

# Skew symmetry proof

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## 1 Proof

**Lemma 1.** *Let  $A_t \in \mathbb{R}^{d \times d}$  be skew symmetric,  $x : \mathbb{R} \rightarrow \mathbb{R}^d$  and  $x(0) = x_0 \in \mathbb{R}^d$ . Suppose  $B \in \mathbb{R}^{d \times d}$  is a constant matrix. Consider the dynamics*

$$\dot{x} = A_t \cdot x$$

$$\dot{A}_t = B \cdot A_t$$

*and suppose that  $A_t$  does not exhibit exponential dynamics. Then both  $B$  and  $B \cdot A_t$  must be skew symmetric matrices.*

*Proof.* As  $A$  is skew symmetric,

$$\dot{A}_t + \dot{A}_t^T = \dot{0} \tag{1}$$

$$= 0 \tag{2}$$

$$= B \cdot A_t + (B \cdot A_t)^T \tag{3}$$

Hence,  $B \cdot A_t$  is skew symmetric. Furthermore, we can write

$$A_t = (a_1, \dots, a_d) \tag{4}$$

Where  $\forall i : a_i \in \mathbb{R}^d$ . Then

$$\dot{A}_t = (\dot{a}_1, \dots, \dot{a}_d) \tag{5}$$

$$= B \cdot (a_1, \dots, a_d) \tag{6}$$

$$= (B \cdot a_1, \dots, B \cdot a_d) \tag{7}$$

is just a vector where each entry corresponds to a differential equation of the form

$$\dot{a}_i = B \cdot a_i \tag{8}$$

So if  $A$  does not exhibit exponential dynamics,  $B$  must be skew symmetric.  $\square$