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# Pyramid Image and Resize Based on Spline Model

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**Abstract:** The paper is based around the formalization of the image model as a linear combination of B-splines, which is close to interpolation. The authors present, on average, its corresponding explicit aspects and low-frequency filtering and scaling operators. The possibility to obtain digital images scaled to an arbitrary, not necessarily integer, number of times is demonstrated in the article and the corresponding algorithm is provided. The article provides with the examples on estimation of the quality of approximation of the indicated spline model. Also there are given grounds for its introduction as an alternative to the well-known image model based on the two-dimensional Gaussian function. It is noted that with the increasing order, B-splines differ little from Gaussian, and their simpler calculation makes the spline model attractive for research and use. Applying the well-known formalization of the approach to the construction of a pyramid of digital images based on Gaussian functions, the authors suggest its extension onto the case of a spline model. The use of image pyramids is conditioned by the task of finding special points in a digital image in order to determine the unambiguous correspondence between the images of the same object in different digital photographs. The paper presents linear operators based on B-splines of 2-6 orders aimed at the construction of a pyramid, it also demonstrates an example of their usage. Based on the convolution of the raster with a mask with variable coefficients the possibility to obtain digital images scaled to an arbitrary, not necessarily integer, number of times is demonstrated in the article and the corresponding algorithm is provided. Image resizing based on the suggested algorithm is also demonstrated by examples. The authors believe that the research conducted in the paper in the future will allow for digital images to obtain more computationally simple algorithms for determining special points and their detectors. Results of paper: 1. The model of a DI has been formalized on the basis of two-dimensional polynomial splines, on the basis of B-splines of the second-sixth orders which are close to interpolation on the average. 2. The convolution operators of low-frequency DI filtering based on the spline model are presented. 3. Provided are the scaling operators used to build image pyramids, in order to further search for special points. 4. An algorithm for scaling the DI to an arbitrary, not necessarily an integer number of times based on a continuous spline approximation has been suggested. 5. Algorithm for scaling a digital image based on a spline model allows you to change the size of the image in any (not necessarily an integer) number of times, differs in that it provides high scaling accuracy and no artifacts due to high approximate properties and smoothness of the spline model;6. The scaling algorithm allows digital image processing at high computational speed due to the optimal computational scheme with a minimum of simpler mathematical operations, compared with models based on the two-dimensional Gaussian function.

**Index Terms:** Digital Image Processing, Image Model, Pyramid Image, B-splines, Low-frequency Filtering, Scaling Operators.

## 1. Introduction

Image formalization model as a linear combination of B-splines. According to the digital images (DI) registration method, the data supplied are averaged. Therefore, upon fixation of the analog image the following occurs. Let the image plane be defined by the axes $T$ and $Q$. By directions $T$, $Q$ image sampling step is the same $h > 0$ (by default $h = 1$), therefore, a uniform division is set $\Delta_{hh} : t_i = ih, q_j = jh,\ i = \overline{0, H-1},\ j = \overline{0, W-1},$ where $H$ and $W$ – are the linear dimensions of the fixed DI. Let $\phi(t, q)$ be the function of the pulse response of the system, recording $p(t, q)$ – the function of the lighting intensity of the objects in the spatial scene (analog image). Then, due to the purely

technical characteristics of the registration systems, the result of the convolution $p(t, q)$ and the response function will be the value, averaged in the sampling area, in particular:

$$(p * \phi)(ih, jh) = \frac{1}{h^2} \int_{ih-\frac{h}{2}}^{ih+\frac{h}{2}} \int_{jh-\frac{h}{2}}^{jh+\frac{h}{2}} p(t, q) \phi(t - ih, q - jh) dt dq = \overline{p}_{i,j} \tag{1}$$

Therefore, the sampled values of light intensity (digital image) can be represented as:

$$p_{i,j} = \overline{p}_{i,j} + \varepsilon_{i,j}, \ i = \overline{0, H - 1}, \ j = \overline{0, W - 1}, \tag{2}$$

where $\varepsilon_{i,j}$ – accidental defect. Concerning the defect $\varepsilon_{i,j}$ we can assume any allocation, e.g., Gauss. Thus, when constructing an image model based on data (1), arises the task to use approximations that account for both the random nature of the data and the physical properties of the registration systems; in particular, the operators that are interpolated on average or close to the ones interpolated on average [1].

## 2. Literature Review Section

Traditionally, the problem of analog image modeling is solved as follows [2; 3]. If the sampling of the analog image has been done with the help of rasterization, then the perfect interpolation recovery $p(t, q)$ is performed using a two-dimensional filter with an orthogonal frequency characteristic obtained by the inverse Fourier transformation:

$$w(t, q) = \frac{\sin(\pi t)}{\pi t} \cdot \frac{\sin(\pi q)}{\pi q}.$$

The filtration product can be determined using a two-dimensional convolution of the DI and this pulse characteristic. After the completion of the convolution takes place:

$$p(t, q) = \sum_i \sum_j p_{i,j} \frac{\sin(\pi(t - i))}{\pi(t - i)} \cdot \frac{\sin(\pi(q - j))}{\pi(q - j)}.$$

The given relation is a two-dimensional variant of the Kotelnikov-Nyquist theorem. It indicates the method of the exact interpolational reproduction of a continuous image according to a known sequence of its two-dimensional readings. That is, for accurate reconstruction as an interpolating function one must use two-dimensional functions of $\sin(x)/x$ sort. This statement is fair if the two-dimensional signal spectrum is finite and the sampling intervals are small enough. The validity of the conclusions is violated if at least one of these conditions is not met. Real images rarely have spectra with pronounced cutoff frequencies. One of the reasons that lead to unlimited spectrum is the limited size of the image.

