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An Algebraic Approach to Weighted Answer Set Programming

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Abstract

Logic programs, more specifically, answer set programs, can be annotated with probabilities on facts to express uncertainty. We address the problem of propagating weight annotations on facts (*e.g.* probabilities) of an answer set program to its stable models, and from there to events (defined as sets of atoms) in a dataset over the program's domain.

We propose a novel approach which is algebraic in the sense that it relies on an equivalence relation over the set of events. Uncertainty is then described as polynomial expressions over variables. We propagate the weight function in the space of models and events, rather than doing so within the syntax of the program. Our approach allows us to investigate weight annotated programs and to determine how suitable a given one is for modeling a given dataset containing events.

KEYWORDS: Answer-Set Programming, Stable Models, Probabilistic Logic Programming

1 Introduction

▶Using a logic program (LP) to model and reason over a real world scenario is often difficult because of uncertainty underlying the problem being worked on. Classic LPs represent knowledge in precise and complete terms, which turns out to be problematic when the scenario is characterized by stochastic or observability factors.|[1] We aim to explore how answer set programs (ASPs) plus weight annotated facts can lead to useful characterizations for this class of problems. *logic+uncertainty*

[1] |insert simple, nice example

To setup a working framework, we make the following assumption:

Assumption 1 (*System Representation, Data and Propagation*)
Consider a *system* whose states are *partially observable* (*i.e.*, observations can miss some state information) or *stochastic* (*i.e.* reported values are affected by random noise). We assume that knowledge about such *system* features a formal specification including *weighted facts* and empirical *data* such that:

(Representation) ▶ The system has a formal *representation*[1] in the form of a special logic program; The program's stable models correspond one-to-one with the system states. *system = representation, SM=state*
(Data) *Data* is a set of observations; a single *observation* (of the system states) results from a set of (boolean) *sensors*.
(Propagation) ▶ The *weights* in *facts* are *propagated* to the stable models of the representation. *weight propagation*

▶ In this setting, data from observations can be used to estimate some parameteres used in the *parameter estimation*

[1] We use 'representation' instead of 'model' to avoid confusion with the *stable models* of answer set programs.

propagation process and, more importantly, to address the question of '*How accurate is the representation of the system?*'.

▸Other probabilistic logic programming (PLP) systems such as `Problog` (De Raedt et al. 2007), `P-log` (Baral et al. 2009) or $\texttt{LP}^{\texttt{MLN}}$ (Lee and Wang 2016), in line with (Kifer and Subrahmanian 1992), derive a probability distribution of the stable models from the *syntax* of an annotated logic program. The more expressive of these systems, according to (Lee and Yang 2017), is $\texttt{LP}^{\texttt{MLN}}$, that can embed the maximum a posteriori (MAP) estimation of the other systems. ▸Since in these PLP systems the probability distribution results from the program's syntax, these systems are limited to handle *a priori* information and are unable to support *posterior* data. Furthermore, these distributions are limited to the stable models (SMs), while we *extend their domain to any event* (*i.e.* any set of atoms).

other PLP

probability from syntax

However, the key feature that we aim to address is concerned with the inherent uncertainty of the SMs in logic programs such as $a : 0.3, b \vee c \leftarrow a$. Intuitively, this program entails three SMs: ac, ab and $\neg a$. We can, for example, assign the probability of $\neg a$ as 0.7 but what about ab and ac? Systems like $\texttt{LP}^{\texttt{MLN}}$ assign the same probability to these SMs (Lee and Yang 2017; Cozman and Mauá 2020).

|[2]We question the underlying assumption of such assignments and propose a method where the distribution that results from the representation includes parameters (*i.e.* variables in the algebraic sense: symbols for unknown quantities) that express the lack of *prior* information concerning cases as above. The values of these parameters can be estimated *a posteriori*, in the presence of data.

[2]|addressing It is not made sufficiently clear why we need another probabilistic formalism based on answer set programming, or why this particular formalisms should fill that niche.

|[3] To frame this setting we extend our assumptions:

[3]|addressing hidden values and default negation.

Assumption 2 (*Sensor Representation and Events*)

(Sensors) ▸The *sensors* of the system's states are associated to some atoms in the representation;

(Events) ▸An *event* is a set of atoms from the representation.|[4]

sensor = atom

event = atoms

[4]|reconsider: is this a definition?

▸More specifically, following the terminology set in (Calimeri et al. 2020), a sensor σ_a can 'activate' the associated (*classical*, *strong*) *atom* a whereas no activation is represented by the (*default*, *weak* or *negation-as-failure (naf)*) *literal* $\sim a$. The same applies to a sensor $\sigma_{\neg a}$ associated to the (classically) negated atom $\neg a$.

strong + week negation

▸Assumption 2 enables a straightforward representation of *hidden* parts of a system as well as faulty (*stochastic*) sensors. For example, in the event

hidden + stochastic

$$\{a, \neg a, \sim b, \sim\neg b, \sim c, \neg c\} \tag{1}$$

both a and $\neg a$ are activated (suggesting *e.g.* a problem in the associated sensors), b is not observed (*i.e.* hidden), and $\sim c, \neg c$ reports the (consistent) activation of $\neg c$ and no activation of c. ▸ While every observation is an event, since some events can contain atoms not associated to sensors, not all events are observations. Furthermore, some events coincide with a stable model but others don't; an event may not uniquely determine a state of the system — how to associate events to stable models and, thus, to system states, is addressed in section 4.

event > observation

▸If we (i) omit the naf-literals; (ii) use $\overline{x}$ to denote the classical $\neg x$; (iii) and use expressions like ab to denote sets of literals such as $\{a, b\}$, then the event in (1) can be shortened to the equivalent form

event short notation

$$a\overline{a}\overline{c}. \tag{2}$$

Here we follow the convention, from (Gelfond and Lifschitz 1988), of denoting a model by the set of true atoms, stressing that '*falsehood*' results only from the default negation *i.e.* $\sim a$ (*i.e.* '`not a`' or '`\+ a`' in logic languages). More precisely, a model denotation can contain atoms such as a or $\overline{b}$ but not literals $\sim a, \sim\overline{b}$. Our choice to represent sensor input using both positive and negative atoms is based on the following points: (i) it can be the case that there are two different sensors, σ_a for the 'positive' values and $\sigma_{\overline{a}}$ for the 'negative' ones; (ii) the case where a single sensor $\sigma_{\hat{a}}$ always yields either 'positive' or 'negative' values can be represented by the rule $a \vee \neg a$; also, (iii) a closed-world assumption, where absence of sensor output means (classical) negation, can be represented by the rules $\overline{a} \leftarrow \sim a, a \leftarrow \sim\overline{a}$.

