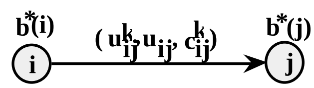
CHAPTER 17

EXERCISE 17.1. Construct the network G = ({s} N_1 N_2, A), in which each node i in N_1 represents a supply site, each node j in N_2 represents a demand site (node j has demand d_j) and each arc (i, j) in A represents a route from i N_1 to j N_2 with a cost of c^k_{ij} (for commodity k.) The arc (i, j) is uncapacitated both in terms of individual commodity flow values as well as bundle capacity flow values. The source node s is connected to every other node i in network by a directed arc (s, i). If i N_1, then the individual commodity flow bound capacity of arc (s, i) is a^k_i units, the bundle capacity is u_i units, and the arc has a cost of zero. If i N_2, then the arc (s, i) is uncapacitated but has a cost of α^k_i. An optimal multicommodity flow in this network will define an optimal schedule of deployment of resources. Figure S17.1 illustrates the formulation for this exercise.



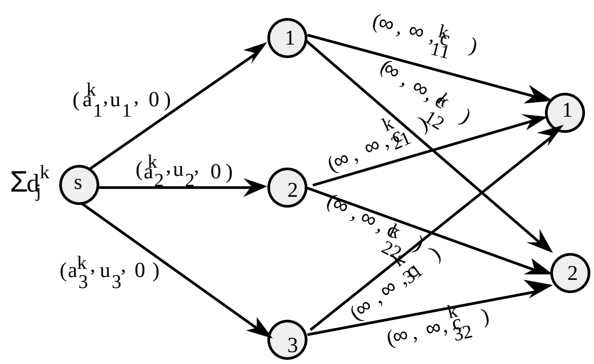


Figure S17.1

EXERCISE 17.3. Use the optimal primal and dual solution given at the end of solution 17.3 of the text in order to verify that the complementary slackness conditions in the arc flow formulation (17.2) are indeed satisfied. The optimal primal variables for the path flow formulation are already given the text. $\mathbf{P}^1 = \{P_1, P_2\}$ and $\mathbf{P}^2 = \{P_3, P_4\}$ where $P_1 = s^1\text{-}t^1$; $P_2 = s^1\text{-}1\text{-}2\text{-}t^1$; $P_3 = s^2\text{-}t^2$; $P_4 = s^2\text{-}1\text{-}2\text{-}t^2$ and $f(P_1) = 5$, $f(P_2) = 5$, $f(P_3) = 5$, $f(P_4) = 15$). The optimal dual variables are $\omega = \{\omega_{s^1t^1}, \omega_{s^11}, \omega_{12}, \omega_{2t^1}, \omega_{2t^2}, \omega_{s^21}, \omega_{s^2t^2}\} = \{12, 0, 2, 0, 0, 0, 0\}$ and $\sigma = \{\sigma^1, \sigma^2\} = \{13, 5\}$. It is easy to verify that this pair of primal and dual solutions satisfies the complementary slackness optimality conditions.

EXERCISE 17.5. Associate the dual variable ω_{ij} with each of the bundle constraints in 17.5(b) and the dual variables σ^k with each of the demand constraints in 17.5(c). Then the dual of the given multicommodity flow problem is:

$$\text{Maximize} \quad \sum_{(i,\ j)\ \ A} u_{ij}\ \omega_{ij} \quad + \sum_{K=1}^{K} d^k\ \sigma^k$$

subject to

$$c^k(P) + \Sigma_{(i,\ j)\ \ P}\ \omega_{ij}\ -\ \sigma^k \geq 0 \text{ for all } k = 1, 2, ..., K \text{ and all } P \quad P^k$$

$$\omega_{ij} \geq 0.$$

It is easy to verified that the complementary slackness conditions for this primal-dual set are the same as those given in 17.6.

