Solution methods for the incompressible Navier-Stokes equations

Elias Balaras

MAE6220

Fall 2017

Navier-Stokes Equations

$$\frac{\partial u_i}{\partial x_i} = 0$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial}{\partial x_j}(u_i u_j) = -\frac{\partial P}{\partial x_i} + \frac{\partial}{\partial x_j}\left(2\nu S_{ij}\right)$$

$$P = p/\rho$$

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

Navier-Stokes Equations

- 4 equations, 4 unknowns: u_i, p.
- No time derivative in the conservation of mass.
- No separate equation for p (or P).

Solution methods for the incompressible Navier-Stokes equations: Explicit methods

Projection method – Fractional time-step method Chorin 1969, Kim & Moin 1985

- Write the momentum equations as

$$\frac{\partial u_j}{\partial t} = \left[F_i^c + F_i^p + F_i^v\right]$$

$$F_i^c = -\frac{\partial}{\partial x_j}(u_i u_j) \qquad \text{Divergence of the convective flux}$$

$$F_i^p = -\frac{\partial}{\partial x_j}(P\delta_{ij}) \qquad \text{Divergence of the pressure flux}$$

$$F_i^v = \frac{\partial}{\partial x_j}\left(\nu\frac{\partial u_i}{\partial x_j}\right) \qquad \text{Divergence of the viscous flux}$$

Fractional time-step method

A time-advancement sequence consists of the following steps:

1. Solve the Helmholtz equation for a predicted velocity.
2. Solve the Poisson equation for the (modified) pressure.
3. Correct the velocity to enforce conservation of mass.

Fractional time-step method

- Divergence-free velocity field at time-step n.
- Explicit Euler scheme.
- Velocity prediction (Helmholtz equation):

$$\frac{v_i - u_i^n}{\Delta t} = F_i^{c\,n} + F_i^{v\,n} \quad \Rightarrow \quad v_i = u_i^n + \Delta t \left(F_i^{c\,n} + F_i^{v\,n} \right).$$

  v_i is not divergence free.
- Velocity correction:

$$\frac{u_i^{n+1} - v_i}{\Delta t} = F_i^{p\,n+1} \quad \Rightarrow \quad u_i^{n+1} = v_i + \Delta t F_i^{p\,n+1}.$$

Fractional time-step method

- Divergence-free velocity field at time-step n.
- Explicit Euler scheme.
- Velocity prediction (Helmholtz equation):

$$\frac{v_i - u_i^n}{\Delta t} = F_i^{c\,n} + F_i^{v\,n} \quad \Rightarrow \quad v_i = u_i^n + \Delta t \left(F_i^{c\,n} + F_i^{v\,n} \right).$$

  v_i is not divergence free.
- Velocity correction:

$$\frac{u_i^{n+1} - v_i}{\Delta t} = F_i^{p\,n+1} \quad \Rightarrow \quad u_i^{n+1} = v_i + \Delta t F_i^{p\,n+1}.$$

- Sum of two steps:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = F_i^{c\,n} + F_i^{p\,n+1} + F_i^{v\,n}$$

Fractional time-step method

- The momentum equation is satisfied.
- We do not know how to compute the pressure.
- Conservation of mass has not been satisfied.

Solution: Take divergence of the correction equation:

Fractional time-step method

- Velocity correction:

$$\frac{u_i^{n+1} - v_i}{\Delta t} = F_i^{p\,n}$$

- Divergence of above:

$$\frac{\partial}{\partial x_i}\left(\frac{u_i^{n+1} - v_i}{\Delta t}\right) = \frac{\partial F_i^{p\,n}}{\partial x_i} \quad \Rightarrow \quad \frac{\partial u_i^{n+1}}{\partial x_i} - \frac{\partial v_i}{\partial x_i} = -\Delta t \frac{\partial^2 P}{\partial x_i \partial x_i}$$

Fractional time-step method

- If the pressure satisfies the Poisson equation:

$$\Delta t \frac{\partial^2 P}{\partial x_i \partial x_i} = \frac{\partial v_i}{\partial x_i}$$

