MATH 5615H Homework 2

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September 15, 2022

1) Let $\epsilon > 0$ be a strictly positive real number.

(i) Let z_0 and w_0 be complex numbers. Prove that for all complex numbers z and w,

$$\text{if} \quad |z - z_0| < \frac{\epsilon}{2} \ \text{ and } \ |w - w_0| < \frac{\epsilon}{2} \quad \text{then} \quad |(z + w) - (z_0 + w_0)| < \epsilon.$$

In doing parts (ii) and (iii) below, the inequality $\big||a| - |b|\big| \leq |a - b|$ for all $a, b \in \mathbb{C}$ will be helpful.

(ii) Let z_0 and w_0 be complex numbers. Prove that, for all complex numbers z and w,

$$\text{if} \quad |z - z_0| < \min\left(1, \frac{\epsilon}{2(1 + |w_0|)}\right) \ \text{ and } \ |w - w_0| < \min\left(1, \frac{\epsilon}{2(1 + |z_0|)}\right) \quad \text{then} \quad |zw - z_0 w_0| < \epsilon.$$

(iii) Let $z_0 \neq 0$ be a non-zero complex number. For all complex numbers z, prove that

$$\text{if} \quad |z - z_0| < \min\left(\frac{|z_0|}{2}, \frac{\epsilon |z_0|^2}{2}\right) \quad \text{then} \quad z \neq 0 \ \text{ and } \ \left|\frac{1}{z} - \frac{1}{z_0}\right| < \epsilon.$$

2) Let a, b, μ, and λ be real numbers > 0, with $\lambda + \mu = 1$. Prove that

$$a^{\lambda} b^{\mu} \leq \lambda a + \mu b.$$

(What in your previous work tells you this inequality is true when, in addition, λ and μ are rational numbers?)