$$* \; \phi = \overline{\phi} + \phi'$$

$$** \; u_i = \overline{u}_i + u_i'$$

$$\frac{\partial \phi}{\partial t} + \frac{\partial}{\partial x_i}(u_i \phi) = \Gamma \frac{\partial^2}{\partial x_i \partial x_i} \phi$$

Substitute (*) and (**):

$$\rightarrow \frac{\partial}{\partial t}(\overline{\phi} + \phi') + \frac{\partial}{\partial x_i}\left[(\overline{u}_i + u_i')(\overline{\phi} + \phi')\right] = \Gamma \frac{\partial^2}{\partial x_i \partial x_i}(\overline{\phi} + \phi')$$

$$\rightarrow \frac{\partial}{\partial t}(\overline{\phi}) + \frac{\partial}{\partial t}(\phi') + \frac{\partial}{\partial x_i}(\overline{u}_i \overline{\phi}) + \frac{\partial}{\partial x_i}(\overline{\phi} u_i') + \frac{\partial}{\partial x_i}(\overline{u}_i \phi') + \frac{\partial}{\partial x_i}(\phi' u_i')$$

$$= \Gamma \frac{\partial^2}{\partial x_i \partial x_i} \overline{\phi} + \Gamma \frac{\partial^2}{\partial x_i \partial x_i} \phi'$$

Apply Reynolds-averaging operator:

$$\rightarrow \underbrace{\frac{\partial}{\partial t}(\overline{\overline{\phi}})}_{a} + \underbrace{\frac{\partial}{\partial t}(\overline{\phi'})}_{b} + \underbrace{\frac{\partial}{\partial x_i}(\overline{\overline{u}_i \overline{\phi}})}_{c} + \underbrace{\frac{\partial}{\partial x_i}(\overline{\overline{\phi} u_i'})}_{d} + \underbrace{\frac{\partial}{\partial x_i}(\overline{\overline{u}_i \phi'})}_{e} + \underbrace{\frac{\partial}{\partial x_i}(\overline{\phi' u_i'})}_{f}$$

$$= \underbrace{\Gamma \frac{\partial^2}{\partial x_i \partial x_i} \overline{\overline{\phi}}}_{g} + \underbrace{\Gamma \frac{\partial^2}{\partial x_i \partial x_i} \overline{\phi'}}_{h}$$

Averaging the averaged part of a variable does not result in any change to that variable. Therefore, a, c and g remain the same.

Averaging the fluctuating part of a variable yields zero.
Therefore, $b = h = 0$.

If $\bar{X}, y'$ are average and fluctuating parts of two variables, then $\overline{\bar{X} y'} = \bar{X}\,\overline{y'} = 0$. In other words, the averaged variable ($\bar{X}$) can be factored out of the averaging operator.
Therefore: $d = e = 0$

The final form of the Reynolds-averaged equation is:

$$\frac{\partial}{\partial t}\bar{\phi} + \frac{\partial}{\partial x_i}(\bar{u}_i \bar{\phi}) + \frac{\partial}{\partial x_i}(\overline{u_i' \phi'}) = \Gamma \frac{\partial^2 \bar{\phi}}{\partial x_i \partial x_i}$$

or

$$\underbrace{\frac{\partial}{\partial t}\bar{\phi}}_{\text{I}} + \underbrace{\nabla.(\bar{u}\bar{\phi})}_{\text{II}} + \underbrace{\nabla.(\overline{u'\phi'})}_{\text{III}} = \underbrace{\Gamma\, \nabla^2 \bar{\phi}}_{\text{IV}}$$

I: Rate of change of mean scalar with respect to time. Note that if averaging is done in time, this term reduces to zero, because time-averaging removes time dependence.

II: $\nabla.(\bar{u}\bar{\phi}) = \bar{u}.(\nabla\bar{\phi}) + \bar{\phi}(\nabla.\bar{u})$.
From continuity we have: $\nabla.\bar{u} = 0$
Therefore II is simply the advection of $\bar{\phi}$ by the mean flow.

III: Turbulent diffusion.

IV: Molecular diffusion