5.2 Elementary Algorithms

It is sufficient to consider systems of 1st order ODEs. To see this, note that an ODE of higher order (we shall assume that it can be transformed into the so-called normal form)

$$y^{(n)} = f(\{y^{(k)}\}_{k=0,\dots,n-1}, x)$$

can be transformed into a system of 1st order equations by defining

$$y_0 = y \ , \quad y'_k = y_{k+1} \ , k = 0, \dots, n-2 \ ,$$

which results in

$$\begin{aligned} y'_0 &= y_1 \\ y'_1 &= y_2 \\ \vdots &= \vdots \\ y'_{n-2} &= y_{n-1} \\ y'_{n-1} &= f(\{y_k\}_{k=0,\dots,n-1}, x) \end{aligned} \tag{5.32}$$

Writing this in vectorial form, one has

$$\boldsymbol{y}' = \boldsymbol{f}(\boldsymbol{y}, x) \ . \tag{5.33}$$

• **Existence and Uniqueness** of solutions of initial value problems (5.33) with $\boldsymbol{y}(x_0) = \boldsymbol{y}_0$ in a neighbourhood of $x_0, \boldsymbol{y}_0$ is granted, if $\boldsymbol{f}(\boldsymbol{y}, x)$ and $\nabla_{\boldsymbol{y}} \boldsymbol{f}(\boldsymbol{y}, x)$ are *continuous* functions of both arguments $\boldsymbol{y}$ and x in a neighbourhood of $x_0, \boldsymbol{y}_0$.

• For simplicity, we shall in the sequel, when discussing algorithms restrict our attention to a single ODE of 1st order

$$y' = f(y, x) \tag{5.34}$$

You are encouraged to convince yourself of the fact that the generalization to systems of 1st order equations is indeed simple.

• All integration routines are based on discretising coordinates. We shall denote by h the step-width of the discretization, and define $x_n = x_0 + nh$, and $y_n = y(x_n)$.

5.2.1 Euler-Integration

The simplest integration routine, Euler integration, estimates the change of the unknown function over a discretization-interval in terms of the first derivative

$$y'(x_n) = f(y_n, x_n)$$

Euler's algorithm (first order Taylor expansion) then results in

$$y_{n+1} = y_n + h \ f(y_n, x_n) + \mathcal{O}(h^2)$$

This results in an error $\mathcal{O}(h^2)$ for each elementary step of size h. If it is desired to propagate the solution via Euler-steps across a finite interval $[a, b] \equiv [x_0, x_N]$, the number N of steps required scales like $N = (b-a)h^-1$ with the discretization step-width h. The *global* error incurred by integrating from a to b (i.e. the error of y_N) thus scales like $\mathcal{O}(h)$, viz. linearly with step-width h.