4512 Homework 2

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Section 2.2 Problem 6

Solve the initial-value problem.

$$2\frac{d^2y}{dt^2} + \frac{dy}{dt} - 10y = 0 \ , \ y(1) = 5, y'(1) = 2$$

We can rewrite y as $e^{\lambda t}$ to give us the equation $2\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - 10e^{\lambda t} = 0$, setting $t = 0$, we get $2\lambda^2 + \lambda - 10 = (2\lambda + 5)(\lambda - 2) = 0$, therefore $\lambda = \frac{-5}{2}, 2$, giving us the general equation for t, $y(t) = c_1 e^{\frac{-5}{2}t} + c_2 e^{2t}$, where $y(1) = 5, y'(1) = 2$. We can rewrite this in matrix notation to solve for c_1 and c_2 plugging 1 in for t.

$$\begin{bmatrix} e^{\frac{-5}{2}} & e^2 \\ \frac{-5}{2}e^{\frac{-5}{2}} & 2e^2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y(1) \\ y'(1) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^{\frac{-5}{2}} & e^2 \\ \frac{-5}{2}e^{\frac{-5}{2}} & 2e^2 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{2e^{-1/2} + \frac{5}{2}e^{-1/2}} \begin{bmatrix} 2e^2 & -e^2 \\ \frac{5}{2}e^{\frac{-5}{2}} & e^{\frac{-5}{2}} \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\Rightarrow c_1 = \frac{2}{9}e^{\frac{1}{2}}\left(10e^2 - 2e^2\right)$$

$$c_2 = \frac{2}{9}e^{\frac{1}{2}}\left(\frac{25}{2}e^{\frac{-5}{2}} + 2e^{\frac{-5}{2}}\right)$$

After simplifying, we get $c_1 = \frac{16}{9}e^{\frac{5}{2}}$, and $c_2 = \frac{29}{9}e^{-2}$. Finally, we can plug these values in to find the solution to the initial value problem.

$$y(t) = \frac{1}{9}\left(16e^{\frac{-5}{2}(t-1)} + 29e^{2t-2}\right)$$

Section 2.2.1 Problem 6

Solve the initial-value problem.

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = 0 \ , \ y(0) = 0, y'(0) = 2$$

Again, We can rewrite y as $e^{\lambda t}$ to give us a new equation $\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + 5e^{\lambda t} = 0$. Setting $t = 0$ gives us a new equation $\lambda^2 + 2\lambda + 5 = 0$. We can plug this equation into the quadratic formula to get $\lambda = \frac{-2 \pm \sqrt{(-2)^2 - 4\cdot 5}}{2} = -1 \pm 2i$, giving us the general equation $y(t) = b_1 e^{(-1+2i)t} + b_2 e^{(-1-2i)t}$. For the first complex root $\lambda_1 = -1 + 2i$, we can apply Euler's formula to get $e^{\lambda_1 t} = e^{(-1+2i)t} = e^{-t}\cos(2t) + ie^{-t}\sin(2t)$ which is a complex-valued solution to $y'' + 2y' + 5y = 0$.

Since, by *Lemma 1* from section 2.2.1 of the textbook, both $\text{Re}\{e^{\lambda_1 t}\} = e^{-t}\cos(2t)$ and $\text{Im}\{e^{\lambda_1 t}\} = e^{-t}\sin(2t)$ are two linearly independent, real-valued solutions to the above equation, we can rewrite the general solution to the initial-value problem as

$$y(t) = e^{-t}[c_1\cos(2t) + c_2\sin(2t)]$$

Plugging 0 in for t we get that $y(0) = e^0[c_1\cos(0) + c_2\sin(0)] = c_1 = 0$, which gives us a new equation $y(t) = c_2 e^{-t}\sin(2t)$. We can now differentiate and plug in 0 to solve for c_2.

$$y'(t) = -c_2 e^{-t}sin(2t) + 2c_2 e^{-t}cos(2t) \ \Rightarrow \ y'(0) = 2c_2 = 2 \ \Rightarrow \ c_2 = 1$$

Now we can plug the values we found for our two constants to give us the solution to the initial value problem.

$$y(t) = e^{-t}sin(2t)$$

Section 2.2.2 Problem 6

Solve the initial-value problem.

