CSCI 2011 HW 4

Fletcher Gornick

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1 3 Problem 18

Prove that 100 cannot be written as the sum of three integers, an even number of which are even.

This problem can be split up into two cases...

Case 1: 0 even integers and 3 odd integers.
Let $i, j, k \in \mathbb{Z}$ be odd integers. This means $\exists a, \exists b, \exists c \in \mathbb{Z}$ such that $i = 2a + 1$, $j = 2b + 1$, and $k = 2c + 1$.
Therefore, $i + j + k = (2a + 1) + (2b + 1) + (2c + 1) = 2a + 2b + 2c + 3 = 2(a + b + c + 1) + 1$
Since $a + b + c + 1$ is an integer, $i + j + k$ must be odd. Hence $i + j + k \neq 100$.

Case 2: 2 even integers and 1 odd integer.
Assume $i, j, k \in \mathbb{Z}$. Let i and j be even integers and k be an odd integer. This means $\exists a, \exists b, \exists c \in \mathbb{Z}$ such that $i = 2a$, $j = 2b$, $k = 2c + 1$.
Therefore $i + j + k = (2a) + (2b) + (2c + 1) = 2a + 2b + 2c + 1 = 2(a + b + c) + 1$.
Since $a + b + c$ is an integer, $i + j + k$ must be odd. Hence $i + j + k \neq 100$.

2 3 Problem 32

Prove that there exist a rational number a and an irrational number b such that a^b is irrational.

Suppose there are two irrational numbers x, y such that $a = x^y$ is rational. We know this is possible because it was proven in result 3.38 of the textbook. With that in mind, Let $b = \frac{1}{y}$, which is irrational, because y is irrational. Therefore $a^b = \left(x^y\right)^{\frac{1}{y}} = x^{\frac{y}{y}} = x$, which is irrational. Hence, a^b is irrational.

3 3 Problem 38

Prove for every integer n that there exist two integers a and b of opposite parity such that $an + b$ is an odd integer.

This problem can be split up into two cases...

Case 1: For all even integers n, there exist two integers a and b of opposite parity such that $an + b$ is an odd integer.
For n to be even, there must exist an integer k such that $n = 2k$. Plugging in $2k$ for n, we get $2ak + b$ is odd. Now let $a = 2$ and $b = 1$, Therefore $an + b = 2(2k) + 1$. Since $2k$ is an integer, $an + b$ is odd.

Case 2: For all odd integers n, there exist two integers a and b of opposite parity such that $an + b$ is an odd integer.
Assuming n is odd, $\exists k \in \mathbb{Z}$ such that $n = 2k + 1$. Now let's assume $a = 1$ and $b = 2$. Therefore $an + b = 1(2k + 1) + 2 = 2k + 3 = 2(k + 1) + 1$. Since $2k + 1$ is an integer, $an + b$ is odd.

4 3.5 Problem 8

Disprove: For every two sets A and B, $\mathcal{P}(A \cup B) = \mathcal{P}(A) \cup \mathcal{P}(B)$.

In order to disprove the claim, one counterexample will suffice. So let $A = \{1\}$ and $B = \{2\}$, so $A \cup B = \{1, 2\}$. Taking the powersets, we get $\mathcal{P}(A) = \big\{\emptyset, \{1\}\big\}$, $\mathcal{P}(B) = \big\{\emptyset, \{2\}\big\}$, and $\mathcal{P}(A \cup B) = \big\{\emptyset, \{1\}, \{2\}, \{1, 2\}\big\}$. Therefore $\mathcal{P}(A) \cup \mathcal{P}(B) = \big\{\emptyset, \{1\}, \{2\}\big\} \neq \big\{\emptyset, \{1\}, \{2\}, \{1, 2\}\big\} = \mathcal{P}(A \cup B)$.

5 3.7 Problem 12

Prove that $\sqrt{2} + \sqrt{3}$ is an irrational number.

Assume, to the contrary, that $\sqrt{2} + \sqrt{3}$ is a rational number, this must mean that there exist two integers, a and b, such that $\sqrt{2} + \sqrt{3} = \frac{a}{b}$, where $b \neq 0$. We can also keep in mind that $\sqrt{2}$ is an irrational number, which was solved in a previous example.
Therefore...

$$\begin{aligned}
\sqrt{2} + \sqrt{3} &= \frac{a}{b} \\
\sqrt{3} &= \frac{a}{b} - \sqrt{2} \\
\left(\sqrt{3}\right)^2 &= \left(\frac{a}{b} - \sqrt{2}\right)^2 \\
3 &= \frac{a^2}{b^2} - \frac{2a}{b}\sqrt{2} + 2 \\
\frac{2a}{b}\sqrt{2} &= \frac{a^2}{b^2} - 3 + 2 \\
\sqrt{2} &= \frac{b\left(\frac{a^2}{b^2} - 1\right)}{2a} \\
\sqrt{2} &= \frac{\frac{a^2}{b} - b}{2a} \\
\sqrt{2} &= \frac{\frac{a^2 - b^2}{b}}{2a} \\
\sqrt{2} &= \frac{a^2 - b^2}{2ab}
\end{aligned}$$

Since $a^2 - b^2$ and $2ab$ are both integers, $\sqrt{2}$ must be rational, which is a contradiction. Therefore $\sqrt{2} + \sqrt{3}$ must be an irrational number.