EXERCISE 17.7. The following table specifies the results of the first five steps of the Lagrangian relaxation algorithm. 1 refers to source-sink pair 1-->4, 2 refers to source-sink pair 5-->8, 3 refers to source-sink pair 9-->12 and 4 refers to source-sink pair 13-->16.

k	μ values	Shortest paths	Shortest path costs	Lower bound	θ_k
0	All zero	1) 1-5-9-13-14-15-16-12-8-4	1) 30	600	1
		2) 5-9-13-14-15-16-12-8	2) 20		
		3) 9-13-14-15-16-12	3) 10		
		4) 13-14-15-16	4) 0		
1	$\mu_{23} = 0$	1) 1-5-9-10-11-12-8-4	1) 35	625	1
	$\mu_{67} = 0$	2) 5-9-10-11-12-8	2)25		
	$\mu_{10,11} = 0$	3) 9-10-11-12	3) 15		
	$\mu_{13,16} = 25$	4) 13-14-15-16	4) 25		
2	$\mu_{23} = 0$	1) 1-5-6-7-8-4	1) 40	650	0.5
	$\mu_{67} = 0$	2) 5-6-7-8	2) 30		
	$\mu_{10,11} = 15$	3) 9-10-11-12	3) 25		
	$\mu_{13,16} = 15$	4) 13-14-15-16	4) 15		
3	$\mu_{23} = 0$	1) 1-5-9-13-14-15-16-12-8-4	1) 40	662.5	1/3
	$\mu_{67} = 2.5$	2) 5-9-13-14-15-16-12-8	2) 30		
	$\mu_{10,11} = 10$	3) 9-13-14-15-16-12	3) 20		
	$\mu_{13,16} = 10$	4) 13-14-15-16	4) 10		
4	$\mu_{23} = 0$	1)* 1-5-9-10-11-12-8-4	1) 40	733.333	0.25
	$\mu_{67} = 0$	2)* 5-9-10-11-12-8	2) 30		
	$\mu_{10,11} = 5$	3) 9-10-11-12	3) 20		
	$\mu_{13,16} = 18.33$	4) 13-14-15-16	4) 18.33		

5	$\mu_{23} = 0$	1) 1-5-6-7-8-4	1) 40	720.83	0.2
	$\mu_{67} = 0$	2) 5-6-7-8	2) 30		
	$\mu_{10,11} = 8.75$	3) 9-10-11-12	3) 23.75		
	$\mu_{13,16} = 17.08$	4) 13-14-15-16	4) 17.08.		

* Note that at this point there is more than one shortest path. The path 1) 1-5-6-7-8-4 also has a cost 40. The path 2) 5-6-7-8 also has a cost 30. The results of iteration 5 depends on which shortest path we choose to send the flow on.

EXERCISE 17.9. The following table shows the optimal basis.

Commodity 1					Commodity 2					
(s^1,t^1)	$(s^1,1)$	(1,2)	$(2,t^1)$	$(2,t^2)$	(s^2,t^2)	$(s^2,1)$	(1,2)	$(2,t^2)$	$(2,t^1)$	
1	5	1	5	1	5	1	1	1	5	Costs
		1					1			(1,2)
1										(s^1,t^1)
1	1									s^1
-1			-1							t^1
										s^2
				-1						t^2
	-1	1								1
		-1	1	1						2
					1	1				s^2
					-1			-1		t^2
										s^1
									-1	t^1
						-1	1			1
							-1	1	1	2

Suppose we use the arcs $(s^1,1)$, (1,2) $(2,t^1)$, and $(2,t^2)$ to form a spanning tree basis for commodity 1 and $(s^2, 1)$, (1,2), $(2,t^2)$, and $(2,t^1)$ to form a spanning tree basis for commodity 2. To form the modified basis B' , we subtract columns 2,3, and 4 from column 1 and columns 7, 8, and 9 from column 6 in the basis. The cost coefficients for $x^1_{s^1t^1}$ becomes 1-5-1-5 = -10 and for $x^2_{s^2t^2}$ becomes 5-1-1-1 = 2.

The working basis W, which is the coefficients of $x^1_{s^1t^1}$ and $x^2_{s^2t^2}$ in the first two rows becomes

$$W = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

To find the simplex multipliers for the two bundle constraints, we solve the system

$$\pi^\circ W = c_W = (-10, 2)$$

and so $\pi^\circ_1 = 12$ and $\pi^\circ_2 = 2$. We then use these values we did at the end of Section 17.3 (with $\pi^\circ = w$) to find the simplex multipliers for the remaining constraints (the values $d^k(j)$ given at the end of that section.)