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0 \ , \ y(2) = 1, y'(2) = -1$$

Again, applying $y = e^{\lambda t}$, we get $\lambda^2 e^{\lambda t} + 2\lambda e^{\lambda t} + e^{\lambda t} = 0$, and plugging 0 in for t, we get $\lambda^2 + 2\lambda + 1 = 0$, which can be simplified to $(\lambda + 1)^2 = 0$, giving us $\lambda = -1$ as our only solution with a multiplicity of 2. As stated in Section 2.2.2 of the textbook, we can yield two linearly independent solutions to the equation by multiplying one of them by t. This gives us the general solution $y(t) = c_1 e^{-t} + c_2 t e^{-t}$. We can now solve for the Wronskian of the matrix at time $t = 2$, and plug in the corresponding values $y(2) = 1, y'(2) = -1$ like so...

$$\begin{bmatrix} e^{-2} & 2e^{-2} \\ -e^{-2} & e^{-2} - 2e^{-2} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y(2) \\ y'(2) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^{-2} & 2e^{-2} \\ -e^{-2} & -e^{-2} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{-e^{-4} + 2e^{-4}} \begin{bmatrix} -e^{-2} & -2e^{-2} \\ e^{-2} & e^{-2} \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow c_1 = e^4(-e^{-2} + 2e^{-2}) = e^2$$

$$\Rightarrow c_2 = e^4(e^{-2} - e^{-2}) = 0$$

Now we can plug the values we found for c_1 and c_2 to give us our final solution.

$$y(t) = e^{2-t}$$

Section 2.4 Problem 6

Solve the initial-value problem.

$$y'' + 4y' + 4y = t^{5/2}e^{-2t} \ , \ y(0) = 0, y'(0) = 0$$

We can write the general solution to this differential equation as $y(t) = c_1y_1(t) + c_2y_2(t) + y_p(t)$, where $c_1y_1(t) + c_2y_2(t)$ is the general solution to $y'' + 4y' + 4y = 0$, otherwise known as the complementary equation. So we can solve for the complementary equation just like we would any normal homogenous differential equation.

$$y'' + 4y' + 4y = 0 \ \Rightarrow \ \lambda^2 e^{\lambda t} + 4\lambda e^{\lambda t} + 4e^{\lambda t} = 0 \ \Rightarrow \ \lambda^2 + 4\lambda + 4 = 0 \ \Rightarrow \ \lambda = -2$$

Since $\lambda = -2$ with a multiplicity of 2, we can now write our general solution to the initail value problem as $y(t) = c_1e^{-2t} + c_2te^{-2t} + y_p(t)$. Now, to find a particular solution y_p that satisfies the differential equation. Since the right side of the equation isn't an easy polynomial multiplied by an exponential, we must use variation of parameters to solve for the particular solution. Let's call the equation on the right-hand side $r(x)$ for simplicity sake. The solution to the particular equation y_p can be written as $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ where $u_1'(t) = \frac{-r(t)y_2(t)}{W[y_1,y_2]}$, and $u_2'(t) = \frac{r(t)y_1(t)}{W[y_1,y_2]}$ as stated in section 2.4. Thus, the formula for the particular solution can be written as

$$y_p(t) = -y_1(t) \int \frac{y_2(t)r(t)}{W[y_1, y_2]}dt + y_2(t) \int \frac{y_1(t)r(t)}{W[y_1, y_2]}dt$$

$W[y_1, y_2]$ represents the Wronskian of the functions y_1 and y_2, so we can solve for W like so...

$$W[y_1, y_2] \ = \ \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} = \begin{vmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{vmatrix} = (1-2t)e^{-4t} + 2te^{-4t} \ = \ e^{-4t}$$

Now we've attained every value needed to solve for our particular solution y_p.

$$y_p(t) = -e^{-2t} \int \frac{te^{-2t} \cdot t^{5/2}e^{-2t}}{e^{-4t}}dt + te^{-2t} \int \frac{e^{-2t} \cdot t^{5/2}e^{-2t}}{e^{-4t}}dt$$

On the next page, we will solve for the particular solution.

