TÉCNICO LISBOA

Flying Tourist Problem

An Integer Linear Programming Approach

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To all my fellow travelers.

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Resumo

O presente trabalho aborda o ”Flying Tourist Problem” (FTP) cujo principal objectivo é determinar o melhor agendamento, rota e conjunto de voos que permitem realizar um itinerário que percorre várias cidades, sem restrições, e realizada exclusivamente com uso de voos comerciais. O trabalho desenvolvido apresenta uma formulação linear inteira com o objectivo de encontrar soluções ótimas para este problema. Esta formulação foi posteriormente integrada no CPLEX e os resultados assim adquiridos foram comparados com sistemas idênticos existentes. Os resultados obtidos mostram que, ao contrário de outros sistemas existentes, este está capacitado para de forma consistente obter o resultado ótimo. Com o objectivo de melhorar a eficiência na resolução de problemas de grande dimensão, este sistema foi posteriormente integrado com um algorítmo de optimização metaheurístico. O FTP foi posteriormente adaptado a um problema idêntico pelo que foi formulado o ”Generalized Flying Tourist Problem” (GFTP). Este problema é uma generalização do FTP cujo principal objectivo é determinar o caminho de um grafo que percorre todos os subconjuntos de cidades constituintes. Comparativamente ao FTP, e em problemas de dimensão semelhante, o GFTP apresenta uma redução no custo total de 46%. Numa fase final, foram analisadas formulações multi-objectivo destes dois problemas, respectivamente o ”Multi-objective Flying Tourist Problem” (MO-FTP) e o ”Multi-objective Generalized Flying Tourist Problem” (MO-GFTP). Ambas formulações apresentam vantagens face ao FTP em termos de tempo de viagem. Em particular, comparativamente ao FTP, o MO-FTP e o MO-GFTP apresentaram, respectivamente, decréscimos de 52% e 80% no tempo de viagem.

Palavras-chave: Problema do Caixeiro Viajante, Programação Linear Inteira, Optimização Combinatória, Optimização Multi-objectivo Combinatória.

Abstract

This work addresses the Flying Tourist Problem (FTP), which aims to find the best schedule, route, and set of flights for a given unconstrained multi-city flight request. The developed work proposes an Integer Linear Programming formulation with the intent of finding optimal solutions. This formulation was implemented in CPLEX and evaluated comparing its results and performance to other similar systems. The obtained results show that, contrary to the existing systems, this optimization system invariably finds the optimal solution. Moreover, to improve the computational performance for large instances, this system is integrated with a metaheuristic optimization algorithm. Furthermore, the FTP was adapted to a similar problem and the Generalized Flying Tourist Problem (GFTP) arised. This problem is a generalization of the FTP whereby it is required to find the best route in a graph which visits all specified subsets of cities. Considering similar size instances, this variation presents a cost decrease of 46% when compared to the FTP. Finally, multi-objective variations of both problems, the Multi-objective Flying Tourist Problem (MO-FTP) and the Multi-objective Generalized Flying Tourist Problem (MO-GFTP), were characterized and formulated. Both present advantages to the FTP concerning the travel time. In fact, compared to the FTP, the MO-FTP and the MO-GFTP presented, for same sized instances, decreases of 52% and 80%, respectively, in the traveling time.

Keywords: Flight Search, Traveling Salesman Problem, Linear Integer Programming, Combinatorial Optimization, Multi-objective Combinatorial Optimization.

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Nomenclature

a	Arival date of flight
c	Cost value of flight
d	Departure date of flight
I	Number of flight alternatives
m	Number of possible intermediate cities
n	Number of intermediate cities or intermediate clusters
s	Stop time
T_{0M}	Maximum start date
T_{0m} or T_{start}	Minimum start date
T_{fM} or T_{end}	Maximum return date
T_{fm}	Minimum return date
u	Integer decision variable for route position
V	Set of city or cluster nodes
w	Binary decision variable for cluster connection
x	Binary decision variable for city connection

Glossary

ACO Ant Colony Optimization.

API Application Programming Interface.

ATSP Asymmetric Traveling Salesman Problem.

B&B Branch and Bound.

B&C Branch and Cut.

COP Combinatorial Optimization Problem.

CSA Client Side Application.

DFJ Dantzig-Fulkerson-Johnson.

DL Desrochers-Laport.

FTP Flying Tourist Problem.

GA Genetic Algorithms.

GFTP Generalized Flying Tourist Problem.

GG Gavish-Graves.

GTSP Generalized Traveling Salesman Problem.

ILP Integer Linear Program or Integer Linear Programming.

KPI Key Performance Indicator.

LP Linear Program or Linear Programming.

MILP Mixed Integer Linear Program.

MO-FTP Multi-objective Flying Tourist Problem.

MO-GFTP Multi-objective Generalized Flying Tourist Problem.

MO-TSP Multi-objective Traveling Salesman Problem.

MOP Multi-objective Problem.

MTZ Miller-Tucker-Zemlin.

SA Simulated Annealing.

SEC Subtour Elimination Constraint.

SLS Stochastic Local Search.

SSA Server Side Application.

STSP Symmetric Traveling Salesman Problem.

TDTSP Time-Dependent Traveling Salesman Problem.

TSP Traveling Salesman Problem.

TSP-TW Traveling Salesman Problem with Time Windows.

1

Introduction

Contents

1.1 Motivation

In the last decades, flying has become a common and affordable means of transportation around the globe. In fact, the International Air Transport Association (IATA) states that there were 3.8 billion air travelers in 2016 and predicts an increase to 8.2 billion by 2037 [1]. Moreover, the IATA also claims that the average cost of flights has reduced to half during the past 2 decades whilst air connectivity has been hastily growing. This also means that the variety of flight options is getting larger, and nowadays the traveler is often faced with multiple decisions when planning a trip. To respond to this variety of options, online travel agencies have surged in an endeavor to ease the process of buying flight tickets. In addition, with the decrease of flight costs, people from different economical backgrounds are becoming more interested in air travel. With this in mind, several new opportunities to develop new products targeting different groups are surging.

To embrace one of those opportunities, the Flying Tourist Problem (FTP), an unconstrained multi-city problem, was defined as a special case of the well known Traveling Salesman Problem (TSP). More specifically, it is related to the Time-Dependent Traveling Salesman Problem (TDTSP) variation due to the existing ticket cost and traveling time variations that depend not only on the chosen route direction but also on the traveling date. Considering a tourist wishing to visit n cities in no particular order, by plane, in a given time period, starting and ending at a given city, the FTP tries to minimize the cost or the traveling time. To the best of our knowledge, this problem is only addressed by one online travel agency, Kiwi [2]. Moreover, R. Marques [3] has recently published an article presenting a web application to assess this problem. Nonetheless, both web services solve this problem in an incomplete manner and hence, do not guarantee the user the best possible solution. In accordance, this thesis aims to extend previous solutions to this particular problem (mainly based on stochastic methods) by exploiting complete optimization methods. Consider the following example: a tourist wishes to depart from Lisbon between the 5th and the 10th of October, visit Paris (for 4 days), Rome (3 days), Berlin (5 days) and then return to Lisbon. The FTP proposes to find the set of flights that the tourist needs to take in order to minimize the total flight cost or traveling time of the trip. A possible solution would be for instance departing on 7th of October and visiting the cities by the following order: Lisbon-Rome-Berlin-Paris-Lisbon.

Furthermore, if the tourist does not possess the time or the resources to visit the initially proposed list of cities, he may be willing to visit some alternative similar cities. With this in mind, the Generalized Flying Tourist Problem (GFTP) arose. A tourist defines n groups of cities (clusters) and the GFTP finds the minimal cost (or traveling time) route, that visits exactly one city of each cluster in no particular order, by plane, in a given time period, starting and ending at a given city. Consider now, that the optimal solution found for the FTP example did not fit the tourist's resources. If the purpose of the trip is, for instance, to experience the cultural background of France, Italy and Ger-

many, the tourist might want to consider alternatives to his initial choices. The tourist might then define the first group of cities as Paris and Marseille, the second group of cities as Rome, Milan and Venice and the third group of cities as Berlin, Munich and Stuttgart. This results in the following list $\{$(Paris,Marseille),(Rome,Milan,Venice),(Berlin,Munich,Stuttgart)$\}$. Accordingly, he also defines the days he wishes to stay at each city $\{$(3,2),(3,2,2),(5,5,3)$\}$. The GFTP proposes to find the set of flights that the tourist needs to take in order to minimize the total flight cost or traveling time of the trip. An example of a solution to this problem would be departing on the 8th of October and visiting this set of cities by the following order: Lisbon-Milan-Berlin-Marseille-Lisbon.

Finally, regarding both the FTP and the GFTP, the tourist may wish to optimize for both the cost and the traveling time. Therefore, the multi-objective formulations of the aforementioned problems were derived and are respectively the Multi-objective Flying Tourist Problem (MO-FTP) and Multi-objective Generalized Flying Tourist Problem (MO-GFTP). Consider that the solutions of the preceding examples were found in a total flight cost minimization problem. If the traveling time of these solutions is too large for the tourist, then, by using these multi-objective formulations, he will be able to make a trade-off between the total flight cost and the total traveling time.

Industry Parallelism: Even though the present section depicts the main motivation behind this thesis, the developed work has a vast number of applications. For instance, the airline industry faces challenges similar to the ones this work aims to resolve. Some examples of such problems are the Fleet Assignment, Aircraft Routing, Crew Pairing and Crew Rostering.

Moreover, the problems addressed by this work can be posed for tourism unrelated issues. Considering the transportation sector, the FTP can be used to minimize the travel costs of a certain cargo ship. Furthermore, the freight transportation by air or ship are not as optimized for small routes as ground transports. Notwithstanding, ground transportation takes longer and cannot deliver overseas. With this in mind, consider the following example: a logistics company has an air cargo to deliver in various cities within Europe. Nonetheless, some of this cities are close to each other (e.g inside Iberian Peninsula) and plane delivery would be too costly. The GFTP shall then be used to plan the air route so that the cargo is delivered in clusters of cities to then ground transports deliver within each cluster.

1.2 Objectives and Contributions

In this thesis, we propose to optimally solve the unconstrained multi-city problem, FTP, i.e to find the guaranteed best solution for a certain user request. With this in mind, an Integer Linear Programming (ILP) formulation for this problem shall be defined. We aim to integrate this formulation in a commercial solver to find optimal solutions for real life sized, complex multi-city flight requests. Furthermore, this system shall be compared to existing systems. Next, since the optimal solution might still not suit every

end user, we propose to formulate variations of the FTP. Three variations of the FTP should be hypothesized: the Generalized Flying Tourist Problem, the MO-FTP and the MO-GFTP. These formulations aim to satisfy end users that pretend, for instance, a shorter trip. These formulations should also be solved using a commercial ILP solver. Finally, the developed work shall be integrated in a web application prototype.

1.3 Thesis Outline

The presented work is divided into two main studies: the study of complete methods to solve the unconstrained multi-city routing problem and its variations, and the evaluation of the results and comparison to the state-of-the-art. Hence, this document is structured as follows.

Chapter 2 presents a literature review on widely known problems that are related to the FTP, in particular the Traveling Salesman Problem, and on complete methods to approach these problems.

Chapter 3 introduces a formal definition, as well as ILP formulations for the FTP, GFTP, MO-FTP and MO-GFTP.

Chapter 4 gives an overview of the development and design choices of the web application that was used during this work, as well as the modifications that were conducted for its adaptability to the new models that were proposed.

Chapter 5 outlines the results achieved with this work, comparing the complete method applied here with the existing work on this problem, as well as the results of the FTP's variations.

Finally, Chapter 6 describes the main conclusions of this work, and addresses the future work that could greatly benefit the aforementioned work in a real time web service.

2

Background

Contents

As it was referred in the previous chapter, this thesis addresses Combinatorial Optimization Problems (COPs), more specifically variations of the Traveling Salesman Problem (TSP). This chapter formally introduces the TSP in Section 2.1 followed by a description of several approaches to its resolution presented in Section 2.2. In Subsection 2.2.1, an exact approach is introduced. Finally, Subsection 2.2.2 presents other alternative techniques.

2.1 The Traveling Salesman Problem

As per Dantzig et al. [4], the traveling salesman problem is described as: *"Find the shortest route (tour) for a salesman starting from a given city, visiting each of a specified group of cities, and then returning to the origin point of departure. More generally, given an n by n symmetric matrix $D = (d_{IJ})$, where d_{IJ} represents the 'distance' from I to J, arrange the points in a cyclic order in such a way that the sum of the d_{IJ} between consecutive points is minimal."*

This problem has been studied by three main scientific branches (Operations Research, Applied Mathematics and Theoretical Computer Science). All these disciplines consider this problem as part of a broader topic: Combinatorial Optimization.

A Combinatorial Optimization Problem, as explained by Papadimitriou and Steiglitz [5], consists of a model $P = (S, \Omega, f)$ where:

- S is a search space, defined by a finite set of decision variables, each with a domain;
- Ω is a set of constraints amongst the decision variables;
- $f : S \rightarrow \mathbb{R}_0$, is an objective function to be minimized.

Moreover, as stated by complexity theory adopted in Computer Science, these problems are divided and classified according to their inherent difficulty. Figure (2.1) illustrates how the complexity of these problems grows. P is the complexity class of problems that are solvable in polynomial time. These polynomial time algorithms are characterized by a computation time that is bounded by $\mathcal{O}(p(n))$, where p is a polynomial function and n states the size of the problem. Any problem that is solvable in a short amount of time is classified as tractable. Hence, most P problems are tractable and vice-versa. However, this relation is not always true, as tractable means "efficiently solvable" which is not true for P problems with large coefficients. On the other hand, NP, or non-deterministic polynomial time, is the class of problems that given an answer, the proof is verifiable in polynomial time. NP-hard problems are commonly described as "at least as hard as the hardest problems in NP". The subclass NP-complete, contained by the NP-hard problems, contains the most difficult problems in NP. If there is an algorithm that is able to solve one of these problems in polynomial time then, all NP problems are solvable in polynomial time. Furthermore, such an algorithm would also answer the Millennium Problem "P vs NP".

Lastly, as in the TSP the salesman is required to visit each city exactly once, then the path travelled is an Hamiltonian path[1]. Moreover, as the salesman is to return to the origin city, this is an Hamiltonian cycle[2]. Determining the existence of an Hamiltonian cycle in a graph is a NP-complete problem, which implies that the TSP is also NP-complete [6].

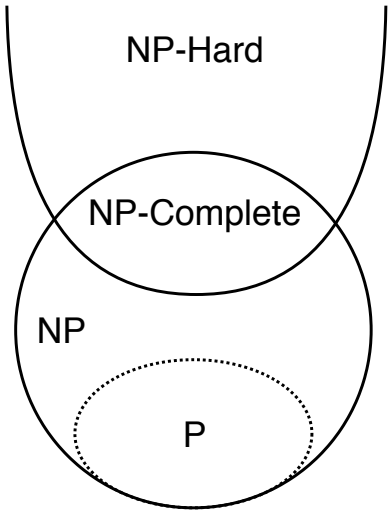


Figure 2.1: P vs NP if $P \neq NP$.

Integer Linear Programming

Integer Linear Programming (ILP) is a mathematical program with integer variables, where both the objective function and the constraints are linear. If not all decision variables are discrete, then the program is known as a Mixed Integer Linear Program (MILP). In the case where all variables are binary, the problem is called a 0-1 ILP [5, 7]. This is defined in the canonical form as:

$$\begin{aligned} \text{min} \quad & \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{c_j x_j} \\ \text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad 1 \leq i \leq m, \\ & x_j \in \mathbb{N}_0 \qquad\qquad 1 \leq j \leq n. \end{aligned} \tag{2.1}$$

[1] In graph theory, an Hamiltonian path is a graph that visits every vertex exactly once.
[2] In graph theory, a cycle is a directed or undirected path that only repeats the first (and last) vertex.

where x_j are the decision variables, c_j and b_i are the elements of coefficient vectors and a_{ij} are the entries of the coefficient matrix. For instance, considering the following problem:

$$\begin{aligned} \min \quad & \mathbf{y} \\ \text{s.t.} \quad & y \geq 3x - 3, \\ & y \leq x + 2, \\ & y \geq -3x + 3 \\ & x, y \in \mathbb{N}_0. \end{aligned} \tag{2.2}$$

The linear programming solution is represented in Figure 2.2. The red, blue and violet lines represent the constraints, respectively. These three lines together also define the polyhedron of the Linear Program relaxation[3]. The black dashed line represents the convex hull that contains all the feasible integer points, shown in black. Considering the minimization problem, the optimization problem hence finds the solution $(1, 0)$. This is also the solution of the Linear Program relaxation. On the other hand, if this was instead a maximization problem, the solution would be $(2, 4)$ which is different than its relaxation, $(2.5, 4.5)$.

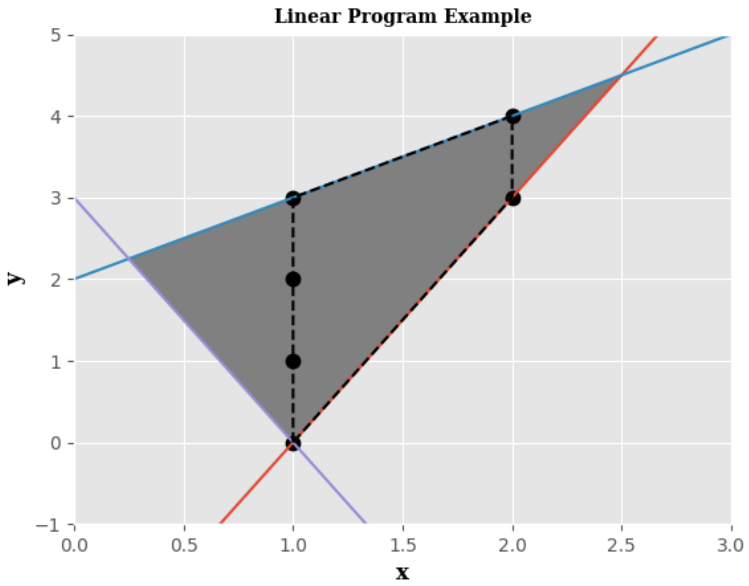


Figure 2.2: Graphical representation of Equation (2.2).

TSP Problem Statement

The TSP can be defined as a graph $G = (N, A)$,[4] with N being the set of nodes of size $n = |N|$ and A the set of arcs connecting the nodes of size n^2. The arcs are hence represented as $(i, j) \in A, i \neq j$

[3] Linear Program relaxation is the Linear Program without integrality constraints. This concept is explained in detail further in this work.

[4] In some literature this is defined as $G = (V, E)$ where V is the set of vertices and E is the set of edges instead.

and have an attached weight matrix of c_{ij}. Problems in which the triangle inequality ($c_{ik} + c_{kj} \geq c_{ij}$ for all $i, j, k \in N$) is satisfied are called *Euclidean problems* [8]. In the literature, this matrix has several designations as it can describe distance, cost, time or even a weighted variable. A is often assumed to be complete, this means that every distinct node is connected by one and one only arc, and the missing arcs are replaced with large weights thus making a connected and weighted graph. Moreover, in the standard TSP, G is also defined as undirected, resulting in the well known Symmetric Traveling Salesman Problem (STSP), where $c_{ij} = c_{ji}, \forall i, j \in N$. On the other hand, if G is directed, the problem is more general and is named Asymmetric Traveling Salesman Problem (ATSP). These problems can be transformed in one another with ease. If one duplicates the arcs of a STSP and considers the undirected graph as a directed one, then the STSP is converted into an ATSP. On the other hand, if instead, the nodes are duplicated, it is possible to transform an ATSP into an equivalent STSP. This transformation can be seen in Figure 2.3. For deeper insight on the transformation between the two, the reader is referred to [9, 10].