▸Like in $\texttt{LP}^{\texttt{MLN}}$, we annotate facts (*i.e.* atoms) with weights (Lee and Yang 2017) instead of probabilities, that result from normalization of the former. ▸ By *propagation* we mean the use of those weights to define a 'weight' function on the stable models and then extended it to all the events in the program domain. ▸The step from facts to SMs is non-deterministic in the sense that a given set of facts may entail zero, one or more SMs (see ex. 1). This is a well-known situation, explained in section 3 and also in (Verreet et al. 2022; Pajunen and Janhunen 2021; Cozman and Mauá 2020; Baral et al. 2009), when propagating weights to stable models: *How to distribute the*

weights + probabilities

propagation

SMs + non-unique/non-deterministic

weight of a fact to all the entailed SMs? ▸ We represent non-unique choices by parameters that can be later estimated from further information, *i.e.* data. This approach enables later refinement from additional evidence and also scoring a program w.r.t. the available data. *parameters*

▸ Answer set programs (ASPs) are logic programs based on the stable model semantics of normal programs (NPs) (Lifschitz 2002; 2008). ASPs represent a problem and the resulting models (*answer sets*) can be found using different approaches, including SAT solving technology (Gebser et al. 2011; Adrian et al. 2018; Niemelä and Simons 1997) or through top-down search (Alberti et al. 2017; Arias et al. 2020; Marple et al. 2017). *ASPs*

▸ The distribution semantics (DS) (Sato 1995; Riguzzi 2022) is the base for probabilistic logic programming (PLP), to extend logic programs with probabilistic reasoning. |[5]We are particularly interested in the following setting and application scenarios of such an extension to logic programs: *DS*

[5] |addressing better examples that show the real usefulness of this approach *e.g.* a toy problem with a biased coin; also see the examples on the other systems.

Partial Observability ▸A system's state can have *hidden variables*, not reported by the sensors. *partial observability*

Sensor Error ▸Information gathered from the sensors can carry *stochastic perturbations*. *sensor error*

Representation Induction ▸Combine representations with data to induce *more accurate representations*. *representation induction*

Probabilistic Tasks ▸Support common probabilistic tasks such as maximum a posteriori (MAP), maximum likelihood estimation (MLE) and Bayesian inference (BI) on the *representation domain i.e.* the set of all events. *probabilistic tasks*

The remainder of this article is structured as follows: section 2 provides necessary context. In section 3 we discuss the syntax and semantics of our proposed language for weighted answer set programs (WASPs). We also define a weight function over total choices and address the issue of how to propagate these probabilities from facts to events, in section 4. This method relies on an equivalence relation on the set of events. Furthermore, we express uncertainty by polynomial expressions over variables which depend on the total choices and on the stable models. By then the *Partial Observability* and *Sensor Error* points are addressed. Some final remarks and ideas for future developments including *Representation Induction* and *Probabilistic Tasks* are presented in section 5.

2 Framework

▸We start by refining assumptions 1 and 2 to set 'representation' as an 'ASP with weights': *WASP*

Assumption 3 (*Representation by Answer set programs with Weigths*)
(Answer set programs and Weights) A *representation* of a system is an *answer set program* that includes *weighted facts*.

▸ A *weighted fact (WF)* or an *annotated fact* has the from '$a : w$' where a is an atom, $w \in [0, 1]$, and defines the disjunctive fact $a \vee \overline{a}$. A model will include either a or $\overline{a}$ but never both. *weighted fact*

▸ Selecting one of $a, \overline{a}$ for each WF in a program will lead to a *total choice (TC)* [2]. ▸ Propagating weights from WFs to TCs is relatively straightforward (see eq. (7)) but propagation to events requires a step trough the program's stable models, addressed in section 4. *total choice* *propagating*

About Propagating Weights from Total choices. ▸ Our goal to propagate weights from TCs to SMs and from there to any event soon faces a non-deterministic problem, illustrated by the program P_1 in ex. 1, section 3, where multiple SMs, ab and ac, result from a single TC, a, but *there is not enough information in the representation to assign a single weight to each one of those SMs*. In section 4.2 we use algebraic variables[3] to describe the lack of information in a representation in order to deterministically propagate the weight to the stable models and events. The values of those variables can be estimated from available data, a contribute to the *Representation Induction* application goal, set in section 1. *propagation + non-unique/non-deterministic*

The lack of a unique stable model from a total choice is also addressed in (Cozman and Mauá 2020) along an approach using credal sets. In another related work (Verreet et al. 2022), epistemic uncertainty (or model uncertainty) is considered as lack of knowledge about the underlying model, that may be mitigated via further observations. This seems to presuppose a bayesian approach to imperfect knowledge in the sense that having further observations allows one to improve or correct

[2] We use the term 'choice' for historical reasons, *e.g.* see (Cozman and Mauá 2020) even though not related to the usual 'choice' elements, atoms or rules from *e.g.* (Calimeri et al. 2020).

[3] We explicitly write 'algebraic' variables to avoid confusion with logic variables.

the model. Indeed, that approach uses Beta distributions on the total choices in order to be able to learn a distribution on the events. This approach seems to be specially fitted to being able to tell when some weight lies beneath some given value. Our approach is similar in spirit, while remaining algebraic in the way the propagation of weights is addressed.

3 Syntax and Semantics of Weighted ASP

We start the formal definition of *weighted answer set program* with the setup and discussion of a minimal syntax and semantics of propositional ASP, without variables, functors or relation symbols, but enough to illustrate our method to propagate weights from annotated facts to events. From now on '$\neg x$' and '$\overline{x}$' denote classical negation and '$\sim x$' default negation.

