CSCI 2011 HW 5

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1 4.1 Problem 10

Let $r \geq 2$ be an integer. Prove that $1 + r + r^2 + ... + r^n = \frac{r^{n+1}-1}{r-1}$ for every positive integer n.

Base Case: Since this claim must hold true for every positive integer n, we can use $n = 1$ as our base case. Therefore $1 + \cdots r^n = 1 + r^1 = 1 + r$. Since $\frac{r^{n+1}-1}{r-1} = \frac{r^2-1}{r-1} = \frac{(r+1)(r-1)}{r-1} = r + 1$, our base case holds.

Inductive Step: Now let's assume for some $k \geq 1$, that $1 + r + r^2 + ... + r^k = \frac{r^{k+1}-1}{r-1}$. We show that $1 + r + r^2 + ... + r^k + r^{k+1} = \frac{r^{k+2}-1}{r-1}$.

$$\begin{aligned}
1 + r + r^2 + ... + r^k + r^{k+1} &= \frac{r^{k+1}-1}{r-1} + r^{k+1} && \text{(by the inductive hypothesis)} \\
&= \frac{r^{k+1} - 1 + r^{k+1}(r-1)}{r-1} \\
&= \frac{r^{k+1}(1 + r - 1)}{r-1} - \frac{1}{r-1} \\
&= \frac{r \cdot r^{k+1} - 1}{r-1} \\
&= \frac{r^{k+2}-1}{r-1}
\end{aligned}$$

Therefore the claim holds for the inductive step as well. Hence, by the principle of mathematical induction, the claim $1 + r + r^2 + ... + r^n = \frac{r^{n+1}-1}{r-1}$ is true for all integers $n \geq 1$.

2 4.2 Problem 14

Prove that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n+1}$ for ever integer $n \geq 3$.

Base Case: Let $n = 3$, $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} \approx 2.284 > 2 = \sqrt{3+1}$. Therefore, our base case holds.

Inductive Step: Assume for some integer $k \geq 3$, that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k+1}$. We show that $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+2}$. By the inductive hypothesis, we know that...
$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1} + \frac{1}{\sqrt{k+1}} = \frac{k+1+1}{\sqrt{k+1}} = \frac{k+2}{\sqrt{k+1}} > \frac{k+2}{\sqrt{k+2}} = \sqrt{k+2}$.

Therefore, by the principle of mathematical induction, the claim $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n+1}$ holds for all integers $n \geq 3$.

3 4.2 Problem 16

Prove for every positive integer n that $2! \cdot 4! \cdot 6! \cdots (2n)! \geq ((n+1)!)^n$.

Base Case: Since n can be any positive integer, we can look at when $n = 1$, so $(2 \cdot 1)! = 2$, and $((n+1)!)^n = ((1+1)!)^1 = 2$. So the base case holds.

Inductive Step: Now we can assume for some integer $k \geq 1$, that $2! \cdot 4! \cdot 6! \cdots (2k)! \geq ((k+1)!)^k$. We show that $2! \cdot 4! \cdot 6! \cdots (2k)! \cdot (2k+2)! \geq ((k+2)!)^{k+1}$

$$\begin{aligned}
2! \cdot 4! \cdot 6! \cdots (2k)! \cdot (2k+2)! &\geq ((k+1)!)^k (2k+2)! && \text{(by the inductive hypothesis)} \\
&\geq ((k+1)!)^k (k+1)!(k+2)^{k+1} && \left(\frac{(2k+2)!}{(k+1)!} > (k+2)^{k+1}\right) \\
&= ((k+1)!)^{k+1}(k+2)^{k+1} \\
&= ((k+2)(k+1)!)^{k+1} \\
&= ((k+2)!)^{k+1}
\end{aligned}$$

Therefore, the claim holds for the inductive step. Hence, by the principle of mathematical induction, the claim $2! \cdot 4! \cdot 6! \cdots (2n)! \geq ((n+1)!)^n$ holds for all integers $n \geq 1$.

4 4.3 Problem 14

A sequence $a_1, a_2, a_3 ...$ is defined recursively by $a_1 = 3$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

(a) **Determine a_2, a_3, a_4 and a_5.**

$a_2 = 2a_1 + 1 = 2 \cdot 3 + 1 = 7$.
$a_3 = 2a_2 + 1 = 2 \cdot 7 + 1 = 15$.
$a_4 = 2a_3 + 1 = 2 \cdot 15 + 1 = 31$.
$a_5 = 2a_4 + 1 = 2 \cdot 31 + 1 = 63$.

(b) **Based on the variables obtained in (a), make a guess for a formula for a_n for every positive integer n and use induction to verify that your guess is correct.**

My best guess for the formula for this equation is $a_n = 2^{n+1} - 1, \forall n \in \mathbb{N}$.

Base Case: First, we must prove for the $n = 1$ case. We know that $a_1 = 3$ from above, and $2^{n+1} - 1 = 2^{1+1} - 1 = 3$, so the base case holds.

Inductive Step: Let's assume for some integer $k \geq 2$, that $a_k = 2a_{k-1} + 1 = 2^{k+1} - 1$. We show that $a_{k+1} = 2a_k + 1 = 2^{k+2} - 1$.

By the inductive hypothesis, we know that $a_{k+1} = 2a_k + 1 = 2(2^{k+1} - 1) + 1$, so $2(2^{k+1} - 1) + 1 = 2 \cdot 2^{k+1} - 2 + 1 = 2^{k+2} - 1$, therefore the claim holds for the inductive step as well.

By the principle of mathematical induction, the claim $a_n = 2a_{n-1} + 1 = 2^{n+1} - 1$ holds for all integers $n \geq 2$.

5 4.3 Problem 22

Use induction to show the following for Fibonacci numbers: $F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$ **for every positive integer** n**.**

Base Case: Assume $n = 1$, we know $F_{2\cdot 1} = 1$. Also $F_{2\cdot 1+1} - 1 = F_3 - 1 = 1$, therefore the claim holds for the base case.

Inductive Step: Now let's assume for some integer $k \geq 1$, that $F_2 + F_4 + \cdots + F_{2k} = F_{2k+1} - 1$. We show that $F_2 + F_4 + \cdots + F_{2k} + F_{2k+2} = F_{2k+3} - 1$

$$\begin{aligned}
F_2 + F_4 + \cdots + F_{2k} + F_{2k+2} &= F_{2k+1} - 1 + F_{2k+2} && \text{(by the inductive hypothesis)} \\
&= F_{2k+1} + F_{2k+2} - 1 \\
&= F_{2k+3} - 1 && \text{(by the definintion of the Fibonacci Sequence)}
\end{aligned}$$

Therefore the claim holds for the inductive step. Hence, by the principle of mathematical induction, the claim $F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$ holds for all intergers $n \geq 1$.