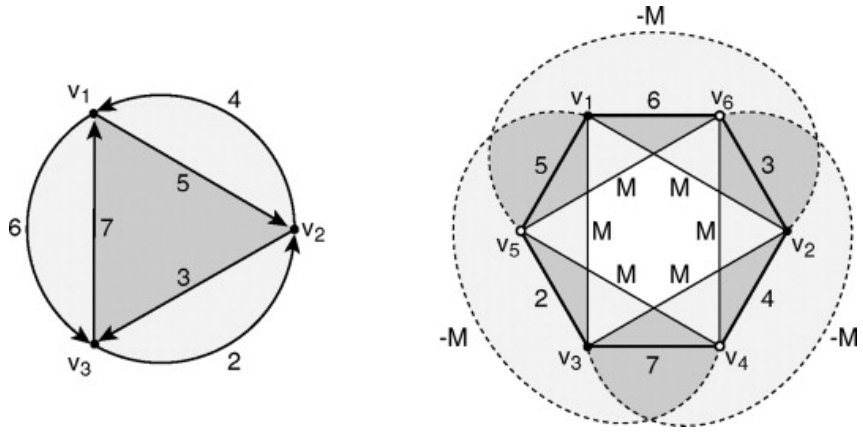


Figure 2.3: Transformation of an ATSP instance (left side) into a STSP instance(right side) [10]. M is a sufficiently large number and represents the weight of the associated arcs.

The TSP has innumerous variations. A brief introduction of the most common variations, as well as a more thorough description for the much relevant variations that underpin the work under development, based on the formulations presented in [11], is presented next.

2.1.1 Dantzig-Fulkerson-Johnson ATSP Model

Integer linear programming (ILP) addresses an optimization problem focused on the minimization of a linear objective function amidst the integer points of a polytope P [12]. In the case of a three dimensional problem the polytope is a polyhedron.

The particular case of the ATSP can be formulated as an ILP model. Dantzig et al. [4]:

$$\min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \tag{2.3}$$

$$\text{s.t.} \sum_{i=1}^{n} x_{ij} = 1 \quad j = 1, \ldots, n, \tag{2.4}$$

$$\sum_{j=1}^{n} x_{ij} = 1 \quad i = 1, \ldots, n, \tag{2.5}$$

$$\sum_{i \in S} \sum_{j \in S} x_{ij} \leq |S| - 1 \quad S \subset V : S \neq \emptyset, \tag{2.6}$$

$$x_{ij} \in \{0, 1\} \quad i, j = 1, \ldots, n. \tag{2.7}$$

Every arc $(i, j) \in A$ is associated with a binary variable x_{ij} and a corresponding weight c_{ij}. The number of variables is n^2 and the decision variable space has a size of 2^{n^2}. An arc is part of the optimal tour, and hence part of the solution, when the associated decision variable is assigned the value 1. Constraints (2.4) and (2.5) state that each vertex has exactly one arc entering and one arc leaving, enforcing the tour to be an Hamiltonian path. Constraints (2.6) are *Subtour Elimination Constraints* (SECs) and impose that subtours, i.e partial circuits, are excluded. The latter group of constraints can be equivalently rewritten as *Connectivity constraints*:

$$\sum_{i \in S} \sum_{j \in V \setminus S} x_{ij} \geq 1 \quad S \subset V : S \neq \emptyset, \tag{2.8}$$

Lastly, constraints (2.7) impose that decision variables are either equal to 0 or 1. The previous ILP model has an exponential number of SECs and a lower bound on the optimal solution may be obtained through LP relaxation by removing the *integrality* constraint of each variable.

Polynomial Formulations

Having an exponential number of SECs would greatly increase the computational time, thus a few polynomial formulations for the ATSP are considered.

Remark 1. Notice that in this section, by polynomial formulations we mean, formulations in which the number of subtour elimination constraints[a] grows with a polynomial factor with the size of the problem.

[a] Subtour elimination constraints forbid the existence of solutions consisting of several subtours.

Roberti, Toth. [13] present a review of polynomial formulations and report that the best formulations to be used are Miller, Tucker and Zemlin [14], Gavish and Graves [15] and Desrochers and Laporte [16] (MTZ, GG and DL, respectively). All of these formulations consist on DFJ model replacing (2.6) with another group of constraints.

MTZ achieves this by creating a group of integer decision variables u_i. These variables represent the order of vertex i in the optimal tour. The constraints are defined as follows:

$$u_i - u_j + (n-1)x_{ij} \leq n-2, \quad i,j = 2,\ldots,n. \tag{2.9}$$

On the other hand, GG creates $n(n-1)$ integer variables g_{ij} that represent the number of arcs from node 1 to arc (i,j) in the optimal tour as well as the following constraints:

$$\sum_{j=1}^{n} g_{ij} - \sum_{j=2}^{n} g_{ji} = 1, \quad i = 2,\ldots,n, \tag{2.10}$$

$$0 \leq g_{ij} \leq (n-1)x_{ij}, \quad i = 2,\ldots,n \quad j = 1,\ldots,n. \tag{2.11}$$

Finally DL is an enhanced MTZ formulation and hence utilizes the same decision variables u_i. The constraints are:

$$u_i - u_j + (n-1)x_{ij} + (n-3)x_{ji} \leq n-2, \quad i,j = 2,\ldots,n, \tag{2.12}$$

$$-u_i p(n-3)x_{i1} + \sum_{j=2}^{n} xji \leq -1, \quad i = 2,\ldots,n, \tag{2.13}$$

$$u_i + (n-3)x_{1i} + \sum_{j=2}^{n} x_{ij} \leq n-1, \quad i = 2,\ldots,n. \tag{2.14}$$

Both GG and DL present formulations having stronger LP relaxation than that of MTZ. A strong, or tight, LP relaxation means that the feasible set of the relaxation is close to the convex hull of integer feasible solutions and thus, the optimal solution of both GG and DL are more likely to be closer to the integer optimum than the optimal solution of the MTZ.

2.1.2 The Time Dependent Traveling Salesman Problem

The Time-Dependent Traveling Salesman Problem (TDTSP) is a generalization of the TSP, where the cost of the arcs connecting the nodes depends on the time at which the salesman traverses the arc. This model has several real-world applications as the one-machine sequencing problem. The TDTSP can be defined as a oriented graph $G = (N, A)$ with the condition that for each arc $(i,j) \in A$ the travel cost at period $k = 1,2,\ldots,n$ is defined as c_{ij}^k. This problem was first addressed by Fox in 1973 [17] and later revisited in 1980 with the help of Gavish and Graves [18]. In 1978, Picard and Queyranne [19] proposed what is now the basis of this problem using $\mathcal{O}(n^3)$ variables and $\mathcal{O}(n^2)$ constraints as the following:

In this formulation, x_{ij}^k are the decision variables. These are assigned the value 1 if the arc (i,j) on time instance k is taken, and 0 otherwise. Constraints 2.16 guarantee that the cycle leaves each city once and once only in the comprised time-window. Constraints 2.17 and 2.19 ensure that the cycle starts and finishes on node 1 on days 1 and n, respectively. Constraints 2.18 assure that the inflow on

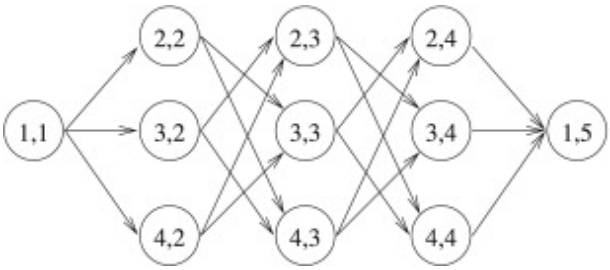


Figure 2.4: Example of a multipartite TDTSP network with $n = 4$ cities [21].

day k at each node is the same as the outflow on day $k+1$, which eliminates subtours[5] and controls the time spent at each node. Lastly, Constraints 2.20 are integrality constraints and secure that the decision variables are binary. Defining now that a state is a pair node-time (i, k), then the decision variables $x_{i,j}^k$ represent the transition from state (i, k) to state $(j, k+1)$. Considering a multipartite network, where the state $(1, 1)$ is the source and $(1, n+1)$ is the sink, then a 4 city TDTSP can be represented as in Figure 2.4.

$$\min \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} c_{ij}^k x_{ij}^k \tag{2.15}$$

$$\text{s.t.} \sum_{i=1}^{n}\sum_{k=1}^{n} x_{ij}^k = 1, \quad j = 1, \ldots, n \tag{2.16}$$

$$\sum_{(1,j)\in A} x_{1j}^1 = 1, \tag{2.17}$$

$$\sum_{(i,j)\in A} x_{ij}^k - \sum_{(j,i)\in A} x_{ji}^{k+1} = 0, \quad t = 2, \ldots, n-1, \quad j = 1, \ldots, n \tag{2.18}$$

$$\sum_{(i,1)\in A} x_{i1}^n = 1, \tag{2.19}$$

$$x_{ij}^k \in \{0, 1\} \quad i, j, k = 1, \ldots, n. \tag{2.20}$$

The objective of this problem is the same as the TSP: find the minimum cost Hamiltonian cycle over the graph $G = (N, A)$.

2.1.3 Traveling Salesman Problem With Time Windows

The Traveling Salesman Problem with Time Windows (TSP-TW) is defined as a generalization of the TSP with changeover times over nodes, represented as setup times t_{ij}, as well as the processing time $p_j \geq 0$, release date $r_j \geq 0$ and deadline $d_j \geq r_j$ for every node $j \in N$. The time window referred for node j is thus $[r_j, d_j]$. The objective of this variation is to find the minimum cost Hamiltonian path (cycle

[5] Subtours are solutions consisting of several disconnected tours [20].

if faced with a closed-tour variation) visiting a node sequence such that for every node, the processing time lies in the time-window. If the time-window is $[0, +\infty]$ then it is *relaxed*. On the other hand, if this interval is characterized by $r_j > 0$ and $d_j < +\infty$ then the time-window is active.

The Asymmetric TSP-TW is very important when dealing with scheduling and routing problems [22]. Considering an Asymmetric TSP-TW instance defined on a complete digraph $G = (V, A)$, the convex hull of the eigenvectors of all the feasible paths contained in such graph can be represented as a polytope. Finding the dimension of such polytope is a NP-complete problem even when all time windows with the exception of one are relaxed [22].

2.1.4 Generalized Traveling Salesman Problem

The Generalized Traveling Salesman Problem (GTSP), or set TSP, is an extension of the TSP. This problem consists on determining the shortest or minimum cost route that passes through each cluster of nodes at least once and never repeats a node. It assumes that N is a set of clusters containing every node and the clusters are formed by at least one node. Let $G = (N, A)$ be a m-node weighted undirected graph where each arc contained in A is associated with a non-negative cost c_{ij}. Furthermore, N is partitioned in n disjoint subsets or clusters S_l, $l = 1, \ldots, n$.

In the following ILP model, the integer variables x_{ij} take the value 1 if the arc between nodes i and j is used and the value 0 otherwise. Variables y_i and $y_i^{'}$ refer to the outflow and inflow of each node i, respectively. For each node i they are assigned the same value, 1 when node i is visited and 0 otherwise.

$$\text{min} \sum_{i \in N} \sum_{j \in N \setminus \{i\}} c_{ij} x_{ij} \tag{2.21}$$

$$\text{s.t.} \sum_{j \in N \setminus \{i\}} x_{ji} = y_i^{'} \quad i \in N, \tag{2.22}$$

$$\sum_{j \in N \setminus \{i\}} x_{ij} = y_i, \quad i \in N, \tag{2.23}$$

$$\sum_{i \in S_l} y_i = \sum_{i \in S_l} y_i^{'} \geq 1, \quad l = 1, \ldots, n, \tag{2.24}$$

$$y_i = y_i^{'}, \quad i \in N, \tag{2.25}$$

$$\sum_{i \in T} \sum_{j \in T \setminus \{j\}} x_{ij} \leq |T| - 1, \quad T \subseteq N, \quad T \cap S_l = \emptyset \text{ for at least but not all } l, \tag{2.26}$$

$$x_{ij}, y_i, y_i^{'} \in \{0, 1\} \quad i, j \in N, \quad i \neq j \tag{2.27}$$

Constraints 2.22 and 2.23 state the inflow and outflow of each node i. Constraints 2.24 force each cluster to be visited at least once. Constraints 2.26 are subtour elimination constraints. Finally, constraints 2.27 are integrality constraints. The reader is referenced to [8] for further insight and more detail

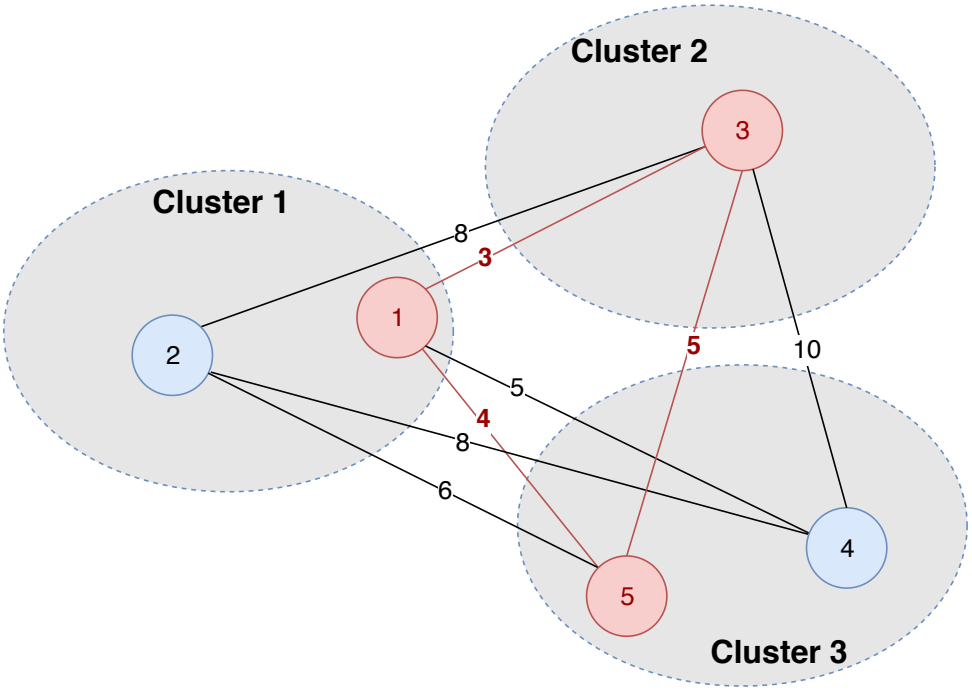


Figure 2.5: Generalized Traveling Salesman Problem.

on the previous formulation and how to improve it. In the literature, other authors also formulate the GTSP more strictly, stating that each cluster should be visited once and once only by imposing that the tour includes exactly one node from each node set [23, 24]. This difference only changes constraints 2.24, which would become: $\sum_{i \in S_l} y_i = \sum_{i \in S_l} y_i^{'} = 1, \quad l = 1, \ldots, n$. A diagram of the latter formulation is shown in Figure 2.5. This diagram represents a GTSP with 3 clusters. The first and third clusters contain 2 cities and the second contains 1 city. The black lines represent all the possible arcs, whilst the red lines represent the shortest Hamiltonian cycle of this problem.

2.1.5 Multi-Objective Traveling Salesman Problem

The Multi-objective Traveling Salesman Problem (MO-TSP) is another generalization of the TSP. However, instead of solely minimizing the distance/cost of the tour, it tries to respond to a set of different objective functions. This problem is often associated with a wider set, the multi-objective COPs [25].

For the formal definition of this problem, several definitions must be understood. Firstly, a Multi-objective Problem (MOP) is defined in [26] as:

$$
\begin{aligned}
\text{min} \quad & \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{c_j^1 x_j} \\
& \dots \\
& \sum_{\mathbf{j=1}}^{\mathbf{n}} \mathbf{c_j^k x_j} \\
\text{s.t.} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad 1 \leq i \leq m, \\
& x_j \in \mathbb{N}_0 \qquad\qquad 1 \leq j \leq n.
\end{aligned}
\tag{2.28}
$$

where each arc has k associated weights c_j^k.

For easiness of explaining the rest of the definitions on this section, $F(\vec{x})$ is defined as the k dimensional objective set that comprises the objective functions $\sum_{j=1}^{n} c_j^1 x_j, \dots, \sum_{j=1}^{n} c_j^k x_j$, where $\vec{x} = (x_1, \dots, x_n)$ is a n-dimensional decision variable vector from some universe Ω.

Moreover, the terminology related to **Pareto concepts** is also of the utmost importance in MOPs. **Pareto Dominance** is defined as follows: A vector $\vec{u} = (u_1, \dots, u_k)$ is said to dominate $\vec{v} = (v_1, \dots, v_k)$, represented as $\vec{u} \preceq \vec{v}$, if and only if $\vec{u}$ is partially less than $\vec{v}$, i.e $\forall i \in \{1, \dots, k\}, u_i \leq v_i \wedge \exists i \in \{1, \dots, k\} : u_i < v_i$. On the other hand, **Pareto Optimality** is stated as: A solution $x \in \Omega$ is said to be Pareto optimal with respect to Ω if and only if there is no $x' \in \Omega$ for which $\vec{v} = F(x') = (f_1(x'), \dots, f_k(x')))$ dominates $\vec{u} = F(x) = (f_1(x), \dots, f_k(x))$. **Pareto Optimal Set** is defined as follows: For a given MOP, the Pareto optimal set (P^*), is defined as:

$$
P^* := \{x \in \Omega \mid \neg \exists\, x' \in \Omega : F(x') \preceq F(x)\}
$$

Lastly, for a given MOP $F(x)$ and Pareto optimal set P^* the **Pareto front** PF^* is defined as:

$$
PF^* := \{\vec{u} = F(\vec{x}) = (f_1(x), \dots, f_k(x)) \mid x \in P^*\}
$$

The previous concepts refer to the Pareto global optimum set. In the case of the MO-TSP, finding this set is NP-hard. With this in mind, it is important to also consider the Pareto local optimum sets instead, as this can have improvements in the computational time. The definitions are very similar to the ones above with the inclusion of a neighbourhood in the evaluation of the Pareto optimum solutions. For more insight refer to [27].

Lastly, understanding all of the above concepts, the MO-TSP definition is as follows: Let $G = (N, A, c)$ be a complete and weighted graph where N is the set of nodes, A is the set of arcs connecting the nodes and c the cost/distance associated with each arc. The multi-objective TSP finds one or more Pareto optimal Hamiltonian tours, or cycle in a closed problem, of the graph visiting each node exactly once.

