CSCI 2011 HW 8

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1 6.2 Problem 8

For the function f defined by $f(n) = \frac{n^2+1}{n+1}$ for each $n \in \mathbb{N}$, show that $f(n) = O(n)$.

for $n \geq 1$, $f(n) = \frac{n^2+1}{n+1} \leq \frac{n^2+n}{n+1} < \frac{n^2+n}{n} = n+1 \leq n+n = 2n$. Therefore $f(n) < 2n$ for $n \geq 1$, and so $f(n) = O(n)$.

2 Chapter 6 Problem 12

Let $f : \mathbb{N} \to \mathbb{R}^+$ and $g : \mathbb{N} \to \mathbb{R}^+$ be defined by $f(n) = 2n^3 + n + 10$ and $g(n) = n^3 + 4n^2 + 1$ for $n \in \mathbb{N}$. Show that $f = \Theta(g)$.

First we show that $f = O(g)$.
for $n \geq 1$, $f(n) = 2n^3 + n + 10 \leq 2n^3 + n^2 + 10 < 10n^3 + 40n^2 + 10 = 10(n^3 + 4n^2 + 1)$. Therefore $f(n) < 10 \cdot g(n)$ for $n \geq 1$, and so $f = O(g)$.

Now we show that $f = \Omega(g)$ or, put more simply, $g = O(f)$.
for $n \geq 1$, $g(n) = n^3 + 4n^2 + 1 \leq n^3 + 4n^3 + 1 = 5n^3 + 1 < 10n^3 + 50 < 10n^3 + 5n + 50 = 5(2n^3 + n + 10)$. Therefore $g(n) < 5 \cdot f(n)$ for $n \geq 1$, and so $g = O(f)$.

Since $f = O(g)$ and $g = O(f)$, it must be the case that $f = \Theta(g)$

3 6.2 Problem 14

Let $f : \mathbb{N} \to \mathbb{R}^+$, $g : \mathbb{N} \to \mathbb{R}^+$ and $h : \mathbb{N} \to \mathbb{R}^+$ be three functions. Prove that if $f = \Theta(g)$ and $g = \Theta(h)$, then $f = \Theta(h)$.

For $n, k \in \mathbb{Z}$, $n \geq k$, $f = \Theta(g)$, so there must exist $c_1, c_2 \in \mathbb{N}$, such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$.
For $n, k \in \mathbb{Z}$, $n \geq k$, $g = \Theta(h)$, so there must exist $d_1, d_2 \in \mathbb{N}$, such that $d_1 h(n) \leq g(n) \leq d_2 h(n)$.

4 7.1 Problem 16

Prove that $4 \mid (3^{2n-1} + 1)$ for every positive integer n.

5 7.2 Problem 8

Prove that every prime except one has the form $a^2 - b^2$ for some positive integers a and b.