MAE 6220

Applied Computational Fluid Dynamics

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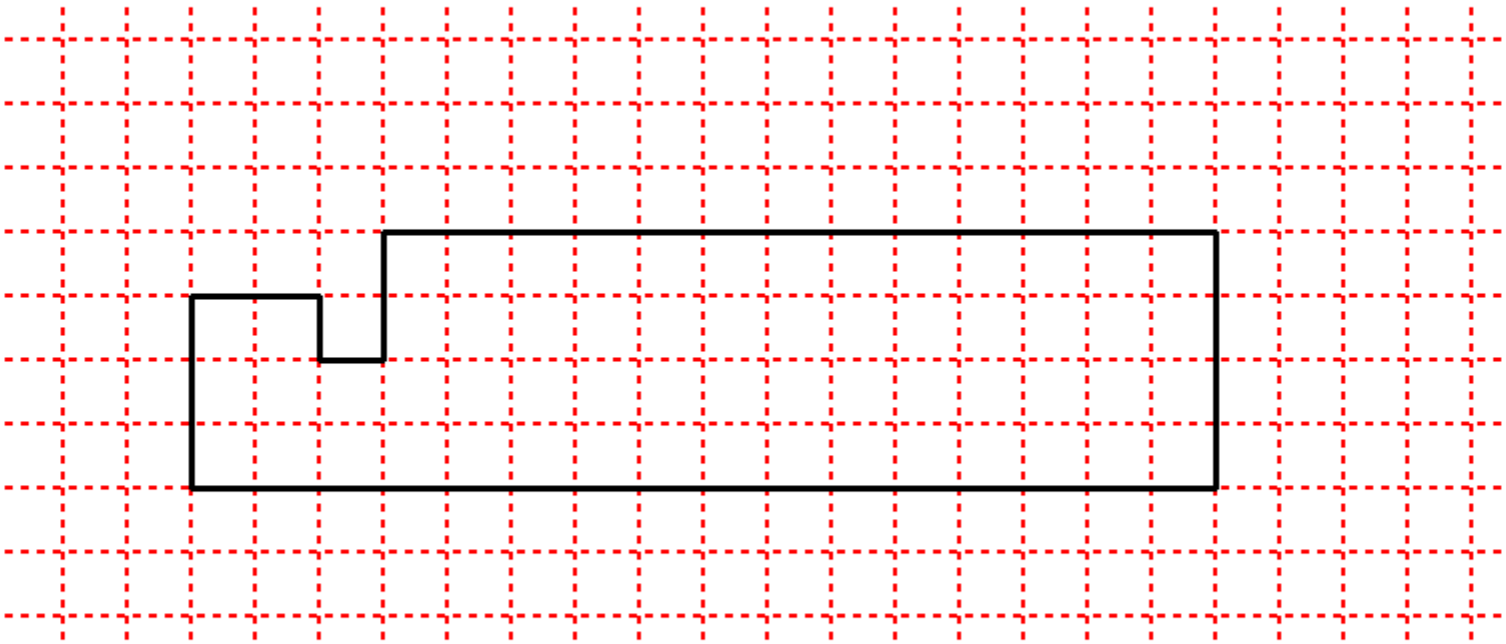
Lecture 4

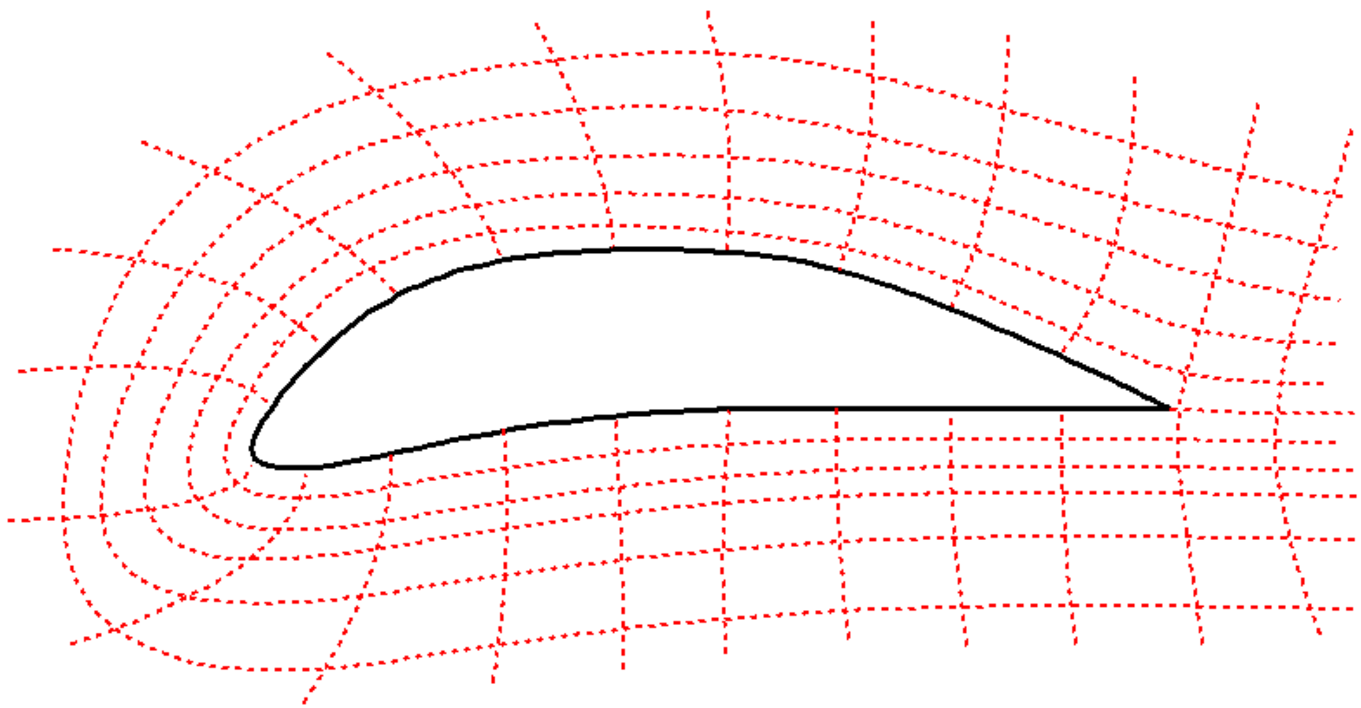
Complex geometries

- Most flows of engineering interest involve complex geometries
- Only rarely can they be represented using Cartesian grids.



