solution to the system (PO) and (D.14) can be computed explicitly. Indeed, define

$$\eta^* := \sup \{ \eta : (a_e - a_h)\eta\bar{\rho}(\eta) = a_e\tilde{\sigma}^2 \}.$$

Then, $\kappa = 1$ for all $\eta \in (\eta^*, 1)$. For $\eta \in (0, \eta^*)$, we compute $\kappa < 1$ as the positive root $\tilde{\kappa}$

from

$$0 = (a_e - a_h)\tilde{\kappa}^2 + [a_h - \eta(a_e - a_h)]\tilde{\kappa} - \eta a_h - \frac{\eta(1 - \eta)(a_e - a_h)\bar{p}(\eta)}{\tilde{\sigma}^2}.$$

After determining κ for all values of η , capital price q can be computed from (PO), as an explicit function of η .

Given $q_t = q(\eta_t)$, the dynamics of q_t are given by $\dot{q}_t = q'(\eta_t)\dot{\eta}_t$, which only depends on η and not q (notice that $\dot{\eta}_t$ also only depends on η and not q). Consequently, the linearized system near steady state takes the form

$$\begin{bmatrix} \dot{\eta} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} m_1 & 0 \\ m_2 & 0 \end{bmatrix} \begin{bmatrix} \eta \\ q \end{bmatrix}$$

for $m_1, m_2 \neq 0$. The eigenvalues of this system are $m_1 < 0$ and 0. □

As a result of Lemma D.3, there is a unique transition path $(\eta_t, q_t)_{t \geq 0}$ to steady state, given an initial condition η_0 . In other words, q_0 is pinned down uniquely. Our sunspot equilibria are not constructed by randomizing over a multiplicity of transition paths that arise due to steady state stability, which is the usual approach (Azariadis, 1981; Cass and Shell, 1983). This can be seen in a relatively transparent way by examining Lemma D.2, which shows how sunspot equilibria can exist in this model (if $\tilde{\sigma}$ is small enough), despite the instability of the steady state.

D.4 Limited commitment as equilibrium refinement

Here, we add a small limited commitment friction, in the spirit of Gertler and Kiyotaki (2010). The result: only equilibria with the property $\lim_{\eta \rightarrow 0} \kappa = 0$ survive, similarly to equilibria with a small amount of idiosyncratic risk (Appendix D.3).

Suppose capital holders can abscond with a fraction $\lambda^{-1} \in (0, 1)$ of their assets and renege on repayment of their short-term bonds. After doing this diversion, the capital holder would have net worth $\tilde{n}_{j,t} := \lambda^{-1} q_t k_{j,t}$.

To prevent diversion, bondholders will impose some limitation on borrowing. To see this, note that diversion delivers utility $\log(\tilde{n}_{j,t}) + \xi_t$, where ξ_t is an aggregate process (independent of the identity j of the diverter). This is the form of indirect utility for a log utility investor in our model, as discussed in Appendix B.2. For diversion to be sub-optimal, it must be the case that $\log(\tilde{n}_{j,t}) + \xi_t \leq \log(n_{j,t}) + \xi_t$. As a result, bondholders

impose the following leverage constraint to ensure non-diversion is incentive compatible:

$$\frac{q_t k_{j,t}}{n_{j,t}} \leq \lambda. \quad (\text{D.15})$$

We will study the equilibrium with constraint (D.15) additionally imposed, and then we will take $\lambda \rightarrow \infty$ so that the limited commitment friction is vanishingly small.

Risk-balance condition (RB) is now replaced by

$$0 = \min \left[1 - \kappa, \lambda\eta - q\kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1 - \eta)} |\sigma_R|^2 \right]. \quad (\text{D.16})$$

The most important feature of equation (D.16) is that leverage constrained experts ($\lambda\eta = q\kappa$) must hold less than the full capital stock ($\kappa < 1$).

Condition (D.16) implies that there exists a threshold $\eta^\lambda := \inf\{\eta : \lambda\eta > q\kappa\}$ below which experts' leverage constraints bind. By combining $\lambda\eta = q\kappa$ with condition (PO) for κ , we obtain an explicit formula for the capital price in this region:

$$q = \frac{1}{2} \left[\frac{a_h}{\bar{\rho}} + \sqrt{(a_h/\bar{\rho})^2 + 4\lambda\eta(a_e - a_h)/\bar{\rho}} \right], \quad \text{if } \eta \leq \eta^\lambda. \quad (\text{D.17})$$

Taking the limit $\eta \rightarrow 0$ in equation (D.17) shows that $q \rightarrow a_h/\rho_h$ and thus $\kappa \rightarrow 0$. This proves that there is no flexibility for coordination on a worst-case capital price, unlike the leverage-unconstrained economy. The equilibrium worst-case capital price must coincide with $\kappa_0 = 0$.

As the limited commitment problem vanishes ($\lambda \rightarrow \infty$), the leverage constraint becomes non-binding at all times (formally $\eta^\lambda \rightarrow 0$).⁴⁵ But along the sequence, $\kappa_0 = 0$ is uniformly required. (And if we focus on equilibria which are Markov in η , the entire equilibrium converges to the W-BSE of Proposition A.1.) We collect these results.

Lemma D.4. *Among all equilibria, only those with the property $\lim_{\eta \rightarrow 0} \kappa = 0$ survive a vanishingly-small limited commitment friction.*

Intuitively, the leverage constraint gives experts an additional motive to sell capital, which forces coordination on maximal selling in response to negative shocks. Said differently: due to the prospect of violating the leverage constraint, losses incurred from retaining capital when others are selling is larger than losses incurred from selling capital when others are retaining it. This property is reminiscent of “risk dominant” equilibria

⁴⁵This intuitive property can be shown easily by taking $\lambda \rightarrow \infty$ in (D.17). For any fixed $\eta \in (0, 1)$, taking this limit implies $q \rightarrow \infty$, which is ruled out by price-output relation (PO).

being selected by strategic uncertainty (Harsanyi and Selten, 1988; Frankel et al., 2003), but the exact modeling is different here.

D.5 General CRRA preferences

We modify the model by generalizing preferences to the CRRA type. In particular, we replace the $\log(c)$ term in utility specification (3) with the flow consumption utility $c^{1-\gamma}/(1-\gamma)$. For simplicity, we consider no type-switching structure ($\delta_e = \delta_h = 0$), but we continue to allow experts' discount rate to exceed households' ($\rho_e \geq \rho_h$). We impose $\sigma = 0$ so that any non-deterministic equilibrium is a sunspot equilibrium. Finally, we restrict attention to W-BSEs, i.e., those equilibria in which experts' wealth share η is the only state variable.

Equilibrium. The key equation (A.1) still holds, repeated here for convenience:

$$\left[1 - (\kappa - \eta) \frac{q'}{q}\right] \sigma_\eta = 0. \quad (\text{D.18})$$

The sunspot equilibrium is associated with the term in brackets being equal to zero. Unlike with logarithmic preferences, this condition does not pin down $q(\eta)$ function, because we can no longer write $\kappa(q, \eta)$ from the goods market clearing condition: the consumption to wealth ratio is not constant anymore, and depends on agents' value functions.