Syntax. We slightly adapt (Calimeri et al. 2020). Let $\mathcal{A}$ be a finite set of symbols, the *positive atoms*. For $a \in \mathcal{A}$, the expressions a and $\neg a$ (the later a *negative atom*, also denoted $\overline{a}$) are *(classical) atoms*. If a is an atom, the expressions a and $\sim a$ are *(naf-)literals*. A *rule* is of the form

def. atom
def. literal
def. rule

$$h_1 \vee \cdots \vee h_n \leftarrow b_1 \wedge \cdots \wedge b_m$$

where the h_i are atoms and the b_j are literals. The symbol '$\leftarrow$' separates the *head* from the *body*. A rule is a *constraint*[4] if $n = 0$, *normal* if $n = 1$, *disjunctive* if $n > 1$, and a *fact* if $m = 0$.

def. head;body
def. constraint;normal;disjunctive;fact

An *answer set program (ASP)* is a set P of facts and rules, denoted, resp. $\mathcal{F}(P)$ and $\mathcal{R}(P)$, or simply $\mathcal{F}$ and $\mathcal{R}$. In a *normal program* all the rules are normal. Notice that programs with constraint or disjunctive rules can be converted into normal programs (Gebser et al. 2022).

def. program

Semantics. The standard semantics of an ASP has a few different, but equivalent, definitions (Lifschitz 2008). A common definition is as follows (Gelfond and Lifschitz 1988): let P be a normal program. The Gelfond/Lifschitz *reduct* of P relative to the set X of atoms results from (i) deleting rules that contain a literal of the form $\sim p$ in the body with $p \in X$ and then (ii) deleting the remaining literals of the form $\sim q$ from the bodies of the remaining rules. Now, M is a *stable model (SM)* of P if it is the minimal model of the reduct of P relative to M. We denote by $\mathcal{M}(P)$, or simply $\mathcal{M}$, the set of stable models of the program P.

def. reduct
def. stable model

Evaluation without Grounding. While the most common form to generate stable models is based on *grounding*, a different approach is the one supported by `s(CASP)`, a system that can evaluate ASP programs with function symbols (functors) and constraints without grounding them either before or during execution, using a method similar to SLD resolution (Marple et al. 2017; Arias et al. 2020). |[6] This enables the generation of human readable explanations of the results of programs and addresses two major issues of grounding-based solvers, that (i) either do not support function symbols or, using finite domains, lead to exponential groundings of a program and (ii) compute the complete model of the grounded program when, in some scenarios, it is desirable to compute only a partial stable model containing a query.

[6] |Improve grounding and propositional cases.

WASPs and their Derived Programs

Weighted answer set programs (WASPs) extend ASPs by adding facts with weight annotations, *i.e. weighted facts (WFs)*. Notice that we have $w \in [0, 1]$ but w is not interpreted as a probability but, instead, as a *balance* between the *choices* a and $\overline{a}$.

def. weighted fact

We denote the set of weighted facts of a program by $\mathcal{W}$, and $\mathcal{A}_\mathcal{W}$ the set of positive atoms in $\mathcal{W}$.

Our definition of WASPs is restricted because our goal is to illustrate the core of a method to propagate weights from total choices to events. Our programs do not feature logical variables, relation symbols, functors or other elements common in standard ASP. Also, weight annotations are not associated to (general) rule heads or disjunctions. However, these last two restrictions do not reduce the expressive capacity of the language because, for the former, a rule with an annotated head can be rewritten as:

$$\alpha : w \leftarrow \beta \qquad \Longrightarrow \qquad \begin{cases} \gamma : w, \\ \alpha \leftarrow \beta \wedge \gamma \end{cases}$$

[4] An '*integrity constraint*' in (Calimeri et al. 2020).

while annotated disjunctive facts

$$\alpha \vee \beta : w \qquad \Longrightarrow \qquad \begin{cases} \gamma : w, \\ \alpha \vee \beta \leftarrow \gamma, \\ \overline{\alpha} \leftarrow \overline{\gamma}, \quad \overline{\beta} \quad \leftarrow \overline{\gamma}. \end{cases}$$

Derived Program. The *derived program* of a WASP is the ASP obtained by replacing each weighted fact $a : w$ by a disjunction $a \vee \overline{a}$. The *stable models* of a WASP program are the stable models of its derived program. So, we also denote the set of SMs of a (derived or) WASP program P by $\mathcal{M}(P)$ or $\mathcal{M}$. **def.** derived

Events. An *event* of a program P is a set of atoms from P. We denote the set of events by $\mathcal{E}(P)$ or simply $\mathcal{E}$. An event $e \in \mathcal{E}$ which includes a set $\{x, \overline{x}\}$ is said to be *inconsistent*; otherwise it is *consistent*. The set of consistent events is denoted by $\mathcal{C}$. **def.** event **def.** (in)consistent

Example 1
Consider the following weighted answer set program :

$$P_1 = \begin{cases} a : 0.3, \\ b \vee c \leftarrow a \end{cases} \tag{3}$$

which has the set $\mathcal{W} = \{a : 0.3\}$ *of weighted facts. This program is transformed into the answer set program*

$$P_1' = \begin{cases} a \vee \overline{a}, \\ b \vee c \leftarrow a, \end{cases} \tag{4}$$

with the set $\mathcal{M} = \{\,\overline{a}, ab, ac\,\}$ *of stable models.*

The atoms of these programs are

$$\mathcal{A} = \left\{a, \overline{a}, b, \overline{b}, c, \overline{c}\right\} \tag{5}$$

and the events are[5]

$$\mathcal{E} = 2^{\mathcal{A}}. \tag{6}$$

Total Choices and their Weights

A disjunctive head $a \vee \overline{a}$ in the derived program represents a single *choice*, either a or $\overline{a}$. A *total choice* of the derived program, and of the WASP program, is $t = \{\hat{a} \mid a : p \in \mathcal{W}\}$ where each $\hat{a}$ is either a or $\overline{a}$. We denote by $\mathcal{T}$ the set of total choices of a WASP or of its derived program. **def.** total choice

The *weight of the total choice* $t \in \mathcal{T}$ is given by the product **def.** $\omega_{\mathcal{T}}$

$$\omega_{\mathcal{T}}(t) = \prod_{\substack{a:w \,\in\, \mathcal{W}, \\ a \,\in\, t}} w \quad \times \quad \prod_{\substack{a:w \,\in\, \mathcal{W}, \\ \overline{a} \,\in\, t}} \overline{w}. \tag{7}$$

Here $\overline{w} = 1 - w$, and we use the subscript in $\omega_{\mathcal{T}}$ to explicitly state that this function concerns total choices. Later we'll use subscripts $\mathcal{M}, \mathcal{E}$ to deal with weight functions of stable models and events, $\omega_{\mathcal{M}}, \omega_{\mathcal{E}}$.

