# Groups acting on trees... Exercises

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This documents contains a set of exercises for reviewing the topics of the lectures. Some of the exercises come **from the past**, meaning we have mentioned them in the class or clarify things we didn't do in detail. Some of the exercises come **from the future**, as they will be useful tools in classes which are still to be held (but involve notions we already discussed). Moreover, we collect many definitions and pieces of notation used in the course.

**Symbols.** Some exercises have a symbol. A book 📘 is for for exercises useful to review and familiarize with notions seen in class. An eye ◉ means that the exercise will be useful for future lectures (we suggest that you read and understand their statements, at least).

## Definitions and notation

We recall some notation and terminology used in the course.

**Data defining a graph.** If $X$ is a graph, we denote by $X^0$ its vertex set and by $X^1$ its set of (oriented) edges. Then, $\alpha_X : X^1 \to X^0$ and $\omega_X : X^1 \to X^0$ denote the "initial vertex" and "terminal vertex" functions. Finally, we have an inversion function $e \leftrightarrow \bar{e}$ which is an involution of $X^1$, with $\bar{e} \neq e$, such that $\alpha_X(\bar{e}) = \omega_X(e)$. Usually we omit the subscript $X$ from the notation, if it is clear from the context.

A *geometric edge* a subset of $X^1$ of the form $\{e, \bar{e}\}$.

**Remark.** The intuition behind this formalization is that we want to describe *unoriented* graphs; for each geometric edge we have two distinct elements of $X^1$ representing the two ways (orientations) of crossing the geometric edge. Loops (edges with $\alpha(e) = \omega(e)$) are allowed (notice that $\bar{e} \neq e$ also for loops!), as well as multiple edges sharing the same endpoints. We do not impose restrictions on the cardinality of the vertex or edge sets.

**Topological realization of a graph.** Let $X$ be a graph. Its topological realization $\text{real}(X)$ is the topological space obtained as follows. Endow $X^0$

and $X^1$ with the discrete topologies, the interval $[0, 1]$ with the Euclidean topology, and consider on the disjoint union

$$X^0 \sqcup (X^1 \times [0, 1])$$

the finest equivalence relation $\sim$ such that: $(e, 0) \sim \alpha(e)$ and $(e, t) = (\bar{e}, 1-t)$ for every $e \in X^1$ and $t \in [0, 1]$. Then, $\mathrm{real}(X)$ is the quotient (as a topological space) of $X^0 \sqcup (X^1 \times [0, 1])$ by this equivalence relation.

**Subgraphs.** A subgraph $Z \subseteq X$ is given by subsets $Z^0 \subseteq X^0$ and $Z^1 \subseteq X^1$ so that inversion and endpoint functions restrict to them; with these restricted functions, $Z$ becomes a graph on its own.

**Morphisms of graphs.** A morphisms $f : X \to Y$ between two graphs consists of functions $f : X^0 \to Y^0$ and $f : X^1 \to Y^1$ such that $\overline{f(e)} = f(\bar{e})$ and $f(\alpha_X(e)) = \alpha_Y(f(e))$. It then also follows that $f(\omega_X(e)) = \omega_Y(f(e)))$. If $Z \subseteq X$ is any subgraph, then $f(Z)$ is defined in the obvious way and is a subgraph of $Y$. A morphism is injective/surjective if it is so at the level of both vertices and edges. It is an isomorphism if it is both injective and surjective. Given a graph $X$, the self-isomorphisms $f : X \to X$ are called automorphisms; they form a group under composition, which we denote by $\mathrm{Aut}(X)$.

**Stars and local properties of morphisms.** The star of a vertex $x \in X^0$ is the set $\mathrm{st}_X(x) = \{e \in X^1 : \alpha(e) = x\}$. The valence of $x$ is the cardinality of its star. A morphism $f : X \to Y$ restricts, for every $x \in X^0$, to a function $f_x : \mathrm{st}_X(x) \to \mathrm{st}_Y(f(x))$. The morphism is locally injective/surjective if for every $x \in X^0$ the function $f_x$ is injective/surjective.

**Covering maps.** We say that a morphism is a *covering map* if it is both locally injective and locally surjective. If $f : X \to Y$ is a covering map, then a covering automorphism of $p$ is an automorphism $\varphi \in \mathrm{Aut}(X)$ such that $f \circ \varphi = f$. The covering automorphisms of $f$ form a subgroup of $\mathrm{Aut}(X)$, which we denote by $\mathrm{Aut}_f(X)$.

