Groups acting on trees. . . Exercises

Giuseppe Bargagnati and Francesco Milizia

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This documents contains a set of exercises for reviewing the topics of the lectures. Some of the exercises come **from the past**, meaning we have mentioned them in the class or clarify things we didn't do in detail. Some of the exercises come **from the future**, as they will be useful tools in classes which are still to be held (but involve notions we already discussed). Moreover, we collect many definitions and pieces of notation used in the course.

Symbols. Some exercises have a symbol. A book 📖 is for for exercises useful to review and familiarize with notions seen in class. An eye 👁 means that the exercise will be useful for future lectures (we suggest that you read and understand their statements, at least).

Definitions and notation

We recall some notation and terminology used in the course.

Data defining a graph. If X is a graph, we denote by X^0 its vertex set and by X^1 its set of (oriented) edges. Then, $\alpha_X : X^1 \to X^0$ and $\omega_X : X^1 \to X^0$ denote the "initial vertex" and "terminal vertex" functions. Finally, we have an inversion function $e \leftrightarrow \overline{e}$ which is an involution of X^1, with $\overline{e} \neq e$, such that $\alpha_X(\overline{e}) = \omega_X(e)$. Usually we omit the subscript X from the notation, if it is clear from the context.

A *geometric edge* a subset of X^1 of the form $\{e, \overline{e}\}$.

Remark. The intuition behind this formalization is that we want to describe *unoriented* graphs; for each geometric edge we have two distinct elements of X^1 representing the two ways (orientations) of crossing the geometric edge. Loops (edges with $\alpha(e) = \omega(e)$) are allowed (notice that $\overline{e} \neq e$ also for loops!), as well as multiple edges sharing the same endpoints. We do not impose restrictions on the cardinality of the vertex or edge sets.

Topological realization of a graph. Let X be a graph. Its topological realization $\mathrm{real}(X)$ is the topological space obtained as follows. Endow X^0

and X^1 with the discrete topologies, the interval $[0,1]$ with the Euclidean topology, and consider on the disjoint union

$$X^0 \sqcup (X^1 \times [0,1])$$

the finest equivalence relation $\sim$ such that: $(e,0) \sim \alpha(e)$ and $(e,t) = (\overline{e}, 1-t)$ for every $e \in X^1$ and $t \in [0,1]$. Then, $\mathrm{real}(X)$ is the quotient (as a topological space) of $X^0 \sqcup (X^1 \times [0,1])$ by this equivalence relation.

Subgraphs. A subgraph $Z \subseteq X$ is given by subsets $Z^0 \subseteq X^0$ and $Z^1 \subseteq X^1$ so that inversion and endpoint functions restrict to them; with these restricted functions, Z becomes a graph on its own.

Morphisms of graphs. A morphisms $f : X \to Y$ between two graphs consists of functions $f : X^0 \to Y^0$ and $f : X^1 \to Y^1$ such that $\overline{f(e)} = f(\overline{e})$ and $f(\alpha_X(e)) = \alpha_Y(f(e))$. It then also follows that $f(\omega_X(e)) = \omega_Y(f(e)))$. If $Z \subseteq X$ is any subgraph, then $f(Z)$ is defined in the obvious way and is a subgraph of Y. A morphism is injective/surjective if it is so at the level of both vertices and edges. It is an isomorphism if it is both injective and surjective. Given a graph X, the self-isomorphisms $f : X \to X$ are called automorphisms; they form a group under composition, which we denote by $\mathrm{Aut}(X)$.

Stars and local properties of morphisms. The star of a vertex $x \in X^0$ is the set $\mathrm{st}_X(x) = \{e \in X^1 : \alpha(e) = x\}$. The valence of x is the cardinality of its star. A morphism $f : X \to Y$ restricts, for every $x \in X^0$, to a function $f_x : \mathrm{st}_X(x) \to \mathrm{st}_Y(f(x))$. The morphism is locally injective/surjective if for every $x \in X^0$ the function f_x is injective/surjective.

Covering maps. We say that a morphism is a *covering map* if it is both locally injective and locally surjective. If $f : X \to Y$ is a covering map, then a covering automorphism of p is an automorphism $\varphi \in \mathrm{Aut}(X)$ such that $f \circ \varphi = f$. The covering automorphisms of f form a subgroup of $\mathrm{Aut}(X)$, which we denote by $\mathrm{Aut}_f(X)$.

