MATH 5615H Homework 3

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1) if $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2 + \sqrt{s_n}}$ $(n = 1, 2, 3, \dots)$, prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, \dots$.

2) Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, \ldots$, by the recursion formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right).$$

(a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.

(b) Put $\epsilon_n = x_n - \sqrt{\alpha}$, and show that

$$\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$$

so that setting $\beta = 2\sqrt{\alpha}$,

$$\epsilon_{n+1} < \beta\left(\frac{\epsilon_1}{\beta}\right)^{2^n} \quad (n = 1, 2, 3, \ldots).$$

(c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For example, if $\alpha = 3$ and $x_1 = 2$, show that $\epsilon/\beta < \frac{1}{10}$ and that therefore

$$\epsilon_5 < 4 \cdot 10^{-16}, \quad \epsilon_6 < 4 \cdot 10^{-32}.$$

3) Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define

$$x_{n+1} = \frac{\alpha + x_n}{1 + x_n} = x_n + \frac{\alpha - x_n^2}{1 + x_n}$$

(a) Prove that $x_1 > x_3 > x_5 > \ldots$.

(b) Prove that $x_2 < x_4 < x_6 < \ldots$.

(c) Prove that $\lim x_n = \sqrt{\alpha}$.

(d) Compare the rapidity of convergence of this process with the one described in the previous problem.

4) As is usual, for each real number t we write $[t]$ for the greatest integer $\leq t$, e.g. $[\pi] = 3$ and $[-\pi] = -4$. Observe that, for all real numbers s and t, $((s - [s]) = (t - [t])) \Leftrightarrow (s - t$ is an integer$)$. In particular, $t - [t]$ is the unique real number $x \in [0, 1)$ such that $t - x$ is an integer.

Fix an irrational number $\alpha \in \mathbb{R}$, and define a sequence $\{x_n\}_{n\geq 1}$ by setting, for each integer $n \geq 1$,

$$x_n = n\alpha - [n\alpha].$$

(i) Check that $\{x_n\}_{n\geq 1}$ is a sequence of distinct (that is, $m \neq n \Rightarrow x_m \neq x_n$) irrational numbers in the interval $(0, 1)$.

Let $N \geq 3$ be an integer, and partition the interval $[0, 1]$ into N consecutive subintervals $I_1^N \cup I_2^N \cup \cdots \cup I_N^N$ of length $1/N$ by setting

$$I_j^N = \left[\left(\frac{j-1}{N}\right), \left(\frac{j}{N}\right)\right] \quad \text{for } 1 \leq j \leq N.$$

Of course, the union $I_1^N \cup I_2^N \cup \cdots \cup I_N^N$ is equal to $[0, 1]$, and for distinct indicies $j < k$, the intersection $I_j^N \cap I_k^N$ is empty unless $k = j + 1$, in which case theis intersection consists of the single rational number j/N. Thus, by (i), each term x_n of our sequence lies in a unique subinterval I_j^N. Indeed, x_n lies in the interior $\left(\frac{j-1}{N}, \frac{j}{N}\right)$ of that interval.

(ii) Prove that at least one of the first $N - 1$ terms $x_1, x_2, \ldots, x_{N-1}$ of our sequence lies either in the first subinterval $I_1^N = \left[0, \frac{1}{N}\right]$ or in the last subinterval $I_1^N = \left[1 - \frac{1}{N}, 1\right]$. (If not, why must some subinterval I_l^N with $2 \leq l \leq N - 1$ contain two distinct terms x_j and x_k with $1 \leq j < k < N - 1$? If this is true, what then can you say about $x_{(k-j)}$?)

(iii) Suppose that $p \geq 1$ is an index such that $x_p \in I_1^N$, and consider the subsequence $\{x_{l_p}\}_{l\geq 1}$ of our original sequence. Prove the following statements about it's behavior: Suppose that the term x_{l_p} lies in the jth subinterval I_j^N. If $j < N$, then the nexy term $x_{(l+1)_p}$ lies either again in I_j^N or the next subinterval I_{j+1}^N. If $j = N$, then $x_{(l+1)_p}$ lies either again in I_N^N or in the first subinterval I_1^N. Moreover, at most $1+[1/(Nx_p)]$ consecutive terms of the subsequence $\{x_{l_p}\}_{l\geq 1}$, can lie in any one subinterval I_k^N. (First observe and explain why $x_{(l+1)_p} = x_{l_p} + x_p - [x_{l_p} + x_p]$.)

(iv) Suppose that $p \geq 1$ is an index such that $x_p \in I_N^N$, and again consider the subsequence $\{x_{l_p}\}_{l\geq 1}$. Prove the following statements: Suppose that the term x_{l_p} lies in the jth subinterval I_j^N. If $j > 1$, then the next term $x_{(l+1)_p}$ lies either again in I_j^N or in the previous subinterval I_{j-1}^N. If $j = 1$, then $x_{(l+1)_p}$ lies either again in I_1^N or in the last subinterval I_N^N. At most $1 + [1/(N(1 - x_p))]$ consecutive terms of the subsequence $\{x_{l_p}\}_{l\geq 1}$ can lie in one subinterval I_k^N. (First observe and explain why $x_{(l+1)_p} = x_{l_p} - (1 - x_p) - [x_{l_p} - (1 - x_p)]$.)

(v) Let $a < b$ be distinct real numbers in $[0, 1]$. Prove that there are infinitely many indicies n for which

$$a < x_n < b.$$

(vi) Prove that there are infinitely many positive integers n such that the first seven digits (counting from the left) of the usual (base 10) expression of 2^n are $7777777\ldots$. (Why is $\log_{10}(2)$ an irrational number? Why does $n \log_{10}(2) - [n \log_{10}(2)]$ specify the digits of 2^n?)

(vii) Also deduce from (ii) the following statement: Let α be an irrational number. Then there exists a sequence $\left\{\frac{p_l}{q_l}\right\}_{l\geq 1}$ of rational numbers ($p_l \in \mathbb{Z}, q_l \in \mathbb{N}^+$) with strictly increasing denominators

$$1 \leq q_1 < q_2 < \ldots < q_l < q_{l+1} < \ldots$$

such that

$$0 < \left|\alpha - \frac{p_l}{q_l}\right| < \frac{1}{q_l(q_l+1)} \quad \text{for all } l \geq 1.$$

5) As is usual, for all strictly positive real exponents $\lambda > 0$, we set $0^\lambda = 0$.

Let α be a real number with $0 < \alpha < 1$. Let $\{x_n\}_{n\geq 0}$ be a sequence of real numbers, all ≥ 0, which converges to a real number $L \geq 0$. Prove that

$$\lim_{n\to\infty} (x_n)^\alpha = L^\alpha.$$

(You may want to treat separately the cases $L = 0$ and $L > 0$.)