2.2 Optimization Techniques

This section introduces two different approaches to solve COPs. In Subsection 2.2.1, Linear Programming (LP) is presented, as well as a specific model where this approach is applied to the ATSP. Afterwards, we present a small overview of other methods that have advantages relative to LP in some situations.

2.2.1 Linear Programming

Linear Programming (LP) is a specific technique of mathematical programming for the optimization of a linear objective function subject to linear equality and inequality constraints. When a feasible problem is considered, this method achieves at least one optimal solution. Infeasibility means that the method proved that there is no solution that respects all the constraints. A LP formulation can also be interpreted as a geometrical problem where the feasible region is a convex polytope with at least one of its vertices being an optimal solution.

2.2.1.A Branch and Cut Algorithms

Branch and Cut (B&C) methods are exact algorithms[6] used in integer programming problems. These methods solve a series of linear programming relaxations of the problem, apply cutting plane methods[7] to further improve the relaxation of the problem, followed by the application of Branch and Bound (B&B) algorithms to solve the problem in a divide-and-conquer[8] approach. These methods are also usually aided by preprocessing, primal heuristics, as well as lifting/strengthen constraint techniques [29]. Even though some problems might be too large to be handled by these methods, the nature of such algorithms make it possible to exploit coarse grain parallel computing, which divides a program in various large tasks to be computed separately by various processors.

In an optimization problem, relaxation is responsible for finding bounds, lower bounds in minimization and upper bounds in maximization, on the optimal value. With this in mind, relaxation bounds are important to calculate the relative gap, how close to the optimum is the suboptimal solution, and to accelerate the search. By defining bounds, if the bounds of a certain node are worse than the incumbent solution[9]., then that node will no longer be branched.

Cutting plane methods are usually very useful when combined with a B&B algorithm. However, very ineffective when used standalone. These methods are based on sequential LP relaxations to better approximate the problem. Developed by R.E.Gomory [30], at first these algorithms seemed to be weak. Nonetheless, with the observed progresses in polyhedral theory, these algorithms were revamped and

[6] An exact algorithm solves a problem to optimality (if faced with a NP-hard problem in exponential time).
[7] Cutting planes method iteratively applies linear inequalities, called *cuts*, refining the feasible set. For more insight [28].
[8] Divide-and-Conquer is a generic framework (algorithm paradigm) that using recursion breaks problems into sub-problems.
[9] Incumbent solution is the best integral feasible solution of the integer program

they are today a cutting edge method in some COPs. The basic structure of such techniques is as follows:

Algorithm 1 Cutting Planes Method

```
1: while True do
2:     Solve the LP Relaxation;
3:     if feasible solution is found then
4:         Optimality achieved;
5:         break While.
6:     else
7:         Apply more cutting planes to separate the solution, given by the relaxation, from the feasible
   points convex hull.
```

Usually, the first relaxation is solved using the primal simplex algorithm. The following are then more commonly solved by applying dual[10] simplex. This happens as the applied cutting planes often make primal simplex infeasible. Cutting plane methods are also of such importance as they return provably optimal solutions or, when an instance cannot be solved to optimality, bounds on the optimal value [32].

B&B is the most used approach when it comes to integer programs. A more thorough description of this algorithm is as follows: by iteratively partitioning the set in subsets, or branches, and creating bounds for each branch, its effectiveness comes from the fact that, in a minimization problem, if a lower bound for the objective value of a certain subproblem is larger than the objective value of an integer feasible solution, then the optimal solution does not lie in the respective subproblem. Another concept to be understood in B&B algorithms is that a relaxation of an optimization problem is also an optimization problem and solving a relaxation of the problem provides a lower bound for the ILP problem. This implementation can be compared to a tree search, where the integer problem is the root. A general assertion of a B&B algorithm is as follows [33]:

1. (Initialization): Form a set of subproblems as the original integer program and initialize the upper bound of these subproblems to ∞.

2. (Termination): If the set of subproblems is empty then the saved solution is optimal. If such solution does not exist, then the problem is infeasible.

3. (Problem selection and relaxation): Select and delete a subproblem from the set. Solve a relaxation of this subproblem and get the optimal objective value of such relaxation.

4. (Fathoming and Pruning): This step deals with the nodes that are no longer necessary to be explored. When all the nodes are fathomed, the search ends.

[10] In simple terms, the dual of a LP is achieved by transforming the variables of the primal into constraints, the constraints of the primal into variables and inversing the objective. If the duality is strong then the optimal value exists for both and is the same. If the duality is weak then the objective value of the dual at any feasible solution is a bound of the primal [31].

(a) If the optimal objective value of the relaxation is greater or equal to the incumbent objective value, return to step 2.

(b) else, if the optimal objective value of the relaxation is less than the incumbent objective value and if the optimal solution of the relaxation is integral, then the optimal objective value of the relaxation is the new incumbent objective value. In addition, delete all the subproblems with lower bounds greater than the new incumbent objective value. Return to step 2.

5. (Partitioning): Considering a partition of the constraint set of a subproblem, add new subproblems, with the feasible region restricted to the new partition of the contraint set, to the whole set of subproblems and establish the lower bounds as the actual incumbent objective value. Return to step 2.

It is important to keep in mind that several partitioning strategies exist. These typically use linear constraints and form two new nodes at each division. During this process, branching variables are selected to create more nodes. Several methods of branching and variable selection exist and their usage mostly depend on the type of integer program. The strategy that generally comes after is the node selection. This affects the chance of node fathoming hence the number of problems to be solved before optimality. Nonetheless, B&B has its flaws. For instance, in very large problems, integer feasible solutions may not be found at first, which leads to an accumulation of several active nodes, thus resulting in memory explosion. To counteract this possibility, cutting planes are often added to the root node to strengthen the LP formulation. Several preprocessing techniques, used either before or during B&B, can also greatly improve these methods. Some examples of such techniques are the removal of empty or dependent rows/columns, aggregation, coefficient reduction, logical implications and probing. We refer the reader to [34] for more details on these techniques. Moreover, heuristic approaches can also be used to obtain solutions in a fast manner. In B&B, heuristics are used to produce a good upper bound for reduced cost fixing at the root, hence reducing the size of the LP. With the purpose of achieving a good result in a small ammount of time, heuristics are underpinned by the following five crucial methods [33]:

- Greediness: Choose the variable based on the best local outcome.
- Local Search: Search in a neighbourhood of a feasible solution for a better objective value. Examples of such methods are for instance Simulated annealing [35] and Hill climbing.
- Randomized enumeration: Based on random factors. Genetic algorithms [36] are good examples of this foundation.
- Primal heuristics: LP-based procedure that solve a modified problem where a good solution is found in a point that fails to satisfy integrality.

- Primal-Dual interplay: Usage of solutions from the dual problem in the primal problem and vice-versa.

Heuristics make use of at least one of the above methods. Even when using all these methods, some large scale problems may present some difficulties. In such contexts, using interior point algorithms, as the LP solver, can have promising results. Interior point methods reach the optimal solution by transversing the interior of the feasible region. Since Karmakar breakthrough in 1984, these methods can be more efficient than the simplex algorithms in many settings [37, 38].

2.2.1.B LP Solvers

In this subsection, an overview of two different mathematical optimization problem solvers is presented as well as a comparison between both. This was conducted in order to achieve the best possible results for the rest of this work. Only two commercial software tools, CPLEX and Gurobi, are presented as these are typically the ones referenced by the state-of-the-art literature. The benchmarks in [39] are referenced in several sources. However, these benchmarks were removed due to requests of the solver's developers. On the other hand, it is important to notice that the problems tackled through this work can be solved with a vast number of other solvers like GLPK, LP_Solve, SCIP or even the comercial solver Xpress.

CPLEX

The ILOG IBM Optimization Studio is a high-performance mathematical programming solver for linear programming, mixed-integer programming and quadratic programming [40]. Presented with an integer linear program, this solver resorts to 3 different algorithms:

- The simplex algorithm is an iterative method that starts with an initial feasible solution (a vertex of the polytope) and moves progressively over the neighbourhood until optimality is achieved.
- The dual simplex algorithm starts with any optimal solution and iterates until feasibility is reached.
- The barrier methods, also known as interior point methods, start somewhere inside the feasibility region and using a predictor-corrector algorithm follow a central path through the interior until an optimal solution is found.

In addition, before using any of these algorithms to solve the model, CPLEX runs a presolve procedure that attempts to reduce the size of the problem by removing redundant constraints.

This optimization software, created by Robert E. Bixby, is currently developed by IBM and it is implemented in C languange. However, as of today, different interfaces to C++, C#, Java and Python languages as well as Matlab also exist. Moreover, this optimizer is also accesible through modeling systems [41].

Gurobi

Gurobi is another modern solver for mathematical optimization problems. Robert E. Bixby is, again, one of the creators of this software implemented in C. This optimizer also supports interfaces for C++, C#, Java, Python, .NET, Matlab and R. Furthermore, it can also solve non-linear problems and has a cloud-server system, where problems can be deployed. As of February 2019, the developers of this solver claim better results than any other mathematical problem solving software in [42].

Comparison of both

According to [41], for large scale problems, free solvers fall behind the comercial ones like CPLEX and Gurobi. When comparing these two, the results are fairly similar and problem dependent. This means that depending on the problem, the performance of one may be better than the other and vice-versa. Nonetheless, authors commonly adress Gurobi as the fastest solver whilst CPLEX as the most robust. For the previously mentioned reasons, these were the only two solvers considered throughout this work.

2.2.2 Other Techniques

Even though the approaches presented in the previous section allow to achieve an exact optimal solution, this can be very demanding. In some cases, complete methods cannot find a feasible solution in a reasonable amount of time. With this in mind, several incomplete methods[11] that find solutions in a faster manner are discussed in this section.

2.2.2.A Heuristic Algorithms

An incomplete method of solving NP-hard problems is through heuristics. An heuristic is any technique that in a small amount of time is able to find a solution for a given problem. Usually these are divided in construction or improvement heuristics. For more insight on heuristics, the reader is referenced to [43].

Held-Karp Lower Bound

As previously explained, complete algorithms allow to prove optimality. Even when that does not happen, an absolute or relative gap towards the best possible solution is easily computed. However, when using heuristics this is not necessarily the case. As a result, the evaluation of the quality of a certain heuristic is not straightforward. With this purpose, a lower bound, called Held-Karp (HK) lower bound, is often computed. This lower bound is achieved either by computing the solution of a linear programming relaxation of the ILP model, computed in polynomial time, or through an iterative algorithm that computes minimum spanning trees [44].

[11] Incomplete methods cannot guarantee the quality of the solution found.

Tour Construction

Tour construction algorithms use heuristics to construct a solution. Next, three different common tour construction algorithms are presented.

The **Nearest Neighbour** (NN) heuristic proceeds in three different steps. First, a random node is selected for the beginning of the path. Step 2 consists on finding the nearest node relative to the previously selected one and adding it to the path. Finally, step 2 is repeated until all nodes are contained in the tour and the last node is connected to the first [45]. This algorithm requires n^2 computations, where n denotes the number of nodes, and generally finds a solution within 25% of the HK lower bound [43].

The **Greedy** algorithm is very similar to the NN and some authors even consider it the same. Steps 2 and 3 are the same as of the NN. Nonetheless, instead of choosing the first node randomly in step 1, the arcs are all sorted according to their weight and the smallest one is chosen. With this change, a better solution can be found (often within 20% of the HK lower bound) with the downside of increasing the computational complexity to $\mathcal{O}(n^2 log_2(n))$ [43].

Insertion heuristics can also be very useful, sometimes presenting improvements relative to the ones presented previously. These heuristics begin with the creation of a subtour with the inclusion of the rest of the nodes by some heuristic. For instance, the **Nearest Insertion** starts by finding the minimum cost arc and forms a subtour. After that, it searches for the node that forms the minimum weight arc with any of the nodes already present on the subtour and inserts it. This final process is repeated until the Hamiltonian cycle is formed [45]. These heuristics have a computational complexity on the order of $\mathcal{O}(n^2)$ and often present a solution with a relative gap to HK of less than 20%.

Tour Improvement

After constructing a tour, one may wish to improve this solution. Keeping this objective in mind, perhaps the most used heuristics with this purpose are the branch exchange heuristics. The 2-opt, k-opt and, later introduced by Lin and Kernighan, the Lin-Kernighan algorithm, are discussed next [46].

The **2-opt** and **k-opt** algorithms work by removing 2 and k branches, respectively, from the tour and reconnecting the paths. This process is only done if the new tours are shorter and is repeated until 2-opt or k-opt improvements are no longer possible, resulting in 2-optimal and k-optimal tours respectively. Notice that a k-optimal tour is also (k-1)-optimal and hence 2-optimal. The 2-opt heuristic generally results in a solution within 5% of the HK lower bound [45].

The **Lin-Kernighan algorithm** [46] finds a feasible solution for the symmetric TSP with a time complexity of approximately $\mathcal{O}(n^{2.2})$. It is a variation of the k-opt heuristic that at each step decides how many paths it should switch to find the minimum weight tour.

2.2.2.B Meta-Heuristic Algorithms

Meta-heuristics are algorithms that are designed to be applied to any COP. These are generally based on concepts from biological evolution, physics or even statistical mechanics and include for example, Genetic Algorithms, Simulated Annealing, Tabu-Search, Ant Colony Optimization or even Neural Newtorks [47]. This subsection introduces some of these algorithms.

Ant Colony Optimization is based on the behaviour of real ants. When traveling from the nest to a food source, ants seem to all take the same path and curiously, the shortest one as well. This does not happen at first but when ants walk they leave a pheromone trail. Ants are also probabilistically more prone to take a trail that has more concentration of that pheromone [48]. Considering various paths, ants pass more often on the same spot in a shorter one. Hence, in this path there will be more concentration of that pheromone, leading to more ants choosing this trail which positively feedbacks the other ants, eventually leading this path to be the only one taken. Simulating a TSP with artificial ants might then have surprising results if one identifies an appropriate representation of the problem. A good heuristic to determine the distance/weight between two cities and a good probabilistic interaction ant/pheromone model should be used [49]. ACOs are hybrid Stochastic Local Search (SLS) approaches with both probabilistic solution construction and normal local search techniques [10]. A very effective ACO algorithm for the TSP is the MAX-MIN Ant System [50].

Simulated Annealing simulates the physical process of heating a metal until the melting point and then reducing the temperature in a controlled way. This results in a particle arrangement that minimizes the energy state (ground state of the solid). An analogy to this physical process was used to develop this metaheuristic. In this analogy, the total weight of the solution corresponds to the energy state and the particle rearrangement to the solution space [35]. This metaheuristic can then be compared to a local search heuristic with the tweak that now up-hill moves are possible (in the analogy this would be the heating process).

Genetic Algorithms are part of the broader class of *Evolutionary Algorithms* and are inspired by the natural selection process. These algorithms are based on a five step process. During the initialization, a population is stochastically created. Then, based on a fitness function, some of the population is selected to breed another generation. This new generation is subject to genetic operators such as crossover and mutation. During this process, other heuristics may also be applied. Finally, this steps are repeated until a termination condition has been satisfied.

3

Problem Formulation and Optimization Models

Contents

In this Chapter, four different ILP models that formalize the Flying Tourist Problem (FTP) and its considered variations are presented. In Section 3.1, the model presented by R. Marques [3, 51] is revisited and a different method to solve it is proposed. This model is then modified to answer three different problems. In Section 3.2, the FTP is modified considering a generalization of the problem, the Generalized Flying Tourist Problem (GFTP). In Section 3.3, the FTP is adapted based on a multi-objective optimization, leading to the Multi-objective Flying Tourist Problem (MO-FTP). In Section 3.4 a final model considering both adaptations, the Multi-objective Generalized Flying Tourist Problem (MO-GFTP), is presented. Lastly, in Section 3.5, these problems are compared to the original TSP and both a graph and a dimensional overview of these models are conducted.

3.1 Flying Tourist Problem

The FTP formulates the problem of a tourist that desires to visit several cities by plane in a specified time-window. The main objective is to minimize the cost or traveling time of the tourist. The solution is then a set of flights that the tourist should buy in order to minimize one of these objectives. This set of flights is a Hamiltonian Cycle.

This problem is very close to the TDTSP, as the tourist is to specify the stop-time at each city, and can be illustrated in a multipartite graph, as shown in Figure 3.1. This is a simple instance, where the departure and arrival cities are the same, respectively $V_0 = V_{n+1} = V_X$ and it considers 3 intermediate nodes (V_A, V_B, V_C), each with a fixed stop time, respectively (1,2,3) time units. The starting date window T_0 is singular with $t = 0$. Moreover, a possible solution to this instance, represented in the figure with the path corresponding to the set of arrows in red, is given by the set of arcs $(A^0_{X,B}, A^2_{B,A}, A^3_{A,C}, A^6_{C,X})$.

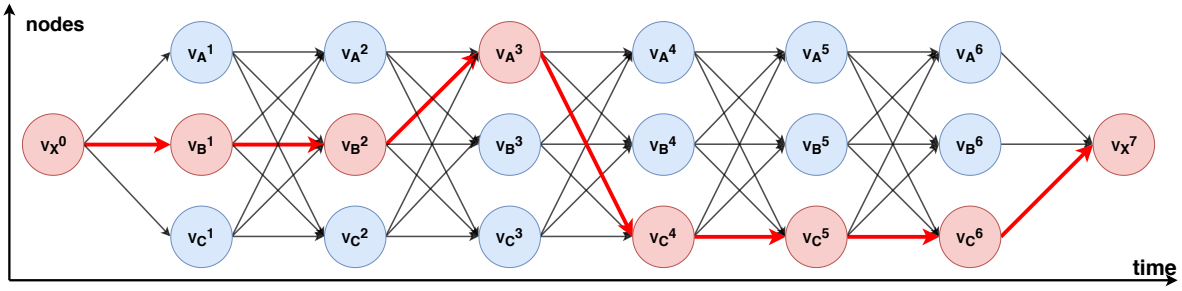


Figure 3.1: Multipartite graph of the Flying Tourist Problem [3].

Problem statement

Assuming that the user wants to visit a set V of n cities connected by the set of arcs A, the objective of this model is to minimize the weight function c_{ijk}, the weight of the arc connecting nodes i and j on time instance k, of a tour that starts on origin city 0, visits exactly once all n cities and finishes at arrival city $n+1$. The weight function is defined by the user itself and it can be the travel cost, the traveling time or a compromise between the two. The user is also the provider of the stop time s_i at each intermediate city and the time horizon during which the travel may start $T_0 = [T_{0m}, T_{0M}]$. Even though (in this work) the stop time will always be considered as an integer, it may be a range of values instead. With T_0 and s_i, it is then possible to define a time window during which each city may be visited, denoted by TW. The set S containing all valid solutions is hereinafter defined and the purpose of this model is to find the global minimum, by considering the objective function of this set.

The present formulation also took in consideration the work developed by Li et al. [52]. They proposed a problem that aims to find itineraries with the lowest cost for travelers visiting multiple cities under the constraints of time horizon and stop time at each city. With this in mind, the travel itinerary problem was raised and it is fairly similar to what we propose in this subsection.

