Question 7:

a)

$$\phi_{i+1} = \phi_i + \frac{\partial \phi}{\partial x} h + \frac{h^2}{2}\frac{\partial^2 \phi}{\partial x^2} + \frac{h^3}{6}\frac{\partial^3 \phi}{\partial x^3} + \frac{h^4}{24}\frac{\partial^4 \phi}{\partial x^4} + O(h^5) \qquad (I)$$

$$\phi_{i+2} = \phi_i + 2h\frac{\partial \phi}{\partial x} + \frac{1}{2}(2h)^2\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{6}(2h)^3\frac{\partial^3 \phi}{\partial x^3} + \frac{1}{24}(2h)^4\frac{\partial^4 \phi}{\partial x^4} + O(h^5) \qquad (II)$$

$$\phi_{i+3} = \phi_i + 3h\frac{\partial \phi}{\partial x} + \frac{1}{2}(3h)^2\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{6}(3h)^3\frac{\partial^3 \phi}{\partial x^3} + \frac{1}{24}(3h)^4\frac{\partial^4 \phi}{\partial x^4} + O(h^5) \qquad (III)$$

$$\phi_{i+4} = \phi_i + 4h\frac{\partial \phi}{\partial x} + \frac{1}{2}(4h)^2\frac{\partial^2 \phi}{\partial x^2} + \frac{1}{6}(4h)^3\frac{\partial^3 \phi}{\partial x^3} + \frac{1}{24}(4h)^4\frac{\partial^4 \phi}{\partial x^4} + O(h^5) \qquad (IV)$$

Multiply equation (I) by A, eq (II) by B, eq III, by C and eq (IV) by D. The coefficient for $\frac{\partial^2 \phi}{\partial x^2}$ must be equal to one and the coefficients for all other derivatives must be zero which gives us the following linear system:

$$\begin{cases} A + 2B + 3C + 4D = 0 \\ A + 4B + 9C + 16D = 2 \\ A + 8B + 27C + 64D = 0 \\ A + 16B + 81C + 256D = 0 \end{cases} \longrightarrow \quad A = -\frac{26}{3}, \quad B = 19/2, \quad C = \frac{-14}{3}, \quad D = \frac{11}{12}$$

$$\rightarrow h^2 \frac{\partial^2 \phi}{\partial x^2} = \frac{-26}{3}\phi_{i+1} + \frac{19}{2}\phi_{i+2} - \frac{14}{3}\phi_{i+3} + \frac{11}{12}\phi_{i+4} + O(h^5)$$

$$\rightarrow \frac{\partial^2 \phi}{\partial x^2} = \left(-\frac{26}{3}\phi_{i+1} + \frac{19}{2}\phi_{i+2} - \frac{14}{3}\phi_{i+3} + \frac{11}{12}\phi_{i+4}\right)/h^2 + O(h^3)$$

The method is 3rd order accurate.

b)

$$\phi_{i-1} = \phi_i - h\frac{\partial \phi}{\partial x} + \frac{1}{2}h^2\frac{\partial^2 \phi}{\partial x^2} + o(h^3) \qquad (I)$$

$$\phi_{i-2} = \phi_i - 2h\frac{\partial \phi}{\partial x} + \frac{1}{2}(-2h)^2\frac{\partial^2 \phi}{\partial x^2} + o(h^3) \qquad (II)$$

Multiply eq (I) by A and eq (II) by B. Coefficient of $\frac{\partial \phi}{\partial x}$ must be one, coefficient of $\frac{\partial^2 \phi}{\partial x^2}$ must be zero.

$$\begin{cases} A + 2B = -1 \\ \frac{1}{2}A + 2B = 0 \end{cases} \longrightarrow \begin{matrix} A = -2 \\ B = \frac{1}{2} \end{matrix}$$

$$\Longrightarrow \frac{\partial \phi}{\partial x} \times h = \frac{3}{2}\phi_i - 2\phi_{i-1} + \frac{1}{2}\phi_{i-2} + o(h^3)$$

$$\longrightarrow \frac{\partial \phi}{\partial x} = \frac{\frac{3}{2}\phi_i - 2\phi_{i-1} + \frac{1}{2}\phi_{i-2}}{h} + o(h^2)$$

Question 2:

a) RHS: $c\frac{\partial \phi}{\partial x}$

$$\phi_{i+1}^{n+1} = \phi_{i+1}^{n} + \frac{\partial \phi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 \phi}{\partial t^2}\Delta t^2 + o(\Delta t^3)$$

$$\phi_{i-1}^{n+1} = \phi_{i-1}^{n} - \frac{\partial \phi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 \phi}{\partial t^2}(-\Delta t)^2 + o(\Delta t^3)$$

$$\Longrightarrow \phi_{i+1}^{n+1} - \phi_{i-1}^{n+1} = \underbrace{\phi_{i+1}^{n} - \phi_{i-1}^{n}}_{(*)} + 2\frac{\partial \phi}{\partial t}\Delta t + o(\Delta t^3) \qquad (I)$$

Central differencing formulation:

$$\frac{\partial \phi}{\partial x} = \frac{\phi_{i+1}^{n} - \phi_{i-1}^{n}}{2\Delta x} + o(\Delta x^2) \Longrightarrow (*) = 2\Delta x\frac{\partial \phi}{\partial x} + o(\Delta x^3)$$

Plug into eq (I)

$$\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1} = 2\Delta x \frac{\partial \phi}{\partial x} + 2\frac{\partial \phi}{\partial t}\Delta t + O(\Delta t^3) + O(\Delta x^3)$$

$$\Rightarrow +C\,\frac{\phi_{i+1}^{n+1} - \phi_{i-1}^{n+1}}{2\Delta x} = +C\left[\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial t}\frac{\Delta t}{\Delta x} + O\left(\frac{\Delta t^3}{\Delta x}\right) + O(\Delta x^2)\right]$$

$$\rightarrow \lim_{\Delta t, \Delta x \to 0} RHS = C\frac{\partial \phi}{\partial x} + \underbrace{\lim_{\Delta t, \Delta x \to 0} \frac{\partial \phi}{\partial t}\frac{\Delta t}{\Delta x}}$$

$\hookrightarrow$ Cannot be determined.