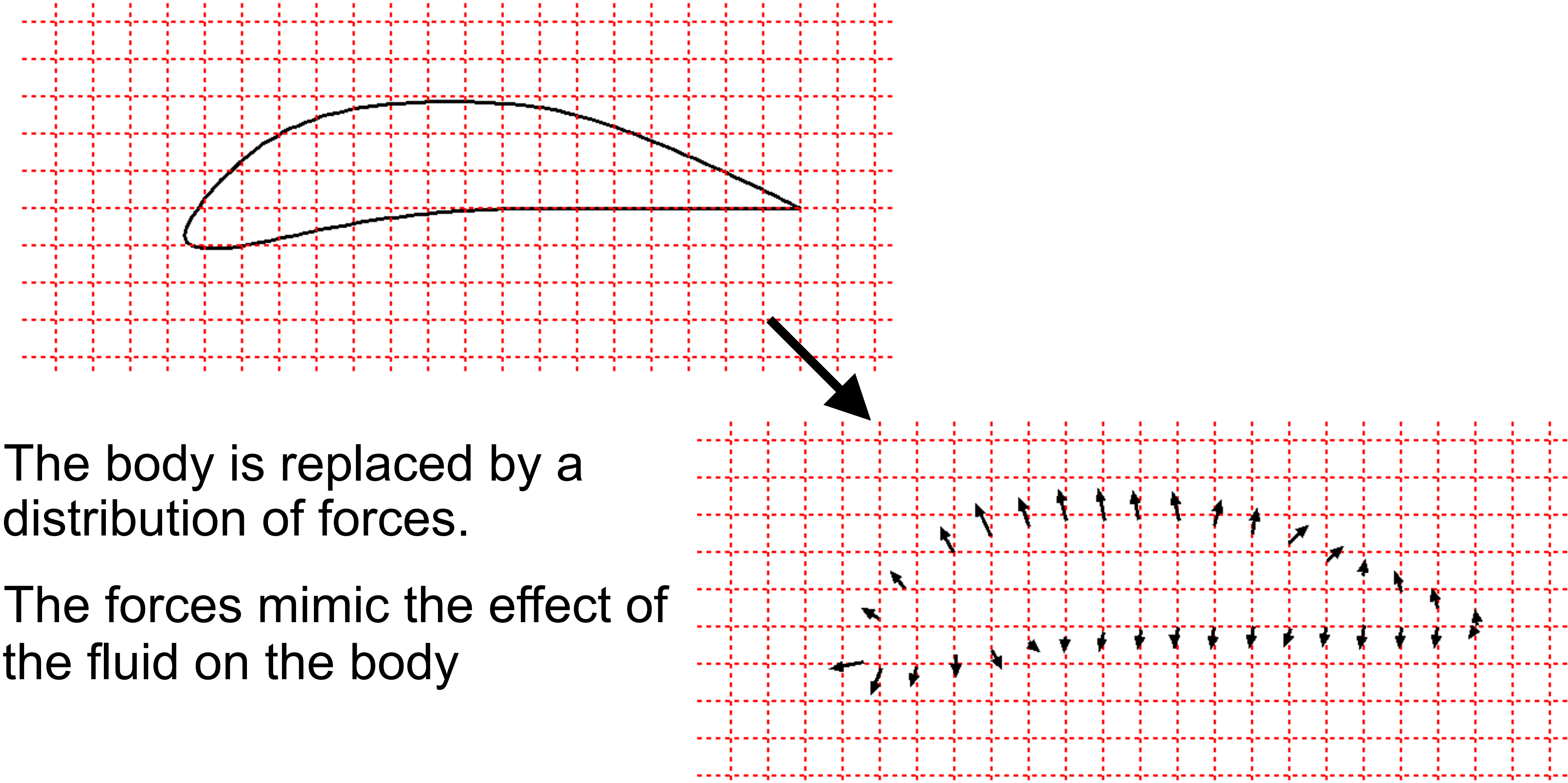
Complex geometries

- One can use a body-fitted grid
- Need to transform equations of motion in generalized coordinates
- Their cost is significantly higher (CPU time, memory) than that of Cartesian-grid calculations.



