5.2.2 Taylor-Expansion Method

The precision of an integration routine can be boosted, by evaluating y_{n+1} starting from y_n via a Taylor-expansion carried to higher order.

$$y_{n+1} = y_n + h\left[f(y_n, x_n) + \frac{h}{2}f'(y_n, x_n) + \ldots + \frac{h^{p-1}}{p!}f^{(p-1)}(y_n, x_n) + \ldots\right] + \mathcal{O}(h^{p+1})$$

Here $f^{(k)}(y_n, x_n)$ denotes the kth derivative of f w.r.t. x, evaluated in x_n. Noting that x appears both explicitly and implicitly through the x dependence of y one gets

$$\begin{aligned}
f^{(1)}(y_n, x_n) &= f_x(y_n, x_n) + f_y(y_n, x_n)\ y'(x_n) \\
&= f_x(y_n, x_n) + f_y(y_n, x_n)\ f(y_n, x_n) \\
f^{(k)}(y_n, x_n) &= f_x^{(k-1)}(y_n, x_n) + f_y^{(k-1)}(y_n, x_n)\ y'(x_n) \\
&= f_x^{(k-1)}(y_n, x_n) + f_y^{(k-1)}(y_n, x_n)\ f(y_n, x_n)
\end{aligned}$$

with $f_x(y_n, x_n) = \frac{\partial}{\partial x} f(y, x)|_{y_n, x_n}$ and similarly for the partial derivative w.r.t. y
The local error of the Taylor-expansion algorithm of order p is $\mathcal{O}(h^{p+1})$, the global error $\mathcal{O}(h^p)$. The main disadvantage of this approach is that it requires recursively computing possibly high partial derivatives of $f(y, x)$

• Euler's-method is nothing but the $p = 1$ case of the Taylor-expansion algorithm.

5.2.3 Runge-Kutta Methods

The Taylor-expansion algorithm and its special case, Euler's method are so-called one-step methods: they propagate a solution in a single step across a discretization interval of length h.

The idea and aim of Runge-Kutta methods is to approximate Taylor-expansion methods, however, by replacing evaluations of higher derivatives of f in terms of evaluations at intermediate steps.

• Illustration on the **2nd order method**: Second order, i.e. curvature, information is contained in variation of the first order derivative along the discretization interval. This observation is exploited by setting

$$y_{n+1} = y_n + h\left[\alpha_1 f(y_n, x_n) + \alpha_2 f(\hat{y}_n, \hat{x}_n)\right]$$

with "intermediate coordinates"

$$\hat{x}_n = x_n + \beta_1 h\ , \qquad \hat{y}_n = y_n + \beta_2 h f(y_n, x_n)$$

and by choosing the constants α_i and β_i such that the Taylor-expansion method at order h^2 is recovered, i.e. such that

$$R = \alpha_1 f(y_n, x_n) + \alpha_2 f(\hat{y}_n, \hat{x}_n) = \alpha_1 f(y_n, x_n) + \alpha_2 f(y_n + \beta_2 h f(y_n, x_n), x_n + \beta_1 h)$$

and

$$T = f(y_n, x_n) + \frac{h}{2}f'(y_n, x_n)$$