Dot product and the Application 2D Ball Collisions, 1999,2011

1 Axioms

bilinear $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

? $\vec{b} \cdot (c \cdot \vec{a}) = \vec{a} \cdot (c \cdot \vec{b})$

symmetric $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

positive $\vec{x}^2 > 0$ $(\vec{x} \neq \vec{0})$

Furthermore, the magnitude of vectors is defined using the dot product: $|\vec{x}|^2 = \vec{x}^2$ and, because $\vec{x}^2 > 0$: $|\vec{x}| = \sqrt{\vec{x}^2}$.

2 Geometric explanation

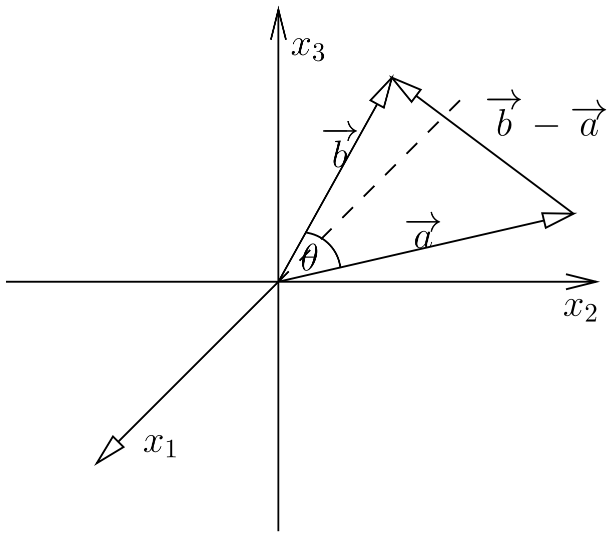

Figure 1: geometric explanation of the dot product

According to the law of cosine (figure 1):

$$|\vec{b} - \vec{a}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}||\vec{b}|\cos(\theta) \Leftrightarrow |\vec{a}|^2 + |\vec{b}|^2 - |\vec{b} - \vec{a}|^2 = 2|\vec{a}||\vec{b}|\cos(\theta) \Leftrightarrow$$
$$a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - a_1^2 + 2a_1b_1 - b_1^2 - a_2^2 + 2a_2b_2 - b_2^2 - a_3^2 + 2a_3b_3 - b_3^2 =$$
$$2|\vec{a}||\vec{b}|\cos(\theta) \Leftrightarrow a_1b_1 + a_2b_2 + a_3b_3 = |\vec{a}||\vec{b}|\cos(\theta)$$

3 2-Dimensional Ball Collision using Scalar Projection

I won't start out explaining 1-dimensional collisions explicitly, because 2D-collisions can be reduced to 1-dimensional collisions using scalar projections of the velocity-vectors[1].
$C_1(-10, 0)$ is the center (of mass) of ball 1, $C_2(11/17)$ that of ball 2. The pre-collision velocity of ball 1 is $\vec{v_{1_i}} = \begin{bmatrix}10 & 5\end{bmatrix}^T \mathrm{m} \cdot \mathrm{s}^{-1}$ and that of ball 2 is $\vec{v_{2_i}} = \begin{bmatrix}-15 & 0\end{bmatrix}^T \mathrm{m} \cdot \mathrm{s}^{-1}$. The mass of ball 1 is 2.2kg and ball 2 weighs 1.8kg.

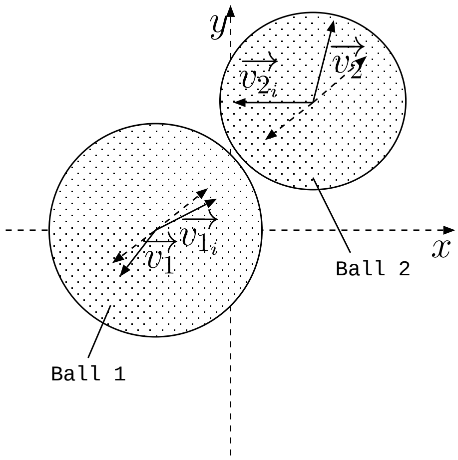

Figure 2: Example of an elastic 2-dimensional Ballcollision as an application of scalar projection