The value function can be written as $V_i = v_i(\eta) K^{1-\gamma}/(1-\gamma)$ where $v_i(\eta)$ is determined in equilibrium. Then, consumption is $c_i/n_i = (\eta_i q)^{1/\gamma-1}/v_i^{1/\gamma}$ where η_i corresponds to the wealth share of sector i . Then, goods market clearing becomes

$$q^{1/\gamma} \left[\left(\frac{\eta}{v_e} \right)^{1/\gamma} + \left(\frac{1-\eta}{v_h} \right)^{1/\gamma} \right] = (a_e - a_h) \kappa + a_h. \quad (\text{D.19})$$

Optimal portfolio decisions imply that

$$0 = \min \left[1 - \kappa, \frac{a_e - a_h}{q} - \left(\frac{v'_h}{v_h} - \frac{v'_e}{v_e} + \frac{1}{\eta(1-\eta)} \right) (\kappa - \eta) \sigma_q^2 \right]. \quad (\text{D.20})$$

The HJB equation for $i \in \{e, h\}$ has the familiar form $\rho_i V_i = u(c) + \mathbb{E}[\frac{dV_i}{dt}]$, which be-

comes

$$\rho_i = \frac{(\eta_i q)^{1/\gamma-1}}{v_i^{1/\gamma}} + \underbrace{\frac{v'_i}{v_i} \mu_\eta + \frac{1}{2} \frac{v''_i}{v_i} \sigma_\eta^2}_{:= \mu^{v,i}} + (1-\gamma)g. \quad (\text{D.21})$$

The dynamics of η satisfy

$$\sigma_\eta = (\kappa - \eta)\sigma_q \quad (\text{D.22})$$

$$\mu_\eta = \eta(1-\eta) \left(\varsigma_e \frac{\kappa}{\eta} \sigma_q - \varsigma_h \frac{1-\kappa}{1-\eta} \sigma_q + \frac{c_h}{n_h} - \frac{c_e}{n_e} \right) - \sigma_\eta \sigma_q \quad (\text{D.23})$$

and agent-specific risk prices satisfy

$$\varsigma_e = -\frac{v'_e}{v_e} \sigma_\eta + \frac{\sigma_\eta}{\eta} + \sigma_q \quad (\text{D.24})$$

$$\varsigma_h = -\frac{v'_h}{v_h} \sigma_\eta - \frac{\sigma_\eta}{1-\eta} + \sigma_q. \quad (\text{D.25})$$

A Markov equilibrium is a set of functions: prices $\{q, \sigma_q, \varsigma_e, \varsigma_h\}$, allocation $\{\kappa\}$, value functions $\{v_h, v_e\}$ and aggregate state dynamics $\{\sigma_\eta, \mu_\eta\}$ that solve the system (D.18)-(D.25).

The fundamental equilibrium corresponds to the solution for (D.18) where $\sigma_\eta = 0$, which implies deterministic economic dynamics. Then, the capital price has no volatility ($\sigma_q = 0$), risk prices are zero ($\varsigma_e = \varsigma_h = 0$), and experts hold the entire capital stock ($\kappa = 1$). The capital price is then solved from (D.19), and the value functions satisfy

$$\rho_i = \frac{(\eta_i q)^{1/\gamma-1}}{v_i^{1/\gamma}} + \frac{v'_i}{v_i} \underbrace{\eta(1-\eta) \left(\frac{c_h}{n_h} - \frac{c_e}{n_e} \right)}_{=\mu_\eta} + (1-\gamma)g.$$

Conversely, the sunspot equilibrium corresponds to the solution for (D.18) with $\frac{q'}{q} = (\kappa - \eta)^{-1}$ (and potentially $\sigma_\eta \neq 0$).

Disaster belief. With logarithmic preferences, we proved that any sunspot equilibrium must satisfy $\sigma_q(0+) = 0$. This allowed us, in Section D.2, to construct sunspot equilibria with $\kappa(0+) = \kappa_0$ for any $\kappa_0 \in [0, 1)$. With CRRA preferences, we attempt to construct the same class of equilibria, with $\sigma_q(0+) = 0$ and $\kappa_0 \in (0, 1)$.

In order to have a non-degenerate stationary distribution, we have the following requirements. Since $\sigma_\eta(0+) = \kappa_0 \sigma_q(0+) = 0$, the state variable avoids the boundary $\{0\}$

if $\mu_\eta(0+) > 0$. Using (D.20) for $\kappa < 1$, we have⁴⁶

$$\frac{a_e - a_h}{q(0+)} = (\varsigma_e(0+) - \varsigma_h(0+))\sigma_q(0+)$$

which allows us to show that⁴⁷

$$\mu_\eta(0+) = \kappa_0 \frac{a_e - a_h}{q(0+)} > 0.$$

In addition, we need $\mu_\eta(\eta^*+) < 0$ where $\eta^* := \inf\{\eta : \kappa(\eta) = 1\}$. This requirement should be satisfied for $\rho_e - \rho_h$ sufficiently large.⁴⁸

Numerical solution. We do not provide an existence proof—which involves the existence of a solution to the ODE system—but construct numerical examples. For tractability, the numerical examples are constructed for $\kappa_0 > 0$, which keeps $q'(0+) = q(0+)/\kappa_0$ bounded.⁴⁹

The numerical strategy is the following. Construct a grid $\{\eta_1, \dots, \eta_N\}$ with limit points arbitrarily close to but bounded away from zero and one. Conjecture value functions $v_h(\eta)$ and $v_e(\eta)$. Impose $\kappa(\eta_1) = \kappa_0$ and use (D.19) to solve for $q(\eta_1)$. At each interior grid point, use $q' = q/(\kappa - \eta)$ and (D.19) to solve for $\kappa(\eta)$ and $q(\eta)$ until $\kappa(\eta^*) = 1$. In this region, recover σ_q from (D.20). For $\eta \in (\eta^*, 1]$ impose $\kappa(\eta) = 1$ and $\sigma_q = 0$, and solve capital price from (D.19). The rest of equilibrium objects are calculated directly from the system above. The guesses of the value functions are updated by augmenting the HJBs (D.21) with a time derivative and moving a small time-step backward, as in Brunnermeier and Sannikov (2016). The procedure terminates when the value functions converge to time-independent functions.

In Figure D.3, we plot the equilibrium objects as functions of η , for different levels of risk aversion γ . In Figure D.4, we make the same plots, for different levels of the disaster belief κ_0 . Higher risk aversion (higher γ) or more pessimism about disasters (lower κ_0) generates sunspot equilibria featuring lower capital prices and higher volatility.

⁴⁶Note that this implies $\varsigma_e(0+) - \varsigma_h(0+)$ diverges.

⁴⁷This expression also assumes that $\varsigma_h(0+)$ remains bounded. This is a mild assumption since households own all capital.

⁴⁸There is an important distinction between the restriction not to reach $\eta = 0$ and $\mu_\eta(\eta^*+) < 0$. Without the first one, the equilibrium for any $\kappa_0 > 0$ unravels, while without the second one, the equilibrium is still valid, but it has a degenerate stationary distribution at some value $\eta^{ss} > \eta^*$.

⁴⁹With logarithmic utility, we obtain a limiting result in Proposition D.2, that as $\kappa_0 \rightarrow 0$, the equilibrium converges to the W-BSE with $\kappa(0) = 0$. With CRRA, we do not prove such a result analytically, but we do observe numerically what looks like convergence as κ_0 becomes small.