Some stable models are entailed from some total choices while other SMs are entailed by other TCs. We write $\mathcal{M}(t)$ to represent the set of stable models entailed by the total choice $t \in \mathcal{T}$.

Our goal can now be rephrased as to know how to propagate the weights of the program's total choices, $\omega_{\mathcal{T}}$, in eq. (7) to the program's events, $\omega_{\mathcal{E}}$ to be defined later, in eqs. (22a) and (22b).

Propagation of Weights. As a first step to propagate weight from total choices to events, consider the P_1 program of eq. (3) and a possible propagation of $\omega_{\mathcal{T}} : \mathcal{T} \to [0, 1]$ from total choices to the stable models, $\omega_{\mathcal{M}} : \mathcal{M} \to [0, 1]$ (still informal, see eq. (17)). |[7] It might seem straightforward, in ex. 1, to set $\omega_{\mathcal{M}}(\overline{a}) = 0.7$ but there is no explicit way to assign values to $\omega_{\mathcal{M}}(ab)$ and $\omega_{\mathcal{M}}(ac)$. We represent this non-determinist by a parameter θ as in

[7]|addressing Definition of $\mathcal{T}$?

$$\begin{aligned} \omega_{\mathcal{M}}(ab) &= 0.3\,\theta, \\ \omega_{\mathcal{M}}(ac) &= 0.3\,(1 - \theta) \end{aligned} \tag{8}$$

[5] 2^X is the *power set* of X: $A \in 2^X \Leftrightarrow A \subseteq X$.

to express our knowledge that ab and ac are models entailed from a specific choice and, simultaneously, the inherent non-determinism of that entailment. In general, it might be necessary to have several such parameters, each associated to a given stable model s (in eq. (8), $s = ab$ in the first line and $s = ac$ in the second line) and a total choice t ($t = a$ above), so we write $\theta_{s,t}$. Obviously, for reasonable $\theta_{s,t}$, the total choice t must be a subset of the stable model s.

Unless we introduce some bias, such as $\theta = 0.5$ as in $\mathtt{LP}^{\mathtt{MLN}}$ (Lee and Wang 2016), the value for $\theta_{s,t}$ can't be determined just with the information given in the program. But it might be estimated with the help of further information, such as an empirical distribution from a dataset. Further discussion of this point is outside the scope of this paper.

Now consider the program

$$\begin{cases} a : 0.3, \\ b \leftarrow a \wedge \sim b \end{cases} \tag{9}$$

that has a single SM, $\overline{a}$. Since the weights are not interpreted as probabilities, there is no need to have the sum on the stable models equal to 1. So the weights in the TCs of eq. (9) only set

$$\omega_{\mathcal{M}}(\overline{a}) = 0.7.$$

In this case, if we were to derive a probability of the SMs, normalization would give $\mathrm{P}(\overline{a}) = 1.0$. |[8]

Also facts without annotations can be transformed into facts with weight 1:

$$a \qquad \Longrightarrow \qquad a : 1.0. \tag{10}$$

[8] |addressing Explain what appens with $a \vee \neg a, b \leftarrow a \wedge \sim b$ that has only the stable model $\neg a$. by stressing **weights** instead of probabilities.

propagation, semantics

▸ The method that we are proposing does not follow the framework of (Kifer and Subrahmanian 1992) and others, where the syntax of the program determines the propagation from probabilities explicitly set either in facts or other elements of the program. Our approach requires that we consider the semantics, *i.e.* the stable models of the program, independently of the syntax that provided them. From there we propagate weights to the program's events and then, if required, normalization provides the final probabilities. Moreover, we allow the occurrence of variables in the weights, in order to deal with the non-determinism that results from the non-uniqueness of SMs entailed from a single TC. These variables can be later estimated from available data.

Related Approaches and Systems

The core problem of setting a semantics for probabilistic logic programs, the propagation of probabilities from total choices to stable models in the case of ASP or to other types in other logic programming systems (*e.g.* to possible worlds in `Problog`) has been studied for some time (Kifer and Subrahmanian 1992; Sato 1995).

For example, the *credal set* approach of (Cozman and Mauá 2020), defines $\mathrm{P}_{\mathcal{T}}$ in a way similar to eq. (7) but then, for $a \in \mathcal{A}, t \in \mathcal{T}$, the probability $\mathrm{P}(a \mid t)$ is unknown but bounded by $\underline{\mathrm{P}}(a \mid t)$ and $\overline{\mathrm{P}}(a \mid t)$, that can be explicitly estimated from the program.

`Problog` (Fierens et al. 2015; Verreet et al. 2022) extends `Prolog` with probabilistic facts so that a program specifies a probability distribution over possible worlds. A *world* is a model of $T \cup R$ where T is a total choice and R the set of rules of a program. The semantics is only defined for *sound* programs (Riguzzi and Swift 2013) *i.e.*, programs for which each possible total choice T leads to a well-founded model that is two-valued or *total*. The probability of a possible world that is a model of the program is the probability of the total choice. Otherwise the probability is 0 (Riguzzi and Swift 2013; Van Gelder et al. 1991).

Another system, based on Markov Logic (Richardson and Domingos 2006), is $\mathtt{LP}^{\mathtt{MLN}}$ (Lee and Wang 2016; Lee and Yang 2017), whose models result from *weighted rules* of the form $a \leftarrow b \wedge n$ where a is disjunction of atoms, b is conjunction of atoms and n is constructed from atoms using conjunction, disjunction and negation. For each model there is a unique maximal set of rules that are satisfied by it and the respective weights determine the weight of that model, that can be normalized to a probability.

Towards Propagating Weights from Total Choices to Events

The program P_1 in eq. (3) from ex. 1 showcases the problem of propagating weights from total choices to stable models and then to events. The main issue arises from the lack of information in the program on how to assign *un-biased* weights to the stable models. This becomes crucial in situations where multiple stable models result from a single total choice.

