CSCI 2011 HW 5

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1 Chapter 4 Problem 16

Let a be a real number. Use induction to prove that $\sum_{i=0}^{n}(a+i) = \frac{1}{2}(n+1)(2a+n)$ for every nonnegative integer n.
For this problem we will proceed by using the principle of mathematical induction.

BASE CASE: When $n = 0$, $\sum_{i=0}^{n}(a+i) = a$, and $\frac{1}{2}(n+1)(2a+n) = \frac{1}{2} \cdot 1 \cdot (2a) = a$, therefore the claim holds for the base case.

INDUCTIVE STEP: Assume, for some $k \in \mathbb{N}$, that $\sum_{i=0}^{k}(a+i) = \frac{1}{2}(k+1)(2a+k)$, we show that $\sum_{i=0}^{k+1}(a+i) = \frac{1}{2}(k+2)(2a+k+1)$.

$\sum_{i=0}^{k+1}(a+i) = \sum_{i=0}^{k}(a+i) + a + k + 1 = \frac{1}{2}(k+1)(2a+k) + a + k + 1$ by the inductive hypothesis. $\frac{1}{2}(k+1)(2a+k)+a+k+1 = ak+\frac{k^2}{2}+a+\frac{k}{2}+a+k+1 = \frac{k^2}{2}+ak+\frac{3k}{2}+2a+1 = \frac{1}{2}(k^2+2ak+3k+4a+2) = \frac{1}{2}(k+2)(2a+k+1)$. Hence, the claim holds for the inductive step as well.

Therefore, by the principle of mathematical induction, the claim $\sum_{i=0}^{n}(a+i) = \frac{1}{2}(n+1)(2a+n)$ is true for every nonnegative integer n.

2 Chapter 4 Problem 20

A sequence $\{a_n\}$ is defined recursively by $a_1 = 2$, $a_2 = 4$, and $a_n = a_{n-1} + 2a_{n-2}$ for $n \geq 3$. Prove that $a_n = 2^n$ for every positive integer n.
For this problem we will proceed by using the principle of strong mathematical induction.

BASE CASE: We will prove this for $n = 1$ and $n = 2$. $a_1 = 2^1 = 2$ and $a_2 = 2^2 = 4$, therefore the claim holds for the base cases $n = 1$ and $n = 2$.

INDUCTIVE STEP: Suppose, for $1 \leq i \leq k$, where $k \geq 3$, that $a_i = 2^i$. We show that $a_{k+1} = 2^{k+1}$.

By definintion, $a_{k+1} = a_k + 2a_{k-1}$, and by our strong inductive hypothesis, $a_k + 2a_{k-1} = 2^k + 2 \cdot 2^{k-1}$. $2^k + 2 \cdot 2^{k-1} = 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$. Hence, the claim holds for our inductive step as well.

Therefore, by the principle of strong mathematical induction, the claim $a_n = 2^n$ is true for every positive integer n.

3 Chapter 4 Problem 28

Prove for every positive integer n and the Fibonacci numbers $F_1, F_2, ...$ that $F_{n+6} = 4F_{n+3} + F_n$. For this problem we will proceed by using the principle of strong mathematical induction.

BASE CASE: We will prove the claim for the $n = 1$ case. $F_{n+6} = F_7 = 13$. $4F_{n+3} + F_n = 4F_4 + F_1 = 4 \cdot 3 + 1 = 12$, so the claim holds for the base case.

INDUCTIVE STEP: Suppose, for $1 \leq i \leq k$, where $k > 1$, that $F_{i+6} = 4F_{i+3} + F_n$. We show that $F_{k+7} = 4F_{k+4} + F_{k+1}$.

By the definition of Fibonacci Series, $F_{k+7} = F_{k+6} + F_{k+5}$, and by our strong inductive hypothesis... $F_{k+6} + F_{k+5} = 4F_{k+3} + F_k + 4F_{k+2} + F_{k-1} = 4(F_{k+3} + F_{k+2}) + F_k + F_{k-1} = 4F_{k+4} + F_{k+1}$, again, by the definition of Fibonacci Series. Hence, the claim holds for the inductive step.

Therefore, by the principle of strong mathematical induction, the claim $F_{n+6} = 4F_{n+3} + F_n$ is true for all positive integers n.

4 Chapter 5.1 Problem 10

The following are relations on the set $\mathbb{R}$ of real numbers. Which of the properties reflexive, symmetric and transitive does each relation below possess?