Paths and connected graphs. A path of length 0 is a vertex; a path of length $n \in \mathbb{N}_{>0}$ is a list of edges $e_1 \dots e_n$ such that $\alpha(e_{i+1}) = \omega(e_i)$ for $i \in \{1, \dots, n-1\}$. Any path has an initial vertex and a terminal vertex defined in the obvious way. A path is closed if its initial and terminal vertices coincide. A path is reduced if it either has length 0, or it has length $n \geq 1$ and $e_{i+1} \neq \overline{e_i}$ for every $i \in \{1, \dots, n-1\}$. A graph X is connected if it is nonempty ($X^0 \neq \emptyset$) and for every $x, y \in X^0$ there exists a path joining x to y (*i.e.*, with initial vertex x and terminal vertex y).

Circuits and trees. For every positive integer n, let C_n be the "standard circle with n vertices": the graph with vertex set $\{1, \dots, n\}$ and edges connecting them in a circular fashion (formally, C_n^1 would be a set with $2n$ elements, *e.g.*, $C_n^1 = \{e \in \mathbb{Z} : 1 \leq |e| \leq n\}$, with endpoint and inversion functions defined appropriately). A circuit in a graph X is a subgraph isomorphic to C_n for some positive integer n; in other words, it is the image of an injective morphism $C_n \to X$. A tree is a connected graph that doesn't have any circuits.

Equivalence relations and quotients. An equivalence relation on a graph X is given by two equivalence relations, one on X^0 and one on X^1, that satisfy the following property (we denote with $\sim$ the relation both for vertices and edges): if $e_1 \sim e_2$ are equivalent edges, then $\alpha(e_1) \sim \alpha(e_2)$ and $\overline{e_1} \sim \overline{e_2}$ (then, it follows that also $\omega(e_1) \sim \omega(e_2)$). The equivalence relation is *without inversions* if no edge is equivalent to its inverse. Given an equivalence relation $\sim$ without inversions, the quotient $X/{\sim}$ is formed in the natural way, with endpoint and inversion functions inherited from X (if there were inversions, the quotient wouldn't be a graph! There would be an edge inverse of itself). There is a natural projection $X \to X/{\sim}$, which is a graph morphism.

Group actions and quotients. A (left) action $G \curvearrowright X$ of a group G on a graph X is given by two (left) actions $G \curvearrowright X^0$ and $G \curvearrowright X^1$ so that endpoint and inversion functions are G-equivariant. In other words, it is a homomorphism $G \to \mathrm{Aut}(X)$. We say that G acts *without inversions* if $g \cdot e \neq \overline{e}$ for every $g \in G$ and $e \in X^1$. An action $G \curvearrowright X$ induces an equivalence relation (the orbit relation) on X, which is without inversions precisely when the action is without inversions. The resulting quotient, which is defined if the action is without inversions, is denoted by $G \backslash X$. An action is *free* if it has no inversions and is free on vertices, *i.e.*, $g \cdot x \neq x$ for every $x \in X^0$ and $g \in G \setminus \{1_G\}$ (it follows that G acts freely also on X^1).

Generating systems and rank of a group. Let G be a group. A generating system for G is given by a set S and a function $[\cdot]_G : S \to G$ such that its image $[S]_G$ generates G as a group, *i.e.*, every element of G is equal to a product of the form $1_G \cdot [s_1]_G^{\varepsilon_1} \cdot \dots \cdot [s_k]_G^{\varepsilon_k}$ for some $k \in \mathbb{N}$, $s_i \in S$ and $\varepsilon_i \in \{\pm 1\}$. When we say "Let S be a generating system for G" we mean that we have fixed a function $S \to G$ as above, which is often kept implicit in the notation (so we can write things like $s_1 s_2 \in G$. In other words, we think of any $s \in S$ as a *name* of an element of G, and in principle an element of G could have multiple different "names" in S). Often, S is just a subset of G (which generates G) and the map is the inclusion: $[s]_G = s$. The rank of G, denoted by $\mathrm{rank}(G)$, is the minimum cardinality of a generating system of

G (*i.e.*, of a set S as above), which is the same as the minimum cardinality of a subset of generators of G.

