CSCI 2011 HW 8

Fletcher Gornick

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1 6.2 Problem 8

For the function f defined by $f(n) = \frac{n^2+1}{n+1}$ for each $n \in \mathbb{N}$, show that $f(n) = O(n)$.

for $n \geq 1$, $f(n) = \frac{n^2+1}{n+1} \leq \frac{n^2+n}{n+1} < \frac{n^2+n}{n} = n+1 \leq n+n = 2n$. Therefore $f(n) < 2n$ for $n \geq 1$, and so $f(n) = O(n)$.

2 Chapter 6 Problem 12

Let $f : \mathbb{N} \to \mathbb{R}^+$ and $g : \mathbb{N} \to \mathbb{R}^+$ be defined by $f(n) = 2n^3 + n + 10$ and $g(n) = n^3 + 4n^2 + 1$ for $n \in \mathbb{N}$. Show that $f = \Theta(g)$.

First we show that $f = O(g)$.
for $n \geq 1$, $f(n) = 2n^3 + n + 10 \leq 2n^3 + n^2 + 10 < 10n^3 + 40n^2 + 10 = 10(n^3 + 4n^2 + 1)$. Therefore $f(n) < 10 \cdot g(n)$ for $n \geq 1$, and so $f = O(g)$.

Now we show that $f = \Omega(g)$ or, put more simply, $g = O(f)$.
for $n \geq 1$, $g(n) = n^3 + 4n^2 + 1 \leq n^3 + 4n^3 + 1 = 5n^3 + 1 < 10n^3 + 50 < 10n^3 + 5n + 50 = 5(2n^3 + n + 10)$. Therefore $g(n) < 5 \cdot f(n)$ for $n \geq 1$, and so $g = O(f)$.

Since $f = O(g)$ and $g = O(f)$, it must be the case that $f = \Theta(g)$

3 6.2 Problem 14

Let $f : \mathbb{N} \to \mathbb{R}^+$, $g : \mathbb{N} \to \mathbb{R}^+$ and $h : \mathbb{N} \to \mathbb{R}^+$ be three functions. Prove that if $f = \Theta(g)$ and $g = \Theta(h)$, then $f = \Theta(h)$.

For $n, k \in \mathbb{Z}$, $n \geq k$, $f = \Theta(g)$, so there must exist $c_1, c_2 \in \mathbb{N}$, such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$.

For $n, k \in \mathbb{Z}$, $n \geq k$, $g = \Theta(h)$, so there must exist $d_1, d_2 \in \mathbb{N}$, such that $d_1 h(n) \leq g(n) \leq d_2 h(n)$.

Since $d_1 h(n) \leq g(n)$, it follows that $c_1 d_1 h(n) \leq c_1 g(n)$. We can let $e_1 = c_1 d_1$.
Also, since $d_2 h(n) \geq g(n)$, it follows that $c_2 d_2 h(n) \geq c_2 g(n)$. We can let $e_2 = c_2 d_2$.

$e_1 h(n) \leq c_1 g(n) \leq f(n) \leq c_2 g(n) \leq e_2 h(n)$ implies that $e_1 h(n) \leq f(n) \leq e_2 h(n)$. Since e_1, e_2 are constants, it follows that $f = \Theta(h)$.

4 7.1 Problem 16

Prove that $4 \mid (3^{2n-1}+1)$ for every positive integer n.
We will proceed using an Inductive proof.

BASE CASE: First we show that $4 \mid (3^{2n-1}+1)$ for the base case, where $n=1$. So $3^{2n-1}+1 = 4q$ for some $q \in \mathbb{Z}$. $3^{2n-1}+1 = 3^{2-1}+1 = 3+1 = 4 = 4(1)$, therefore the claim holds for the base case.

INDUCTIVE STEP: Suppose $4 \mid (3^{2k-1}+1)$ is true for some $k \geq 2$, we show that $4 \mid (3^{2k+1}+1)$. Also keep in mind that $4 \mid (3^{2k-1}+1)$ implies that $3^{2k-1}+1 = 4q$ for some $q \in \mathbb{Z}$.

$$3^{2k+1}+1 = 9 \cdot 3^{2k-1}+1 = 9 \cdot 3^{2k-1}+9-8 = 9(3^{2k-1}+1)-8 = 9(4q)-8 = 4(9q-2)$$

Keep in mind that the second to last step was from our inductive hypothesis. Since $9q-2$ is an integer, the claim holds for our inductive step as well.

By the principle of mathematical induction, the claim $4 \mid (3^{2n-1}+1)$ holds for all positive integers n.

5 7.2 Problem 8

Prove that every prime except one has the form a^2-b^2 for some positive integers a and b.

First, we can express a^2-b^2 as $(a+b)(a-b)$, so in order to yield a prime, we can assume that $|a| > |b|$ (otherwise $a-b \leq 0$, which cannot yield a prime).

The two smallest positive integers that fit our rule is $a=2$ and $b=1$ because they must both be positive, and cannot be equal. $(2+1)(2-1) = 3 > 2$. Since the smallest number that can be represented by this equation is 3, it must follow that 2 is the one prime that cannot be expressed by a^2-b^2.

Now since we are looking for a prime outcome, and primes must be a multiple of themself and 1, and there are no two positive integers that add up to 1, it must be the case that $a-b=1$ and $a+b$ is equal to the prime number. Since $a-b=1$, it follows that $a=b+1$, so prime number $p=2b+1$, so $b=\frac{p-1}{2}$.

Since p is prime and $p \neq 2$, p must be odd, so $p-1$ is even and $\frac{p-1}{2}$ must be an integer. Also, $b=a-1$, so $2a-1=p \Rightarrow a=\frac{p+1}{2}$. Again since all primes are odd except two, which is the exception, $\frac{p+1}{2}$ must be an integer.

Since a and b are integers if p is a prime (except 2), and $p=a+b$, it follows that p can be expressed as a^2-b^2.