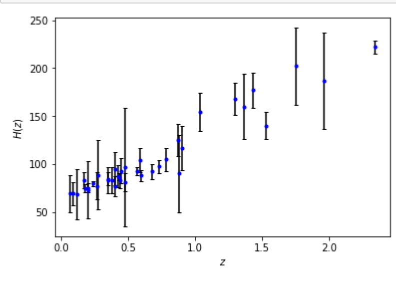
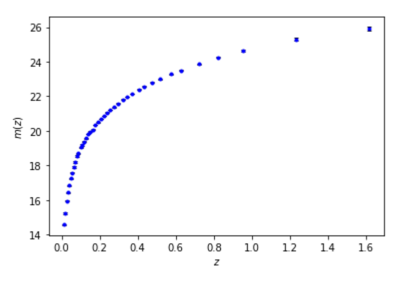
```
In [4]: import numpy as np
        import matplotlib.pyplot as plt
        from time import time
        import pandas as pd
        from scipy.constants import c
        from scipy.integrate import quad

        #Nested sampling package
        import ultranest
        import corner
```

I. Import data

```
In [5]: # Import Hubble H(z) data

        hubble_data = pd.read_csv('hubble_data.csv', header=0)

        z_H = np.array(hubble_data['z'])
        H = np.array(hubble_data['H'])
        dH = np.array(hubble_data['dH'])

        plt.figure()
        plt.errorbar(z_H, H, yerr=dH, marker = '.', color='blue', ecolor='black', capsize=2, ls='none')
        plt.ylabel(r'$H(z)$')
        plt.xlabel(r'$z$')
        plt.show()

        # Import apparent magnitude m(z) data
        m_data = pd.read_csv('m_data.txt', sep = ' ', header = 0)
        m_sys_unc = pd.read_csv('m_sys_unc.txt', sep = ' ', header = 0)
        m_sys_unc = np.array(m_sys_unc['40']).reshape(40, 40)

        tot = m_sys_unc + np.diag(m_data['dmb']**2)

        z_m = np.array(m_data['zcmb'])
        m = np.array(m_data['mb'])
        dm = np.sqrt(np.diag(tot))

        plt.figure()
        plt.errorbar(z_m, m, yerr=dm, marker = '.', color='blue', ecolor='black', capsize=2, ls='none')
        plt.ylabel(r'$m(z)$')
        plt.xlabel(r'$z$')
        plt.show()

        # Combine redshifts for likelihood computation later
        combined_z = []
        combined_z.append(z_H)
        combined_z.append(z_m)

        # Combine data
        combined_data = []
        combined_data.append(H)
        combined_data.append(m)

        # Combine uncertainties
        combined_unc = []
        combined_unc.append(dH)
        combined_unc.append(dm)
```

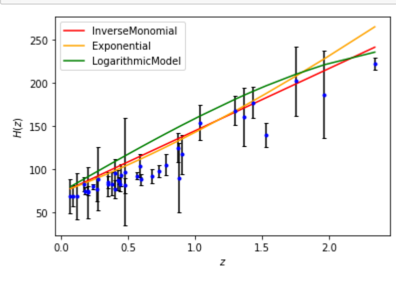


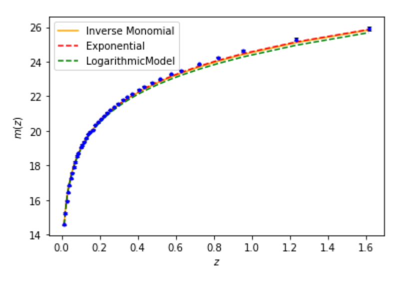


II. Define models

```
In [36]: 'Define Inverse Monomial model'
         def InverseMonomial(z, params):
             H0 = params[0]
             B = params[1]

             return H0*np.sqrt((1/(1+B*z))*(1+z)**3)

         'Define Exponential Model'
         def Exponential(z, params):
             H0 = params[0]
             B = params[1]

             return H0*np.sqrt(np.exp(B*((1/(1+z))-1))*(1+z)**3)

         'Define apparent magnitude function'
         def ApparentMagnitude(z, Hubble, params):

             def integrand_dl(z, Hubble, params):
             #integrand of luminosity distance formula
                 return params[0]/Hubble(z, params)

             def dl(z, Hubble, params):
             #dimensionless luminosity distance at redshift z (input array)
                 rz_array = np.zeros(len(z))
                 for i in np.arange(len(z)):
                     rz_each = quad( integrand_dl, 0, z[i],
                                     args = (Hubble, params) )[0]
                     rz_array[i] = rz_each