The notation used in our model is as follows:

- V is the set of city nodes,
- i is the index of the departure city of a certain flight,
- j is the index of the arrival city of a certain flight,
- k is the index of the transport alternative used,
- n is the number of intermediate cities,
- I_{ij} is the number of transport alternatives from city i to city j,
- c_{ijk} is the cost value of the flight from city i to city j, using the transport alternative k,
- d_{ijk} is the departure date and time from city i to city j, using the transport alternative k,
- a_{jik} is the arrival date and time at city i from city j using the transport alternative k,
- s_i is the stop time at city i,
- T_{start} or T_{0m} is the minimum start date,
- T_{0M} is the maximum start date,
- T_{fm} is the minimum return date,

- T_{end} or T_{fM} is the maximum return date,

- x_{ijk} is a binary decision variable that carries the value 1 if the flight k from city i to city j is taken.

- u_i is an integer decision variable that carries the non-negative value associated to the position of city i in the tour.

Remark 2. This work focuses on flight travel only and for that reason, index k is instead the flight number. In addition, the implemented prototype discards everything but the lowest weight flight from city i to city j at each day for memory reasons. For this reason, when this prototype is used, index k is degenerated to the day of the flight, thus I_{ij} is transformed into I and takes the same value for every connection (ij). Nonetheless the model is maintained with further work in mind.

Remark 3. In future work, the end time T_{end} can also be an input provided by the user. For now, this variable will take the value of the last possible day of start time horizon plus the sum of all stop times s_i.

The constructed ILP model is hence composed by an objective function and a few constraint equations and inequations. The total weight of the tour, and thus the function aimed to be minimized, is:

$$\sum_{i=0}^{n+1}\sum_{j=0}^{n+1}\sum_{k=1}^{I_{ij}} c_{ijk}x_{ijk} \tag{3.1}$$

In what concerns the constraints, they are divided in 4 different sections (degree, time, subtour elimination and integrality). Degree constraints ensure that each city is visited exactly once. With this in mind, there are out-degree and in-degree constraints. The former imply that there is exactly one flight leaving from each city, while the latter imply that there is exactly one flight arriving to each city. Respectively:

$$\begin{aligned} \sum_{j=1}^{n+1}\sum_{k=1}^{I_{ij}} x_{ijk} = 1, \quad i = 0, 1, \ldots, n, \\ \sum_{i=0}^{n}\sum_{k=1}^{I_{ij}} x_{ijk} = 1, \quad j = 1, 2 \ldots, n+1, \end{aligned} \tag{3.2}$$

It is also important to bear in mind that there should not be any trip arriving at the origin city or departing from the arrival city. These are called the source and the sink respectively.

$$\begin{aligned} \sum_{i=1}^{n+1}\sum_{k=1}^{I_{ij}} x_{i0k} = 0, \\ \sum_{j=0}^{n}\sum_{k=1}^{I_{ij}} x_{(n+1)jk} = 0, \end{aligned} \tag{3.3}$$

Remark 4. If the origin city and the destination city are the same, then we are facing a cycle and the degree constraints are still consistent. In this case, the origin/destination city is both the source and the sink of the problem.

Time constraints assure that the start time T_{start}, stop times s_i and end time T_{end} provided by the user are respected. For now, s_i is an integer. Nonetheless, in future work, it may be implemented as a time window. Constraints 3.4 (represented below) state that the departure from the origin city must not happen before T_{start}. On its counterpart, constraints 3.6 force that the arrival at the final destination city must not happen after T_{end}. It is important to notice that since s_i are fixed and since T_{end} is defined by the latest T_{start} plus the sum of s_i then, the latest start date T_{start} is also indirectly forced. Lastly, constraints 3.5 impose that the departure and arrival from an intermediate city must be spaced s_i days.

$$\sum_{j=1}^{n+1}\sum_{k=1}^{I_{ij}} d_{0jk}x_{0jk} \geq T_{start}, \tag{3.4}$$

$$\sum_{j=1}^{n+1}\sum_{k=1}^{I_{ij}} d_{ijk}x_{ijk} - \sum_{j=0}^{n+1}\sum_{k=1}^{I_{ij}} a_{jik}x_{jik} = s_i \quad i = 1, 2, \ldots, n, \tag{3.5}$$

$$\sum_{i=0}^{n+1}\sum_{k=1}^{I_{ij}} a_{i(n+1)k}x_{i(n+1)k} \leq T_{end}, \tag{3.6}$$

Assuming now that the user may provide a trip where the number of intermediate cities n is greater than 1, there is the need of eliminating any subtours that exist, as these are not part of the desired solution space. To guarantee a single tour, the model is given a set of subtour elimination constraints based on the polynomial formulations reported in 2.1.1.

$$u_i - u_j + (n+1)x_{ij} \leq n, \quad i, j = 1, 2, \ldots, n+1, \tag{3.7}$$

Lastly, in order to ensure that all decision variables are binary, integrality constraints are applied:

$$x_{ijk} \in \{0, 1\} \quad i, j = 0, 1, \ldots, n+1, \quad k = 1, 2, \ldots I_{ij}. \tag{3.8}$$

Bearing all of the above, the complete model is presented in Figure 3.2.

3.2 Generalized Flying Tourist Problem

The application of this problem is dedicated to travelers that wish to visit an enormous quantity of cities but do not have the resources or time to do so.

This problem is an adaptation of the FTP to the GTSP presented in Section 2.1.4. The user defines n groups of cities and wishes to visit each group exactly once. Considering an example in which the traveller wishes to visit three different cities in one trip, if these three cities are, for instance, $\{$(London), (Barcelona), (Rome)$\}$ then the FTP model that was presented in the previous subsection will return the

$$
\begin{aligned}
\text{min} \quad & \sum_{i=0}^{n+1}\sum_{j=0}^{n+1}\sum_{k=1}^{I_{ij}} c_{ijk}x_{ijk} \\
\text{s.t.} \quad & \sum_{j=1}^{n+1}\sum_{k=1}^{I_{ij}} d_{0jk}x_{0jk} \geq T_{start}, \\
& \sum_{j=1}^{n+1}\sum_{k=1}^{I_{ij}} d_{ijk}x_{ijk} - \sum_{j=0}^{n+1}\sum_{k=1}^{I_{ij}} a_{jik}x_{jik} = s_i, && i = 1,2,\ldots,n \\
& \sum_{i=0}^{n}\sum_{k=1}^{I_{ij}} a_{i(n+1)k}x_{i(n+1)k} \leq T_{end}, \\
& \sum_{j=1}^{n+1}\sum_{k=1}^{I_{ij}} x_{ijk} = 1, && i = 0,1,\ldots,n \\
& \sum_{i=0}^{n}\sum_{k=1}^{I_{ij}} x_{ijk} = 1, && j = 1,2,\ldots,n+1 \\
& \sum_{i=1}^{n+1}\sum_{k=1}^{I_{ij}} x_{i0k} = 0, \\
& \sum_{j=0}^{n}\sum_{k=1}^{I_{ij}} x_{(n+1)jk} = 0, \\
& u_i - u_j + (n+1)\sum_{k=1}^{I_{ij}} x_{ijk} \leq n, && i \neq j; \quad i,j = 1,2,\ldots,n+1 \\
& x_{ijk} \in \{0,1\}, && i,j = 0,1,\ldots,n+1; \quad k = 1,2,\ldots,I_{ij} \\
& u_i \geq 0. && i = 1,2,\ldots,n+1
\end{aligned}
\tag{3.9}
$$

Figure 3.2: ILP model of the FTP.

optimal solution for a specific start date window and stop time at each of the cities. On the other hand, if the main purpose of the traveller is, for example, to experience distinct cultural backgrounds, a few adaptations to improve the objective can be made as there are other cities where the user will face identical experiences. Let each of the cities {(London), (Barcelona), (Rome)} be considered as a cluster instead. If we add cities to each of the clusters in a way that we have a group of clusters in which at least one of them has more than one city, then the FTP model will no longer be able to return a feasible solution. In this case, let us consider the example {(London, Liverpool), (Barcelona), (Rome, Florence, Venice)}. This new problem has necessarily a better or at least equal optimal solution than the previous one, since the solution of the former is also a solution of the latter. On the other hand, the solution space is at least as large as the one of the FTP and is generally larger. With the purpose of enabling the user to postulate such problems, the Generalized Flying Tourist Problem (GFTP) is introduced.

Consider the following vertex partitioning: {(London, Liverpool), (Barcelona), (Rome, Florence, Venice)} $\equiv$

$\equiv \{(V_A, V_B), (V_C), (V_D, V_E, V_F)\}$. Similarly to the FTP, the GFTP can also be represented in a multipartite graph. This model is illustrated in Figure 3.3. This graph represents a simple instance, where the departure and arrival cities are the same, respectively $V_0 = V_X$ and $V_{n+1} = V_X$, and it considers 3 intermediate clusters with different cities $\{(V_A, V_B), (V_C), (V_D, V_E, V_F)\}$. Each city must be associated with a fixed stop time. Nonetheless let, without any loss of generality, each of these cities be associated with respectively {(2,2),(1),(3,3,3)} time units. The starting date window T_0 is singular with $t = 0$. Finally, a possible solution to this instance, represented in the figure with the set of arrows in red, is given by the set of arcs $(A^0_{X,C}, A^2_{C,A}, A^3_{A,F}, A^6_{F,X})$.

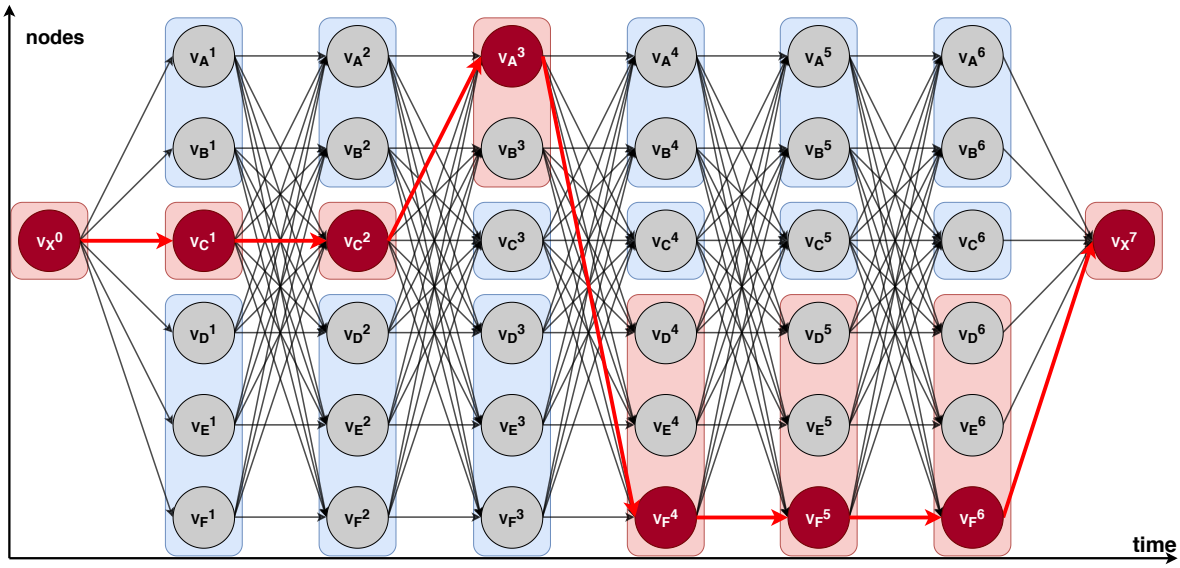


Figure 3.3: Multipartite graph of the Generalized Flying Tourist Problem.

Problem statement

This problem is a generalization of the FTP. In this case, the user wishes to visit n sets of cities (clusters). Each of the clusters is to be visited exactly once. This model is also based on the node based formulation by Imdat Kara et al. [53]. The notation used in this formulation is similar to the one previously used in FTP with the following differences:

- V_p is the subset of city nodes $\in$ cluster p.
- n is the number of intermediate clusters (instead of intermediate cities),
- m is the number of possible intermediate cities,

- p is the index of the departure cluster,

- q is the index of the arrival cluster,

- w_{pq} is a binary decision variable that carries the value 1 if the connection between cluster p and q is done and the value 0 otherwise.

- u_p is an integer decision variable that carries the positive value related to the position of cluster p in the tour.

Let $G = (V, A)$ be a graph where V is the set of nodes and A is the set of arcs connecting these nodes. The weight (usually the cost or distance) of each arc performed by a specific means of transport, k, is represented as c_{ijk}. Here, the means of transport are hence, any flight that can make the arc. Nonetheless, in this work only flights are considered and Remark 2.1.1 should be considered. Moreover, the set V is composed by the disjoint subsets $V_0, V_1, \ldots, V_{n+1}$ of cities. Then, the size of $\mid M \mid = m$ represents the total number of cities and $\mid V \mid = n$ the number of clusters. Since this GFTP model (represented in Figure 3.4) is very similar to the FTP model, only the different equations will be addressed. It is important to mention that V_0 and V_{n+1} are usually of unit size as the user starts and finishes in a specific city.

Several different constraints should be taken into consideration in this model. Firstly, the constraints that already existed on the FTP model are now relative to clusters instead of cities. Then, a new auxiliary binary variable is needed and is described by the following constraints:

$$w_{pq} = \sum_{i \in V_p} \sum_{j \in V_q} \sum_{k=1}^{I_{ij}} x_{ijk}, \quad p \neq q; \quad p, q = 0, 1, \ldots, n+1, \tag{3.11}$$

This variable, w_{pq} takes the value 1 if the traveler goes from cluster p to cluster q and the value 0 otherwise. Lastly, we created a new set of constraints that equalizes the inflow and outflow of each cluster as defined below:

$$\sum_{i \in V_p} \sum_{j \in V_q} \sum_{k=1}^{I_{ij}} (-x_{ijk} + \sum_{r \in V \setminus \{V_0, V_p, V_q\}} x_{jr(a_{ijk}+s_j)}) \geq 0 \quad p = 0, 1, \ldots, n, \quad q = 1, 2, \ldots, n, \tag{3.12}$$

If there is an arc from city $i \in V_p$ to city $j \in V_q$, then there must be an arc leaving city j, s_j days after to a cluster that is neither V_p or V_q. On the other hand, if there is not an arc from cluster V_p to city $j \in V_q$, the arc leaving this city might still exist since it is possible that the arc entering this city comes from a different cluster than V_p.

$$
\begin{aligned}
\text{min} \quad & \sum_{i\in V}\sum_{j\in V\setminus\{i\}}\sum_{k=1}^{I_{ij}} c_{ijk}x_{ijk} \\
\text{s.t.} \quad & \sum_{j\in V\setminus V_0}\sum_{k=1}^{I_{ij}} d_{ijk}x_{ijk} \geq T_{start}, && i\in V_0 \\
& \sum_{j\in V\setminus V_0}\sum_{k=1}^{I_{ij}} d_{ijk}x_{ijk} - \sum_{j\in V\setminus V_{n+1}}\sum_{k=1}^{I_{ji}} a_{jik}x_{jik} = s_i, && i\in V\setminus\{V_0,V_{n+1}\} \\
& \sum_{i\in V_p}\sum_{j\in V\setminus V_p}\sum_{k=1}^{I_{ij}} x_{ijk} = 1, && p=0,1,\ldots,n, \\
& \sum_{i\in V\setminus V_p}\sum_{j\in V_p}\sum_{k=1}^{I_{ij}} x_{ijk} = 1, && p=1,2,\ldots,n+1, \\
& \sum_{i\in V\setminus V_0}\sum_{k=1}^{I_{ij}} x_{ijk} = 0, && j\in V_0, \\
& \sum_{j\in V\setminus V_{n+1}}\sum_{k=1}^{I_{ij}} x_{ijk} = 0, && i\in V_{n+1}, \\
& \sum_{i\in V_p}\sum_{j\in V_q}\sum_{k=1}^{I_{ij}}(-x_{ijk} + \sum_{r\in V\setminus\{V_0,V_p,V_q\}} x_{jr(a_{ijk}+s_j)}) \geq 0 && p=0,1,\ldots,n, \quad q=1,2,\ldots,n, \\
& w_{pq} = \sum_{i\in V_p}\sum_{j\in V_q}\sum_{k=1}^{I_{ij}} x_{ijk}, && p\neq q; \quad p,q=0,1,\ldots,n+1, \\
& u_p - u_q + (n+1)w_{pq} \leq n, && p\neq q; \quad p,q=1,2,\ldots,n+1, \\
& x_{ijk}\in\{0,1\}, && \forall(i,j)\in A, \quad k=1,2,\ldots I_{ij}, \\
& u_p \geq 0, && p=1,2,\ldots,n+1, \\
& w_{pq}\in\{0,1\}. && p\neq q; \quad p,q=0,1,\ldots,n+1.
\end{aligned}
\tag{3.10}
$$

Figure 3.4: ILP model of the GFTP.

3.3 Multi-objective Flying Tourist Problem

The application of this problem is dedicated to tourists that do not only value the cost of the trip but also the traveling time.

More often than not, the cheapest flights that are available usually imply several connections in their route, usually known as layovers. On the other hand, this decrease in the flight costs imply a consequent increase in the traveling time, due to additional layovers. In this particular context, the problem that is herein formulated is an adaptation of the FTP to the MO-TSP, presented in Section 2.1.5 and focuses on minimizing both the cost of the trip and the implicit traveling time. This has inevitably a downside: the cost of the trip is generally higher and at most as good as the cost from the solution of the FTP, since now the time is also being minimized. On the other hand, the advantage of such model is that the traveling time is at most as large as the one from the FTP but generally lower. With the intent of giving the user the possibility of deciding on this trade-off, the Multi-objective Flying Tourist Problem (MO-FTP) is now introduced.

Problem statement

This problem considers that the user wishes not only to optimize the cost of the trip, but also the corresponding traveling time, thus a multi-objective formulation based on Subsection 2.1.5 should be considered. Despite its similarities with formulation 3.1, its main difference lays on the number of objective functions to be considered. In this problem, a set of 2 objectives will be taken in consideration: $F(x) = (f_1(x), f_2(x))$, where the overall objective is defined as min $F(x)$ and:

$$\begin{aligned} f_1(x) &= \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=1}^{I_{ij}} c_{ijk} x_{ijk} \\ f_2(x) &= \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=1}^{I_{ij}} t_{ijk} x_{ijk} \end{aligned} \tag{3.13}$$

Thus, the complete formulation is present in Figure 3.5.