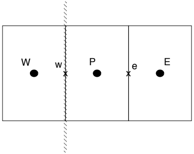
then u_i^{n+1} satisfies

$$\frac{\partial u_i^{n+1}}{\partial x_i} = 0$$

Fractional time-step method
Explicit Euler Implementation

1. Velocity prediction (Helmholtz equation):
$$v_i = u_i^n + \Delta t \left(F_i^{c\,n} + F_i^{v\,n} \right)$$
2. Poisson solution:
$$\nabla^2 P^{n+1} = \frac{1}{\Delta t} \frac{\partial v_i}{\partial x_i}$$
3. Velocity correction:
$$u_i^{n+1} = v_i + \Delta t F_i^{p\,n+1}$$

Boundary conditions



- Notice that in explicit methods the boundary conditions for the velocity normal to the boundary are irrelevant. All that matters are the b.c.s for the parallel component.
- In 1-D the pressure equation at the boundary is

$$\frac{1}{\Delta x}\left(\left.\frac{\partial p}{\partial x}\right|_e - \left.\frac{\partial p}{\partial x}\right|_w\right) = -\frac{1}{\Delta t}\frac{v_e - v_w}{\Delta x}$$

- The pressure gradient at w can be obtained from the correction step:

$$\left.\frac{\partial p}{\partial x}\right|_w = \frac{u_w^{n+1} - v_w}{\Delta t} \Rightarrow \frac{1}{\Delta x}\left(\left.\frac{\partial p}{\partial x}\right|_e - \frac{u_w^{n+1} - v_w}{\Delta t}\right) = \frac{v_e - v_w}{\Delta x \Delta t}$$

$\Rightarrow v_w$ cancels out and does not affect the solution.

Solution methods for the incompressible Navier-Stokes equations: Semi-implicit methods

Semi-implicit schemes

- In some cases the viscous time-step limit is too restrictive:

$$\frac{\Delta t_c}{\Delta t_v} = \frac{\mathrm{CFL}}{\sigma} \frac{\nu}{|u|\Delta x} \sim Re_c^{-1}$$

- If the grid is very fine or ν is large, $Re_c << 1$.
- In such cases it is desirable to make the viscous term implicit.

Explicit + Implicit Euler – 1

- Helmholtz step:

$$
\begin{aligned}
\frac{v_i - u_i^n}{\Delta t} &= F_i^{c\,n} + \widetilde{F}_i^{vn+1} \\
\text{where } \widetilde{F}_i^{vn+1} &= \frac{\partial}{\partial x_j}\left(\nu \frac{\partial v_i}{\partial x_j}\right) \\
\left[v_i - \Delta t \widetilde{F}_i^{vn+1}\right] &= u_i^n + \Delta t F_i^{c\,n} \\
\left[1 - \Delta t \frac{\partial}{\partial x_j}\left(\nu \frac{\partial}{\partial x_j}\right)\right] v_i &= u_i^n + \Delta t F_i^{c\,n} \\
A v_i &= u_i^n + \Delta t F_i^{c\,n}
\end{aligned}
$$

- The matrix A depends on the discretization operators. Typically, A is penta-diagonal in 2D, hepta-diagonal in 3D.
- u and v equations are decoupled.

Explicit + Implicit Euler – 2

- Poisson solution:

$$\nabla^2 P^{n+1} = \frac{1}{\Delta t}\frac{\partial v_i}{\partial x_i}$$

- Velocity correction:

$$u_i^{n+1} = v_i + \Delta t F_i^{p\,n+1}$$

Explicit + Implicit Euler – 3

- Sum of first and third steps:

$$\begin{aligned}\frac{u_i^{n+1} - u_i^n}{\Delta t} &= F_i^{p\,n+1} + F_i^{c\,n} + \widetilde{F}_i^{vn+1} \\ &= -\frac{\partial P^{n+1}}{\partial x_i} - \frac{\partial}{\partial x_i}(u_i^n u_j^n) + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial v_i}{\partial x_j}\right)\end{aligned}$$

  $\Rightarrow$ momentum equation is **not** satisfied.