EXERCISE 17.11. Suppose the cost function for any arc (i, j) is piecewise linear, having p linear segments in the intervals $[u^0_{ij}, u^1_{ij}]$, $[u^1_{ij}, u^2_{ij}]$ ... $[u^{p-1}_{ij}, u^p_{ij}]$ with respective cost slopes $c^1_{ij} \le c^2_{ij} \ldots \le c^p_{ij}$. We replace the arc (i,j) with the parallel arcs $(i, j)_1, (i, j)_2 \ldots (i, j)_p$: the cost (for any commodity k) and the bundle capacity of arc $(i, j)_r$ are c^r_{ij} and $u^r_{ij} - u^{r-1}_{ij}$ respectively. An optimal multicommodity flow in the transformed network will yield an optimal flow in the original problem.

EXERCISE 17.13. This problem can be formulated as:

$$\text{Minimize } \sum_{1\le k\le K} \Big(\sum_{W \quad Wk} c^k (W) f(W) + \sum_{P \quad Pk} c^k (P) f(P)\Big)$$

subject to

$$\sum_{1\le k\le K} \Big(\sum_{W \quad Wk} \delta_{ij} (W) f(W) + \sum_{P \quad Pk} \delta_{ij} (P) f(P)\Big) \le u_{ij}$$

$$\sum_{P \quad Pk} f(P) = d^k$$

$$f(P), f(W) \ge 0.$$

We determine the arc prices ω_{ij} and the commodity variables σ^k by solving the following system:

$\sum_{(i,\, j) \quad P} (c^k_{ij} + \omega_{ij}) = \sigma^k$ for every path P in the basis.

$\sum_{(i,\, j) \quad W} (c^k_{ij} + \omega_{ij}) = 0$ for every cycle W in the basis.

The optimality conditions are $c^{\alpha,\omega}_P \ge 0$ for every path P P^k and $c^{\omega}_W \ge 0$ for every cycle W W^k. We now apply the revised simplex method. In order to determine the entering variable, we solve a shortest path problem for every source-sink pair $[s^k, t^k]$ (as discussed in text) and a negative cycle detection algorithm for each commodity k (as discussed in the solution of the previous exercise.)

EXERCISE 17.15. (a) First find a maximum flow x^1 from s to t^1. Then starting with $G(x^1)$, find a maximum flow x^2 from s to t^2 while keeping the net flow into t^1 constant. We claim that x^2 is optimal for the single commodity problem. If this claim is true, we can turn x^2 into a multicommodity flow using flow decomposition. x^2 decomposes into flows from s to t^1, s to t^2 and flows around cycles. If we eliminating flows around cycles, the remaining flows are the solution to the multicommodity flow problem.

We now claim that $G(x^2)$ contains no augmenting paths from s to t^1. Otherwise, some augmentation leads to a larger flow from s to t^1, contradicting the optimality of x^1. In addition, $G(x^2)$ contains no augmenting paths into t^2 except those that decrease the flow into node t^1, by assumption.

Now consider the network obtained by adding arcs (t^1,s) and (t^2,s) with costs -3 and -2 respectively. A minimum cost circulation for this problem is equivalent to an optimal flow for the original problem. Let x'^2 be obtained from x^2 by sending an appropriate amount of flow in arcs (t^1,s) and (t^2,s) to make it a circulation. We claim that x'^2 has no negative cost circuits, and is thus optimal. Note that any negative cost circuit must contain arc (t^1,s) or arc (t^2,s). There is no 0 cost path in $G(x'^2)$ from s to t^1 or from s to t^2. This still leaves the case of a path using arcs (t^1,s) and (s,t^2). But this would imply a path from t^2 to t^1 which would, in turn, imply that the flow into node t^1 was not maximum, a contradiction.

(b) First, solve the maximum flow problem to send flow from s to t^1. Let x^1 denote the maximum flow. In general, solve the maximum flow problem from s to t_j starting with the residual network $G(x^{j-1})$ for j = 2 to K.

We first claim that the flow is optimal for the single commodity flow problem. If this claim is true, then we can turn it into an optimal flow for the multicommodity flow problem using flow decomposition.

The proof that it is optimal for the single commodity flow problem follows the form of the proof of part (a). In general, $G(x_j)$ contains no path from node t_j to node t_i for $i < j$. And, it contains no path from node s to node t_j for $j \le i$.

To establish optimality, we appends arcs (t_j,s) for j = 1 to K with cost $-c_j$, and then show that there can be no negative cost cycle.