In order to find the momentum of each ball after the collision, you have to find the amount of each ball's velocity which acts towards the center of mass of the other ball. And here is where the dot product comes in handy: We are looking for the component of $\overrightarrow{v_{1_i}}$or $\overrightarrow{v_{2_i}}$that is parallel to the connection of the centers of mass ("line of sight"), given by $\overrightarrow{C_1C_2} = \begin{bmatrix}21 & 17\end{bmatrix}^T$ m. This is equivalent to the scalar projection of $\overrightarrow{v_{1_i}}$ and $\overrightarrow{v_{2_i}}$ on $\overrightarrow{C_1C_2}$.
The scalar projection of $\overrightarrow{a}$ onto $\overrightarrow{n}$ is defined as $\text{comp}_{\overrightarrow{n}}\overrightarrow{a}$ or $\overrightarrow{a} \cdot \overrightarrow{n}^0$. This can be understood easily by looking at the geometric definition of the dot product (section 2). Note that the "axis"-vector (the one that will be projected on) must be normalized. Thus we must find the unit-vector of $\overrightarrow{C_1C_2}$ ($\overrightarrow{C_1C_2}^0$) which is $\frac{los}{\sqrt{730}}$ (the length should be $r_1 + r_2$, though).
I define v_{1_i} as the part of the velocity of ball 1 acting *in line of sight* (**not** $|\overrightarrow{v_{1_i}}|$**!**):
$v_{1_i} = \text{comp}_{\overrightarrow{C_1C_2}^0}\overrightarrow{v_{1_i}} = \begin{bmatrix}10 & 5\end{bmatrix}^T \cdot \begin{bmatrix}21 & 17\end{bmatrix}^T \frac{1}{\sqrt{730}} = \frac{295}{\sqrt{730}} = 10.9184\text{m} \cdot \text{s}^{-1}$. Similarly, v_{2_i} is the velocity of ball 2 in line of sight: $v_{2_i} = \text{comp}_{\overrightarrow{C_1C_2}^0}\overrightarrow{v_{2_i}} = \begin{bmatrix}-15 & 0\end{bmatrix}^T \cdot \begin{bmatrix}21 & 17\end{bmatrix}^T \frac{1}{\sqrt{730}} = \frac{-315}{\sqrt{730}} = -11.6587\text{m} \cdot \text{s}^{-1}$ (the negative value is okay since this is a projection).
Having done this, we can now consider the collision as being one-dimensional if we choose the line of sight ($\overrightarrow{C_1C_2}$) to be the one axis (which we just did). This implies that we need to express the post-collision velocities in this 1D-space before we can translate the results back to 2-space. Thus, I define v_1 and v_2 the post-collision velocity- components in line of sight (and again: **not** $|\overrightarrow{v_1}|/|\overrightarrow{v_2}|$).
In the one-dimensional case, two conditions hold:

$$\begin{aligned} m_1 \cdot v_{1_i} + m_2 \cdot v_{2_i} = m_1 \cdot v_1 + m_2 \cdot v_2 \Leftrightarrow \\ m_1 \cdot (v_{1_i} - v_1) = m_2 \cdot (v_2 - v_{2_i}) \quad \text{(conservation of linear momentum)} \end{aligned} \tag{1}$$

$$\begin{aligned} \frac{1}{2} \cdot m_1 \cdot v_{1_i}^2 + \frac{1}{2} \cdot m_2 \cdot v_{2_i}^2 = \frac{1}{2} \cdot m_1 \cdot v_1^2 + \frac{1}{2} \cdot m_2 \cdot v_2^2 \Leftrightarrow \\ m_1 \cdot \left(v_{1_i}^2 - v_1^2\right) = m_2 \cdot \left(v_2^2 - v_{2_i}^2\right) \quad \text{(conservation of kinetic energy)} \end{aligned} \tag{2}$$

(1) and (2) yield a linear system with two equations and two unknowns (v_1 and v_2):

$$\begin{aligned} m_1 = m_2 \frac{v_2 - v_{2_i}}{v_{1_i} - v_1} \wedge m_1 = m_2 \frac{v_2^2 - v_{2_i}^2}{v_{1_i}^2 - v_1^2} \Leftrightarrow \frac{v_2 - v_{2_i}}{v_{1_i} - v_1} = \frac{(v_2 - v_{2_i}) \cdot (v_2 + v_{2_i})}{(v_{1_i} - v_1) \cdot (v_{1_i} + v_1)} \Leftrightarrow \\ v_{1_i} + v_1 = v_{2_i} + v_2 \Leftrightarrow v_1(v_2) = v_2 + v_{2_i} - v_{1_i} \wedge v_2(v_1) = v_1 + v_{1_i} - v_{2_i} \end{aligned}$$

Note that I assumed that $v_{1_i} - v_1 \neq 0$, $v_{1_i}^2 - v_1^2 = (v_{1_i} - v_1) \cdot (v_{1_i} + v_1) \neq 0$ and $v_2 - v_{2_i} \neq 0$ (which is okay, because no collision would take place if the pre- and post-collision velocities were identical) and $m_2 \neq 0$.