- Solution: replace P with ϕ such that

$$\frac{\partial \phi}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\nu \Delta t \frac{\partial^2 \phi}{\partial x_i \partial x_j}\right] = \frac{\partial P}{\partial x_i}$$

Explicit + Implicit Euler – 4

- Helmholtz step:

$$\frac{v_i - u_i^n}{\Delta t} = F_i^{c\,n} + \widetilde{F}_i^{vn+1}$$

- Poisson solution:

$$\nabla^2 \phi = \frac{1}{\Delta t} \frac{\partial v_i}{\partial x_i}$$

- Velocity correction:

$$u_i^{n+1} = v_i + \Delta t \frac{\partial \phi}{\partial x_i}$$

Explicit + Implicit Euler – 5

- Sum of first and third steps:

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \widetilde{F}_i^{vn+1} + F_i^{c\,n} - \frac{\partial \phi}{\partial x_i}$$

but

$$\begin{aligned}
\frac{\partial \phi}{\partial x_i} &= \frac{\partial P^{n+1}}{\partial x_i} - \frac{\partial}{\partial x_j}\left[\nu \Delta t \frac{\partial^2 \phi}{\partial x_i \partial x_j}\right] \\
&= \frac{\partial P^{n+1}}{\partial x_i} - \frac{\partial}{\partial x_j}\left[\nu \frac{\partial}{\partial x_j}\left(\Delta t \frac{\partial \phi}{\partial x_i}\right)\right] \\
&= \frac{\partial P^{n+1}}{\partial x_i} - \frac{\partial}{\partial x_j}\left[\nu \frac{\partial}{\partial x_j}\left(u_i^{n+1} - v_i\right)\right]
\end{aligned}$$

Explicit + Implicit Euler – 6

- Sum of first and third steps:

$$
\begin{aligned}
\frac{u_i^{n+1} - u_i^n}{\Delta t} &= \widetilde{F}_i^{vn+1} + F_i^{c\,n} - \frac{\partial P^{n+1}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\nu \frac{\partial}{\partial x_j}\left(u_i^{n+1} - v_i\right)\right] \\
&= \frac{\partial}{\partial x_j}\left(\nu \frac{\partial v_i}{\partial x_j}\right) + F_i^{c\,n} - \frac{\partial P^{n+1}}{\partial x_i} \\
&\quad + \frac{\partial}{\partial x_j}\left[\nu \frac{\partial}{\partial x_j}\left(u_i^{n+1} - v_i\right)\right] \\
&= \frac{\partial}{\partial x_i}(u_i^n u_j^n) - \frac{\partial P^{n+1}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\nu \frac{\partial u_i^{n+1}}{\partial x_j}\right]
\end{aligned}
$$

Boundary conditions

- If $\partial p/\partial n = 0$ on the boundary, b.c.s for the predicted velocity are the same as for the actual velocity.
- Otherwise, Leveque-Oliger splitting.

J. Kim and P. Moin. "Application of a fractional step method to incompressible Navier-Stokes equations." *J. Comput. Phys.* **59** 308, 1985.

Solution methods for the incompressible Navier-Stokes equations: Fully implicit methods

Fully implicit methods

- In some cases (steady-state problems) we are not interested in the transient at all and we want to speed up convergence as much as possible.
- This can be obtained by making the whole equation implicit.

Crank-Nicolson time-advancement

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{1}{2}\left[-\frac{\delta}{\delta x_j}\left(u_i^{n+1}u_j^{n+1} + u_i^n u_j^n\right) - \frac{\delta}{\delta x_i}\left(P^{n+1} + P^n\right) + \nu\frac{\delta^2}{\delta x_j \delta x_j}\left(u_i^{n+1} + u_i^n\right)\right]$$

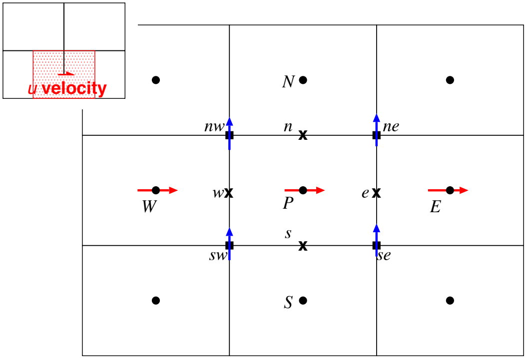
Lumping together u^{n+1} terms and dropping the $n+1$...