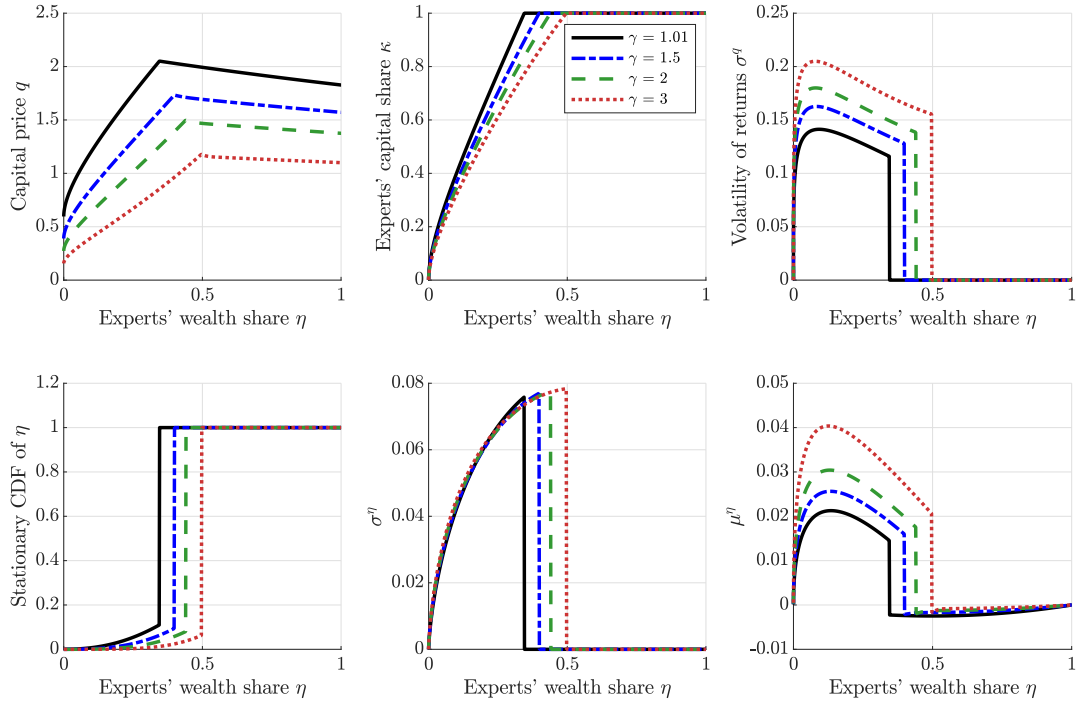


Figure D.3: Sunspot equilibrium for different risk aversion γ . The disaster belief is set to $\kappa_0 = 0.001$. Other parameters: $a_e = 0.11$, $a_h = 0.03$, $\rho_e = 0.06$, $\rho_h = 0.05$, $g = 0.02$.

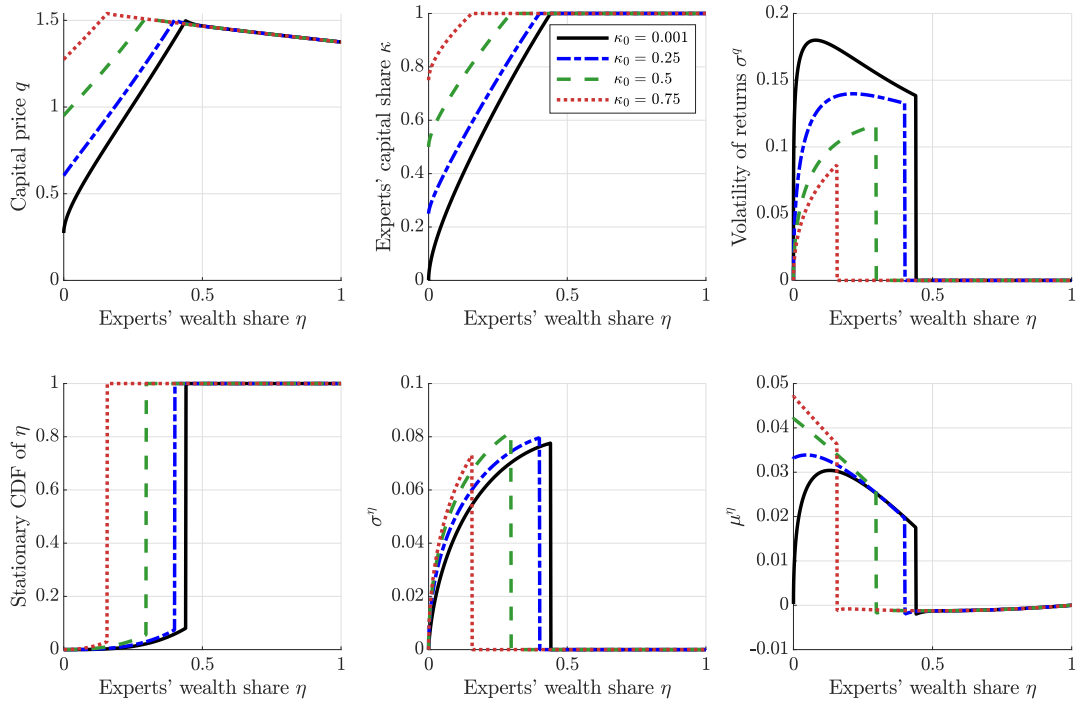


Figure D.4: Sunspot equilibrium for different disaster beliefs κ_0 . Risk aversion is set to $\gamma = 2$. Other parameters: $a_e = 0.11$, $a_h = 0.03$, $\rho_e = 0.06$, $\rho_h = 0.05$, $g = 0.02$.

D.6 Correlation between sentiment and fundamentals

What happens if sentiment shocks are correlated with fundamental shocks? To model this, we allow

$$ds_t = \mu_{s,t}dt + \sigma_{s,t}^{(1)}dZ_t^{(1)} + \sigma_{s,t}^{(2)}dZ_t^{(2)}.$$

In Section 3.1, we restricted attention to $\sigma_{s,t}^{(1)} = 0$. Without this assumption, equations (23) and (22) are modified to read:

$$0 = \min \left[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1 - \eta)} \left(\frac{(\sigma + \sigma_s^{(1)}\partial_s \log q)^2 + (\sigma_s^{(2)}\partial_s \log q)^2}{(1 - (\kappa - \eta)\partial_\eta \log q)^2} \right) \right]$$

$$\sigma_q = \frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix}(\kappa - \eta)\sigma\partial_\eta \log q + \sigma_s\partial_s \log q}{1 - (\kappa - \eta)\partial_\eta \log q}.$$

The rest of the equilibrium restrictions are identical.

For the present illustration, we additionally assume that $\sigma_{s,t}^{(2)} = 0$, i.e., sentiment shocks *only* load on fundamental shocks. What emerges is the possibility that sentiment shocks “hedge” fundamental shocks: we can have $\sigma_s^{(1)}\partial_s \log q < 0$, which lowers return volatility and raises asset prices. In the extreme case, if $\sigma_s^{(1)}\partial_s \log q \rightarrow -\sigma$, the economy will converge to the W-BSE of Section A.1. At the other end, if $\sigma_s^{(1)}\partial_s \log q \rightarrow 0$, the economy resembles the Fundamental Equilibrium (FE) with positive fundamental shocks (this FE was q^{FE} in our baseline construction in Section 3.1). Thus, by constructing our conjectured capital price function as a convex combination of the W-BSE and the FE, with weight $1 - s$ on the W-BSE and s on the FE, we can ensure that $\sigma_s^{(1)}\partial_s \log q$ endogenously emerges negative. Figure D.5 displays the equilibrium constructed this way.