The solution to the indicated problem can be the construction of a model of the DI. However it should find itself on the basis of finite functions that are close in properties to the properties of the analog image in the spectral domain. For example, linear combinations of *B*-splines [4-8] are a computational tool for processing sequences of samples of functions, which has a number of valuable properties: computational simplicity, the ability to take into account local "features" of the signal, smoothing properties and others. Therefore, it may be important to consider the possibility of using the afore mentioned splines in case an analog image model is to be built [1].

## 3. Research Methodology Section

In the monograph [9], to approximate the function $p(t,q)$ by values of type (1) in the partition nodes $\Delta_{h,h}$, linear combinations of B-splines are presented, which, on average, are close to the interpolation ones. For example, spline-operators of zero- and first-degree of refinement on the basis of B-splines of the second order are as follows:

$$S_{2,0}(p,t,q) = \sum_{i \in Z} \sum_{j \in Z} p_{i,j} B_{2,h}(t - ih) B_{2,h}(q - jh), \tag{3}$$

$$S_{2,1}(p,t,q) = \sum_{i \in Z} \sum_{j \in Z} \left( p_{i,j} - \frac{1}{6}\left(\Delta_i^2 p_{i,j} + \Delta_j^2 p_{i,j}\right) + \frac{1}{36}\Delta_{ij}^2 p_{i,j} \right) B_{2,h}(t - ih) B_{2,h}(q - jh), \tag{4}$$

where (precise up to independent variable)

$$B_{2,h}(t) = \begin{cases} 0, & t \notin \left[-3h/2; 3h/2\right], \\ \left(3 + 2t/h\right)^2 / 8, & t \in \left[-3h/2; -h/2\right], \\ 3/4 - \left(2t/h\right)^2 / 4, & t \in \left[-h/2; h/2\right], \\ \left(3 - 2t/h\right)^2 / 8, & t \in \left[h/2; 3h/2\right]; \end{cases} \tag{5}$$

$$\Delta_i^2 p_{i,j} = p_{i-1,j} - 2p_{i,j} + p_{i+1,j}; \quad \Delta_j^2 p_{i,j} = p_{i,j-1} - 2p_{i,j} + p_{i,j+1};$$

$$\Delta_{ij}^2 p_{i,j} = \Delta_i^2 p_{i,j-1} - 2\Delta_i^2 p_{i,j} + \Delta_i^2 p_{i,j+1} = \Delta_j^2 p_{i-1,j} - 2\Delta_j^2 p_{i,j} + \Delta_j^2 p_{i+1,j}.$$

The rationale for choosing the considered splines as an analog image model can be the considerations corresponding to those set out in the learned work [10] for modeling analog one-dimensional signals with finite energy based on similar one-dimensional linear combinations of B-splines.

In particular it is acceptable to use the introduced in the learned work [9] two-dimensional local polynomial splines which are close to interpolation on average functioning as a model of the DI. This is based on the position that the basis of B-splines is Riesz basis as well as the fact that fundamental splines based on B-splines [11] go to zero exponentially fast when digressing from the local $(i, j)$ area of approximation.

If we choose splines (3), (4) as the image model $p(t,q)$, then such an evaluation is actually asymptotically accurate under certain conditions. In particular, if $p(t,q) \in C^{2,2}$, $\left| \varepsilon_{i,j} \right| < \varepsilon$, $i, j \in Z$ and $\forall \varepsilon > 0$, then the fair estimate [9] is as follows:

$$\left\| p(t,q) - S_{2,0}(p,t,q) \right\| \le \frac{h^2}{6} \left\| p''_{t^2}(t,q) \right\| + \frac{h^2}{6} \left\| p''_{q^2}(t,q) \right\| + \frac{h^4}{36} \left\| p^{(4)}_{t^2 q^2}(t,q) \right\| + \varepsilon \cdot \left\| p(t,q) \right\| + o\left(h^4\right), \tag{6}$$

for $\forall p(t,q) \in C^{3,3}$ and $\forall \varepsilon > 0$ just is the inequality

$$\left\| p(t,q) - S_{2,1}(p,t,q) \right\| \le \frac{h^3}{12\sqrt{3}} \left\| p'''_{t^3}(t,q) \right\| + \frac{h^3}{12\sqrt{3}} \left\| p'''_{q^3}(t,q) \right\| + \frac{h^6}{432} \left\| p^{(6)}_{t^3 q^3}(t,q) \right\| + \varepsilon \cdot \frac{16}{9} \left\| p(t,q) \right\| + o\left(h^6\right) \tag{7}$$

