CSCI 2011 HW 5

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1 Chapter 5.3 Problem 28

Let $A = \{1, 2, 3\}$, $B = \{1, 2, 3, 4, 5\}$ and $C = \{1, 2, 3, 4\}$. Also let $f : A \to B$ and $g : B \to C$, where $f = \{(1, 4), (2, 5), (3, 1)\}$ and $g = \{(1, 3), (2, 3), (3, 2), (4, 4), (5, 1)\}$,

(a) **Determine $(g \circ f)(1)$, $(g \circ f)(2)$ and $(g \circ f)(3)$.**
$(g \circ f)(1) = 4$ because $f(1) = 4$ and $g(4) = 4$.
$(g \circ f)(2) = 1$ because $f(2) = 5$ and $g(5) = 1$.
$(g \circ f)(3) = 3$ because $f(3) = 1$ and $g(1) = 3$.

(b) **Determine $g \circ f$.**
Since in the previous example, we found all possible values of $g \circ f$, we know $g \circ f = \{(1, 4), (2, 1), (3, 3)\}$.

2 Chapter 5.4 Problem 24

Prove or disprove each of the following.

(a) **There exists functions $f : A \to B$ and $g : B \to C$ such that f is not one-to-one and $g \circ f : A \to C$ is one-to-one.**

Suppose $g \circ f$ is injective, and we want to show that f is not. There must exist elemtents a and b such that $a \neq b$ but $f(a) = f(b)$ in order for f to not be injective. Therefore $g(f(a)) = g(f(b))$ because $f(a) = f(b)$. Since $g(f(x)) = (g \circ f)(x)$, this means that $(g \circ f)(a) = (g \circ f)(b)$, meaning that $g \circ f$ is not injective, which contradicts our supposition. Therefore, by contradictive proof, if f is not one-to-one, it cannot be the case that $g \circ f$ is.

(b) **There exists functions $f : A \to B$ and $g : B \to C$ such that f is not onto and $g \circ f : A \to C$ is onto.**

Suppose $A = \{a\}$, $B = \{b, c\}$ and $C = \{d\}$. We can also assume $f(a) = b$ and $g(b) = d$. Therefore f is not onto, because you cannot link element c in set B to any element in set A through f. But we do know that $g \circ f$ is onto, because $(g \circ f)(a) = g(f(a)) = g(b) = d$, and there's only one element in C that links to $a \in A$. Therefore, by proof of existence, there exists functions f and g, such that $g \circ f$ is onto but f is not.

3 Chapter 5.5 Problem 12

Prove or disprove: The set $S = \{(a, b) : a, b \in \mathbb{R}\}$ of all points in the plane is uncountable.

we can take a subset of S by making b constant and leaving a as an element in $\mathbb{R}$. So we have a set A such that $A \subseteq S$, and $A = \{(a, 0) : a \in \mathbb{R}\}$. We can now create a bijective function $f : \mathbb{R} \to A$, where $f(x) = (x, 0), \forall x \in \mathbb{R}$. We know this function is bijective because for every distinct value of x, we have a distinct $f(x)$ (therefore it's onto), and we know that every value in the co-domain of f can be mapped to it's domain ($(x, 0) \to x$, so it's onto as well). Since A has the same cardinality of $\mathbb{R}$, and the set of real numbers is uncountable, we know that S is uncountable because $|A| = |\mathbb{R}|$ and $A \subseteq S$.

4 Chapter 5 Problem 32

Prove that the function $f : \mathbb{R} - \{3\} \to \mathbb{R} - \{1\}$ defined by $f(x) = \frac{x}{x-3}$ is bijective.

First, we must show the function is one-to-one.
Suppose there exists two numbers $a, b \in \mathbb{R}$ and $a, b \neq 3$, such that $f(a) = f(b)$, therefore $\frac{a}{a-3} = \frac{b}{b-3}$, which means $ab - 3a = ab - 3b \Rightarrow 3a = 3b \Rightarrow a = b$. This means that f is one-to-one.

Next, we show the function is onto.
Suppose $y = f(x)$ and $y \neq 1$ as stated in the definition. therefore y $= \frac{x}{x-3}$, so we can manipulate this equation to find a function mapping the co-domain to the domain.
$y = \frac{x}{x-3} \Rightarrow yx - 3y = x \Rightarrow yx - x = 3y \Rightarrow x(y-1) = 3y \Rightarrow x = \frac{3y}{y-1}$, and since $y \neq 1$, f is onto.

Since the function is both one-to-one and onto, f is bijective.

5 Chapter 5 Problem 40

Determine, with explanation, whether the following is true or false. If A and B are disjoint sets such that A is countable and B is uncountable, then $A \cup B$ is uncountable.

Since B is uncountable, and $B \subseteq A \cup B$, $A \cup B$ must also be uncountable, because by theorem 5.81 from the textbook, every set that contains an uncountable set is itself uncountable.