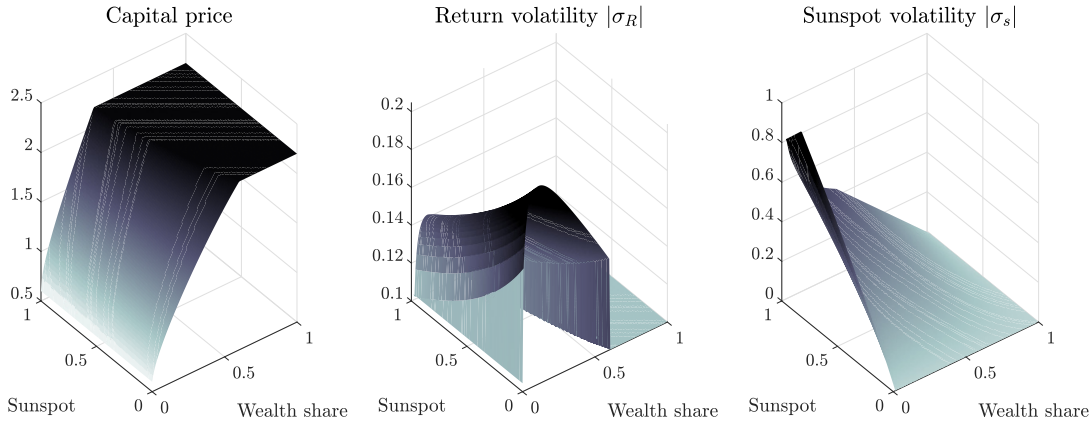


Figure D.5: Capital price q , volatility of capital returns $|\sigma_R|$, and sunspot shock volatility $|\sigma_s|$. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.10$.

D.7 Exogenous sunspot dynamics

In Section 3.1, we solved for a Markov S-BSE that featured endogenous sunspot dynamics, i.e., (σ_s, μ_s) could potentially depend on η . Here, we show that sunspot equilibria can be built on top of exogenous sunspot dynamics as well. As we will show, this construction can be naturally viewed as the limit of equilibria in which the variable s has a vanishing contribution to fundamentals. With that in mind, we actually start from a more general setting in which s can impact fundamental volatility, and then we take the limit as this impact becomes vanishingly small.

Consider the following stochastic volatility model:

$$\begin{aligned}\frac{dK_t}{K_t} &= gdt + \sigma\sqrt{1 + \omega s_t}dZ_t \\ ds_t &= \mu_s(s_t)dt + \vartheta\sqrt{1 + \omega s_t}dZ_t\end{aligned}$$

where $\vartheta > 0$ is an exogenous parameter and $\omega \in \mathbb{R}$ measures the impact of s_t on capital growth volatility. Thus, the diffusion of s_t , namely $\sigma_s(s) := \vartheta\sqrt{1 + \omega s}$, is specified exogenously. Also, $\mu_s(s)$ is an exogenous function that is specified to ensure that $s_t \in (s_{\min}, s_{\max})$, for some pre-specified interval satisfying $s_{\min} \geq 0$ and $\omega s_{\max} > -1$. Such a choice can always be made, e.g., by putting $\mu_s(s) = -(s_{\max} - s)^{-(1+\beta)} + (s - s_{\min})^{-(1+\beta)}$. Note that s_t becomes a sunspot when $\omega = 0$. When $\omega < 0$, the state s_t is an inverse measure of capital's volatility.

For simplicity, we assume there is a single aggregate shock, i.e., Z is a one-dimensional Brownian motion; this can easily be generalized to multiple shocks. Also for simplicity of expressions, we assume here that $\rho_e = \rho_h = \rho$. Then, an equilibrium capital price function $q(\eta, s)$ must satisfy the PDE defined by the following system

$$\begin{aligned}\rho q &= \kappa a_e + (1 - \kappa)a_h \\ 0 &= \min \left[1 - \kappa, \frac{a_e - a_h}{q} - \frac{(\kappa - \eta)(1 + \omega s)}{\eta(1 - \eta)} \left(\frac{\sigma + \vartheta \partial_s \log q}{1 - (\kappa - \eta) \partial_\eta \log q} \right)^2 \right].\end{aligned}$$

Technically, the multiplicity arises from the selection of the boundary conditions on $q(\eta, s_{\min})$ and $q(\eta, s_{\max})$, which are not pinned down by any equilibrium restriction.

We perform two exercises. First, we show that there are multiple equilibria for a given set of parameters. We use $\omega < 0$ here, along with $s_{\min} = 0$ and $s_{\max} = 2$. In this case, the “natural” and intuitive solution is for q to increase with s , because volatility decreases. In Figure D.6, we pick a “low” boundary condition for $q(\eta, 0)$ and the solution follows

this intuition.⁵⁰

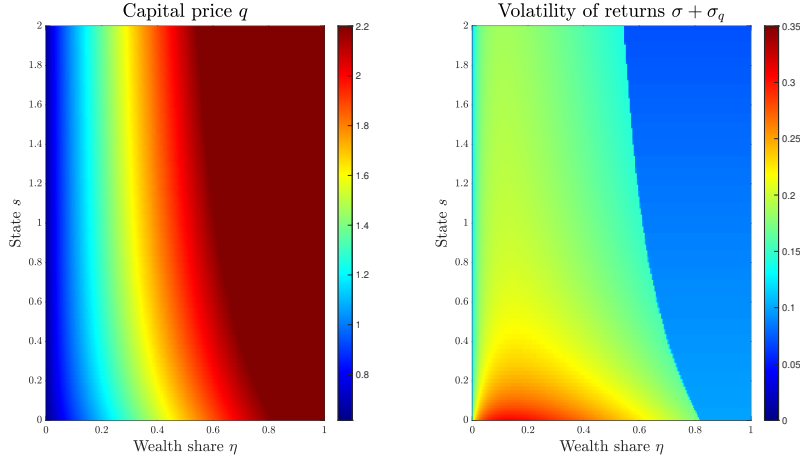


Figure D.6: Equilibrium with $\omega = -0.25$, and the “low” boundary condition for $q(\eta, 0)$, which is a 50% weighted-average of the fundamental equilibrium and the infinite-volatility equilibrium. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$, $\vartheta = 0.25$. The boundary condition at $\eta = 0$ is set so that $\kappa(0, s) = 0.01$ for all s .

However, agents could equally well coordinate on a “high” boundary condition, which results in the solution of Figure D.7.⁵¹ Notice the capital price and return volatility exhibit a non-monotonicity in s . At low values of s , q is decreasing in s , while return volatility increases. The very different behavior in Figures D.6 and D.7 is made possible by coordination on the different boundary conditions.

Our second exercise considers the limit $\omega \rightarrow 0$. Figure D.8 shows the solution for $\omega = -10^{-6}$, again equipped with the “low” boundary condition for $q(\eta, 0)$. There remains a tremendous amount of variation in the equilibrium as s varies, illustrating convergence to a sunspot equilibrium. Thus, as promised, we are able to construct sunspot equilibria even if the dynamics (σ_s, μ_s) are specified exogenously. In fact, it appears that the amount of price volatility is relatively insensitive to the real effects s has (i.e., the size of ω), which is reminiscent of the “volatility paradox” of Brunnermeier and San-nikov (2014) but one level deeper. Their paradox is that total volatility is only modestly sensitive to exogenous fundamental volatility; our paradox is that total volatility is only modestly sensitive to the *exogenous impact of s on fundamental volatility*.

