3283W Homework 10

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8.1 - 5(f)

Find the sum of the series. $\displaystyle\sum_{\mathbf{n=1}}^{\infty} \frac{\mathbf{1}}{\mathbf{(2n-1)(2n+1)}}$

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \sum_{n=1}^{\infty} \Big[\frac{A}{2n-1} + \frac{B}{2n+1}\Big] \qquad \text{by partial fractions}$$

$$\Rightarrow A(2n+1) + B(2n-1) = 1$$

$$\Rightarrow A = \frac{1}{2}, \quad B = -\frac{1}{2} \qquad \text{by setting } n = \frac{1}{2}, -\frac{1}{2}$$

$$\sum_{n=1}^{\infty} \frac{1}{2}\Big[\frac{1}{2n-1} - \frac{1}{2n+1}\Big] = \frac{1}{2}\Big[\sum_{n=1}^{\infty} \frac{1}{2n-1} - \sum_{n=1}^{\infty} \frac{1}{2n+1}\Big] \qquad \text{linearity property of summations}$$

$$= \frac{1}{2}\Big[\sum_{n=1}^{\infty} \frac{1}{2n-1} - \sum_{n=2}^{\infty} \frac{1}{2n-1}\Big] \qquad 2n+1 = 2(n+1)-1$$

$$= \frac{1}{2}\Big[\frac{1}{2(1)-1} + \sum_{n=2}^{\infty} \frac{1}{2n-1} - \sum_{n=2}^{\infty} \frac{1}{2n-1}\Big] \qquad \text{pull out } n = 1 \text{ term}$$

$$= \frac{1}{2}\Big[\frac{1}{2(1)-1} + 0\Big] = \frac{1}{2}$$

Therefore, we have that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)}$ sums to $\dfrac{1}{2}$.

8.1 - 11

Prove that if $\sum |\mathbf{a_n}|$ converges, and $(\mathbf{b_n})$ is a bounded sequence, then $\sum \mathbf{a_n b_n}$ converges.

We know (b_n) is bounded, therefore there exists a value M such that $M > |b_n|$ for all $n \in \mathbb{N}$. We also know $\sum |a_n|$ converges, therefore it must satisfy the Cauchy criterion, meaning that for any $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n \geq m \geq N$, $|a_m + a_{m+1} + \ldots a_n| < \varepsilon$

Since $M > 0$, we know $\frac{\varepsilon}{M} > 0$, and by the cauchy criterion, there exists $N \in \mathbb{N}$ such that for all $n \geq m \geq N$, $\displaystyle\sum_{k=m}^{n} |a_k| < \frac{\varepsilon}{M}$.

Now we will show $\sum |a_n|$ converges by using the Cauchy criterion to show that for any $\varepsilon > 0$, $\displaystyle\left|\sum_{k=m}^{n} a_k b_k\right| < \varepsilon$

$$
\begin{aligned}
\left|\sum_{k=m}^{n} a_k b_k\right| &\leq \sum_{k=m}^{n} |a_k b_k| && \text{triangle inequality} \\
&= \sum_{k=m}^{n} |a_k||b_k| && \text{multiplicative property of absolute values} \\
&< \sum_{k=m}^{n} |a_k| M && (M > |b_n| \quad \forall\, n \,\in\, \mathbb{N}) \\
&= M \sum_{k=m}^{n} |a_k| && \text{linearity property of summations} \\
&< M \cdot \frac{\varepsilon}{M} && \text{defined above} \\
&= \varepsilon
\end{aligned}
$$

Therefore, by the Cauchy criterion for series, $\sum a_n b_n$ must converge.

8.2 - 3(o)

Does following series converge or diverge? justify your answer. $\sum \frac{(\mathbf{n}!)^{\mathbf{2}}}{(\mathbf{2n})!}$

We can proceed using the ratio test. Setting $a_n = \frac{(n!)^2}{(2n)!}$, we get

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \left|\frac{[(n+1)!]^2(2n)!}{(n!)^2(2n+2)!}\right| \\
&= \frac{[(n+1)(n!)]^2(2n)!}{(n!)^2(2n+2)(2n+1)(2n)!} \\
&= \frac{(n+1)^2(n!)^2}{(n!)^2(2n+2)(2n+1)} \\
&= \frac{(n+1)(n+1)}{(2n+1)(2n+2)}
\end{aligned}$$

We can use theorem 2.1 from chapter 4 of the book to get

$$\lim_{x\to\infty} \frac{(n+1)(n+1)}{(2n+1)(2n+2)} = \frac{1}{4}$$

Therefore, since $\lim_{x\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{4} < 1$, The series $\sum \frac{(n!)^2}{(2n)!}$ must converge by the ratio test.

8.2 - 15

Prove that if a series is conditionally convergent, then the series of negative terms is divergent.

We can represent our conditionally convergent series as $\sum a_n$. We can also let b_n denote all the positive terms and c_n denote all the negative terms. Note that

$$\sum_{n=0}^{\infty} b_n = \sum_{n=0}^{\infty} \frac{a_n + |a_n|}{2}, \qquad \sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} \frac{a_n - |a_n|}{2}$$

This is true because for b_n, if $a_n \leq 0$, then $\frac{a_n+|a_n|}{2} = 0$, and if $a_n > 0$, then $\frac{a_n+|a_n|}{2} = a_n$. Same goes for c_n. So now we can prove that the series of the negative terms must be divergent via proof by contradiction.

Assume, to the contrary, that $\sum b_n$ converges to some value b, and since $\sum a_n$ is conditionally convergent, $\sum a_n$ must converge to some value a. Since $\sum a_n$ is equal to the sum of all it's positive and negative terms, it must be the case that $\sum a_n = \sum b_n + \sum c_n$, meaning $a = b + \sum c_n$, and since a and b are both finite real numbers, $\sum c_n$ must also converge to some finite value c.

Since the series of positive terms converge, and the series of negative terms converge, it must be the case that $\sum |a_n|$ converges to $|b| + |c| = b - c \in \mathbb{R}$ by theorem 1.4 from 8.1 of the textbook. But $\sum a_n$ is conditionally convergent, meaning $\sum |a_n|$ must diverge, this is a contradiction. Therefore, it must be the case that the series of negative terms of a conditionally convergent series diverges.