CSCI 2011 HW 3

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1 3.1 Problem 14

Consider the following quantified statement: For every even integer a and every odd integer b, $a+b$ is odd.

(a) **Express this quantified statement in symbols.**

Let S be the set of all even integers and let T be the set of all odd integers.

$\forall a \in S, \forall b \in T : a + b \in T$

(b) **Express the negation of this quantified statement in symbols.**

$\exists a \in S, \exists b \in T : a + b \notin T$

(c) **Express the negation of this quantified statement in words.**

There exists an even integer a, and an odd integer b, such that $a+b$ is not odd.

2 3.1 Problem 18

State the negation of the quantified statement below.
For every integer a, there exists an integer b such that $|\frac{a+1}{2} - b| \leq 1$.

There exists an integer a such that for every integer b, $|\frac{a+1}{2} - b| \nleq 1$.

3 3.2 Problem 16

Prove that if a and b are positive integers, then $\frac{a}{b} + \frac{b}{a} \geq 2$.
Assume that $\frac{a}{b} + \frac{b}{a} \geq 2$, it follows that...

$$\begin{aligned}
\frac{a}{b} + \frac{b}{a} \geq 2 &\Rightarrow \frac{a^2+b^2}{ab} \geq 2 \\
&\Rightarrow a^2 + b^2 \geq 2ab \\
&\Rightarrow a^2 - 2ab + b^2 \geq 0 \\
&\Rightarrow (a-b)^2 \geq 0
\end{aligned}$$

We know that $(a-b)^2 \geq 0$, because any number squared is greater than or equal to zero (even negative). Therefore it must be the case that $\frac{a}{b} + \frac{b}{a} \geq 2$.

4 3.2 Problem 18

Prove the following:

(a) **If a and b are even integers, then $a + b$ is even.**

Let $a = 2k$, and $b = 2l$, for $k, l \in \mathbb{Z}$ (definition of an even number), it follows that $a + b = 2k + 2l = 2(k + l)$. Since $k + l$ is an integer, $2(k + l)$ must be an even integer (by definition again). Therefore $a + b$ is even.

(b) **If c and d are even integers, do we know that $c + d$ is even?**

Yes, because the previous proof applies to all even integers, not just a and b.

(c) **For integers x and y, if we know $x + y$ is even, do we know that x and y are even?**

No. For example, say $x = 1$, and $y = 3$. $1 + 3 = 4$, which is even, but both x and y are odd.

(d) **If a and b are integers that are not both even, do we know that $a + b$ is not even?**

No. Just like in the previous example, say $a = 1$ and $b = 3$. a and b are integers that are not both even in this case, but their sum is 4, which is even.

5 3.3 Problem 8

Give a proof of

Let $n \in \mathbb{Z}$. Then $n - 3$ Is even if and only if $n + 4$ is odd.

using

(a) **two direct proofs.**

- Assume $n + 4$ is odd. This would mean that for some integer k, $n + 4 = 2k + 1$. It follows that $n - 3 = 2k - 6 = 2(k - 3)$. Since $k - 3$ is an integer, and $n - 3 = 2(k - 3)$, $n - 3$ must be even.

- Now let's assume $n - 3$ is even. This means that for some integer k, $n - 3 = 2k$. It follows that $n + 4 = 2k + 7 = 2(k + 3) + 1$. Since $k + 3$ is an integer, and $n + 4 = 2(k + 3) + 1$, $n + 4$ must be odd.

(b) **one direct proof and one proof by contrapositive**

- DIRECT PROOF: Assume $n+4$ is odd. This would mean that for some integer k, $n+4 = 2k+1$. It follows that $n - 3 = 2k - 6 = 2(k - 3)$. Since $k - 3$ is an integer, and $n - 3 = 2(k - 3)$, $n - 3$ must be even.

- PROOF BY CONTRAPOSITIVE: Now let's assume that $n + 4$ is not odd, or simply, $n + 4$ is even. This means that there exists an integer k such that $n + 4 = 2k$. It follows that $n - 3 = 2k - 7 = 2(k - 4) + 1$. Since $k - 4$ is an integer, and $n - 3 = 2(k - 4) + 1$, $n - 3$ must be odd.

(c) **two proofs by contrapositive.**

- Assume that $n + 4$ is not odd, or simply, $n + 4$ is even. This means that there exists an integer k such that $n + 4 = 2k$. It follows that $n - 3 = 2k - 7 = 2(k - 4) + 1$. Since $k - 4$ is an integer, and $n - 3 = 2(k - 4) + 1$, $n - 3$ must be odd.

- Now let's assume that $n - 3$ is odd. This means that there exists an integer k such that $n - 3 = 2k + 1$. It follows that $n + 4 = 2k + 8 = 2(k + 4)$. Since $k + 4$ is an integer, and $n + 4 = 2(k + 4)$, $n + 4$ must be even.

6 3.3 Problem 12

Let a, b and m be integers. Prove that if $2a + 3b \geq 12m + 1$, then $a \geq 3m + 1$ or $b \geq 2m + 1$.

We will proceed by examining the contrapositive...

Let a, b and m be integers. Prove that if $a < 3m + 1$ and $b < 2m + 1$, then $2a + 3b < 12m + 1$.

Since a, b, and m are integers, the statement can be rewritten as..

If $a \leq 3m$ and $b \leq 2m$, then $2a + 3b < 12m + 1$.

Therefore $2a + 3b \leq 2(3m) + 3(2m) \leq 12m < 12m + 1$.

Thus, either $a \geq 3m + 1$ or $b \geq 2m + 1$, if $2a + 3b \geq 12m + 1$.