In the context of the MO-FTP, the cost of the flights is represented as c_{ijk} and the traveling time is defined as t_{ijk}. In addition, the concept of Key Performance Indicator (KPI) is of the utmost importance to the understanding of the optimization process on this model. KPIs are performance measurements that evaluate the success of a certain activity. In this specific problem, the lower the KPIs of $f_1(x)$ and $f_2(x)$ are, the better, since this is a minimization problem. However, some performance indicators can be more important than others, leading to the definition of **priority**. Defining different priorities determines the order in which KPIs are approached. If various KPIs have the same priority, then these are blended together and are treated as a single objective. This is useful when various objectives are equally important or when various objectives have the same unit. Moreover, when this is the case,

$$
\begin{aligned}
\min \quad & \sum_{i=0}^{n+1}\sum_{j=0}^{n+1}\sum_{k=1}^{I_{ij}} c_{ijk}x_{ijk} \\
& \sum_{i=0}^{n+1}\sum_{j=0}^{n+1}\sum_{k=1}^{I_{ij}} t_{ijk}x_{ijk} \\
\text{s.t.} \quad & \sum_{j=1}^{n+1}\sum_{k=1}^{I_{ij}} d_{0jk}x_{0jk} \geq T_{start}, \\
& \sum_{j=1}^{n+1}\sum_{k=1}^{I_{ij}} d_{ijk}x_{ijk} - \sum_{j=0}^{n+1}\sum_{k=1}^{I_{ij}} a_{jik}x_{jik} = s_i, && i = 1,2,\ldots,n \\
& \sum_{i=0}^{n}\sum_{k=1}^{I_{ij}} a_{i(n+1)k}x_{i(n+1)k} \leq T_{end}, \\
& \sum_{j=1}^{n+1}\sum_{k=1}^{I_{ij}} x_{ijk} = 1, && i = 0,1,\ldots,n \\
& \sum_{i=0}^{n}\sum_{k=1}^{I_{ij}} x_{ijk} = 1, && j = 1,2,\ldots,n+1 \\
& \sum_{i=1}^{n+1}\sum_{k=1}^{I_{ij}} x_{i0k} = 0, \\
& \sum_{j=0}^{n}\sum_{k=1}^{I_{ij}} x_{(n+1)jk} = 0, \\
& u_i - u_j + (n+1)\sum_{k=1}^{I_{ij}} x_{ijk} \leq n, && i \neq j; \quad i,j = 1,2,\ldots,n+1 \\
& x_{ijk} \in \{0,1\}, && i,j = 0,1,\ldots,n+1; \quad k = 1,2,\ldots,I_{ij} \\
& u_i \geq 0. && i = 1,2,\ldots,n+1
\end{aligned}
\tag{3.14}
$$

Figure 3.5: Formulation model of the MO-FTP.

blending the objectives with different **weights** can be useful. In this particular case, if the priorities of $f_1(x)$ and $f_2(x)$ are the same and if we denote w_1 and w_2 as each respective weight, then the objective function $F(x)$ wished to be minimized will be $F(x) = w_1 f_1(x) + w_2 f_2(x)$. On the other hand, in the case of different priorities, **tolerances** can be applied. These define how much a KPI may be degraded to comprise for better optimal values of lower priority KPIs. Regarding again the present problem, consider that $f_1(x)$ has higher priority than $f_2(x)$. The problem is solved by considering $f_1(x)$ as the objective and a solution is found. After that, an absolute or relative tolerance is applied to demean the first objective and a solution minimizing $f_2(x)$ is found while still respecting $f_1(x)$.

3.4 Multi-Objective Generalized Flying Tourist Problem

Finally, the motivation of this specific problem is to enable travellers to attain the advantages of both the GFTP and the MO-FTP.

In Subsection 3.2, we claim that the GFTP has the advantage of returning lower optimal values when compared to the FTP. Moreover, in Subsection 3.3 we suggest that the MO-FTP has the advantage of improving the traveling time at the expense of increasing the cost of the trip. The model suggested in this subsection is an adaptation of the FTP to both the GTSP and the MO-TSP. By joining the previously presented models, the capacity of the generalized problem in achieving better optimal values can diminish, negate or even reverse the increase in the cost of the trip, caused by the multi-objective variation. With this objective in mind, the Multi-objective Generalized Flying Tourist Problem (MO-GFTP) is introduced.

Problem statement

By following the same methodology as in the last model, the MO-GFTP is based on Subsection 2.1.5, applied this time on the model proposed in Subsection 3.2. In this problem, a set of 2 objectives will be taken in consideration: $F(x) = (f1(x), f2(x))$, where the overall objective is defined as min $F(x)$ and:

$$\begin{aligned} f_1(x) &= \sum_{i \in V} \sum_{j \in V \setminus \{i\}} \sum_{k=1}^{I_{ij}} c_{ijk} x_{ijk} \\ f_2(x) &= \sum_{i \in V} \sum_{j \in V \setminus \{i\}} \sum_{k=1}^{I_{ij}} t_{ijk} x_{ijk} \end{aligned} \tag{3.15}$$

Hence, the complete formulation is represented in Figure 3.6.

Where, again, c_{ijk} represents the cost of the flights whilst t_{ijk} depicts the traveling time. Key Performance Indicators (KPIs) are also used to evaluate the these objectives. Finally, a problem with this formulation that conceives a solution of the same size of a solution from a MO-FTP problem is, in most cases, computationally heavier. This occurs since the number of nodes is necessarily larger which implies more arcs and thus a larger search space. For this reason, of the four models concerned in this work, this is the most complex.

3.5 Final Considerations

In the last section of this chapter, some final considerations on the models previously presented are given. First, in Subsection 3.5.1, a relation of the defined models with the Traveling Salesman Problem is explained. Then, in Subsection 3.5.2, both a graph and a dimensional analysis of these models is given.

$$
\begin{aligned}
\text{min} \quad & \sum_{i\in V}\sum_{j\in V\setminus\{i\}}\sum_{k=1}^{I_{ij}} c_{ijk}x_{ijk} && \\
& \sum_{i\in V}\sum_{j\in V\setminus\{i\}}\sum_{k=1}^{I_{ij}} t_{ijk}x_{ijk} && \\
\text{s.t.} \quad & \sum_{j\in V\setminus V_0}\sum_{k=1}^{I_{ij}} d_{ijk}x_{ijk} \geq T_{start}, && i \in V_0 \\
& \sum_{j\in V\setminus V_0}\sum_{k=1}^{I_{ij}} d_{ijk}x_{ijk} - \sum_{j\in V\setminus V_{n+1}}\sum_{k=1}^{I_{ji}} a_{jik}x_{jik} = s_i, && i \in V\setminus\{V_0, V_{n+1}\} \\
& \sum_{i\in V_p}\sum_{j\in V\setminus V_p}\sum_{k=1}^{I_{ij}} x_{ijk} = 1, && p = 0,1,\ldots,n, \\
& \sum_{i\in V\setminus V_p}\sum_{j\in V_p}\sum_{k=1}^{I_{ij}} x_{ijk} = 1, && p = 1,2,\ldots,n+1, \\
& \sum_{i\in V\setminus V_0}\sum_{k=1}^{I_{ij}} x_{ijk} = 0, && j \in V_0, \\
& \sum_{j\in V\setminus V_{n+1}}\sum_{k=1}^{I_{ij}} x_{ijk} = 0, && i \in V_{n+1}, \\
& \sum_{i\in V_p}\sum_{j\in V_q}\sum_{k=1}^{I_{ij}}(-x_{ijk} + \sum_{r\in V\setminus\{V_0,V_p,V_q\}} x_{jr(a_{ijk}+s_j)}) \geq 0 && p = 0,1,\ldots,n, \quad q = 1,2,\ldots,n, \\
& w_{pq} = \sum_{i\in V_p}\sum_{j\in V_q}\sum_{k=1}^{I_{ij}} x_{ijk}, && p \neq q; \quad p,q = 0,1,\ldots,n+1, \\
& u_p - u_q + (n+1)w_{pq} \leq n, && p \neq q; \quad p,q = 1,2,\ldots,n+1, \\
& x_{ijk} \in \{0,1\}, && \forall (i,j) \in A, \quad k = 1,2,\ldots I_{ij}, \\
& u_p \geq 0, && p = 1,2,\ldots,n+1, \\
& w_{pq} \in \{0,1\}. && p \neq q; \quad p,q = 0,1,\ldots,n+1.
\end{aligned}
\tag{3.16}
$$

Figure 3.6: Formulation of the MO-GFTP.

3.5.1 Relation to the Traveling Salesman Problem

The GFTP intends to find the best route, schedule and set of flights to visit a set of clusters. These clusters are composed by one or more cities. If these clusters are reduced to one-city nodes, then the problem is reduced to the FTP and thus, the GFTP is a generalization of the former.

Consider now the set of allowances to be applied to the previously formulated FTP:

1. The FTP considers the possibility of starting and ending the trip in different cities. Consider now

the depot to be singular, $V_0 \equiv V_{n+1}$.

2. Consider also an allowance on the time specification:

   (a) The starting time is now a time unit.

   (b) Each city shall be visited at any given position of the arc, which means that the definition of Time Window is now ignored.

   (c) Each city is now solely a passing point and hence the stop time is not considered.

   (d) The weight matrix is no longer time-dependent, meaning that the cost of connecting cities i and j is constant throughout time.

By applying the above permissions to any FTP, one ends up with an Asymmetric Traveling Salesman Problem. If reduction 2d is not applied, the FTP is, on the other hand, reduced to the Time-Dependent Traveling Salesman Problem. As one can transform any ATSP into a Symmetric Traveling Salesman Problem, then it is proved that both the GFTP and the FTP are generalizations of the TSP. Moreover, since the TSP is classified as a NP-hard problem and since a generalization of a NP-hard problem is itself NP-hard, then both the FTP and the GFTP are NP-hard. Finally, since the MO-FTP and the MO-GFTP are generalizations of these problems, then these are also generalizations of the TSP and thus are NP-hard as well.

3.5.2 Graph and Dimensional Overview

This subsection will first introduce a graph analysis of the introduced models. Finally, using the information conveyed in this analysis, a dimensional overview is possible and it will be herein described.

The nature of commercial flights implies that it is possible for several flights to connect two cities in a particular moment. Furthermore, the cost of a flight connecting two cities varies with the moment the flight occurs, hence the relation with the TDTSP. For this reason, the weight matrix can be represented by a triplet (i, j, k). In this triplet, i is relative to the city from which the flight departs, j is the city to which the flight arrives and k is the number of the flight. On the other hand, the computational time optimizing this 3-dimensional matrix could rapidly escalate in cases where multiple connections between i and j occurring at the same time period exist. Furthermore, this could also raise memory problems. For these reasons, as the objective is to minimize the total weight of the trip, we only keep the lowest weight flight between two cities at each time instant (or day).

Moreover, not all entries need to be filled and these pieces of memory can also be spared. This is done using different sets of arcs: initial, intermediate and final arcs that together form the solution. In the case of the FTP, initial arcs correspond to the arcs connecting node V_0 with the rest of the nodes during the start time window, $k \in T_0 = [T_{0m}, T_{0M}]$. The final arcs are the arcs connecting the arrival city,

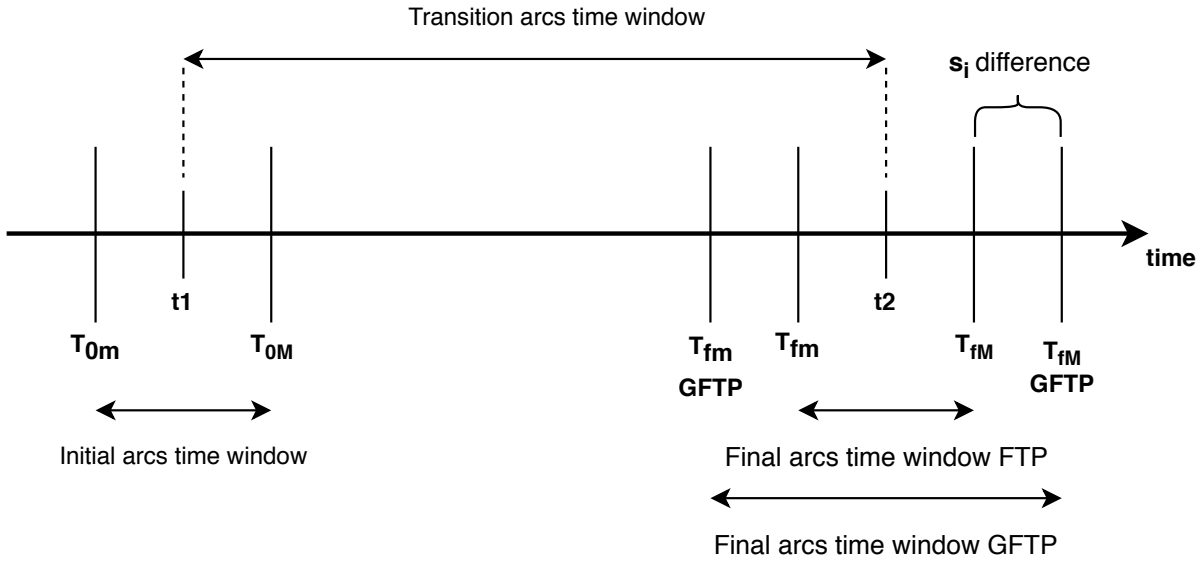


Figure 3.7: Timeframe of initial, intermediate and final arcs of both the FTP and the GFTP [51].

node V_{n+1} during the end time window. This end window can be found as explained next and visualized in Figure 3.7. It starts on the day which corresponds to the sum of stop times with the first possible start day and ends on the day that corresponds to the sum of stop times with the last possible start day, hence $k \in [T_{fm}, T_{fM}]$ where $T_{fm} = T_{0m} + \sum(s_i)$ and $T_{fM} = T_{0M} + \sum(s_i)$. Finally, intermediate arcs connect every intermediate node $V_1, \ldots, V_n$ from the day corresponding to the minimum stop time of every city summed with the first possible start day, until the day corresponding to the minimum stop time of every city subtracted from the last possible end day. This means that these arcs occur from $t_1 = T_{0m} + min(s_i)$ to $t_2 = T_{fM} - min(s_i)$.

In the GFTP, initial arcs are the same as in the FTP. On the other hand, the end time window starts in the sum of the minimum stop times at each cluster with the first possible start day and ends in the sum of the maximum stop times at each cluster with the last possible start day. This is: $T_{fm} = T_{0m} + \sum_{p=1}^{n} min(s_i, i \in V_p)$ and $T_{fM} = T_{0M} + \sum_{p=1}^{n} max(s_i, i \in V_p)$ respectively. Lastly, intermediate arcs are the same as in the FTP.

The matrix produced is hence of size $(|V|, |V|, |T_0| + \sum(s_i)$, where $|V| = n+2$ in the FTP, $|V| = m+2$ in the GFTP and T_0 is the length of the problem. Thus the number of entries of the matrix is given by: $n_e = (T_0 + \sum(s_i))(n+2)^2$ or $n_e = (T_0 + \sum(s_i))(m+2)^2$ respectively. Considering that each entry is a weight, it can be represented by a simple 32 bit integer that occupies only 4 bytes, hence $memory = 4n_e$ bytes [51]. The same is observed in the MO-FTP and in the MO-GFTP.

3.6 Summary

The aforementioned chapter presents an Integer Linear Programming formulation for the Flying Tourist Problem, a problem that to the best of our knowledge has not yet been subject to a complete optimization method. Moreover, three variations of this problem are proposed: the Generalized Flying Tourist Problem in which the tourists only visits a subset of cities, the Multi-objective Flying Tourist Problem in which the tourist wishes not only to minimize the cost but also the traveling time and finally the Multi-objective Generalized Flying Tourist Problem that considers a tourist that requests again a minimization of both objective functions and only visits a subset of cities.

Finally, the defined models shall now be used to find solutions for complex real world cases. For such purpose, Chapter 4 presents the implementation details of these formulations on a commercial solver and the integration of this optimization module in a web application system.

4

Prototype Implementation

Contents

This chapter presents the design choices that were used in the flight search application developed by R. Marques [3, 51], as well as the modifications that were subsequently conducted to adopt this infrastructure to the new models proposed in Chapter 3.

4.1 Prototype

The considered infrastructure consists in a web-based system that enables the end user to use the work developed during this dissertation. The system's structure is represented in Figure 4.1. The prototype system is composed of two different applications: the Client Side Application (CSA) and the Server Side Application (SSA). These communicate with one another via requests denoted as *resources* (when CSA sends information to SSA) and *responses* (when SSA sends information to CSA) developed in Asynchronous Javascript and XML (AJAX). AJAX allows the user to keep interacting with the application when the SSA is processing the response (in Figure 4.1, the time comprised between number 2 and 5).

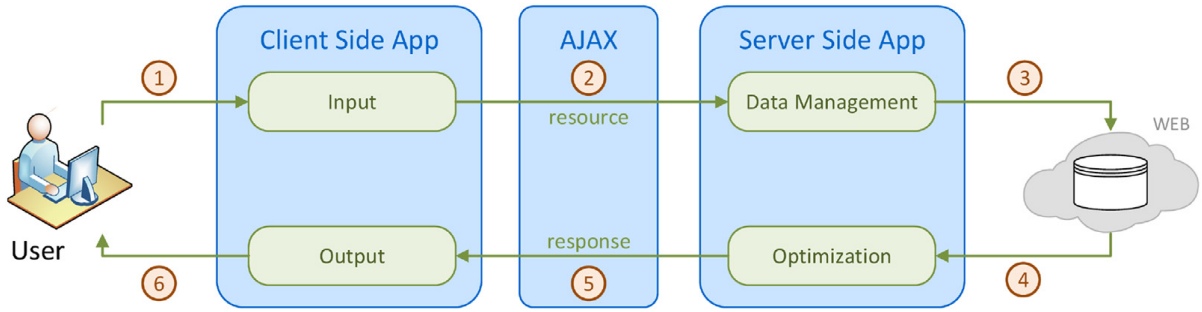


Figure 4.1: Data flow on the different applications of the prototype [3]

4.1.1 Client Side Application

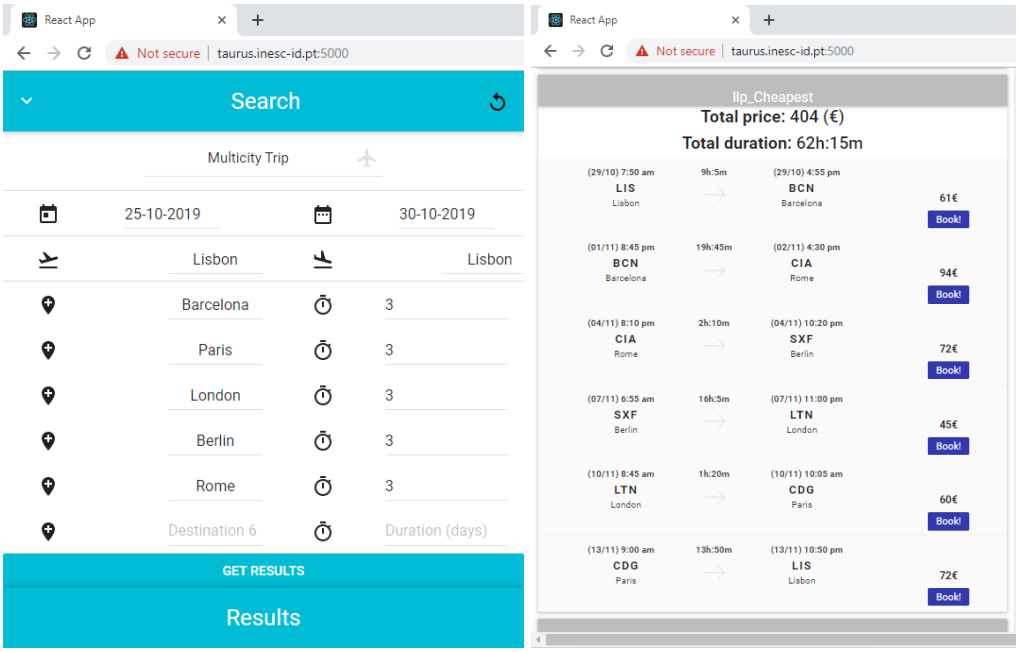
The web application was designed solely with the purpose of interacting with the end user in two different steps:

1. The user defines the flight requests,

2. The CSA presents the final solutions to the end user.

This application was developed under React [54] and Redux [55][1] environments. In this environment, the user defines the departure and arrival cities, the list of intermediate cities, the waiting periods associated to each city and the start time window of the trip (Figure 4.2(a), represented in Figure 4.1 by number 1).