Plug $v_1(v_2)$ into (1) and solve for v_2:

$$m_1 \cdot (v_2 + v_{2_i} - 2v_{1_i}) = m_2 \cdot (v_{2_i} - v_2) \Leftrightarrow m_1 v_2 + m_1 v_{2_i} - 2m_1 v_{1_i} = m_2 v_{2_i} - m_2 v_2 \Leftrightarrow$$
$$v_2 \cdot (m_1 + m_2) = v_{2_i} \cdot (m_2 - m_1) + 2m_1 v_{1_i}$$

Repeating the same procedure for v_1, you get:

$$v_1(v_{1_i}, v_{2_i}, m_1, m_2) = v_{1_i}\frac{m_1 - m_2}{m_1 + m_2} + v_{2_i}\frac{2m_2}{m_1 + m_2} \quad (3)$$
$$v_2(v_{1_i}, v_{2_i}, m_1, m_2) = v_{2_i}\frac{m_2 - m_1}{m_1 + m_2} + v_{1_i}\frac{2m_1}{m_1 + m_2} \quad (4)$$

At this point, a special case becomes obvious: What if both balls have equal mass (i.e pool, if you neglect non-linear momentum and friction) ? In that case, $v_1 = v_{2_i}$ and $v_2 = v_{1_i}$.
For our example, plugging into (3) results in:
$v_1 = \frac{295}{\sqrt{730}}\mathrm{m \cdot s^{-1}} \cdot \frac{1}{10} - \frac{315}{\sqrt{730}}\mathrm{m \cdot s^{-1}} \cdot \frac{9}{10} = \frac{-254}{\sqrt{730}}\mathrm{m \cdot s^{-1}} = -9.40096\mathrm{m \cdot s^{-1}}$ and from (4):
$v_2 = -\frac{315}{\sqrt{730}}\mathrm{m \cdot s^{-1}} \cdot -\frac{1}{10} + \frac{295}{\sqrt{730}}\mathrm{m \cdot s^{-1}} \cdot 1.1 = \frac{356}{\sqrt{730}}\mathrm{m \cdot s^{-1}} = 13.1762\mathrm{m \cdot s^{-1}}$ (again, negative signs are valid, because v_1 and v_2 only represent the "intensity" of each ball's velocity in line of sight).
In a 1D-collision, I'd be done now: v_1 would be the resulting velocity of ball 1 and v_2 that of ball 2. But the 1D-case is a special case in this respect (see below); for the two- or three-dimensional case, you have to find the difference between pre- and post-collision velocity in 1D ("change in line of sight -velocity"), project that onto $\overrightarrow{C_1C_2}$ to get the 2-dimensional change in velocity, and add to the pre-collision velocity to get the final, post-collision velocity (in the 1D-case, this step is unnecessary because $v_{1_i} + \triangle v_1 = v_{1_i} + v_1 - v_{1_i} = v_1$).
$\overrightarrow{C_1C_2}^0 = \begin{bmatrix} 21 \cdot 730^{-0.5} \\ 17 \cdot 730^{-0.5} \end{bmatrix} = \begin{bmatrix} 0.7772 & 0.6292 \end{bmatrix}^T$. Consequently, the change in line of sight-velocity of ball 1 projected on $\overrightarrow{C_1C_2}$ is: $\overrightarrow{\triangle v_1} = \overrightarrow{C_1C_2}^0 \cdot (v_1 - v_{1_i}) = \begin{bmatrix} -15.79221 & -12.78494 \end{bmatrix}^T$, and therefore the post-collision velocity of ball 1 is: $\overrightarrow{v_1} = \overrightarrow{v_{1_i}} + \overrightarrow{\triangle v_1} = \begin{bmatrix} -5.79221 & -7.78494 \end{bmatrix}^T \mathrm{m \cdot s^{-1}}$. The post-collision velocity of ball 2 can be calculated in the same way: $\overrightarrow{\triangle v_2} = \overrightarrow{C_1C_2}^0 \cdot (v_2 - v_{2_i}) = \begin{bmatrix} 19.30168 & 15.62612 \end{bmatrix}^T$ und $\overrightarrow{v_2} = \overrightarrow{v_{2_i}} + \overrightarrow{\triangle v_2} = \begin{bmatrix} 4.30168 & 15.62612 \end{bmatrix}^T \mathrm{m \cdot s^{-1}}$.

References

[1] http://www.mcasco.com/p1lmc.html