4512 Homework 1

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Section 1.4 Question 3

Find the general solution: $\frac{dy}{dt} = 1 - t + y^2 - ty^2$.

$$\frac{dy}{dt} = 1-t+y^2-ty^2 \Rightarrow \frac{dy}{dt} = (y^2+1)(1-t) \Rightarrow \int \frac{dy}{1+y^2} = \int (1-t)\ dt \Rightarrow \arctan(y) = -\frac{1}{2}t^2+t+C$$

so $y = tan(-\frac{1}{2}t^2 + t + C)$ in the interval $-k\frac{\pi}{2} < -\frac{1}{2}t^2 + t + C < k\frac{\pi}{2}$ for some $k \in \mathbb{N}$.

Section 1.5 Question 12

given $\frac{dp}{dt} = bp^2 - ap, \quad a, b > 0$, show that $p(t)$ approaches 0 as $t \to \infty$ if $p_0 < a/b$.

First thing to do is separate this equation...

$$\frac{dp}{dt} = bp^2 - ap \ \Rightarrow \ \int_{p_0}^{p} \frac{dp}{bp^2 - ap} = \int_{t_0}^{t} dt$$

Now to solve the first integral, we can first note that $bp^2 - ap$ can be rewritten as $p(bp - a)$, then we can separate these by finding values A and B such that $\frac{1}{p(bp-a)} = \frac{A}{p} + \frac{B}{bp-a}$...

$$A(bp-a) + Bp = 1 \quad \Rightarrow \quad A = \frac{-1}{a},\ B = \frac{b}{a}$$

Now we can plug these values in and solve our integral...

$$\frac{1}{a}\int_{p_0}^{p} \frac{-1}{p} + \frac{b}{bp-a}\ dp \ = \ \frac{1}{a}\ln\left|\frac{p_0(bp-a)}{p(bp_0-a)}\right| \ = \ t - t_0 \ \Rightarrow \ \frac{p_0(bp-a)}{p(bp_0-a)} \ = \ e^{a(t-t_0)}$$

After simplifying, we get $p(t) = \dfrac{ap_0}{bp_0 + (a - bp_0)e^{a(t-t_0)}}$. As you can see, as $t \to \infty$, the denominator gets larger, and $p(t)$ approaches 0.

Section 1.8 Question 13

Find the orthogonal trajectory: $y^2 - x^2 = c$.

$$F(x,y,c) = y^2 - x^2 - c = 0 \quad \Rightarrow \quad \frac{\partial}{\partial x}F(x,y,c) = 2y\frac{dy}{dx} - 2x = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{2x}{2y} = \frac{x}{y}$$

The orthogonal slope is the oppposite reciprocal of the original slope, so $\dfrac{dy}{dx} = \dfrac{-y}{x}$. Now we can separate and integrate to solve for y.

$$\frac{dy}{dx} = \frac{-y}{x} \quad \Rightarrow \quad -\int \frac{dy}{y} \; = \; \int \frac{dx}{x} \quad \Rightarrow \quad -\ln|y| \; = \; \ln|x| + k_1 \quad \Rightarrow \quad \frac{1}{y} \; = \; k_2 x \quad \Rightarrow \quad y(x) \; = \; \frac{k}{x}$$

Section 1.8 Question 18

$y(t)$ = number of living bacteria at time t.

$T(t)$ = number of toxins at time t.

Also, production of toxins begin at time $t = 0$, so $T(0) = 0$.

Finally, $\frac{dT}{dt} = c$.

(a) **Find a first-order differential equation satisfied by $y(t)$**

$\frac{dy}{dt}$ = (bacteria birth) - (bacteria death).

Bacteria grows proportionally to the amount present y with a proportionality constant b.

Bacteria dies proportionally to both the amount present y and the number of toxins present T. The proportionality constant for bacteria death is a.

Therefore, $\frac{dy}{dt} \propto y - yT = by - ayT = y(b - aT)$.

(b) **Solve the differential equation to obtain $y(t)$. What happens to $y(t)$ as $t \to \infty$?**

First let's find $T(t)$...

$$\frac{dT}{dt} = c \Rightarrow T(t) = \int c \, dt = ct + C$$

Since $T(0) = 0$, we know $C = 0$, so $T(t) = ct$. Now we can plug this into our equation for $\frac{dy}{dt}$.

$$\frac{dy}{dt} = y(b - aT) = y(b - act) \Rightarrow \int \frac{dy}{y} = \int (b - act) \, dt \Rightarrow \ln|y(t)| = bt - \frac{1}{2}act^2 + C_1$$

Ultimately giving us...

$$y(t) = Ce^{bt - \frac{1}{2}act^2}$$

As $t \to \infty$, the exponent in the equation is becoming more and more negative. This is because $-\frac{1}{2}act^2$ is growing much faster than bt, so the exponent approaches $-\infty$, making the whole equation approach zero. This basically means that the bacteria is slowly getting wiped out by the toxins.

Section 1.10 Question 5

Show that the solution $y(t)$ of the initial value problem exists on the given interval...

$$y' = 1 + y + y^2 \cos t, \quad y(0) = 0, \quad 0 \le t \le \frac{1}{3}$$

Existence Theorem: Consider the initial value problem $\{*\}$: $\frac{dy}{dt} = f(t, y)$, $y(t_0) = y_0$, Suppose f is continuous on $R = [t_0, t_0 + a] \times [y_0 - b, y_0 + b]$ ($\frac{\partial f}{\partial y}$ doesn't need to be continuous because we're not proving for uniqueness). Then there exists a solution of $\{*\}$ defined on $[t_0, t_0 + \alpha]$, where $\alpha = \min\{a, \frac{b}{M}\}$, and $M = \max|f(t, y)|$.

First, we can note that $f(t, y) = 1 + y + y^2 \cos t$ is continuous for all $y, t \in \mathbb{R}$. Since f is continuous on $R = [0, a] \times [-b, b]$, we can show that there exists a solution on $[0, \frac{1}{3}]$.

$$M = \max|f(t, y)| = 1 + b + b^2$$

$$\alpha = \min\left\{a, \frac{b}{M}\right\} = \min\left\{a, \frac{b}{1 + b + b^2}\right\}$$

Since f is continuous for all values of t, a can be any number, so $\alpha = \min\left\{a, \frac{b}{M}\right\}$ is dependent on b. The largest value of α is when $b = 1$, so $\alpha = \frac{1}{1+1+1^2} = \frac{1}{3}$, Therefore, by the Existence Theorem, there must exist a solution to the initial value problem on the interval $0 \le t \le \frac{1}{3}$.