**Paths and connected graphs.** A path of length 0 is a vertex; a path of length $n \in \mathbb{N}_{>0}$ is a list of edges $e_1 \dots e_n$ such that $\alpha(e_{i+1}) = \omega(e_i)$ for $i \in \{1, \dots, n-1\}$. Any path has an initial vertex and a terminal vertex defined in the obvious way. A path is closed if its initial and terminal vertices coincide. A path is reduced if it either has length 0, or it has length $n \geq 1$ and $e_{i+1} \neq \bar{e_i}$ for every $i \in \{1, \dots, n-1\}$. A graph $X$ is connected if it is nonempty ($X^0 \neq \emptyset$) and for every $x, y \in X^0$ there exists a path joining $x$ to $y$ (*i.e.*, with initial vertex $x$ and terminal vertex $y$).

**Circuits and trees.** For every positive integer $n$, let $C_n$ be the "standard circle with $n$ vertices": the graph with vertex set $\{1, \dots, n\}$ and edges connecting them in a circular fashion (formally, $C_n^1$ would be a set with $2n$ elements, *e.g.*, $C_n^1 = \{e \in \mathbb{Z} : 1 \le |e| \le n\}$, with endpoint and inversion functions defined appropriately). A circuit in a graph $X$ is a subgraph isomorphic to $C_n$ for some positive integer $n$; in other words, it is the image of an injective morphism $C_n \to X$. A tree is a connected graph that doesn't have any circuits.

**Equivalence relations and quotients.** An equivalence relation on a graph $X$ is given by two equivalence relations, one on $X^0$ and one on $X^1$, that satisfy the following property (we denote with $\sim$ the relation both for vertices and edges): if $e_1 \sim e_2$ are equivalent edges, then $\alpha(e_1) \sim \alpha(e_2)$ and $\overline{e_1} \sim \overline{e_2}$ (then, it follows that also $\omega(e_1) \sim \omega(e_2)$). The equivalence relation is *without inversions* if no edge is equivalent to its inverse. Given an equivalence relation $\sim$ without inversions, the quotient $X/\sim$ is formed in the natural way, with endpoint and inversion functions inherited from $X$ (if there were inversions, the quotient wouldn't be a graph! There would be an edge inverse of itself). There is a natural projection $X \to X/\sim$, which is a graph morphism.

**Group actions and quotients.** A (left) action $G \curvearrowright X$ of a group $G$ on a graph $X$ is given by two (left) actions $G \curvearrowright X^0$ and $G \curvearrowright X^1$ so that endpoint and inversion functions are $G$-equivariant. In other words, it is a homomorphism $G \to \mathrm{Aut}(X)$. We say that $G$ acts *without inversions* if $g \cdot e \ne \overline{e}$ for every $g \in G$ and $e \in X^1$. An action $G \curvearrowright X$ induces an equivalence relation (the orbit relation) on $X$, which is without inversions precisely when the action is without inversions. The resulting quotient, which is defined if the action is without inversions, is denoted by $G \backslash X$. An action is *free* if it has no inversions and is free on vertices, *i.e.*, $g \cdot x \ne x$ for every $x \in X^0$ and $g \in G \setminus \{1_G\}$ (it follows that $G$ acts freely also on $X^1$).

**Generating systems and rank of a group.** Let $G$ be a group. A generating system for $G$ is given by a set $S$ and a function $[\cdot]_G : S \to G$ such that its image $[S]_G$ generates $G$ as a group, *i.e.*, every element of $G$ is equal to a product of the form $1_G \cdot [s_1]_G^{\varepsilon_1} \cdot \dots \cdot [s_k]_G^{\varepsilon_k}$ for some $k \in \mathbb{N}$, $s_i \in S$ and $\varepsilon_i \in \{\pm 1\}$. When we say "Let $S$ be a generating system for $G$" we mean that we have fixed a function $S \to G$ as above, which is often kept implicit in the notation (so we can write things like $s_1 s_2 \in G$. In other words, we think of any $s \in S$ as a *name* of an element of $G$, and in principle an element of $G$ could have multiple different "names" in $S$). Often, $S$ is just a subset of $G$ (which generates $G$) and the map is the inclusion: $[s]_G = s$. The rank of $G$, denoted by $\mathrm{rank}(G)$, is the minimum cardinality of a generating system of

$G$ (*i.e.*, of a set $S$ as above), which is the same as the minimum cardinality of a subset of generators of $G$.

