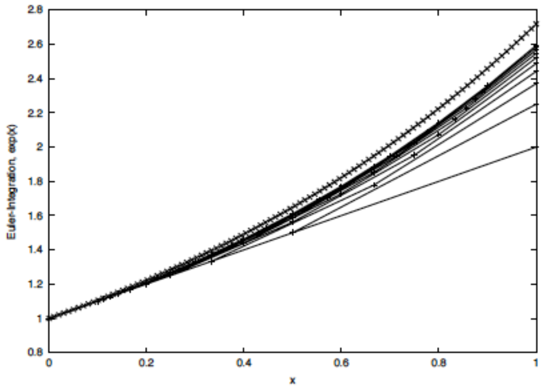


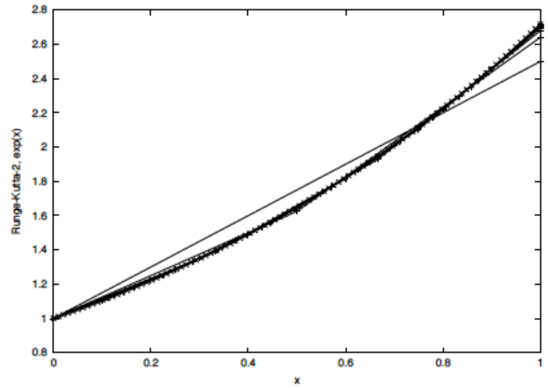


Figure 8: Euler integration (left) and 2nd order Runge-Kutta integration (right) over a unit interval, for step-widths $h = 1$,1/2, 1/3 ..., 1/10, compared with exact solution $e^x$ (crosses).

$$
\begin{aligned}
k_3 &= h\ f(y_n + \frac{k_2}{2}, x_n + \frac{h}{2}) \\
k_4 &= h\ f(y_n + k_3, x_n + h) \\
y_{n+1} &= y_n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6}\ , \qquad x_{n+1} = x_n + h
\end{aligned}
$$

It requires 4 evaluations per discretization interval, and wins two orders of $h$ in precision compared to RK2; i.e. the gain of efficiency of RK4 relative to RK 2 is like that of RK2 relative to Euler.

## 5.3 Adaptive Step-size Control; Bulirsch-Stoer Algorithm

• **Adaptive Step-size Control:** Further efficiency is gained by choosing step-widths large, where functions are smooth, and small, only when higher derivatives become large. Drawback: the user no longer controls him/her-self, at which points the r.h.s – the function $f(y, x)$ – is evaluated.

• **Bulirsch-Stoer Algorithm:** The Bulirsch-Stoer algorithm integrates the ODE using a sequence of successively smaller step-widths (e.g. by factors of 2) and extrapolates the results to $h = 0$.

The underlying idea is either Richardson-Extrapolation, or extrapolation of a polynomial of given order that can be fitted through the results obtained for different $h$

The idea of Richardson-Extrapolation is as follows: Let $y(h)$ denote the value of the solution computed at a certain point using a discretization step of size $h$. Let us assume that we use RK4 and have a dominant error of order $h^4$, and let us further assume that for the case in question the subdominant error is of order $h^6$:

$$y(h) = y(0) + c_4 h^4 + c_6 h^6 + \dots\ .$$

one then has

$$y(h/2) = y(0) + \frac{1}{16} c_4 h^4 + \frac{1}{64} c_6 h^6 + \dots$$