$$u_{i,P} + \frac{\Delta t}{2}\frac{\delta}{\delta x_j}\left[u_i u_j - \nu\frac{\delta u_i}{\delta x_j}\right] = Q_i - \frac{\Delta t}{2}\frac{\delta P}{\delta x_j}$$

where

$$Q_i = u_i^n + \frac{\Delta t}{2}\left[-\frac{\delta}{\delta x_j}\left(u_i^n u_j^n\right) - \frac{\delta P^n}{\delta x_i} + \nu\frac{\delta^2 u_i^n}{\delta x_j \delta x_j}\right]$$

CN advancement, Staggered grid



$$u_{i,P} + \frac{\Delta t}{2}\frac{\delta}{\delta x_j}\left[u_i u_j - \nu \frac{\delta u_i}{\delta x_j}\right] = Q_i - \frac{\Delta t}{2}\frac{\delta P}{\delta x_j}$$

CN advancement, Staggered grid

- Applying the discretization to the u equation...

$$
\begin{aligned}
u_P + \frac{\Delta t}{2} \Bigg[& \frac{(u_P + u_E)^2/4 - (u_W + u_P)^2/4}{\Delta x} \\
& + \frac{(u_P + u_N)(v_{ne} + v_{nw})/4 - (u_P + u_S)(v_{se} + v_{sw})/4}{\Delta y} \\
& - \nu \left(\frac{u_E - 2u_P + u_W}{\Delta x^2} + \frac{u_N - 2u_P + u_S}{\Delta y^2} \right) \Bigg] \\
= Q_1 - \frac{\Delta t}{2} \frac{P_e - P_w}{\Delta x}
\end{aligned}
$$

CN advancement, Staggered grid

This equation can be recast in the form

$$A_{i,P}u_{i,P}+\sum_{l}A_{i,l}u_{i,l}=Q_i-\frac{\delta P}{\delta x_i}$$

with the coefficients $A_{i,P}$ and $A_{i,l}$ functions of $u_{i,P}$ and $u_{i,l}$:

$$\begin{aligned}A_{1,P} \quad = \quad & 1+\frac{\Delta t}{2}\left[\frac{u_P+2u_E}{4\Delta x}-\frac{u_P+2u_W}{4\Delta x}\right.\\ & +\frac{v_{ne}+v_{nw}}{4\Delta y}-\frac{v_{se}+v_{sw}}{4\Delta y}\\ & \left.-\nu\left(\frac{2}{\Delta x^2}+\frac{2}{\Delta y^2}\right)\right]\end{aligned}$$

CN advancement, Staggered grid

- Since the coefficients $A_{i,P}$ and $A_{i,l}$ depend on $u_{i,P}$ and $u_{i,l}$, this system must be solved iteratively.
- We distinguish the Outer iteration (index n) from the Inner iteration (index m) that is carried out, within each time-step, to solve the system

$$A_{i,P} u_{i,P} + \sum_{l} A_{i,l} u_{i,l} = Q_i - \frac{\delta P}{\delta x_i}.$$

Inner iteration

- The new guess u_i^{m*} can be obtained by solving the linear system

$$A_{i,P} u_{i,P}^{m*} + \sum_l A_{i,l} u_{i,l}^{m*} = Q_i - \frac{\delta P^{m-1}}{\delta x_i}.$$

  The $*$ indicates that u_i^{m*} does not satisfy conservation of mass.
- Rewrite this as

$$u_{i,P}^{m*} = \frac{Q_i - \sum_l A_{i,l} u_{i,l}^{m*}}{A_{i,P}} - \frac{1}{A_{i,P}} \frac{\delta P^{m-1}}{\delta x_i} = \widetilde{u}_{i,P}^{m*} - \frac{1}{A_{i,P}} \frac{\delta P^{m-1}}{\delta x_i}.$$

