Problem 1:

$$\phi_{i-2} = \phi_i - \frac{\partial \phi}{\partial x}(2h) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(2h)^2 - \frac{1}{6}\frac{\partial^3 \phi}{\partial x^3}(2h)^3 + \frac{1}{24}\frac{\partial^4 \phi}{\partial x^4}(2h)^4 + O(h^5)$$

$$\phi_{i-1} = \phi_i - \frac{\partial \phi}{\partial x} h + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2} h^2 - \frac{1}{6}\frac{\partial^3 \phi}{\partial x^3} h^3 + \frac{1}{24}\frac{\partial^4 \phi}{\partial x^4} h^4 + O(h^5)$$

$$\phi_{i+1} = \phi_i + \frac{\partial \phi}{\partial x} h + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2} h^2 + \frac{1}{6}\frac{\partial^3 \phi}{\partial x^3} h^3 + \frac{1}{24}\frac{\partial^4 \phi}{\partial x^4} h^4 + O(h^5)$$

$$\phi_{i+2} = \phi_i + \frac{\partial \phi}{\partial x}(2h) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(2h)^2 + \frac{1}{6}\frac{\partial^3 \phi}{\partial x^3}(2h)^3 + \frac{1}{24}\frac{\partial^4 \phi}{\partial x^4}(2h)^4 + O(h^5)$$

1

$$\mathrm{I}: \phi_{j-2} - \phi_{j+2} = -4\frac{\partial \phi}{\partial x} h - \frac{1}{3}\frac{\partial^3 \phi}{\partial x^3}(2h)^3 + O(h^5) = -4\frac{\partial \phi}{\partial x} h - \frac{8}{3}\frac{\partial^3 \phi}{\partial x^3} h^3 + O(h^5)$$

$$\mathrm{II}: 8(\phi_{j+1} - \phi_{j-1}) = 8\left[2\frac{\partial \phi}{\partial x} h + \frac{1}{3}\frac{\partial^3 \phi}{\partial x^3} h^3 + O(h^5)\right] = 16\frac{\partial \phi}{\partial x} h + \frac{8}{3}\frac{\partial^3 \phi}{\partial x^3} + O(h^5)$$

$\mathrm{I} + \mathrm{II}$:

$$-\phi_{j+2} + 8\phi_{j+1} - 8\phi_{j-1} + \phi_{j-2} = 12\frac{\partial \phi}{\partial x} h + O(h^5)$$

$$\Longrightarrow \frac{\partial \phi}{\partial x} = \frac{-\phi_{j+2} + 8\phi_{j+1} - 8\phi_{j-1} + \phi_{j-2}}{12h} + O(h^4)$$

②

$$2\phi_{j+1}+3\phi_j-6\phi_{j-1}+\phi_{j-2}=2\phi_j+2\frac{\partial\phi}{\partial x}h+\frac{\partial^2\phi}{\partial x}h^2+\frac{1}{3}\frac{\partial^3\phi}{\partial x^3}h^3+\frac{1}{12}\frac{\partial^4\phi}{\partial x^4}h^4+$$

$$3\phi_j-6\phi_j+6\frac{\partial\phi}{\partial x}h-3\frac{\partial^2\phi}{\partial x^2}h^2+\frac{\partial^3\phi}{\partial x^3}h^3-\frac{1}{2}\frac{\partial^4\phi}{\partial x^4}h^4+$$

$$\phi_j-2\frac{\partial\phi}{\partial x}h+2\frac{\partial^2\phi}{\partial x^2}h^2-\frac{4}{3}\frac{\partial^3\phi}{\partial x^3}h^3+\frac{2}{3}\frac{\partial^4\phi}{\partial x^4}h^4$$

$\Rightarrow$

$$6h\frac{\partial\phi}{\partial x}=2\phi_{j+1}+3\phi_j-6\phi_{j-1}+\phi_{j-2}-\underbrace{\left(\frac{1}{12}\frac{\partial^4\phi}{\partial x^4}h^4-\frac{1}{2}\frac{\partial^4\phi}{\partial x^4}h^4+\frac{2}{3}\frac{\partial^4\phi}{\partial x^4}h^4\right)}_{O(h^4)}$$

$$\rightarrow\frac{\partial\phi}{\partial x}=\frac{2\phi_{j+1}+3\phi_j-6\phi_{j-1}+\phi_{j-2}}{6h}+O(h^3)$$