Cayley graphs. Let G be a group and S be a generating system of G. Let $\overline{S}$ be a disjoint copy of S (a set of formal "inverses"), with a bijection $S \leftrightarrow \overline{S}$, which we write as $s \mapsto \overline{s}$; we extend the function $s \mapsto [s]_G \in G$ to the disjoint union $S \sqcup \overline{S}$ by setting $[\overline{s}]_G = [s]_G^{-1}$. The Cayley graph of G with respect to S, which we denote by $\Gamma(G, S)$, is the graph with:

- $\Gamma(G, S)^0 = G$;
- $\Gamma(G, S)^1 = G \times (S \sqcup \overline{S})$;
- $\alpha((g, s)) = g$, $\omega((g, s)) = g \cdot [s]_G$, for every $(g, s) \in G \times (S \sqcup \overline{S})$;
- Inversion function $\overline{(g, s)} = (g \cdot [s]_G, \overline{s})$.

There is a standard action $G \curvearrowright \Gamma(G, S)$, given by $g \cdot x = gx$ for vertices $x \in G$, and by $g \cdot (x, s) = (gx, s)$ for edges $(x, s) \in G \times (S \sqcup \overline{S})$.

Bases and free groups. Let G be a group. A subset $S \subseteq G$ is a basis of G if for every $g \in G$ we can write in a *unique* way

$$g = 1_G \cdot s_1^{\varepsilon_1} \cdot \dots \cdot s_k^{\varepsilon_k}$$

with $k \in \mathbb{N}$, $s_i \in S$, $\varepsilon_i \in \{\pm 1\}$ with $\varepsilon_{i+1} = \varepsilon_i$ whenever $s_{i+1} = s_i$. A group is free if it admits a basis. Given any set S, there is a canonical construction of a free group F_S and an injective function $S \hookrightarrow F_S$ so that the image of S is a basis of F_S. We usually think of S as included into F_S, so that S itself is a basis. In general, we say that a generating system S of a group G is a basis if the function $S \to G$ is injective and its image is a basis.

Group presentations. If S is a set and R is a subset of the free group F_S, we denote by $\langle S \mid R \rangle$ the quotient $F_S/\mathrm{ncl}(R)$, where $\mathrm{ncl}(R)$ is the normal closure of R in F_S. If G is a group, a presentation of G is an isomorphism $G \cong \langle S \mid R \rangle$ for some S and R as above. In other words, it is a package containing:

- A set S;
- A function $S \to G$ inducing a surjective map $F_S \to G$;
- A subset $R \subseteq F_S$ whose normal closure coincides with the kernel of the map of the previous point.

If we say "let $\langle S \mid R \rangle$ be a presentation of G", it means in particular that we have fixed a function $S \to G$ so that we are in the situation above. To have a friendlier notation, we may write some elements of R as *equations* of the form $w_1 = w_2$, with $w_i \in F_S$, which we interpret as the element $w_1 w_2^{-1}$.

Amalgamated products. Let G be a group. Let A be a subgroup of G and $\{G_i\}_{i\in I}$ be a nonempty family of subgroups of G, with $A < G_i$ for every i. For $n \in \mathbb{N}$ and $\underline{i} = (i_1, \dots, i_n) \in I^n$ such that $i_m \neq i_{m+1}$ for $m \in \{1, \dots, n-1\}$, we say that a *reduced word* of length n and type $\underline{i}$ is a sequence

$$ag_1 \dots g_n$$

with $a \in A$ and $g_m \in G_{i_m} \setminus A$. We say that G is an amalgamated product of the G_i's along A if for every $g \in G$ there exist unique $n \in \mathbb{N}$ and $\underline{i} \in I^n$ as above such that

$$g = ag_1 \dots g_n$$

for some reduced word of type $\underline{i}$, and any other such reduced word is of the form

$$(aa_1^{-1})(a_1g_1a_2^{-1}) \dots (a_n^{-1}g_n)$$

for some $a_1, \dots a_n \in A$. We write $G = *_A^{i\in I} G_i$. If $A = \{1_G\}$ we say that G is a free product of the G_i's and write $G = *^{i\in I} G_i$. When I has cardinality 2 we write $G = G_1 *_A G_2$, or $G = G_1 * G_2$ if $A = \{1_G\}$.

There is a **standard construction** that takes

- a group A;
- a family of groups $\{G_i\}_{i\in I}$ with injective homomorphisms $j_i : A \to G_i$

and produces a group G with injective homomorphisms $\varphi : A \to G$ and $\varphi_i : G_i \to G$ such that:

- $\varphi = \varphi_i \circ j_i$ for every $i \in I$;
- $G = *_{\varphi(A)}^{i\in I} \varphi(G_i)$.

