Prob. 1:

$$\mathcal{L} = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$$

$$L = \frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2}$$

Taylor series expansion:

$$u_i^{n+1} = u_i^n + \frac{\partial u_i}{\partial t}\Delta t + O(\Delta t^2)$$

$$u_{i+1}^n = u_i^n + \frac{\partial u_i}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 u_i}{\partial x^2}\Delta x^2 + \frac{1}{6}\frac{\partial^3 u_i}{\partial x^3}\Delta x^3 + O(\Delta x^4)$$

$$u_{i-1}^n = u_i^n - \frac{\partial u_i}{\partial x}\Delta x + \frac{1}{2}\frac{\partial^2 u_i}{\partial x^2}\Delta x^2 - \frac{1}{6}\frac{\partial^3 u_i}{\partial x^3}\Delta x^3 + O(\Delta x^4)$$

$$\Longrightarrow L = \mathcal{L} + O(\Delta t, \Delta x^2)$$

$$\Delta t, \Delta x \longrightarrow 0 \qquad L \rightarrow \mathcal{L}$$

Prob. 2

$$\frac{\partial u_i}{\partial t} = \frac{u_{i+1} + u_{i-1} - 2u_i}{h^2}$$

$$\frac{\partial u_1}{\partial t} = \frac{1}{h^2}\left(u_2 - 2u_1 + u_0\right)$$

$$\frac{\partial u_2}{\partial t} = \frac{1}{h^2}\left(u_3 - 2u_2 + u_1\right)$$

$$\frac{\partial u_3}{\partial t} = \frac{1}{h^2}\left(u_4 - 2u_3 + u_2\right)$$

$$\Longrightarrow \frac{\partial}{\partial t}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{1}{h^2}\underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}}_{A}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{u_0}{h^2} \\ 0 \\ u_4/h^2 \end{bmatrix}}_{f}$$

If the real parts of the eigenvalues of matrix A are less than zero, the scheme is stable.

$$A x = \lambda x \rightarrow (A - \lambda I)x = 0$$

$$\rightarrow |A - \lambda I| = 0 \quad \Longrightarrow \begin{cases} \lambda_1 = -2-\sqrt{2} \\ \lambda_2 = -2 \\ \lambda_3 = -2+\sqrt{2} \end{cases}$$

Therefore the solution is stable.