⁵⁰This “low” boundary condition is a weighted average between the solution with infinite volatility and the fundamental equilibrium solution. The fundamental equilibrium, which is the capital price solution that keeps $s = 0$ fixed forever, is discussed in Online Appendix E. The infinite-volatility solution has $\kappa = \eta$, hence $q = (\eta a_e + (1 - \eta) a_h) / \bar{p}(\eta)$.

⁵¹This “high” boundary condition is a weighted average between the W-BSE of Section A.1 (which is a potential solution to the equilibrium with $\sigma = 0$) and the fundamental equilibrium solution.

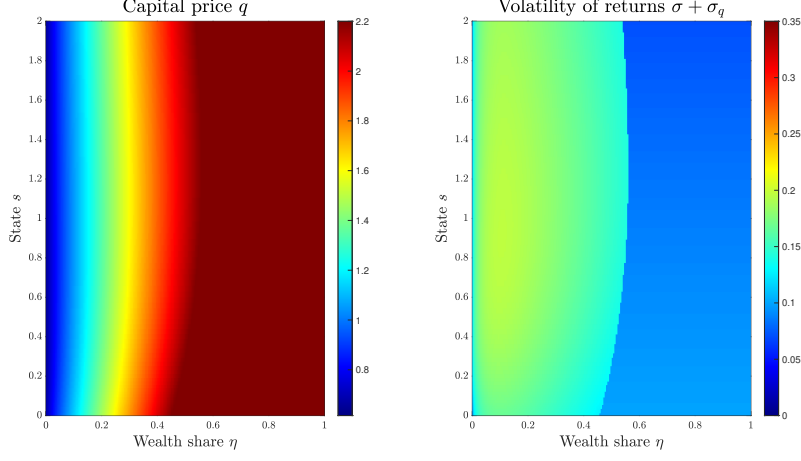


Figure D.7: Equilibrium with $\omega = -0.25$, and the “high” boundary condition for $q(\eta, 0)$, which is a 50% weighted-average of the fundamental equilibrium and a W-BSE. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$, $\vartheta = 0.25$. The boundary condition at $\eta = 0$ is set so that $\kappa(0, s) = 0.01$ for all s .

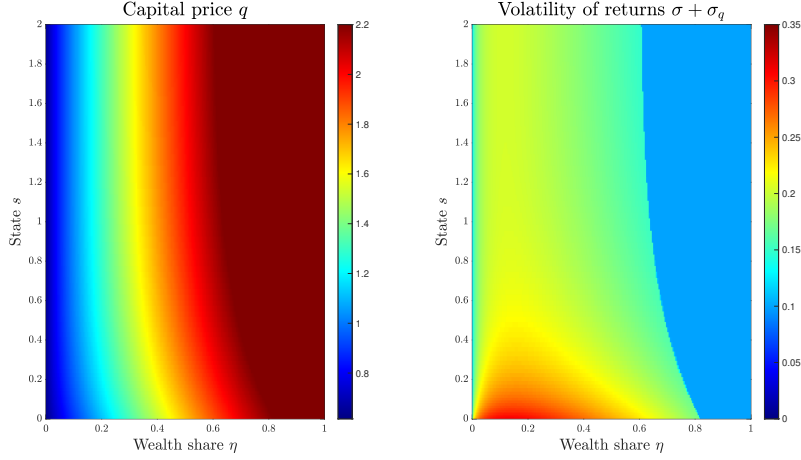


Figure D.8: Equilibrium with near-sunspot $\omega = -10^{-6}$ and the “low” boundary condition for $q(\eta, 0)$, which is a 50% weighted-average of the fundamental equilibrium and the infinite-volatility equilibrium. Other parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.1$, $\vartheta = 0.25$. The boundary condition at $\eta = 0$ is set so that $\kappa(0, s) = 0.01$ for all s .

E Fundamental Equilibria

In this section, we investigate properties of equilibria where sunspot shocks $Z^{(2)}$ are irrelevant and experts’ wealth share η serves as the only state variable, i.e., fundamental equilibria. We illustrate previously undocumented multiplicity along two dimensions: the disaster belief κ_0 and the sign of the sensitivity of capital returns to fundamental shocks $\sigma + \sigma_q$. The key equations describing FEs are (PO), (A.6), and (A.5), restated here

for convenience:

$$q\bar{\rho} = \kappa a_e + (1 - \kappa)a_h \quad (\text{E.1})$$

$$0 = \min \left[1 - \kappa, \frac{a_e - a_h}{q} - \frac{\kappa - \eta}{\eta(1 - \eta)}(\sigma + \sigma_q)^2 \right]. \quad (\text{E.2})$$

$$\sigma_q = \frac{(\kappa - \eta)q'/q}{1 - (\kappa - \eta)q'/q} \sigma. \quad (\text{E.3})$$

Also, wealth share dynamics are given in (13)-(14), restated here for convenience:

$$\mu_\eta = -\eta(1 - \eta)(\rho_e - \rho_h) + \mathbf{1}_{\{\kappa < 1\}}(\kappa - 2\kappa\eta + \eta^2)\frac{a_e - a_h}{q} + \delta_h - (\delta_e + \delta_h)\eta \quad (\text{E.4})$$

$$\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q). \quad (\text{E.5})$$

We define a fundamental equilibrium as follows, analogously to Lemma 1.

Definition 4. Given $\eta_0 \in (0, 1)$, a *Markov fundamental equilibrium* consists of adapted processes $(\eta_t, q_t, \kappa_t, r_t)_{t \geq 0}$ such that (E.1)-(E.3) and (11) hold, and (E.4)-(E.5) describe dynamics of η_t .

Note that the interest rate r_t can be simply set from (11), given the other variables, and it affects no other equilibrium equation. Similarly, the dynamics of η_t are set from (E.4)-(E.5), and they affect none of (E.1)-(E.3). Hence, below, we will often refer to a fundamental equilibrium simply by reference to (q, κ) .

E.1 Properties of the non-sunspot solution with fundamental shocks

We describe here some properties of fundamental equilibria with fundamental volatility $\sigma > 0$, where we additionally impose the full-deleveraging condition $\kappa(0) = 0$.

Lemma E.1. *Assuming it exists, suppose (q, κ) is a fundamental equilibrium in η in the sense of Definition 4. Assume $\kappa(0+) = 0$. Define $\eta^* := \inf\{\eta : \kappa = 1\}$. Then, the following hold:*

- (i) $(\bar{\rho}q - \eta a_e - (1 - \eta)a_h)\frac{q'}{q} = a_e - a_h - \sigma\sqrt{q\frac{\bar{\rho}q - \eta a_e - (1 - \eta)a_h}{\eta(1 - \eta)}}$, for all $\eta \in (0, \eta^*)$.
- (ii) $\eta a_e + (1 - \eta)a_h < \bar{\rho}q < a_e$, for all $\eta \in (0, \eta^*)$.
- (iii) $\frac{q'(0+)}{q(0+)} = \frac{a_e}{a_h} - \frac{\rho_e}{\rho_h} + \rho_h\left(\frac{a_e - a_h}{\sigma a_h}\right)^2$.
- (iv) If σ is sufficiently small, then $q' > \frac{a_e - a_h}{\bar{\rho}}$, for $\eta \in (0, \eta^*)$.
- (v) If σ is sufficiently small, then $\frac{\rho_h}{\rho_e}\left(\frac{1 - a_h/a_e}{\sigma^2} - 1 + \frac{\rho_h}{\rho_e}\right)^{-1} < \eta^* < 1$.