Then, we usually think of the groups A and G_i as subgroups of G, and write $G = *_A^{i\in I} G_i$.

Colimits. Let $\{G_i\}_{i\in I}$ be a family of groups and $\{F_{ij}\}_{i,j\in I}$ be a family of homomorphisms between those groups, with $F_{ij} \subseteq \mathrm{Hom}(G_i, G_j)$; let us call such data a *system of groups.* The colimit of such a system of groups is a group G with homomorphisms $\varphi_i : G_i \to G$, with $\varphi_j \circ f = \varphi_i$ for every $f \in F_{ij}$, such that the following universal property holds: if H is a group and $h_i : G_i \to H$ are homomorphisms (one for every $i \in I$) such that $h_j \circ f = h_i$ for every $f \in F_{ij}$, then there *exist a unique* homomorphism $h : G \to H$ such that $h_i = h \circ \varphi_i$ for every $i \in I$.

There is a **standard construction** that takes a system of groups and associates to it its colimit (*i.e.*, the colimit always exists; its universal property implies its uniqueness up to unique isomorphisms).

Graphs of groups. A graph of groups consists of a *connected* graph X, groups G_p and G_y associated to vertices $p \in X^0$ and edges $y \in X^1$, with $G_{\overline{y}} = G_y$, and *injective* homomorphisms $G_y \to G_{\omega(y)}$, one for every edge $y \in X^1$. Notice that we also have homomorphisms $G_y \to G_{\alpha(y)}$, since $G_y = G_{\overline{y}}$ and $\alpha(y) = \omega(\overline{y})$. We may denote such a graph of groups by (X, G); formally G would be the collection of groups and homomorphisms defining the graph of groups, while X is the underlying graph.

Trees of groups. A tree of group is a graph of groups (G, X) where X is a tree. Its collection of groups and homomorphisms determines a system of groups, and we denote its colimit (which is a group) by G_X. Basic useful properties of G_X are collected in Exercise 6.2.

Fundamental group of a graph of groups. Let (Y, G) be a graph of groups. Denote by $a \mapsto a^y$ the map $G_y \to G_{\omega(y)}$, which is part of the graph of groups structure, for any edge y. Then, define the auxiliary group

$$F(Y,G) = \frac{(*_{P \in Y^0}\ G_P)\ *\ F_{Y^1}}{y^{-1} = \overline{y},\quad y\ a^y\ \overline{y} = a^{\overline{y}}},$$

where, in the denominator (which contains relations defining $F(Y,G)$ as a quotient of a free product), y varies in Y^1 and a varies in G_y. Then, the *fundamental group* of (Y, G) can be defined in two ways:

- With respect to a spanning tree $T \subseteq Y$. Define

  $$\pi_1(Y, G; T) = \frac{F(Y,G)}{y = 1 \text{ for } y \in T^1}.$$

- With respect to a base vertex $P_0 \in Y^0$. Given any path $c = (y_1, \dots y_n)$ in Y, with vertices $P_0, \dots, P_n$, we say that a word of type c is a sequence $\mu = (\mu_0, \dots, \mu_n)$ with $\mu_i \in G_{P_i}$. We may also directly say that (c, μ) is a word, meaning that c is a path and μ is as above. Given a word (c, μ) as above, we associate to it the element

  $$\mu_0 y_1 \mu_1 \dots y_n \mu_n \in F(Y, G).$$

  Then, define

  $$\pi_1(Y, G; P_0)$$

  as the subgroup of $F(Y, G)$ consisting of elements associated to words (c, μ) whose type c is a path which starts and ends at P_0.

The two definitions are related by this fact: For any spanning tree T and vertex P_0, the projection $F(Y, G) \to \pi_1(Y, G; T)$ restricts to an isomorphism $\pi_1(Y, G; P_0) \cong \pi_1(Y, G; T)$.

1 First lecture (November 6)

Exercise 1.1. What are the regular graphs of valence 2?

Definition 1.2 (Standard lines)**.** For every $n \in \mathbb{N}$, including $n = 0$, denote by P_n the graph X with vertex set $\{0, \dots, n\}$, and edges "connecting the vertices in a line, in increasing order". The formal description would require P_n^1 to consist of $2n$ elements, *e.g.*, $P_n^1 = \{e \in \mathbb{Z} : 1 \leq |e| \leq n\}$, and endpoint and inversion functions defined appropriately. Also define P_∞ (the standard infinite ray) with vertex set $\mathbb{N}$ and edges connecting consecutive numbers.