**Cayley graphs.** Let $G$ be a group and $S$ be a generating system of $G$. Let $\overline{S}$ be a disjoint copy of $S$ (a set of formal "inverses"), with a bijection $S \leftrightarrow \overline{S}$, which we write as $s \mapsto \overline{s}$; we extend the function $s \mapsto [s]_G \in G$ to the disjoint union $S \sqcup \overline{S}$ by setting $[\overline{s}]_G = [s]_G^{-1}$. The Cayley graph of $G$ with respect to $S$, which we denote by $\Gamma(G, S)$, is the graph with:

- $\Gamma(G, S)^0 = G$;

- $\Gamma(G, S)^1 = G \times (S \sqcup \overline{S})$;

- $\alpha((g, s)) = g$, $\omega((g, s)) = g \cdot [s]_G$, for every $(g, s) \in G \times (S \sqcup \overline{S})$;

- Inversion function $\overline{(g, s)} = (g \cdot [s]_G, \overline{s})$.

There is a standard action $G \curvearrowright \Gamma(G, S)$, given by $g \cdot x = gx$ for vertices $x \in G$, and by $g \cdot (x, s) = (gx, s)$ for edges $(x, s) \in G \times (S \sqcup \overline{S})$.

**Bases and free groups.** Let $G$ be a group. A subset $S \subseteq G$ is a basis of $G$ if for every $g \in G$ we can write in a *unique* way

$$g = 1_G \cdot s_1^{\varepsilon_1} \cdot \ldots \cdot s_k^{\varepsilon_k}$$

with $k \in \mathbb{N}$, $s_i \in S$, $\varepsilon_i \in \{\pm 1\}$ with $\varepsilon_{i+1} = \varepsilon_i$ whenever $s_{i+1} = s_i$. A group is free if it admits a basis. Given any set $S$, there is a canonical construction of a free group $F_S$ and an injective function $S \hookrightarrow F_S$ so that the image of $S$ is a basis of $F_S$. We usually think of $S$ as included into $F_S$, so that $S$ itself is a basis. In general, we say that a generating system $S$ of a group $G$ is a basis if the function $S \to G$ is injective and its image is a basis.

# 1 First lecture (November 6)

📖 **Exercise 1.1.** What are the regular graphs of valence 2?

**Definition 1.2** (Standard lines). For every $n \in \mathbb{N}$, including $n = 0$, denote by $P_n$ the graph $X$ with vertex set $\{0, \ldots, n\}$, and edges "connecting the vertices in a line, in increasing order". The formal description would require $P_n^1$ to consist of $2n$ elements, *e.g.*, $P_n^1 = \{e \in \mathbb{Z} : 1 \leq |e| \leq n\}$, and endpoint and inversion functions defined appropriately. Also define $P_\infty$ (the standard infinite ray) with vertex set $\mathbb{N}$ and edges connecting consecutive numbers.

📖 **Exercise 1.3.** Prove that a path of length $n$ in a graph $X$ "is the same" as a morphism $P_n \to X$, and that reduced paths correspond to locally injective morphisms.

📖 **Exercise 1.4.** If two vertices of a graph are joined by a path, then they are joined by an injective (hence, reduced) path.

🐦 **Exercise 1.5.** Let $X$ be a connected graph in which each vertex has finite valence (we say that $X$ is *locally finite*). Prove that, if $X$ does not contain an infinite injective ray (*i.e.*, there is no injective morphism $P_\infty \to X$, see Definition 1.2), then $X$ is finite.

📖👁 **Exercise 1.6.** Let $T$ be a tree. Show that any closed reduced path in $T$ has length 0. Show that a path in $T$ is reduced if and only if it is a length-minimizing path, *i.e.*, it minimizes the length among paths with the same endpoints (hint: induction on the minimum length of paths joining the endpoints; the base case — length 0 — corresponds to the first sentence).

Recall that these paths are called geodesics; for a graph which is not necessarily a tree, we call geodesic any length-minimizing path.

📖👁 **Exercise 1.7.** Let $X$ be a connected graph. Show that $X$ is a tree if and only if any reduced path in $X$ is length-minimizing, if and only if any closed reduced path in $X$ has length 0.

📖 **Exercise 1.8.** Every nonempty graph has a maximal subtree with respect to inclusion (we have seen in the lecture that, if the graph is connected, such a maximal subtree contains all the vertices, and is called a spanning tree).

📖👁 **Exercise 1.9** (Sets hanging from a tree). Let $T$ be a tree. Suppose we have a (possibly infinite) sequence of sets $V_1, V_2, \ldots$ and functions

$$T^0 \xleftarrow{\ f_1\ } V_1 \xleftarrow{\ f_2\ } V_2 \xleftarrow{\ f_3\ } \ldots .$$

Form a new graph $X$ whose vertex set is the disjoint union of $T^0$ and all the $V_i$'s, and having the same edges as $T$, plus one geometric edge (formally, two edges inverses of each other) for each $v \in V_i$, joining $v$ and $f_i(v)$. Prove that $X$ is a tree. This is a very natural construction of a tree, which often occurs in the special case where $T$ has just one vertex.