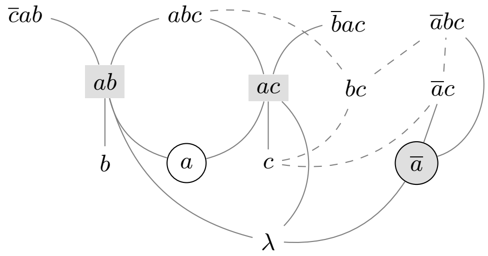

Fig. 1. This (partial sub-/super-set) diagram shows some events related to the stable models of the program P_1. The circle nodes are total choices and shaded nodes are stable models. Solid lines represent relations with the stable models and dashed lines some sub-/super-set relations with other events. The set of events contained in all stable models, denoted by Λ, is $\{\lambda\}$ in this example, because $\overline{a} \cap ab \cap ac = \emptyset = \lambda$.

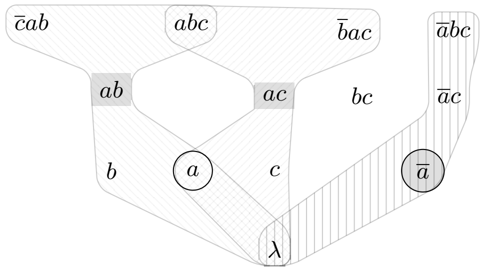

Fig. 2. Classes of (consistent) events related to the stable models of P_1 are defined through sub-/super-set relations. In this picture we can see, for example, that $\{\overline{c}ab, ab, b\}$ and $\{a, abc\}$ are part of different classes, represented by different fillings. As before, the circle nodes are total choices and shaded nodes are stable models. Notice that bc is not in a filled area.

Our assumptions 1 to 3 enunciate that a WASP program represents a *system*; the *states* of that system, which are partially observable and stochastic, are associated to the program's stable models; and state *observations* are encoded as events, *i.e.* sets of atoms of the program. Then:

1. With a weight set for the stable models, we extend it to any event in the program domain.
2. In the case where some statistical knowledge is available, for example, in the form of a distribution relating some atoms, we consider it as 'external' knowledge about the parameters, that doesn't affect the propagation procedure described below.
3. However, that knowledge can be used to estimate the parameters $\theta_{s,t}$ and to 'score' the program.
4. In that case, if a program is but one of many possible candidates, then that score can be used, *e.g.* as fitness, by algorithms searching (optimal) programs of a dataset of events.
5. If events are not consistent with the program, then we ought to conclude that the program is wrong and must be changed accordingly.

Next, we address the problem of propagating a weight, possibly using parameters (*i.e.* algebraic variables) such as θ in ex. 1, defined on the stable models of a program, $\mu_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}$, to the events of that program: $\mu_{\mathcal{E}} : \mathcal{E} \to \mathbb{R}$. The latter function can then be normalized and set a probability $\mathrm{P}_{\mathcal{E}} : 2^{\mathcal{E}} \to [0, 1]$. This way probabilistic reasoning is consistent with the ASP program and follows our interpretation of stable models as the states of an observable system.

4 Propagating Weights

The diagram in fig. 1 illustrates the problem of propagating weights from total choices to stable models and then to general events in an *edge-wise* process, *i.e.* where the value in a node is defined from the values in its neighbors. This quickly leads to interpretation issues concerning weight, with no clear systematic approach. For example, notice that bc is not directly related with any stable model. Propagating values through edges would assign a value ($\neq 0$) to bc hard to explain in terms of the semantics of the program. Instead, we propose to settle such propagation on the relation an event has with the stable models.

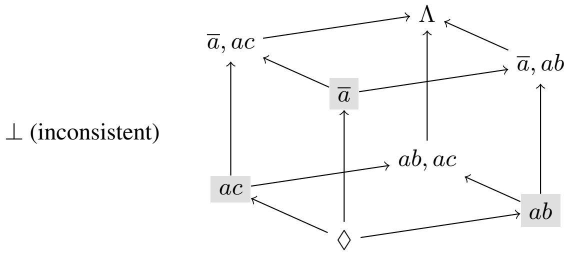

Fig. 3. Lattice of the stable cores from ex. 1. In this diagram the nodes are the different stable cores that result from the stable models, plus the *inconsistent* class ($\bot$). The bottom node ($\lozenge$) is the class of *independent* events, those that have no sub-/super-set relation with any SM and the top node (Λ) represents events related with all the SMs *i.e.* the *consequences* of the program. As in previous diagrams, shaded nodes represent the SMs.

4.1 An Equivalence Relation

Our path to propagate weights starts with the perspective that stable models play a role similar to *prime factors* or *principal ideals*. The stable models of a program play a role akin to 'irreducible events' entailed from that program and any event must be considered under its relation with the stable models.

From ex. 1 (*i.e.* P_1) and fig. 2 consider the stable models $\overline{a}, ab, ac$ and events a, abc and c. While a is related with (i.e. contained in) both ab, ac, the event c is related only with ac. So, a and c are *related with different sets of stable models.* On the other hand, abc contains both ab, ac. Therefore a and abc are *related with the same set of stable models.* We proceed to formalize this relation.

The *stable core (SC)* of the event $e \in \mathcal{E}$ is **def.** $[\![e]\!]$

$$[\![e]\!] := \{\, s \in \mathcal{M} \mid s \subseteq e \vee e \subseteq s \,\} \tag{11}$$

where $\mathcal{M}$ is the set of stable models.

Notice that the minimality of stable models implies that either e is a stable model or at least one of $\exists s\,(s \subseteq e)\,, \exists s\,(e \subseteq s)$ is false *i.e.,* no stable model contains another.

We now define an equivalence relation so that two events are related if either both are inconsistent or both are consistent and, in the latter case, with the same stable core.