Problem 2.

$$\phi_{j+1} = \phi_j + \frac{\partial \phi}{\partial y} h_1 + \frac{1}{2}\frac{\partial^2 \phi}{\partial y^2}(h_1^2) + \frac{1}{6}\frac{\partial^3 \phi}{\partial y^3} h_1^3 + O(h_1^4)$$

$$\phi_{j+2} = \phi_j + \frac{\partial \phi}{\partial y}(h_1+h_2) + \frac{1}{2}\frac{\partial^2 \phi}{\partial y^2}(h_1+h_2)^2 + \frac{1}{6}\frac{\partial^3 \phi}{\partial y^3}(h_1+h_2)^3 + O([h_1+h_2]^4)$$

$$\phi_{j+3} = \phi_j + \frac{\partial \phi}{\partial y}(h_1+h_2+h_3) + \frac{1}{2}\frac{\partial^2 \phi}{\partial y^2}(h_1+h_2+h_3)^2 + \frac{1}{6}\frac{\partial^3 \phi}{\partial y^3}(h_1+h_2+h_3)^3 + O([h_1+h_2+h_3]^4)$$

$$\phi_{j+4} = \phi_j + \frac{\partial \phi}{\partial y}(h_1+h_2+h_3+h_4) + \frac{1}{2}\frac{\partial^2 \phi}{\partial y^2}(h_1+h_2+h_3+h_4)^2 + \frac{1}{6}\frac{\partial^3 \phi}{\partial y^3}(h_1+h_2+h_3+h_4)^3 + O([h_1+h_2+h_3+h_4]^4)$$

We need to combine these four equations in such way that second and third order derivatives ($\frac{\partial^2 \phi}{\partial x^2}$ and $\frac{\partial^3 \phi}{\partial x^3}$) along with ϕ_js disappear. If we multiply each equation in a, b, c, and d respectively, and add them together the coefficient for ϕ_j, $\frac{\partial^2 \phi}{\partial x^2}$ and $\frac{\partial^3 \phi}{\partial x^3}$ should be zero while the coefficient for $\frac{\partial \phi}{\partial x}$ should be one.

This will give us the following linear equations:

$$a+b+c+d=0$$

$$ah_1^2 + b(h_1+h_2)^2 + c(h_1+h_2+h_3)^2 + d(h_1+h_2+h_3+h_4)^2 = 0$$

$$ah_1^3 + b(h_1+h_2)^3 + c(h_1+h_2+h_3)^3 + d(h_1+h_2+h_3+h_4)^3 = 0$$

$$ah_1 + b(h_1+h_2) + c(h_1+h_2+h_3) + d(h_1+h_2+h_3+h_4) = 1$$

Four equations, four unknowns.

$$a = \frac{-3h_1^2 - 6h_1h_2 - 4h_1h_3 - 2h_4h_1 - 3h_2^2 - 4h_2h_3 - 2h_4h_2 - h_3^2 - h_4h_3}{h_2(h_2+h_3)(h_2+h_3+h_4)}$$

$$b = \frac{3h_1^2 + 4h_1h_2 + 4h_1h_3 + 2h_4h_1 + h_2^2 + 2h_2h_3 + h_4h_2 + h_3^2 + h_4h_3}{h_2h_3(h_3+h_4)}$$

$$c = \frac{-4h_1h_2 - 2h_1h_3 - 2h_1h_4 - h_2h_3 - h_2h_4 - 3h_1^2 - h_2^2}{h_4(h_3^2 + h_2h_3)}$$

$$d = \frac{3h_1^2 + 4h_1h_2 + 2h_3h_4 + h_2^2 + h_3h_2}{h_3^2h_4 + 2h_3h_4^2 + h_2h_3h_4 + h_4^3 + h_2h_4^2}$$

All of the coefficients are $o(h^{-1})$ which means when multiplied by $o(h^4)$, the accuracy of the formulation is $o(h^3)$.