Expression (4) provides a high-precision approximation, and the clarifying spline itself (and other similar ones [9]) are operators close to interpolation on average in the asymptotic sense. If as an image approximation you choose expression (3) or any other combination of B–splines of this type:

$$S_{r,0}(p,t,q) = \sum_{i \in Z} \sum_{j \in Z} p_{i,j} B_{r,h_t}(t - ih_t) B_{r,h_q}(q - jh_q), \quad r = 2,3,\ldots, \tag{8}$$

then we obtain a model with the properties of a pulsed non-recursive low-pass filter [12], where, for example (precise up to independent variable)

$$B_{6,h}(t) = \begin{cases} 0, & t \notin \left[-\dfrac{7h}{2}, \dfrac{7h}{2}\right], \\[2mm] \dfrac{1}{720}\left(\dfrac{t}{h}+\dfrac{7}{2}\right)^6, & t \in \left[-\dfrac{7h}{2}, -\dfrac{5h}{2}\right], \\[2mm] -\dfrac{1}{120}\left(\dfrac{t}{h}\right)^6 - \dfrac{7}{60}\left(\dfrac{t}{h}\right)^5 - \dfrac{21}{32}\left(\dfrac{t}{h}\right)^4 - \dfrac{133}{72}\left(\dfrac{t}{h}\right)^3 - \dfrac{329}{128}\left(\dfrac{t}{h}\right)^2 - \dfrac{1267}{960}\left(\dfrac{t}{h}\right) + \dfrac{1379}{7680}, & t \in \left[-\dfrac{5h}{2}, -\dfrac{3h}{2}\right], \\[2mm] \dfrac{1}{48}\left(\dfrac{t}{h}\right)^6 + \dfrac{7}{48}\left(\dfrac{t}{h}\right)^5 + \dfrac{21}{64}\left(\dfrac{t}{h}\right)^4 + \dfrac{35}{288}\left(\dfrac{t}{h}\right)^3 - \dfrac{91}{256}\left(\dfrac{t}{h}\right)^2 + \dfrac{7}{768}\left(\dfrac{t}{h}\right) + \dfrac{7861}{15360}, & t \in \left[-\dfrac{3h}{2}, -\dfrac{h}{2}\right], \\[2mm] -\dfrac{1}{36}\left(\dfrac{t}{h}\right)^6 + \dfrac{7}{48}\left(\dfrac{t}{h}\right)^4 - \dfrac{77}{192}\left(\dfrac{t}{h}\right)^2 + \dfrac{5887}{11520}, & t \in \left[-\dfrac{h}{2}, \dfrac{h}{2}\right], \\[2mm] \dfrac{1}{48}\left(\dfrac{t}{h}\right)^6 - \dfrac{7}{48}\left(\dfrac{t}{h}\right)^5 + \dfrac{21}{64}\left(\dfrac{t}{h}\right)^4 - \dfrac{35}{288}\left(\dfrac{t}{h}\right)^3 - \dfrac{91}{256}\left(\dfrac{t}{h}\right)^2 - \dfrac{7}{768}\left(\dfrac{t}{h}\right) + \dfrac{7861}{15360}, & t \in \left[\dfrac{h}{2}, \dfrac{3h}{2}\right], \\[2mm] -\dfrac{1}{120}\left(\dfrac{t}{h}\right)^6 + \dfrac{7}{60}\left(\dfrac{t}{h}\right)^5 - \dfrac{21}{32}\left(\dfrac{t}{h}\right)^4 + \dfrac{133}{72}\left(\dfrac{t}{h}\right)^3 - \dfrac{329}{128}\left(\dfrac{t}{h}\right)^2 + \dfrac{1267}{960}\left(\dfrac{t}{h}\right) + \dfrac{1379}{7680}, & t \in \left[\dfrac{3h}{2}, \dfrac{5h}{2}\right], \\[2mm] \dfrac{1}{720}\left(\dfrac{t}{h}-\dfrac{7}{2}\right)^6, & t \in \left[\dfrac{5h}{2}, \dfrac{7h}{2}\right]. \end{cases} \tag{9}$$

In particular, in the learned work [11], provided is the proof that like the Gaussian function, any B-spline of the order above the first one can be used to determine the short-time Fourier transformation (STFT). So, if $B_r(t)$, $r \geq 2$ – is a B-spline of order $r$, then [13]:

$$\hat{B}_r(\omega) = \left(\frac{e^{i\omega/2} - e^{-i\omega/2}}{i\omega}\right)^{r+1} = \left(\frac{\sin(\omega/2)}{(\omega/2)}\right)^{r+1}.$$