Complex geometries

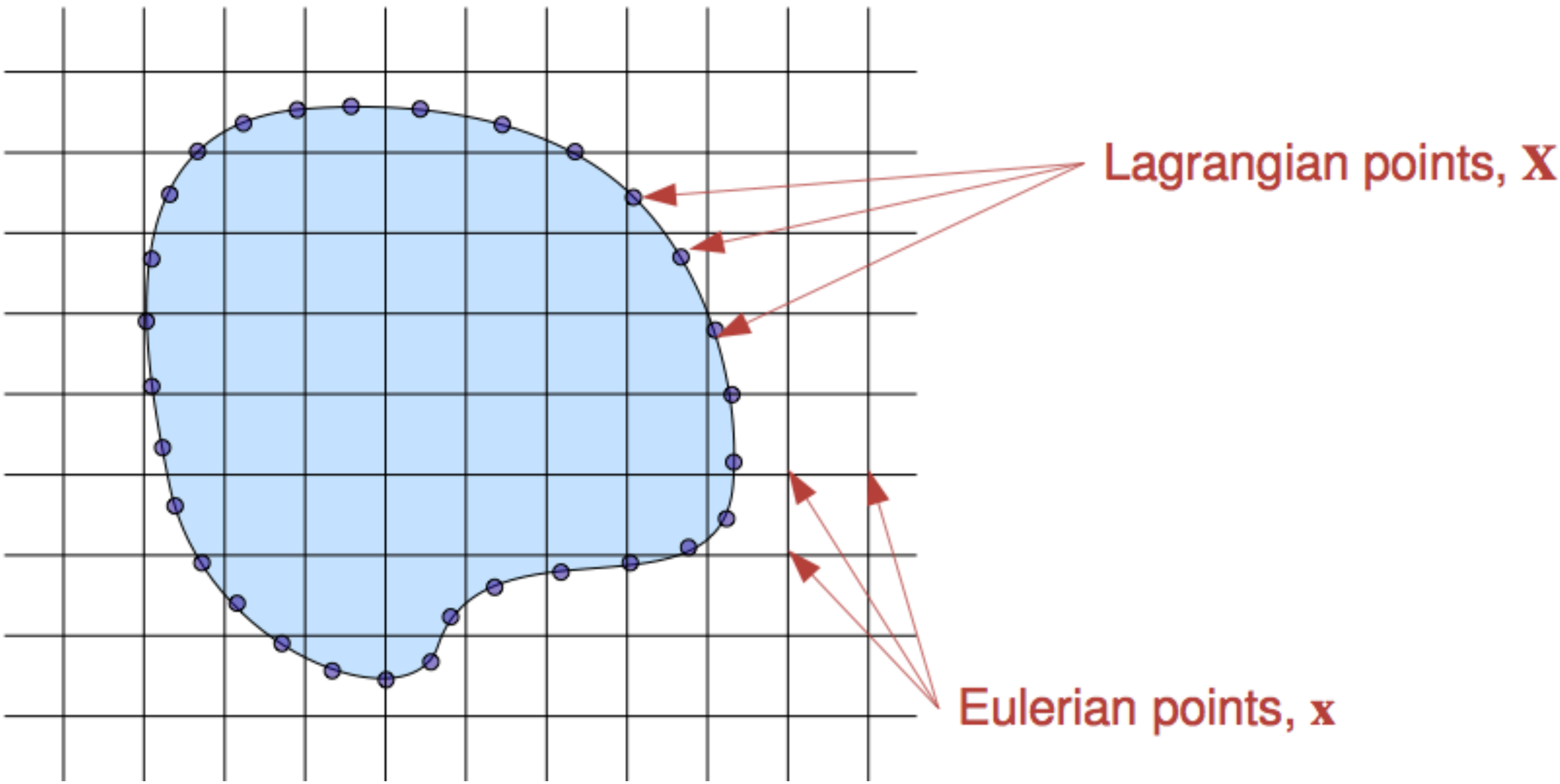
- Or one utilize a technique that allows the use of structured meshes to calculate the flow around complex geometries: **The immersed-boundary method (IBM)**



- The body is replaced by a distribution of forces.
- The forces mimic the effect of the fluid on the body

The IB method

- The immersed object is usually described by a set of Lagrangian points embedded in an Eulerian grid.



The IB method

- The immersed object is usually described by a set of Lagrangian points embedded in an Eulerian grid.
- Force terms are added at the Eulerian points to model the effect of the immersed body on the flow.
- The Navier-Stokes equations with the distributed force field f

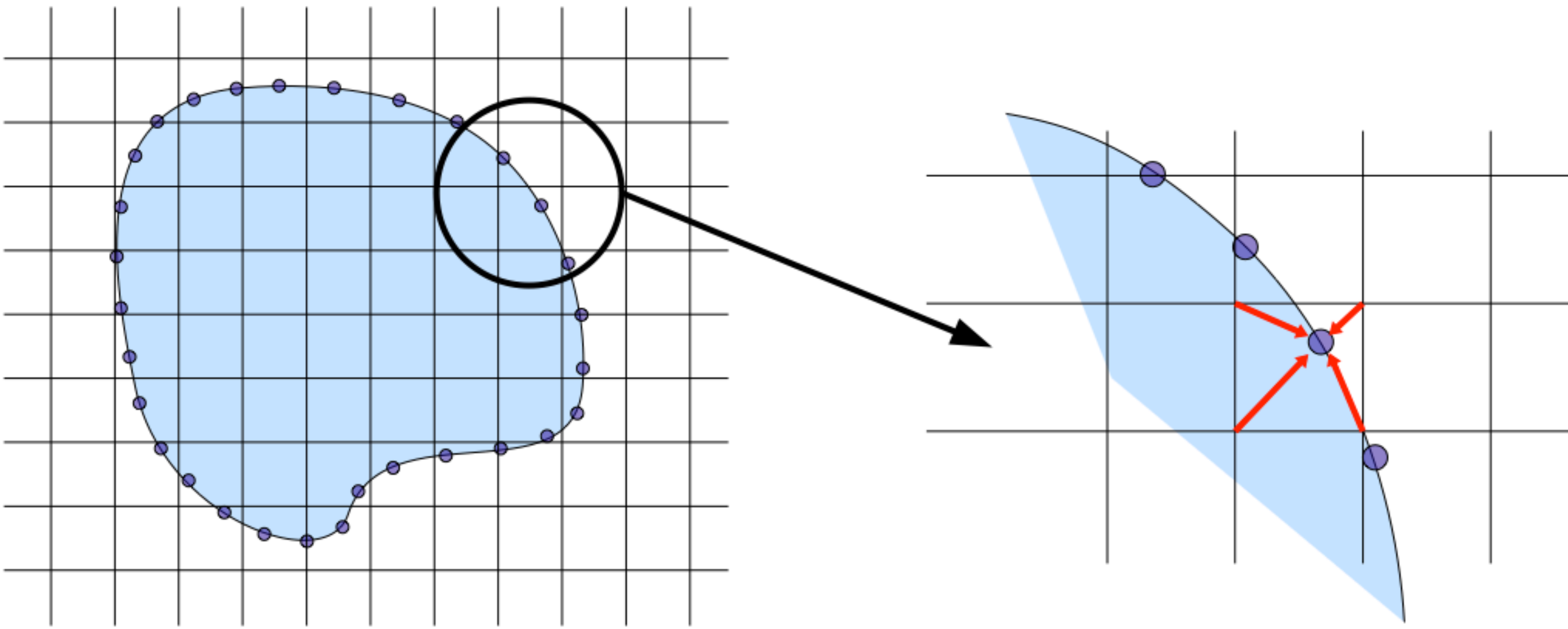
$$\frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \nu \nabla^2 \mathbf{u} - \rho^{-1} \nabla p + \mathbf{f}, \qquad \nabla \cdot \mathbf{u} = 0$$