EXERCISE 17.17. For each source node i, let b(i) denote $\sum_j d_{ij}$. For each sink node j, let $b(j) = - \sum_i d_{ij}$. Let $B = \sum_{i,j} d_{ij}$. First, solve a single commodity flow problem from the source nodes to node v, where b(v) = -B, and the supply of node i is b(i). Let us call this flow x'. Next solve a single commodity flow problem from node v to the sink nodes, where b(v) = B, and the demand of node is -b(j). Let us call the optimal flow x^*. (If neither of these two problems has a feasible solution, then the original problem is also infeasible.)

Any optimal solution to the original single commodity flow problem can be decomposed into sums of flows from source nodes to v and sums of flows from v to sink nodes plus flows around cycles. Thus the optimal solution to the single commodity flow problem is the solution $x' + x^*$.

Next decompose x' into flows from a source node into node v plus flows around cycles. Eliminate all flows around cycles, and label each path flow into v by its source node.

Next decompose x^* into flows from v into a sink node, plus flows around cycles. Eliminate the flows around cycles, and label each path flow from v by its sink node.

Now carry out the following procedure to determine flows from source nodes to sink nodes.

begin

while there is any remaining path in the flow decomposition do

    begin

        select a source node i and a sink node j whose current flow is less than d_{ij}

        select a path P_i from node i to node v and a path P_j from node v to node j

        send $\delta = \min(\text{Capacity}(P_i), \text{Capacity}(P_j), d_{ij})$ units of flow on the two paths

    decrease the capacities of P_i and P_j by δ and decrease d_{ij} by δ

end

end

At the end, the sum of the costs of all of the multicommodity flows is the same as the costs of $x' + x^*$, and thus the flow is optimal. We know that there is a feasible solution since for each source node i, the supply of b(i) at each source is enough to satisfy all of the demands d_{ij}.

EXERCISE 17.19. Let x^{-k} be the subvector corresponding to the columns of **N** of $\mathbf{N}x^k = b^k$ in the submatrix $\mathbf{M}^k$. As indicated in the hint, since **B** is a basis, we can solve $\mathbf{M}^k x^{-k} = b^k$ for any vector b^k. Let A^1 be the arcs in the underlying graph corresponding to the arcs of (columns in) $\mathbf{M}^k$. If the graph G' corresponding to the arcs A' is disconnected we can set $b^k_i = +1$ and $b^k_j = -1$ for nodes i and j in different components and $b^k_q = 0$ otherwise. But then the system $\mathbf{M}^k x^{-k} = b^k$ has no solution since any solution can send flow only in the arcs A' and we cannot send a unit flow from i to j using these arcs. This same argument implies that the graph G' must contain every node of the underlying graph G. But since G' contains all nodes and is connected, it must contain a spanning tree of G.

EXERCISE 17.21. (a) Every unit of flow between nodes i and j in the undirected network must flow on the paths

i - i' - j' - j or j - i' - j'- i in the directed figure. The arc (i', j') implies that this flow cannot exceed u_{ij}; moreover, each unit incurs a cost of c_{ij}. Note that since $c_{ij} \geq 0$, we can eliminate the flow on any directed cycle in the directed figure.

(b) We use the indicated transformation on each capacitated arc, replacing each capacitated arc with two directed parallel arcs (i,j) and (j,i). For a network with |A| arcs and $|A^1|$ capacitated arcs, the transformed network will have $2(|A| - |A^1|) + 4\,|A^1| = 2\,|A| + 2\,|A^1|$ arcs and $n + 2\,|A^1|$ nodes.

EXERCISE 17.23. (a) In any given feasible solution x^k_{ij}, let v^k denote the flow from s^k to t^k and $v = \Sigma_{1\leq k\leq K} v^k$ be the total flow. These variables satisfy the mass balance equations

$$\Sigma_j\ x^k_{ij} + \Sigma_j\ x^k_{ji} = \begin{cases} v^k \text{ if } i = s^k \\ -v^k \text{ if } i = t^k \\ 0 \text{ otherwise.} \end{cases} \quad \text{for all } k = 1, 2, ..., K$$