                 return (1+z)*rz_array

             return 5*np.log10((c*100/params[0])*dl(z, Hubble, params)) - 19.25
```

```
In [43]: params_InverseMonomial = [72.21, 1]
         params_Exponential = [73.37, 1.5]

         plt.figure()
         plt.errorbar(z_H, H, yerr=dH, marker = '.', color='blue', ecolor='black', capsize=2, ls='none')
         plt.plot(z_H, InverseMonomial(z_H, params_InverseMonomial), color='red', label='InverseMonomial')
         plt.plot(z_H, Exponential(z_H, params_Exponential), color='orange', label='Exponential')
         plt.legend(loc='best')
         plt.ylabel(r'$H(z)$')
         plt.xlabel(r'$z$')
         plt.show()

         plt.figure()
         plt.errorbar(z_m, m, yerr=dm, marker = '.', color='blue', ecolor='black', capsize=2, ls='none')
         plt.plot(z_m, ApparentMagnitude(z_m, InverseMonomial, params_InverseMonomial), color='orange', label='I
         nverse Monomial')
         plt.plot(z_m, ApparentMagnitude(z_m, Exponential, params_Exponential), color='red', ls='--', label='Exp
         onential')
         plt.legend(loc='best')
         plt.ylabel(r'$m(z)$')
         plt.xlabel(r'$z$')
         plt.show()
```





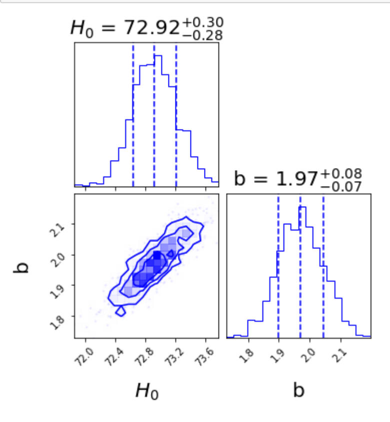
III. Nested Sampling: Inverse Monomial

```
In [77]: def Prior_InverseMonomial(cube):

             #H0 Prior: [0,100]
             H0_min = 0
             H0_max = 100

             #B Prior: [0,4]
             B_min = 0
             B_max = 4

             #M Prior: [0,1]
             #M_min = -21
             #M_max = -18

             #Extract values
             H0prime = cube[0]
             Bprime = cube[1]

             #OLprime = cube[2]
             #Mprime = cube[3]

             H0 = H0prime*(H0_max-H0_min) + H0_min
             B = Bprime*(B_max-B_min) + B_min
             #OL = OLprime*(OL_max-OL_min) + OL_min
             #M = Mprime*(M_max-M_min) + M_min

             return np.array([H0, B])

         def LogLikelihood_InverseMonomial(params):


             # calculate the model
             hubble_model = InverseMonomial(z_H, params)
             apparent_magnitude_model = ApparentMagnitude(z_m, InverseMonomial, params)

             #calculate the likelihood
             residual_H = H - hubble_model
             residual_m = m - apparent_magnitude_model
             sig_H = 1/dH
             sig_m = 1/dm

             lnL_H = -0.5*np.sum((residual_H*sig_H)**2)
             lnL_m = -0.5*np.sum((residual_m*sig_m)**2)

             return  lnL_H + lnL_m
```

```
In [57]: t_i = time()
         sampler_InverseMonomial = ultranest.ReactiveNestedSampler(['H0', 'B'], LogLikelihood_InverseMonomial, P
         rior_InverseMonomial)
         result_InverseMonomial = sampler_InverseMonomial.run()
         sampler_InverseMonomial.print_results()
         t_f = time()

         print('Sampling time: {} s'.format(t_f-t_i))

         [ultranest] Sampling 400 live points from prior ...