Definition 1 (*Equivalence Relation on Events*)
For a given program, let $u, v \in \mathcal{E}$. The equivalence relation $u \sim v$ is defined by **def.** $\sim$

$$u, v \notin \mathcal{C} \vee \big(u, v \in \mathcal{C} \wedge [\![u]\!] = [\![v]\!]\big)\,. \tag{12}$$

This equivalence relation defines a partition on the set of events, where each class holds a unique relation with the stable models. In particular we denote each class by: **def.** $[e]_\sim$

$$[e]_\sim = \begin{cases} \bot := \mathcal{E} \setminus \mathcal{C} & \text{if } e \in \mathcal{E} \setminus \mathcal{C}, \\ \big\{ u \in \mathcal{C} \,\big|\, [\![u]\!] = [\![e]\!] \big\} & \text{if } e \in \mathcal{C}. \end{cases} \tag{13}$$

where $\bot$ denotes the set $\mathcal{E} \setminus \mathcal{C}$ of *inconsistent* events, *i.e.* events that contain $\{x, \overline{x}\}$ for some atom x. **def.** $\bot$

Proposition 1 (*Class of the Program's Consequences*) **def.** λ, Λ
Let λ be the empty set event (notice that $\lambda = \emptyset \in \mathcal{E}$) [6], and Λ the *consequence class* of (consistent) events related with all the stable models. Then |[9]

[9] |Removed '$= [\![\mathcal{M}]\!]$' from this equation.

$$[\lambda]_\sim = \Lambda. \tag{14}$$

Proof
Let $x \in [\lambda]_\sim$ be consistent. Then $x \sim \lambda$. Since $[\![\lambda]\!] = \mathcal{M}$ also $[\![x]\!] = \mathcal{M}$. Hence $x \in \Lambda$.

Now, let $x \in \Lambda$. So x is consistent and, for each $s \in \mathcal{M}$, either $s \subseteq x$ or $s \supseteq x$. So, $[\![x]\!] = \mathcal{M} = [\![\lambda]\!]$. By definition 1, $x \sim \lambda$ *i.e.* $x \in [\lambda]_\sim$. $\square$

The combinations of stable models, *i.e.* the stable cores, together with the set of inconsistent events ($\bot$) forms a set of representatives for the equivalence relation $\sim$. Since all events within a consistent equivalence class have the same stable core, we are interested in functions (including weight assignments), that are constant within classes. A function $f : \mathcal{E} \to Y$, where Y is any set, is

[6] We adopt the notation 'λ' for *empty word*, from formal languages, to distinguish '$\emptyset \in \mathcal{E}$' from '$\emptyset \subset \mathcal{E}$'.

said to be *coherent* if

$$\forall e \in \mathcal{E}\, \forall u \in [e]_\sim \left(f(u) = f(e)\right). \tag{15}$$

Considering coherent functions, in the specific case of eq. (3), instead of dealing with the $2^6 = 64$ events, we need to consider only the $2^3 + 1 = 9$ classes, well defined in terms of combinations of the stable models, to define coherent functions. In general, a program with n atoms and m stable models has 2^{2n} events and $2^m + 1$ stable cores — but easily $m \gg n$.

4.2 From Total Choices to Events

The 'propagation' phase, traced by eq. (7) and eqs. (17) to (22b), starts with the weight of total choices, $\omega_\mathcal{T}(t)$, propagates it to the stable models, $\omega_\mathcal{M}(s)$, and then, within the equivalence relation from eq. (12), to a coherent weight of events, $\omega_\mathcal{E}(e)$. So we are specifying a sequence of functions

$$\omega_\mathcal{T} \longrightarrow \omega_\mathcal{M} \longrightarrow \omega_\mathcal{R} \longrightarrow \omega_\mathcal{E} \tag{16}$$

on successive larger domains

$$\mathcal{T} \longrightarrow \mathcal{M} \longrightarrow [\mathcal{E}]_\sim \longrightarrow \mathcal{E}$$

such that the last function ($\omega_\mathcal{E}$) is a finite coherent function on the set of events and thus, as a final step, it can easily be used to define a probability distribution of events by normalization: $\omega_\mathcal{E} \longrightarrow \mathrm{P}_\mathcal{E}$.

Total choices and Stable models

Let's start by looking into the first two steps of the sequence of functions eq. (16): $\omega_\mathcal{T}$ and $\omega_\mathcal{M}$.

The weight $\omega_\mathcal{T}$ of the total choice $t \in \mathcal{T}$ is already given by eq. (7).

Recall that each total choice $t \in \mathcal{T}$, together with the rules and the other facts of a program, defines the set $\mathcal{M}(t)$ of stable models associated with that choice. Given a total choice $t \in \mathcal{T}$, a stable model $s \in \mathcal{M}$, and formal variables or values $\theta_{s,t} \in [0, 1]$ such that $\sum_{s\in\mathcal{M}(t)} \theta_{s,t} = 1$, we define

def. $\omega_\mathcal{M}(s,t)$

$$\omega_\mathcal{M}(s,t) := \begin{cases} \theta_{s,t} & \text{if } s \in \mathcal{M}(t) \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

The $\theta_{s,t}$ parameters in eq. (17) express the *program's* lack of information about the weight assignment, when a single total choice entails more than one stable model. We address this issue by assigning a possibly unknown parameter, *i.e.* a formal algebraic variable ($\theta_{s,t}$) associated with a total choice (t) and a stable model (s). This allows the expression of a quantity that does not result from the program's syntax but can be determined or estimated given more information, *e.g.* observed data.

As sets, two stable models can have non-empty intersection. But because different SMs represent different states of a system —which are *disjoint events*— we assume that the algebra of the stable models is σ-additive

Assumption 4 (*Stable models as Disjoint events*)
For any set X of stable models and any total choice t,

$$\omega_\mathcal{M}(X,t) = \sum_{s\in X} \omega_\mathcal{M}(s,t). \tag{18}$$

Equation (18) is the basis for eq. (20a) and effectively extends $\omega_\mathcal{M} : \mathcal{M} \times \mathcal{T} \to \mathbb{R}$ to $\omega_\mathcal{M} : 2^\mathcal{M} \times \mathcal{T} \to \mathbb{R}$. Now the pre-condition of eq. (17) can be stated as $\omega_\mathcal{M}\left(\mathcal{M}(t), t\right) = 1$.

Classes

Consider the next step in sequence eq. (16), the function $\omega_\mathcal{R}$ on $[\mathcal{E}]_\sim$. Each class of the equivalence relation $\sim$ (eq. 12) is either the inconsistent class ($\perp$) or is associated with a stable core, *i.e.* a set of stable models. Therefore, $\omega_\mathcal{R}$ is defined considering the following two cases:

def. $\omega_\mathcal{R}$

Inconsistent class. This class contains events that are (classically) inconsistent, thus should never be observed and thus have weight zero:

$$\omega_\mathcal{R}(\perp, t) := 0.^{7} \tag{19}$$

[7] This weight being zero is independent of the stable models.

