

Density of \mathbb{Q} in \mathbb{R}

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December 4, 2021

Theorem 1. *The set of rationals \mathbb{Q} is dense in the set of real numbers \mathbb{R} . meaning that between any two real numbers, there exists a rational number.*

Proof. Take two arbitrary real numbers x and y such that $x < y$. Since $x < y$, we know $y - x > 0$. Then, by the Archimedean Principle, there exists an $n \in \mathbb{N}$ such that

$$0 < \frac{1}{n} < y - x \Leftrightarrow x < x + \frac{1}{n} < y$$

Take the set $S = \{z \in \mathbb{Z} : \frac{z}{n} \leq x\}$. Since $x \geq \frac{z}{n}$ for all $z \in S$, xn is an upper bound.

Now, since S is a nonempty subset of \mathbb{Z} and is bounded from above, by *Lemma 2*, S contains a maximum value. Call this maximum value $m \in \mathbb{Z}$. Since xn is an upper bound of S , $\frac{m}{n} \leq x$, and since $\frac{m}{n}$ is the maximum value satisfying the inequality, it must also be the case that $\frac{m+1}{n} > x$, giving us...

$$\frac{m}{n} \leq x < \frac{m+1}{n}$$

Applying the fact that $x + \frac{1}{n} < y$, we have...

$$x < \frac{m+1}{n} = \frac{m}{n} + \frac{1}{n} \leq x + \frac{1}{n} < y$$

Thus, for any two real numbers, there exists a rational number in between them, and therefore \mathbb{Q} is dense in \mathbb{R} .

□

Lemma 2. Take a set S , if $S \neq \emptyset$, $S \subseteq \mathbb{Z}$, and S is bounded above, then S has a maximum element.

Proof. Since S is bounded from above, by the completeness axiom, S contains a supremum $x \in \mathbb{R}$. We will show that $x \in S$, that is, the supremum is the largest element in S .

Suppose, to the contrary, that x is a supremum, but $x \notin S$. This means that x is strictly greater than every element in S , otherwise it would equal an element in S , and therefore be in S . And since the supremum is not in the set, the set has no largest value.

Now, since x is the smallest value greater than every element of S , there must exist an $s \in S$ such that $s > x - 1$, because $x > x - 1$, so $x - 1$ can't be an upper bound, otherwise x wouldn't be our supremum. So we get

$$x - 1 < s < x$$

And since our set has no largest element, there must exist an element t that's bigger than s but less than x . This gives us

$$x - 1 < s < t < x$$

And since $S \subseteq \mathbb{Z}$ the smallest value that t can be that's strictly greater than s is $s + 1$, but this would imply that $s + 1 < x$. Since we already know $x - 1 < s \Rightarrow x < s + 1$, we have a contradiction.

Therefore, our assumption must be false, meaning that the supremum of our set S is in S . Therefore, given a set S , where $S \neq \emptyset$, $S \subseteq \mathbb{Z}$, and S is bounded above, S must contain a largest element.

□