The scheme is inconsistent.

b)

$$\frac{\partial \phi_i}{\partial t} - C\,\frac{\phi_{i+1} - \phi_i}{\Delta x} = 0$$

$$\left.\begin{aligned} \frac{\partial \phi_0}{\partial t} &= C\,\frac{\phi_1 - \phi_0}{h} \\ \frac{\partial \phi_1}{\partial t} &= C\,\frac{\phi_2 - \phi_1}{h} \\ \frac{\partial \phi_3}{\partial t} &= C\,\frac{\phi_3 - \phi_2}{h} \end{aligned}\right\} \rightarrow \frac{\partial}{\partial t}\begin{bmatrix}\phi_0\\ \phi_1\\ \phi_2\end{bmatrix} = \frac{c}{h}\underbrace{\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1\end{bmatrix}}_{A}\begin{bmatrix}\phi_0\\ \phi_1\\ \phi_2\end{bmatrix} + \begin{bmatrix}0\\ 0\\ q/h\end{bmatrix}$$

We need to inspect the eigenvalues of matrix A to determine the stability of the scheme.

$$|A - \lambda I| = 0 \rightarrow \begin{vmatrix}-1-\lambda & 1 & 0\\ 0 & -1-\lambda & 1\\ 0 & 0 & -1-\lambda\end{vmatrix} = 0 \Rightarrow (-1-\lambda)^3 = 0$$

$$\Rightarrow \begin{cases}\lambda_1 = -1\\ \lambda_2 = -1\\ \lambda_3 = -1\end{cases}$$

The real part of all eigenvalues are negative, therefore the scheme is stable.

Question 3:

(a)

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right)$$

$T = \bar{T} + T'$, $u = \bar{u} + u'$, $v = \bar{v} + v'$ $\longrightarrow$ plug into the equation.

$$\frac{\partial}{\partial t}(\bar{T}+T') + (\bar{u}+u')\frac{\partial}{\partial x}(\bar{T}+T') + (\bar{v}+v')\frac{\partial(\bar{T}+T')}{\partial y} = \frac{k}{\rho c_p}\left(\frac{\partial^2}{\partial x^2}(\bar{T}+T') + \frac{\partial^2}{\partial y^2}(\bar{T}+T')\right)$$

$$\frac{\partial}{\partial t}\bar{T} + \frac{\partial T'}{\partial t} + \bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{u}\frac{\partial T'}{\partial x} + u'\frac{\partial \bar{T}}{\partial x} + u'\frac{\partial T'}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} + \bar{v}\frac{\partial T'}{\partial y} + v'\frac{\partial \bar{T}}{\partial y} + v'\frac{\partial T'}{\partial y}$$
$$= \frac{k}{\rho c_p}\left[\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2} + \frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2}\right]$$

Now average both sides of the equation. The following terms vanish:

$$\overline{\frac{\partial T'}{\partial t}} = 0, \quad \overline{\bar{u}\frac{\partial T'}{\partial x}} = 0, \quad \overline{u'\frac{\partial \bar{T}}{\partial x}} = 0, \quad \overline{\bar{v}\frac{\partial T'}{\partial y}} = 0, \quad \overline{v'\frac{\partial \bar{T}}{\partial y}} = 0$$

$$\overline{\frac{\partial^2 T'}{\partial x^2}} = 0, \quad \overline{\frac{\partial^2 T'}{\partial y^2}} = 0$$

Finally

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} + \underbrace{\overline{u'\frac{\partial T'}{\partial x}} + \overline{v'\frac{\partial T'}{\partial y}}}_{\text{using the chain rule we can rewrite these terms}} = \frac{k}{\rho c_p}\left[\frac{\partial^2 \bar{T}}{\partial x^2} + \frac{\partial^2 \bar{T}}{\partial y^2}\right]$$

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} + \frac{\partial}{\partial x}\overline{(u'T')} + \frac{\partial}{\partial y}\overline{(v'T')} - \overline{T'\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)} = \frac{k}{\rho c_p}\nabla^2 \bar{T}$$

From continuity equation:

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0$$

$$\longrightarrow \underbrace{\frac{\partial \overline{T}}{\partial t}}_{\text{Rate of changing in time}} + \underbrace{\bar{u}\frac{\partial \overline{T}}{\partial x} + \bar{v}\frac{\partial \overline{T}}{\partial y}}_{\text{Convection}} + \underbrace{\frac{\partial}{\partial x}(\overline{u'T'}) + \frac{\partial}{\partial y}(\overline{v'T'})}_{\text{Reynolds stress energy transport.}} = \underbrace{\frac{k}{\rho c_p}\nabla^2 \overline{T}}_{\text{diffusion}}$$

(b) $\overline{u'T'}$ and $\overline{v'T'}$ need to be modeled.

One strategy is to write transport equations for them but will run into the closure problem.

Another strategy is to assume a K_t analogous to eddy viscosity ν_t to account for turbulent effects.