are solved on the Cartesian grid.

The IB method

- Key elements of IBMs:
  - *Interpolation: from Eulerian grid to Lagrangian points*

$$f(\mathbf{x}) \rightarrow F(\mathbf{X})$$

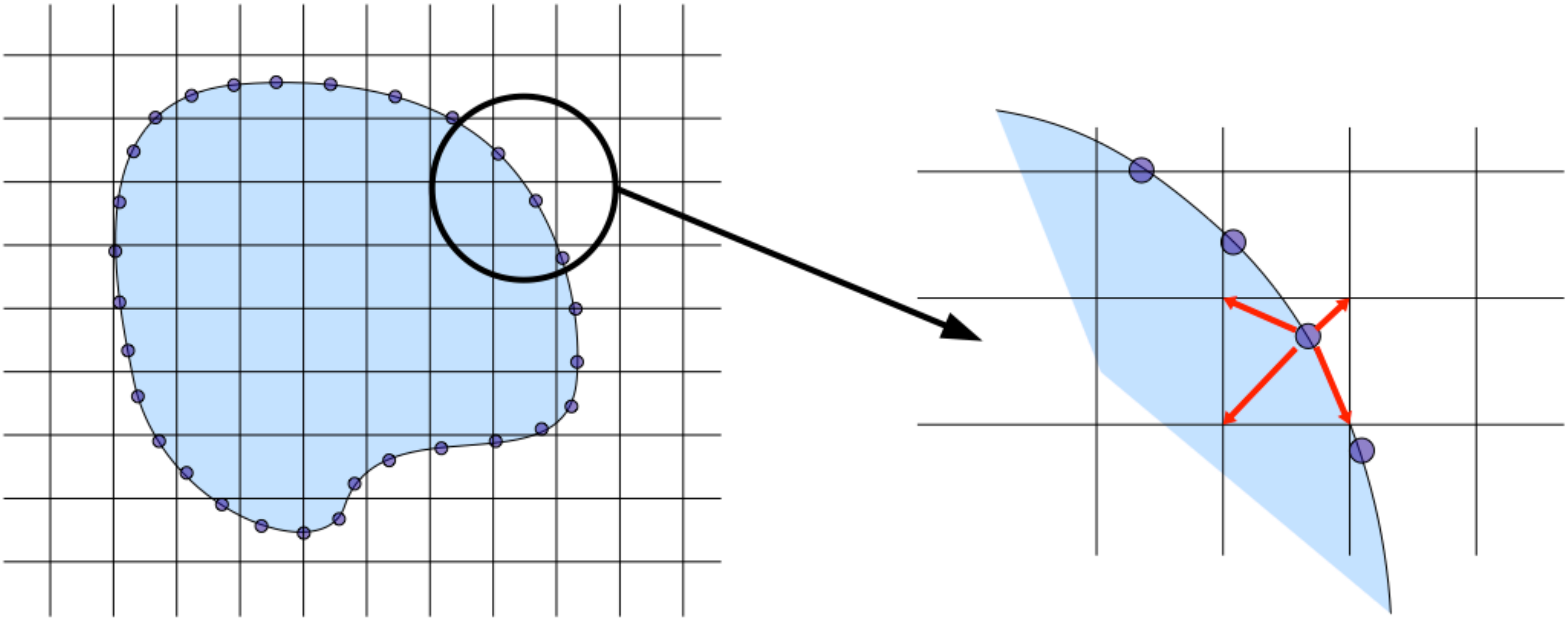


The IB method

- Key elements of IBMs:
  - *Interpolation: from Eulerian grid to Lagrangian points*

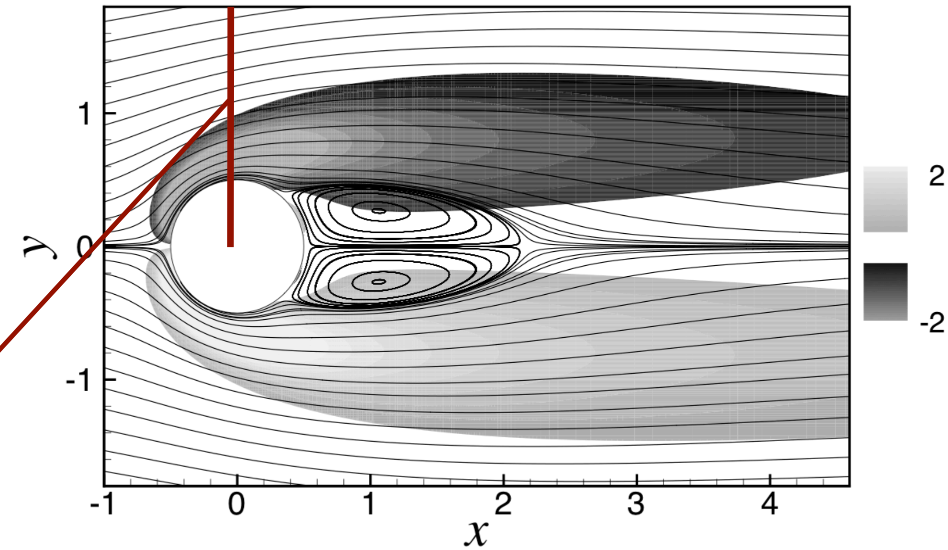
$$f(\mathbf{x}) \rightarrow F(\mathbf{X})$$