(vi) On $\eta \in (0, \eta^*)$, the solution q is infinitely-differentiable.

PROOF OF LEMMA E.1. Since a fundamental equilibrium is assumed to exist, we make use of equations (E.1) and (E.2). Recall that $\bar{\rho} := \eta\rho_e + (1 - \eta)\rho_h$. By analogy, let $\bar{a} := \eta a_e + (1 - \eta)a_h$.

- (i) Start from equation (E.2), and rearrange to obtain the result, where we have implicitly selected the solution with $1 > (\kappa - \eta)\frac{q'}{q}$.
- (ii) The first inequality, which is equivalent to $\kappa > \eta$, is a direct implication of equation (E.2). The second inequality, equivalent to $\kappa < 1$, is a restatement of the definition of η^* .
- (iii) Start from equation (E.2). Taking the limit $\eta \rightarrow 0$, and using $\kappa(0+) = 0$, delivers an equation for $\kappa'(0+)$. Differentiating (E.1), we may then substitute $\kappa'(0+) = \frac{\rho_h q'(0+) + (\rho_e - \rho_h)q(0+)}{a_e - a_h}$. Rearranging, we obtain the desired result.
- (iv) By part (iii), there exists $\eta^\circ > 0$ and $\bar{\sigma} > 0$ such that uniformly for all $\sigma < \bar{\sigma}$, we have $q' > \frac{a_e - a_h}{\bar{\rho}}$ on the set $\{\eta < \eta^\circ\}$. On the set $\{\eta^\circ \leq \eta < \eta^*\}$, we know that $\kappa - \eta$ is bounded away from zero, uniformly for all $\sigma < \bar{\sigma}$. Using the expression in part (i), the fact that q is bounded by $a_e/\bar{\rho}$ uniformly for all σ , and the previous fact about $\kappa - \eta = \bar{\rho}q - \bar{a}$, we can write

$$q' = \frac{a_e - a_h}{\bar{\rho}q - \bar{a}}q - o(\sigma), \quad \eta \in (\eta^\circ, \eta^*).$$

Therefore,

$$q' + o(\sigma) = \frac{a_e - a_h}{\bar{\rho}q - \bar{a}}q = \frac{a_e - a_h}{\bar{\rho}} \frac{q}{q - \bar{a}/\bar{\rho}} > \frac{a_e - a_h}{\bar{\rho}}, \quad \eta \in (\eta^\circ, \eta^*),$$

where the last inequality is due to $\bar{\rho}q > \bar{a}$ [part (ii)]. Taking σ is small enough implies the result on (η°, η^*) , which we combine with the result on $(0, \eta^\circ)$ to conclude.

- (v) Consider the function $\tilde{q} := \bar{a}/\bar{\rho}$, whose derivative is $\tilde{q}' = \frac{a_e - a_h}{\bar{\rho}} - \frac{\bar{a}}{\bar{\rho}} \frac{\rho_e - \rho_h}{\bar{\rho}} < \frac{a_e - a_h}{\bar{\rho}}$. Combining this result with part (iv), we obtain $q' > \tilde{q}'$. If \tilde{q} was the capital price, then equation (E.1) implies the associated capital share $\tilde{\kappa} = \eta$. On the other hand, the fact that $q' > \tilde{q}'$ implies $\kappa' > \tilde{\kappa}' = 1$, which implies $\eta^* < 1$.

Next, consider $\eta \in (\eta^*, 1)$ so that $\kappa = 1$. By equation (E.2), with $q = a_e/\bar{\rho}$, we must have

$$\sigma^2 \leq \eta \bar{\rho} \frac{a_e - a_h}{a_e} \left(1 + (1 - \eta) \frac{\rho_e - \rho_h}{\bar{\rho}} \right)^2, \quad \eta \geq \eta^*.$$

This is equivalent to

$$1 \leq \eta \frac{\rho_e}{\rho_h} \left(\frac{a_e - a_h}{a_e \sigma^2} \rho_e - 1 + \frac{\rho_h}{\rho_e} \right), \quad \eta \geq \eta^*.$$

Substituting $\eta = \eta^*$, and rearranging, we obtain the first inequality. There is no contradiction with $\eta^* < 1$, due to the assumption that σ can be made small enough.

- (vi) Note that $F(\eta, q) := q \left[\frac{a_e - a_h}{\bar{\rho}(\eta)q - \bar{a}(\eta)} - \sigma \left(\frac{\eta(1-\eta)(\bar{\rho}(\eta)q - \bar{a}(\eta))}{q} \right) \right]$ is infinitely differentiable in both arguments on $\{(\eta, q) : \eta \in (0, 1), \bar{\rho}(\eta)q > \bar{a}(\eta)\}$. Thus, the result is a simple consequence of differentiating part (i), noting that by part (ii) we have $\bar{\rho}(\eta)q(\eta) > \bar{a}(\eta)$, and then using induction. \square

Although the existing literature always imposes $\kappa(0+) = 0$, this is actually not a necessary feature of a fundamental equilibrium.⁵² If we let $\kappa_0 \in (0, 1)$ be a given “disaster belief” about experts’ deleveraging and we suppose $\kappa(0+) = \kappa_0$ (similar to Appendix D.2 for the sunspot case with $\sigma = 0$), there is no inherent contradiction to equilibrium. Existence of such an equilibrium boils down simply to existence of a solution to a first-order ODE. Thus, a variety of fundamental equilibria could exist, and indeed we provide a numerical example after the following lemma and proof.

Lemma E.2. *A fundamental equilibrium with disaster belief $\kappa_0 \in (0, 1)$ exists if the free boundary problem*

$$(\bar{\rho}q - \eta a_e - (1 - \eta)a_h) \frac{q'}{q} = a_e - a_h - \sigma \sqrt{q \frac{\bar{\rho}q - \eta a_e - (1 - \eta)a_h}{\eta(1 - \eta)}}, \quad \text{on } \eta \in (0, \eta^*), \quad (\text{E.6})$$

$$\text{subject to } q(0) = \frac{\kappa_0 a_e + (1 - \kappa_0)a_h}{\rho_h} \quad \text{and} \quad q(\eta^*) = \frac{a_e}{\bar{\rho}(\eta^*)}, \quad (\text{E.7})$$

⁵²Brunnermeier and Sannikov (2014) justify $\kappa_0 = 0$ in their online appendix: “because in the event that η_t drops to 0, experts are pushed to the solvency constraint and must liquidate any capital holdings to households.” This is technically not needed; as shown in Lemma E.2 of Appendix E.1, the dynamics of η_t will not allow it to ever reach 0, so there is no contradiction to equilibrium with both $\kappa_0 > 0$ and $\sigma > 0$. Although we do not prove an existence result, Appendix E.1 presents several numerical examples. The continuum of fundamental equilibria, indexed by κ_0 , may be of independent theoretical interest.

In some sense, the literature has picked the worst possible fundamental equilibrium (minimal-price, maximal-volatility) by imposing $\kappa_0 = 0$. This can be partly justified by the refinement results of Sections D.3 and D.4, which carry over to the case with $\sigma > 0$, i.e., only the belief $\kappa_0 = 0$ survives vanishingly-small idiosyncratic risk or a vanishingly-small limited commitment friction.

has a solution.