Consistent classes. For the propagation function to be coherent, it must be constant within a class and its value dependent only on the stable core:

$$\omega_{\mathcal{R}}\left([e]_{\sim}, t\right) := \omega_{\mathcal{M}}\left(\llbracket e \rrbracket, t\right) = \sum_{s \in \llbracket e \rrbracket} \omega_{\mathcal{M}}(s, t)\,. \tag{20a}$$

and we further define the following:

$$\omega_{\mathcal{R}}\left([e]_{\sim}\right) := \sum_{t \in \mathcal{T}} \omega_{\mathcal{T}}(t)\, \omega_{\mathcal{R}}\left([e]_{\sim}, t\right) \tag{20b}$$

Equation (20a) states that the weight of a class $[e]_{\sim}$ is the weight of its stable core ($\llbracket e \rrbracket$) and eq. (20b) *averages* eq. (20a) over the total choices. Notice that eq. (20a) also applies to the independent class, $\Diamond = \left\{ e \mid \llbracket e \rrbracket = \emptyset \right\}$, because events in this class are not related with any stable model: **def.** $\Diamond$

$$\omega_{\mathcal{R}}(\Diamond, t) = \sum_{s \in \emptyset} \omega_{\mathcal{M}}(s, t) = 0. \tag{21}$$

Events and Probability

Each consistent event $e \in \mathcal{E}$ is in the class defined by its stable core $\llbracket e \rrbracket$. So, denoting the number of elements in X as $\#X$, we set: **def.** $\omega_{\mathcal{E}}$

$$\omega_{\mathcal{E}}(e, t) := \begin{cases} \frac{\omega_{\mathcal{R}}\left([e]_{\sim}, t\right)}{\#[e]_{\sim}} & \text{if } \#[e]_{\sim} > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{22a}$$

and, by averaging over the total choices:

$$\omega_{\mathcal{E}}(e) := \sum_{t \in \mathcal{T}} \omega_{\mathcal{T}}(t)\, \omega_{\mathcal{E}}(e, t)\,. \tag{22b}$$

The eq. (22b) is the main goal of this paper: propagate the weights associated to facts of an WASP to the set of all events of that program. In order to get a probability from eq. (22b), concerning the *Probabilistic Tasks* goal, we define the *normalizing factor*:

$$Z := \sum_{e \in \mathcal{E}} \omega_{\mathcal{E}}(e) = \sum_{[e]_{\sim} \in [\mathcal{E}]_{\sim}} \omega_{\mathcal{R}}\left([e]_{\sim}\right), \tag{23}$$

and now eq. (22b) provides a straightforward way to define the *probability of a single event* $e \in \mathcal{E}$: **def.** $\mathrm{P}_{\mathcal{E}}$

$$\mathrm{P}_{\mathcal{E}}(e) := \frac{\omega_{\mathcal{E}}(e)}{Z}. \tag{24}$$

Equation (24) defines a coherent *prior*[8] probability of events and, together with external statistical knowledge, can be used to learn about the *initial* probabilities of the atoms.

The effect of propagation. One way to assess the effect of propagating weights through the SMs to the events is to compare $\mathrm{P}_{\mathcal{E}}$ with a 'probability' induced syntactically from the weights of facts. It is sufficient to consider what appens with the total choices: if we syntactically induce a probability for the TCs, say $\mathrm{P}_{\mathcal{T}}$, is it true that $\mathrm{P}_{\mathcal{E}}(t) = \mathrm{P}_{\mathcal{T}}(t)$ for all $t \in \mathcal{T}$?

The weight assignment of the total choices can also be normalized into a probability distribution. For $t \in \mathcal{T}$, **def.** $\mathrm{P}_{\mathcal{T}}$

$$\mathrm{P}_{\mathcal{T}}(t) = \frac{\omega_{\mathcal{T}}(t)}{\sum_{\tau \in \mathcal{T}} \omega_{\mathcal{T}}(\tau)} \tag{25}$$

And now we ask if these probabilities coincide in $\mathcal{T}$:

$$\forall t \in \mathcal{T}\left(\mathrm{P}_{\mathcal{E}}(t) = \mathrm{P}_{\mathcal{T}}(t)\right)?$$

It is easy to see that, in general, this cannot be true. While the domain of $\mathrm{P}_{\mathcal{T}}$ is the set of total choices, for $\mathrm{P}_{\mathcal{E}}$ the domain is much larger, including all the events. Except for trivial programs, some events other than total choices will have non-zero weight.

[8] In the Bayesian sense that future observations might update this probability.

Proposition 2 (*Two Distributions*)
If a program has a consistent event $e \in \mathcal{C} \setminus \mathcal{T}$ such that $\mathrm{P}_{\mathcal{E}}(e) \neq 0$ then there is at least one $t \in \mathcal{T}$ such that

$$\mathrm{P}_{\mathcal{T}}(t) \neq \mathrm{P}_{\mathcal{E}}(t)\,. \tag{26}$$

Proof
Suppose towards a contradiction that $\mathrm{P}_{\mathcal{T}}(t) = \mathrm{P}_{\mathcal{E}}(t)$ for all $t \in \mathcal{T}$ and η as above.
Then

$$1 = \sum_{t \in \mathcal{T}} \mathrm{P}_{\mathcal{T}}(t) = \sum_{t \in \mathcal{T}} \mathrm{P}_{\mathcal{E}}(t)\,.$$

Therefore, $\mathrm{P}_{\mathcal{E}}(\eta) = 0$ for all $\eta \in \mathcal{C} \setminus \mathcal{T}$, in contradiction with the hypothesis on e. □

The essential, perhaps *counter-intuitive*, conclusion of prop. 2 is that we are dealing with *two distributions*: $\mathrm{P}_{\mathcal{T}}$, restricted to the total choices, results *syntactically* from the annotations of the program, while $\mathrm{P}_{\mathcal{E}}$, extended to events, results from both the annotations and the program's *semantics*, *i.e.* the stable models.

For ex. 1:

$$\begin{aligned} \mathrm{P}_{\mathcal{T}}(a) &= 0.3 && \text{from the program } P_1, \\ \mathrm{P}_{\mathcal{E}}(a) &= \frac{3}{64} && \text{from eq. (29).} \end{aligned}$$

Now $\mathrm{P}_{\mathcal{E}} : \mathcal{E} \to [0,1]$ can be extended to $\mathrm{P}_{\mathcal{E}} : 2^{\mathcal{E}} \to [0,1]$ by abusing notation and setting, for $X \subseteq \mathcal{E}$,

$$\mathrm{P}_{\mathcal{E}}(X) = \sum_{x \in X} \mathrm{P}_{\mathcal{E}}(x)\,. \tag{27}$$

It is straightforward to verify that the latter satisfies the Kolmogorov axioms of probability.