- Now correct the velocity to make it divergence-free:

$$\frac{u_i^m - u_i^{m*}}{\Delta t} = -\frac{1}{A_{i,P}} \frac{\delta (P^m - P^{m-1})}{\delta x_i}$$

  which gives

$$u_i^m = u_i^{m*} - \frac{\Delta t}{A_{i,P}} \frac{\delta P^m}{\delta x_i} + \frac{\Delta t}{A_{i,P}} \frac{\delta P^{m-1}}{\delta x_i} = \widetilde{u}_{i,P}^{m*} - \frac{\Delta t}{A_{i,P}} \frac{\delta P^m}{\delta x_i}.$$

Inner iteration

$$u_i^m = \widetilde{u}_{i,P}^{m*} - \frac{\Delta t}{A_{i,P}} \frac{\delta P^m}{\delta x_i}.$$

- We want

$$\frac{\delta u_i^m}{\delta x_i} = 0 \qquad \Rightarrow \qquad \frac{\delta \widetilde{u}_i^{m*}}{\delta x_i} = \frac{\delta}{\delta x_i}\left[\frac{1}{A_{i,P}} \frac{\delta P^m}{\delta x_i}\right]$$

- Which gives us the Poisson equation:

$$\frac{\delta}{\delta x_i}\left[\frac{1}{A_{i,P}} \frac{\delta P^m}{\delta x_i}\right] = \frac{\delta \widetilde{u}_i^{m*}}{\delta x_i}$$

Setup

Given the velocity and pressure fields at time-step n:

- Compute the known source term Q_i containing the known velocity and pressure.
- Set $m = 0$, u_i^m and P^m equal to the velocity and pressure at the previous time-step n.
- Begin the inner iteration to find $u_i = u_i^{n+1}$ and $P = P^{n+1}$.

Inner iteration

1. Update $m = m + 1$.
2. Assemble the coefficient matrices $A_{i,P}$.
3. Assemble and solve the system

$$A_{i,P} u_{i,P}^{m*} + \sum_l A_{i,l} u_{i,l}^{m*} = Q_i - \frac{\delta P^{m-1}}{\delta x_i}.$$

4. Assemble and solve the pressure correction

$$\frac{\delta}{\delta x_i}\left[\frac{1}{A_{i,P}}\frac{\delta P^m}{\delta x_i}\right] = \frac{\delta \widetilde{u}_i^{m*}}{\delta x_i}; \qquad \widetilde{u}_{i,P}^{m*} = \frac{Q_i - \sum_l A_{i,l} u_{i,l}^{m*}}{A_{i,P}}$$

5. Correct the velocity by

$$u_{i,P}^{m} = u_{i,P}^{m*} - \frac{1}{A_{i,P}}\frac{\delta P^m}{\delta x_i}.$$

6. Calculate the norm of the residual

$$||e|| = \left|\left| A_{i,P} u_{i,P} + \sum_l A_{i,l} u_{i,l} - Q_i + \frac{\delta P}{\delta x_i} \right|\right|.$$

7. If $||e|| > \epsilon$, return to (a). Otherwise exit the inner iteration.

Pressure correction

- Some methods use a pressure correction instead of the actual pressure. Let

$$u_i^m = u_i^{m*} + u_i'; \qquad P^m = P^{m-1} + P'$$

- Substituting $u_i^m = u_i^{m*} + u'$ and $P^m = P^{m-1} + P'$ into

$$A_{i,P} u_{i,P} + \sum_l A_{i,l} u_{i,l} = Q_i - \frac{\delta P}{\delta x_i}.$$

gives

$$A_{i,P} u_{i,P}^{m*} + \sum_l A_{i,l} u_{i,l}^{m*} + A_{i,P} u_{i,P}' + \sum_l A_{i,l} u_{i,l}' = Q_i - \frac{\delta P^{m-1}}{\delta x_i} - \frac{\delta P'}{\delta x_i}.$$