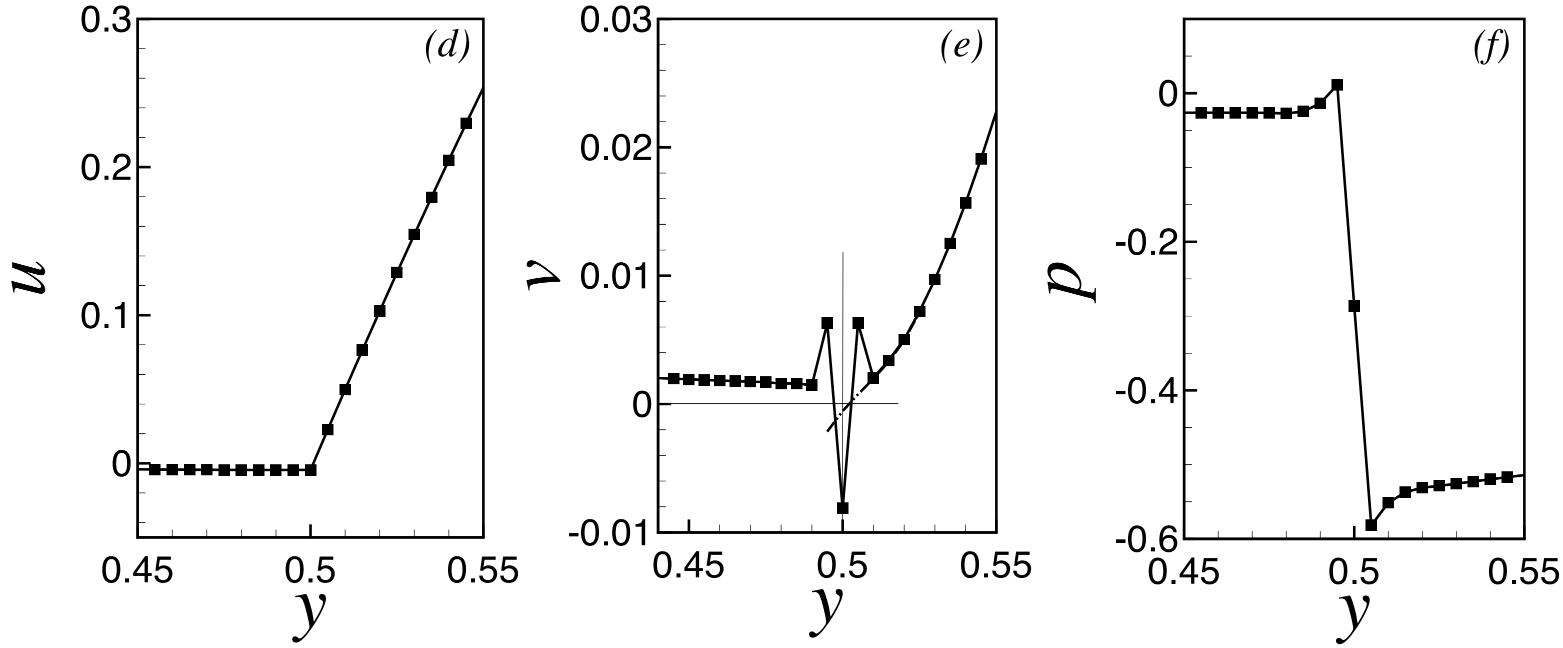
  - *Spreading (convolution): from Lagrangian points to Eulerian grid*



The IB method

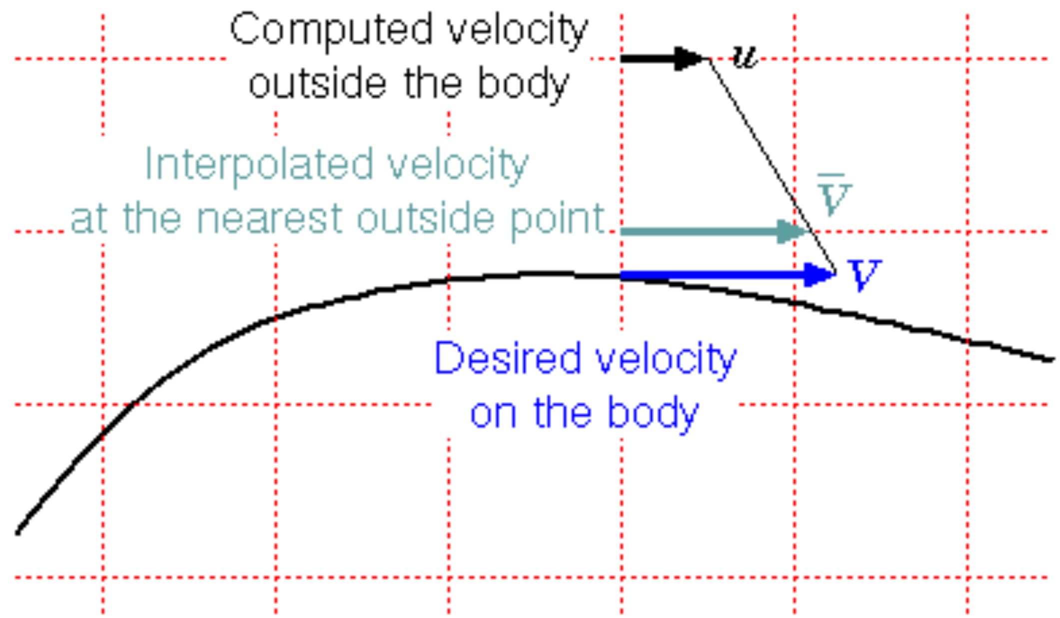
- The form of the spreading and interpolation operators determines the form of the interface
  - *Diffused interface methods*