We shall also mention that starting already from the order $r = 5$, both the B-spline and the Gaussian in the frequency domain actually hardly differ. In addition to that the calculation of the fifth-order B-spline [14] requires less computational costs. Therefore, if there is a need to obtain a digital low-frequency filter of a DI, it is sufficient to determine in model (8) the value of the spline in the nodes of the partition $\Delta_{h,h}$ [15]. For example, if you enter a replacement

$$x = \frac{2}{h}(t - ih), \; |x| \leq 1, \; y = \frac{2}{h}(q - jh), \; |y| \leq 1, \tag{10}$$

we can present (3) in the expanded form

$$S_{2,0}(p,t,q) = \frac{1}{64}\Big((1-x)^2(1-y)^2 p_{i-1,j-1} + (1-x)^2\big(6-2y^2\big)p_{i-1,j} +$$
$$+ (1-x)^2(1+y)^2 p_{i-1,j+1} + \big(6-2x^2\big)(1-y)^2 p_{i,j-1} + \big(6-2x^2\big)\big(6-2y^2\big)p_{i,j} +$$
$$+ \big(6-2x^2\big)(1+y)^2 p_{i,j+1} + (1+x)^2(1-y)^2 p_{i+1,j-1} + (1+x)^2\big(6-2y^2\big)p_{i+1,j} +$$
$$+ (1+x)^2(1+y)^2 p_{i+1,j+1}\Big) \tag{11}$$

Next, inserting into (10) $x = 0$, $y = 0$, we obtain a linear low-pass filtering operator $L\left(p^{i,j}\right)$:

$$L\left(p^{i,j}\right) = S_{2,0}(p, ih, jh) = \left(p_{i-1,j-1} + 6p_{i-1,j} + p_{i+1,j+1} + \right.$$
$$\left. + 6p_{i,j-1} + 36p_{i,j} + 6p_{i,j+1} + p_{i+1,j-1} + 6p_{i+1,j} + p_{i+1,j+1}\right)\Big/64, \ i,j \in \mathbb{Z}$$

Using the record in the form of a discrete convolution sequence $p_{i,j}$, $i,j \in \mathbb{Z}$, with a filter mask $\gamma$, low-pass filtering operators based on linear combinations of B-splines of the 2nd order can be written as follows:

$$L\left(p^{i,j}\right) = \sum_{ii=i-1}^{i+1} \sum_{jj=j-1}^{j+1} \gamma_{ii-i,jj-j}^{(r)} p_{ii,jj}, \ i,j \in \mathbb{Z},$$

where $r = \{2,3\}$,

$$\gamma^{(2)} = \frac{1}{64}\begin{pmatrix} 1 & 6 & 1 \\ 6 & 36 & 6 \\ 1 & 6 & 1 \end{pmatrix}; \tag{12}$$

Using, for example, a sixth-order B-spline will provide the following operator:

$$L\left(p^{i,j}\right) = \sum_{ii=i-3}^{i+3} \sum_{jj=j-3}^{j+3} \gamma_{ii-i,jj-j}^{(6)} p_{ii,jj}, \ i,j \in \mathbb{Z},$$

$$\gamma^{(6)} = \frac{1}{21233664}\begin{pmatrix} 0,01 & 7,22 & 105,43 & 235,48 & \cdots \\ 7,22 & 5212,84 & 76120,46 & 170016,56 & \cdots \\ 105,43 & 76120,46 & 1111548,49 & 2482665,64 & \cdots \\ 235,48 & 170016,56 & 2482665,64 & 5545083,04 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{13}$$

Note that the mask (13) has dimensions (7x7), and the values that are missing in the representation are determined taking into account the symmetry.

## 4. Construction of an Image Pyramid Using Linear Operators based on Linear Combinations of B-Splines

Image models with the characterstics of a pulsed, non-recursive low-pass filter such as (8) are traditionally used in DI processing for large-scale (multiple-scale) analysis and in the construction of image pyramids in methods that require the determination of certain points, such as SIFT-like recognition methods [16; 17].

Among the various kernels in the composite image model in the dimension of the space-scale-position, the Gaussian function has traditionally been the most widespread. So let's allow a deviation from the introduced model, based on a linear combination of B-splines to set out the known provisions on the use of Gaussian functions in scaling of the DI. Therefore, the space-scale-position for the original image $p(t,q)$ is determined by the function

$$L(t,q,\sigma) = \int_{(\xi,\eta)} p(t - \xi, q - \eta) G(\xi, \eta, \sigma) d\xi d\eta, \tag{14}$$

which is a convolution of a Gaussian function of a variable scale

$$G(t,q,\sigma) = \frac{1}{2\pi\sigma} \exp\left(-\frac{t^2 + q^2}{2\sigma}\right),$$

with light intensity function $p(t,q)$:

$$L(t,q,\sigma) = G(t,q,\sigma) * p(t,q),$$

where '*' – is a convolution operation on coordinates $t$ and $q$.

From a computational point of view in order to effectively find lasting (stable) keypoint locations $(t,q,\sigma)$ in the scale-space extension the learned work [17] suggests the extremes in the scale-space extension defined for $D(t,q,\sigma)$ – convolution of differences of Gaussian functions of the two nearest scales , divided by constant factors k, with the original image $p(t,q)$:

$$D(t,q,\sigma) = \left(G\left(t,q,k\sigma\right) - G\left(t,q,\sigma\right)\right) * p\left(t,q\right) = L\left(t,q,k\sigma\right) - L\left(t,q,\sigma\right) \tag{15}$$