(a) **$x\ R_1\ y$ if $|x - y| \leq 1$.**
Reflexive: the relation is reflexive because $|x - x| = 0 \leq 1$.
Symmetric: the relation is symmetric because $|x - y| = |y - x|$.
Transitive: the relation is not transitive. Take for example the relations $1\ R\ 2$ and $2\ R\ 3$. $1\ R\ 2 = |1 - 2| = 1 \leq 1$. $2\ R\ 3 = |2 - 3| = 1 \leq 1$. Now if we take $1\ R\ 3$, we get $|1 - 3| = 2 \not\leq 1$, therefore the relation is NOT transitive.

(b) **$x\ R_2\ y$ if $y \leq 2x + 1$.**
Reflexive: the relation is not reflexive because if we take for example $x = -10$, we get $-10 \leq 2(-10) + 1 = -19$ which isn't true.
Symmetric: the relation is also not symmetric because take for example $x = 10, y = 1$, which gives us $1 \leq 20(10) + 1 = 21$ which is true, but if we switch x and y, we get $10 \leq 2(1) + 1$ which is not true.
Transitive: the relation is also not transitive. Take, for example, $x = 5$, $y = 10$, and $z = 20$. $y = 10 \leq 11 = 2(5) + 1 = 2x + 1$, so this relation is true. $z = 20 \leq 21 = 2(10) + 1 = 2y + 1$, this relation is true as well. But when we relate x to z, we get $z = 20 \leq 2(5) + 1 = 2x + 1$ which is not true.

(c) **$x\ R_3\ y$ if $y = x^2$.**
Reflexive: this relation is not reflexive. Take for example $x = 2$, since $x^2 = 4 \neq 2$ the relation cannot be reflexive.
Symmetric: this relation is not symmetric. Take for example $x = 2$ and $y = 4$, $x^2 = 4 = y$ so $x\ R_3\ y$ is true, but if we look at $y\ R_3\ x$, we get $x = 2 = 16 = y^2$ which is not true.
Transitive: this relation is not transitive. Take for example $x = 2$, $y = 4$, and $z = 16$. $x^2 = 4 = y$, so $x\ R_3\ y$ is true. $y^2 = 16 = z$, so $y\ R_3\ z$ is true. But looking at $x\ R_3\ z$, we get $x^2 = 4 = z$ which is not true.

(d) **$x\ R_4\ y$ if $x^2 + y^2 = 9$.**
Reflexive: the relation is not reflexive. Take for example $x = 2$ and $y = \sqrt{5}$, so $x^2 + y^2 = 4 + 5 = 9$. But if we take just x, we get $x^2 + x^2 = 4 + 4 = 8 \neq 9$, so this relation cannot be reflexive.
Symmetric: The relation is symmetric because addition is commutative ($x^2 + y^2 = y^2 + x^2 = 9$).
Transitive: The relation is not transitive. Take for example $x = 2$, $y = \sqrt{5}$, and $z = 2$. $x^2 + y^2 = 4 + 5 = 9$, and $y^2 + z^2 = 5 + 4 = 9$, but $x^2 + z^2 = 4 + 4 = 8 \neq 9$, so the relation cannot be commutative.

5 Chapter 5.2 Problem 14

A relation R is defined on the set $\mathbb{R}^+$ of positive real numbers by $a\ R\ b$ if the arithmetic mean (the average) of a and b equals the geometric mean of a and b, that is, if $\frac{a+b}{2} = \sqrt{ab}$.

(a) **Prove that R is an equivalence relation.**
Reflexive: a R $a = \frac{a+a}{2} = \frac{2a}{2} = a = \sqrt{a \cdot a}$, this shows the relation is reflexive.
Symmetric: Since addition and multiplication are both commutative properties, this means $\frac{a+b}{2} = \frac{b+a}{2} = \sqrt{ab} = \sqrt{ba}$, therefore the relation is symmetric.
Transitive: For this case, we can assume that it is given that $\frac{a+b}{2} = \sqrt{ab}$ and $\frac{b+c}{2} = \sqrt{bc}$. Now we show that $\frac{a+c}{2} = \sqrt{ac}$...

$$\frac{a+b}{2} = \sqrt{ab} \Rightarrow a+b = 2\sqrt{ab} \Rightarrow (a+b)^2 = (2\sqrt{ab})^2 \Rightarrow a^2+b^2+2ab = 4ab \Rightarrow a^2+b^2-2ab = 0 \Rightarrow (a-b)^2 = 0 \Rightarrow a = b$$

Since a must equal b, the relation b R c is equivalent to the relation a R c, therefore, the relation is transitive.

Since the relation is reflexive, symmetric, and transitive, this means it's an equivalence relation.

(b) **Describe the distinct equivalence classes resulting from R.**
As we proved in part b, in order for the relation to be true, the relation variables a and b must be equivalent. Therefore, the only equivalence class is Class $[a] = \{a\}$, $\forall a \in \mathbb{R}^+$.