[1] Both are Javascript libraries.



(a) Example of a end user's request on the CSA.

(b) Example of a response on the CSA.

Figure 4.2: Example of user interaction with the CSA [51]. On the left, an example of a request. On the right, the response to the same request. This request is sent to the SSA as:*http:/taurus.inesc-id.pt:8000/flights/flyFrom=Lisbon&returnTo=Lisbon&minDate=26/10/2019&maxDate=30/10/2019&duration=3,3,3,3,3&cities=bcn,cdg,ltn,sxf,cia*.

The user's input is processed and sent to the SSA via a *resource* (Figure 4.1, number 2). Later, the CSA receives a response from the SSA, which contains the output information to be shown to the end user (Figure 4.2(b), represented in Figure 4.1 by numbers 5 and 6, respectively). The response contains at least one multi-city tour, containing hyperlinks where these can be bought, a map of the trajectories (created using Google Maps Platform show in Figure 4.3) taken by each of the group of flights and some information such as duration, cost, departure/arrival dates of the flights and number of layovers.

Even though the application architecture was maintained, the CSA was subject to some modifications in order to make it compatible with the proposed models. The GFTP model receives a list of clusters of cities and the associated stop times, instead of just a list of cities and respective stop times. This will allow users to use the GFTP model formulated in Section 3.2. In addition, the considered type of search must also provide an option for multi-objective optimization, which will solve the MO-FTP when used together with a list of cities, or the MO-GFTP when used together with a list of clusters.

Accordingly, the updated Client Side Application has now the option for the end user to choose between six different possible problems:

1. Single Flight search,

Figure 4.3: Example of the map with the flight trajectories corresponding to the response to the request represented in Figure 4.2(a).

2. Round Flight search,

3. Multi-city Flight search:

   (a) Flying Tourist Problem with Random Search, Nearest Neighbour and two Metaheuristics previously implemented, as well as the ILP model proposed in Section 3.1,

   (b) Generalized Flying Tourist Problem with the ILP model proposed in Section 3.2,

   (c) Multi-objective Flying Tourist Problem with the multi-objective model proposed in Section 3.3,

   (d) Multi-objective Generalized Flying Tourist Problem with the multi-objective model proposed in Section 3.4.

4.1.2 Server Side Application

The Server Side Application (SSA) is divided in two different modules developed in a Python [56] environment: the Data Management module and the Optimization module.

Data Managment Module: The SSA receives the input information that was introduced by the user from the CSA (Figure 4.1, number 2). Next, it creates a graph with all the possible arcs linking the chosen cities, collects the data corresponding to all the flights connecting all the nodes during that time-window from Kiwi API [2] and creates the objective matrices. These matrices are constructed based on the objectives selected by the user. For example, if the user wishes to minimize the cost, then the matrix contains the cheapest flights. Generally, two matrices are kept. The first matrix contains the cheapest

flights at each day for every arc whilst the second one contains, for each day, the flights for every arc that have the least traveling time.

Through the study of this module it was possible to understand how it could still be improved. One of the flaws was that all the flights connecting all the nodes during that time-window were being requested. This was slowing the data acquisition process, since unnecessary flights were being requested and by increasing the search space, the optimization process was also being slowed. To overcome these unnecessary requests, Figure 3.7 was taken into consideration and the intermediate arcs are now only requested if they occur inside the transition arcs time window $[t1, t2]$. Furthermore, initial arcs are only requested if they occur inside the initial arcs time window $[T_{0m}, T_{0M}]$ and final arcs inside the final arcs time window $[T_{fm}, T_{fM}]$. Lastly, since the Generalized Flying Tourist Problem has a different final arc time window, this module was adapted to only acquire final arcs inside this time window when faced with a GFTP request.

Remark 5. Keeping only the cheapest or the flight with the least traveling time for each connection at each day can, however, compromise the provability of optimal solutions. Let us now consider the following weight matrix:

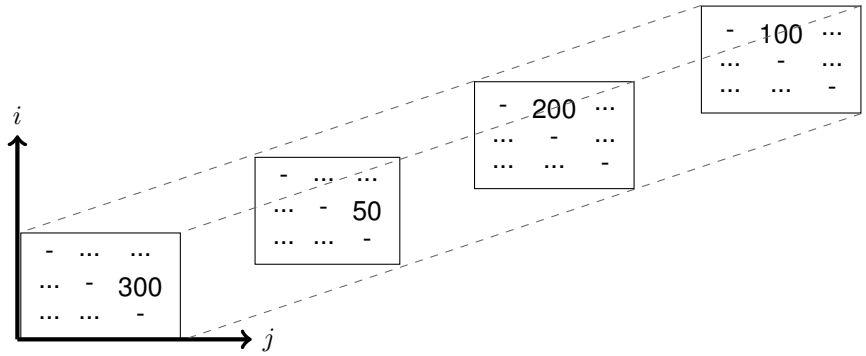


Figure 4.4: Weight Matrix. Each 2-dimensional matrix is a flight number from city i to city j each in its respective axis.

Consider that the matrix above does not follow this strategy. Consider as well that the flights that cost 200 and 100 connect the same cities (1,2), both on day 1, and that the flights that cost 50 and 300 connect the cities (2,3) in different days. If the flight that costs 200 enables you to catch either of the flights that connect (2,3) and if the flight that costs 100 only allows you to then catch the flight that costs 300 then, keeping only the flight with the cost 100 will never be able to globally minimize the cost of this tour as a tour with this flight will cost $100 + 300 = 400$, whilst the minimum cost tour with the other flight weights $200 + 50 = 250$.

The flight information managed by the management module is received from the Kiwi API in the *JSON* format as a response to the Hypertext Transfer Protocol (HTTP) request that was previously sent

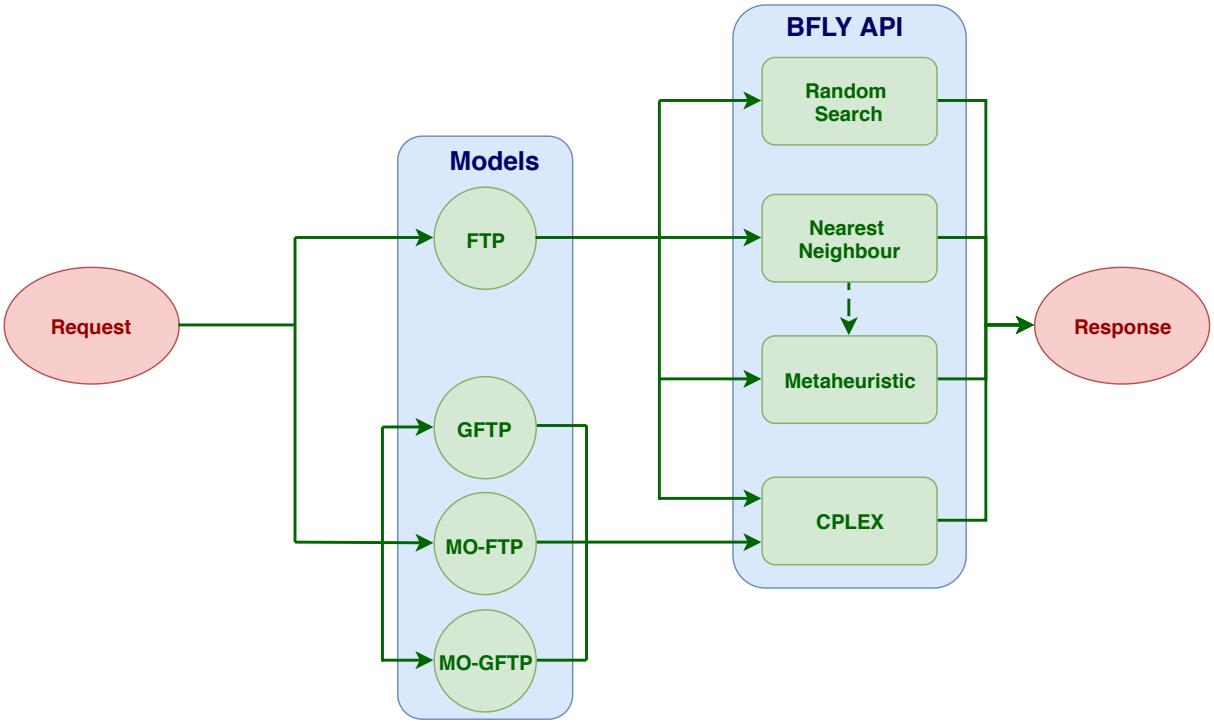


Figure 4.5: Simplified optimization system. Depending on the type of problem, different optimization systems are used. The FTP is solved using 4 different methods while the remaining problems are solved only by CPLEX.

(Figure, 4.1 number 3). Several other flight search APIs exist. Nonetheless, they usually only allow single requests. This means that for each day, a request between every two cities would be necessary. With Kiwi API this is not the case, as there is the possibility to request a time window. As a result, one request is enough to fill the entries of the matrix relative to two cities for the entire time frame. Afterwards, the graph and the objective matrix are sent to the Optimization Module (Figure 4.1, number 4) that will find a solution to the request.

Optimization Module: This module solves the models presented in Chapter 3 together with the alternative stochastic optimization models developed by R. Marques in [51]. The system is built upon a modular approach, so that different optimization techniques can be used. The Multi-city flight request is illustrated in Figure 4.5.

Figure 4.5 depicts the request and response blocks, corresponding to number 4 and 5 in Figure 4.1, respectively. The request carries the information of which model and optimization technique are to be chosen. If no optimization technique is chosen, then all of the possible methods of that specified model are executed in a sequential manner. First, the problem is solved using a Random Search, then utilizing the Nearest Neighbour, proceeding to one of the Metaheuristic methods, either ACO or SA, and finally

utilizing the models presented in Chapter 3. It is important to notice that both Metaheuristic approaches are dependent on the solution derived from the Nearest Neighbour algorithm. In Figure 4.5, the CPLEX solver was used without loss of generalization, as there are diverse solvers[2] able to solve the proposed models. Nonetheless, CPLEX is the solver used by in this module. The reason behind this choice is grounded on Subsection 2.2.1.B. Since this is a web service to serve a user on real time requests, the choices for LP solvers were reduced to the two commercial solvers (CPLEX and Gurobi) as these reportedly present better optimization times than the other options. As most references on this report use CPLEX, this was the final option.

This solver was integrated through the Python framework that CPLEX offers, the IBM Decision Optimization CPLEX Modeling for Python, also known as DOcplex. This is an object oriented API and integrated library that offers both mathematical programming modeling and constraint programming modeling. Using the mathematical programming modeling capability the formulations presented in Chapter 3 represented in Figures 3.2,3.4,3.5 and 3.6 are implemented programmatically. First, all the parameters received through the Data Management Module are stored in Python objects, i.e arrays or dictionaries. Then the model is constructed and the optimization is run. With the solution, we traceback the flight information to be within the response. Finally, the response, which carries the chosen arcs that belong to the optimized solution, is directed to the CSA.

In summary, the SSA operates in the following sequence:

1. Generation of the list of considered flights;
2. Data request to Kiwi;
3. Construction of Weight Matrices;
4. Optimization module;
5. Construction of the solution to be sent to CSA.

4.2 Deployment

The implemented prototype was deployed in two different servers. One server handles the CSA as a web service, to allow user interaction, whilst the other holds the SSA that communicates exclusively with the CSA. The deployment was previously hosted using Heroku [57], a cloud application platform that allows applications to be built, deployed, monitored and scaled. However, using this platform, in a free of charge manner, has the disadvantage of serving a limited number of queries[3]. Since this would put a limitation on the number of served users as it would greatly increase the difficulty of retrieving

[2] E.g. Gurobi, Lpsolve, Glpk, Scip
[3] In this context, by queries, we refer to user web requests

enough data for the analysis that will be presented in Chapter 5, Heroku is no longer used. Instead, the prototype is hosted in a physical server, *taurus*, that belongs to INESC-ID. The CSA is publicly accessible through PORT 8000 and both the CSA and Kiwi API communicate with the SSA through PORT 3000. The CSA utilizes Node.js [58] to migrate the frontend code to the server hosted in *taurus* whilst the web development relative to SSA was written in Python along with the framework Django [59]. Denoting now the CSA and the SSA as BFly Application and BFly API, respectively, Figure 4.6 depicts the technology bundle used in this application.

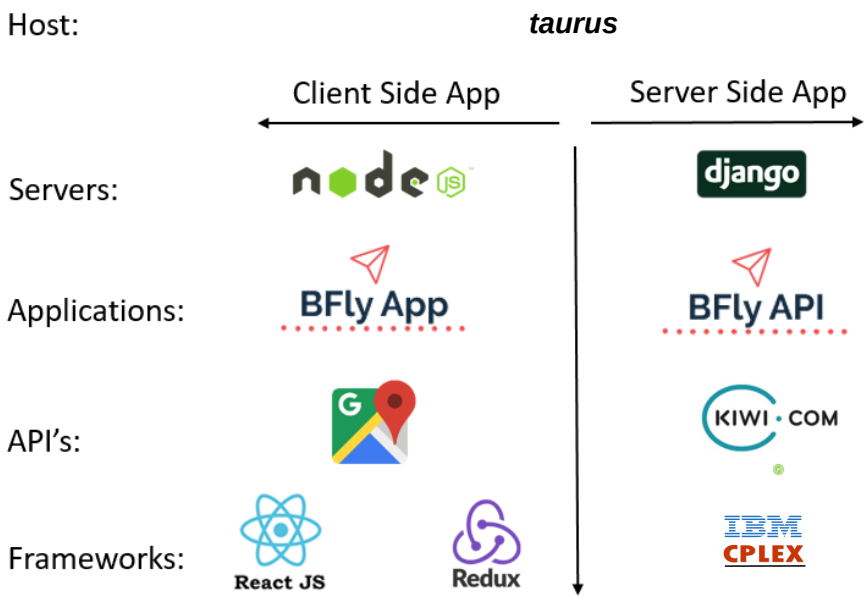


Figure 4.6: Technology stack used in the application [51].

5

Evaluation and Experimental Results

Contents

In this chapter some experimental results regarding the optimization models described in Chapter 3 are presented. For this evaluation, 50 different cities were considered as intermediate nodes see (Table 5.1) and Lisbon was considered as the departure and arrival city in all considered trips.

On the 1st of August of 2019, the prototype described in Chapter 4 was used to construct a matrix containing the cheapest flights between these cities for each day during a period of 65 days. This period starts on the 1st of October of 2019 and ends on the 5th of December of 2019. The aforementioned analysis does not take into account the time for the data acquisition since we want to evaluate the optimization module. For this reason, this analysis considers the hypothesis where the prototype has a preemptive database with all necessary flights. Moreover, the evaluation does not take into account any special event or holiday, even though these could have impact on the results. All the results were obtained on a 2.3 GHz AMD Opteron(TM) 6276 Processor and the code was developed using Python programming language. In Sections 5.1, 5.2, 5.3 and 5.4, the achieved results, by applying the FTP, GFTP, MO-FTP and MO-GFTP formulations, respectively, are presented and commented.

LHR	London	**DUB**	Dublin	**HEL**	Helsinki	**TIA**	Tirana	**ALA**	Almaty
CDG	Paris	**ZRH**	Zurich	**OTP**	Bucharest	**EVN**	Erevan	**LUX**	Luxembourg
AMS	Amsterdam	**CPH**	Copenhagen	**MIL**	Milan	**VIE**	Wien	**SKP**	Skopje
FRA	Frankfurt	**OSL**	Oslo	**IEV**	Kiev	**GYD**	Baku	**KIV**	Chișinău
IST	Istanbul	**ARN**	Stockholm	**KEF**	Reykjavík	**SJJ**	Sarajevo	**PRN**	Pristinaa
MAD	Madrid	**BRU**	Brussels	**RIX**	Riga	**SOF**	Sofia	**TIV**	Tivat
BCN	Barcelona	**ATH**	Athens	**MLA**	Valletta	**ZAG**	Zagreb	**EDI**	Edinburgh
MUC	Munich	**WAW**	Warsaw	**BEG**	Belgrade	**LCA**	Nicosia	**CWL**	Cardiff
DME	Moscow	**BUD**	Budapest	**VNO**	Vilnius	**TLL**	Tallinn	**BFS**	Belfast
CIA	Rome	**PRG**	Prague	**MSQ**	Minsk	**TBS**	Tiblissi	**LJU**	Ljubjana

Table 5.1: List of the considered cities and the respective main Airport's IATA (The International Air Transport Association) codes.

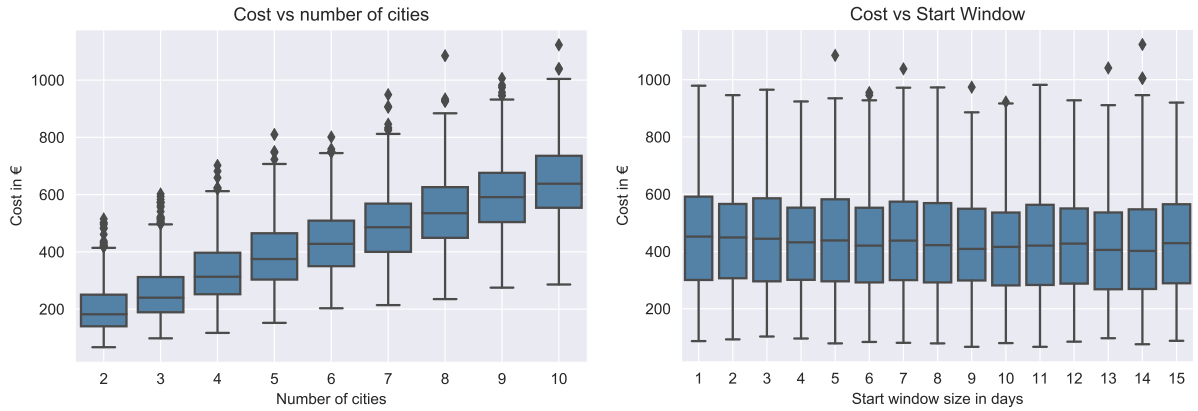
5.1 Flying Tourist Problem

Having the full weight matrix properly filled in, it is possible to evaluate the performance of the developed FTP model. For such purpose, a script choosing random requests was created and respecting the following rules:

- the requested trip must start and end in Lisbon,
- the tourist wishes to visit between two and ten cities,
- at each city, the user wishes a stop time between two and five days,
- the trip must start during the first fifteen days of October.

With the purpose of analyzing the FTP, this script was used to create 9000 different requests with a uniform distribution of cities. All the instances relative to the FTP achieved optimality, i.e the solution that was found is the global optimum of each specific problem.

However, considering a tourist that wishes to minimize the flights cost of such trip, it is important to understand which requested parameters influence the cost the most. With this in mind, Figures 5.1(a) and 5.1(b) represent the cost variation with the number of cities visited and with the size of the start window, respectively.