Adding these equations over all nodes i   S for any set S containing s^k but not t^k, shows that

$$v^k = \Sigma_{i\quad S}\ \Sigma_{j\quad N-S}\ x^k_{ij}.$$

Summing over all k for the directed case shows that

$$v = \Sigma_{1\leq k\leq K}\, v^k = \Sigma_{i\quad S}\ \Sigma_{j\quad N-S}\ \Sigma_{1\leq k\leq K}\ x^k_{ij} \leq \Sigma_{i\quad S}\ \Sigma_{j\quad N-S}\ u^k_{ij}$$

and for the undirected case shows that

$$v = \Sigma_{1\le k\le K} v^k = \Sigma_{i \in S} \Sigma_{j \in N-S} \Sigma_{1\le k\le K} x^k_{ij} \le \Sigma_{i \in S} \Sigma_{j \in N-S} \Sigma_{1\le k\le K} |x^k_{ij}| \le \Sigma_{i \in S} \Sigma_{j \in N-S} u^k_{ij}.$$

In both cases, these inequalities imply that the maximum flow is always less than or equal to the minimum cut.

(b) In Figure 17.15 since the capacity of the arc (1,2) is one unit and the unique path from both sources to their destination nodes passes through this arc, the maximum flow is 1 unit. Any cut separating both sources from both destinations must contain two arcs, though, and so has a capacity of 2 units. Therefore, the maximum flow is less than the minimum cut.

(c) Suppose that we convert any undirected problem into a directed model using the transformation given in Exercise 17.21. As we saw in that exercise, the two problems are equivalent so the maximum flow in both of them is the same. Consider any finite capacity source-sink cut [S, N - S] in the directed problem. Using the notation in Figure 17.14, if $i \in S$ then $i' \in S$ and if $j \in N - S$ then $j' \in N - S$; otherwise the cut has an infinite capacity. But then, either both i and j belong to S or to N - S, or arc (i', j') is in the cut, contributing a capacity of u_{ij}. Therefore, every finite capacity cut in the directed formulation corresponds to a cut, with the same capacity, in the undirected formulation. When applied to the directed version example in Figure 17.15, this result implies that the maximum flow is 1 unit, but the minimum capacity is 2 units.

Exercise 17.25. For each commodity k = 1, 2, ... , K, we add a return arc (t^k, s^k), with an infinite capacity, to the maximum flow network. If we then set each $b^k = 0$, and the arc costs to be $c_{t^k s^k} = -1$ for all k = 1, 2, ... , K and $c_{ij} = 0$ otherwise, then a minimum cost flow in the transformed network, which is in the form of the model (17.1), is a maximum flow.

EXERCISE 17.27#. This theorem is incorrectly stated. DEMAND(S) is the sum of the demands of the nodes whose origin nodes are in S and whose destination nodes are in N-S. Let θ' = CAP(S)/DEMANDS(S). Then the total flow in an optimum concurrent flow x^* from source nodes in S to destination nodes in N-S is at most θ' DEMANDS(S). But then for at least one origin node s^k in S, the flow in x^* from s^k to t^k is at most θ' D_k, and the result follows.

EXERCISE 17.29#. Minimizing the amount of time it takes to send flows for each commodity is equivalent to maximizing the proportion of flow sent per unit of time. This can be approximated by the concurrent flow problem in which there is a bundle constraint u_{ij} on the flow in arc (i,j) and the demand for flow from i to j is d_{ij}. Time can be essentially ignored in a computer network as the time to send one unit of flow from one node to another node is negligible compared to the time it takes to send all of the flow.

EXERCISE 17.31. We make the transformation indicated in Exercise 17.30 with node i taking the place of node 1. In the new network the bundle constraint becomes

$$x_{1^*,j1} + x_{1^*,j2} + \dots + x_{1^*,jK} + s = u_{ij}.$$

In this expression, s is a slack variable for the single bundle constraint. If we now subtract this equation from the mass balance constraint for node 1^*, then each variable in the modified system will appear in exactly two equations, once with a coefficient of +1 and once with a coefficient of -1. Therefore, the system is a network flow problem.

(Note that this transformation corresponds to adding a new node j^* in the network, connecting node 1^* to node j^* with an arc with capacity u_{ij}, and replacing every arc $(1^*, j^*)$ by the arc $(1^*, j^k)$.)

EXERCISE 17.33. (a) Let $f_a(\theta)$ be the excess in arc a if θ units of flow are sent around the cycle W. Let c_a be the unit cost of arc a in the residual network. Then the increase in the cost of sending flow around the cycle W is

$$g(\theta) = \Sigma_{a \quad W} \; c_a\theta + f_a(\theta)^2 - f_a(0)^2$$

Since f_a is convex in θ, $f_a(\theta)^2$ is convex in θ, from which it follows that $\Sigma_{a \quad W} \; c_a \theta + f_a(\theta)^2 - f_a(0)^2$ is convex in θ.