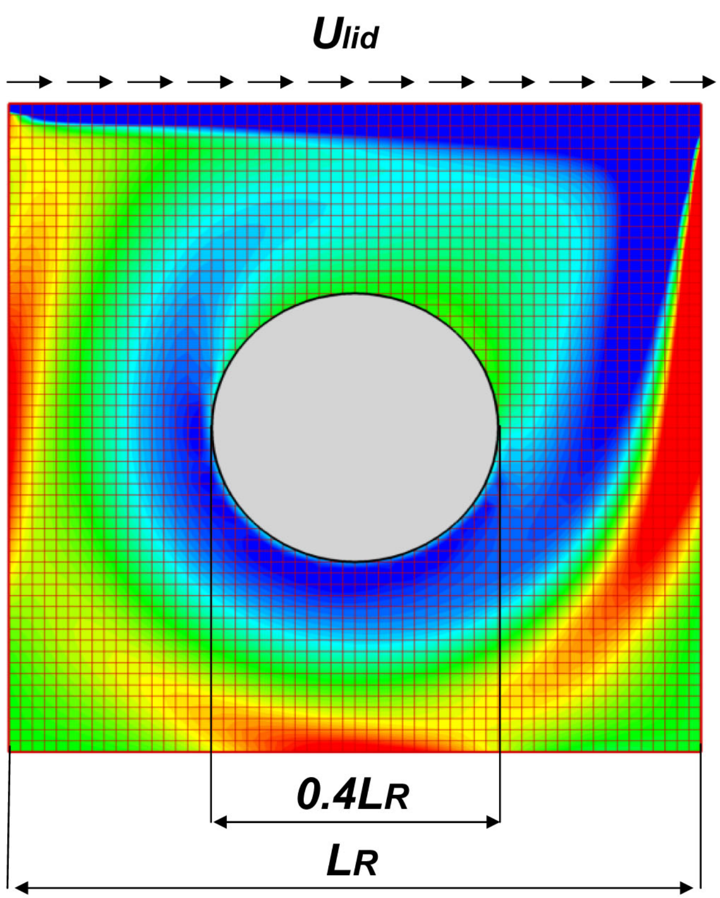
THE IB METHOD: DIRECT RECONSTRUCTION

- Solve the momentum equations everywhere for the predicted velocity.
- Apply the boundary conditions at all the (true) boundary points.
- Apply the boundary conditions at all the immersed boundary points.
  - *Consider the nearest point outside the body.*
  - *Set the velocity there to* $\bar{V}_i$

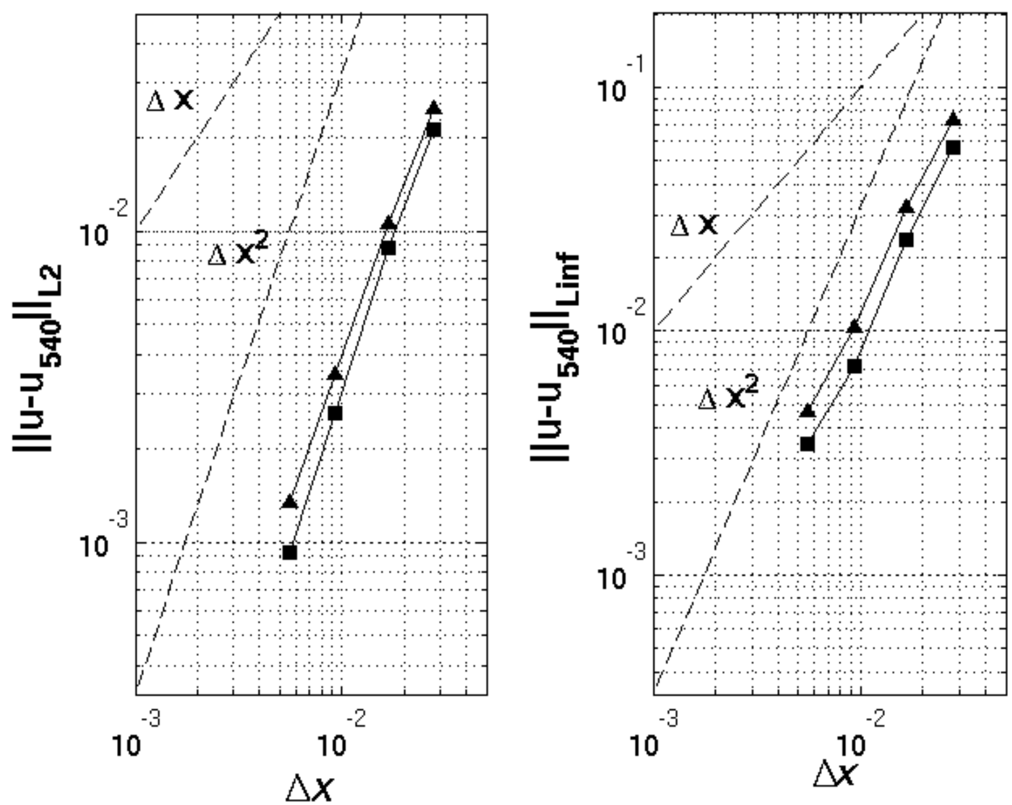


THE IB METHOD: DIRECT RECONSTRUCTION

Accuracy study:

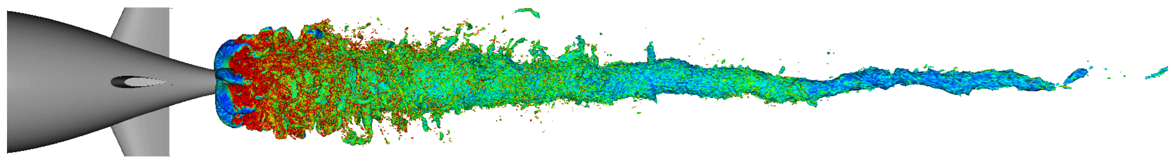


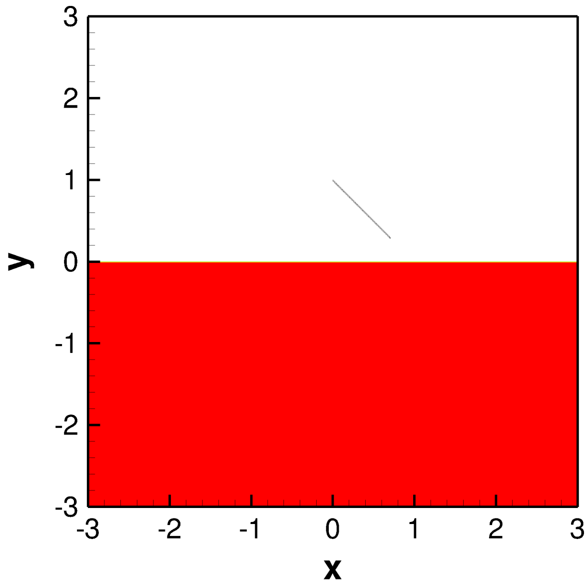
Cylinder in a cavity, Re=1000

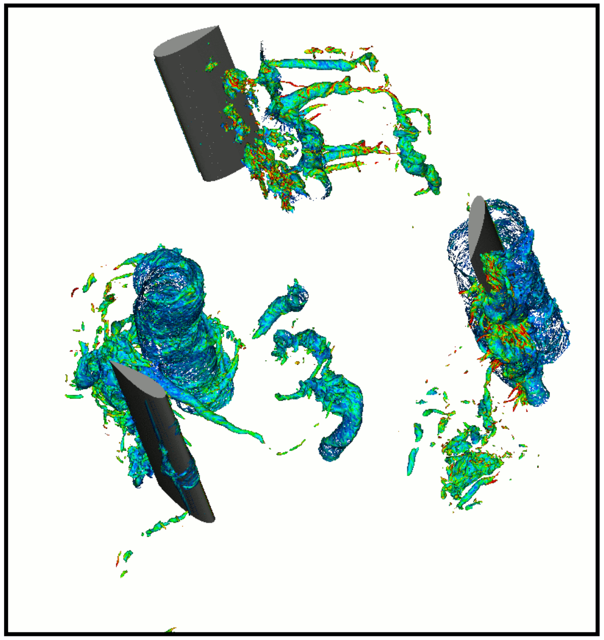


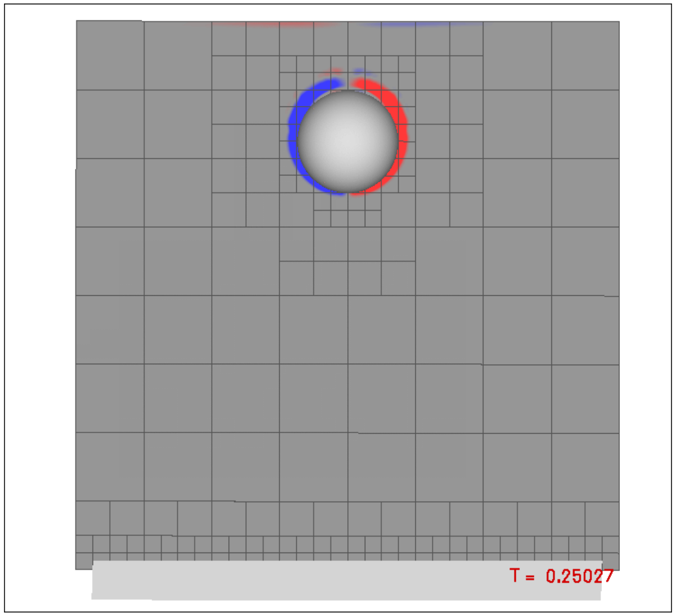
Error norms

The IB method: examples



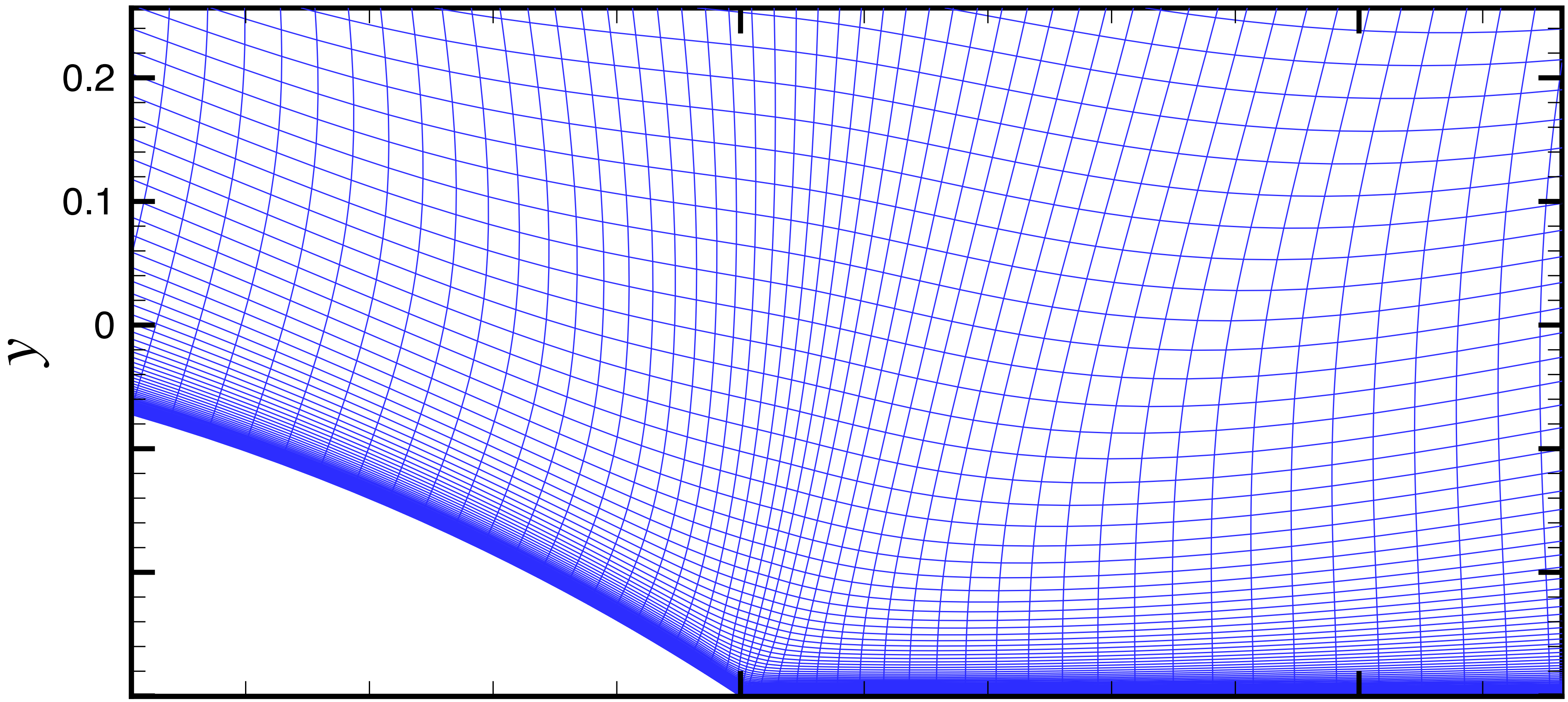






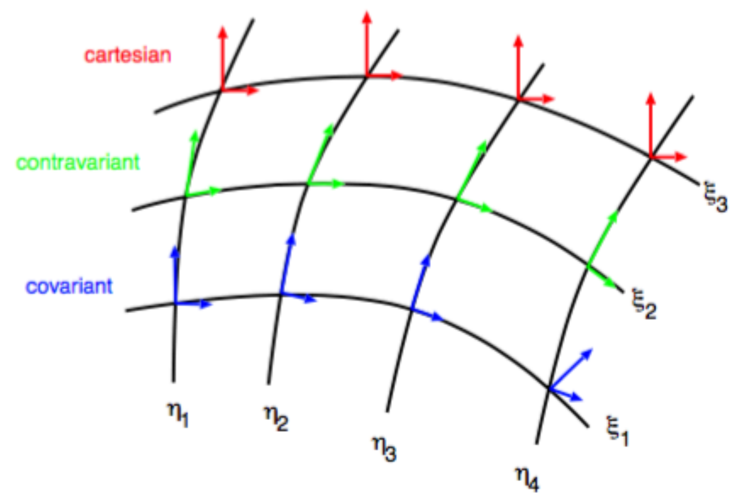
Body-fitted grids

- One can use a body-fitted grid
- Need to transform equations of motion in generalized coordinates



Velocity components

- The definition of velocity components is not unique if the grid is non-orthogonal:
  - Cartesian components (along the cartesian axes).
  - Contravariant components (along the curvilinear axes).
  - Covariant components (normal to the curvilinear axes).



- In orthogonal grids, co-variant and contra-variant components coincide.

Body-fitted grids

- General case for curvilinear coordinates:
  - Let the coordinate transformation be given by $\xi = \xi(x, y)$, $\eta = \eta(x, y)$.
  - The transformation is characterized by the Jacobean:

$$\mathcal{J} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi}$$

  - We define as β^{ij} the cofactor of $\partial x_i / \partial \xi_j$ in $\mathcal{J}$.

Body-fitted grids

- The derivatives are obtained by applying the chain rule, so that:
$$\frac{\partial \phi}{\partial x} = \frac{1}{\mathcal{J}}\left(\frac{\partial \phi}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial \phi}{\partial \eta}\frac{\partial y}{\partial \xi}\right)$$