(a) Cost growth with number of cities visited.

(b) Cost growth with start window size of the request.

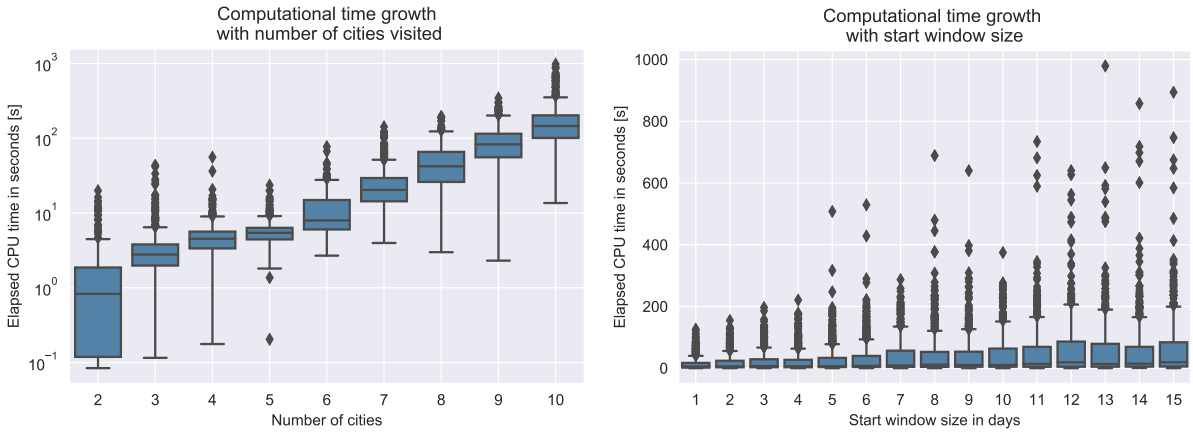
Figure 5.1: Cost of flights variation with two different parameters. Number of cities on the left. Start window size (in days) on the right.

In Figure 5.1, it is possible to observe that the number of visited cities is the parameter that influences the cost the most. This cost growth was expected, since visiting more cities means catching more flights and hence a higher overall cost. On the other hand, it was also expected that the cost would decrease with the size of the start window, as shown by R. Marques [51], since a larger start window means, in most cases, more possible flights. This was not the case and Figure 5.1(b) indicates that there is no relation, or in case there is, it is very small, between the cost and the start window size. Furthermore, in what concerns the relation between the cost and the number of cities, even though the cost grows with the number of visited cities, the mean cost of each taken flight per visited cities slightly decreases as seen in Figure 5.2. This relation was expected, since visiting more cities, increases the number of feasible arcs.

Now that the relations between the objective and the parameters are known, it is important to understand the feasibility of implementing this model in the web application, i.e if it is possible to achieve a good solution in a reasonable amount of time. Furthermore, it is also important to understand which parameters influence the computational time the most. With this in mind, the relations between the

Figure 5.2: Mean cost of flights taken per number of cities visited.

elapsed CPU time and the input parameters are depicted in Figure 5.3.



(a) Computational time growth with the number of visited cities (in a logarithmic scale).

(b) Computational time variation with the size of the start window in days.

Figure 5.3: Computational time analysis of the FTP.

Figure 5.3(a) shows an exponential growth of the computational time with the increase of the number of visited cities, whilst Figure 5.3(b) illustrates a small linear growth of the computational time with the increase of the start window size. This implies that, even though the computational time increases with both parameters, the input that mostly influences the CPU time is the number of cities.

To evaluate whether or not this model is reliable on real time requests, the wall time should also be computed. The wall time is the real time of the process and it is possible to find a relation between this and the CPU time in normal conditions. This relation is illustrated in Figure 5.4 and one can observe that most requests were answered in less than 10 seconds. In particular, 91.82% of the requests were answered in less than 10 seconds and 99.07% in less than 20 seconds. This wall time is considerably lower

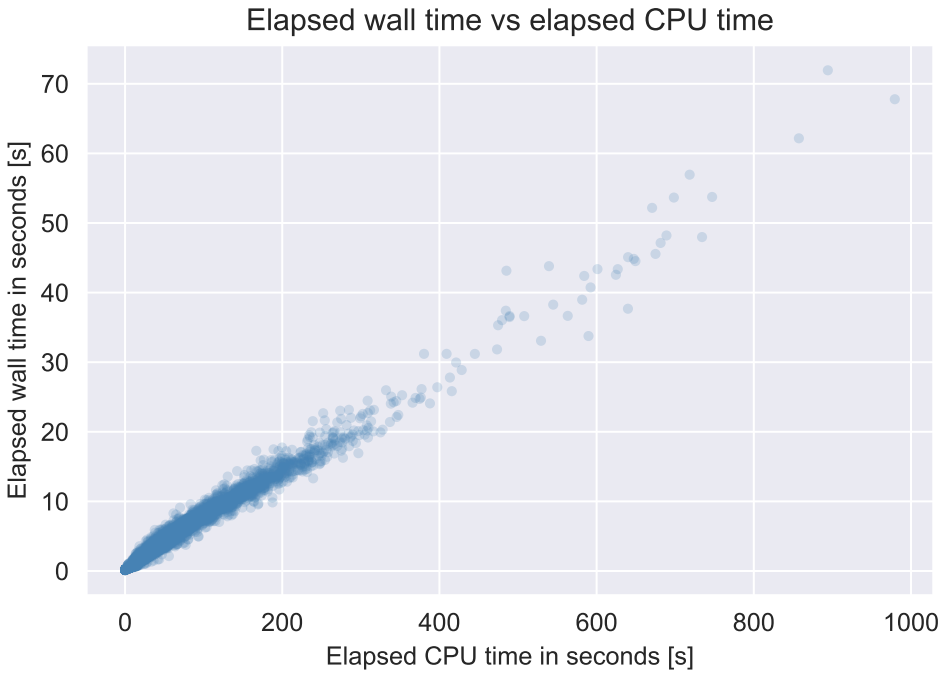


Figure 5.4: Wall time compared to CPU time.

that the CPU time since CPLEX makes use of parallel computing and the server where the optimizations were run has 16 cores each capable of running 2 threads simultaneously.

Moreover, since to the best of our knowledge the only work concerning the same exact problem is R. Marques [3, 51], a more detailed comparison should be conducted as presented in Table 5.2.

	Number of cities visited	2		3		4		5		6	
Cost in €	**SA**	182.0		*256.0*		317.0		*385.0*		446.0	
	FTP ILP	182.0	(+ 0.0%)	240.0	(-6.25%)	313.0	(-1.26%)	375.0	(-2.60%)	428.0	(-4.03%)
Elapsed CPU time [s]	**SA**	1.04		1.34		1.70		2.05		2.46	
	FTP ILP	0.83	(-20.19%)	2.78	(+107.46%)	4.53	(+164.47%)	5.43	(+164.87%)	7.95	(+223.17%)
Elapsed Wall time [s]	**SA**	1.04		1.34		1.70		2.05		2.46	
	FTP ILP	0.24	(-77.04%)	0.33	(-75.27%)	0.43	(-74.84%)	0.52	(-74.81%)	0.69	(-71.87%)

	Number of cities visited	**7**		**8**		**9**		**10**		**Entire set**	
Cost in €	**SA**	502.5		560.5		604.5		665.0		62.37 per flight	
	FTP ILP	486.0	(-3.28%)	535.0	(-4.56%)	591.0	(-2.23%)	638.0	(-4.06%)	60.27 per flight	(-3.35%)
Elapsed CPU time [s]	**SA**	2.88		3.38		3.88		4.39		2.46	
	FTP ILP	20.34	(+606.25%)	42.12	(+1146.15%)	82.67	(+2030.67%)	145.65	(+3217.77%)	8.23	(+234.86%)
Elapsed Wall time [s]	**SA**	2.88		3.38		3.88		4.39		2.46	
	FTP ILP	1.60	(-44.70%)	3.69	(+8.79%)	7.02	(+80.54%)	11.20	(+154.32%)	0.70	(-71.80%)

Table 5.2: Comparison between the SA algorithm used by R.Marques [3, 51] and the FTP ILP formulation (solved with CPLEX) implemented in this work. For each number of cities, the results indicate the median of the mean cost, elapsed CPU time and elapsed Wall time of each request. According to the experiments described throughout this section.

Table 5.2 compares the formulation proposed in 3.1 with the Simulated Annealing (SA) optimization algorithm used by R. Marques in [3, 51]. All the results in the table represent the median value of the mean of each parameter. Three parameters are compared since we want to evaluate the response in two ways. First, the main objective of each approach should be analyzed in order to understand if there are significant improvements in using a complete method in this problem. Secondly, the feasibility of applying these methods in a real-time web application needs to be assessed. For such purpose, both the Elapsed CPU time and the Elapsed Wall time are evaluated. From Table 5.2 we can observe that the achieved improvements in the cost come at the expense of a large increase in the Elapsed CPU time. However, since CPLEX makes use of coarse grained parallel computing, the real computing time in this machine largely decreases in normal conditions, outperforming the SA algorithm in most cases of requests with less than seven cities.

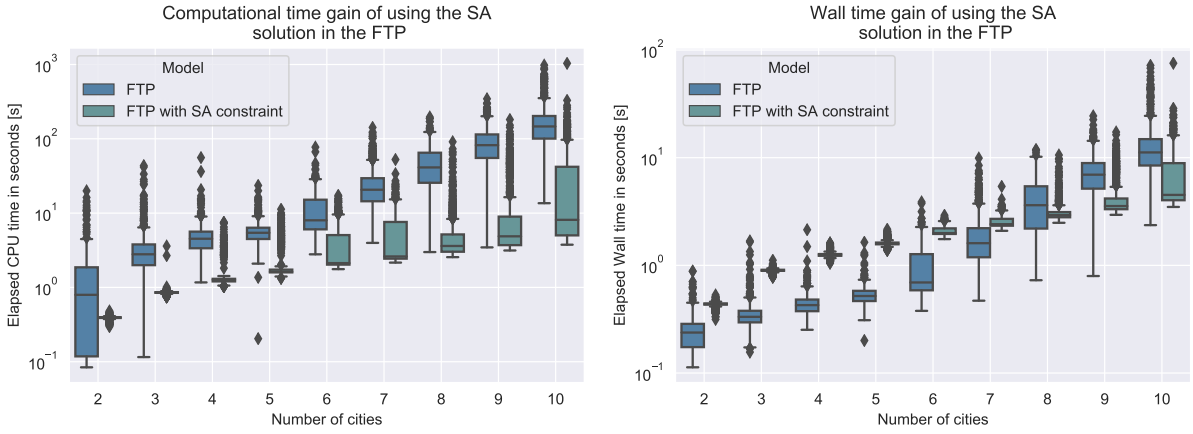
These results emphasize that the SA algorithm is faster in achieving a good solution but cannot, in most cases, find the optimal solution. On the other hand, using CPLEX, optimality was achieved at all cases. Furthermore, considering the parallel computing capability of CPLEX, one concludes that this method provides great advantages. With this in mind, the idea of using the result from the SA algorithm to build an additional constraint for the ILP formulation, arised. The new constraint is hence written as:

$$c_{sa}(x) = \sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{k=1}^{I_{ij}} c_{ijk} x_{ijk} \leq SA_{solution} \tag{5.1}$$

This constraint will act an aditional cut, during the B&C algorithm, and has the purpose of accelerating the process. The whole data set of 9000 requests was analyzed. With this new ILP formulation, all requests were solved to optimality. Nevertheless, this new formulation has the disadvantage of requiring the SA solution beforehand. Figure 5.5 compares the new formulation with the FTP presented in 3.1.

From Figure 5.5(a), we can conclude that using the SA solution as an additional constraint to the ILP formulation, the computational time required to solve the requests to optimality can be greatly reduced. More specifically, the median of the global CPU time was enhanced in 70.80%. With this change, instead of an increase of 2.35 times in the CPU time (vs the SA algorithm), the FTP only has an increase of 19.35%. However, because of the non-parallel implementation of the SA algorithm, when analyzing the wall time, one observes that this parameter only presents improvements in requests of more than 8 cities, as seen in Figure 5.5(b). Hence, considering the fact that the prototype presented in Chapter 4 is able to progressively update the responses to the end user, it is concluded that a good solution for large requests is presenting the SA solution first. Then, use it as a constraint to the ILP formulation to search for optimality to retrieve the final and optimal response.

Next, the results of the remaining variations of this model are presented, commented and compared with the results shown in this section.



(a) Comparison of the computational time of the FTP vs the FTP with the SA solution as a constraint.

(b) Comparison of the wall time of the FTP vs the FTP with the SA solution as a constraint.

Figure 5.5: Impact of using the SA solution on the FTP formulation. The times relative to the FTP with SA constraint also include the time required to execute the SA algorithm.

5.2 Generalized Flying Tourist Problem

Similarly to what was described in Section 5.1, a script to construct 9000 random requests for the analysis of this model was also implemented. The script respects the following rules:

- the requested trip must start and end in Lisbon,
- the tourist wishes to visit between two and nine clusters, each containing between two and five cities,
- at each city, the user wishes a stop time between two and five days,
- the trip must start during the first fifteen days of October.

An additional restriction was considered by limiting the number of visited cities to 9, since at this point some requests to visit 9 clusters could not be solved in less than 3600 CPU seconds. More specifically, in 11.26% of the requests with 9 clusters, no solution was found in this time span. This corresponds to 1.32% of the total data set. Moreover, some other requests were also not solved to optimality in this time span, as seen in Table 5.3. All the requests not shown in the table, with less than 6 clusters, were solved to optimality in all cases.

It is now necessary to conduct an analysis of the impacts of this the modification of one of the parameters and the addition of a new one, the number of cities and number of clusters respectively. The number of cities is no longer equal to the number of cities visited. However, this parameter still influences the size of the search space and hence, the computational time to solve the problem. On the

number of clusters	7	8	9
requests solved	100%	96.80%	83.98%
requests solved to optimality	90.38%	42.77%	12.65%

Table 5.3: Requests of the GFTP that did not achieve optimality in less than CPU seconds.

other hand, the number of clusters should now be the parameter that represents the number of cities to be visited. With the purpose of analyzing the impact of both parameters in the computational time, the chart depicted in Figure 5.6 was drawn. If we analyze a specific set with a constant number of clusters, it can be observed that the computational time increases with the increase of the number of cities. Likewise, if we analyze a specific set with a constant number of cities, it can be observed that the computational time increases with the increase of the number of clusters. It is concluded then, that both parameters influence the computational time.

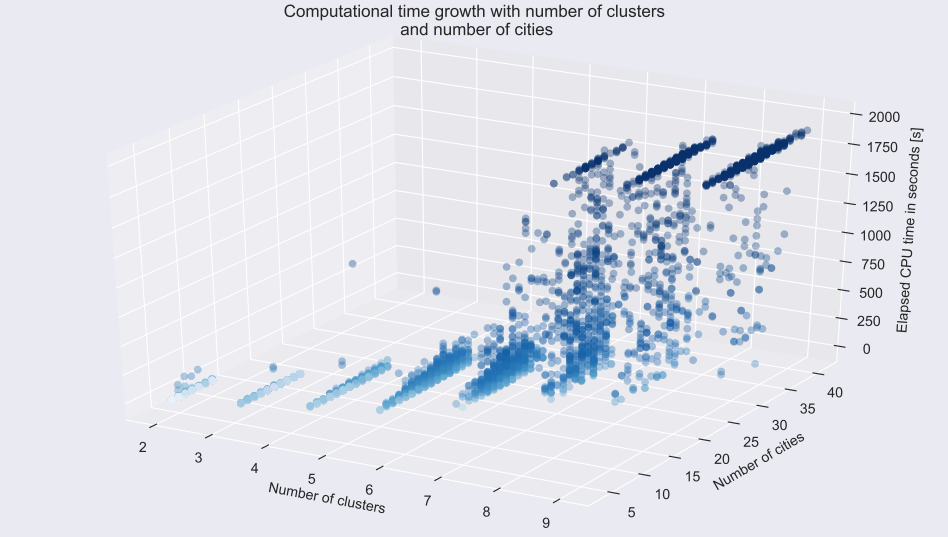
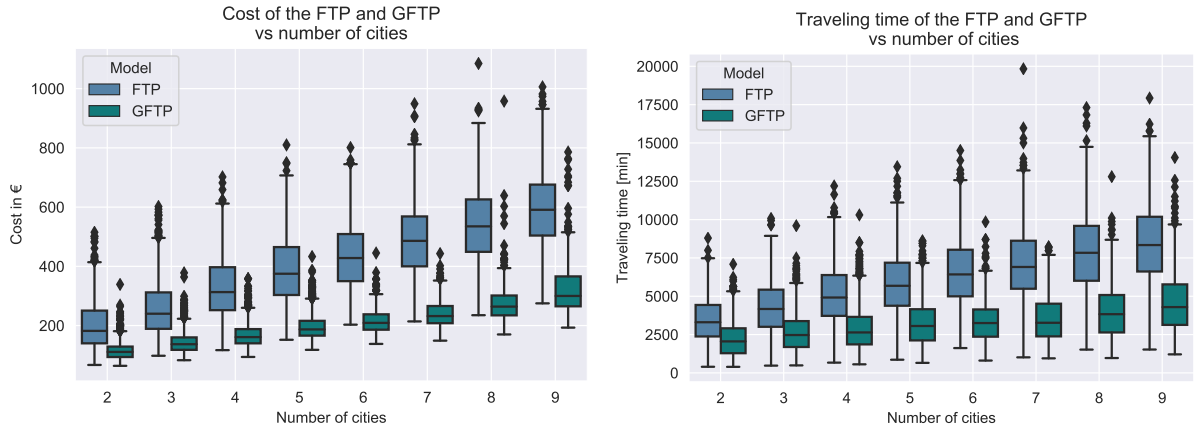


Figure 5.6: Computational time exponential growth with the number of cities and number of clusters requested.

Moreover, one of these parameters should be chosen in order to compare this problem with the FTP. With this purpose, the number of clusters is adopted as it is the equivalent to the number of visited cities.

In particular, in Section 3.2 it was postulated that the GFTP would necessarily have a better or at least equal optimal solution when compared to the FTP. Figure 5.7 indicates that this is verified and that GFTP has, in fact, a much cheaper solution. Moreover, it illustrates that the gap of the objective increases with the number of visited cities, which might imply that this model is better suited for users wishing to visit a large quantity of cities. Not only is it possible to observe that the cost decreased as expected, but the traveling time also increased accordingly.

Nevertheless, this gain occurs at the expense of having a solution space generally larger than that of



(a) Cost growth with the number of cities visited of both the FTP and the GFTP.

(b) Traveling time growth with the number of cities visited of both the FTP and the GFTP.

Figure 5.7: Objective function variation in the FTP and GFTP with the number of visited cities.

the FTP. For this reason, requesting large quantities of cities to be visited might lead to a computationally heavy request. Figure 5.8 depicts that the exponential growth of the CPU time associated to this new model has a larger growth ratio than that of the FTP ILP model.