PROOF OF LEMMA E.2. A fundamental equilibrium in state variable η exists if and only if equations (E.1), (E.2), and (E.3) hold, and if the time-paths $(\eta_t)_{t \geq 0}$ induced by dynamics (σ_η, μ_η) avoid $\eta = 0$ almost-surely. We will demonstrate these conditions.

Suppose (E.6)-(E.7) has a solution (q, η^*) corresponding to $\kappa_0 \in (0, 1)$. If there are multiple solutions, we pick the one such that $q(\eta) < a_e/\bar{\rho}(\eta)$ for all $\eta \in (0, \eta^*)$, which is always possible because the boundary conditions (E.7) imply $\bar{\rho}(0)q(0) < \bar{\rho}(\eta^*)q(\eta^*)$. Set $q(\eta) = a_e/\bar{\rho}(\eta)$ for all $\eta \geq \eta^*$. Define $\kappa = \frac{\bar{\rho}q - a_h}{a_e - a_h}$. Note that (E.1) is automatically satisfied. Note that (E.3) is also satisfied automatically, by applying Itô's formula to the solution $q(\eta)$ and using $\sigma_\eta = (\kappa - \eta)(\sigma + \sigma_q)$.

We show (E.2) holds separately on $(0, \eta^*)$ and $[\eta^*, 1)$. Using (E.1) and (E.3) in the ODE (E.6) and rearranging, we show that (E.2) holds when $\kappa < 1$. The boundary condition $q(\eta^*) = a_e/\bar{\rho}(\eta^*)$ is equivalent to $\kappa(\eta^*) = 1$, which shows that $\kappa(\eta) < 1$ for all $\eta < \eta^*$. Therefore, we have proven that (E.2) holds on $(0, \eta^*)$.

If $\eta^* = 1$, then we are done verifying (E.2); otherwise, we need to verify (E.2) on $[\eta^*, 1)$. On this set, $\kappa = 1$, so we need to verify

$$\eta \frac{a_e - a_h}{q} \geq (\sigma + \sigma_q)^2 \quad \text{for all } \eta \geq \eta^*. \quad (\text{E.8})$$

First, we show that it suffices to verify this condition exactly at η^* . Indeed, on $(\eta^*, 1)$, we have $\kappa = 1$ and $q = a_e/\bar{\rho}$. Substituting these and (E.3) into (E.8), we obtain

$$(\text{E.8}) \text{ holds} \Leftrightarrow \left(\frac{a_e - a_h}{a_e \sigma^2} \rho_e - \frac{\rho_e - \rho_h}{\rho_e} \right) \eta \geq \frac{\rho_h}{\rho_e} \quad \text{for all } \eta \geq \eta^*.$$

But since the left-hand-side is increasing in η , if it holds at $\eta = \eta^*$, it holds for all $\eta > \eta^*$.

Now, writing (E.8) at η^* , using (E.3) to replace σ_q , and using ODE (E.6) to replace $\eta^* \frac{a_e - a_h}{q(\eta^*)} = \sigma[1 - (1 - \eta^*)q'(\eta^*-)/q(\eta^*)]^{-1}$, we need to verify

$$(\text{E.8}) \text{ holds} \Leftrightarrow \frac{\sigma}{1 - (1 - \eta^*)q'(\eta^*-)/q(\eta^*)} \geq \frac{\sigma}{1 - (1 - \eta)q'(\eta^+)/q(\eta^*)} \Leftrightarrow q'(\eta^*-) \geq q'(\eta^*+).$$

We clearly have $q'(\eta^*-) \geq q'(\eta^*+)$ by the simple fact that $q < a_e/\bar{\rho}$ for $\eta < \eta^*$ and $q = a_e/\bar{\rho}$ for $\eta \geq \eta^*$.

Finally, it remains to verify that η_t almost-surely never reaches the boundary 0. Near

$\eta = 0$, the dynamics in (E.4)-(E.5) become

$$\begin{aligned}\mu_\eta(\eta) &= \kappa_0 \frac{a_e - a_h}{q(0+)} + \delta_h + o(\eta) \\ \sigma_\eta^2(\eta) &= \kappa_0 \frac{a_e - a_h}{q(0+)} \eta + o(\eta).\end{aligned}$$

By the same analysis as in Theorem 1, the boundary 0 is unattainable. \square

What happens in an equilibrium of Lemma E.2 in which $\kappa_0 > 0$? Behavior at the boundary $\eta = 0$ is substantially different than the $\kappa_0 = 0$ case, because equation (E.2) can only hold there if $\sigma_q \rightarrow -\sigma$ as $\eta \rightarrow 0$. Capital prices “hedge” fundamental shocks to capital, in a brief region of the state space $(0, \eta^{\text{hedge}})$. Said differently, given the formula (E.3), the fact that $\sigma_q(0+) = -\sigma$ implies $q'(0+) = -\infty$, so that prices rise as experts lose wealth in a region of the state space. The hedging region is exactly what incentivizes experts to take so much leverage (indeed, expert leverage κ/η blows up near 0). For $\eta > \eta^{\text{hedge}}$, this behavior reverses, and the equilibrium behaves very much like the equilibrium with $\kappa_0 = 0$. Overall, there is no inconsistency with equilibrium even though $q' < 0$ in the region $(0, \eta^{\text{hedge}})$.⁵³

Figure E.1 displays several examples of equilibria with different choices of $\kappa_0 > 0$. The solid black lines, which are equilibrium outcomes with $\kappa_0 = 0.001$, corresponds approximately to the equilibrium choice made by Brunnermeier and Sannikov (2014). The other curves, with higher disaster beliefs κ_0 , are new to the literature. Similar to the the sunspot results of Section D.2, more optimistic disaster beliefs raise capital prices and reduce capital price volatility.

E.2 The “hedging” equilibrium

The equilibria described in Appendix E.1 are “normal” in the sense that a positive exogenous shock increases asset prices and experts’ wealth share.⁵⁴ But technically, agents

⁵³One may think that $q'(0+) = -\infty$, and more generally that $q' < 0$ in some region of the state space, could imply that κ hits η at some point. However, this cannot happen. Indeed, since $\kappa_0 > 0$, we have that $q(0+) > \tilde{q}(0+)$, where $\tilde{q}(\eta) := ((a_e - a_h)\eta + a_h)/\bar{\rho}$ is the price function consistent with $\kappa = \eta$. Now, assume there is an $\hat{\eta} \in (0, 1)$ such that $\kappa(\hat{\eta}) = \hat{\eta}$ (or equivalently, $q(\hat{\eta}) = \tilde{q}(\hat{\eta})$). If there is more than one, consider the minimum among them, so $q(\eta) > \tilde{q}(\eta)$ for all $\eta \in (0, \hat{\eta})$. From the $\tilde{q}(\eta)$ definition, we have $\tilde{q}'(\eta) = (a_e - a_h)/\bar{\rho} - ((a_e - a_h)\hat{\eta} + a_h)(\rho_e - \rho_h)/\bar{\rho}^2 < \infty$, while from (E.6) it must be that $q'(\hat{\eta}-) \rightarrow \infty$. But this implies that q crosses \tilde{q} from below, contradicting $q(\eta) > \tilde{q}(\eta)$ on $\eta \in (0, \hat{\eta})$.

⁵⁴Except, as we have discussed above, if full-deleveraging does not hold ($\kappa_0 > 0$), then there is a (small) region $\eta \in (0, \eta^{\text{hedge}})$ on which $q' < 0$, so negative shocks reduce experts’ wealth share but increase asset prices. But broadly speaking, especially if accounting for the stationary distribution, the equilibria feature $q' > 0$ and thus the “normal” behavior (see Figure E.1).

