2.1 Linear model

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- $y = \beta_0 x_1^{\beta}$ can be linearized by taking logs:

$$\log y = \log(\beta_0 x_1^{\beta}) \Leftrightarrow \log y = \log \beta_0 + \log x_1^{\beta}$$

$$\Leftrightarrow \log y = \log \beta_0 + \beta \log x_1$$

$$\Leftrightarrow y' = \beta_0' + \beta x_1'$$

- $y = \beta_0 x_1^{\beta} \varepsilon$:

$$\log y = \log \beta_0 + \beta \log x_1 + \log \varepsilon \Leftrightarrow y' = \beta_0' + \beta x_1' + \varepsilon'$$

2.2 MATRIX REPRESENTATION

Data:

y_1	x_{11}	x_{12}	x_{13}
y_2	x_{21}	x_{22}	x_{23}
$\ldots$		$\ldots$	
y_n	x_{n1}	x_{n2}	x_{n3}

$$E(\varepsilon) = 0$$

Equation: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_n x_n + \varepsilon$

Matrix equation:

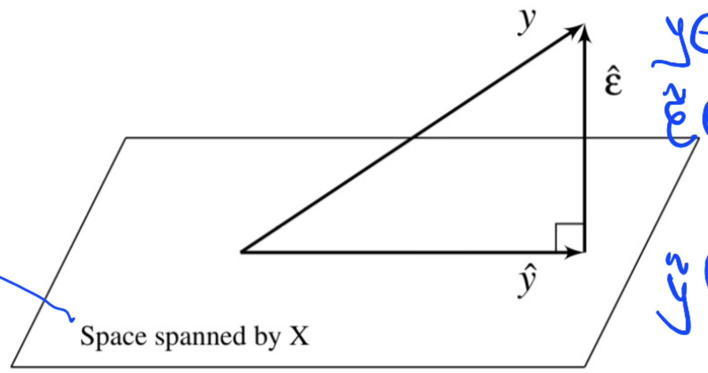
$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}_{n \times 1} = \underbrace{\begin{bmatrix} 1 & x_{11} & x_{12} & \cdots & x_{1p} \\ 1 & x_{21} & x_{22} & \cdots & x_{2p} \\ & & \vdots & & \\ 1 & x_{n1} & x_{n2} & \cdots & x_{np} \end{bmatrix}}_{n \times (p+1)} \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_p \end{bmatrix}}_{(p+1) \times 1} + \underbrace{\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}}_{n \times 1}$$

$$Y = X\beta + \varepsilon$$

Null model: $y = \mu + \varepsilon$ $\quad E(\varepsilon) = 0$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \mu + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_n \end{bmatrix}$$

2.3 Estimating β



$\mathbb{R}^K$, with $K \le p$

$\underset{\sim}{y} \in \mathbb{R}^n$

$\underset{\sim}{\hat\varepsilon} \in \mathbb{R}^n$

$\underset{\sim}{\hat y} \in \mathbb{R}^n$

Figure 2.1 Geometrical representation of the estimation β. The data vector Y is projected orthogonally onto the model space spanned by X. The fit is represented by projection $\hat{y} = X\hat{\beta}$ with the difference between the fit and the data represented by the residual vector $\hat{\varepsilon}$.

$$\underset{\sim}{\hat y} = \beta_0 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} + \beta_1 \begin{bmatrix} x_{11} \\ x_{21} \\ \vdots \\ x_{n1} \end{bmatrix} + \cdots + \beta_p \begin{bmatrix} x_{1p} \\ x_{2p} \\ \vdots \\ x_{np} \end{bmatrix}$$

These p+1 vectors are in $\mathbb{R}^n$, but it may happen that they do not span $\mathbb{R}^n$. Actually, if $p+1 < n$, they span an at most $(p+1)$-dimensional space.

A small example: $n=3$, $p=1$ (simple regression)

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \qquad Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$\boxed{\hat{Y} = \beta_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \beta_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}$$

$\hat{Y}$ lives in a plane with equations

$$\begin{cases} x = r + s \\ y = r + 2s \\ z = r + 3s \end{cases}$$

where parameters r, s represent all possible values of β_0, β_1, resp.

$$\beta_1 = \frac{\sum (x_i - \bar{x}) y_i}{\sum (x_i - \bar{x})^2}$$

$$\bar{x} = 2$$

$$\sum (x_i - \bar{x})^2 = (-1)^2 + 0^2 + 1^2 = 2$$

$$\beta_1 = \frac{-y_1 + y_3}{2}$$

Try $y_3 = 8$, $y_1 = 4$

Then $\beta_1 = 2$

$$\beta_0 = \bar{y} - \beta_1 \bar{x} = \bar{y} - 2.2 = \bar{y} - 4$$

Try $\bar{y} = 5$. Then $y_2 = 3$. So $Y = \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix}$

$$\hat{Y} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

$$\begin{array}{ccccc} & \tilde{Y} & & Y & \\ \hat{\varepsilon} = & \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix} & - & \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix} & = \boxed{\begin{bmatrix} -1 \\ +2 \\ -1 \end{bmatrix}} \end{array}$$

