CSCI 2011 HW 9

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1 Chapter 7.3 Problem 18

Let $n \in \mathbb{Z}$. Prove that $3 \mid (2n^2 + 1)$ if and only if $3 \nmid n$.

First, we show that if $3 \mid (2n^2 + 1)$, then $3 \nmid n$. We will use the contrapositive for this proof. We show that $3 \mid n \Rightarrow 3 \nmid (2n^2 + 1)$. By definition, $n = 3s$ for some $s \in \mathbb{Z}$. Therefore...
$2n^2 + 1 = 2(3s)^2 + 1 = 18s^2 + 1 = 3(6s^2) + 1$, and since $6s^2$ is an integer, $3 \nmid (2n^2 + 1)$.

Next, we show that if $3 \nmid n$, then $3 \mid (2n^2 + 1)$. We can break this up into two separate cases, either $n = 3t + 1$ or $n = 3t + 2$ for some $t \in \mathbb{Z}$. Let's look at the first case, where $n = 3t + 1$...
$2n^2 + 1 = 2(3t + 1)^2 + 1 = 18t^2 + 12t + 3 = 3(6t^2 + 4t + 1)$, and since $6t^2 + 4t + 1$ is an integer, $3 \mid (2n^2 + 1)$.

Now we look at the case where $n = 3t + 2$...
$2n^2 + 1 = 2(3t + 2)^2 + 1 = 18t^2 + 24t + 9 = 3(6t^2 + 8t + 3)$, and since $6t^2 + 8t + 3$ is an integer, $3 \mid (2n^2 + 1)$.
Therefore the biconditional must be true.

2 Chapter 7.4 Problem 14

Let $a \in \mathbb{Z}$. Prove that if $a^2 \not\equiv a \pmod 3$, then $a \not\equiv 0 \pmod 3$ and $a \not\equiv 1 \pmod 3$.

Let's look at the contrapositive. If $a \equiv 0 \pmod 3$ or $a \equiv 1 \pmod 3$, then $a^2 \equiv a \pmod 3$. Now all we must do is prove that the implication is true for either $a \equiv 0 \pmod 3$ or $a \equiv 1 \pmod 3$, for this instance, I'll choose $a \equiv 0 \pmod 3$. We show that $0^2 \equiv 0 \pmod 3$. Therefore $3 \mid (0^2 - 0)$ by definition, and since $0^2 - 0 = 0$ and $3 \mid 0$ we know this statement must be true.

3 Chapter 7.6 Problem 6

Let a and b be integers not both 0, and let $d = gcd(a, b)$. Prove that if $a = da'$ and $b = db'$ for some integers a' and b', then $gcd(a', b') = 1$.

By definition, $d = sa + tb$ for some integers $s, t \in \mathbb{Z}$. And since $a = da'$ and $b = db'$, it must be the case that $d = sda' + tdb'$, factoring out d, we get $sa' + tb' = 1$. By Theorem 7.47 of the textbook, we know that $gcd(a', b')$ is the smallest positive integer that's a linear combination of a' and b', and since 1 is a linear combination of a' and b', it must follow that $gcd(a', b') = 1$, because 1 is the smallest positive integer in the set of all integers.

4 Chapter 7 Problem 12

Let x and y be integers such that $x + y \equiv 0 \pmod 3$. Prove that if $a, b \in \mathbb{Z}$ such that $a \equiv b \pmod 3$, then $ax + by \equiv 0 \pmod 3$.

By definition, $x + y \equiv 0 \pmod 3$ implies that $x + y = 3q$ for some $q \in \mathbb{Z}$. Also $a = 3r + b$ for some $r \in \mathbb{Z}$.

Since $x + y = 3q$, we know $x = 3q - y$, now solving for $ax + by$, we have...
$ax + by = (3r + b)(3q - y) + by = 9rq - 3ry + 3bq - by + by = 9rq - 3ry + 3bq = 3(3rq - ry + bq)$, and since $3rq - ry + bq$ is an integer, it follows that $3 \mid (ax + by)$, and therefore $ax + by \equiv 0 \pmod 3$ by definition.

5 Chapter 8.1 Problem 14

A man leaves for work on a rainy morning. He has a choice of three raincoats, four umbrellas and two hats. Assuming that he must take a coat and an umbrella (but not necessarily a hat), how many possibilities for raingear does he have?

If he chooses not to bring a hat, then he has $3 \cdot 4 = 12$ options, and if he does choose to bring a hat, then he has $3 \cdot 4 \cdot 2 = 24$ options. All together, he has $12 + 24 = 36$ options.