Exercise 1.3. Prove that a path of length n in a graph X "is the same" as a morphism $P_n \to X$, and that reduced paths correspond to locally injective morphisms.

Exercise 1.4. If two vertices of a graph are joined by a path, then they are joined by an injective (hence, reduced) path.

Exercise 1.5. Let X be a connected graph in which each vertex has finite valence (we say that X is *locally finite*). Prove that, if X does not contain an infinite injective ray (*i.e.*, there is no injective morphism $P_\infty \to X$, see Definition 1.2), then X is finite.

Exercise 1.6. Let T be a tree. Show that any closed reduced path in T has length 0. Show that a path in T is reduced if and only if it is a length-minimizing path, *i.e.*, it minimizes the length among paths with the same endpoints (hint: induction on the minimum length of paths joining the endpoints; the base case — length 0 — corresponds to the first sentence).

Recall that these paths are called geodesics; for a graph which is not necessarily a tree, we call geodesic any length-minimizing path.

Exercise 1.7. Let X be a connected graph. Show that X is a tree if and only if any reduced path in X is length-minimizing, if and only if any closed reduced path in X has length 0.

Exercise 1.8. Every nonempty graph has a maximal subtree with respect to inclusion (we have seen in the lecture that, if the graph is connected, such a maximal subtree contains all the vertices, and is called a spanning tree).

Exercise 1.9 (Sets hanging from a tree)**.** Let T be a tree. Suppose we have a (possibly infinite) sequence of sets $V_1, V_2, \dots$ and functions

$$T^0 \xleftarrow[f_1]{} V_1 \xleftarrow[f_2]{} V_2 \xleftarrow[f_3]{} \dots .$$

Form a new graph X whose vertex set is the disjoint union of T^0 and all the V_i's, and having the same edges as T, plus one geometric edge (formally, two edges inverses of each other) for each $v \in V_i$, joining v and $f_i(v)$. Prove that X is a tree. This is a very natural construction of a tree, which often occurs in the special case where T has just one vertex.

2 Second lecture (November 10)

Exercise 2.1 (Morphisms on topological realizations)**.** Describe how a graph morphism $f : X \to Y$ induces a continuous map $\mathrm{real}(f) : \mathrm{real}(X) \to \mathrm{real}(Y)$; show that the induced map is injective/surjective if and only if f is so.

Exercise 2.2 (Combinatorial vs. topological conditions)**.** A graph X is connected if and only if its topological realization is path-connected; it is finite if and only if its topological realization is compact.

Exercise 2.3 (Trees are contractible)**.** Show that the topological realization of a tree is contractible. (Use geodesics to retract to a chosen vertex.)

Exercise 2.4. Let X be a tree and let S be a nonempty set of vertices of X, with diameter n. Then, the subtree generated by S (which is defined as the smallest subtree of X containing S) has diameter equal to n.

Exercise 2.5 (Relations and surjective morphisms)**.** Let $p : X \to Y$ be a surjective graph morphism. Show that the relation $\sim$ on X defined as $x_1 \sim x_2 \iff f(x_1) = f(x_2)$ for vertices, and with the analogous definition for edges, is an equivalence relation on X without inversions, and that the quotient $X/{\sim}$ is isomorphic to Y, with $p : X \to Y$ corresponding to the quotient projection. In the opposite direction, if X is a graph and $\sim$ is an equivalence relation on X without inversions, then the projection $X \to X/{\sim}$ is a surjective morphism.

Exercise 2.6 (Lifting trees[1])**.** Let $f : X \to Y$ be a locally surjective morphism of graphs, $T \subset Y$ be a subtree, and $x_0 \in X^0$ such that $f(x_0) \in T^0$. Then, there exists a subtree $L \subseteq X$ containing x_0 that lifts T, *i.e.*, f restricts to an isomorphism $L \cong T$.

Exercise 2.7. Show that the conclusion of Exercise 2.6 doesn't necessarily hold if f is a surjective morphism.

Exercise 2.8. If a group G acts on a graph X without inversions, then the projection $X \to G\backslash X$ is a locally surjective morphism (therefore, subtrees can be lifted as in Exercise 2.6).