To see whether there is a negative cost cycle, we need only check to see whether some cycle satisfies the condition $g'(0) < 0$, where $g'(0)$ is the right derivative of g evaluated at 0. The right derivative of $f_a(\theta)^2$ at $\theta = 0$ is $2 f_a(0)$, i.e., it is twice the current excess. Thus, suppose we look for a negative cost cycle in the residual network, where the cost associated with arc a is $c_a + 2 f_a(0)$. This network has a negative cost cycle if and only if there is a unit flow y and a value θ such that $F(x + \theta y) < F(x)$.

(b) One algorithm is to search for a negative cost cycle and send an optimal amount of flow around the cycle. (An optimal amount of flow is one that minimizes $F(x + \theta y)$ over all θ such that the flow is feasible. This is an easy one dimensional problem to solve, in principle). An alternative is to find a cycle whose cost is sufficiently negative, such as the one that minimizes the minimum cycle mean. We could also develop alternative approaches for identifying and sending flow around negative cost cycles.

EXERCISE 17.35. (a) If x^k is any feasible solution to the formulation (17.12), define $r^k = \mathbf{A}x^k$. Then r^k and x^k are feasible in the resource-allocation formulation. Conversely, if r^k and x^k are feasible in the resource-allocation formulation, then x^k is feasible in the formulation (17.12). Therefore, the two models are equivalent.

(b) The algorithm is essentially the same. We now define the kth subproblem as

$$z^k(r^k) = \min c^k x^k$$

subject to

$$\begin{aligned} \mathbf{D}^k x^k &\leq b \\ \mathbf{A}^k x^k &\leq r^k \\ x^k &\geq 0. \end{aligned}$$

$z(r) = \Sigma_{1 \leq k \leq K} z^k(r^k)$ is still piecewise linear and convex. If $\pi \leq 0$ and $\mu^k \leq 0$ are optimal dual variables to this linear program, then by linear programming duality, $z^k(r^k) = \pi b + \mu^k r^k$. For any other value r^{-k} of the resource allocation, linear programming duality implies that $z^k(r^{-k}) \geq \pi b + \mu^k r^{-k}$. Therefore, $z^k(r^{-k}) \geq z^k(r^k) + \mu^k (r^{-k} - r^k)$, and so μ^k is a subgradient of $z^k(r^k)$. Using this value of the subgradient, we can solve the problem by the procedure discussed in the text.

EXERCISE 17.37. For any choice of multipliers μ on the constraints $\Sigma_{1 \le k \le K} \mathbf{A}^k x^k \le b$, the subproblem decomposes into k subproblems:

Minimize $(c^k - \mu \mathbf{A}^k)x^k$

subject to (17.12c) and (17.12d).

Let $x^{k,1}, x^{k,2}, \ldots, x^{k,q_k}$ be any set of solutions of the k^{th} master problem. The restricted master problem becomes

$$\min \quad \Sigma_{1 \le k \le k} c^k (\Sigma \lambda^{k,j} x^{k,j})$$

subject to

$$\Sigma_{1 \le k \le K} \mathbf{A}^k (\Sigma_{1 \le j \le q_k} x^{k,j} \lambda^{k,j}) \le b \quad \text{(i)}$$

$$\Sigma_{1 \le j \le q_k} \lambda^{k,j} = 1 \text{ for all } k \quad \text{(ii)}$$

$$\lambda^{k,j} \ge 0 \text{ for all } k \text{ and } 1 \le j \le q_k.$$

This is a linear program in the variables $\lambda^{k,j}$. Once we solve it, we determine a new set of multipliers as the optimal dual variables for the constraints (i). Using these dual variables, we solve the subproblems, generating an optimal solution $\bar{x}^k$ for each k. If σ^k is the optimal dual variable for the constraint (ii), we check the optimality condition.

$$c^k \bar{x}^k - \mu \mathbf{A}^k \bar{x}^k - \sigma < 0.$$

If $\bar{x}^k$ satisfies this condition, we set q_k to q_k+1, $x^{k,\, q_k+1} = \bar{x}^k$ and add $x^{k,\, q_k+1}$ to the restricted master problem.