61 - 62 : 32, 33, 36, 37, 41

64: 43, 44, 45

32. a) Show that R is symmetric IFF $R^{-1} \subseteq R$

Suppose that R is symmetric. Then for $x, y \in A$ s.t. xRy, we have yRx.

$R^{-1} = \{(x,y) \mid \exists x \,\&\, (x,y) \in R\}$. However, we ~~know~~ symmetry gives us that all such (y,x) will be in R. For the converse, if $R^{-1} \subseteq R$, then ~~eq~~ $\{(y,x) \mid (x,y) \in R\} \in R$. But this ~~set~~ ~~is~~ ~~equal~~ ~~to~~ is equivalent to the condition of symmetry, giving us our result.

b) Assume that R is transitive. Then for (x,y), $(y,z) \in R$, $(x,z) \in R$. Note that the set $R \circ R := \{(x,z) \mid \exists x, y, z \;\; (x,y), (y,z) \in R\}$.

Thus $R \circ R \subseteq R$.

For the converse, suppose that $R \circ R \subseteq R$. We then know that $\left\{ \begin{matrix} (x,z) \mid \exists x,y,z \in R \\ (x,y), (y,z) \in R \end{matrix} \right\} \subseteq R$.

This is equivalent to transitivity.

33. Suppose that R is symmetric & transitive. It then follows that for $(x,y) \in R$, $(y,x) \in R$, and that for (a,b), $(b,c) \in R$, $(a,c) \in R$.

We know that $R^{-1} \circ R \subseteq R$, for, since R is transitive & symmetric, for $(a,b) \in R$, $(b,a) \in R^{-1}$, $(a,b) \circ (b,a) \in R$ by transitivity. Additionally, symmetry gives us that $R \subseteq R^{-1} \circ R$

For the converse, assume that $R^{-1} \circ R = R$.

Note that $R \subseteq R^{-1} \circ R$. Thus $a \in R$ will be of the form $\langle x, y\rangle \circ \langle y, x\rangle$. Additionally, since $R \supseteq R^{-1} \circ R$, all elements of R will be of this form. Transitivity & symmetry follow.

36. Note that $\{\langle f(x), f(y)\rangle\} \subseteq R$, which is an equivalence relation on B. We know that $x\,Q\,x \in Q$, thus Q is a reflexive relation on A. Additionally, if $\langle x, y\rangle \in Q$, then $\langle f(x), f(y)\rangle \in R \to \langle f(y), f(x)\rangle \in R \to \langle y, x\rangle \in Q$. Thus we have symmetry. Lastly, if $\langle a, b\rangle$, $\langle b, c\rangle$ are in Q, then $\langle f(a), f(b)\rangle \in R$, $\langle f(b), f(c)\rangle \in R$, $\langle f(a), f(c)\rangle \in R$. This gives us transitivity, as this implies that $\langle a, b\rangle$ & $\langle b, c\rangle \in Q$ gives us that $\langle a, c\rangle \in Q$. Thus Q is an equivalence relation on A.

37. Let Π be a partition of a set A. Suppose that $x \in B$, $B \in \Pi$. $x R_\Pi y$ since a partition is disjoint. Additio $x R_\Pi y \to y R_\Pi x$. For transitivity, suppose $a R_\Pi b \wedge b R_\Pi c$. Then $a \in B$, $b \in B$. It follows from the disjoint property of partitions that $c \in B$. Thus $a R_\Pi c$ holds.

42. a) R is obviously reflexive as $u+v = u+v$.

$u+v = x+y \to x+y = u+v \to$ symmetric

$u+v + x+y = x+y + a+b \to u+v = a+b \to$ transitive

b)

43. Suppose that R is a linear ordering on A.

We know that R satisfies a trichotomy on A & R is transitive.

From this we know that for $x, y \in A$, either $x R y$, $y R x$, or $x = y$.

In all cases, the same conditions would be satisfied on R^{-1}.

For transitivity, we suppose $x R y$, and $y R z$. Then $x R z$. We have that $y R x$, $z R y$, $z R x \in R^{-1}$. From this we have that $y R x$, $z R y \to z R x$, i.e., R^{-1} is transitive. Thus R^{-1} defines a total ordering on A.

44. Let $<$ be a linear ordering on A.

Suppose that $f(x) = f(y)$. We must then have that $x = y$. Now suppose $f(x) < f(y)$.

Suppose for contradiction that $\neg (x < y)$.

Either $y < x$ or $x = y$. We know that the later cannot hold since $f(x) \neq f(y)$.

Suppose $y < x$. Then $f(y) < f(x)$. We thus have that $x < y$ is not the case, contradicting $f(x) < f(y)$. Thus f is one-to-one & $f(x) < f(y) \to x < y$.

+5. Suppose $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$ ~~[illegible]~~

~~[illegible]~~ and $\langle a_2, b_2 \rangle <_L \langle a_3, b_3 \rangle$.

~~Take~~ either $a_1 <_A a_2$ or $\{a_1 = a_2$ & $b_1 < b_2\}$.

Additionally, either $a_2 <_A a_3$ or $\{a_2 = a_3$ & $b_2 < b_3\}$

$$\text{case 1.}\quad \left.\begin{array}{l} a_1 <_A a_2 \\ \& \ a_2 <_A a_3 \end{array}\right\} \to \begin{array}{l} b_1 = b_3 \\ a_1 \leq_A a_3 \end{array}$$

$$\text{case 2.}\quad \left.\begin{array}{l} a_1 <_A a_2,\ b_2 = b_1 \\ \& \ a_2 = a_3 \ \& \ b_2 <_B b_3 \end{array}\right\} \to \begin{array}{l} a_1 = a_3 \\ b_1 = b_2 <_B b_3 \end{array}$$

$$\text{case 3.}\quad \left.\begin{array}{l} a_1 = a_2 \ \& \ b_2 <_B b_2 \\ \& \ a_2 <_A a_3 \end{array}\right\} \to \begin{array}{l} b_1 = b_3 \\ a_1 <_A a_3 \end{array}$$

$$\text{case 4.}\quad \left.\begin{array}{l} a_1 = a_2 \ \& \ b_1 <_B b_2 \\ \& \ a_2 = a_3 \ \& \ b_2 <_B b_3 \end{array}\right\} \begin{array}{l} a_1 = a_3 \\ b_1 <_B b_3 \end{array}$$

In all cases, $<_L$ is transitive. It remains for us to prove that $<_L$ defines a trichotomy on $A \times B$.

~~Express~~ consider $\langle a_1, b_1 \rangle$, $\langle a_2, b_2 \rangle \in A \times B$.

~~Suppose that $<_L$ does not form a trichotomy on $A \times B$. $\langle a_1, b_1 \rangle \not<_L \langle a_2, b_2 \rangle$ $\langle a_2, b_2 \rangle \not<_L \langle a_1, b_1 \rangle$ $\langle a_1, b_1 \rangle \neq \langle a_2, b_2 \rangle$~~

~~[illegible]~~

Since $<_A$, $<_B$ are linear orderings, they define trichotomies on A & B, respectively. If $\langle a_1, b_1 \rangle <_L \langle a_2, b_2 \rangle$ nor $\langle a_2, b_2 \rangle <_L \langle a_1, b_1 \rangle$ do not hold, then $a_1 = a_2$, $b_1 = b_2$, and $\langle a_1, b_1 \rangle = \langle a_2, b_2 \rangle$, giving that $<_L$ forms a total ordering on $A \times B$.