Density of $\mathbb{Q}$ in $\mathbb{R}$

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Theorem 1. *The set of rationals $\mathbb{Q}$ is dense in the set of real numbers $\mathbb{R}$. Meaning that between any two real numbers, there exists a rational number.*

Proof. Let us first take two arbitrary real numbers x and y such that $x < y$. Since $x < y$, we know $y - x > 0$. Then, by the Archimedean Principle, there exists an $n \in \mathbb{N}$ such that

$$0 < \frac{1}{n} < y - x \quad \Rightarrow \quad x < x + \frac{1}{n} < y$$

Now we have the addition of a real number and a rational number exists between two reals, our next step is to show there exists a rational number between x and y.

Now we can take the set $S = \{z \in \mathbb{Z} \ : \ \frac{z}{n} \leq x\}$ where x is our real number and n is our natural number previously defined. And since $x \geq \frac{z}{n}$ for all $z \in \mathbb{Z}$, xn is an upper bound.

Now, since our set S is a nonempty subset of $\mathbb{Z}$, and is bounded from above, by *Lemma 2*, the set contains a maximum value. We can call this max value $m \in \mathbb{Z}$. And since xn is an upper bound of S, $\frac{m}{n} \leq x$, and since $\frac{m}{n}$ is the maximum value satisfying the inequality, it must also be the case that $\frac{m+1}{n} > x$, giving us

$$\frac{m}{n} \ \leq \ x \ < \ \frac{m+1}{n}$$

We can take the fact that $x + \frac{1}{n} < y$ to finally show that there must exist a rational number between x and y.

$$x < \frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} \leq x + \frac{1}{n} < y$$

So take our rational number q to be $\frac{m+1}{n}$, giving us

$$x < q < y$$

Thus, we've proved that for any two given real numbers, there exists a rational number in between, and therefore, the set of rationals $\mathbb{Q}$ is dense in the set of real numbers $\mathbb{R}$.

$\square$

Lemma 2. *Take a set S, if $S \neq \emptyset$, $S \subseteq \mathbb{Z}$, and S is bounded above, then S has a maximum element.*

Proof. Since S is bounded from above, by the completeness axiom, S contains a supremum $x \in \mathbb{R}$. We will show that $x \in S$, that is, the supremum is the largest element in S.

Suppose, to the contrary, that x is a supremum, but $x \notin S$. This means that x is strictly greater than every element in S, otherwise it would equal an element in S, and therefore be in S. And since the supremum is not in the set, the set has no largest value.

Now, since x is the smallest value greater than every element of S, there must exist an $s \in S$ such that $s > x - 1$, because $x > x - 1$, so $x - 1$ can't be an upper bound, otherwise x wouldn't be our supremum. So we get

$$x - 1 < s < x$$

And since our set has no largest element, there must exist an element t that's bigger than s but less than x. This gives us

$$x - 1 < s < t < x$$

And since $S \subseteq \mathbb{Z}$ the smallest value that t can be that's strictly greater than s is $s + 1$, but this would imply that $s+1 < x$. Since we already know $x-1 < s \Rightarrow x < s+1$, we have a contradiction.

Therefore, our assumption must be false, meaning that the supremum of our set S is in S. Therefore, given a set S, where $S \neq \emptyset$, $S \subseteq \mathbb{Z}$, and S is bounded above, S must contain a largest element.

$\square$