         [ultranest] Explored until L=-5e+01   [-53.9015..-53.9015]*| it/evals=5294/7293 eff=76.8026% N=400
               00
         [ultranest] Likelihood function evaluations: 7296
         [ultranest]   logZ = -62.53 +- 0.0995
         [ultranest] Effective samples strategy satisfied (ESS = 1561.6, need >400)
         [ultranest] Posterior uncertainty strategy is satisfied (KL: 0.46+-0.07 nat, need <0.50 nat)
         [ultranest] Evidency uncertainty strategy is satisfied (dlogz=0.20, need <0.5)
         [ultranest]   logZ error budget: single: 0.14 bs:0.10 tail:0.01 total:0.10 required:<0.50
         [ultranest] done iterating.

         logZ = -62.531 +- 0.179
           single instance: logZ = -62.531 +- 0.139
           bootstrapped   : logZ = -62.532 +- 0.179
           tail           : logZ = +- 0.010
         insert order U test : converged: True correlation: inf iterations

             H0                  72.93 +- 0.30
             B                   1.970 +- 0.076
         Sampling time: 46.60527491569519 s
```

```
In [58]: points_InverseMonomial= np.array(result_InverseMonomial["weighted_samples"]["points"])
         weights_InverseMonomial= np.array(result_InverseMonomial["weighted_samples"]["weights"])
         scaledweights_InverseMonomial = weights_InverseMonomial / weights_InverseMonomial.max()
         mask_InverseMonomial = np.random.rand(len(scaledweights_InverseMonomial)) < scaledweights_InverseMonomi
         al

         samples_InverseMonomial = points_InverseMonomial[mask_InverseMonomial, :]
```

```
In [59]: # Open saved samples
         def open_samples(file):

             chains = pd.read_csv(file)
             nparams = len(chains.columns)

             if nparams == 2:
                 samples = np.vstack((np.array(chains.iloc[:, 0]), np.array(chains.iloc[:, 1]))).T
             if nparams == 3:
                 samples = np.vstack((np.array(chains.iloc[:, 0]), np.array(chains.iloc[:, 1]), np.array(chains.
         iloc[:, 2]))).T
             if nparams == 4:
                 samples = np.vstack((np.array(chains.iloc[:, 0]), np.array(chains.iloc[:, 1]), np.array(chains.
         iloc[:, 2]), np.array(chains.iloc[:, 3]))).T

             return samples

         # Extract and compile chains
         InverseMonomial_chain_1 = samples_InverseMonomial[:,0] # extract chain of H0 values
         InverseMonomial_chain_2 = samples_InverseMonomial[:,1] # extract chain if OM values
         #lcdm_chain_3 = samples_LCDM[:,2] # extract chain if OL values
         InverseMonomial_samples = np.vstack((InverseMonomial_chain_1, InverseMonomial_chain_2)).T

         # Save chains and evidence (do not forget)
         np.savetxt('InverseMonomial_chains.csv', InverseMonomial_samples, delimiter=",")
         np.savetxt('InverseMonomial_Z.csv', [[result_InverseMonomial['logz'] , result_InverseMonomial['logzerr'
         ]]])

         # Open saved chains
         #samples_open =  open_samples('InverseMonomial_chains_Aug11_9PM.csv')

         fig = corner.corner(InverseMonomial_samples, labels=[r"$H_0$", "b"],color='blue', label_kwargs={"fontsi
         ze": 20},quantiles=[0.16, 0.5, 0.84],
                                    show_titles=True, title_kwargs={"fontsize": 20})
         fig.savefig('InverseMonomial.pdf')
```



IV. Nested Sampling: Exponential Model

```
In [60]: def Prior_Exponential(cube):

             #H0 Prior: [0,100]
             H0_min = 0
             H0_max = 100

             #B Prior: [0,4]
             B_min = 0
             B_max = 4

             #M Prior: [0,1]
             #M_min = -21
             #M_max = -18

             #Extract values
             H0prime = cube[0]
             Bprime = cube[1]

             #OLprime = cube[2]
             #Mprime = cube[3]

             H0 = H0prime*(H0_max-H0_min) + H0_min
             B = Bprime*(B_max-B_min) + B_min
             #OL = OLprime*(OL_max-OL_min) + OL_min
             #M = Mprime*(M_max-M_min) + M_min

             return np.array([H0, B])

         def LogLikelihood_Exponential(params):


             # calculate the model
             hubble_model = Exponential(z_H, params)
             apparent_magnitude_model = ApparentMagnitude(z_m, Exponential, params)

             #calculate the likelihood
             residual_H = H - hubble_model
             residual_m = m - apparent_magnitude_model
             sig_H = 1/dH
             sig_m = 1/dm

             lnL_H = -0.5*np.sum((residual_H*sig_H)**2)
             lnL_m = -0.5*np.sum((residual_m*sig_m)**2)

             return  lnL_H + lnL_m
```

```
In [61]: t_i = time()
         sampler_Exponential = ultranest.ReactiveNestedSampler(['H0', 'B'], LogLikelihood_Exponential, Prior_Exp
         onential)
         result_Exponential = sampler_Exponential.run()
         sampler_Exponential.print_results()
         t_f = time()

         print('Sampling time: {} s'.format(t_f-t_i))

         [ultranest] Sampling 400 live points from prior ...