The calculation of the original image's convolution with Gaussian functions (smoothing of Gaussian images) $L(t,q,\sigma)$ (14) is necessary for scale-space feature description. However, after their calculation, the convolution $D(t,q,\sigma)$ (15) can be determined by a simple subtraction of Gaussian images (14). In addition, the difference of Gaussian functions gives a closed approximation of the normalized with a scale Laplacian $\sigma^2 \nabla^2 G$:

$$\nabla^2_{norm}L = t\left(L_{xx} + L_{yy}\right) \tag{16}$$

(normalized multiplayer $\sigma^2$ is required for invariance with respect to scale). The connection between $D(t,q,\sigma)$ (15) and $\sigma^2 \nabla^2 G$ follows from the diffusion equation

$$\frac{\partial G}{\partial \sigma} = \sigma \nabla^2 G \tag{17}$$

From (17) we see that $\nabla^2 G$ can be calculated from the finite-difference approximation of the derivative $\partial G / \partial \sigma$ when using the difference of Gaussian functions to connect the scales $k\sigma$ and $\sigma$:

$$\sigma \nabla^2 G = \frac{\partial G}{\partial \sigma} \approx \frac{G\left(t,q,k\sigma\right) - G\left(t,q,\sigma\right)}{k\sigma - \sigma}$$

and therefore

$$G\left(x,y,k\sigma\right) - G\left(x,y,\sigma\right) \approx \left(k-1\right)\sigma^2 \nabla^2 G \tag{18}$$

Formula (18) shows that if the difference of Gaussian function is calculated with scales that differ by a constant factor $k$, this difference contains $\sigma^2$ – scale normalization, required for the scale-invariant Laplacian (16). In formula (18), the factor $(k-1)$ is a constant for all scales and, accordingly, does not affect the position of the extrema. The approximation error goes to zero at $k \to 1$, but in practice it turns out that this approximation does not affect the stability of detection or localization of extrema, such as $k = \sqrt{2}$ .

Gaussian images separated by a constant factor $k$ are calculated in the scale-space extension. Each octave in the scale-space extension (i.e., Gaussian images before doubling the variation) is divided into an integer number of $s$ intervals, therefore $k = 2^{1/s}$ .

For each octave, placed in the stack are $(s+3)$ blurred images, so the final extrema detection encloses full octave. The images of adjacent scales are subtracted, giving $(s+1)$ the difference of Gaussian images $D(\cdot)$ (15). After forming each complete octave from $n = 1,2,...$ Gaussian image $L(t,q,2^n\sigma)$, $n = 1,2,...$, which has twice the variation

of the original value for this octave $2^{n-1}\sigma$, $n=1,2,...$ is halved so that every second pixel remains in each row and each column. This corresponds to the transition to the next level of the Gaussian pyramid [18].

If you are building a pyramid of images, the model of which is given in the form (8), to reduce the size of the image it is advisable to use operators such as low-pass filters with masks (12)-(13). Suppose preset is a raster, each pixel of which is matched by two indexes $\{(i,j)\}_{i,j\in Z}$ that determine its location. Without reducing the universality, let's denote $\{p_{i,j,0}\}$ the notation of the computational scheme when working with sequences of color components – red, green and blue. To recurrently double the zoom (double diminution of horizontal and vertical size) of the image, it is necessary to quadruple diminish the number of pixels at each recurrence $\kappa$ ($\kappa=1,2,...$) step, freeing up the space in the new raster from the three old pixels to the right, top, and top-oblique from each $(i,j)$ pixel of the $(\kappa-1)$ raster. That is, if $\{p_{i,j,\kappa}\}_{i,j\in Z}$ is a sequence of color components of the $\kappa$ reduced raster, then

$$p_{i,j,\kappa} = p_{2i,2j,\kappa-1}, \tag{19}$$

at the same time memory for placement of sizes $p_{2i+1,2j,\kappa-1}$, $p_{2i,2j+1,\kappa-1}$, $p_{2i+1,2j+1,\kappa-1}$, can be disentangled.

In addition to the trivial definition of the members of the sequence $\{p_{i,j,\kappa}\}_{i,j\in Z}$ according to problem (19), implemented is the increase of the scale of the image with smoothing, contrast, directional filtering, etc., depending on the specific needs. In this case, values $p_{i,j,\kappa}$ are determined on the basis of some linear functional $p_{i,j,\kappa} = A\left(p^{\kappa-1,2i,2j}\right)$, $i,j \in Z$, which is based on the data of the previous step of recursion. For example, the reduction with smoothing using a low-pass filter with a mask (12), can be implemented as follows:

$$p_{i,j,\kappa} = \frac{1}{64}\Big( p_{2i-1,2j-1,\kappa-1} + 6p_{2i-1,2j,\kappa-1} + p_{2i-1,2j+1,\kappa-1} + 6p_{2i,2j-1,\kappa-1} + 36p_{2i,2j,\kappa-1} +$$
$$+6p_{2i,2j+1,\kappa-1} + p_{2i+1,2j-1,\kappa-1} + 6p_{2i+1,2j,\kappa-1} + p_{2i+1,2j+1,\kappa-1} \Big).$$

In a general case, when zooming in, described by the sline model (8) can be represented as follows:

$$p_{i,j,\kappa} = \sum_{ii=2i-1}^{2i+1} \sum_{jj=2j-1}^{2j+1} \gamma_{L,ii-2i,jj-2j}^{(2)} p_{ii,jj,\kappa-1}, \tag{20}$$

for operators with a mask (12) or, as an example,

$$p_{i,j,\kappa} = \sum_{ii=2i-3}^{2i+3} \sum_{jj=2j-3}^{2j+3} \gamma_{L,ii-2i,jj-2j}^{(6)} p_{ii,jj,\kappa-1}, \tag{21}$$

for an operator with a mask (13).