$RSS = 6$ $\qquad$ $RSE = \sqrt{6}$

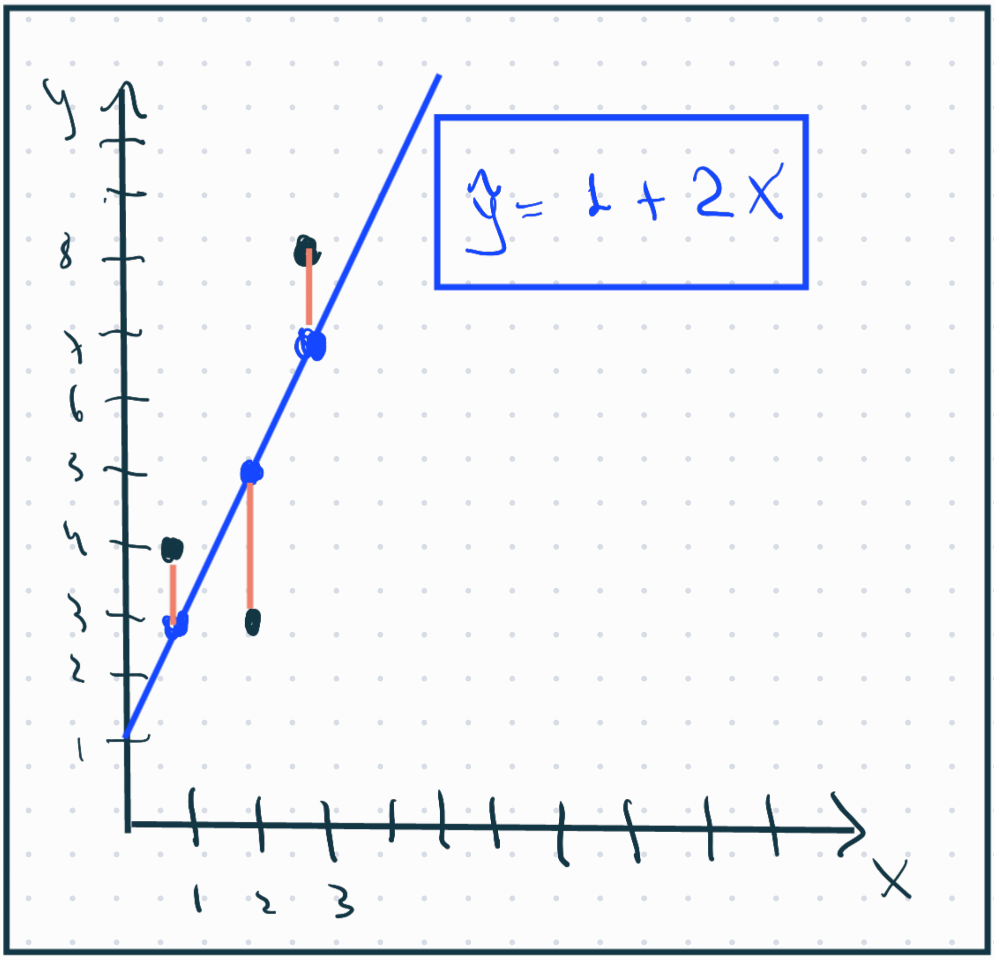
$$\beta_1 = 2 = r \frac{\sigma_Y}{\sigma_X} \quad \therefore \quad r = \frac{2\sigma_X}{\sigma_Y}$$

$$\sigma_X^2 = 1$$

$$\sigma_Y^2 = \frac{(-1)^2 + (-2)^2 + 3^2}{2} = 7$$

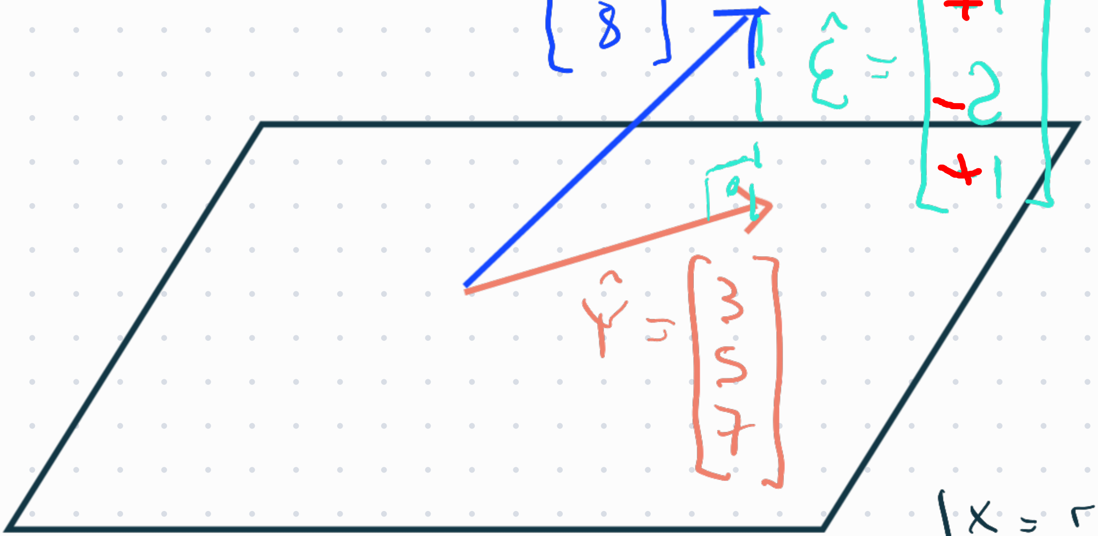
$$r = 2/\sqrt{7}$$

$$r^2 = 4/7 \simeq 0{,}57$$



$$Y = \begin{bmatrix} 4 \\ 3 \\ 8 \end{bmatrix}$$

$$\hat{\varepsilon} = \begin{bmatrix} +1 \\ -2 \\ +1 \end{bmatrix}$$



$$\hat{Y} = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$$

$$\Pi : \begin{cases} x = r + s \\ y = r + 2s \\ z = r + 3s \end{cases}$$