Minimum Enclosing Circle

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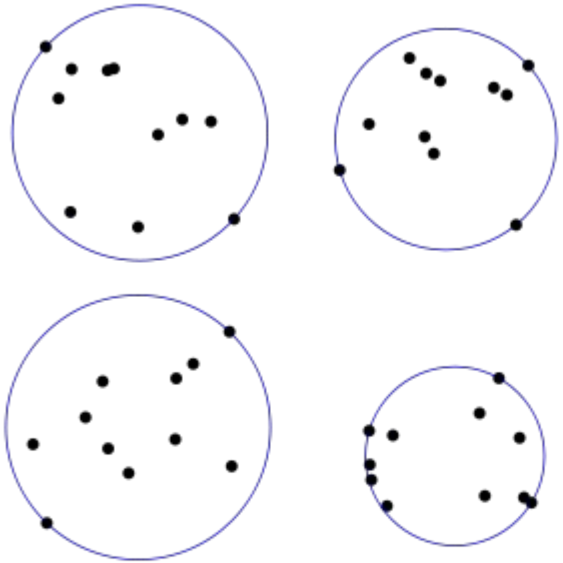

Problem Formulation

Let the given set of n points be denoted by $P := \{p_1, p_2, \ldots, p_n\}$, where each $p_i \in \mathbb{R}^2$. Let $C(P)$ denote the minimal enclosing circle of the set P. As a circle can be defined by two parameters: the radius r and the centre p_o, the minimal enclosing circle $C(P)$ of a set of points P is

$$C(P) := argmin_{r,p_o} r^2 \quad s.t. \quad \|p_i - p_o\|^2 \leq r^2 \quad \forall i \in \{1, 2, \ldots, n\} \tag{1}$$

Brute Force

Two facts to note:

- Minimum enclosing circle always intersects at least two points A and B
- If line AB is not diameter, circle diameter can be shifted towards AB until it intersects a third point

Brute Force Approach:

- Try every pair and triplet to obtain a circle defined by those points → $O(n^2)$ pairs and $O(n^3)$ triplets
- Check if all other points lie within the circle, and record minimum area so far → $O(n)$
- Overall running time is $O(n^3 + n^4) \rightarrow O(n^4)$

Welzl's Randomized Algorithm

A randomized recursive algorithm that runs in O(n) time !!

The idea is for a set P containing (p_1 ... p_n), we randomly and uniformly choose a point p_i and recursively compute the MEC for P - { p_i }.

- If p_i is within the computed MEC, we are done
- Else, p_i must lie on the boundary of the MEC

Base case is reached if P becomes empty or an MEC can be found from the remaining points, i.e. There are 3 or less points.

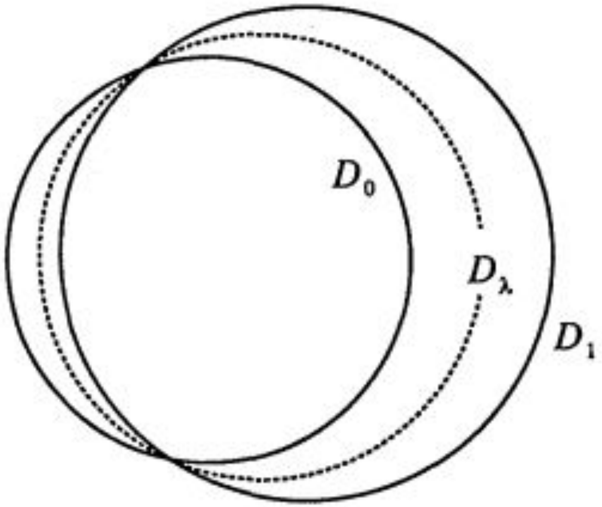

Figure 1: Continous deformation of disk D_0 into disk D_1 in proof of Lemma 1.

```
function procedure B_MINIDISK(P, R); comment:  returns b_md(P, R)
    if P = ∅ [or |R| = 3] then
        D := b_md(∅, R)
    else
        choose random p ∈ P;
        D := B_MINIDISK(P − {p}, R);
        if [D defined and] p ∉ D then
            D := B_MINIDISK(P − {p}, R ∪ {p});
    return D;
```

Implementation & Experiments

- For comparison, generate points on a plane using
  - Gaussian distribution
  - Multimodal
- Find MEC using both algorithms (implemented in python)
- Track growth rate of both algorithms

- We expect
  - Welzl’s to always find the solution much faster
  - Brute force to be only tractable on a much smaller n than Welzl’s

References

Welzl E. (1991) Smallest enclosing disks (balls and ellipsoids). In: Maurer H. (eds) New Results and New Trends in Computer Science. Lecture Notes in Computer Science, vol 555. Springer, Berlin, Heidelberg. https://doi.org/10.1007/BFb0038202

Thank you!!