[1] During the lecture, an imprecise version was stated, which does not hold, see Exercise 2.7 (and Exercise 2.5); this exercise amends that statement, and the proof given in class actually works as a solution.

3 Third lecture (November 13)

Exercise 3.1. Let G be a free group with a basis S. Then, $\mathrm{rank}(G)$ is equal to the cardinality of S.

Exercise 3.2. Let $f : X \to Y$ be a locally surjective morphism between connected graphs. Show that f is surjective.

Exercise 3.3. A graph morphism $f : X \to Y$ is a covering map if and only if the induced continuous function $\mathrm{real}(f) : \mathrm{real}(X) \to \mathrm{real}(Y)$ is a covering map in the topological sense.

Exercise 3.4. Let $f : X \to Y$ be a covering map of graphs. Show that any covering automorphism (in the topological sense) of $\mathrm{real}(f)$ is induced by a unique covering automorphism of p. This gives an isomorphism between $\mathrm{Aut}_f(X)$ and the group of (topological) covering automorphisms of $\mathrm{real}(f)$.

Exercise 3.5. Consider the generating system of $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ given by the set $S = \{a, b\}$ whose elements are sent to G as $a \mapsto ([1], [0])$, $b \mapsto ([0], [1])$. Draw $\Gamma(G, S)$. Then, find a basis of the kernel of $F_S \twoheadrightarrow G$.

Exercise 3.6. Let F_2 be a free group of rank 2, with basis $S = \{a, b\}$, and let $f : F_2 \to \mathbb{Z} \times \mathbb{Z}$ be the homomorphism sending $a \mapsto (1, 0)$ and $b \mapsto (0, 1)$. What is the rank of the kernel of f? Find a basis of it.

Exercise 3.7. Let S be a generating system of a group G. Show that the Cayley graph $\Gamma(G, S)$ is a tree if and only if S is a basis of G, meaning that $S \to G$ is injective and its image is a basis of G.

4 Fourth lecture (November 24)

Exercise 4.1. Show that

$$\langle x_1, x_2, \dots \mid x_k = x_{k+1}^{k+1} \text{ for } k \in \mathbb{N}_{\geq 1} \rangle$$

is isomorphic to the group of rational numbers.

Exercise 4.2. Let G be a group, and consider the map $f : G * \mathbb{Z} \to \mathbb{Z}$ which is trivial on G and the identity on $\mathbb{Z}$. Denote by t a generator of $\mathbb{Z} < G * \mathbb{Z}$. Show that $\ker(f) = *^{n \in \mathbb{Z}}(t^n G t^{-n})$.

Exercise 4.3. Let $f : H \to G_1 *_A G_2$ be a surjective homomorphism. Show that $H = f^{-1}(G_1) *_{f^{-1}(A)} f^{-1}(G_2)$. Hint: use the action on the associated Bass-Serre tree.

5 Fifth lecture (November 27)

Recall that, if $x = a/b$ and $y = c/d$ are rational numbers expressed as reduced integer fraction with $b, c > 0$, the *mediant* of x and y is defined as $(a + c)/(b + d)$.

Exercise 5.1. Show that the "mediant binary search", starting from an interval with rational endpoints, and with the "goal number" being a rational number contained in the interval, terminates in a finite number of steps. (Hint: show that if $0 < a/b < x/y < c/d$ are reduced fractions, then $(bc - ad)(x + y) \geq a + b + c + d$.)

We have seen that $SL_2(\mathbb{Z})$ acts on a tree, which is called the *Farey tree*, and that this action corresponds to an amalgamated product decomposition $SL_2(\mathbb{Z}) \cong C_6 *_{C_2} C_4$.

Exercise 5.2. From the action on the Farey tree, deduce that $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm\mathrm{Id}\}$ is a free product of two finite groups.

Exercise 5.3. Show that the matrix

$$\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} \in SL_2(\mathbb{Z}),$$

with $k \in \mathbb{Z} \setminus \{0\}$, acts on the Farey tree as a hyperbolic automorphism. What is its axis? (If you know about the action of $SL_2(\mathbb{Z})$ on the upper half-plane, notice that the same matrix acts *parabolically*, i.e., fixing one point at infinity, on it!)