## 2 Second lecture (November 10)

📖 **Exercise 2.1** (Morphisms on topological realizations)**.** Describe how a graph morphism $f : X \to Y$ induces a continuous map $\mathrm{real}(f) : \mathrm{real}(X) \to \mathrm{real}(Y)$; show that the induced map is injective/surjective if and only if $f$ is so.

📖 **Exercise 2.2** (Combinatorial vs. topological conditions)**.** A graph $X$ is connected if and only if its topological realization is path-connected; it is finite if and only if its topological realization is compact.

◉ **Exercise 2.3** (Trees are contractible)**.** Show that the topological realization of a tree is contractible. (Use geodesics to retract to a chosen vertex.)

**Exercise 2.4.** Let $X$ be a tree and let $S$ be a nonempty set of vertices of $X$, with diameter $n$. Then, the subtree generated by $S$ (which is defined as the smallest subtree of $X$ containing $S$) has diameter equal to $n$.

📖 **Exercise 2.5** (Relations and surjective morphisms)**.** Let $p : X \to Y$ be a surjective graph morphism. Show that the relation $\sim$ on $X$ defined as $x_1 \sim x_2 \iff f(x_1) = f(x_2)$ for vertices, and with the analogous definition for edges, is an equivalence relation on $X$ without inversions, and that the quotient $X/\sim$ is isomorphic to $Y$, with $p : X \to Y$ corresponding to the quotient projection. In the opposite direction, if $X$ is a graph and $\sim$ is an equivalence relation on $X$ without inversions, then the projection $X \to X/\sim$ is a surjective morphism.

📖 ◉ **Exercise 2.6** (Lifting trees[1])**.** Let $f : X \to Y$ be a locally surjective morphism of graphs, $T \subset Y$ be a subtree, and $x_0 \in X^0$ such that $f(x_0) \in T^0$. Then, there exists a subtree $L \subseteq X$ containing $x_0$ that lifts $T$, *i.e.*, $f$ restricts to an isomorphism $L \cong T$.

📖 **Exercise 2.7.** Show that the conclusion of Exercise 2.6 doesn't necessarily hold if $f$ is a surjective morphism.

◉ **Exercise 2.8.** If a group $G$ acts on a graph $X$ without inversions, then the projection $X \to G\backslash X$ is a locally surjective morphism (therefore, subtrees can be lifted as in Exercise 2.6).

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[1]During the lecture, an imprecise version was stated, which does not hold, see Exercise 2.7 (and Exercise 2.5); this exercise amends that statement, and the proof given in class actually works as a solution.

# 3 Third lecture (November 13)

**Exercise 3.1.** Let $G$ be a free group with a basis $S$. Then, rank($G$) is equal to the cardinality of $S$.

**Exercise 3.2.** Let $f : X \to Y$ be a locally surjective morphism between connected graphs. Show that $f$ is surjective.

**Exercise 3.3.** A graph morphism $f : X \to Y$ is a covering map if and only if the induced continuous function real$(f) :$ real$(X) \to$ real$(Y)$ is a covering map in the topological sense.

**Exercise 3.4.** Let $f : X \to Y$ be a covering map of graphs. Show that any covering automorphism (in the topological sense) of real$(f)$ is induced by a unique covering automorphism of $p$. This gives an isomorphism between $\text{Aut}_f(X)$ and the group of (topological) covering automorphisms of real$(f)$.

**Exercise 3.5.** Consider the generating system of $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ given by the set $S = \{a, b\}$ whose elements are sent to $G$ as $a \mapsto ([1], [0])$, $b \mapsto ([0], [1])$. Draw $\Gamma(G, S)$. Then, find a basis of the kernel of $F_S \twoheadrightarrow G$.

**Exercise 3.6.** Let $F_2$ be a free group of rank 2, with basis $S = \{a, b\}$, and let $f : F_2 \to \mathbb{Z} \times \mathbb{Z}$ be the homomorphism sending $a \mapsto (1, 0)$ and $b \mapsto (0, 1)$. What is the rank of the kernel of $f$? Find a basis of it.

**Exercise 3.7.** Let $S$ be a generating system of a group $G$. Show that the Cayley graph $\Gamma(G, S)$ is a tree if and only if $S$ is a basis of $G$, meaning that $S \to G$ is injective and its image is a basis of $G$.