We can now properly state the following property about *certain facts* such as $a : 1.0$.

Proposition 3 (*Probability of Certain Facts*)
Consider a program A with the weighted fact $a : 1.0$ and A^* that results from A by replacing that fact by the deterministic fact a. Let $\mathrm{P}_{\mathcal{E}}$ be as eq. (24) for A and $\mathrm{P}^*_{\mathcal{E}}$ for A^*. Then

$$\forall e \in \mathcal{E}\left(\mathrm{P}_{\mathcal{E}}(e) = \mathrm{P}^*_{\mathcal{E}}(e)\right). \tag{28}$$

Proof
[10]
□

[10] do this

Example 2 (*Probability of Events*)
The coherent prior *probability of events of program P_1 in ex. 1 is*

$$\begin{array}{c|cccccccccc} [\![e]\!] & \bot & \Diamond & \overline{a} & ab & ac & \overline{a},ab & \overline{a},ac & ab,ac & \Lambda \\ \hline \mathrm{P}_{\mathcal{E}}(e) & 0 & 0 & \frac{7}{207} & \frac{1}{23}\theta & \frac{1}{23}\overline{\theta} & 0 & 0 & \frac{3}{46} & \frac{10}{23} \end{array} \tag{29}$$

This table can be used to compute the probability of any single event $e \in \mathcal{E}$ by looking at the column of the event's stable core.

For example:

$\omega_{\mathcal{E}}(ab) = \frac{\theta}{23}$*, because ab is the only* SM *related with ab so $[\![ab]\!] = \{ab\}$ and the weight value is found in the respective column of eq.* (29).

$\omega_{\mathcal{E}}(abc) = \frac{3}{46}$ *because $abc \supset ab$ and $abc \supset ac$. So $[\![abc]\!] = \{ab, ac\}$.*

$\omega_{\mathcal{E}}(bc) = 0$ *because, since there is no* SM *s that either $s \subset bc$ or $bc \subset s$, $[\![bc]\!] = \emptyset$* i.e. $bc \in \Diamond$.

$\omega_{\mathcal{E}}(\overline{a}b) = \frac{7}{207}$ *because $[\![\overline{a}b]\!] = \{\overline{a}\}$.*

$\omega_{\mathcal{E}}(\overline{a}) = \frac{7}{207}$ *and $\omega_{\mathcal{E}}(a) = \frac{3}{46}$. Notice that $\omega_{\mathcal{E}}(\overline{a}) + \omega_{\mathcal{E}}(a) \neq 1$. This highlights the fundamental difference between $\omega_{\mathcal{E}}$ and $\omega_{\mathcal{T}}$* (cf. *prop.* 2*), where the former results from the lattice of the stable cores and the latter directly from the explicit assignment of probabilities to literals.*

Related with the last case above, consider the complement of a consistent event e, denoted by $\complement e$. To calculate $\mathrm{P}_{\mathcal{E}}(\complement e)$ we look for the classes in $[\mathcal{E}]_{\sim}$ that are not $[e]_{\sim}$, i.e. *the complement of e's class* **def.** $\complement e$

within $[\mathcal{E}]_\sim$[9], $\complement[e]_\sim$. *Considering that* $[\mathcal{E}]_\sim$ *is in a one-to-one correspondence with the stable cores plus* $\bot$,

$$[\mathcal{E}]_\sim \simeq \left\{\bot, \Diamond, \{\overline{a}\}, \{ab\}, \{ac\}, \{\overline{a}, ab\}, \{\overline{a}, ac\}, \{ab, ac\}, \Lambda\right\}.$$

In particular, for $\omega_\mathcal{E}(\complement a)$, *since* $[\![a]\!] = \{ab, ac\}$ *then* $\complement[a]_\sim = [\mathcal{E}]_\sim \setminus [a]_\sim$ *and* $\mathrm{P}_\mathcal{E}(\complement a) = \mathrm{P}_\mathcal{E}([\mathcal{E}]_\sim \setminus [a]_\sim) = 1 - \mathrm{P}_\mathcal{E}(a)$. *Also,* $\mathrm{P}_\mathcal{E}(\complement\overline{a}) = 1 - \mathrm{P}_\mathcal{E}(\overline{a})$.

While not illustrated in our examples, this method also applies to programs that have more than one probabilistic fact, like

$$\begin{cases} \quad a : 0.3, & b \quad : 0.6, \\ c \vee d \leftarrow a \wedge b. \end{cases}$$

Our approach generalizes to Bayesian networks in a way similar to (Cozman and Mauá 2020; Raedt et al. 2016) and (Kießling et al. 1992; Thöne et al. 1997) as follows. On the one hand, any acyclic propositional program can be viewed as the specification of a Bayesian network over binary random variables. So, we may take the structure of the Bayesian network to be the dependency graph. The random variables then correspond to the atoms and the probabilities can be read off of the probabilistic facts and rules. Conversely, any Bayesian network over binary variables can be specified by an acyclic non-disjunctive WASP.

5 Discussion and Future Work

This work is a first venture into expressing weight assignments using algebraic expressions derived from a logical program, in particular an ASP. We would like to point out that there is still much to explore concerning the full expressive power of logic programs and ASP programs. So far, we have not considered recursion, logical variables or functional symbols.

The theory, methodology, and tools, from Bayesian Networks can be adapted to our approach. The connection with Markov Fields (Kindermann and Snell 1980) is left for future work. An example of a 'program selection' application (as mentioned in item 4, section 3) is left for future work. Also, there is still little effort concerning the *Probabilistic Tasks* and to articulate with the related fields of probabilistic logical programming, machine learning, inductive programming, *etc.*

The equivalence relation from definition 1 identifies the $s \subseteq e$ and $e \subseteq s$ cases. Relations that distinguish such cases might enable better relations between the representations and processes from the stable models.

Furthermore, we decided to set the weight of inconsistent events to 0 but, maybe, in some cases, we shouldn't. For example, since observations may be affected by noise, one can expect to occur some inconsistencies.

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[9] All the usual set operations hold on the complement. For example, $\complement\complement X = X$.

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