or, in general,
$$\frac{\partial \phi}{\partial x_i} = \frac{\beta^{ij}}{\mathcal{J}}\frac{\partial \phi}{\partial \xi_j}$$
where β^{ij} is the cofactor of $\partial x_i / \partial \xi_j$.

Body-fitted grids

- The generic transport equation

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x_j}\left(u_j \phi - \Gamma \frac{\partial \phi}{\partial x_j}\right) = q$$

becomes

$$\mathcal{J}\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial \xi_j}\left(U_j \phi - \frac{\Gamma}{\mathcal{J}} \frac{\partial \phi}{\partial \xi_m} B^{mj}\right) = \mathcal{J} q$$

where

$$B^{mj} = \beta^{kj}\beta^{km}$$

and

$$U_j = u_k \beta^{kj}$$

are the covariant velocity components.

Body-fitted grids

- Notice that
  - The flux term has the transport velocity normal to the surface.
  - Cross-derivatives appear in the diffusion term:

$$\frac{\partial}{\partial \xi_1}\left(-\frac{\Gamma}{\mathcal{J}}\frac{\partial \phi}{\partial \xi_2}B^{21}\right)$$

    etc., which are zero in orthogonal systems.
    These terms must be treated explicitly to avoid making the matrix very complex. Therefore, they must be much smaller than the normal diffusion terms (grid not-too-skewed).
  - The derivatives $\partial x_i/\partial \xi_j$ must be evaluated numerically using the same order of accuracy, even if they are known analytically.
  - In the transformed space, one can use $\Delta \xi_i = 1$.