$$y_p(t) = -e^{-2t} \int \frac{te^{-2t} \cdot t^{5/2}e^{-2t}}{e^{-4t}} dt + te^{-2t} \int \frac{e^{-2t} \cdot t^{5/2}e^{-2t}}{e^{-4t}} dt$$

$$= -e^{-2t} \int \frac{t^{7/2}e^{-4t}}{e^{-4t}} dt + te^{-2t} \int \frac{t^{5/2}e^{-4t}}{e^{-4t}} dt$$

$$= -e^{-2t} \int t^{7/2} dt + te^{-2t} \int t^{5/2} dt$$

$$= \frac{-2}{9} t^{9/2}e^{-2t} + \frac{2}{7} t^{9/2}e^{-2t}$$

$$= \frac{4}{63} t^{9/2}e^{-2t}$$

So now we have our particular solution $y_p(t) = \frac{4}{63}t^{9/2}e^{-2t}$, so we can plug this into our general equation to get $y(t) = c_1 e^{-2t} + c_2 te^{-2t} + \frac{4}{63}t^{9/2}e^{-2t}$. The last thing we need to do is solve for our constants c_1 and c_2. We can use the same method from previous pages to do this.

$$\begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} e^{-2t} & te^{-2t} \\ -2e^{-2t} & (1-2t)e^{-2t} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{e^{-4t}} \begin{bmatrix} (1-2t)e^{-2t} & -te^{-2t} \\ 2e^{-2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow c_1 = 0$$

$$c_2 = 0$$

So finally, with our complementary equation and particular equation, we have that the solution to the non-homogenous initial-value problem is

$$y(t) = \frac{4}{63}t^{9/2}e^{-2t}$$

Section 2.6 Problem 1

It is found experimentally that a 1 kg mass stretches a spring 49/320 m. If the mass is pulled down an additional 1/4 m and released, find the amplitude, period, and frequency of the resulting motion, neglecting air resistance (use g $=$ 9.8 m/s^2)

We know by Newton's second law, $F = mg = 9.8 \cdot 1 = 9.8$. Also from equation (1) in section 2.6 of the textbook, $F(t) = m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky$. Since the spring is stretched $49/320 = 0.153$ m, $y = 0.153$, and $\frac{d^2y}{dt^2} = \frac{dy}{dt} = 0$. So now we have this equation $F(t) = ky \Rightarrow 9.8 = 0.153k \Rightarrow k = 64$.

Since there's no air resistance, our damping coefficient $c = 0$. Also after the initial displacement of the spring, there is no external force, so we have the equation $m\frac{d^2y}{dt^2} + ky = 0$. This equation can be rewritten as $\frac{d^2y}{dt^2} + \omega_0^2 y = 0$, where $\omega_0^2 = k/m$. From section 2.6 equation (3), we know this equation has the general form $y(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$. We can also solve for ω_0, so we get $\omega_0 = \sqrt{k/m} = \sqrt{64/1} = 8$, thus our general equation is now $y(t) = c_1 \cos(8t) + c_2 \sin(8t)$

Before finding our amplitude, period, and frequency, we must first solve for constants c_1 and c_2. This spring at time $t = 0$ is initially displaced 0.25m, so $0.25 = c_1 \cos(0) + c_2 \sin(0) \Rightarrow c_1 = 0.25$. Also the initial velocity of the spring at time $t = 0$ is 0, so $0 = -8c_1 \sin(0) + 8c_2 \cos(0) \Rightarrow c_2 = 0$. This gives us our final equation $y(t) = 0.25 \cos(8t)$.

By Lemma 1 in section 2.6 of the textbook, we know our previous function $y(t)$ can be rewritten as $y(t) = R\cos(\omega_0 + \delta)$, where $R = \sqrt{c_1^2 + c_2^2} = \frac{1}{4}$, and $\delta = \arctan(\frac{c_2}{c_1}) = \arctan(0) = 0$. In this case, R represents the amplitude and ω_0 represents the freqency, and thus the period $T = 2\pi/\omega_0$.

Therefore, we have an amplitude $A = R = \frac{1}{4}$, a frequency $f = \omega_0 = 8$, and a period $T = \frac{2\pi}{\omega_0} = \frac{\pi}{4}$.