5.3 Multi-Objective Flying Tourist Problem

This model was analyzed in 9000 different random requests that follow the same construction rules as in Section 5.1 with the difference that now, instead of using the ILP formulation of the FTP, we use the multi-objective formulation introduced in Section 3.3, the MO-FTP. This means that we are now optimizing both the cost and the traveling time of the flights. We remind that we are using the CPLEX solver, which has two options to approach the multi-objective optimization. The first option is a lexicographic multi-objective optimization, where some objective functions are incomparably more important than others and so, this optimization is processed in a sequential level of importance. The second option reduces the multi-objective problem to a single-objective optimization through the merge of both objective functions with the associated weights. Since the former implies a substantially greater optimization effort, the computational time would substantially increase as two single-objective optimizations are performed. For this reason, we only conduct the latter method, both in this section as well as in 5.4.

Consider a user that does not prioritize any of the objectives, i.e considers the cost and the traveling time to be of the same importance when deciding on a trip. The weights of the objective functions were not normalized since the trade-off between the cost and the traveling time is strictly subjective to each user's personal preference. Nonetheless, since the main objective of this formulation is to present the improvements in terms of the traveling time when compared to the FTP, the used weights were both set

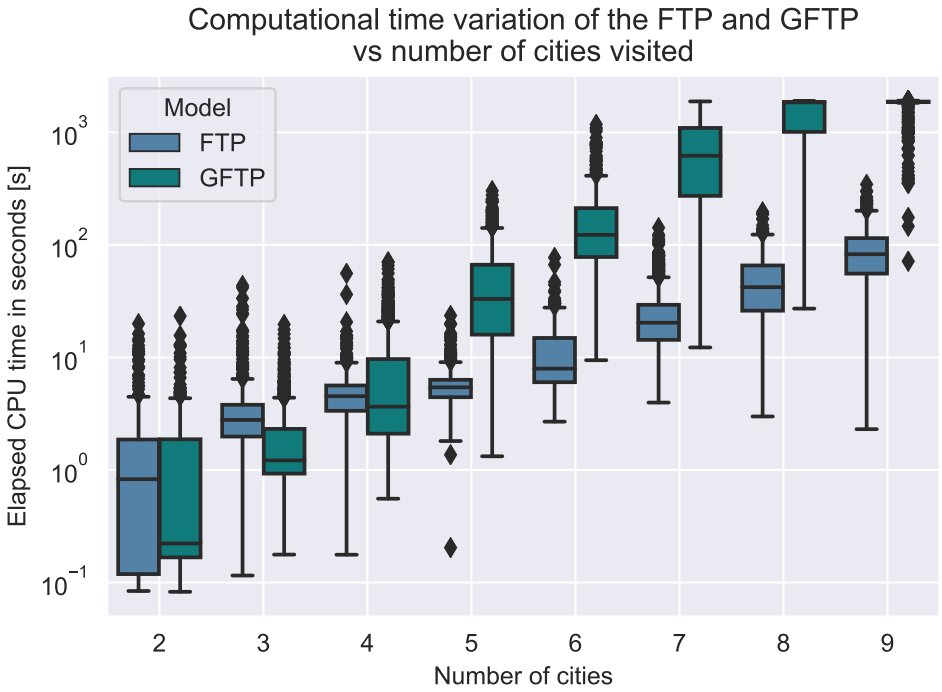


Figure 5.8: Computational time exponential growth of both the FTP and the GFTP with the number of visited cities.

to the value 1. However, since magnitude of the traveling time of a flight (in minutes) is usually larger than its cost (in euros), the traveling time presents as a larger term in the single-objective function.

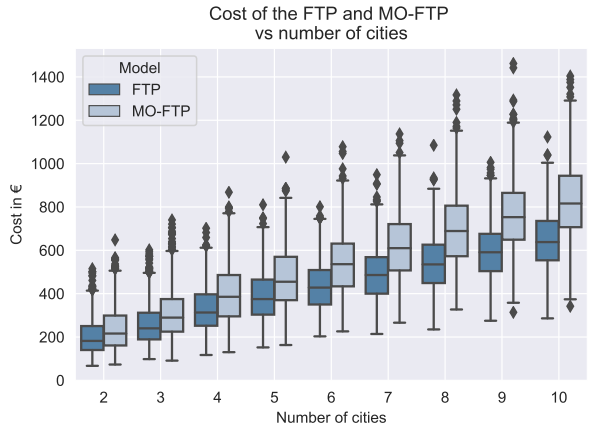
Figure 5.9 illustrates the comparison of the results between the FTP and the MO-FTP, when considering both the cost and the traveling time objectives to have the same weight on the latter formulation.

In section 3.3, it was assumed that the MO-FTP would certainly achieve a better (or at least equal) traveling time, when compared to the FTP. However, this improvement is achieved at the expense of penalizing the flight cost. Figure 5.9 verifies these assumptions, and the average increase of 33.59% in cost results, on average, on the reduction in 44.74% of the traveling time. Furthermore, since this optimization treated both objective functions as a singular objective function, then the computational time remains, approximately, the same as in the FTP, as clarified in Figure 5.10.

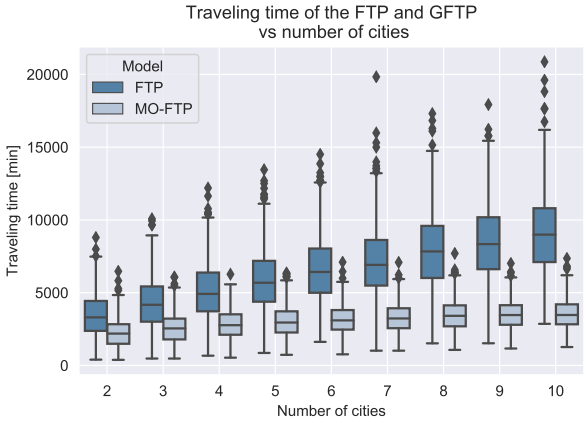
5.4 Multi-Objective Generalized Flying Tourist Problem

This formulation was used in 9000 random requests that follow the same construction rules as described in Section 5.2. However, instead of utilizing the ILP formulation of the GFTP, we use the Multi-objective formulation described in Section 3.4, the MO-GFTP.

Considering both the cost and the traveling time objective functions to be of the same importance,



(a) Cost increase with the number of cities of both the FTP and the MO-FTP.



(b) Traveling time increase with the number of cities of both the FTP and the MO-FTP.

Figure 5.9: Objective function variation (cost on the left and traveling time on the right) in the FTP and MO-FTP with the number of cities. For the MO-FTP, both objectives have the same priority and weight.

they are given the same priority and hence, merged together.

Figure 5.11 depicts the comparison of the MO-GFTP with both the MO-FTP and the GFTP when considering both objective functions to have the same weight. In Section 3.4, we theorized that the MO-GFTP would attain the advantages of both the MO-FTP and the GFTP, i.e achieve better traveling times through the multi-objective optimization, although not penalizing the flight cost as much by introducing more arcs as a generalized, or set, problem. Figure 5.11 confirms these predictions. Furthermore, the addition of more arcs, by using the MO-GFTP, allowed even better traveling times. However, the results concerning the MO-GFTP take longer to process, since this is heavier computational problem. Considering a timeframe of 3600 CPU seconds, a solution for 2.48% of the requests was not found and 5.08% of the results are not proved to be optimal.

Even though this is a multi-objective formulation (reduced to a single-objective problem), where both objective functions have the same priority, it is possible to evaluate how the results vary with the weight of each objective function. With this in mind, a Pareto front analysis was conducted. Consider a problem consisting on 5 clusters, each with 5 cities and a stop time associated to each city of 3 days. For the construction of the Pareto front, we use interval variations of the weight of both objective functions of 0.01. Naturally, since this variation is not introduced for every point in the infinite continuous domain of possible weights, then the set of found solutions are merely an approximation of the Pareto Front and the found points can be dominated by an unknown solution. However, this is improbable considering that 100 different weights were considered. Considering that the traveling time function weight is given by $Weight_{timef} = 1 - Weight_{costf}$, Figure 5.12 illustrates the obtained approximation of the Pareto front.

The obtained approximation conveys only five points and suggest that at low traveling times, a small

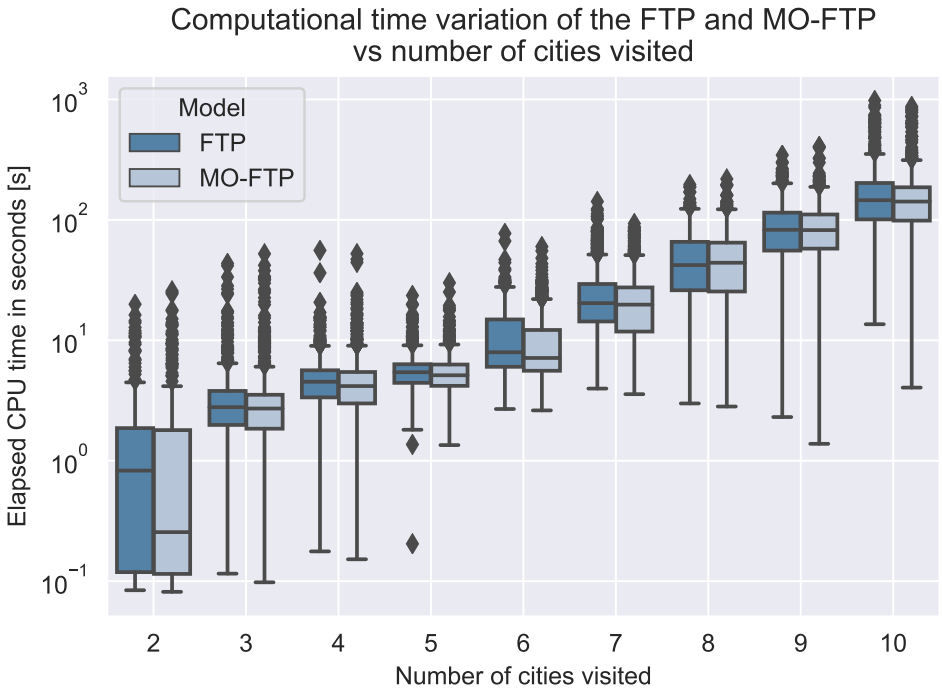
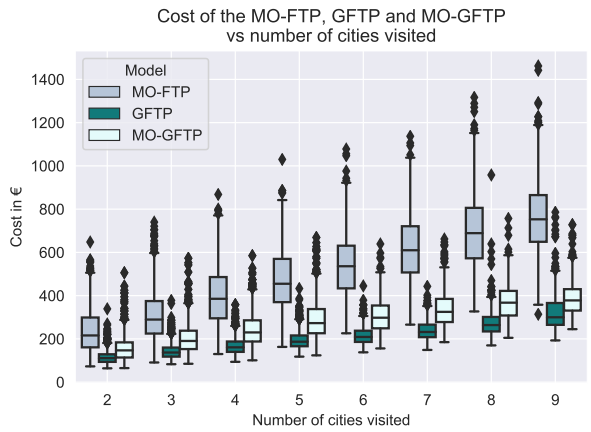
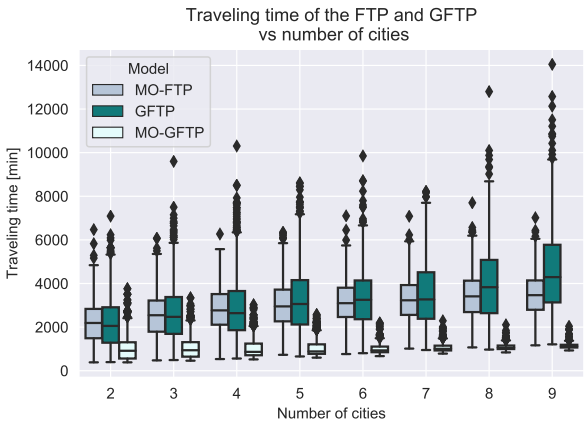


Figure 5.10: Computational time exponential growth of both the FTP and the MO-FTP with the number of cities. For the MO-FTP, the objectives (cost and traveling time) have the same priority and weight.



(a) Cost growth with the number of cities of the GFTP, the MO-FTP and the MO-GFTP.



(b) Traveling time growth with the number of cities of the GFTP, the MO-FTP and the MO-GFTP.

Figure 5.11: Objective function variation, cost on the left and traveling time on the right, in the GFTP, the MO-FTP and the MO-GFTP with the number of cities. For the MO-FTP and MO-GFTP, both objectives have the same priority and weight.

improvement in this objective implies a large increase in cost. Likewise, at low costs, a small improvement in this objective implies a large increase in traveling time.

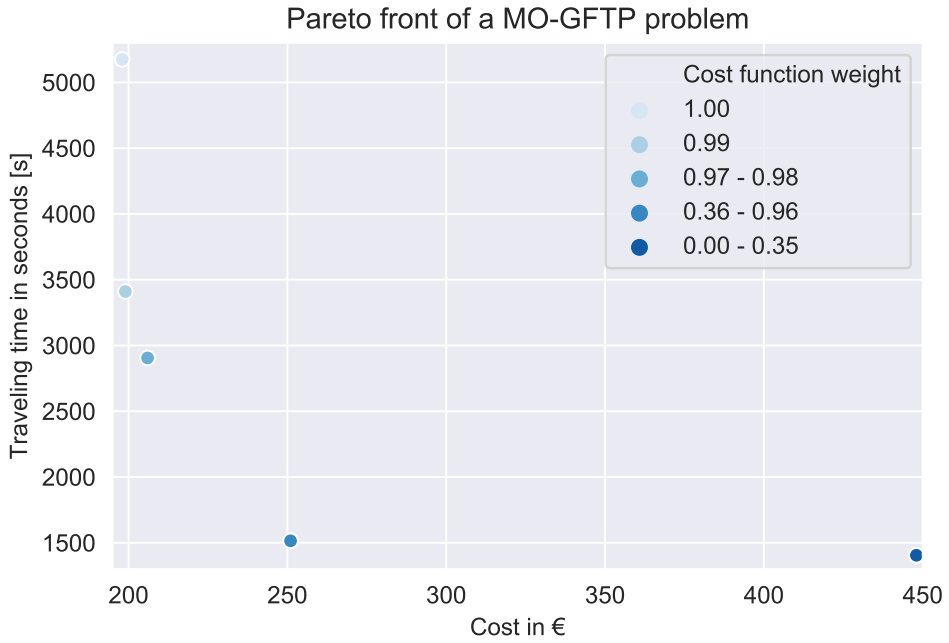


Figure 5.12: Obtained approximation of Pareto front of a MO-GFTP considering 5 clusters and 5 cities per cluster.

5.5 Summary

The foregoing chapter presents an evaluation of the models presented in Chapter 3. In Section 5.1, we introduced the overall results of the an ILP formulation of FTP concluding that the number of cities visited is the parameter that influences this formulation the most. We then compare the results of this optimization method with the SA algorithm used by R. Marques [51] and conclude that this formulation presents advantages and disadvantages versus the stochastic method. Even though it always achieves optimality, this comes at the expense of larger computational times. With this in mind, we implemented an optimization method that finds the aimed solution in a sequential manner, i.e first we solve the problem with the SA algorithm and use the solution retrieved as a constraint to the ILP formulation, which reduces the computational time to reach optimality. Next, we demonstrate that the results predicted for the

	Cost per flight [€]		**Traveling time per flight [min]**	
FTP	60.27		912.82	
GFTP	30.71	(-46.2%)	533.87	(-41.5%)
MO-FTP	75.20	(+24.8%)	435.81	(-52.3%)
MO-GFTP	45.83	(-24.0%)	179.33	(-80.4%)

Table 5.4: Comparison of the objective results of the aforementioned models. The results indicate the median, per flight taken, of the mean cost and traveling time of each request.

considered variations of the FTP are verified. Table 5.4 resumes and compares the median of the results of the proposed models. First, the GFTP presents great improvements both in the cost and in the traveling time. For this reason this model is ideal in case the end user wishes to reduce both of these parameters. Second, if the end user wishes to maintain the original set of cities but wishes to save traveling time at the expense of an increase in cost, the MO-FTP is recommended. Lastly, if the traveling time of the GFTP is still not the desirable, then the MO-GFTP joins both benefits.

6

Conclusions

Contents

6.1 Summary

The increase on flights over the last decade has posed several new challenges. A vast variety of online travel agencies have started to attempt to make flight choices easier for the users. A large quantity of these web services already offer simple solutions as finding an optimal tour connecting two cities. Nonetheless, the ways of traveling have changed throughout the last two decades [1]. Nowadays, flying has become a common and affordable means of transportation and travelers have today very different purposes for flying. Therefore, these web applications have started to tailor their services in order to provide new creative services that fit every user's needs. Throughout this work, we developed different models that shape peculiar situations we believe to be of the user's interest.

When accounting for more complex problems, for instance an unconstrained multi-city search, these search engines aim to present solutions in a small time frame. Notwithstanding, this implies the use of incomplete methods and therefore there is no guarantee that these solutions are optimal. With this in mind, the main contributions of this work were the formulations of complete methods to assure the quality of four different scenarios:

1. The Flying Tourist Problem, which minimizes the total flight cost or traveling time of an end user wishing to visit multiple cities in no preemptive order.
2. The Generalized Flying Tourist Problem, that minimizes the total flight cost or traveling time of an end user wishing to visit one city per cluster, having multiple cluster in no prescribed order.
3. The Multi-objective Flying Tourist Problem, that aims to optimize a trade-off between the total flight cost and traveling time of the Flying Tourist Problem.
4. The Multi-objective Generalized Flying Tourist Problem, which optimizes the total flight cost and traveling time of the Generalized Flying Tourist Problem.

After the problem statements, Integer Linear Programming (ILP) formulations for each of the problems were proposed and implemented using a commercial solver, CPLEX, as an optimization module. This module was later integrated in a web application previously developed by R.Marques [51] in order to solve real world complex routing problems. This application was subject to several design modifications to adapt it to the new proposed models.

The results of the ILP formulation of the FTP were compared to an incomplete method, the Simulated Annealing (SA) algorithm. The complete method used in this work outperformed the SA algorithm in what concerns the solution, since it consistently found an optimal solution in less than 1800 seconds for a maximum of ten cities. Moreover, even though the ILP formulation involved larger computational optimization times, this method outperformed the SA algorithm in what concerns the wall clock optimization time for less than eight cities. Comparing the other formulations with the obtained results of

the FTP, we concluded that the GFTP managed to consistently reduce the total flight cost in half, the MO-FTP achieved a large reduction in the traveling time at the expense of the total flight cost and finally, the MO-GFTP attained both benefits.

6.2 Future Work

Considering that the work developed proposes to resolve complex routing problems in near real-time, there is the need to consistently improve and find innovative techniques for the foregoing system.

- Considering the FTP, we suggest the integration of other means of transportation such as buses or trains. We believe that this solution starts by constructing a database, for instance with the software MongoDB. This strategy could also improve the efficiency of the data acquisition system.
- Since the developed optimization modules of the GFTP, MO-FTP and MO-GFTP only comprise complete methods, the first answer provided to the Client Side Application will take a long time. To negate this, we suggest the implementation of heuristics and meta-heuristics to be solved before the formulations here presented so that solutions are given in a sequential manner to the user.
- Considering the multi-objective formulations we recommend exploring the multi-objective evolutionary algorithms in order to compare the quality of the solutions here presented. With this in mind we suggest using framework MOEA/D and the NSGA II Algorithm.

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