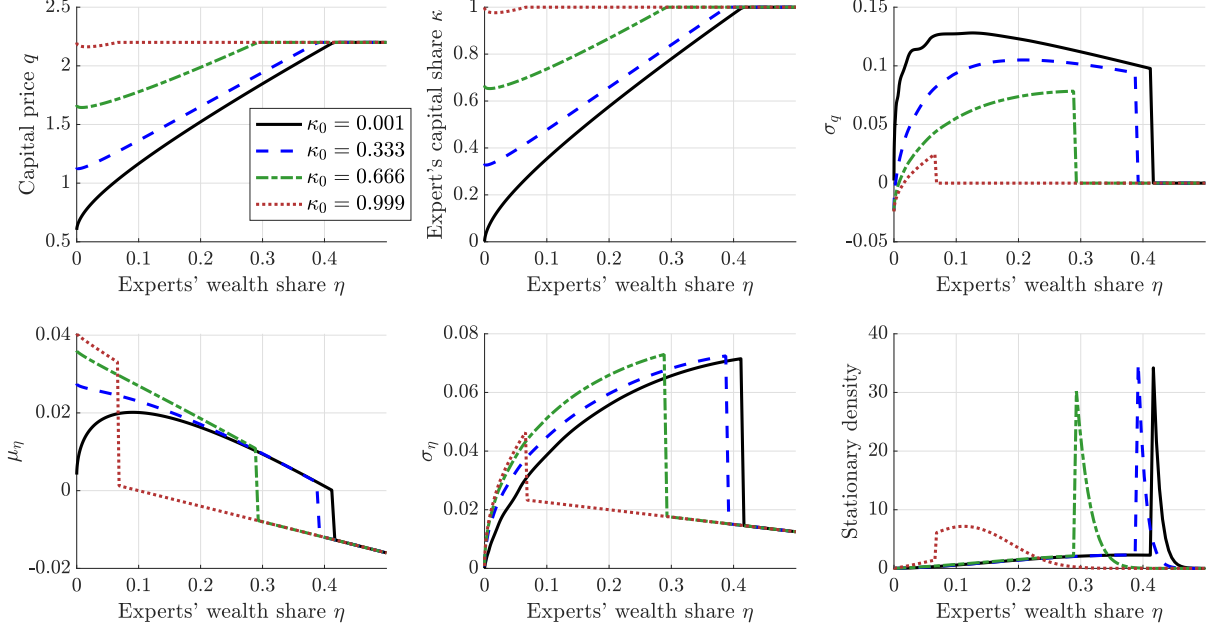


Figure E.1: Fundamental equilibria with different disaster beliefs κ_0 . Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. Type-switching parameters: $\delta_h = 0.004$ and $\delta_e = 0.036$.

do not care about the direction prices move when they make their portfolio choices. They only care about risk which is measured in return variance; this can be seen in the optimality condition (E.2) where $(\sigma + \sigma_q)^2$ appears. An immediate implication is that two types of equilibria are possible: the “normal” one has $\sigma + \sigma_q > 0$; an alternative equilibrium has $\sigma + \sigma_q < 0$.⁵⁵

We term this latter equilibrium the “hedging” equilibrium because asset price movements move oppositely to exogenous shocks. In fact, asset price responses are so strong in opposition that experts actually gain in wealth share upon a negative fundamental shock. This can only happen because of coordination: experts and households simply believe negative shocks are good news for asset prices, so they rush to purchase capital, which percolates through equilibrium relationships to justify beliefs about price increases. Such coordination stands in contrast to the normal equilibrium, in which negative shocks beget fire sales that push down asset prices.

Mathematically, we need only solve a slightly different capital price ODE. Whereas ODE (E.6) holds in the normal equilibrium, the hedging equilibrium requires

$$(\bar{\rho}q - \eta a_e - (1 - \eta)a_h) \frac{q'}{q} = a_e - a_h + \sigma \sqrt{q \frac{\bar{\rho}q - \eta a_e - (1 - \eta)a_h}{\eta(1 - \eta)}}, \quad \text{on } \eta \in (0, \eta^*). \quad (\text{E.9})$$

⁵⁵For a conjecture of this specific type of indeterminacy, see footnote 16 of [Kiyotaki and Moore \(1997\)](#).

The difference between (E.9) and (E.6) is merely the sign in front of σ , which ensures different signs for σ_q . Finally, note that just like the normal equilibria, hedging equilibria could exist for $\kappa_0 \neq 0$. Figure E.2 compares a normal equilibrium to a hedging equilibrium.

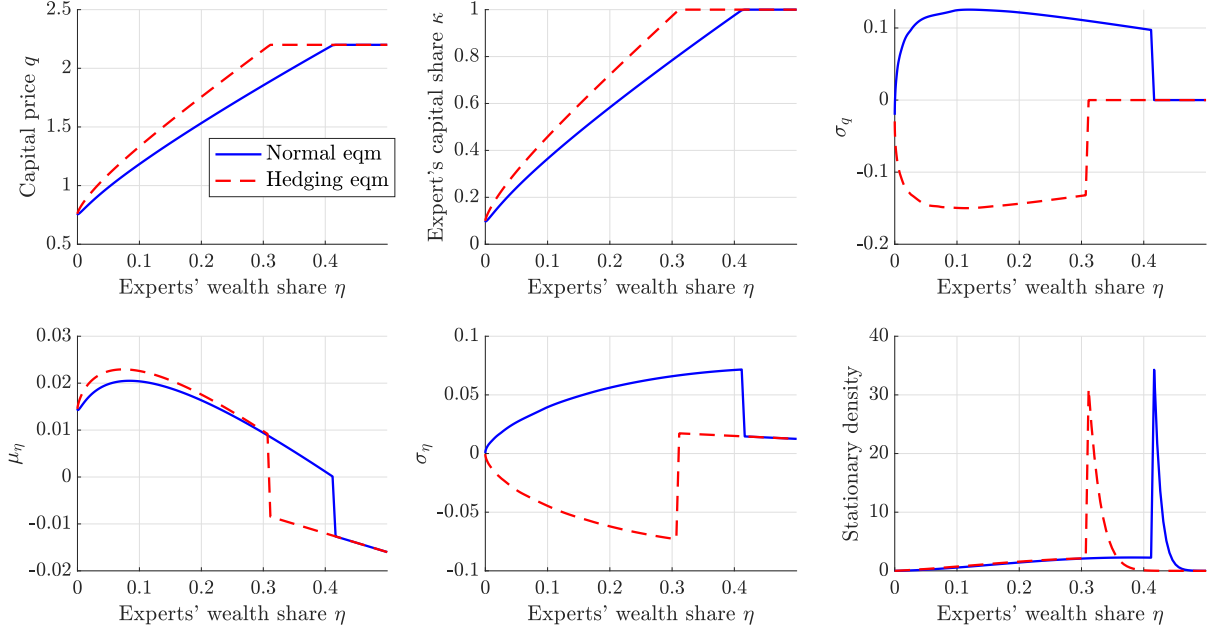


Figure E.2: Two equilibria (normal versus hedging) both with disaster belief $\kappa_0 = 0.1$. Parameters: $\rho_e = \rho_h = 0.05$, $a_e = 0.11$, $a_h = 0.03$, $\sigma = 0.025$. Type-switching parameters: $\delta_h = 0.004$ and $\delta_e = 0.036$.