Exercise 5.4. Show that $M \in SL_2(\mathbb{Z})$ acts as a hyperbolic automorphism of the Farey tree if and only if it is not $\pm\mathrm{Id}$ and has trace $\notin \{-1, 0, 1\}$. (Hint: if the trace is 0, 1 or -1, what are the eigenvalues of M?)

Exercise 5.5. Describe an algorithm that takes as input a matrix $M \in SL_2(\mathbb{Z})$ and produces a reduced word (with respect to the amalgamated product $SL_2(\mathbb{Z}) \cong C_6 *_{C_2} C_4$) representing M.

6 Sixth lecture (December 4)

Exercise 6.1. If $G = G_1 *_A G_2$, and both G_1 and G_2 are torsion-free groups, then G is torsion-free.

Exercise 6.2. Show that, if (G, T) is a tree of groups, then:

- For every $P \in T^0$, the homomorphism $G_P \to G_T$ is injective;
- For every $y \in T^1$, the homomorphism $G_y \to G_T$ is injective;
- For every subtree $S \subseteq T$, there is a natural map $G_S \to G_T$, which is injective (here, G_S denotes the colimit of the groups corresponding to vertices and edges belonging to S);
- For any $y \in T^1$, let T_α and T_ω be the subtrees of T obtained by erasing the geometric edge $\{y, \overline{y}\}$; then, $G_T = G_{T_\alpha} *_{G_y} G_{T_\omega}$ (notice that, by the previous points, G_{T_α}, G_{T_ω} and G_y can all be considered as subgroups of G_T).

Hint: show simultaneously all items, for finite trees T, by induction on the number of vertices of T. Then deduce that they hold also for infinite trees, by a “limit” argument.

7 Ninth lecture (January 12)

Exercise 7.1. Check that, if (Y, G) is a graph of group and Y is a tree, then the fundamental group of (Y, G) is the colimit of the associated system of groups. Show that the same conclusion *does not hold* if Y is not a tree.

Exercise 7.2. This was part of a proof in the lecture. Let (Y, G) be a graph of groups over the loop graph Y (with one vertex P and one geometric edge $\{y, \overline{y}\}$). Let L be the "infinite line" graph, with vertex set $\mathbb{Z}$, and consider the graph of groups (L, H) where all edge groups are equal to G_y, the vertex groups are $H_n = y^n G_P y^{-n}$ (each of those is a subgroup of $G_P * \mathbb{Z}$, with $\mathbb{Z} = \langle y \rangle$), and the inclusion maps corresponding to the edge with endpoints $n-1$ and n are:

$$\begin{aligned} a \in G_y &\mapsto y^{n-1} a^{\overline{y}} y^{-(n-1)} \in H_{n-1} \\ a \in G_y &\mapsto y^n a^y y^{-n} \in H_n. \end{aligned}$$

Prove that $F(L, H)$ is canonically isomorphic to the kernel of the natural map $p_{\mathbb{Z}} : F(Y, G) \to \mathbb{Z}$. Moreover, write how we can naturally associate, to a word (c, μ) over (Y, G) of "winding number zero", a closed "lifted" word $(\tilde{c}, \tilde{\mu})$ over (L, H), and check that the respective associated elements in $F(Y, G)$ and $F(L, H)$ correspond to each other via the identification $(\ker p_{\mathbb{Z}}) \cong F(L, H)$.

Exercise 7.3. Let (Y, G) and (Z, H) be two graphs of groups, and let $f : (Z, H) \to (G, Y)$ be a morphism of graphs of groups, i.e., a morphism of graphs $f : Z \to Y$ augmented with homomorphisms $H_P \to G_{f(P)}$ and $H_e \to G_{f(e)}$, for vertices $P \in Z^0$ and edges $e \in Z^1$, such that the relevant diagrams with $H_e \to H_{\omega(e)}$ and $G_{f(e)} \to G_{\omega(f(e))}$ commute. Show that f induces a homomorphism $\pi_1(Z, H; P_0) \to \pi_1(Y, G; f(P_0))$, for any $P_0 \in Z^0$.

Exercise 7.4. Let (Y, Z) be a graph of group. Let $Z \subseteq Y$ be a connected subgraph. Notice that the inclusion of Z in Y gives a morphism $(Z, G_{\upharpoonright Z}) \to (Y, G)$, in the sense of Exercise 7.3. Let $P_0 \in Z^0$. Show that the induced map $\pi_1(Z, G \upharpoonright Z; P_0) \to \pi_1(Y, G; P_0)$ is injective.