4512 Final

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1) Find the general solution of $\mathbf{\dfrac{d^4y}{dt^4} + y = 0}$.

2) We know that the SIR model is given by the equations

$$\frac{\mathbf{d}}{\mathbf{dt}}\begin{pmatrix} S \\ I \\ R \end{pmatrix} = \begin{pmatrix} -\beta SI \\ \beta AI - \delta I \\ \delta I \end{pmatrix}$$

where the removal rate δ and the infection rate β are positive constants. We want to modify this model by changing the assumption that the infected population keeps it's immunity. So we assume that the rate of change of the population susceptible to infection is increased proportionally to R. The new model is

$$\frac{\mathbf{d}}{\mathbf{dt}}\begin{pmatrix} S \\ I \\ R \end{pmatrix} = \begin{pmatrix} -\beta SI \\ \beta AI - \delta I \\ \delta I \end{pmatrix} + \begin{pmatrix} \mu R \\ 0 \\ -\mu R \end{pmatrix}, \quad \mu > 0.$$

Find the equilibrium points and determine if they are stable or not. Plot the phase portrait. What happens to $(S(t), I(t), R(t))$ as t goes to infinity? (Consider only $S, I, R \geq 0$).

3) Consider the autonomous system

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x(a-by) \\ y(-c+dx) \end{bmatrix}, \quad a, b, c, d > 0.$$

Find the equilibrium points (only for $x, y \geq 0$) and determine (if possible) their stability. Plot the phase portrait. Argue that, if $x(t = 0)$ and $y(t = 0)$ are positive, the corresponding orbit is periodic. (Hint: show that the orbits lie on the curves of the form $dx - c\ln x + by - a\ln y =$ constant). Plot the phase portrait.

4) Use the Poincaré-Bendixson theorem to prove that a periodic orbit of the system

$$\frac{\mathbf{d}}{\mathbf{dt}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x - y - \frac{1}{2}x(x^2+y^2) \\ x + \frac{1}{2}y - \frac{1}{2}y(x^2+y^2) \end{bmatrix}$$

lies on the circle $x^2 + y^2 = 1$. Sketch the phase portrait of the system. What happens to $(x(t), y(t))$ as t goes to infinity?