- And, subtracting

$$A_{i,P} u_{i,P}^{m*} + \sum_l A_{i,l} u_{i,l}^{m*} = Q_i - \frac{\delta P^{m-1}}{\delta x_i},$$

we obtain the equation for u_i':

$$A_{i,P} u'_{i,P} + \sum_l A_{i,l} u'_{i,l} = -\frac{\delta P'}{\delta x_i},$$

which can be rewritten as

$$u'_{i,P} = -\frac{\sum_l A_{i,l} u'_{i,l}}{A_{i,P}} - \frac{1}{A_{i,P}} \frac{\delta P'}{\delta x_i}$$

$$\Rightarrow \qquad u'_{i,P} = \widetilde{u}'_{i,P} - \frac{1}{A_{i,P}} \frac{\delta P'}{\delta x_i}.$$

- The pressure equation can be obtained by combining

$$\frac{\delta u_i^m}{\delta x_i} = 0 \qquad \Rightarrow \qquad \frac{\delta u_i^{m*}}{\delta x_i} = -\frac{\delta u_i'}{\delta x_i}$$

and

$$u'_{i,P} = \widetilde{u}'_{i,P} - \frac{1}{A_{i,P}} \frac{\delta P'}{\delta x_i}.$$

to yield

$$\frac{\delta u_i^{m*}}{\delta x_i} = -\frac{\delta \widetilde{u}_i'}{\delta x_i} + \frac{\delta}{\delta x_i}\left[\frac{1}{A_{i,P}} \frac{\delta P'}{\delta x_i}\right].$$

or

$$\frac{\delta}{\delta x_i}\left[\frac{1}{A_{i,P}} \frac{\delta P'}{\delta x_i}\right] = \frac{\delta \widetilde{u}_i'}{\delta x_i} + \frac{\delta u_i^{m*}}{\delta x_i}$$

(Notice that u_i' is not yet known at this point).

- After P' is determined, u' can be obtained from

$$u'_{i,P} = \widetilde{u}'_{i,P} - \frac{1}{A_{i,P}} \frac{\delta P'}{\delta x_i}.$$

- And u and P can be updated:

$$u_i^m = u_i^{m*} + u_i'; \qquad P^m = P^{m-1} + P'$$

Setup

Given the velocity and pressure fields at time-step n:

- Compute the known source term Q_i containing the known velocity and pressure.
- Set $m = 0$, u_i^m and P^m equal to the velocity and pressure at the previous time-step n.
- Begin the inner iteration to find $u_i = u_i^{n+1}$ and $P = P^{n+1}$:

Inner iteration

- Update $m = m + 1$.
- Assemble the coefficient matrices $A_{i,P}$. Assemble and solve the system

$$A_{i,P} u_{i,P}^{m*} + \sum_l A_{i,l} u_{i,l}^{m*} = Q_i - \frac{\delta P^{m-1}}{\delta x_i}.$$

- Assemble and solve the pressure correction

$$\frac{\delta}{\delta x_i}\left[\frac{1}{A_{i,P}}\frac{\delta P'}{\delta x_i}\right] = \frac{\delta \widetilde{u}_i'}{\delta x_i} + \frac{\delta u_i^{m*}}{\delta x_i}$$

- Correct the velocity by

$$u_{i,P}^{m} = u_{i,P}^{m*} - \frac{1}{A_{i,P}}\frac{\delta P'}{\delta x_i}.$$

- Calculate the norm of the residual

$$||e|| = \left|\left| A_{i,P} u_{i,P} + \sum_l A_{i,l} u_{i,l} - Q_i + \frac{\delta P}{\delta x_i} \right|\right|.$$

- If $||e|| > \epsilon$, return to (a). Otherwise exit the inner iteration.

Methods for the pressure correction

- The pressure correction equation

$$\frac{\delta}{\delta x_i}\left[\frac{1}{A_{i,P}}\frac{\delta P'}{\delta x_i}\right] = \frac{\delta \widetilde{u}_i'}{\delta x_i} + \frac{\delta u_i^{m*}}{\delta x_i}$$

  contains an unknown term $\delta \widetilde{u}_i'/\delta x_i$.
- There are various ways to deal with this problem:
  1. SIMPLE
  2. SIMPLEC
  3. PISO

SIMPLE scheme

- Ignore it altogether and solve:

$$\frac{\delta}{\delta x_i}\left[\frac{1}{A_{i,P}}\frac{\delta P'}{\delta x_i}\right] = \frac{\delta u_i^{m*}}{\delta x_i}$$

- Calculate u' from

$$u'_{i,P} = \widetilde{u}'_{i,P} - \frac{1}{A_{i,P}}\frac{\delta P'}{\delta x_i}.$$

- Use under-relaxation to stabilize the system:

$$u_i^m = u_i^{m,*} + \alpha_i u'; \qquad P^m = P^{m-1} + \alpha_p P'.$$

$0 < \alpha < 1.$

- Slow convergence.