The picture (Fig. 1) shows an example of a pyramid with four levels of DI, which was obtained after $\kappa=3$ steps of recursion with the help of the operator (21). In DI processing the use of a pyramidal data structure reduces the image processing time, low-pass filtering to suppress high-frequency oscillations of the light intensity function, and thus obtain global features (special points) that are characteristic of images at all levels – more accurate initial approximations of features for lower levels procesessing according to the results of the upper levels of the pyramid.

Fig. 1. Four-level pyramid for a test image

Operators such as (20) - (21) can be used to implement the construction of the octave according to (15). In essence, the difference between the low-frequency components of the image, which are obtained by filtering the image by operators with masks (12) - (13) results in a high-frequency component of the image. For example, the operator mask based on the difference of low-pass filters based on B-splines of the 3rd and 2nd order is as follows:

$$\delta L_{3,2} = \gamma_L^{(3)} - \gamma_L^{(2)} = \frac{1}{576}\begin{pmatrix} 7 & 10 & 7 \\ 10 & -68 & 10 \\ 7 & 10 & 7 \end{pmatrix} \tag{22}$$

Similarly, using filters based on B-splines of 3, 4, 5 and 6 orders, you can enter the following masks:

$$\delta L_{4,3} = \gamma_L^{(4)} - \gamma_L^{(3)} = \frac{1}{147456}\begin{pmatrix} 1 & 76 & 230 & 76 & 1 \\ 76 & 1680 & 1096 & 1680 & 76 \\ 230 & 1096 & -12636 & 1096 & 230 \\ 76 & 1680 & 1096 & 1680 & 76 \\ 1 & 76 & 230 & 76 & 1 \end{pmatrix} \tag{23}$$

$$\delta L_{5,4} = \gamma_L^{(5)} - \gamma_L^{(4)} = \frac{1}{33177600}\begin{pmatrix} 2079 & 42804 & 100314 & \cdots \\ 42804 & 257904 & 20664 & \cdots \\ 100314 & 20664 & -1866276 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{24}$$

$$\delta L_{6,5} = \gamma_L^{(6)} - \gamma_L^{(5)} = \frac{1}{21233664}\begin{pmatrix} 0,01 & 7,22 & 105,43 & 235,48 & \ldots \\ 7,22 & 3738,28 & 37781,9 & 72695,6 & \ldots \\ 105,43 & 37781,9 & 114745,93 & -47679,32 & \ldots \\ 235,48 & 72695,6 & -47679,32 & -878100,32 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{25}$$

Thus, to build a pyramid of images of the positions of key points with the operators close to (15) one can use the following expressions:
based on mask (22)

$$p_{i,j,\kappa} = \sum_{ii=2i-1}^{2i+1} \sum_{jj=2j-1}^{2j+1} \delta L_{3,2}\, p_{ii,jj,\kappa-1}, \tag{26}$$

based on masks (23) - (24)

$$p_{i,j,\kappa} = \sum_{ii=2i-2}^{2i+2} \sum_{jj=2j-2}^{2j+2} \delta L_{rr}\, p_{ii,jj,\kappa-1}, \quad rr = \left\{"4,3","5,4"\right\} \tag{27}$$

and using mask (25)

$$p_{i,j,\kappa} = \sum_{ii=2i-3}^{2i+3} \sum_{jj=2j-3}^{2j+3} \delta L_{6,5} p_{ii,jj,\kappa-1} \cdot \tag{28}$$

For the models (8) of order four differential invariants with respect to local rotations can be constructed - gradient magnitude $\left|\nabla S_{r,0}\right|$, Laplacian $\nabla^2 S_{r,0}$, Hessian determinant $\det H_{r,0}$ and the curvature of the scaling curve $\tilde{k}_{r,0}$ (up to the notation of operators of different order):

$$\left|\nabla S\right| = S_t'^2 + S_q'^2, \tag{29}$$

$$\nabla^2 S = S_{tt}'' + S_{qq}'', \tag{30}$$

$$\det H = S_{tt}'' S_{qq}'' - S_{tq}''^2, \tag{31}$$

$$\tilde{k} = S_t'^2 S_{qq}'' + S_q'^2 S_{tt}'' - 2 S_t' S_q' S_{tq}'', \tag{32}$$