         [ultranest] Explored until L=-4e+01   [-38.0629..-38.0629]*| it/evals=5360/7377 eff=76.8238% N=400
           0   00 0
         [ultranest] Likelihood function evaluations: 7396
         [ultranest]   logZ = -46.88 +- 0.07376
         [ultranest] Effective samples strategy satisfied (ESS = 1553.8, need >400)
         [ultranest] Posterior uncertainty strategy is satisfied (KL: 0.46+-0.06 nat, need <0.50 nat)
         [ultranest] Evidency uncertainty strategy is satisfied (dlogz=0.17, need <0.5)
         [ultranest]   logZ error budget: single: 0.14 bs:0.07 tail:0.01 total:0.07 required:<0.50
         [ultranest] done iterating.

         logZ = -46.883 +- 0.168
           single instance: logZ = -46.883 +- 0.140
           bootstrapped   : logZ = -46.883 +- 0.168
           tail           : logZ = +- 0.010
         insert order U test : converged: True correlation: inf iterations

             H0                  73.31 +- 0.26
             B                   2.119 +- 0.050
         Sampling time: 39.31010365486145 s
```

```
In [63]: points_Exponential = np.array(result_Exponential["weighted_samples"]["points"])
         weights_Exponential = np.array(result_Exponential["weighted_samples"]["weights"])
         scaledweights_Exponential = weights_Exponential/ weights_Exponential.max()
         mask_Exponential = np.random.rand(len(scaledweights_Exponential)) < scaledweights_Exponential

         samples_Exponential = points_Exponential[mask_Exponential, :]
```

```
In [64]: # Open saved samples
         def open_samples(file):

             chains = pd.read_csv(file)
             nparams = len(chains.columns)

             if nparams == 2:
                 samples = np.vstack((np.array(chains.iloc[:, 0]), np.array(chains.iloc[:, 1]))).T
             if nparams == 3:
                 samples = np.vstack((np.array(chains.iloc[:, 0]), np.array(chains.iloc[:, 1]), np.array(chains.
         iloc[:, 2]))).T
             if nparams == 4:
                 samples = np.vstack((np.array(chains.iloc[:, 0]), np.array(chains.iloc[:, 1]), np.array(chains.
         iloc[:, 2]), np.array(chains.iloc[:, 3]))).T

             return samples

         # Extract and compile chains
         Exponential_chain_1 = samples_Exponential[:,0] # extract chain of H0 values
         Exponential_chain_2 = samples_Exponential[:,1] # extract chain if OM values
         #lcdm_chain_3 = samples_LCDM[:,2] # extract chain if OL values
         Exponential_samples = np.vstack((Exponential_chain_1, Exponential_chain_2)).T

         # Save chains and evidence (do not forget)
         np.savetxt('Exponential_chains.csv', Exponential_samples, delimiter=",")
         np.savetxt('Exponential_Z.csv', [[result_Exponential['logz'] , result_Exponential['logzerr']]])

         # Open saved chains
         #samples_open =  open_samples('InverseMonomial_chains_Aug11_9PM.csv')

         fig = corner.corner(Exponential_samples, labels=[r"$H_0$", "b"],color='blue', label_kwargs={"fontsize":
          20},quantiles=[0.16, 0.5, 0.84],
                                    show_titles=True, title_kwargs={"fontsize": 20})
         fig.savefig('Exponential.pdf')
```