SIMPLEC scheme

- Express $u'_{i,P}$ in terms of neighboring values:

$$u'_{i,P} = \frac{\sum_l A_{i,l} u'_{i,l}}{\sum_l A_{i,l}}.$$

- Substitute into the definition of $\widetilde{u}'_i$:

$$\widetilde{u}'_{i,P} = -\frac{\sum_l A_{i,l} u'_{i,l}}{A_{i,P}} = -u'_{i,P} \frac{\sum_l A_{i,l}}{A_{i,P}}.$$

to give

$$u'_{i,P} = \widetilde{u}'_{i,P} - \frac{1}{A_{i,P}} \frac{\delta P'}{\delta x_i} = -u'_{i,P} \frac{\sum_l A_{i,l}}{A_{i,P}} - \frac{1}{A_{i,P}} \frac{\delta P'}{\delta x_i}$$

$$\left[\frac{A_{i,P} + \sum_l A_{i,l}}{A_{i,P}}\right] u'_{i,P} = -\frac{1}{A_{i,P}} \frac{\delta P'}{\delta x_i} \Rightarrow u'_{i,P} = -\left[\frac{1}{A_{i,P} + \sum_l A_{i,l}}\right] \frac{\delta P'}{\delta x_i}$$

- Take the divergence of the above to yield:

$$\frac{\delta}{\delta x_i}\left[\frac{1}{A_{i,P} + \sum_l A_{i,l}} \frac{\delta P'}{\delta x_i}\right] = -\frac{\delta u'_{i,P}}{\delta x_i}.$$

- But

$$\frac{\delta u'_{i,P}}{\delta x_i} + \frac{\delta u^{m*}_{i,P}}{\delta x_i} = \frac{\delta u^{m}_{i,P}}{\delta x_i} = 0 \qquad \Rightarrow \qquad -\frac{\delta u'_{i,P}}{\delta x_i} = \frac{\delta u^{m*}_{i,P}}{\delta x_i}$$

- And we obtain the new Poisson equation

$$\frac{\delta}{\delta x_i}\left[\frac{1}{A_{i,P}+\sum_l A_{i,l}}\frac{\delta P'}{\delta x_i}\right]=\frac{\delta u_{i,P}^{m*}}{\delta x_i}$$

- The only difference between this equation and the one solved in the SIMPLE method is the denominator of the pressure gradient term; however, this method has much faster convergence.

PISO scheme

- Calculate $u'_{i,P}$ as in the SIMPLE method: solve

$$\frac{\delta}{\delta x_i}\left[\frac{1}{A_{i,P}}\frac{\delta P'}{\delta x_i}\right] = \frac{\delta u_i^{m*}}{\delta x_i}$$

then obtain $u'_{i,P}$:

$$u'_{i,P} = -\frac{1}{A_{i,P}}\frac{\delta P'}{\delta x_i}.$$

(the term $\widetilde{u}'_{i,P}$ was ignored).

- Now solve the pressure equation again:

$$\frac{\delta}{\delta x_i}\left[\frac{1}{A_{i,P}}\frac{\delta P''}{\delta x_i}\right] = \frac{\delta \widetilde{u}'_i}{\delta x_i} + \frac{\delta u_i^{m*}}{\delta x_i}$$

(since $\widetilde{u}'_{i,P}$ can now be computed).

- Calculate a new velocity correction

$$u''_{i,P} = u'_{i,P} - \frac{1}{A_{i,P}}\frac{\delta P''}{\delta x_i}.$$

- get u_i^m:

$$u_i^m = u_i^{m,*} + u''_i; \qquad P^m = P^{m-1} + P''.$$