At different $r = 2,3,\dots$ the authors obtain explicit types of partial derivatives of model (1). So, for work with DI at $r = 3$ discrete analogs $S_{3,0}'(p,t,q)_t$, $S_{3,0}'(p,t,q)_q$ considering $h_t = h_q = 1$, can be filed as follows [17]:

$$S_{3,0,l}' = \sum_{ii=i-1}^{i+1} \sum_{j=i-1}^{j+1} \gamma_{l,ii-i,jj-j}' \cdot p_{ii,jj} \tag{33}$$

where

$$l = \{t,q\}; \quad \gamma_t' = \frac{1}{12}\begin{pmatrix} -1 & -4 & -1 \\ 0 & 0 & 0 \\ 1 & 4 & 1 \end{pmatrix} \quad \gamma_q' = \frac{1}{12}\begin{pmatrix} -1 & 0 & 1 \\ -4 & 0 & 4 \\ -1 & 0 & 1 \end{pmatrix} \cdot$$

Discrete convolutions of second-order differentiation operators $r = 3$ such :

$$S_{3,0,l}'' = \sum_{ii=i-1}^{i+1} \sum_{j=i-1}^{j+1} \gamma_{l,ii-i,jj-j}'' \cdot p_{ii,jj} \tag{34}$$

where

$$l = \{tt,qq,tq\}; \quad \gamma_{tt}'' = \frac{1}{6}\begin{pmatrix} 1 & 4 & 1 \\ -2 & -8 & -2 \\ 1 & 4 & 1 \end{pmatrix};$$

$$\gamma_{qq}'' = \frac{1}{6}\begin{pmatrix} 1 & -2 & 1 \\ 4 & -8 & 4 \\ 1 & -2 & 1 \end{pmatrix}; \quad \gamma_{tq}'' = \frac{1}{4}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \cdot$$

Similarly, first-order and second-order partial derivatives are obtained for higher-order.

In the author's work [16] the method of determining the features on the basis of operators (29) - (32), obtained with the help of expressions like (33), (34), is widely described. In contrast to the well-known method SIFT [19] and its like, it is proposed to use the distribution of differential invariants, in particular to identify which of them are present on «tails» distributions when selecting features. This approach has been analysed in more detail in the work [18], but such research can be continued as described in this publication.

According to [18], the possibility of using invariants (29) - (32) to identify objects in the production of aerial photography was considered (Fig. 2).

c)                                          d)





e)                                          f)

Fig.2. Test examples of terrain elements according to aerial photography

It was proposed to use a linear operator to determine the detectors to the reference image (Fig. 2). The scaling operation (28) shows (Table 1) that the correlation of the values of operators (29) and (32) increases significantly, which in itself may be useful to take into account in further studies the issue of selecting the positions of singular points.

Table 1. Correlation of invariant values

|      | a)     | b)     | c)     | d)     | e)     | f)     |
|------|--------|--------|--------|--------|--------|--------|
| (2)  | -0,818 | -0,798 | -0,792 | -0,855 | -0,832 | -0,668 |
| (3)  | -0,988 | -0,991 | -0,992 | -0,989 | -0,991 | -0,988 |
| (4)  | 0,962  | 0,973  | 0,974  | 0,957  | 0,96   | 0,935  |
| (5)  | -0,786 | -0,871 | -0,877 | -0,876 | -0,875 | -0,686 |

As a result of the conducted experimental researches it is established that application to initial DI of the operator (28) when finding singular points has no priority in comparison with the definition of features on the basis of differential invariants (29) – (32) before the approach, was investigated in this work, namely - the calculation directly behind the raster. In both cases, the distributions of all invariants have a similar shape and, despite the different scales along the abscissa, there is a significant correlation of their values.

## 5. Continuous Scaling of a Digital Image

The recurrent scheme of the construction of the pyramid of images on the basis of operators of type (20) -(21) or (26) - (28), when at each level we receive the image halved in linear sizes, is the most optimum from the computational side. However, if there is a need to obtain a new image scaled to an arbitrary (not necessarily integer) number of times based on the skeleton models (8) or (4), explicit aspects of spline operators such as (11) the corresponding algorithm can be used. Thus, identical expressions (3) and (11) can be represented as a convolution with a mask with variable coefficients

$$S_{2,0}(p,t,q) = \frac{1}{64} \sum_{ii=i-1}^{i+1} \sum_{jj=j-1}^{j+1} \gamma_{ii-i,jj-j}^{2,0} p_{ii,jj} ,$$

Where $x$ and $y$ are defined at (10), $|x| \leq 1$; $|y| \leq 1$;

$$\gamma^{2,0} = \begin{pmatrix} (1-x)^2(1-y)^2 & (1-x)^2\left(6-2y^2\right) & (1-x)^2(1+y)^2 \\ \left(6-2x^2\right)(1-y)^2 & \left(6-2x^2\right)\left(6-2y^2\right) & \left(6-2x^2\right)(1+y)^2 \\ (1+x)^2(1-y)^2 & (1+x)^2\left(6-2y^2\right) & (1+x)^2(1+y)^2 \end{pmatrix}. \tag{35}$$

To build high-precision approximations of images based on (4), one should use a spline $S_{2,1}(p,t,q)$, which in the form of a convolution is presented as follows:

$$S_{2,1}(p,t,q) = \frac{1}{2304} \sum_{ii=i-2}^{i+2} \sum_{jj=j-2}^{j+2} \gamma^{2,1}_{ii-i,jj-j} p_{ii,jj},$$

