MATH 5615H Homework 1

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1) Let $x < y$ be distinct real numbers. Prove that there exists an irrational number α with $x < \alpha < y$. (No cardinality argument allowed!)

2) Sketch the subsets of the (x, y) plane $\mathbb{R}^2$ specified by each of the following inequalities. Explain your reasoning clearly.

(i) $x^2 + y^2 - 5 \leq 4x$

(ii) $|x^2 + y^2 - 5| \leq 4x$

(iii) $x^2 + y^2 - 5 \leq |4x|$

(iv) $|x^2 + y^2 - 5| \leq |4x|$

3) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function such that, for all $x, y \in \mathbb{R}$, we have $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x) \cdot f(y)$. Prove that either:

(i) For all $x \in \mathbb{R}$, $f(x) = 0$; or

(ii) For all $x \in \mathbb{R}$, $f(x) = x$

(At some point in your argument, the fact that every positive real number has a real square root will be essential.)

4) For each finite list $x_1, x_2, \ldots, x_N$ of strictly positive real numbers, we set

$$A_N(x_1, x_2, \ldots, x_N) = \frac{1}{N}(x_1 + x_2 + \cdots + x_N) \quad \text{the "arithmetic mean" and}$$
$$G_N(x_1, x_2, \ldots, x_N) = \frac{1}{N}(x_1 \cdot x_2 \cdot \ldots \cdot x_N)^{\frac{1}{N}} \quad \text{the "geometric mean".}$$

It is always true that $G_N(x_1, x_2, \ldots, x_N) \leq A_N(x_1, x_2, \ldots, x_N)$ and equality holds if and only if $x_1 = x_2 = \ldots = x_N$. Two proofs of this fact are developed below.

(i) *Proof.* For each list $x_1, x_2, \ldots, x_N$, we let $d(x_1, x_2, \ldots, x_N)$ be the number of indicies l for which $x_l \neq A_N(x_1, x_2, \ldots, x_N)$. If $d(x_1, x_2, \ldots, x_N) > 0$, then there must be two indicies i and j such that $x_i < A_N(x_1, x_2, \ldots, x_N) < x_j$. Why? If we select two such indicies i and j, and form a new list $x'_1, x'_2, \ldots, x'_N$ by setting $x'_l = x_l$ for $l \neq i, j$ and $x'_i = A_N(x_1, x_2, \ldots, x_N)$ and $x'_j = x_i + x_j - A_N(x_1, x_2, \ldots, x_N)$ then...

(a) $x'_1, x'_2, \ldots, x'_N$ is again a list of strictly positive real numbers.

(b) For the two indicies i and j we chose, $x'_i + x'_j = x_i + x_j$ and $x'_i x'_j > x_i x_j$.

Therefore,

$$A_N(x'_1, x'_2, \ldots, x'_N) = A_N(x_1, x_2, \ldots, x_N)$$
$$G_N(x'_1, x'_2, \ldots, x'_N) > G_N(x_1, x_2, \ldots, x_N)$$
$$d_N(x'_1, x'_2, \ldots, x'_N) < d_N(x_1, x_2, \ldots, x_N).$$

$\square$

Explain why (a) and (b) are true, then use this to fashion one proof, using "complete induction" on the size of $d(x_1, x_2, \ldots, x_N)$.

(ii) *Proof.* A second proof establishes our result first for lists whose length is a power of 2, and then deduces the general case.

(a) Check directly that if x_1 and x_2 are positive, then $\frac{1}{2}(x_1 + x_2) \geq \sqrt{x_1 x_2}$, and that equality holds if and only if $x_1 = x_2$. This is our result for lists of length 2.

(b) Now let $k \geq 1$ be an integer. For each list $x_1, x_2, \ldots, x_{2^{k+1}}$, check that

$$A_{2^{k+1}}(x_1, x_2, \ldots, x_{2^{k+1}}) = A_2(A_{2^k}(x_1, \ldots, x_{2^k}), A_{2^k}(x_{2^k+1}, \ldots, x_{2^{k+1}})) \quad \text{and}$$
$$G_{2^{k+1}}(x_1, x_2, \ldots, x_{2^{k+1}}) = G_2(G_{2^k}(x_1, \ldots, x_{2^k}), G_{2^k}(x_{2^k+1}, \ldots, x_{2^{k+1}}))$$

Use this and (a) to prove, by induction on l, that for all $l \geq 1$, and lists $x_1, x_2, \ldots, x_{2^l}$

$$A_{2^l}(x_1, x_2, \ldots, x_{2^l}) \geq G_{2^l}(x_1, x_2, \ldots, x_{2^l})$$

with equality if and only if $x_1 = x_2 = \ldots = x_{2^l}$.

(c) Let $x_1, x_2, \ldots, x_N$ be a list of arbitrary positive length N, and select a positive integer l such that $2^l > N$. By considering the list $x'_1, x'_2, \ldots, x'_{2^l}$ of length 2^l formed by setting $x'_j = x_j$ for $1 \leq j \leq N$ and $x'_j = A_N(x_1, \ldots, x_N)$ for $N + 1 \leq j \leq 2^l$, deduce the general result.

$\square$