

Skew symmetry proof

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1 Proof

Lemma 1. Let $A_t \in \mathbb{R}^{d \times d}$ be skew symmetric, $x : \mathbb{R} \rightarrow \mathbb{R}^d$ and $x(0) = x_0 \in \mathbb{R}^d$. Suppose $B \in \mathbb{R}^{d \times d}$ is a constant matrix. Consider the dynamics

$$\dot{x} = A_t \cdot x$$

$$\dot{A}_t = B \cdot A_t$$

and suppose that A_t does not exhibit exponential dynamics. Then both B and $B \cdot A_t$ must be skew symmetric matrices.

Proof. As A is skew symmetric,

$$\dot{A}_t + \dot{A}_t^T = \dot{0} \tag{1}$$

$$= 0 \tag{2}$$

$$= B \cdot A_t + (B \cdot A_t)^T \tag{3}$$

Hence, $B \cdot A_t$ is skew symmetric. Furthermore, we can write

$$A_t = (a_1, \dots, a_d) \tag{4}$$

Where $\forall i : a_i \in \mathbb{R}^d$. Then

$$\dot{A}_t = (\dot{a}_1, \dots, \dot{a}_d) \tag{5}$$

$$= B \cdot (a_1, \dots, a_d) \tag{6}$$

$$= (B \cdot a_1, \dots, B \cdot a_d) \tag{7}$$

is just a vector where each entry corresponds to a differential equation of the form

$$\dot{a}_i = B \cdot a_i \tag{8}$$

So if A does not exhibit exponential dynamics, B must be skew symmetric. \square