Where

$$\gamma^{2,1} = \begin{pmatrix} (1-x)^2(1-y)^2 & -(1-x)^2\left(2-16y+10y^2\right) & -(1-x)^2\left(46-18y^2\right) & -(1-x)^2\left(2+16y+10y^2\right) & (1-x)^2(1+y)^2 \\ -\left(2-16x+10x^2\right)(1-y)^2 & \left(2-16x+10x^2\right)\left(2-16y+10y^2\right) & \left(2-16x+10x^2\right)\left(46-18y^2\right) & \left(2-16x+10x^2\right)\left(2+16y+10y^2\right) & -\left(2-16x+10x^2\right)(1+y)^2 \\ -\left(46-18x^2\right)(1-y)^2 & \left(46-18x^2\right)\left(2-16y+10y^2\right) & \left(46-18x^2\right)\left(46-18y^2\right) & \left(46-18x^2\right)\left(2+16y+10y^2\right) & -\left(46-18x^2\right)(1+y)^2 \\ -\left(2+16x+10x^2\right)(1-y)^2 & \left(2+16x+10x^2\right)\left(2-16y+10y^2\right) & \left(2+16x+10x^2\right)\left(46-18y^2\right) & \left(2+16x+10x^2\right)\left(2+16y+10y^2\right) & -\left(2+16x+10x^2\right)(1+y)^2 \\ (1+x)^2(1-y)^2 & -(1+x)^2\left(2-16y+10y^2\right) & -(1+x)^2\left(46-18y^2\right) & -(1+x)^2\left(2+16y+10y^2\right) & (1+x)^2(1+y)^2 \end{pmatrix}. \tag{36}$$

Suppose that using a local spline (3) or (4) it is necessary to change the linear dimensions of a DI with an intensity in pixels $P = \left\{ p_{i,j}, i = \overline{1,W}, j = \overline{1,H} \right\}$ at $k$ times. Then the following algorithm takes place.

**Algorithm.** Zoom of a DI any number of times.

**Step 1.** Determine the new linear dimensions of the DI

$$N = \left[ k \cdot W \right], \; M = \left[ k \cdot H \right],$$

where $\left[ \cdot \right]$ – is the whole part.

So we need to define a digital image afterwards

$$P^* = \left\{ p^*_{i\_H, j\_H}, i\_H = \overline{1,N}, j\_H = \overline{1,M} \right\}.$$

**Step 2.** For any $\left( i\_H, j\_H \right)$- pixel $i\_H = \overline{1,N}$, $j\_H = \overline{1,M}$ of the newly formed raster we determine the values

$$xx = \frac{i\_H}{k}, \; yy = \frac{j\_H}{k}.$$

**Step 3.** Specific values $x$ and $y$ in masks (33), (34) are calculated as follows:

$$x = 2\left( xx - i \right), \; y = 2\left( yy - j \right),$$

where

$$i = round\left( xx \right); \; j = round\left( yy \right);$$

$round\left( \cdot \right)$ – the result of rounding to the nearest whole number;

$i, j$ – pixel indices of the original image's raster $P$.

**Step 4.** The calculated value of the spline with a mask (35) or (36) determines the intensity of the $\left(i\_н, j\_н\right)$-pixel of the newly formed DI.
**Step 5.** Repeat steps 2-4 to determine all pixels of the new DI.

The pictures (Fig. 3, 4) show the result of applying the above algorithm in case of using a spline operator with a mask (35).



Fig. 3. Initial image and image

which linear dimensions have been changed by 0.84 times



Fig.4. Initial image and image

which linear dimensions have been changed by 1.37 times
Therefore, if it is necessary to obtain an approximation of the operator like (15) for a DI scaled to an arbitrary, not necessarily an integer number of times, it can be suggested to first perform the scaling based on the above algorithm and spline with a mask (36), and then directly approximate the operator (15) with a linear filter, with any of the masks (22) - (25). We would also like to note that the degree of smoothing of the scaled image can be increased by using recursive low-pass filters with masks (12) - (13), preceding this operation before applying filtering operators with masks (22) - (25).

## 6. Conclusions Section

Based on the conducted research, the following conclusions can be drawn.

1. The model of a DI has been formalized on the basis of two-dimensional polynomial splines, on the basis of B-splines of the second-sixth orders which are close to interpolation on the average.
2. The convolution operators of low-frequency DI filtering based on the spline model are presented.
3. Provided are the scaling operators used to build image pyramids, in order to further search for special points.
4. An algorithm for scaling the DI to an arbitrary, not necessarily an integer number of times based on a continuous spline approximation has been suggested.
5. Algorithm for scaling a digital image based on a spline model allows you to change the size of the image in any (not necessarily an integer) number of times, differs in that it provides high scaling accuracy and no artifacts due to high approximate properties and smoothness of the spline model;
6. The scaling algorithm allows digital image processing at high computational speed due to the optimal computational scheme with a minimum of simpler mathematical operations, compared with models based on the two-dimensional Gaussian function.
7. Future work is to obtain methods and algorithms for determining the characteristics of digital images (SIFT methods).

8. Further work involves the application of scaling algorithms, which can be promising in multi-stream real-time video processing in single-board computers (unmanned aerial stations), in applications for mobile devices, in devices with memory limitations, and so on.

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